W24

EECS 203 Discussion 2

Predicates, Quantifiers, and Intro to Proofs

Important Forms

- Two beginning-of-semester surveys on Canvas
  - **FCI BoT** Survey and **Better Belonging in Computer Science (BBCS) Entry** Survey
  - **Due:** Friday, Feb. 2nd @11:59pm
- Exam Date Confirmation Survey
  - **Due:** Friday, Feb. 2nd @11:59pm
  - Please fill this out, even if you don’t have an exam conflict!
- They are each worth a few points, so make sure to fill them out!

Upcoming Homework

- Homework/Groupwork 2 will be due **Feb. 1st**
  - **Don’t forget to match pages!**
  - Please note as soon as you press submit you’ve successfully submitted by the deadline. **You can still match pages** with no rush without adding to your submission time.
- Groupwork
  - Groupwork can be done alone, but the problems tend to be more difficult, and the goal is for you to puzzle them out with others!
  - Your discussion section is a great place to find a group!
  - There is also a pinned Piazza thread for searching for homework groups.

Predicates & Quantifiers

Predicates & Quantifiers

- **Predicate:** A sentence or mathematical expression whose truth value depends on a parameter, and becomes a proposition when the parameter is specified.
    - **Example:** **$x > 10$** predicate that depends on parameter x

- **Universal Quantifier:** Denoted by **∀** and translated as **“for all"**, it specifies that the following propositional function is true for all possible parameters in the domain.
    - **Example:** Let x be a positive integer. **$\forall x \ [x > 0]$**

- **Existential Quantifier:** Denoted by **∃** and translated as **“there exists"**, it specifies that the following propositional function is true for at least one parameter in the domain.
    - **Example:** Let x be an integer. **$\exists x \ [x = 3]$**

Quantifiers Continued

- **Nested Quantifiers:** A nested quantifier is a quantifier that involves the use of two or more quantifiers to quantify a compound proposition P(x,y). In nested quantifiers, order matters…
  - **P(x,y):** some statement about x and y
  - **Example:** $\forall x \exists y\, P(x,y)$ is different from $\exists y \forall x\, P(x,y)$
    - **$\forall x \exists y\, P(x,y)$:** "For all x, there exists y such that…"
    - **$\exists y \forall x\, P(x,y)$:** "There exists y such that for all x…"

- **De Morgan's Laws for Quantifiers:**
  - $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$
  - $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

Problem 1

1. Quantifiers and Negations $\star$

Find the negation of each of these propositions. Simplify so that your answers do not include the negation symbol.

a) $\exists x[-4 < x \leq 1]$

b) $\forall z \exists x \exists y[x^3 + y^3 = z^3]$

Problem 2

2. Quantified Statement Counterexamples

Find a counterexample, if possible, to these quantified statements, where the domain for all variables is integers.

a) $\forall x \exists y (x = 1/y)$

b) $\forall x \exists y (y^2 - x < 100)$

c) $\forall x \forall y (x^2 \neq y^3)$

Problem 3

3. Quantifier Translations ⋆

Let $P(x)$ be "x is perfect"; let $F(x)$ be "x is your friend"; and let the domain of quantifiers be all people. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.

b) Not everyone is perfect.

c) All your friends are perfect.

d) At least one of your friends is perfect

e) Everyone is your friend and is perfect.

f) Not everybody is your friend or someone is not perfect.

Intro to Direct Proofs

Introduction to Proofs

A **proof** (or *argument*) for a statement **S** is a sequence of statements ending with **S** (the **conclusion**).

A proof starts with some beginning statements you assume are true, called the **premises**.

A proof is **valid** if every statement after the premises is implied ($\rightarrow$) by the some combination of the statements before it.

Then, whenever the premises are true, the conclusion **must be true**.

