Lorentzian manifolds and the definition of space-time in Special Relativity

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October 7, 2020

1 Introduction

This essay is attempting to have a brief interruption on how space-time is modeled in Special Relativity and the ultimate goal of this essay is to reach one of the famous multiplier in Relativity

$$D = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where v is the speed of a particle and c is the speed of light. This multiplier is used, for example, in the high school physics courses when calculating the mass of a rocket when speed of it is relatively large. One would be lacking in enough mathematical tools to imply this formula at that moment, but not now.

2 Tensors

Semi-Riemannian geometry involves a particular kind of (0, 2) tensor on tangent spaces. Therefore, to define a (0, 2) tensor for a Semi-Riemannian manifold M, we are going to define a bilinear form for each point $p \in M$. In general, let V be a real vector space with finite dimensional.

Definition 2.1. *A symmetric bilinear form on V is an R-bilinear map* $b : V \times V \to R$, *s.t.,* $b(v,w) = b(w,u)$ *for all* u, w.

Definition 2.2. *A symmetric bilinear form b on V is*
(1) positive [negative] definite if $u \neq 0$ *implies* $b(u, v) > 0$ $[< 0]$,
(2) positive [negative] semidefinite if $b(v, v) \geq 0$ $[\leq 0]$ *for all* $u \in v$.
(3) nondegenerate if $b(v, w) = 0$ *for all* $w \in V$ *implies* $u = 0$.

If b is a symmetric bilinear form on V then for any subspace W of V the restriction $b|_{(WxW)}$, denoted merely by $b|_W$, is again symmetric and bilinear. If b is [semi-] definite, so is $b|_W$.

Definition 2.3. *The index* v *of a symmetric bilinear form b on V is the largest integer that is the dimension of a subspace* $W \subset V$ *on which* $b|_W$ *is negative definite.*

Thus $0 \leq v \leq dimV$, and $v = 0$ if and only if b is positive semidefinite. The function $q : V \to R$ given by $q(v) = b(v, u)$ is the associated quadratic form of b, and

$$b(v,w) = \frac{1}{2}[q(u+w) - q(u) - q(w)].$$

Definition 2.4. *A scalar product g on a vector space V is a nondegenerate symmetric bilinear form on V. The space V equipped with a scalar product g is called scalar product space.*

Definition 2.5. *a non-zero vector $v \in V$ s.t., $q(v) = 0$ is called $null$. Evidently null vector exist if and only if g is indefinite.*

Definition 2.6. *Vectors $v, w \in V$ are $orthogonal$, denoted as $v \perp w$, if $g(v, w) = 0$. Subsets $A, B \in V$ are $orthogonal$, denoted as $A \perp B$, if $g(A, B) = 0$*

Then let W be a subspace of V. Denote

$$W^{\perp} = \{v \in V : v \perp W\}.$$

We have $W^{\perp} + W \neq V$ but following properties:

Lemma 2.1. *If W is a subspace of a scalar product space (V, g), then*
(1) $dim\ W + dim\ W^{\perp} = dim\ V$.
(2) $(W^{\perp})^{\perp} = W$.

Lemma 2.2. *A subspace W of V is nondegenerate if and only if $V = M + M^{\perp}$.*

Lemma 2.3. *A scalar product space $V \neq 0$ has an orthonormal basis.*

Lemma 2.4. *Let $e_1, \ldots, e_n$ be an orthonormal basis for V, with $\epsilon_i = g(e_i, e_i)$. Then each $v \in V$ has a unique expression*

$$v = \sum_i \epsilon_i g(v, e_i) e_i.$$

The $orthogonal\ projecttion$ π of V onto a nondegenerate subspace W is the linear transformation that sends $W^{\perp}$ to 0 and leaves each vector of W fixed. Note that V is not necessarily the direct sum of W and $W^{\perp}$. Then given an orthonormal basis $e_1, \ldots, e_k$ for W, we have

$$\pi(v) = \sum_{i=1}^{k} \epsilon_i g(v, e_i) e_i$$

Lemma 2.5. *For any orthonormal basis $e_1, \ldots, e_n$ for V the number of negative signs in the signature $(\epsilon_1, \ldots, \epsilon_n)$ is the index ν of V.*

