EECS 203: Discrete Mathematics
Winter 2024
FoF Discussion 4 Notes

1 Definitions

- **Types of Proofs:**
  - **Direct Proof:** Prove that if some proposition p is true, then another proposition q is true "directly". Start by assuming that p is true, then make some deductions and eventually arrive at the conclusion that q must be true.

    $$p \to q$$

  - **Proof by Contraposition:** Prove that "if p is true, then q is true" by proving that if q is false, then p is false (since these are logically equivalent).

    $$\neg q \to \neg p$$

  - **Proof by Contradiction:** Prove p is true by assuming it is false, and arriving at a contradiction, i.e. a conclusion that we know is false.
    When using a proof by contradiction to prove "if p is true then q is true", we assume that p is true and that q is false, and derive a contradiction. This shows us that if p is true, then q is true.

    $$\neg(p \to q) \equiv (p \wedge \neg q) \to F \to \neg(p \wedge \neg q) \equiv (p \to q)$$

    A simpler way to view this: Assume p is true and show that

    $$\neg q \to \mathrm{F} \to q$$

  - **Proof by Cases:** Prove by considering all possibilities, or all categories of possibilities (i.e., cases), and showing that in each of those cases, the proposition you're trying to prove is true.

- **Set:** A set is an unordered collection of distinct objects

- **Universe:** In set theory, a universe is a collection that contains all the entities one wishes to consider in a given situation.

- **Set Operations:**
  - **Union** $S \cup T$: The set containing those elements that are in S or T
    $S \cup T = \{x \mid x \in S \vee x \in T\}$
  - **Intersection** $S \cap T$: The set containing those elements that are in S and T
    $S \cap T = \{x \mid x \in S \wedge x \in T\}$
  - **Complement** $\overline{S}$: The set containing those elements that are in the universe U but not in S.
    $\overline{S} = \{x \mid x \in U \wedge x \notin S\}$
  - **Minus** $S - T$: The set containing those elements that are in S but not in T
    $S - T = \{x \mid x \in S \wedge x \notin T\}$

- **Inclusion−Exclusion Principle:** The inclusion-exclusion principle states the the size of the union of two sets is equal to the sum or their sizes minus the size of their intersection:
  $$|A \cup B| = |A| + |B| - |A \cap B|$$

- **Subset:** The set A is a subset of B if and only if every element of A is also an element of B. Denoted $A \subseteq B$. Note that A and B may be the same set.
  $A \subseteq B \quad \text{iff} \quad \forall x\, [x \in A \rightarrow x \in B]$

- **Proper Subset:** The set A is a proper subset of B if and only if A is a subset of B and $A \neq B$. That is, A is a subset of B and there is at least one element of B that is not in A. Denoted $A \subsetneq B$.
  $A \subsetneq B \quad \text{iff} \quad \forall x\, [x \in A \rightarrow x \in B] \;\wedge\; A \neq B$

- **Disjoint:** The sets A and B are disjoint if and only if they do not share any elements.

- **Power Set:** The power set of a set S is the set of all subsets of S. $P(S)$ denotes the power set of S.
  $P(S) = \{T \mid T \subseteq S\}$

- **Cardinality:** The number of elements in a set. The cardinality of a set S is denoted by $|S|$.

- **Cartesian Product:** $A \times B$ is the set of all ordered pairs of elements (a, b) where $a \in A$ and $b \in B$.
  $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

- **Empty Set:** The empty set, denoted $\emptyset$ or $\{\}$, is the unique set having no elements.

2 Exercises

1. Associativity of minimum

Use proof by cases to show that $min(a, min(b, c)) = min(min(a, b), c)$ whenever a, b, c are real numbers.
Let a, b and c be arbitrary real numbers.
Case 1: a is the smallest ($a < b$ and $a < c$)

Case 2: b is the smallest

Case 3: c is the smallest

Since it is true in all the cases, we have thus shown through proof by cases that $min(a, min(b, c)) = min(min(a, b), c)$.

2. Proof by Cases/Contradiction $\star$

Prove that there is no rational solution to the equation $x^3 + x + 1 = 0$. **Hint:** Use the fact that 0 is an even number.
You can use the following lemmas without proving:

- Odd $\times$ Even $=$ Even
- Odd $\times$ Odd $=$ Odd
- Even $\times$ Even $=$ Even
- Odd $+$ Even $=$ Odd
- Odd $+$ Odd $=$ Odd
- Even $+$ Even $=$ Even

3. Prime Proof $\star$

Show that for any prime number p, $p^2 + 11$ is composite (not prime). Recall that a prime p is defined to be a natural number ≥ 2 such that p and 1 are the only factors that divide p.

4. Proving the Triangle Inequality

Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).

5. Set Exploration $\star$

a) What is $|\emptyset|$?

b) Let $A = \{1, 2, 3\}$, $B = \{\emptyset\}$, $C = \{\emptyset, \{\emptyset\}\}$, $D = \{4, 5\}$, and $E = \{\emptyset, 5\}$.

i. Is $\emptyset \in A$?

ii. Is $\emptyset \subseteq A$?

iii. Is $\emptyset \in B$?

iv. Is $\emptyset \subseteq B$?

v. Is $\emptyset \in C$?

vi. Is $\emptyset \subseteq C$?

vii. What is $A \cap D$?

viii. What is $B \cap C$?

ix. What is $B \cap E$?

x. What is $|B|$, $|C|$, $|E|$?

c) Let A and C be the sets defined above.

i. What is $P(A)$?

ii. What is $P(C)$?

iii. Find a formula for the size of the power set of S, $|P(S)|$, in terms of $|S|$.

iv. What is $C \times A$?

v. What is A^2? ($A^2 = A \times A$)

vi. Find a formula for the size of the Cartesian product of A and B, $|A \times B|$ in terms of $|A|$ and $|B|$.

6. Double Subset Equality $\star$

Prove the set equivalence: $A - (B \cap C) = (A - B) \cup (A - C)$

7. Subset Proofs

Let A, B, and C be sets. Prove that

a) $(A \cap B \cap C) \subseteq (A \cap B)$

b) $(A - B) - C \subseteq A - C$

8. Power Sets

Can you conclude that $A = B$ if A and B are two sets with the same power set?

9. More Power Sets $\star$

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$

b) $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

10. Power Set of a Cartesian Product

Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

Inclusion−Exclusion Principle:

The inclusion-exclusion principle states the the size of the union of two sets is equal to the sum or their sizes minus the size of their intersection:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

11. Expanding to 3 sets

Come up with the inclusion exclusion principle for the union of 3 sets: $|A \cup B \cup C|$.

12. Three sets

Suppose there is a group of 120 U of M students. Here's what you know:

- There are 31 in Engineering.
- There are 65 in LSA.
- There are 44 in Ross.
- There are 20 that are not in any of these 3 schools.
- There are 15 in Engineering and Ross.
- There are 17 in Engineering and LSA.
- There are 18 in LSA and Ross.

How many are in all 3 schools?