EECS 203: Discrete Mathematics
Winter 2024
Discussion 9 Notes

1 Counting

1.1 Combinations and Permutations

- **Product Rule:** Suppose you want to count the ways of doing one thing, and then another, and then another.... If there are n_1 ways to do the first thing, n_2 ways to do the second thing, ..., and n_k ways to do the last thing, then in total, there are

  $$n_1 \cdot n_2 \cdots n_k$$

  ways to do the entire thing.

- **Sum Rule:** Suppose you want to count the number of ways of choosing to do one thing, or another thing, or ... (but you can't do more than one). If there are n_i ways to do the ith thing, then there are

  $$n_1 + n_2 + \cdots + n_k$$

  ways to do the entire thing.

- **Subtraction Rule:** If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

- **Division Rule:** If there are N ways to choose an object, and each object can be chosen in exactly k ways, there are N/k objects.

- **Permutation:** $P(n, k)$ is the number of ways to choose k things out of n things, where the selection order matters.

  $$P(n, k) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

  Note that we are selecting objects **without replacing** them. (So if I draw a 2 of hearts, that card is no longer in the deck when I draw the next one.)

- **Combination:** $C(n,k)$ is the number of ways to choose k things (order doesn't matter) out of n things.

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Note that we are selecting objects **without replacing** them. (So if I draw a 2 of hearts, that card is no longer in the deck when I draw the next one.)

- **Distinguishable:** Different from each other, "labeled"
- **Indistinguishable:** Considered identical, "unlabeled"
- **Binomial Theorem:** Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

1.1.1 Basic Permutations and Combinations

a. How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

b. How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select a set of 47 cards from a standard deck of 52 cards? The order of the hand of five cards and the order of the set of 47 cards do not matter.

c. How many permutations of the letters ABCDEFGH contain the string ABC?

d. How many bit strings of length n contain exactly r 1's?

Solution:

a. Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is $P(100,3) = 100 \cdot 99 \cdot 98 = 970,200$.

b. Because the order in which the five cards are dealt from a deck of 52 cards does not matter, there are $C(52,5) = \frac{52!}{5!47!} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$ different hands of five cards that can be dealt. Note that there are $C(52,47) = \frac{52!}{47!5!}$ different ways to select 47 cards from a standard deck of 52 cards. We do not need to compute this value because $C(52,47) = C(52,5)$.

c. Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters D, E, F, G, and H. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters ABCDEFGH in which ABC occurs as a block.

d. The positions of r 1s in a bit string of length n form an r-combination of the set $1, 2, 3, ..., n$. Hence, there are $C(n, r)$ bit strings of length n that contain exactly r 1s.

1.1.2 Standing in Line $\star$

How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [Hint: First position the men and then consider possible positions for the women.]

Solution:

$$P(8,8) \cdot P(9,5) = (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (9 \cdot 8 \cdot 7 \cdot 6 \cdot 5)$$

First, we position the eight men in a line; we can count the ways to do this using a permutation, which is $P(8,8)$ or $8!$. Then, we position the women. There are 9 spots between the men for the first woman to go. Then, there are 8 spots for the next woman since no two women stand next to each other, and so on, giving $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ or $P(9,5)$. Finally, we combine these two numbers with the product rule.

1.1.3 Forming a Committee

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

Solution: $\binom{10}{3}\binom{15}{3}$

There must be 3 women and 3 men. There are $\binom{10}{3}$ ways to choose 3 men from 10 to be in the committee, and there are $\binom{15}{3}$ ways to choose 3 women. By product rule, the number of ways to choose the committee is $\binom{10}{3}\binom{15}{3}$.

$\binom{10}{3} \cdot \binom{15}{3} = \dfrac{10!}{(10-3)!3!} \cdot \dfrac{15!}{(15-3)!3!}$

1.2 Permutations with Objects of Different Types

Permutations with Objects of Different Types: This is sort of like permutations with *partial* repetition, in that each type has a different number of repetitions. The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ... , and n_k indistinguishable objects of type k, is:

$$\begin{aligned}\binom{n}{n_1, n_2, \ldots, n_k} &= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} \\ &= \frac{n!}{n_1!n_2!\cdots n_k!}\end{aligned}$$

The first term is called a **multinomial coefficient**.

1.2.1 Permutations with Objects of Different Types

How many different strings can be made by reordering the letters of the word SUCCESS?

