EECS 203 Exam 2 Review

Day 1

Today’s Review Topics

- Modular Arithmetic
- Induction
    - Weak Induction
    - Strong Induction

Divisibility and Modular Arithmetic

Divisibility Recap

- Divisibility: a | b iff $\exists c\ (b = ac)$ 0|3? 3|0?
- Prime Number p>1: p is only divisible by 1 and itself

Two types of "mods"

- **$a \equiv b \pmod{m}$** is a predicate involving three numbers. Sometimes we leave out the parens; **$\equiv$** is the important part
- **a mod m** is the remainder after dividing a by m. This is always an integer between 0 and m–1. (a%m in C++)

Modular

- We can write b = na + r (n is some int and 0<=r<a)
- a ≡ b (mod m) “a and b have same remainder upon division by m”?
- More about modular arithmetic:

- Suppose a ≡ b (mod m) and c ≡ d (mod m).
- **Claim:** a+c ≡ b+d (mod m) (***Addition works!***)
- **Claim:** a–c ≡ b–d (mod m) (***Subtraction works!***)
- **Claim:** ac ≡ bd (mod m) (***Multiplication works!***)
    - Simplify the bases of exponents and constant addition/multiplication terms
    - Split exponents using exponent rules

Mods Question 1

Let $x \equiv 3 \pmod{12}$, $y \equiv 11 \pmod{21}$, and $z \equiv 3 \pmod{4}$. Which of the following statements must be true?

(a) $x + y \equiv 2 \pmod{3}$

(b) $x + z \equiv 3 \pmod{4}$

(c) $x - y \equiv -8 \pmod{12}$

(d) $x \cdot y \equiv 12 \pmod{21}$

(e) $x \cdot z \equiv 1 \pmod{4}$

Solution: a,e. a: 12 and 21 are both multiples of 3, so we know $x \equiv 3 \pmod 3$ and $y \equiv 11 \pmod 3$, so $x + y \equiv 3 + 11 \equiv 2 \pmod 3$
b,e: 12 and 4 are both multiples of 4, so we know $x \equiv 3 \pmod 4$ and $z \equiv 3 \pmod 4$, so $x + z \equiv 3 + 3 \equiv 2 \pmod 4$. We also know that $x \cdot z \equiv 3 \cdot 3 \equiv 1 \pmod 4$.
c,d: We don't know what y is mod 12 or x is mod 21, so c and d cannot be guaranteed.

Alternate Solution: (a), (e)

Using the definition of mods:

- $x = 12k_1 + 3$
- $y = 21k_2 + 11$
- $z = 4k_3 + 3$

for some integers k_1, k_2, and k_3.

a) $x + y$
$= (12k_1 + 3) + (21k_2 + 11)$
$\equiv (0 + 0) + (0 + 2) \pmod 3$
$\equiv 2 \pmod 3 \rightarrow$ True

b) $x + z$
$= (12k_1 + 3) + (4k_3 + 3)$
$\equiv (0 + 3) + (0 + 3) \pmod 4$
$\equiv 6 \pmod 4$
$\equiv 2 \pmod 4 \rightarrow$ False; $2 \neq 3$

a)

b)

c) $x - y$
$= (12k_1 + 3) - (21k_2 + 11)$
$\equiv (0 + 3) - (9k_2 + 11) \pmod{12}$
$\equiv -9k_2 - 8 \pmod 3 \rightarrow$ False; value depends on k_2 which depends on y

d) $x * y$
$= (12k_1 + 3) * (21k_2 + 11)$
$\equiv (12k_1 + 3) * (0 + 11) \pmod{21}$
$\equiv 132k_1 + 33 \pmod{21}$
$\equiv 6k_1 + 12 \pmod{21} \rightarrow$ False; value depends on k_1 which depends on x

e) $x * z$
$= (12k_1 + 3) * (4k_3 + 3)$
$\equiv (0 + 3) * (0 + 3) \pmod 4$
$\equiv 9 \pmod 4$
$\equiv 1 \pmod 4 \rightarrow$ True

