EECS 203: Discrete Mathematics
Winter 2024
Homework 7

Due **Thursday, Mar. 21**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $8 + 2$ Total Points: $100 + 18$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Growing your Growth Mindset [5 points]

(a) Watch the linked video about developing a growth mindset. This is a different video than the one you saw in lecture.

(b) Rewrite the last two fixed mindset statements as growth mindset statements.

(c) Write down one of your recurring fixed mindset thoughts, then write a thought you can replace it with that reflects a growth mindset.

Video: Developing a Growth Mindset (tinyurl.com/eecs203growthMindset)

What to submit: Your three pairs of fixed and growth mindset statements (the two from the table, and one that you came up with on your own).

Fixed Mindset Statement	**Growth Mindset Statement**
When I have to ask for help or get called on in lecture, I get anxious and feel like people will think I'm not smart.	The question I have is likely the same question someone else in lecture may have. It's important for me to ask so I can better understand what I am learning.
I'm jealous of other people's success.	I am inspired and encouraged by other people's success. They show me what is possible.
I didn't score as high on the exam as I expected. I'm not going to do well in this class and should drop it.	I learned from my mistakes on exam 1, and exam 2 will be a new opportunity for me to practice what I've learned.
This class is hard for me, so I am not fit for this major.	[FILL IN YOUR OWN]
Either I'm good at Discrete Math, or I'm not.	[FILL IN YOUR OWN]
[FILL IN YOUR OWN]	[FILL IN YOUR OWN]

Solution:

I am learning to be better at Discrete Math, and I will get better with practice.
I can be good at Discrete Math if I put in the effort to learn and understand the material.
I am not smart enough to understand this material.

The content is hard but it is possible for me to understand it with practice and effort as well as learning from other people.

2. Sketchy Compositions [15 points]

Consider $f\colon X \to Y$ and $g\colon Y \to X$. **Prove or disprove** each of the following statements.

(a) If $f \circ g$ is one-to-one, then g must be one-to-one.

(b) If $g \circ f$ is one-to-one, then g must be one-to-one.

Solution:

a) Proof by contradiction:

Seeking contradiction, assume the negation of the statement:

$f \circ g$ is one-to-one $\wedge g$ is $\neg$ one-to-one.

$\exists x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $g(x_1) = g(x_2)$	Definition of one-to-one
Since $x_1 \neq x_2 \wedge f(g(x_1)) = f(g(x_2))$	Definition of $f \circ g$
$f \circ g$ is not one-to-one	Definition of one-to-one

This contradicts the assumption that $f \circ g$ is one-to-one.

Therefore, if $f \circ g$ is one-to-one, then g must be one-to-one.

Thus, the original statement holds by proof by contradiction.

b) Proof by contrapositive:

The contropostive is $g \neg$ one-to-one $\rightarrow g \circ f \neg$ one-to-one.

Asseme $g \neg$ one-to-one:

$\exists y_1, y_2 \in Y[y_1 \neq y_2 \wedge g(y_1) = g(y_2)]$	Definition of one-to-one
Thus, $\exists f(y_1), f(y_2)[g(f(y_1)) = g(f(y_2)) \wedge f(y_1) \neq f(y_2)]$	Definition of $g \circ f$
Which means $g \circ f \neg$ one-to-one.	Definition of one-to-one

Thus the original statement holds by contrapositive.

3. Flippy Function Fun! [15 points]

A function $f\colon A \to A$ is said to be *flippy* if for all $a \in A$, $f(f(a)) = a$. **Prove or disprove** each of the following statements

(a) If $f\colon A \to A$ is flippy, then f is bijective. (Either prove f is both onto and one-to-one using their respective definitions, or provide a counterexample.)

(b) If $f\colon A \to A$ and $g\colon A \to A$ are flippy, then $f \circ g$ must be flippy.