Definition 2.7. *A linear isomorphism $T : V \to W$ that preserves scalar products is called a linear isometry.*

Lemma 2.6. *Scalar product spaces V and W have the same dimension and index if and only if there exists a linear isometry from V to W.*

3 Semi-Riemannian Manifolds

Now we are going to define a (0, 2) tensor g on a Semi-Riemannian manifold (M, g) for T_pM for each point $p \in M$. The prototype of this tensor is the natural inner product $\langle \cdot, \cdot \rangle$ in Riemannian manifold $(R^n, \langle \cdot, \cdot \rangle)$.

Definition 3.1. *A metric tensor g on a smooth manifold M is a symmetric nondegenerate (0, 2) tensor field on M of constant index ν_p for all $p \in M$.*
We sometimes use $\langle \cdot, \cdot \rangle$ as an alternative notation for g, i.e., $g(v, w) = \langle v, w \rangle \in \boldsymbol{R}$.

Definition 3.2. *A semi-Riemannian (or, pseudo-Riemannian) manifold is a smooth manifold M furnished with a metric tensor g. The common value ν of index g_p on a semi-Riemannian manifold (M,g) is called the index of M, $0 \leq \nu \leq dimM$.*

M is a *Riemannian manifold* if v = 0. and each g_p is a (positive definite) inner product on $T_p(M)$.

Definition 3.3. *M is called a Lorentz manifold if $v = 1$ and $n \geq 2$. The tangent space of a Lorentz manifold at each point p equipped with the scalar product is called the Lorentz Vector Space.*

Definition 3.4. *Let M and N be semi-Riemannian manifolds with metric tensors g_M and g_N. An isometry from M to N is a diffeomorphism $\phi : M \to N$ that preserves metric tensors, i.e., $\phi^*(g_N) = g_M$ (i.e., $\phi^*(q_N) = q_M$).*

Note that since the pull back ϕ is defined point wisely, this definition is still consistent with the definition 2.7.

Example 3.1. *Recall that for each $p \in \mathbb{R}^n$ there is a canonical linear isomorphism from $\mathbb{R}$ to $T_p\mathbb{R}^n$ in terms of natural coordinates, Thus the dot product on $\mathbb{R}^n$ gives rise to a metric tensor called inner product*

$$\langle v_p, w_p \rangle = v_p \cdot w_p = \sum v_i w_i.$$

And ($\mathbb{R}$, $\langle \cdot, \cdot \rangle$ is called Euclidean n-space. Similarly, define another metric tensor

$$\langle v_p, w_p \rangle^* = -\sum_{i=1}^{v} v_i w_i + \sum_{i=1}^{n} v_i w_i$$

with index v. Then $(\mathbb{R}^n, \langle \cdot, \cdot \rangle^*)$, denoted as $\mathbb{R}^n_v$, is called semi-Euclidean space. For $v = 1$ and $n \geq 2$, $(\mathbb{R}^n, \langle \cdot, \cdot \rangle^*)$ is called Minkowski n-space. Note that if $n = 4$ it is the simplest example of a relativistic spacetime and if $v = 0$ it is a Euclidean space.

3.1 Levi-Civita Connection and Parallel Translation

So far, we have defined what a semi-Riemannian manifold (M, g) is and how vectors are measure by metric tensor g. We have also descried the prototypes of Riemannian Manifolds and Semi-Riemannian Manifolds.

Next let X, Y be vector fields on a semi-Riemannian manifold (M, g), we are going to define a new vector field called *covariant derivative* $D_x Y$ on M whose value at each point p is the vector rate of change of Y in the X_p direction.

Definition 3.5. *Let $(\mathbb{R}^n, x^1, \ldots, x^n)$ be the natural chart about $p \in \mathbb{R}^n_v$. Let X, Y be vector fields on $\mathbb{R}^n_v$ s.t., $X = \sum a^i \frac{\partial}{\partial x^i}$ and $Y = \sum b^i \frac{\partial}{\partial x^i}$. Then define*

$$\nabla_X Y = \sum X(b^i) \frac{\partial}{\partial x^i}.$$

We call it the covariant derivative of Y with respect to X.

