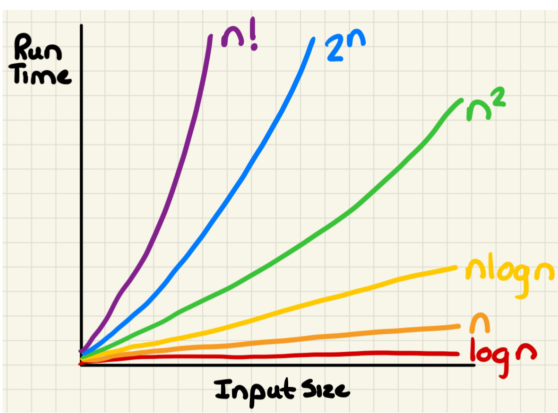
EECS 203: Discrete Mathematics
Winter 2024
FOF Discussion 11 Notes

1 Run Time Analysis

1.1 Definitions

- **Big-O:** f is $O(g)$ means f grows no faster than g...
  $f(n) = O(g(n)) \implies \exists c, k > 0 \; \forall n > k \; \Big[f(n) \leq c|g(n)| \Big]$
- **Big-Ω:** f is $\Omega(g)$ means f grows at least as fast as g...
  $f(n) = \Omega(g(n)) \implies \exists c, k \; \forall n > k \; \Big[|f(n)| \geq c|g(n)| \Big]$
- **Big-Θ:** f is $\Theta(g)$ means f grows at the same rate as g...
  $f(n) = \Theta(g(n)) \implies \exists k, c_1, c_2 \; \forall n > k \; \Big[c_1|g(n)| \leq |f(n)| \leq c_2|g(n)| \Big]$
  - f is $\Theta(g)$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
- **Run Time of Standard Functions:**



- **Properties for Combining Functions:** Consider positive-valued functions $f_1(n) = \Theta(g_1(n))$ and $f_2(n) = \Theta(g_2(n))$.
  - **Addition:** $(f_1 + f_2)(n) = \Theta(\max(g_1(n), g_2(n)))$
  - **Scalar Multiplication:** $af(n) = \Theta(f(n))$
  - **Product:** $(f_1 \cdot f_2)(n) = \Theta(g_1(n) \cdot g_2(n))$

- **Divide and Conquer Algorithm:** An algorithm which divides a problem into smaller **non-overlapping** sub-problems, solves each of those sub-problems, and then combines the results of the sub-problems into the final result.
- **Sub-problem:** A smaller-version of a problem that can be solved by the same algorithm for the larger problem.
- **Master Theorem:** If the runtime for an algorithm can be modeled by a recurrence relation of the form $T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^d)$ where $a > 0$, $b > 1$, and $d \geq 0$, then

$$T(n) \text{ is } \begin{cases} \Theta(n^d) & \text{if } \frac{a}{b^d} < 1 \\ \Theta(n^d \log n) & \text{if } \frac{a}{b^d} = 1 \\ \Theta(n^{\log_b a}) & \text{if } \frac{a}{b^d} > 1 \end{cases}$$

- **Rules for Logarithms:** You should keep the following log rules in mind:
  - $\log(xy) = \log(x) + \log(y)$
  - $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$
  - $\log(x^y) = y\log(x)$
  - $\log_b(a) = \dfrac{\log_c(a)}{\log_c(b)}$ for all $a, b, c \in \mathbb{R}^+$ where $b > 1$ and $c > 1$
  - $\log_b(b) = 1$ for all $b \in \mathbb{R}^+$ where $b > 1$

1.2 Exercises

1.2.1 Big-O

Give a big-O estimate for each of these functions. Use a simple function of the smallest order.

(a) $n \cdot \log(n^2 + 1) + n^2 \cdot \log(n)$

(b) $(n \cdot log(n) + 1)^2 + (log(n) + 1)(n^2 + 1)$

(c) $n^{2^n} + n^{n^2}$

Solution:

(a) $O(n^2 \cdot log(n))$

(b) $O(n^2(log(n))^2)$

(c) $O(n^{2^n})$

1.2.2 Big-Ω, Big-Θ

For each function, determine whether that function is $\Omega(x^2)$ and whether it is $\Theta(x^2)$.

