EECS 203: Discrete Mathematics
Winter 2024
FoF Discussion 2 Notes

1 Predicates

Predicate: A sentence or mathematical expression whose truth value depends on a parameter, and becomes a proposition when the parameter is specified. For example, "$x > 10$" is a predicate that depends on the parameter x.

Universal quantifier: Denoted by $\forall$ and read as "for all", it specifies that the following propositional function is true for all possible parameters in the domain.

Existential quantifier: Denoted by $\exists$ and read as "there exists", it specifies that the following propositional function is true for at least one of the possible parameters in the domain.

1.1 Instantiation and Quantification

1.1.1 Predicates into Propositions

Let $Q(x)$ be the statement "$x + 1 > 2x$", where the domain of x is all integers. For each of the following parts, write the full proposition and decide if it is true or false.

a) $Q(0)$

> **Solution:** $0 + 1 > 2(0)$; True.

b) $Q(-1)$

> **Solution:** $(-1) + 1 > 2(-1)$; True.

c) $Q(1)$

> **Solution:** $1 + 1 > 2(1)$; False.

d) $\exists x\, Q(x)$

Solution: $\exists x\, [x + 1 > 2x]$; True.
To prove an exists statement, we just need to find one element of the domain for which the statement holds. Let's consider $x = -10$. $Q(-10)$ is the statement "$-10 + 1 > 2(-10)$". This is true, since

$$-10 + 1 = -11 > -20 = 2(-10).$$

There are many other values of x for which the statement holds as well (any integer less than 1 will work). In fact, in a) and b), we already found two others!

e) $\forall x\, Q(x)$

Solution: $\forall x\, [x + 1 > 2x]$; False.
To prove a for all statement, we need to show the statement is true for all x in the domain. Alternatively, we would prove it false by finding a counterexample. In this case, we can provide a counterexample, $x = 1$. This disproves the statement, so it is false.

f) $\exists x\, \neg Q(x)$

Solution: $\exists x\, [x + 1 \leq 2x]$; True.
To prove this statement, we only need to find one example where $Q(x)$ is false. We have found this example in part (c). Note that the negation of $>$ is $\leq$, not $<$.

g) $\forall x \neg Q(x)$

Solution: $\forall x\, [x + 1 \leq 2x]$; False.
To prove this statement false, we only need one example to show $Q(x)$ is true, and we can use examples such as $x = 0$ or $x = -1$.

1.1.2 Rewriting in English

Translate these statements into English where the predicate $C(x, y)$ is "x is the color y", where the domain for x is all objects and the domain for y is all possible colors, respectively.

a) $\exists x C(x, \text{yellow})$

Solution: There exists some object such that the object is the color yellow. More naturally, there is a yellow object.

b) $\forall x(C(x, \text{black}) \lor C(x, \text{white}))$

> **Solution:** For all objects, the object is either black or white. More naturally, every object is either black or white (or both).

c) $\forall x(C(x, \text{black})) \lor \forall x(C(x, \text{white}))$

> **Solution:** Either for all objects, the object is black, or for all objects, the object is white. More naturally, either every object is black or every object is white.

d) $\exists x C(x, \text{yellow}) \lor \exists x C(x, \text{black})$

> **Solution:** There exists an object such that the object is yellow, or there exists an object such that the object is black. More naturally, some object is yellow or some object is black.

e) $\exists x \neg C(x, \text{purple})$

> **Solution:** There exists an object such that the object is not purple. More naturally, some object isn't purple.

f) $\forall x \neg C(x, \text{white})$

> **Solution:** For all objects, the object is not white. More naturally, no object is white. Alternatively, every object is not white.
>
> Note that "not every object is white" is not a valid answer. This would allow some objects to be white, so long as one of them isn't.

1.1.3 Rewriting in Logic

Let $P(x)$ be "x is perfect"; let $F(x)$ be "x is your friend"; and let the domain of quantifiers be all people. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.

b) Not everyone is perfect.

c) All your friends are perfect.

d) At least one of your friends is perfect

e) Everyone is your friend and is perfect.

f) Not everybody is your friend or someone is not perfect.

Solution: Let $P(x)$ be "x is perfect"; let $F(x)$ be "x is your friend"; and let the domain (universe of discourse) be all people.

a) This means that everyone has the property of being not perfect: $\forall x \neg P(x)$. Alternatively, we can write this as $\neg \exists x P(x)$, which says that there does not exist a person who is perfect.

b) This is just the negation of "Everyone is perfect": $\neg \forall x P(x)$. Alternatively we could write $\exists x \neg P(x)$ (i.e. there exists someone who is not perfect).

c) If someone is your friend, then that person is perfect: $\forall x(F(x) \rightarrow P(x))$. Note the use of conditional statement with universal quantifiers.

d) $\exists x(F(x) \wedge P(x))$. Note the use of conjunction ($\wedge$) with existential quantifiers to restrict the domain. (Note this allows the possibility that more than one of your friends are perfect.)

e) The expression is $\forall x(F(x) \wedge P(x))$. Note that here we did use a conjunction with the universal quantifier because $\forall$ distributes over $\wedge$. We can also split up the expression into two quantified statements and write $(\forall x F(x)) \wedge (\forall x P(x))$.

f) The correct expression is $(\neg \forall x F(x)) \vee (\exists x \neg P(x))$.