We have sort of shown some of the properties of $\nabla_X Y$ is assignment 5 and assignment 7. However, since we may not have a nice global chart and respective geometry on a semi-Riemannian manifold (M, g), to extend this prototype to (M, g) we will begin by its key properties.

Definition 3.6. *A connection D on a smooth manifold M is a map $D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ s.t.,*
(1) $D_X Y$ is $C^\infty(M)$-linear in X.
(2) $D_X Y$ is $\mathbb{R}$-linear in Y.
(3) $D_X(fY) = (Xf)Y + fD_X Y$, for any smooth function $f \in C^\infty(M)$.
And $D_X Y$ is called the covariant derivative of Y with respect to X for the connection D.

Even though there could be many connections in a semi-Riemannian manifold (M, g), the next lemma will show that there is a unique connection sharing some important properties with the natural connection $D_X Y$ on $\mathbb{R}^n_v$.

Lemma 3.1. *Let (M, g) be a semi-Riemannian manifold. Let $X \in \mathfrak{X}(M)$, we define a 1-form X^* by $X^*(Y) = g(X, Y)$. Then the natural map $\Phi : X \longmapsto X^*$ is a $C^\infty(M)$-linear isomorphism from $\mathfrak{X}(M)$ to $\Omega^1(M)$.*

Proof. Since $X^* \in \Omega^1(M)$ is a one-form thus $C^\infty(M)$-linear, we know Φ is $C^\infty(M)$-linear.

Φ is isomorphism by two facts
(a) if $g(X,Z) = g(Y,Z)$ for all $Z \in \mathfrak{X}(M)$, then $X = Y$.
(b) Given any one-form $\theta \in \Omega^1(M)$, there is a unique vector field $Z \in \mathfrak{X}(M)$ s.t., $\theta(X) = g(Z,X)$ for all $X \in \mathfrak{X}(M)$. ☐

Theorem 3.2. *Let (M,g) be a semi-Riemannian manifold. Then there is a unique connection D on M if it satisfies:*
(4) $[X,Y] = D_X Y - D_Y X$,
(5) $Z(g(X,Y)) = g(D_Z X, Y) + g(X, D_Z Y)$,
for all $X,Y,Z \in \mathfrak{X}(M)$. This unique connection D is called Levi-Civita connection of M, and it will characterized by the Koszul formula:

$$\begin{aligned} 2g(D_Z X, Y) =& X(g(X,Y)) + Y(g(Y,Z)) - Z(g(Z,X)) \\ & - g(Z,[X,Y]) + g(X,[Z,Y]) + g(Y,[Z,X]) \end{aligned}$$

Proof. When (4) and (5) are satisfied, the Koszul formula can be shown by simple computation. Then give the fact (a) in the previous proof, it is easy to show $g(D_Z X, Y) = g(D'_Z X, Y)$, thus showing D is unique.

We can check the Right-Hand-Side of the Koszul formula for (1) -(5) to show the existence. ☐

Definition 3.7. *A vector field X is parallel if its covariant derivatives $D_Z X = 0$ for all $Z \in \mathfrak{X}(M)$.*

Example 3.2. *The natural connection ∇ is the Levi-Civita connection of semi-Euclidean space $\mathbb{R}^n_\nu$ for any $\nu = 0,1,\ldots,n$. Further, we have*
(1) $g(\partial_i, \partial_j) = \sigma_{ij}\epsilon_j$, where $\epsilon_j = -1$ for $1 \le j \le \nu$ and $+1$ for $\nu + 1 \le j \le n$.
(2) $\Gamma^k_{ij} = 0$, for $\nabla_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k$.
for all $1 \le i,j,k, \le n$. Here, the function Γ^k_{ij} is called the Christoffel symbols for thee coordinate chart. It can be extended to a local chart for a semi-Riemannian manifold.
Then we know that any coordinate vector field in $\mathbb{R}^n_\nu$ is parallel.