(a) $f(x) = 17x + 11$

(b) $f(x) = x\log x$

(c) $f(x) = 2^x$

(d) $f(x) = x^2 + 1000$

(e) $f(x) = x^4/2$

(f) $f(x) = \lfloor x \rfloor \cdot \lceil x \rceil$

Solution:

(a) Neither $\Omega(x^2)$ nor $\Theta(x^2)$

(b) Neither

(c) $\Omega(x^2)$ but not $\Theta(x^2)$

(d) $\Omega(x^2)$ and $\Theta(x^2)$

(e) $\Omega(x^2)$ but not $\Theta(x^2)$

(f) $\Omega(x^2)$ and $\Theta(x^2)$

1.2.3 Algorithms

Give the tightest big-O estimate for the number of operations (where an operation is arithmetic, a comparison, or an assignment) used in each of the following algorithms:

(a)

procedure $findMax(a_1, a_2, ..., a_N$: real numbers)
  $max := 0$
  for $i := 1$ to N
    if $a_i > max$
      $max = a_i$
  return max

(b)

procedure $sumOddIndices(a_1, a_2, ...a_N$: real numbers)
  $i := 1$
  $oddIndexSum := 0$
  while $i \leq N$
    $oddIndexSum := oddIndexSum + a_i$
    $i := i + 2$
  return $oddIndexSum$

(c)

procedure $findMinPowerAboveN(N$: positive integer)
  $i := 1$
  while $i \leq N$
    $i := i * 2$
  return i

(d)

procedure $findMaxDifference(a_1, a_2, ..., a_N$: real numbers)
  $maxDiff := 0$
  for $i := 1$ to N
    for $j := 1$ to N
      if $a_i - a_j > maxDiff$
        $maxDiff := a_i - a_j$
  return $maxDiff$

(e)

procedure $countElementsGreaterThanMean(a_1, a_2, ..., a_N$: real numbers)
  $sum := 0$
  $numGreaterThanMean := 0$

for $i := 1$ to N
    $sum := sum + a_i$
$mean := sum/N$
for $j := 1$ to N
    if $a_j > mean$
        $numGreaterThanMean := numGreaterThanMean + 1$
return $numGreaterThanMean$

Solution:

(a) $O(N)$. For each element in the list, there is a constant amount of work being done (one comparison, sometimes one assignment). The number of operations ranges from roughly N to $2N$ (depending on how many times it enters the if statement, not counting loop variable arithmetic), which is $O(N)$.

(b) $O(N)$. There is a constant amount of work done for every other element in the list, which is $\frac{N}{2}$ elements, so multiplying this by a constant is still $O(N)$.

(c) $O(\log N)$. Even though this has the same loop bound as the previous problem and the same amount of work within the loop, the way the loop variable is updated affects how many times the loop body is executed. Since the loop variable is multiplied by 2 each time, the number of times the loop is executed is how many times 1 needs to be multiplied by 2 to reach N, which is $\log_2 N$, or $O(\log N)$.

(d) $O(N^2)$. There are two nested loops, and within a single iteration of the outer loop, there is a constant amount work done for each element of the list, which is $O(N)$. Since the list executes N times, the overall complexity is $O(N^2)$.

(e) $O(N)$. Even though there are two loops here as in option (d), they are not nested. So the first loop does a constant amount of work for each element in the list which is $O(N)$, and the second loop also does a constant amount of work for each element in the list, which is $O(N)$. Since the second loop is not done for each iteration of the first loop, but rather done after the first loop is done, we add these instead of multiplying. So we get roughly $2N$ operations, which is still $O(N)$.

1.2.4 Master Theorem

Consider the function f such that

$$f(n) = 2f(\frac{n}{4}) + n, f(1) = 2$$

a) Find $f(16)$.

b) Use the master theorem to find the tightest big-O estimate of f

Solution:

(a) $f(4) = 2f(1) + 4 = 2(2) + 4 = 8$
$f(16) = 2f(4) + 16 = 2(8) + 16 = 32$
$f(16) = 32$

(b) a = 2, b = 4 and d = 1. Since $2 < 4^1$, $f = O(n^d) = O(n)$

2 Exam Review

2.1 Distributing Objects into Bins

For each of the following identify whether the objects/bins are indistinguishable or distinguishable. Then solve the problem.

