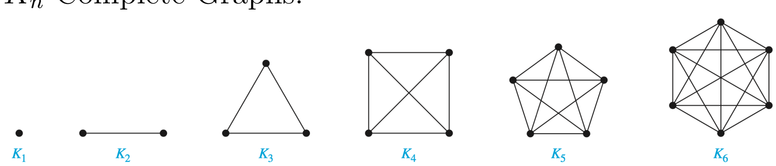
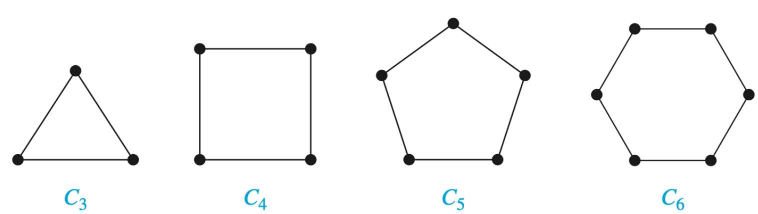
EECS 203: Discrete Mathematics
Winter 2024
FOF Worksheet 8

1 Graphs

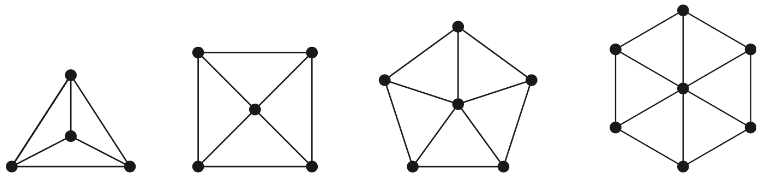
- **Graph, $G = (V, E)$:** A graph $G = (V, E)$ consists of V, a set of vertices, and E, a set of edges. Each edge has either one or two vertices associated with it, where there is only one if the edge is a loop.
- **Simple Graph:** A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices.
- **Directed Graph:** A graph where each edge is associated with an ordered pair of vertices (a directed edge), (u, v), and the edge is said to start at u and end at v.
- **Multigraph:** Edges can have multiplicity. Ex: $\{u, v\}$ and $\{u, v\}$ undirected, (u, v) and (u, v) directed
- **Loops:** An edge that connects a vertex to itself.
- **Adjacent Vertices:** Two vertices are adjacent if there is an edge that connects them. This edge is said to connect the two vertices.
- **Degree, $deg(v)$:** In an undirected graph, the degree of a vertex, v, is the number of edges attached to v (except that a loop at a vertex contributes twice to the degree of that vertex).
- **Special Simple Undirected Graphs:**
  - K_n Complete Graphs:



  - C_n Cycles:

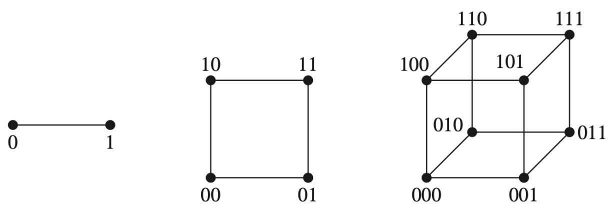


– W_n Wheels:



W_3 W_4 W_5 W_6

– Q_n Hypercubes:



Q_1 Q_2 Q_3

1.1 Graphs Intro

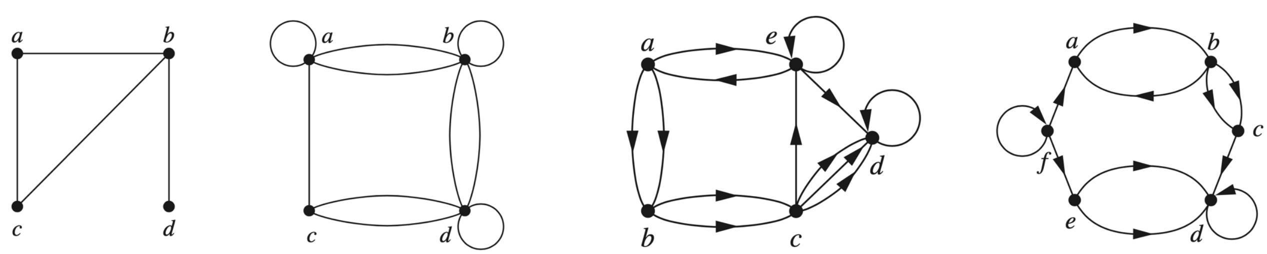
For the following graphs:

a) Identify whether the graph has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops.

b) For each undirected graph, identify whether or not it is simple. If it is not simple, find a set of edges to remove to make it simple.

c) Find $deg(b)$ or if the graph is directed, find $deg^{-}(b)$ and $deg^{+}(b)$.

d) Write out its degree sequence. For this part, treat the directed graphs as if they were undirected.