1.2 Nested Quantifiers

Nested Quantifiers: There's nothing stopping us from using multiple quantifiers at a time. When one quantifier is inside another, we can call it *nested*. By default, we should never assume that we can rearrange things; in particular, rearranged quantifiers can change the meaning of a statement. For example, $\forall x \exists y\, P(x, y)$ is different from $\exists y \forall x\, P(x, y)$.

1.2.1 Rewriting in English

Let P(x, y) be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

a) $\exists x \exists y P(x, y)$

b) $\exists x \forall y P(x, y)$

c) $\forall x \exists y P(x, y)$

d) $\exists y \forall x P(x, y)$

e) $\forall y \exists x P(x, y)$

f) $\forall x \forall y P(x, y)$

Solution:

a) There is a student in your class has taken a computer science course [at your school].

b) There is a student in your class who has taken every computer science course.

c) Every student in your class has taken at least one computer science course.

d) There is a computer science course that every student in your class has taken.

e) Every computer science course has been taken by at least one student in your class.

f) Every student in your class has taken every computer science course.

1.2.2 Rewriting in Logic

Let $K(x, y)$ be the statement "x knows y". Translate each of the following English statements into logical expressions using predicates, quantifiers, and logical connectives

a) Everybody knows everybody

b) Somebody knows somebody

c) Nobody knows anybody

d) Nobody knows everybody

e) Everybody knows somebody

f) There is somebody who everybody knows

Solution:

a) $\forall x \forall y K(x, y)$ (Or $\forall y \forall x K(x, y)$)

b) $\exists x \exists y K(x, y)$ (Or $\exists y \exists x K(x, y)$)

c) $\forall x \forall y \neg K(x, y)$ (Or $\neg \exists x \exists y K(x, y)$. In either case the quantifiers can be flipped)

d) $\neg \exists x \forall y K(x, y)$ (Or $\forall x \neg \forall y K(x, y)$, or $\forall x \exists y \neg K(x, y)$. Note that the quantifiers cannot be flipped in any of these cases)

e) $\forall x \exists y K(x, y)$.

f) $\exists y \forall x K(x, y)$. Note that this is the answer from part (e) with the quantifiers flipped; clearly it is a different statement

1.3 Quantifiers and Negation

Find the negation of each of these propositions. Simplify so that your answers do not include the negation symbol.

a) $\exists x[-4 < x \leq 1]$

b) $\forall z \exists x \exists y[x^3 + y^3 = z^3]$

Solution:

a) $\neg(\exists x(-4 < x \leq 1))$
$\equiv \forall x(\neg(-4 < x \leq 1))$
$\equiv \forall x((x \leq -4) \vee (x > 1))$

b) $\neg(\forall z \exists x \exists y(x^3 + y^3 = z^3))$
$\equiv \exists z \neg(\exists x \exists y(x^3 + y^3 = z^3))$
$\equiv \exists z \forall x \neg(\exists y(x^3 + y^3 = z^3))$
$\equiv \exists z \forall x \forall y(\neg(x^3 + y^3 = z^3))$
$\exists z \forall x \forall y(x^3 + y^3 \neq z^3)$

1.4 Quantified Statement Counterexamples

Find a counterexample, if possible, to these quantified statements, where the domain for all variables is integers.

a) $\forall x \exists y(x = 1/y)$

b) $\forall x \exists y(y^2 - x < 100)$

c) $\forall x \forall y(x^2 \neq y^3)$

Solution:

a) Consider $x = 2$, then there is no y among the integers such that $2 = 1/y$, since the only solution of this equation is $y = 1/2$.

b) Consider $x = -200$. The statement claims there exists a y such that $y^2 + 200 < 100$. This would require our y^2 to be negative, which is not possible in the domain of integers.

c) Consider $x = y = 0$. $x^2 = 0$ and $y^3 = 0$, so $x^2 = y^3$.

1.5 Domain Restrictions

1.5.1 Math

Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators.

a) The product of two negative real numbers is positive.

b) The difference of a real number and itself is zero.

c) Every positive real number has exactly two square roots.

d) A negative real number does not have a square root that is a real number.