(a) How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

(b) How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

(c) How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

Solution:

(a) **Distinguishable Objects (cards) and Distinguishable Bins (people)**
The first player can be dealt 5 cards in $C(52,5)$ ways. The second player can be dealt 5 cards in $C(47,5)$ ways, because only 47 cards are left. The third player can be dealt 5 cards in $C(42,5)$ ways. Finally, the fourth player can be dealt 5 cards in $C(37,5)$ ways. Hence, the total number of ways to deal four players 5 cards each is

$$C(52,5)C(47,5)C(42,5)C(37,5) = \frac{52!}{47!5!} \cdot \frac{47!}{42!5!} \cdot \frac{42!}{37!5!} \cdot \frac{37!}{32!5!} = \frac{52!}{5!^4 \cdot 32!}$$

(b) **Distinguishable Objects (employees) and Indistinguishable Bins (offices)**
We will solve this problem by enumerating all the ways these employees can be placed into the offices. We represent the four employees by A, B, C, and D. We can distribute employees into offices in the following groups of numbers:

- 4
- 3,1
- 2,2
- 2,1,1

Now since the employees are distinguishable we must considered the different ways employees (A,B,C,D) can be assigned to these groups.

- 4 (1 way): ABCD
- 3,1 (4 ways): $C(4,1)$ choose the person who's on their own
- 2,2 (3 ways): pair A with someone, other pair follows AB, AC, AD
- 2,1,1 (6 pairs): choose two to be the pair $C(4,2)$

Thus, there are 14 ways to put four different employees into three indistinguishable offices.

(c) **Indistinguishable Objects (book copies) and Indistinguishable Bins (identical boxes)**
We will enumerate all ways to pack the books. For each way to pack the books, we will list the number of books in the box with the largest number of books, followed by the numbers of books in each box containing at least one book, in order of decreasing number of books in a box. The ways we can pack the books are
6
5,1
4,2
4,1,1
3,3
3,2,1
3,1,1,1
2,2,2
2,2,1,1
For example, 4,1,1 indicates that one box contains four books, a second box contains a single book, and a third box contains a single book (and the fourth box is empty). We conclude that there are **nine** allowable ways to pack the books, because we have listed them all and the order of the boxes does not matter (they are indistinguishable).

2.2 Counting

How many ways are there for a horse race with three horses to finish if ties are possible? [Note: Two or three horses may tie.]

Solution: Breaking this down into cases, there are 4 cases: $P(3,3)$ outcomes where there are no ties, $\binom{3}{2}\binom{1}{1}$ outcomes where 2 horses tie for first and 1 horse receives last, $\binom{3}{1}\binom{2}{2}$ outcomes where 1 horse finishes first and 2 horses tie for last, and $\binom{3}{3}$ outcomes where all three horses finish at the same time. In total, we have $P(3,3)+\binom{3}{2}\binom{1}{1}+\binom{3}{1}\binom{2}{2}+\binom{3}{3}=13$ total outcomes.

2.3 Poker Hands

(a) Find the probability that a hand of five cards in poker contains at least 2 Aces.

(b) Find the probability a hand of five cards in poker has exactly one of every face card(Jack, Queen, King).

Solution:

(a) There are 3 cases: our hand contains exactly 2 Aces, our hand contains exactly 3 Aces, and our hand contains exactly 4 Aces.

- 2 Aces:
  First, we have $\binom{4}{2}$ Aces to pick. Next, we have $\binom{48}{3}$ cards to pick(because we can't pick the other 2 Aces). Therefore, we have $\binom{4}{2}\binom{48}{3}$ ways to get a hand with exactly 2 Aces.

- 3 Aces:
  There are $\binom{4}{3}$ Aces to pick. Next, we have $\binom{48}{2}$ cards to pick. Therefore, we have $\binom{4}{3}\binom{48}{2}$ ways to get a hand with exactly 3 Aces.

- 4 Aces:
  There are $\binom{4}{4}$ Aces to pick. Next, we have $\binom{48}{1}$ cards to pick. Therefore, we have $\binom{4}{4}\binom{48}{1}$.

There are $\binom{52}{5}$ ways of picking a hand of 5 cards and each hand has an equal probability of being drawn. Therefore, the probability that we get at least 2 Aces is $\frac{\binom{4}{2}\binom{48}{3}+\binom{4}{3}\binom{48}{2}+\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$.

Alternate Solution: We can also answer this question using subtraction. First, we will find the probability that a hand of five cards in poker contains 1 Ace or 0 Aces.

- 0 Aces:
  Therefore, since our hand contains 0 Aces, we must pick 5 cards from the remaining 48 cards. Therefore, we have $\binom{48}{5}$ ways to pick a hand with 0 Aces.

