W24

EECS 203 Discussion 5b

Strong Induction & Recurrence Relations

Admin Notes:

- Homework/Groupwork 5 will be due **Mar. 7th – AFTER SPRING BREAK**
  - **Don't forget to match pages!**
  - Please note as soon as you press submit you've successfully submitted by the deadline. **You can still match pages** with no rush without adding to your submission time.

- Exam 1:
  - **Grades will release before the end of the day! (Friday, Feb. 23)**

Weak Induction

Recall Mathematical (Weak) Induction:

We want to show some statement **P(n)** is true for all integers **n ≥ c**.

- **Base Case**
  - First, show that the statement **P(c)** is true for some initial value **c**.
- **Inductive Step**
  - Next, show that if **P(k)** is true for an arbitrary integer **k ≥ c**, then **P(k+1)** is also true.
  - In other words, we want to prove the implication **P(k)** → **P(k+1)**.
  - Since k is arbitrary, we start this step by assuming that **P(k)** is true.
  - When you assume **P(k)**, it's called the **inductive hypothesis**.
- **That's it!**
  - You've proven that **∀(n ≥ c) P(n)**, as desired.
  - Since **P(c)** is true and **P(k)** implies **P(k+1)**, we therefore have:
    P(c) → **P(c+1)** → **P(c+2)** → **P(c+3)** → **P(c+4) . . .**

Problem 1

1. Mathematical Induction - Sets Edition

Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

Solution

1. Mathematical Induction - Sets Edition

Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

Inductive Step:
Let k be an arbitrary integer greater than or equal to 2. Assume $P(k)$: a set with k elements has $k(k-1)/2$ two-element subsets. We want to show $P(k+1)$: a set with $k+1$ elements has $(k+1)k/2$ two-element subsets

Now we consider a set of $k+1$ elements. Take k elements of this set. We know that this set of elements has $k(k-1)/2$ two-element subsets (by our inductive hypothesis). Now if we add in the $(k+1)$th element, it adds k two-element subsets ($(k+1)$th element + one of the k elements that we originally took). So a set of size $k+1$ has $k(k-1)/2+k$ two-element subsets.
$k(k-1)/2+k = k(k-1)/2+2k/2 = [k(k-1)+2k]/2 = k(k-1+2)/2 = (k+1)k/2$.
Therefore, $P(k+1)$ is true.

Base Case:
Prove $P(2)$: a set with 2 elements contains $(2 \cdot 1)/2$ two-element subsets. A set with 2 elements only has one two-element subset: the set itself.
$1 = 2/2 = 2(2-1)/2$ ✓

Alternatively, we could complete this proof with a base case of $P(1)$ or even $P(0)$. Because the problem didn't specify any domain, any base case that allows the inductive step to prove the statement for everything 2 and up is sufficient.

Thus, by mathematical induction, a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

Strong Induction

Strong Induction

As before, we want to show some statement **P(n)** is true for all integers **n ≥ c**.

- **Inductive Step**
  - Show that if P(j) is true for **c ≤ j ≤ k** , then **P(k+1)** is true
    P(c), … , P(k) → P(k+1)
- **Base Case**
  - Show P(c) and any other base cases that are needed…
    P(c), P(c+1), … , P(s)
- **Now, you've shown ∀n ≥ c P(n)** because **P(c),...,P(s)** are true, and:

$$\left.\begin{array}{c} P(c) \\ P(c+1) \\ \vdots \\ P(s) \end{array}\right\} \longrightarrow P(s+1) \longrightarrow P(s+2)\ \dots$$

Problem 2

2. Faulty Induction

Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

Base Case: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all non-negative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k+1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

Solution

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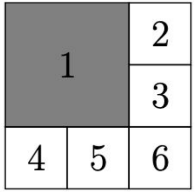
Solution: The proof is invalid for $k = 4$. We cannot increase the postage from 4 cents to 5 cents by either of the replacements indicated, because there is no 3-cent stamp present and there is only one 4-cent stamp present. There is also a minor flaw in the inductive step, because the condition that $j \geq 3$ is not mentioned.

Problem 3

3. Squares Strong Induction$\star$

Prove that a square can be subdivided into any number of squares $n \geq 6$. Note that sub-squares don't need to be the same size. For example, here's how you would subdivide a square into 6 squares.