Lemma 3.3. *For a curve $\alpha : I \to M$, let $t \in I$ and $z \in T_{\alpha(t)}M$, then there is a unique parallel vector field Z on α s.t., $Z(\alpha(t)) = z$.*

In the notation of the lemma, let $s \in I$, then the map

$$P = P^s_t(\alpha) : z \longmapsto Z(s)$$

from T_pM to T_qM is called parallel translation along α from $p = \alpha(t)$.

Lemma 3.4. *Parallel translation is a linear isometry.*

3.2 Geodesic and Exponential Map

Next, we are going to develop a significant tool to describe Lorentz manifolds and Special Relativity.

Intentionally, a geodesic is a curve representing in some sense the shortest path between two points in a semi-Riemannian manifold with its metric tensor. It is a generalization of the notion of a "straight line" to a more general setting. In the semi-Riemannian language, it is parallel.

Definition 3.8. *A geodesic in a semi-Riemannian manifold (M,g) is a curve $\gamma : I \to M$, s.t., γ' is parallel. Equivalently, the acceleration of the curve $\gamma'' = 0$.*

Lemma 3.5. *Let γ be a non-constant geodesic. A re-parametrization $\gamma \circ h : J \to M$ is a geodesic if and only if h has the form $h(t) = at + b$.*

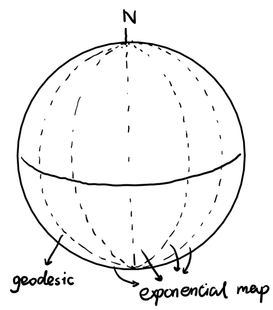

Figure 1: Exponential Map on Earth

Let a semi-Riemannian manifold M be given. At each point $o \in M$ we want to collect the geodesics starting at o into a single mapping. The prototype of this map is the exponential map of the Earth in Fig 1. Need to note that a nice analogy between geodesics and the exponential map is maximal integral curves and flow[1].

Definition 3.9. *Let $o \in M$ and $\mathscr{Q}_o$ represent the set of vectors $v \in T_oM$ s.t., the maximal geodesic γ_v is defined at least on [0,1]. The exponential map of M at o is defined by*

$$exp_o : \mathscr{Q}_o \to M$$

s.t., $exp_o(v) = \gamma_v(1)$ for all $v \in \mathscr{Q}$.

Example 3.3 (Exponential Maps for $\mathbb{R}^n_v$.)**.** *The geodesic with initial velocity $v_p \in T_pM$ is the straight line $t \longmapsto p + tv$. Thus the exponential map at p sends v_p to $p + v$. It thus follows that exp_p is a diffeomorphism due to the canonical isomorphism $T_p(\mathbb{R}^n_v) \simeq \mathbb{R}^n_v$.*

Lemma 3.6. *If $(U, x^1, \ldots, x^n)$ is a chart at $o \in M$, then for all i,j,k, $g_o(\partial_i, \partial_j) = \sigma_{ij}\epsilon_j$ and $\Gamma^k_{ij} = 0$.*

Proof. This is the application of the parallel translation and the property of the Levi-Civita connection on semi-Euclidean space $\mathbb{R}^n_v$. ☐

Intuitively speaking, the exponential map takes a given tangent vector to the manifold, runs along the geodesic starting at that point and goes in that direction for a unit time. Since v corresponds to the velocity vector of the geodesic, the actual (Riemannian) distance traveled will be dependent on that.

4 Lorentz Geometry

4.1 Gauss lemma

There are two geometries specifically important for spacetime. Riemannian geometry with $v = 0$ and Lorentz geometry $v = 1$. Given the exponential map exp_o carries curves $t \longmapsto tx$ in T_pM, the following lemma implies that orthogonality to radial directions is also preserved.