Mods Question 2

Find c with $0 \leq c < 11$ such that $c \equiv 14^6 + 22^{203} \pmod{11}$

Mods Question 2 Solution

Find c with $0 \leq c < 11$ such that $c \equiv 14^6 + 22^{203} \pmod{11}$

$$
\begin{aligned}
c &\equiv 14^6 + 22^{203} && \pmod{11} \\
&\equiv 3^6 + 0^{203} && \pmod{11} \\
&\equiv 3^6 && \pmod{11} \\
&\equiv (3^2)^3 && \pmod{11} \\
&\equiv (9)^3 && \pmod{11} \\
&\equiv (-2)^3 && \pmod{11} \\
&\equiv -8 && \pmod{11} \\
&\equiv 3 && \pmod{11}
\end{aligned}
$$

So $c = 3$

Induction

Cheat Sheet

//	Induction	Strong Induction
Define Predicate	Let P(n) be the statement...	Let P(n) be the statement...
Basis Step	Form your base case $P(x)$ (it can be more than one)	Form your base case(s) $P(x), P(x+1), \ldots$ (usually more than one)
Inductive Hypothesis	$P(x)$ is true	$P(j)$ is true for all j such that smallest base case $\leq j \leq k$
Inductive Step	$P(x) \rightarrow P(x+1)$	$P(i) \wedge P(i+1) \wedge \cdots P(k) \rightarrow P(k+1)$ i =smallest base case $P(j) \rightarrow P(k+1)$, base$\leq j \leq k$

Induction Recap

- Two types of Induction
  - Weak Induction
  - Strong Induction
- Base Case(s), Inductive Hypothesis, Inductive Step
- “Mathematical ladder”

Weak Induction

Weak Induction

1. Show that the expression/statement is true for the base case (often in the form of $n = 0$ or $n = 1$).
2. Assume that the expression is true for some arbitrary element k in the domain appropriate for the problem.*
3. Show that the statement is true for $P(k+1)$ when $P(k)$ is true. (i.e $P(k) \rightarrow P(k+1)$)

* The domain is often $\mathbf{Z}^+$, but it may be different.

Induction 1

Prove that the following equality holds for all positive integers n:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Induction 1 Solution

Solution: Let $P(n)$ be $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots \frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$.

Inductive Step:

We assume that $P(k)$ is true for an arbitrary positive integer k such that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots \frac{1}{k\cdot(k+1)}=\frac{k}{k+1}$. It must be shown that $P(k+1)$ follows from this assumption.

Induction 1 Solution Continued

$$\begin{aligned}
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k\cdot(k+1)}+\frac{1}{(k+1)\cdot((k+1)+1)} &= \frac{k}{k+1}+\frac{1}{(k+1)\cdot((k+1)+1)}\\
&= \frac{k}{k+1}+\frac{1}{(k+1)\cdot(k+2)}\\
&= \frac{k(k+2)}{(k+1)(k+2)}+\frac{1}{(k+1)\cdot(k+2)}\\
&= \frac{k^2+2k+1}{(k+1)(k+2)}\\
&= \frac{(k+1)^2}{(k+1)(k+2)}\\
&= \frac{k+1}{k+2}\\
&= \frac{k+1}{(k+1)+1}
\end{aligned}$$

This shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. Note that the equality in line 1 is true by the inductive hypothesis.

Base Case:
Our base case of $P(1)$ is true since $LHS = \frac{1}{1\cdot 2} = \frac{1}{2} = \frac{1}{1+1} = RHS$.

Therefore, since $P(1)$ and $\forall k P(k) \rightarrow P(k+1)$ are both true, then by mathematical induction, the claim is proven.

“Inequality” Induction

- Hardest part is substituting using inequality
- Manipulate the expressions to reduce them to desired form
- Consider things like “is the product greater than the sum?”

Induction 2

Prove using induction that $1 + (3^0 + 3^1 + 3^2 + \cdots + 3^{n-1}) < 3^n$ for all $n \geq 1$.

Induction 2 Solution

Let $P(n) : 1 + (3^0 + 3^1 + 3^2 + \cdots + 3^{n-1}) < 3^n$.
Claim: $P(n)$ holds for all $n \geq 1$.

Base Case: need to prove $P(1) : 1 + 3^{1-1} < 3^1$.
Proof: $1 + (3^0 \cdots 3^{1-1}) = 1 + 3^0 = 1 + 1 = 2 < 3 = 3^1$.

Alternate proof of Base Case: $1 + 3^0 = 2$ and $3^1 = 3$. $2 < 3$, so thus $1 + 3^0 < 3^1$.

Inductive Step:
Assume $P(k)$: $1 + (3^0 + 3^1 + 3^2 + \cdots + 3^{k-1}) < 3^k$ holds, for some $k \geq 1$.
Want to show $P(k+1)$: $1 + (3^0 + 3^1 + 3^2 + \cdots + 3^{k-1} + 3^k) < 3^{k+1}$.