Solution:

a) Proof by direct proof:
Assume arbitary $f\colon A \to A$ is flippy.
We need to show that f is both onto and one-to-one.
Proof one-to-one:
Let a, b be arbitary elements $\in A$
Assume $f(a) = f(b)$
Then $f(f(a)) = f(f(b))$
Since f is flippy, $f(f(a)) = a$ and $f(f(b)) = b$
Thus $a = b$
Therefore, f is one-to-one.
Proof onto:
Let b be an arbitary element $\in A$
Let $a = f(b)$
Then $f(a) = f(f(b))$
Since f is flippy, $f(f(b)) = b$
Thus $f(a) = b$
Therefore, f is onto.
Thus, f is bijective.
b) Disproof by counterexample:
Will show $f : A \to A \wedge g : A \to A$ are flippy $\to f \circ g$ is not flippy for set $A = \{1, 2, 3, 4\}$
Consider flippy function $f : A \to A$
Let $f(1) = 1, f(2) = 4, f(3) = 3, f(4) = 2$
Consider flippy function $g : A \to A$
Let $g(1) = 4, g(2) = 3, g(3) = 2, g(4) = 1$
Consider $x = 1$
$f(g(f(g(1)))) = f(g(f(4))) = f(g(2)) = f(3) = 3 \neq 1$
Thus $f \circ g$ is not flippy.
Therefore, the original statement is disproof by counterexample.

4. A Hairy Situation [12 points]

Assume that nobody on Earth has more than 1,000,000 hairs on their head. Assume that the population of New York City in 2024 is 8,468,000 people. As of 2024, what is the maximum number of people in New York City that we can guarantee all have the same number of hairs on their heads?

Your explanation should use the Pigeonhole Principle. Make sure to state what the pigeons and holes are, as well as how many of each you have.

Solution:

By the Pigeonhole Principle, we can guarantee that at least 9 people in New York City have the same number of hairs on their heads.
Pigeons: People in New York City, 8,468,000
Holes: Number of hairs on a person's head, could be 0 to 1,000,000, thus 1,000,001
$\lceil \frac{8{,}468{,}000}{1{,}000{,}001} \rceil = 9$

5. A Pairy Situation [14 points]

Suppose that 52 integers are chosen among the set of natural numbers less than 100. In other words, suppose that 52 integers are chosen from $\{0, 1, 2, 3, \ldots, 99\}$. **Prove or disprove** that there must exist at least one pair of integers among those chosen whose difference is equal to 7.

Your proof or disproof should use the Pigeonhole Principle. Make sure to state what the pigeons and holes are, as well as how many of each you have.

Solution:

Proof:
Pigeons: 52 integers chosen from $\{0, 1, 2, 3, \ldots, 99\}$
Holes:
We have the following $7 \cdot 7 + 2 = 51$ grid of numbers, where each row represents a set of numbers that differ by 7:
$\{0, 7\}, \{1, 8\}, \{2, 9\}, \{3, 10\}, \{4, 11\}, \{5, 12\}, \{6, 13\}$
$\{14, 21\}, \{15, 22\}, \{16, 23\}, \{17, 24\}, \{18, 25\}, \{19, 26\}, \{20, 27\}$
$\{28, 35\}, \{29, 36\}, \{30, 37\}, \{31, 38\}, \{32, 39\}, \{33, 40\}, \{34, 41\}$
$\{42, 49\}, \{43, 50\}, \{44, 51\}, \{45, 52\}, \{46, 53\}, \{47, 54\}, \{48, 55\}$
$\{56, 63\}, \{57, 64\}, \{58, 65\}, \{59, 66\}, \{60, 67\}, \{61, 68\}, \{62, 69\}$
$\{70, 77\}, \{71, 78\}, \{72, 79\}, \{73, 80\}, \{74, 81\}, \{75, 82\}, \{76, 83\}$
$\{84, 91\}, \{85, 92\}, \{86, 93\}, \{87, 94\}, \{88, 95\}, \{89, 96\}, \{90, 97\}$
$\{98\}, \{99\}$
Since we have 52 pigeons and 51 holes, by the Pigeonhole Principle, at least one hole must contain at least two pigeons.
Thus, there must exist at least one pair of integers among those chosen whose difference is equal to 7.