Lemma 4.1 (Gauss lemma)**.** *Let $o \in M$ and non-zero vector $x \in T_pM$. if $v_x, w_x \in T_x(T_pM)$ with v_x radial, i.e., v_x is a scalar multiple of x, then*

$$g(d(exp_o(v_x)), d(exp_o(w_x))) = g(v_x, w_x)$$

Proof. Trivial.[2]. ☐

[1] I am not sure about this, but this is based on my feeling

[2] (Joke) In fact, I am not able to follow the proof at all. But this lemma is so important and need to be used later on

In particular, the Gauss lemma describes the exponential map expo as a kind of partial isometry whose principal distortions are in directions orthogonal to radial directions in T_oM.
Using the Gauss lemma we can set up a detailed comparison between a neighborhood of $o \in M$ and the corresponding neighborhood of $0 \in T_oM$.

4.2 Riemannian Distance

Now given the metric tensor g and geodesics defined, we want to know how to measure the distance between in a semi-Riemannian manifold.

Definition 4.1. *Denote that* $||\alpha|| = \sqrt{|\langle \alpha, \alpha \rangle|}$.
For any points p and q on M, the Riemannian distance d(p,q) is the greatest lower bound of $\{L(\alpha) : \alpha \in \Omega(p,q)\}$*, where* $L(\alpha)$ *is the arc-length of* $\alpha : [a,b] \to M$ *defined by*

$$L(\alpha) = \int_a^b ||\alpha'(s)|| ds$$

and $\Omega(p,q)$ *is the set of all piece-wise smooth curve segments in M from p to q.*

Such definition makes sure that $d(\cdot,\cdot)$ satisfies positive definiteness, symmetry and triangle inequality, as what its prototype - distance between p and q in $\mathbb{R}^n$.

4.3 Lorentz Causal Character and Timecone

Having defined the distance of semi-Riemannian Manifold, the next step is to define what we mean by time. Denote $\mathcal{T}$ as the set of all timelike vectors in a Lorentz vector space V.

Definition 4.2. *A vector* $v \in T_pM$ *is*
(1) spacelike if $\langle v, v \rangle > 0$ *or* $v = 0$,
(2) null if $\langle v, v \rangle = 0$ *and* $v \neq 0$,
(3) timelike if $\langle v, v \rangle < 0$.
The set of all null vectors in T_pM *is called the nullcone at* $p \in M$*. The category which a given tangent vector falls is called its causal character.*

Definition 4.3. *Let* $u \in \mathcal{T}$*. Then*

$$C(u) = \{v \in \mathcal{T}, \quad g(u,v) < 0\}$$

is called the timecone of V containing u. Note that since $u^\perp$ *is spacelike,* $\mathcal{T} = C(u) \cup C(-u)$.

Lemma 4.2. *Let v and w be timelike vectors in a Lorentz vector space. Then if v and w are in the same timecone of V, there is a unique number* $\varphi \geq 0$ *s.t.,* $\langle u, v \rangle = -||v|| \cdot ||w|| cosh\varphi$.
This unique number φ *is called Lorentz angle between v and w.*

Timecone is the extension of orientation defined in vector space. Similarly, v, w in a Lorentz vector space are in the same timecone if and only if $g(v,w) < 0$.

Then it follows that

Definition 4.4. *A function* τ *is called time-orientation of M if it assigns to each point p a timecone* τ_p *and smooth on M.*

and similarly in oriented-manifold in our lectures,

Lemma 4.3. *A Lorentz manifold M is time-orientable if and only of there exists a timelike vector field* $X \in \mathfrak{X}(M)$.

5 Special Relativity

Now we have set up the stage for special relativity. The next step is going to build up the special relativity model starting from the following definitions.

Definition 5.1. *A spacetime is a connected time-oriented Lorentz manifold, i.e., the index on a semi-Riemannian manifold M for each $p \in M$ $\nu_p = 1$ and $dimM \geq 2$.*

Definition 5.2. *A Minkowski spacetime M is a spacetime that is isometric to Minkowski 4-space $\boldsymbol{R}_1^4$.*

From now on, M will denote as a Minkowski spacetime (M, g) for simplicity. And the positive time-orientation of M is called the future and its negative is called the past. A tangent vector in a future causal cone XXX is *future-pointing*.