Starting with the LHS of $P(k+1)$ and working our way to the RHS, we have:

$$
\begin{aligned}
1 + (3^0 + 3^1 + 3^2 + \cdots + 3^{k-1} + 3^k) &= \big(1 + (3^0 + 3^1 + 3^2 + \cdots + 3^{k-1})\big) + 3^k \\
&< 3^k + 3^k && \text{(by IH)} \\
&= 2 \cdot 3^k \\
&< 3 \cdot 3^k \\
&= 3^{k+1}
\end{aligned}
$$

Thus, by mathematical induction, we have that $P(n)$ holds for all $n \geq 1$.

Induction 3

Prove using induction that

$$n^2 + n < 2^n, \qquad \text{for all integers } n \geq 5.$$

Every inequality in your proof should be justified by one of the following:

- The inductive hypothesis (IH)
- $k^i < k^j$ when $i < j$ because $k > 1$ (e.g., $k^2 < k^4$)
- $c \leq k$ when $c \leq 5$ because $k \geq 5$ (e.g., $3 \leq k$)

Induction 3 Solution

Solution: Let $P(n)$ be the predicate $n^2 + n < 2^n$. We want to show $\forall n \geq 5\ P(n)$.

Base case: Let $n = 5$. Then $5^2 + 5 = 30 < 32 = 2^5$. So $P(5)$ is true.

Inductive step:
Inductive hypothesis: Assume $P(k)$ is true for an arbitrary $k \geq 5$. That is, assume $k^2 + k < 2^k$ for an arbitrary integer $k \geq 5$.
We want to show that $P(k+1)$ holds, that is, $(k+1)^2 + (k+1) < 2^{k+1}$.

$$
\begin{aligned}
(k+1)^2 + (k+1) &= (k^2 + 2k + 1) + (k+1) \\
&= (k^2 + k) + (2k + 2) \\
&< (k^2 + k) + (k \cdot k + k) && (2 < k) \\
&= (k^2 + k) + (k^2 + k) \\
&< 2^k + 2^k && \text{(Inductive hypothesis)} \\
&= 2^{k+1}
\end{aligned}
$$

Induction 4

Prove that for all $n \geq 1$, the sum of the squares of the first 2n positive integers is given by the formula

$$1^2 + 2^2 + 3^2 + ... + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$$

Induction 4 Solution

Let $P(n)$ be $1^2 + 2^2 + 3^2 + ... + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$

Base Case: $P(1)$, $1^2 + 2^2 = \frac{1 \cdot (2(1)+1) \cdot (4(1)+1)}{3} = \frac{3 \cdot 5}{3} = 5$

Inductive Hypothesis: Assume $P(k)$ is true, $1^2 + 2^2 + 3^2 + ... + (2k)^2 = \frac{k(2k+1)(4k+1)}{3}$ holds

Inductive Step: Prove that $1^2+2^2+3^2+...+(2(k+1))^2 = \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3}$ holds

Induction 4 Solution Continued

$$\begin{aligned}
1^2 + 2^2 + 3^2 + ... + (2(k+1))^2 &= 1^2 + 2^2 + 3^2 + ... + (2k+2)^2 \\
&= 1^2 + 2^2 + 3^2 + ... + (2k)^2 + (2k+1)^2 + (2k+2)^2 \\
&= \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2 + (2k+2)^2 \\
&= \frac{k(2k+1)(4k+1)}{3} + \frac{3(2k+1)^2 + 3(2k+2)^2}{3} \\
&= \frac{k(2k+1)(4k+1) + 3(2k+1)^2 + 3(2k+2)^2}{3} \\
&= \frac{(8k^3 + 6k^2 + k) + (12k^2 + 12k + 3) + (12k^2 + 24k + 12)}{3} \\
&= \frac{8k^3 + 30k^2 + 37k + 15}{3} \\
&= \frac{(2k^2 + 5k + 3)(4k+5)}{3} \\
&= \frac{(k+1)(2k+3)(4k+5)}{3} \\
&= \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3}
\end{aligned}$$

Therefore we have proved that $P(k) \to P(k+1)$ for all $k \geq 1$
By mathematical induction, P(n) is true for all $n \geq 1$
QED

Strong Induction

Strong Induction

- Similar to Weak Induction
- Major Differences
  - Possibly multiple base cases
  - Assumes all previous steps to be true
- Still has the same format as weak induction

Strong Induction 1

Prove that every integer $n \geq 12$ can be written as $n = 4a + 5b$ for some non-negative integer a, b using strong induction.