Proof Methods

- **Direct Proof:** Proves $p \rightarrow q$ by showing: $\mathbf{p \rightarrow stuff \rightarrow q}$
- **Proof by Contraposition:** *next week!*
- **Proof by Contradiction:** *next week!*
- **Proof by Cases:** *next week!*

Some Useful Definitions

***Note:** **iff** stands for if and only if ($\leftrightarrow$)

- **Even:** An integer **x** is even iff there exists an integer k such that **x = 2k**
  - **Mod Definition:** An integer x is even iff $x \equiv 0 \pmod{2}$

- **Odd:** An integer **x** is odd iff there exists an integer k such that **x = 2k + 1**
  - **Mod Definition:** An integer x is odd iff $x \equiv 1 \pmod{2}$

Problem 4

4. Odd Proof

Prove or disprove: The product of two odd numbers is odd.

Problem 5

5. Even Proof

Prove (using a direct proof) that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even.

Disproof

To **disprove** a statement means to **prove the negation** of that statement:

Disprove $P(x) \equiv$ Prove $\neg P(x)$

Note that if the statement you are trying to disprove is a for-all statement, all you need to disprove it is a singular counter example since $\neg \forall x P(x) \equiv \exists x \neg P(x)$.

Example: Disprove it's raining today ≡ **Prove** it's not raining today 🌞

Example: Disprove $P \rightarrow Q \equiv$ **Prove** $\neg(P \rightarrow Q) \equiv \neg(\neg P \vee Q) \equiv (P \wedge \neg Q)$

More Useful Definitions

*Note: **iff** stands for if and only if ($\leftrightarrow$)

- **Rational:** A number **x** is rational iff it can be written as the quotient of two integers. **$x = p/q$**
- **Irrational:** Not rational–cannot be written as the quotient of two integers
- **Prime:** A prime number **p** is a number greater than 1 whose only factors are 1 and itself. $\forall x\ [x|p \rightarrow (x=1 \lor x=p)]$
- **Composite:** A whole number **p** is composite if it has at least one divisor other than 1 and itself. $\exists x\ [x\neq 1 \land x\neq p \land x|p]$

Problem 6

6. Negation Station

For each of the following statements, write the statement's negation. Then, determine which is true: the original statement or the negated statement? (You do not need a rigorous proof.)

Reminder: Two numbers, x and y, are **multiplicative inverses** if $xy = 1$.

a. For all real numbers x and y, if $x + y = 0$, then one of them is negative and the other is positive.

b. For all nonzero rational numbers x and y, if they are multiplicative inverses, then $x \neq y$.

c. Each non-zero rational number has a rational multiplicative inverse.

d. Each non-zero integer has an integer multiplicative inverse.

Proving "For All" and "There Exists" Statements

<u>Claim:</u> For all x, P(x).

- Start with an arbitrary domain element
- End with the statement inside the "for all"

<u>Sample Language:</u>

Let x be an **arbitrary** domain element

… (make some deductions) …

Thus, P(x).

Therefore, P(x) holds for all x in the domain.

<u>Claim:</u> There exists an x such that P(x).

- Name a specific domain element
- Show that the named value satisfies the claim

<u>Sample Language:</u>

Consider x = __ [specific domain element]

… show that P(x) holds for that value of x.

Problem 7

7. Quantifier Proofs

Building on the last question, prove or disprove each of the following statements.
(If you find it helpful to translate the statements to logical connectives and symbols first, you can, but it's not required that you; you can just work with the English statements directly.)

a. For all nonzero rational numbers x and y, if they are multiplicative inverses, then $x \neq y$.

b. Each non-zero rational number has a multiplicative inverse that is also a rational number.

More Useful Definitions

*Note: **iff** stands for if and only if ($\leftrightarrow$)

- **Divisibility:**
  - The statement **n|a** means "**n divides a**".
  - *In other words*, "**a is divisible by n**".
  - *In other words*:
    - **n|a** iff **$\exists$ k (nk = a)**, *where n, a, and k are integers.*

Problem 8

8. Divides Proof

Prove that if n is odd, then $4|(n^2 - 1)$.