Definition 5.3. *A material particle in M is a timelike future-pointing curve $\gamma : I \to M$ s.t., $|\gamma'(\tau)| = 1$ for all $\tau \in I$. The parameter τ is called the proper time of the particle.*

Definition 5.4. *A lightlike particle is a future-pointing null geodesic $\gamma : I \to M$, i.e., $\langle \gamma', \gamma' \rangle = 0$.*

Similarly with Newtonian space-time, a point $p \in M$ is called *event* and image $\gamma(I)$ is called *worldlines* of γ.

Intuitively, a material particle is a historical curve passing through the spacetime, similarly to the Newtonian space-time. But here we assumed that each particle comes equipped with a clock measuring its proper time, i.e., the logical consequence of taking these postulates together is the inseparable joining together of the four dimensions.

However, notice that the lightlike particle s.t., $\langle \gamma', \gamma' \rangle = 0$ means that it is distinct with material particle. That is to say, they cannot carry a clock. This suits the intuition behind "geodesic", meaning moving under the influence of gravity along.

Definition 5.5. *A Lorentz chart in M is a time-orientation preserving isometry $\ell : M \to \mathbb{R}_1^4$.*

Lemma 5.1. *Given a frame e_0, e_1, e_2, e_3 in T_pM s.t., e_0 is future-pointing, there is a unique Lorentz chart ℓ s.t., $\frac{\partial}{\partial x^i}|_p = e_i$, for all i.*

Since Minkowski spacetime is isometric to $\mathbb{R}_1^4$, we know that M has the similar geometry with R_1^4. That is,

1. There exists a unique geodesic σ s.t., $\sigma(0) = p$ and $\sigma(1) = q$.
2. There exists a isometry $T_pM \simeq T_qM$.
3. Each exponential map $exp_o : T_pM \to M$ is isometry.

Remark: M is a normal neighborhood of each of its points. Thus for all $p, q \in M$ the displacement vector $\vec{pq} = \sigma'(a)$ where σ is the geodesic as in (1) above. Note that $exp(\vec{pq}) = q$.

That is, for an event $p \in M$ the future timecone of p is $\{q \in M$: $\vec{pq}$ is timelike and future-pointing$\}$. This is a solid cone whose boundary for p is the future lightcone of p, $\{q \in M$: $\vec{pq}$ is null and future-pointing$\}$. The union of these two sets is the future causal cone of p. Past analogues are defined similarly.

Now, the term causal becomes clear. It is natural to say that an event p can influence an event q if and only if there is a particle from p to q. By the definition of particles (material and lightlike), it follows from Lemma **??** that (1) The only events that can be influenced by an event p are those in its future causal cone. (2) The only events that can influence an event p are those in its past causal cone.

The biggest difference about relativistic causality between special relativity and Newtonian causality is that given a point $p = (x_0, t_0)$, the past and future fill the whole space-time except for 3-plane $t = t_0$. The consequence would be, the speed of Newtonian rockets can go from x_0 to any distant x in arbitrarily short time $t - t_0$. But now it is restricted in within the future causal cone of p!

Definition 5.6. *For $p, q \in M$ the number $pq = ||\vec{pq}|| \geq 0$ is called the separation between p and q. In terms of Lorentz chart, we have*

$$pq = [| - (x^0 q - x^0 p)^2 + \sum_1^3 (x^i q - x^i p)^2 |]^{\frac{1}{2}}.$$

In M there is no natural way to deifne either time or space, but Lorentz chart effect an artificial decomposition as follows.

Definition 5.7. *Let ℓ be a Lorentz chart in M. For each event $p \in M$ the number $x^0(p)$ is called the $\ell - time$ of p and the point $\vec{p} = (x^1(p), x^2(p), x^3(p))$ is called the $\ell - position$ of p.*

Now let $\alpha : I \to M$ be a particle, either material or lightlike, the $\ell - time$ of $\alpha(s)$ is $t = x^0(\alpha(s))$ and $\ell - position$ of $\alpha(s)$ is $(x^1(\alpha(s)), x^2(\alpha(s)), x^3(\alpha(s)))$. Since α is causal and future pointing, we have

$$\frac{d(x^0 \circ \alpha)}{ds} = -\langle \alpha', \frac{\partial}{\partial x^0} \rangle > 0$$

i.e., x^0 is a diffeomorphism of I onto some interval $J \subset \mathbb{R}^1$. Let $u : J \to I$ be the inverse function, then at the $ell - time\ t \in J$, the $\ell - position$ of α is

$$\vec{\alpha}(t) = (x^1 \alpha u(t), x^2 \alpha u(t), x^3 \alpha u(t)).$$

Thus measurements of the particle α in M produce a curve $\vec{\alpha} : J \to \mathbb{R}^3$ is called the $\ell - associated$ *Newtonian particle of* α. And this is exactly what a observer ω observes of α.