Solution

Solution:
We need to show that for any $n \geq 6$, a square can be subdivided into n subsquares.

Notice that we can split any given square into 4 smaller squares by dividing it horizontally and vertically. In doing so, we increase the total number of individual squares by 3 $(1 \to 4)$. This can also be thought of as starting with a small square, then creating a bigger square by adding 3 more squares outside of it, as illustrated below. We can use this process to guide our thinking for the inductive step.

1	4
2	3

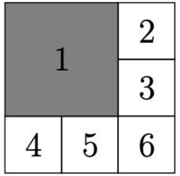
Inductive Step: Assume for $6 \leq i < k$ where $k > 8$, a square can be split into i subsquares. We want to use this to prove that a square can be split into k subsquares. We know that we can split a square into $(k-3)$ subsquares, by our inductive hypothesis since $(k-3) \geq 6$. Thus, we can demonstrate the existence of a square which can be split into k subsquares by starting with the entire square of composed of $(k-3)$ total subsquares (where this entire square is shown below as the shaded square S) and adding 3 squares outside of it, just as we showed above:

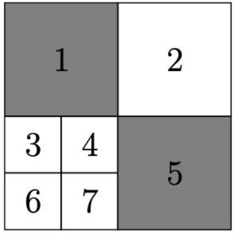
S	3
1	2

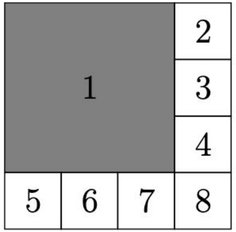
Solution

Base Cases:

Since we showed that $P(k-3)$ implies $P(k)$ for all $k > 8$ and we want to show $P(c)$ for all $n \geq 6$, we need to bridge the gap by showing the base cases $P(6)$, $P(7)$, $P(8)$...







Thus, we've shown by strong induction that for any $n \geq 6$, a square can be subdivided into n subsquares.

Problem 4

4. Jigsaw Puzzle Induction

A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n-1$ moves are required to assemble a puzzle with n pieces.

Solution

4. Jigsaw Puzzle Induction

A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n-1$ moves are required to assemble a puzzle with n pieces.

Solution: Let $P(n)$ be the statement that exactly $n-1$ moves are required to assemble a puzzle with n pieces. Now $P(1)$ is trivially true. Consider a puzzle with k pieces for some $k \geq 1$, and assume that $P(j)$ is true for all $0 < j < k$ where $j \in \mathbb{Z}^+$. The final move must be the joining of two blocks, of size i and $k-i$ for an arbitrary integer i such that $1 \leq i \leq k-1$. (Note that we can view "adding" a piece to a block as joining a the block to another block of size 1.) By the inductive hypothesis, it required $i-1$ moves to construct the one block, and $k-i-1$ moves to construct the other. Therefore $1+(i-1)+(k-i-1) = k-1$ moves are required in all, so $P(k)$ is true.

Thus, by strong induction, $P(n)$ is true for all $n \geq 1$.

Problem 5

5. Forming Discussion Groups 1$\star$

Tom is trying to do a group activity in his next discussion session. He wants to form groups of size 5 or 6.

(a) Show Tom that if there are 23 students attending his discussion, he will be able to split the students into groups of 5 or 6.

(b) In fact, there is some cutoff $p \in \mathbb{N}$ where $\forall n \geq p$, n students can be split into groups of 5 or 6. Find the smallest possible value of p.

(c) Now prove to Tom that if at least p students attends his discussion, he can successfully split the students in to groups of 5 or 6.

Solution

Solution:

(a) $23 = 5 + 6 + 6 + 6$

(b) The smallest value of p is 20.

(c) Proof by strong induction
Let $P(n)$ be the statement "n students can be split into groups of 5 or 6".

Inductive step: Assume that $P(j)$ is true for all $20 \leq j \leq k$, for some $k \geq 24$. Want to show that $k+1$ students can be split into groups of 5 or 6.
Since $k \geq 24$, $k - 4 \geq 20$, thus, by our IH, $k-4$ students can be split into groups of 5 or 6. Thus, we can create an other group of 5 from the remaining 5 students. In other words, if we can create a groups of 5 and b groups of 6 with $k-4$ students, we will be able to create $a+1$ groups of 5 and b groups of 6 with $k+1$ students.