Solution: Because some of the letters of SUCCESS are the same, the answer is not given by the number of permutations of seven letters. This word contains three Ss, two Cs, one U, and one E. To determine the number of different strings that can be made by reordering the letters, first note that the three Ss can be placed among the seven positions in $C(7,3)$ different ways, leaving four positions free. Then the two Cs can be placed in $C(4,2)$ ways, leaving two free positions. The U can be placed in $C(2,1)$ ways, leaving just one position free. Hence E can be placed in $C(1,1)$ way. Consequently, from the product rule, the number of different strings that can be made is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!}$$

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We can also solve by overcounting and applying the division rule. There are 7 letters, so there are $P(7,7) = 7!$ permutations of 7 letters – but this overcounts since there are 3 S's and 2 C's. We overcount due to S's by a factor of $P(3,3) = 3!$, and due to C's by a factor of $P(2,2) = 2!$. So, applying the division rule, we get

$$\frac{P(7,7)}{P(3,3)P(2,2)} = \frac{7!}{3! \cdot 2!}$$

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1.2.2 Hanging Jerseys

Robert went out to the store and bought 10 Michigan basketball jerseys and 15 Michigan football jerseys to adorn his walls due to Michigan's recent successes. Each jersey has a different player's name so he could tell them apart. However, once he got back to his room he realized that he only had room to hang up 6 jerseys! If he doesn't care where each jersey is positioned on his walls, how many ways are there to select the jerseys that will be put up if:

(a) he would like to hang up more or equal number of football jerseys than basketball jerseys

(b) he would like to hang up an equal number of football and basketball jerseys, but he can't hang up Surya's basketball jersey without also hanging up Ashu's football jersey? **Note: he could hang up Ashu's jersey without hanging up Surya's jersey.**

Provide a brief justification for each part.

Solution:

(a) There are four possible ways to hang up at least as many football jerseys as basketball jerseys: 6 football jerseys and no basketball jerseys, 5 football jerseys and 1 basketball jersey, 4 football jerseys and 2 basketball jerseys, and 3 of each. For each case, we take the total combinations for each decoration and multiply them together to make combinations of both decorations. For example, in case 1, there are $\binom{15}{6}$ football jerseys and $\binom{10}{0}$ basketball jerseys, yielding $\binom{15}{6}\binom{10}{0}$ total combinations for that case. Adding up each case together gives us our total:

$$\binom{15}{6}\binom{10}{0}+\binom{15}{5}\binom{10}{1}+\binom{15}{4}\binom{10}{2}+\binom{15}{3}\binom{10}{3}$$

Alternatively, this problem can be done by subtracting the number of combinations that don't meet our criteria (i.e. more basketball jerseys than football jerseys) from the total number of combinations:

$$\binom{25}{6}-\binom{15}{2}\binom{10}{4}-\binom{15}{1}\binom{10}{5}-\binom{15}{0}\binom{10}{6}$$

(b) There are two possible cases: either he hangs up Surya's jersey (and therefore Ashu's) or he doesn't. In the first case, he will place Surya's and Ashu's jerseys at any spot on the wall, leaving four open spots for two of the other 14 football jerseys and two of the other 9 basketball jerseys: $\binom{14}{2}\binom{9}{2}$. In the other case, there are still 6 open spots: 3 for the 15 football jerseys and 3 for the 9 basketball jerseys which don't have Surya's name (Note: while Surya's jersey cannot be hung up without Ashu's, Ashu's jersey can be hung up without Surya's): $\binom{15}{3}\binom{9}{3}$. The total number of combinations is just the sum of both cases because each case is disjoint:

$$\binom{14}{2}\binom{9}{2}+\binom{15}{3}\binom{9}{3}$$

Alternate Solution:
We can consider 3 cases: Hang up Surya and Ashu ($\binom{14}{2}\binom{9}{2}$), Hang up Ashu without Surya ($\binom{14}{2}\binom{9}{3}$), and neither ($\binom{14}{3}\binom{9}{3}$). The total number of combinations would be the sum of these three cases:

$$\binom{14}{2}\binom{9}{2}+\binom{14}{2}\binom{9}{3}+\binom{14}{3}\binom{9}{3}$$

1.3 More Counting

1.3.1 Letters With Repeats

How many different strings can be made from the letters in ATREYATATA, using all the letters?

Solution: The letter "A" is repeated 4 times and "T" is repeated 3 times.
Number of ways =

$$\frac{10!}{4!3!}$$

Alternate soln: Out of the 10 spots, we need to choose 4 spots for the A, 3 spots for the T and one spot each for Y, E and R = $\binom{10}{4}\binom{6}{3}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \binom{10}{4}\binom{6}{3}3!$

1.3.2 Tea Party

You are having a tea party and have 5 unique tea bags and three mugs. Each mug will have exactly one tea bag in it

(a) How many ways are there to distribute the tea bags if the three mugs are identical?

(b) How many ways are there to distribute the tea bags if the three mugs are unique?

Solution:

(a) Since the three mugs are identical the placement of the three teabags selected does not matter. Therefore our answer is C(5,3)

(b) In this case since the three mugs are now unique we can consider their placement to matter and instead will need to use permutations P(5,3)

2 Probability

- **Distinguishable:** different from each other, "labeled"
- **Indistinguishable:** considered identical, "unlabeled"
- **Experiment:** Procedure that yields an outcome
- **Sample Space:** Set of all possible outcomes in an experiment, usually denoted by S.
- **Event:** A subset of the sample space, usually denoted by E.
- **Probability of an Event (Equally Likely Outcomes):** The probability of an event $E \subseteq S$ is $P(E) = \dfrac{|E|}{|S|}$ given all elements in S are equally likely.