Lemma 5.2. *Let γ be a lightlike particle in M. Given a Lorentz coordinate chart, the associated Newtonian particle $\vec{\gamma}$ of γ is a straight line in $\mathbb{R}^3$ with speed 1.*

Proof. Since γ is a geodesic in M, $\ell \circ \gamma$ is a geodesic in $\mathbb{R}^4_1$. Thus γ has affine coordinates

$$x^i \gamma(s) = a_i s + b_i$$

for all i. Hence the $\ell - position$ of γ $\vec{\gamma(s)}$ is a straight line in $\mathbb{R}^3$. Also, since dy/ds is null and dt/ds is positive by the time-orientation, we have

$$v = ||\frac{d\vec{\gamma}}{dt}|| = \frac{||d\vec{\gamma}/ds||}{ds} = 1$$

□

Need to remark that light has the same constant speed 1 relative to every freely falling observer ω. That is, the model fit our hypothesis. The speed of light is 1 (or, $3 \times 10^8 m/s$) whether we observe it or not. Now we are interested in how the material particle be observed, i.e., the parameter s become the proper time τ (i.e., inner-clock) of the material particle.

Lemma 5.3 (High school memorized formula!). *Let ℓ be a Lorentz coordinate chart in M. If $\vec{\alpha}$ is the associated Newtonian particle of a material particle $\alpha : I \to M$, then*

1. *The scalar speed $v = ||d\vec{\alpha}/dt||$ of $\vec{\alpha}$ is $v = tanh\varphi$ where φ is the Lorentz angle between $\alpha' = d\alpha/dt$ and the coordinate vector ∂_0 of ℓ. Note that $0 \leq v < 1$.*
2. *The time τ of α and its $\ell - time\ t$ are related by*

$$\frac{dt}{d\tau} = \frac{d(x^0 \circ \alpha)}{d\tau} = cosh\varphi = \frac{1}{\sqrt{1 - v^2}} \geq 1.$$

Here v and φ are functions of the parameter of α.

Proof. Since α' and ∂_0 are timelike and future-pointing, there is a unique Lorenz angle $\varphi \geq 0$ between them by lemma 4.2, determined by

$$cosh\varphi = \frac{-\langle \alpha', \partial_0 \rangle}{||\alpha'|| \cdot ||\partial_0||} = -\langle \alpha', \partial_0 \rangle \geq 1.$$

Since in the Lorentz coordinate chart ℓ, $\alpha' = \sum \frac{d(x^i \circ \alpha)}{d\tau} \partial_i$, we have

$$\frac{dt}{t\tau} = \frac{d(x^0 \circ \alpha)}{d\tau} = -\langle \alpha', \partial_0 \rangle = cosh\varphi.$$

Therefore, given

$$\langle \alpha', \alpha' \rangle = -(\frac{dt}{d\tau})^2 + ||\frac{d\vec{x}}{d\tau}||^2 = -1,$$

we have

$$||\frac{d\vec{\alpha}}{d\tau}|| = \sqrt{cosh^2\varphi - 1} = sinh\varphi \geq 0.$$

Thus, the observed speed of α relative to ω $v = \vec{a}'$ is

$$v = ||\frac{d\vec{\alpha}}{dt}|| = \frac{||d\vec{\alpha}/d\tau||}{dt/d\tau} = \frac{sinh\varphi}{cosh\varphi} = tanh\varphi.$$