1.2 The Handshake Theorem

The Handshake Theorem: Let $G = (V, E)$ be an undirected graph with m edges. Then:

$$2m = \sum_{v \in V} deg(v)$$

Handshake Theorem Equivalent for Directed Graphs:

$$|E| = \sum_{v \in V} deg^+(v) = \sum_{v \in V} deg^-(v)$$

Corollary of Handshake Theorem: Every graph has an even number of vertices with odd degree.

1.2.1 Edges and Vertices

Suppose a graph has 21 edges, and 3 vertices of degree 4. All other vertices have degree 2. How many vertices are in the graph?

1.3 Trees

Acyclic graph: An acyclic graph is a graph having no cyclic subgraphs. **Tree:** A tree is a connected, acyclic graph. **Tree Theorems (2):**

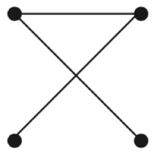
- If $T = (V, E)$ and $u, v \in V$, there is a unique simple path from u to v.
- Every tree on n vertices contains $(n - 1)$ edges.

Spanning Tree: Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.

1.3.1 Trees

Which of the following graphs are trees? If it is not a tree, are you able to construct a spanning tree of the graph?

a)



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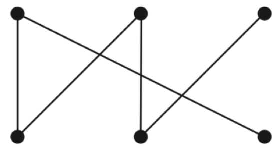
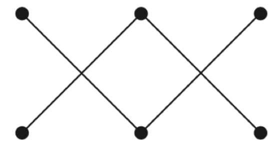
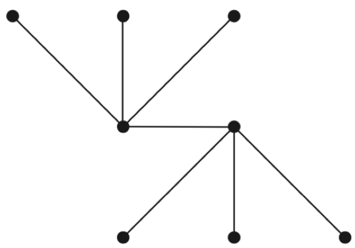


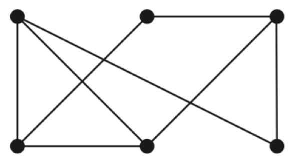
c)



d)



e)



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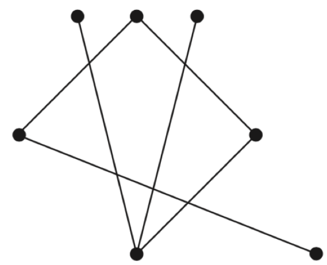

1.4 Isomorphic Graphs

Isomorphism: Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $f : V_1 \to V_2$ such that

$$\forall u, v \big[\{u, v\} \in E_1 \leftrightarrow \{f(u), f(v)\} \in E_2\big]$$

Graph Invariant: A graph invariant is a property preserved by isomorphism of graphs.

- Examples: number of vertices, number of edges, degrees of vertices, existence of subgraphs, path properties

1.4.1 Isomorphic Graphs

Determine whether each given pair of graphs is isomorphic. Exhibit an isomorphism or provide an argument that none exists.

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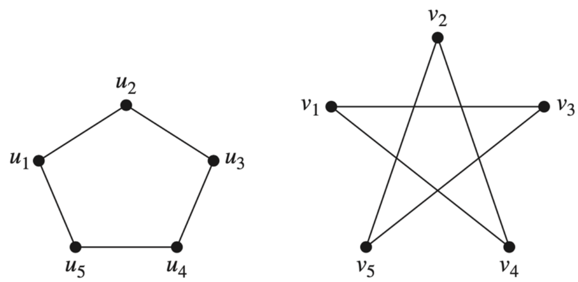

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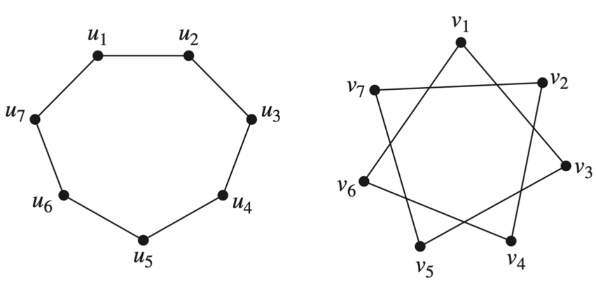

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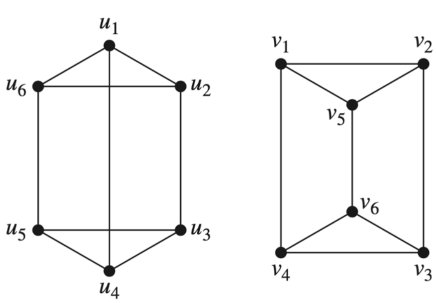