Strong Induction 1 Solution

Let $P(n)$ be $n = 4a + 5b$ for some nonnegative integer a, b

Base Cases:

$P(12)$, $12 = 4 \cdot 3 + 5 \cdot 0$
$P(13)$, $13 = 4 \cdot 2 + 5 \cdot 1$
$P(14)$, $14 = 4 \cdot 1 + 5 \cdot 2$
$P(15)$, $15 = 4 \cdot 0 + 5 \cdot 3$

Inductive Step:

Let $k \geq 15$

Inductive hypothesis: assume $P(j)$ is true for $12 \leq j \leq k$. That is, $j = 4a + 5b$ for some nonnegative integer a, b holds

We want to prove that $k + 1 = 4a + 5b$ holds for some nonnegative integer a, b

From our inductive hypothesis: $12 \leq k - 3 \leq k$

Hence, we know that $P(k-3)$ is true, or $k - 3 = 4a + 5b$ for some nonnegative integers a, b

Thus, we have

$$\begin{aligned} k + 1 &= (k - 3) + 4 \\ &= 4a + 5b + 4 \\ &= 4(a + 1) + 5b \end{aligned}$$

Showing that $P(k+1)$ is true

By strong induction, $P(n)$ is true for $n \geq 12$

Strong Induction 2

Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. [Hint: For the inductive step, separately consider the case where $k+1$ is even and where it is odd. Note that when $(k+1)$ is even, $(k+1)/2$ is an integer.]

Strong Induction 2 Solution

Solution: The basis step is to note that $1 = 2^0$. Notice for subsequent steps that $2 = 2^1$, $3 = 2^1 + 2^0$, $4 = 2^2$, $5 = 2^2 + 2^0$, and so on. Indeed this is simply the representation of a number in binary form (base two).

Assume the inductive hypothesis, that every positive integer up to k can be written as a sum of distinct powers of 2 for some $k \geq 1$. We must show that $k+1$ can be written as a sum of distinct powers of 2. Consider the case where k is even. Because the only odd power of 2 is $2^0 = 1$, and for 2^0 to be part of the numbers summing to k we would therefore have to have 2^0 twice (but these are *distinct* powers of 2), we know that when k is even, k can be written as a sum of distinct powers of 2 without a 2^0 term. If $k+1$ is odd, then k is even, so 2^0 was not part of the sum for k. Therefore the sum for $k+1$ is the same as the sum for k with the extra term 2^0 added. If $k+1$ is even, then $(k+1)/2$ is a positive integer, so by the inductive hypothesis $(k+1)/2$ can be written as a sum of distinct powers of 2. Increasing each exponent by 1 doubles the value and gives us the desired sum for $k+1$.

Strong Induction 2

Let the sequence a_n be defined as $a_1 = a_2 = a_3 = 1$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Prove that

$$a_n < 2^n \qquad (*)$$

holds for all $n \in \mathbb{Z}_+$.

Strong Induction 2 Solution

Solution: We will prove by strong induction.

Base step: For $n = 1, 2, 3$, a_n is equal to 1, whereas the right-hand side of (*) is equal to $2^1 = 2$, $2^2 = 4$, and $2^3 = 8$, respectively. Thus, $(*)$ holds for $n = 1, 2, 3$.

Induction step: Let $k \geq 3$ be given and suppose (*) is true for all $n = 1, 2, \ldots, k$. Then

$$\begin{aligned}
a_{k+1} &= a_k + a_{k-1} + a_{k-2} && \text{(by definition of } a_n\text{)} \\
&< 2^k + 2^{k-1} + 2^{k-2} && \text{(by strong induction hypothesis with } n = k, k-1, \text{ and } k-2\text{)} \\
&= 2^{k+1}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \\
&= 2^{k+1} \cdot \frac{7}{8} \\
&< 2^{k+1}
\end{aligned}$$

Thus, (*) holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: By the strong induction principle, it follows that (*) is true for all $n \in \mathbb{Z}_+$.

Have a great rest of the weekend!

5 Minute Break!

https://joshdata.me/iceberger.html

Iceberger

Draw an iceberg and see how it will float.

(Inspired by a tweet by @GlacialMeg)