Solution: Let the domain of all variables be the real numbers.

a) $\forall x \forall y[(x<0) \wedge (y<0) \implies (xy>0)]$

b) $\forall x(x-x=0)$

c) $\forall x \exists a \exists b \, \{a \neq b \wedge \forall c \, [c^2 = x \iff (c=a \vee c=b)]\}$

d) $\forall x[(x<0) \implies \neg \exists y(x=y^2)]$. Equivalently $\forall x \neg[(x<0) \wedge (\exists y(x=y^2))]$, or $\neg \exists x[(x<0) \wedge (\exists y(x=y^2))]$.

2 Introduction to Proofs

2.1 Even and Odd

Integer: A positive or negative whole number (including 0)
Even: An integer x is even if there exists an integer k with $x = 2k$.
Odd: An integer x is odd if there exists an integer k with $x = 2k + 1$.

2.1.1 Even Examples

Come up with three examples of even numbers. Prove that they are even.

Solution: There are many examples. Here are a few:

- 0 is even, since $0 = 2 \cdot 0$
- 6 is even, since $6 = 2 \cdot 3$
- -8 is even, since $-8 = 2 \cdot -4$

2.1.2 Odd Examples

Come up with three examples of odd numbers. Prove that they are odd.

Solution: There are many examples. Here are a few:

- 1 is odd, since $1 = 2 \cdot 0 + 1$
- 7 is odd, since $7 = 2 \cdot 3 + 1$
- -7 is odd, since $-9 = 2 \cdot -4 + 1$

2.1.3 Odd Proof

Prove that the sum of an even and an odd integer is always odd.

Solution: Without loss of generality (WLOG), let x be an **arbitrary** even integer and y be an **arbitrary** odd integer. By definition, then, x and y can be written as $x = 2n$ and $y = 2m + 1$ for some integers n and m. Looking at their sum, we have

$$\begin{aligned} x + y &= 2n + 2m + 1 \\ &= 2(n + m) + 1. \end{aligned}$$

Since $x + y = 2c + 1$, where c is the integer $n + m$, then by definition, $x + y$ is odd. Therefore, this relation holds for all even x and odd y, and we have proved that the sum of an even and an odd integer is odd.

Note that without loss of generality (WLOG) is used when the same argument can be made for multiple cases, and there is some symmetry between variables.

2.1.4 Even Proof

Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even.

Solution: Using a Direct Proof,

- Let $m + n$ and $n + p$ be **arbitrary** even integers, $m + n = 2a$ and $n + p = 2b$, for some integers a and b.

- Since $m + n = 2a$, we have that $m = 2a - n$
- Likewise, since $n + p = 2b$, we have that $p = 2b - n$
- Therefore, $m + p = (2a - n) + (2b - n) = 2(a + b - n)$
- Since a, b and n are all integers, $(a + b - n)$ is an integer
- Since $m + p$ is equal to two times an integer, $m + p$ is even

Therefore, the statement "if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even" holds for all even $m + n$ and $n + p$.

2.2 Divisibility

Divisibility: Given integers n and a, we say that n *divides* a, written $n \,|\, a$, when there exists an integer k so that $nk = a$.

2.2.1 Divides Proof

Prove that if n is odd, then $4 \,|\, (n^2 - 1)$.

Solution:

- Assume n is odd. So $n = 2k + 1$ for some integer k.
- So $n^2 - 1 = (2k + 1)^2 - 1 = (4k^2 + 4k + 1) - 1 = 4k^2 + 4k = 4(k^2 + k)$.
- Since k in an integer, $(k^2 + k)$ is also an integer.
- Thus $(n^2 - 1)$ is 4 times an integer, i.e., $4 \,|\, (n^2 - 1)$.

2.2.2 Divides Proof 2

Prove that if $a \,|\, c$ and $b \,|\, d$, then $ab \,|\, cd$ where a, b, c, and d are all integers.

Solution:

- Assume that a and b are arbitrary integers and c and d are integers such that $a|c$ and $b|d$.
- By the definition of divisibility, there exists an integer k_1 such that $c = ak_1$.
- By the definition of divisibility, there exists an integer k_2 such that $d = bk_2$.
- $cd = ak_1 \cdot bk_2 = ab(k_1k_2)$.
- The product of two integers is an integer, so k_1k_2 is an integer. Thus, by the definition of divisibility, $ab|cd$.

This proves that if $a|c$ and $b|d$ then $ab|cd$.

2.3 Disproofs

Disproof: To disprove a statement means to prove the negation of that statement.

$$\text{Disprove } P(x) \equiv \text{Prove } \neg P(x)$$

Note that if the statement you are trying to disprove is a for all statement, all you need to disprove it is a singular counterexample (since $\neg \forall x P(x) \equiv \exists x \neg P(x)$).

Rational Numbers: A number is rational if it can be written as the ratio of two integers: $\frac{p}{q}$.

Prime Numbers A prime number is a number greater than 1 whose only factors are 1 and itself.