Base cases:

- $n = 20 : 20 = 5 + 5 + 5 + 5$
- $n = 21 : 21 = 5 + 5 + 5 + 6$
- $n = 22 : 22 = 5 + 5 + 6 + 6$
- $n = 23 : 23 = 5 + 6 + 6 + 6$
- $n = 24 : 24 = 6 + 6 + 6 + 6$

Recurrence Relations

Recurrence Relations

Recurrence Relation: an equation that defines a sequence based on a rule that gives the next term as a function of previous terms.

Example:

A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years. Find a recurrence relation for *L(n)*, where *L(n)* is the number of lobsters caught in year *n*.

$$L(n) = \big(L(n-1) + L(n-2)\big) / 2$$

Recurrence Relations

Recurrence Relation: an equation that defines a sequence based on a rule that gives the next term as a function of previous terms.

Example:

- If we're searching an ordered list of length *n* for a particular number, how many total comparisons will we need to make?

$$S_n = S_{n/2} + 1$$

- We do this by checking the middle of the list each time, recursively narrowing the range we're looking at to half of the previous iteration.

Problem 6

6. Forming Discussion Groups 2⋆

In the previous question, we proved that Tom can split a total of n students into groups of 5 or 6 when $n \geq 20$ using induction.

(a) Give a recurrence relation for the minimum number of groups, $G(n)$ that needs to be formed for a class of n students to be split into groups of 5 or 6.

(b) What are the initial conditions?

Solution

Solution:

(a) $G(n) = min(G(n-5), G(n-6)) + 1$. We can form groups with n total students, by either forming one group of 5, and grouping the remaining $n-5$ students in $G(n-5)$ groups. Alternatively, we can form groups with n total students, by either forming one group of 6, and grouping the remaining $n-5$ students in $G(n-6)$ groups. Because $G(n-6)$ and $G(n-5)$ will not always have the same value, we need to take the minimum of the two in our calculation of $G(n)$.

(b) When looking at the earlier inductive proof we found that students can always be split into groups for $n \geq 20$, so 20 must be the first initial condition.

- $n = 20 : 20 = 5 + 5 + 5 + 5$
- $n = 21 : 21 = 5 + 5 + 5 + 6$
- $n = 22 : 22 = 5 + 5 + 6 + 6$
- $n = 23 : 23 = 5 + 6 + 6 + 6$
- $n = 24 : 24 = 6 + 6 + 6 + 6$
- $n = 25 : 25 = 5 + 5 + 5 + 5 + 5$

Our initial conditions are: $G(20) = G(21) = G(22) = G(23) = G(24) = 4$ and $G(25) = 5$, and the recurrence relation is used for $n \geq 26$

Problem 7

7. Lobster Recurrence

A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years. Find a recurrence relation for $L(n)$, where $L(n)$ is the numbers of lobsters caught in year n, under the assumption for this model.

Solution

7. Lobster Recurrence

A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years. Find a recurrence relation for $L(n)$, where $L(n)$ is the numbers of lobsters caught in year n, under the assumption for this model.

Solution: The number of lobsters caught in the previous year is $L(n-1)$ and the number 2 years prior is $L(n-2)$. The average of the two is therefore $\frac{L(n-1)+L(n-2)}{2}$. Thus, the recurrence relation is, $L(n) = \frac{L(n-1)+L(n-2)}{2}$

Problem 8

8. Stair Climbing

(a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.

(b) What are the initial conditions?

Solution

8. Stair Climbing

(a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.

(b) What are the initial conditions?

Solution:

(a) Let $a(n)$ be the number of ways to climb n stairs. In order to climb n stairs, a person must either start with a step of one stair and then climb $n-1$ stairs (and this can be done in $a(n-1)$ ways), start with a step of two stairs and then climb $n-2$ stairs (and this can be done in $a(n-2)$ ways), or start with a step of three stairs and then climb $n-3$ stairs (and this can be done in $a(n-3)$ ways). From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3 : a(n) = a(n-1) + a(n-2) + a(n-3)$.

(b) $a(0) = 1, a(1) = 1, a(2) = 2$. There is one way to climb up zero stairs (do nothing), one way to go up one stair (one step), and two ways to go up two stairs (one step twice or two step once).