Groupwork 8 Problems

1. Commit Tea Party [15 points]

Two committees are having a meeting. If there are 12 people in each committee, how many different ways can they sit around a table given the following restrictions? Note that two orderings are considered equal if each person has the same two neighbors (without distinguishing their left and right neighbors).

(a) There are no restrictions on seating.

(b) Two people in the same committee cannot be neighbors.

(c) Everybody must have exactly two neighbors from their committee.

(d) Everybody must have exactly one neighbor from their committee.

Solution:

There are 24 people in total and 2 committees.
a) No restrictions on seating.
Therefore, it is a combination problem.
$C(24, 12) = \frac{24!}{12!12!} = 2704156$ ways.
b) Two people in the same committee cannot be neighbors.
Therefore, it is a permutation problem.
$P(24, 2) = 24 \cdot 23 = 552$ ways.
c) Everybody must have exactly two neighbors from their committee.
Therefore, it is a permutation problem.
$P(24, 3) = 24 \cdot 23 \cdot 22 = 12144$ ways.
d) Everybody must have exactly one neighbor from their committee.
Therefore, it is a permutation problem.
$P(24, 2) = 24 \cdot 23 = 552$ ways.

2. Hiking Extravaganza [15 points]

Prove that every complete n-node weighted graph (with all possible edges) with $n \geq 1$ and all distinct edge weights has a (possibly non-simple) path of $n - 1$ edges along which the edge weights are strictly increasing.

Hint: Start by placing a hiker on each node. Try to show that the hikers can walk paths of *total* length $n(n - 1)$, each along increasing-weight paths.

Solution:

Let G be a complete n-node weighted graph with all distinct edge weights.
Let $v_1, v_2, \ldots, v_n$ be the nodes of G.
Let w_{ij} be the weight of the edge between v_i and v_j.
Let H be a complete n-node weighted graph with all distinct edge weights.
Let $u_1, u_2, \ldots, u_n$ be the nodes of H.
Let x_{ij} be the weight of the edge between u_i and u_j.
Let h_{ij} be the weight of the edge between v_i and u_j.
Let $h_{ij} = w_{ij} + \sum_{k=1}^{j-1} x_{k(k+1)}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$.