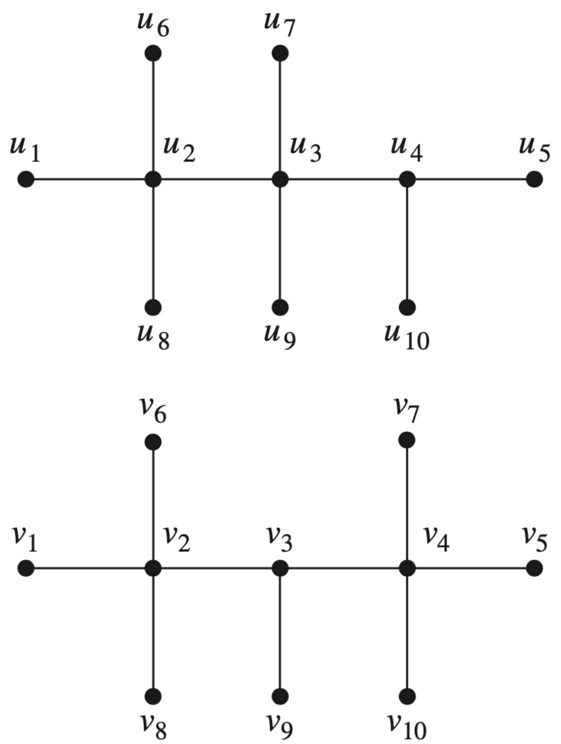
- 1 Ace:
  Therefore, since our hand contains 1 Ace, we must pick $\binom{4}{1}$ Ace and then choose 4 other cards that are not Aces, which is $\binom{48}{4}$. Therefore, we have $\binom{4}{1}\binom{48}{4}$ ways to pick a hand with 1 Ace.

The total number of hands that we can have is $\binom{52}{5}$ and each hand has an equal probability of being drawn, which means that the probability that we do not get at least two Aces in our hand is $\frac{\binom{48}{5}+\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$. Therefore, using subtraction rule, we know that the probability that we do get at least two Aces in our hand is $1-\frac{\binom{48}{5}+\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$.

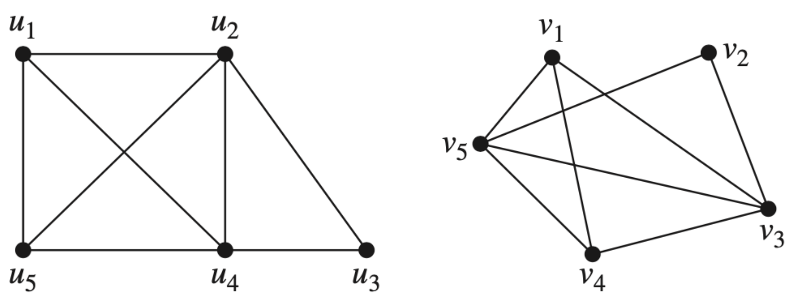
(b) There are $\binom{4}{1}$ ways to choose a King card, $\binom{4}{1}$ ways to choose a Queen card, $\binom{4}{1}$ ways to choose a Jack card, and $\binom{40}{2}$ way to choose the non-face cards (can not be a face card). Therefore, in total, there are $\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{40}{2}$ ways to get exactly one of each face card in a hand. Additionally, we know that there are $\binom{52}{5}$ total hands which all have equal probability of being chosen. Therefore, the probability that the hand contains exactly one of each face card is $\frac{\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{40}{2}}{\binom{52}{5}}$.

2.4 Graph Isomorphisms

Determine whether or not the following graphs are isomorphic and thoroughly justify your answers.



(a)



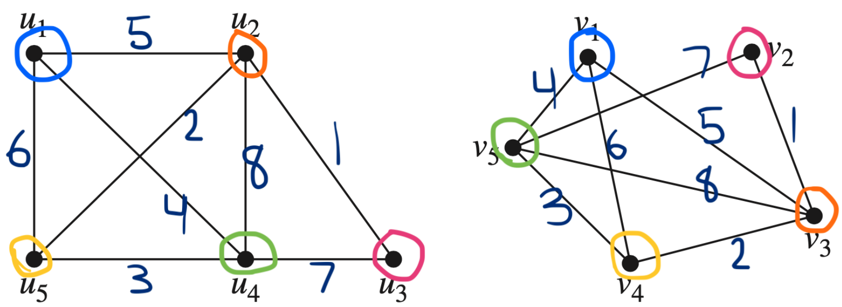
(b)

Solution:

(a) Not isomorphic. The first graph contains an edge between two vertices of degree 4, the second graph does not.