Composite Numbers: A composite number is a number which has at least one factor other than 1 and itself (ie not a prime number). Note that 1 is neither prime nor composite.

2.3.1 Two Sides of the Same Coin

Disprove each of the following statements.

a) For all real numbers x and y, if they sum to zero, one of them is negative and the other is positive.

Solution: To disprove a "for all", we prove an "exists", so we need to find a counterexample (some values of x and y that make the statement false).

Consider $x = 0$ and $y = 0$.

We have that $x+y = 0$, so x and y sum to 0. However, $x = y = 0$ is neither negative nor positive.

Therefore, it is not true for all real numbers x and y that, if they sum to zero, one of them is negative and the other is positive.

b) For all nonzero rational numbers x and y, if they are multiplicative inverses, $x \neq y$. (Two numbers are multiplicative inverses if their product is 1.)

Solution: Consider $x = 1$ and $y = 1$.
We have that $x \cdot y = 1$, so x and y are multiplicative inverses. However, $x = y$.
Therefore, it is not true for all nonzero rational numbers x and y that, if they are multiplicative inverses, then $x \neq y$.

2.4 Connecting Logic and Proofs

2.4.1 Negation Station

For each of the following statements, write the statement's negation. Then, determine which is true: the original statement or the negated statement? (You do not need a rigorous proof.)

Reminder: Two numbers, x and y, are **multiplicative inverses** if $xy = 1$.

a. For all real numbers x and y, if $x + y = 0$, then one of them is negative and the other is positive.

b. For all nonzero rational numbers x and y, if they are multiplicative inverses, then $x \neq y$.

c. Each non-zero rational number has a rational multiplicative inverse.

d. Each non-zero integer has an integer multiplicative inverse.

Solution:

a. "There exist real numbers x and y such that $x+y = 0$ and they do not have opposite signs."

The **negated** statement is true. (Consider $x = y = 0$.)

b. "There exist non-zero rational numbers x and y such that they are multiplicative inverses and $x = y$."

The **negated** statement is true. (Consider $x = y = 1$.)

c. "There exists a non-zero rational number that does not have a rational multiplicative inverse."

The **original** statement is true. (Any non-zero rational number $x = a/b$ has a multiplicative inverse of b/a, which is also rational, where a and b are both integers.)

d. "There exists a non-zero integer that does not have an integer multiplicative inverse."

The **negated** statement is true. (Consider $x = 2$, whose multiplicative inverse is $1/2$, which is not an integer.)

2.4.2 Quantifier Proofs

Building on the last question, prove or disprove each of the following statements.
(If you find it helpful to translate the statements to logical connectives and symbols first, you can, but it's not required that you; you can just work with the English statements directly.)

a. For all nonzero rational numbers x and y, if they are multiplicative inverses, then $x \neq y$.
b. Each non-zero rational number has a multiplicative inverse that is also a rational number.

Solution:

a. **Disprove:**
We will prove the negation: "There exist non-zero rational numbers x and y such that they are multiplicative inverses and $x = y$."
Consider the non-zero rational numbers $x = 1$ and $y = 1$. They are multiplicative inverses since $xy = 1 \cdot 1 = 1$, and they are equal to each other, i.e., $x = y$.

Alternate disproof (for logical expression lovers):
Let x and y come from the domain of all non-zero rational numbers. We can restate the claim as:

$$\forall x \forall y [xy = 1 \rightarrow x \neq y]$$

We can disprove with a counterexample. Note that in symbolic notation, the negation of the original statement is

$$\exists x \exists y [xy = 1 \land x = y]$$

We will prove the negation is true. Consider $x = 1$ and $y = 1$. Then $xy = 1 \cdot 1 = 1$, and $x = y = 1$. Therefore, the negation of the original statement is true, which means the original statement must be false.

b. Prove:
Let x be an arbitrary non-zero rational number. Because $x \cdot 1/x = 1$, the multiplicative inverse of x is $1/x$. Since x is rational, we have $x = a/b$ for some integers a and b, which gives us $y = 1/x = b/a$. So y is rational.

Alternate proof (for logical expression lovers): Let x and y come from the domain of all non-zero rational numbers.

$$\forall x \exists y [xy = 1]$$

Prove:

- Take an arbitrary non-zero rational number x.
- By definition of rational numbers, $x = \frac{p}{q}$ for some integers p and q where $q \neq 0$.
- Since $x \neq 0$, $p \neq 0$.
- Let $y = \frac{q}{p}$. (We can do this since $p \neq 0$)
- y is a rational number by definition
- Since $q \neq 0$, $y \neq 0$.
- $xy = \frac{p}{q} \cdot \frac{q}{p} = 1$

Thus, for all non-zero rational numbers x there exists an inverse rational number y such that $xy = 1$.