6. Super Sets [15 points]

Let A be the set of prime numbers less than 203. The universe of discourse is $\mathbb{R}$. State whether each of the following sets are empty, finite but nonempty, countably infinite, or uncountable. Briefly justify your answers.

(a) $\mathbb{Z} \times \mathbb{Z}$

(b) $(\mathbb{Z} \times \mathbb{Z}) - (\mathbb{Q} \times \mathbb{Q})$

(c) $\mathbb{R} - \mathbb{Q}$

(d) $\mathbb{Q} - \mathbb{R}$

(e) $A \cap \mathbb{Q}$

(f) $\overline{A} \cap \overline{\mathbb{Q}}$

Solution:

a) Countably infinite
Each element in $\mathbb{Z} \times \mathbb{Z}$ can be mapped to a unique element in $\mathbb{N}$, thus $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.
b) Empty
$\mathbb{Z} \subseteq \mathbb{Q}$, thus $\mathbb{Z} \times \mathbb{Z} \nsubseteq \mathbb{Q} \times \mathbb{Q}$, thus $(\mathbb{Z} \times \mathbb{Z}) - (\mathbb{Q} \times \mathbb{Q})$ is empty.
c) Uncountable infinite
$\mathbb{R} - \mathbb{Q} =$Irrational numbers, which are uncountable infinite.
, thus $\mathbb{R} - \mathbb{Q}$ is uncountable infinite.
d) Empty
$\mathbb{Q} \subseteq \mathbb{R}$, thus $\mathbb{Q} - \mathbb{R}$ is empty.
e) Finite but nonempty
Since $A \subseteq \mathbb{Q}$, $A \cap \mathbb{Q}$ is finite but nonempty because A is all prime numbers less than 203, which is finite but nonempty.
finite but nonempty.
f) Uncountable infinite
By compliment, $\overline{A} \cap \overline{\mathbb{Q}} = \mathbb{R} - A \cap \mathbb{R} - \mathbb{Q}$, which is the set of union of real numbers that are not prime numbers less than 203 and irrational numbers, which is uncountable infinite.

7. Cardinal Construction [12 points]

For each part, give *uncountable* sets A and B such that $A - B$ is

(a) uncountable.

(b) countably infinite.

(c) finite but nonempty.

(d) empty.

Solution:

a) $A = \mathbb{R}_{\geq 0}, B = [0, 1]$
$A - B = \mathbb{R}_{>1}$
b) $A = \mathbb{R} \cup \mathbb{Z}, B = \mathbb{R}$
$A - B = \mathbb{Z}$
c) $A = \mathbb{R}_{\geq 0}, B = \mathbb{R}_{\leq 0}$
$A - B = 0$
d) $A = \mathbb{R}, B = \mathbb{R}$
$A - B = \emptyset$

8. Interesting Intervals [12 points]

Prove that $|[0, 3]| = |(2, 5) \cup (6, 7)|$. If you construct functions in your solution with certain properties, you may assert that they have those properties without proof.

Solution:

Let $A = [0, 3]$ and $B = (2, 5) \cup (6, 7)$.
Let $f(x) = \frac{x}{5} + 2$ for $x \in A$ be one-to-one and $g(x) = \frac{x}{1000} - 2.0001$ for $x \in B$ be one-to-one
Thus, $f : A \to B$ and $g : B \to A$ are both one-to-one.
Since f is one-to-one, $|A| \leq |B|$.
Since g is one-to-one, $|B| \leq |A|$.
Thus, $|A| = |B|$ by Schröder-Bernstein Theorem.
Therefore $|[0, 3]| = |(2, 5) \cup (6, 7)|$.