EECS 203: Discrete Mathematics
Winter 2024
Discussion 9 Notes

1 Counting

1.1 Combinations and Permutations

- **Product Rule:** Suppose you want to count the ways of doing one thing, and then another, and then another.... If there are n_1 ways to do the first thing, n_2 ways to do the second thing, ..., and n_k ways to do the last thing, then in total, there are

$$n_1 \cdot n_2 \cdots n_k$$

  ways to do the entire thing.

- **Sum Rule:** Suppose you want to count the number of ways of choosing to do one thing, or another thing, or ... (but you can't do more than one). If there are n_i ways to do the ith thing, then there are

$$n_1 + n_2 + \cdots + n_k$$

  ways to do the entire thing.

- **Subtraction Rule:** If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

- **Division Rule:** If there are N ways to choose an object, and each object can be chosen in exactly k ways, there are N/k objects.

- **Permutation:** $P(n, k)$ is the number of ways to choose k things out of n things, where the selection order matters.

$$P(n, k) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

  Note that we are selecting objects **without replacing** them. (So if I draw a 2 of hearts, that card is no longer in the deck when I draw the next one.)

- **Combination:** $C(n,k)$ is the number of ways to choose k things (order doesn't matter) out of n things.
$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$
Note that we are selecting objects **without replacing** them. (So if I draw a 2 of hearts, that card is no longer in the deck when I draw the next one.)

- **Distinguishable:** Different from each other, "labeled"

- **Indistinguishable:** Considered identical, "unlabeled"

- **Binomial Theorem:** Let x and y be variables, and let n be a nonnegative integer. Then
$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

1.1.1 Basic Permutations and Combinations

a. How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

b. How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select a set of 47 cards from a standard deck of 52 cards? The order of the hand of five cards and the order of the set of 47 cards do not matter.

c. How many permutations of the letters ABCDEFGH contain the string ABC?

d. How many bit strings of length n contain exactly r 1's?

1.1.2 Standing in Line $\star$

How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [Hint: First position the men and then consider possible positions for the women.]

1.1.3 Forming a Committee

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

1.2 Permutations with Objects of Different Types

Permutations with Objects of Different Types: This is sort of like permutations with *partial* repetition, in that each type has a different number of repetitions. The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ... , and n_k indistinguishable objects of type k, is:

$$\begin{aligned}\binom{n}{n_1, n_2, \ldots, n_k} &= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} \\ &= \frac{n!}{n_1!n_2!\cdots n_k!}\end{aligned}$$

The first term is called a **multinomial coefficient**.

1.2.1 Permutations with Objects of Different Types

How many different strings can be made by reordering the letters of the word SUCCESS?

1.2.2 Hanging Jerseys

Robert went out to the store and bought 10 Michigan basketball jerseys and 15 Michigan football jerseys to adorn his walls due to Michigan's recent successes. Each jersey has a different player's name so he could tell them apart. However, once he got back to his room he realized that he only had room to hang up 6 jerseys! If he doesn't care where each jersey is positioned on his walls, how many ways are there to select the jerseys that will be put up if:

(a) he would like to hang up more or equal number of football jerseys than basketball jerseys

(b) he would like to hang up an equal number of football and basketball jerseys, but he can't hang up Surya's basketball jersey without also hanging up Ashu's football jersey? **Note: he could hang up Ashu's jersey without hanging up Surya's jersey.**

Provide a brief justification for each part.

1.3 More Counting

1.3.1 Letters With Repeats

How many different strings can be made from the letters in ATREYATATA, using all the letters?

1.3.2 Tea Party

You are having a tea party and have 5 unique tea bags and three mugs. Each mug will have exactly one tea bag in it

(a) How many ways are there to distribute the tea bags if the three mugs are identical?

(b) How many ways are there to distribute the tea bags if the three mugs are unique?

2 Probability

- **Distinguishable:** different from each other, "labeled"
- **Indistinguishable:** considered identical, "unlabeled"
- **Experiment:** Procedure that yields an outcome
- **Sample Space:** Set of all possible outcomes in an experiment, usually denoted by S.
- **Event:** A subset of the sample space, usually denoted by E.
- **Probability of an Event (Equally Likely Outcomes):** The probability of an event $E \subseteq S$ is $P(E) = \frac{|E|}{|S|}$ given all elements in S are equally likely.

- **Probability of Events:**
  $P(E) = \sum_{s \in E} p(s)$
  Note that $p \in [0, 1]$ is the probability of an element. The probabilities of all elements in a sample space add up to 1:
  $\sum_{s \in S} p(s) = 1$
- **Conditional Probability:** The probability of E_1 given E_2, denoted $P(E_1|E_2)$, is:

  $$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

- **Independence:** Events E_1 and E_2 are independent if

  $$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

  Note that this is NOT the same thing as two events being mutually exclusive.
- **Conditional Probability and Independence:** If $P(E_1) = P(E_1|E_2)$, then E_1 and E_2 are independent (since E_2 doesn't give you any information on E_1).
  More generally,
  Events E_1, E_2, and E_k are independent if
  $P(E_1 \cap E_2 \cap ... \cap E_k) = P(E_1) \cdot P(E_2) \cdots P(E_k)$

2.0.1 Probability Intro

What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 8?

2.0.2 Poker Hands

a. Find the probability that a hand of five cards in poker contains four cards of one rank.

b. What is the probability that a poker hand contains a full house, that is, three of one rank and two of another rank?

2.0.3 Lottery

Find the probability of selecting none of the correct six integers in a lottery, where the order in which these integers are selected does not matter, from the positive integers not exceeding 40.

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2.0.4 Independent vs Mutually Exclusive Events

Two six-sided dice, Dice A and Dice B, are rolled. Using these dice, provide examples of:

(a) A pair of independent events.

(b) A pair of mutually exclusive events.

Recall that events E and F are independent if $P(E \cap F) = P(E) \cdot P(F)$, and they are mutually exclusive if $P(E \cap F) = 0$.

2.0.5 Conditional Probability

A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

2.0.6 Independent Events

Assume that each of the four ways a family can have two children is equally likely. Are the events E, that a family with two children has two boys, and F, that a family with two children has at least one boy, independent?