- **Probability of Events:**
  $P(E) = \sum_{s \in E} p(s)$
  Note that $p \in [0, 1]$ is the probability of an element. The probabilities of all elements in a sample space add up to 1:
  $\sum_{s \in S} p(s) = 1$

- **Conditional Probability:** The probability of E_1 given E_2, denoted $P(E_1|E_2)$, is:

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

- **Independence:** Events E_1 and E_2 are independent if

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

  Note that this is NOT the same thing as two events being mutually exclusive.

- **Conditional Probability and Independence:** If $P(E_1) = P(E_1|E_2)$, then E_1 and E_2 are independent (since E_2 doesn't give you any information on E_1).
  More generally,
  Events E_1, E_2, and E_k are independent if
  $P(E_1 \cap E_2 \cap ... \cap E_k) = P(E_1) \cdot P(E_2) \cdots P(E_k)$

2.0.1 Probability Intro

What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 8?

Solution: There are a total of 36 equally likely possible outcomes when two dice are rolled. (The product rule can be used to see this; because each die has six possible outcomes, the total number of outcomes when two dice are rolled is $6^2 = 36$.) There are five successful outcomes, namely, (2, 6), (3, 5), (4, 4), (5, 3), and (6, 2), where the values of the first and second dice are represented by an ordered pair. Hence, the probability that an eight comes up when two fair dice are rolled is 5/36.

2.0.2 Poker Hands

a. Find the probability that a hand of five cards in poker contains four cards of one rank.

b. What is the probability that a poker hand contains a full house, that is, three of one rank and two of another rank?

Solution:

a. By the product rule, the number of hands of five cards with four cards of one rank is the product of the number of ways to pick one rank, the number of ways to pick the four of this rank out of the four in the deck of this rank, and the number of ways to pick the fifth card. This is

$$C(13,1)C(4,4)C(48,1).$$

There are $C(52,5)$ different hands of five cards. Hence, the probability that a hand contains four cards of one rank is

$$\frac{C(13,1)C(4,4)C(48,1)}{C(52,5)} = \frac{13 \cdot 1 \cdot 48}{C(52,5)}$$

Note: This problem can also be solved by treating poker hands as ordered. There are many possible equivalent expressions from solving this way: one is

$$\frac{C(13,1)C(4,4)C(48,1) \cdot 5!}{P(52,5)}$$

b. By the product rule, the number of hands containing a full house is the product of the number of ways to pick two ranks in order, the number of ways to pick three out of four suits for the first kind, and the number of ways to pick two out of four suits for the second rank. (Note that the order of the two ranks matters, because, for instance, three queens and two aces is different from three aces and two queens.) We see that the number of hands containing a full house is

$$P(13, 2)C(4, 3)C(4, 2) = 13 \cdot 12 \cdot 4 \cdot 6$$

Because there are $C(52, 5)$ poker hands, the probability of a full house is

$$\frac{P(13, 2)C(4, 3)C(4, 2)}{C(52, 5)}$$

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2.0.3 Lottery

Find the probability of selecting none of the correct six integers in a lottery, where the order in which these integers are selected does not matter, from the positive integers not exceeding 40.

Solution: There are $C(40, 6)$ total possible entries. If we wish to avoid all the winning numbers, then we must make our choice from the $40 - 6$ nonwinning numbers, and this can be done in $C(34, 6)$ ways. Therefore, since the winning numbers are picked at random, the probability is $C(34, 6)/C(40, 6)$

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2.0.4 Independent vs Mutually Exclusive Events

Two six-sided dice, Dice A and Dice B, are rolled. Using these dice, provide examples of:

(a) A pair of independent events.

(b) A pair of mutually exclusive events.

Recall that events E and F are independent if $P(E \cap F) = P(E) \cdot P(F)$, and they are mutually exclusive if $P(E \cap F) = 0$.

Solution:

(a) Some possible pairs of independent events are:

- $E :=$ Die A rolls a 1; $F :=$ Die B rolls a 1
- $E :=$ Die A is even; $F :=$ Die B is odd

(b) Some possible pairs of mutually exclusive events are:

- $E :=$ Die A rolls a 1; $F :=$ Die A rolls a 2.
- $E :=$ Die A and Die B add to an even number;
  $F :=$ Die A and Die B add to an odd number.

2.0.5 Conditional Probability

A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

Solution: Let E be the event that a bit string of length four contains at least two consecutive 0s, and let F be the event that the first bit of a bit string of length four is a 0. The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 0, equals

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

Because $E \cap F = 0000, 0001, 0010, 0011, 0100$, we see that $p(E \cap F) = 5/16$. Because there are eight bit strings of length four that start with a 0, we have $p(F) = 8/16 = 1/2$. Consequently,

$$P(E|F) = \frac{5/16}{1/2} = 5/8.$$

2.0.6 Independent Events

Assume that each of the four ways a family can have two children is equally likely. Are the events E, that a family with two children has two boys, and F, that a family with two children has at least one boy, independent?

Solution: $P(E) = 1/4$, $P(F) = 3/4$, and $P(E \cap F) = 1/4$. $P(E)P(F) = (1/4)(3/4) = 3/16$. Since $P(E \cap F) \neq P(E)P(F)$, E and F are NOT independent.