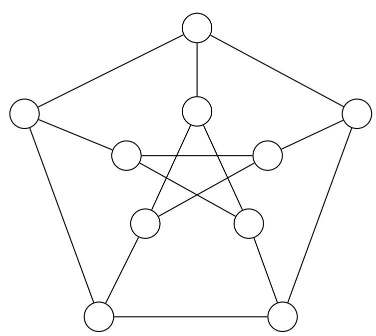
(b) Isomorphic

42.

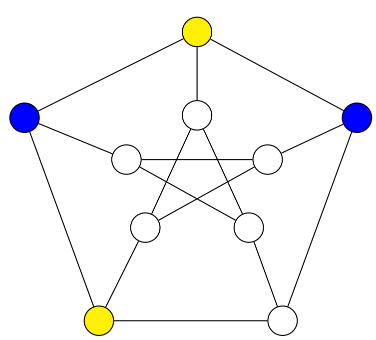


2.5 Coloring

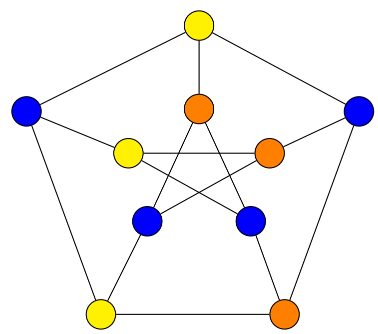
Show that the following graph is 3-colorable and that this is the smallest number of colors needed to color it. Do so by explaining why it is not two colorable and then giving a three coloring.



Solution: The above graph is a representation the famous Petersen graph,suppose the Petersen graph was two colorable, WLOG suppose that the uppermost vertex is yellow(or 1), and that the other color is blue(or 2), then following through coloring properly gives the following



There is a vertex adjacent to both a blue(2) and a yellow(1) vertex and thus cannot be colored either if we require a proper coloring. Alternatively, there are many cycles of odd length, taking the outer pentagon to be explicit. We know that a graph is bipartite iff it has no odd cycles and therefore this graph is not bipartite.
Here is a three coloring:



Note that the notion of the least number of colors required to color a graph is called the *chromatic number*, denoted by χ or $\chi(G)$, of a graph and is quite an important notion in graph theory.

2.6 Exam Scores

The final exam of a discrete mathematics course consists of 50 true/false questions, each worth two points, and 25 multiple-choice questions, each worth four points. The probability that Linda answers a true/false question correctly is 0.9, and the probability that she answers a multiple-choice question correctly is 0.8. What is her expected score on the final?

Solution: Since we want to know the number of points we can use a random variable that assign each outcome to the number of points. So for the true/false getting a question right is assigned to a 2 and getting a question wrong is assigned to a 0. Getting a multiple choice question right is assigned to a 4 and getting a question wrong is assigned to a 0. We end up with...

$50 \cdot X(TF) \cdot p(TF) + 25 \cdot X(MC) \cdot p(MC) = 50(2)(0.9) + 25(4)(0.8) = 170$

2.7 Hat Check Problem

Each of n customers gives a hat to a hat-check person at a restaurant. The hat-check person gives the hats back to the customers in a random order. What is the expected number of customers who get back their own hat?

Solution: Let X be the number of customers who get back their own hat and X_i be the indicator random variable that customer i gets his hat back. The probability that an individual gets his hat back is $1/n$.

$E(I_A) = p(A)$ (formula for indicator variables)
$n \cdot E(I_A) = n \cdot p(A)$ (linearity of expectation)
$n \cdot E(I_A) = n \cdot \frac{1}{n} = 1$

So the expected number of customers who get back their hat is 1.

2.8 Predicting Success

An electronics company is planning to introduce a new camera phone. The company commissions a marketing report for each new product that predicts either the success or the failure of the product. Of new products introduced by the company, 60% have been successes. Furthermore, 70% of their successful products were predicted to be successes, while 40% of failed products were predicted to be successes. Find the probability that this new camera phone will be successful if its success has been predicted.

Solution: Observe we can't use the usual conditional formula and that were given stats about events GIVEN events, thus we use Bayes. Let S be the event "Product is a success", and let A be the event that "the product is predicted to be successful." Then we calculate the probability that the product is a success given that the it was predicted to be a success:

$$\begin{aligned} P(A) =& P(A \mid S) \cdot P(S) + P(A \mid \bar{S}) \cdot \bar{S} \\ =& 0.7 \cdot 0.6 + 0.4 \cdot 0.4 \\ =& 0.58 \\ P(S \mid A) =& \frac{P(A \mid S) \cdot P(S)}{P(A)} \\ =& \frac{0.7(0.6)}{0.58} \\ =& 0.724 \end{aligned}$$