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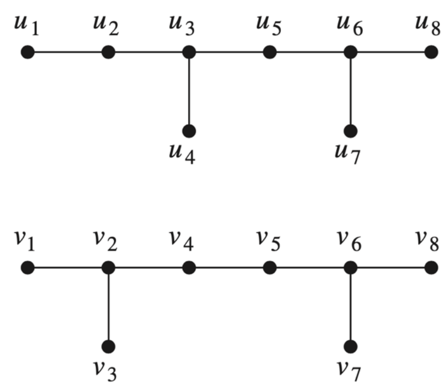

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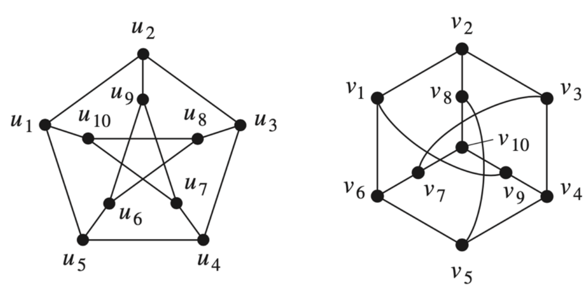

2 Exam 2 Review

2.1 Induction

2.1.1 Inductive Conclusions

Suppose that $P(n)$ is an unknown predicate. Determine for which positive integers n the statement $P(n)$ must be true, and justify your answer, if

a) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then P(n + 2) is true.

b) $P(1)$ and $P(2)$ are true, and for all positive integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

c) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(2n)$ is true.

d) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.

2.1.2 Equality Weak Induction

Use induction to prove the summation of cubes formula. That is, for $n \geq 1$ prove:

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

2.1.3 Strong Induction

Prove that every positive integer is either a power of 2, or can be written as the sum of distinct powers of 2.

2.2 Recurrence

2.2.1 Initial Conditions

What is the minimum number of initial conditions required for the following recurrence relation?

$$R(n) = R(n-4) + R(n-3) + 2log(n)$$

(a) 1

(b) 2

(c) 3

(d) 4

2.2.2 Recurrence Relations

Izzy wants to tile an $n \times 2$ table. She has 3×1 and 2×1 tiles. Note that the 2×1 tiles can be placed in two orientations on the table. What is a recurrence relation for the number of distinct ways to tile the table, so that the table is entirely covered in tiles?

2.3 Functions

If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

2.4 Countability

Which of the following sets are uncountably infinite? (select all that apply)

(a) $\mathbb{R}^-$

(b) $\mathbb{R} \cap \mathbb{Z}$

(c) $\mathbb{Q} - \mathbb{Z}$

(d) $\mathbb{R} \cup \mathbb{N}$

2.5 Schröder–Bernstein

Use the Schröder–Bernstein theorem to show $|[0,1)| = |(-2,0) \cup (0,2)|$.

2.6 Mod

Let $a \equiv 38 \pmod{15}$, $b \equiv 2 \pmod{15}$, and $c \equiv 3 \pmod{5}$. Compute the following if possible:

a) $d \equiv a^{24} \pmod{15}$

b) $e \equiv a^3 b^7 + b^{13} \pmod{15}$

c) $g \equiv a + c \pmod{15}$

d) $h \equiv a + c \pmod{5}$

2.7 Pigeonhole Practice

a) How many integers do we need to select to guarantee one of them is divisible by 7?

b) How many integers do we need to select to guarantee that we'll have a pair whose difference is divisible by 7?

3 Additional Graph Definitions

- **In-Degree,** $deg^{-}(v)$**:** n a directed graph, the in-degree of a vertex, v, is the number of edges with v as their terminal vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

- **Out-Degree,** $deg^{+}(v)$**:** In a directed graph, the out-degree of a vertex, v, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

- **Neighborhood,** $N(v)$**:** The neighborhood of a vertex, v, is the set of all adjacent (or neighbor) vertices of that vertex and is denoted as $N(v)$. For a set of vertices, A, the neighborhood of A, denoted $N(A)$, is the set of all neighbor vertices to any vertex within the set A.

- **In-Neighborhood,** $N^{-}(u)$**:** The in-neighborhood of a vertex, u, is the set of all vertices v adjacent to u such that the edge between them is directed towards u.

$$N^{-}(u) = \{v|(v, u) \in E\}$$

- **Out-Neighborhood,** $N^{+}(u)$**:** The out-neighborhood of a vertex, u, is the set of all vertices v adjacent to u such that the edge between them is directed towards v (away from u).

$$N^{-}(u) = \{v|(u, v) \in E\}$$

- **Degree Sequence:** The degree sequence of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order.