And by MAT137 we know,

$$\frac{dt}{t\tau} = cosh\varphi = \frac{1}{\sqrt{1-v^2}}.$$

□

Example 5.1 (The famous Twin Paradox!)**.** *On their 21st birthday Peter leaves his twin Paul behind on their freely falling spaceship and departs at constant relative speed $u = 24/25$ for a free fall of seven years of his proper time. Then he turns and comes back symmetrically in another seven years. Upon his arrival he is 35 years old but Paul is 71.*

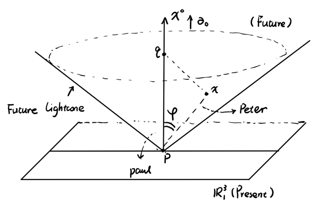

Figure 2: Twin Paradox

Proof. Refer to Fig 2, we know that

$$\frac{1}{2}pq = px \cdot cosh\varphi = \frac{7}{\sqrt{1-(24/25)^2}} = 25,$$

where px represents the proper time τ_{Paul} and pq represents the proper time τ_{Peter}, which is also the observed time t of Peter observed by observer Paul. Therefore, we obtained our result. □

6 Conclusion & Some Thoughts

6.1 Summary

Intuitively, the special relativity, or specifically the Minkowski Spacetime, is modeled out of the perspective of a observer, but the perspective of God (Fig 3), just like the difference between the image accepting by our eyeballs and the image accepting by others' eyeballs but observed from us. And the bound of our eyeballs is built by the basic hypothesis that the speed of light does not change in any direction. Therefore, any material particle we observed is curved and extended, and curved heavier the faster it is, thus different from the proper value of it in the God perspective.

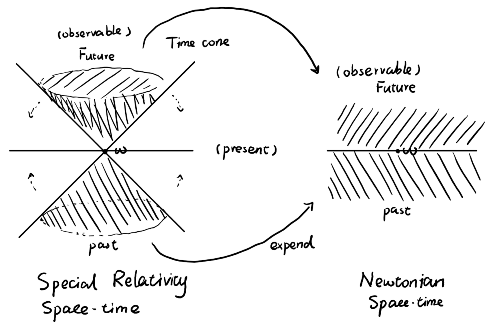

Figure 3: Special Relativity Spacetime v.s. Newtonian Spacetime

6.2 Optional reading

My primary motivation for choosing this topic, and in fact, choosing MAT367, is to prepare for understanding the two biggest names in Physics – Quantum Mechanism & General Relativity. And this is my first attempt to rigorously understand the non-Newtonian model. Having left my high-school physics world for nearly 4 years and majored in Economics, this dream has never left my mind. Therefore, I really appreciate this chance you gave me.
Honestly, I originally thought that after learning through the special relativity, I would be able to get the answer "WHY" the speed of time does not change, "WHY" the time of my high-speed brother will become slow, and "WHY" my high-speed brother will become heavier.However, instead of answering those questions, the special relativity just hypothesis that the time does not change and then making a model fit it. I am really surprised and disappointed with it at the beginning.
But then I realized that this is the same as Newtonian space-time. People generalized the outcome of all sorts of experiments via formulas and parameters, and then models the world to explain those experiments outcome. The big difference is that this time, they are modeling on a different point of view from the usual one.

Could this model or the tools in MAT367 be used in Economics? I guess not. But this is the question I nearly asked myself everyday in the summer. Economics are not complex in terms of models we use. But I guess this kind of philosophy can be used in my future academic life. Clearly, any Economic Prof will claim that an Euclidean space would be enough to model most of the phenomenon in our society, yet those Newtonian in 18 centuries had the same thoughts. Perhaps there is some weird, unexplainable phenomenon in social science that need to be model by jumping out from the usual perspective, just like what we did in the Minkowski spacetime? And when that day comes, there will be a new Economics called "Tianhaonian Economics" and I would be named after Adam Smith, John Forbes Nash and Karl Marx.

$\Sigma(\circ_{\triangle}\circ||) \leftarrow$ wake up from the daylight dream[1]

[1] Exactly 10 pages! I did not intent to do so, but now this is the return for those long assignments haha.(Joke)

7 Reference

1. "Semi-Riemannian geometry with applications to relativity" by O'Neil.
2. "Light cone, Exponential map and Spacetime" by Wiki.