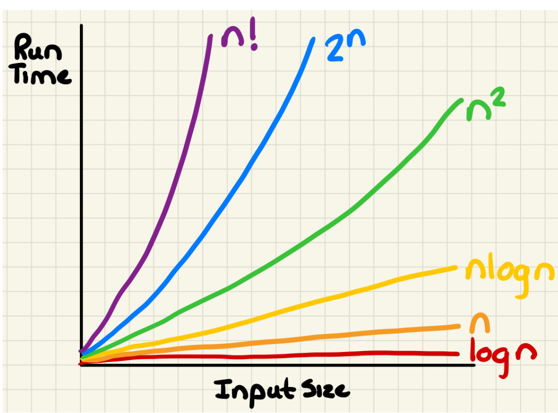
EECS 203: Discrete Mathematics
Winter 2024
FOF Discussion 11 Notes

1 Run Time Analysis

1.1 Definitions

- **Big-O:** f is $O(g)$ means f grows no faster than g...
  $f(n) = O(g(n)) \implies \exists c, k > 0 \; \forall n > k \; \Big[f(n) \leq c|g(n)| \Big]$
- **Big-Ω:** f is $\Omega(g)$ means f grows at least as fast as g...
  $f(n) = \Omega(g(n)) \implies \exists c, k \; \forall n > k \; \Big[|f(n)| \geq c|g(n)| \Big]$
- **Big-Θ:** f is $\Theta(g)$ means f grows at the same rate as g...
  $f(n) = \Theta(g(n)) \implies \exists k, c_1, c_2 \; \forall n > k \; \Big[c_1|g(n)| \leq |f(n)| \leq c_2|g(n)| \Big]$
  - f is $\Theta(g)$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
- **Run Time of Standard Functions:**



- **Properties for Combining Functions:** Consider positive-valued functions $f_1(n) = \Theta(g_1(n))$ and $f_2(n) = \Theta(g_2(n))$.
  - **Addition:** $(f_1 + f_2)(n) = \Theta(\max(g_1(n), g_2(n)))$
  - **Scalar Multiplication:** $af(n) = \Theta(f(n))$
  - **Product:** $(f_1 \cdot f_2)(n) = \Theta(g_1(n) \cdot g_2(n))$

- **Divide and Conquer Algorithm:** An algorithm which divides a problem into smaller **non-overlapping** sub-problems, solves each of those sub-problems, and then combines the results of the sub-problems into the final result.
- **Sub-problem:** A smaller-version of a problem that can be solved by the same algorithm for the larger problem.
- **Master Theorem:** If the runtime for an algorithm can be modeled by a recurrence relation of the form $T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^d)$ where $a > 0$, $b > 1$, and $d \geq 0$, then

$$T(n) \text{ is } \begin{cases} \Theta(n^d) & \text{if } \frac{a}{b^d} < 1 \\ \Theta(n^d \log n) & \text{if } \frac{a}{b^d} = 1 \\ \Theta(n^{\log_b a}) & \text{if } \frac{a}{b^d} > 1 \end{cases}$$

- **Rules for Logarithms:** You should keep the following log rules in mind:
  - $\log(xy) = \log(x) + \log(y)$
  - $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$
  - $\log(x^y) = y \log(x)$
  - $\log_b(a) = \dfrac{\log_c(a)}{\log_c(b)}$ for all $a, b, c \in \mathbb{R}^+$ where $b > 1$ and $c > 1$
  - $\log_b(b) = 1$ for all $b \in \mathbb{R}^+$ where $b > 1$

1.2 Exercises

1.2.1 Big-O

Give a big-O estimate for each of these functions. Use a simple function of the smallest order.

(a) $n{\cdot}\log(n^2 + 1) + n^2{\cdot}\log(n)$

(b) $(n \cdot log(n) + 1)^2 + (log(n) + 1)(n^2 + 1)$

(c) $n^{2^n} + n^{n^2}$

1.2.2 Big-Ω, Big-Θ

For each function, determine whether that function is $\Omega(x^2)$ and whether it is $\Theta(x^2)$.

(a) $f(x) = 17x + 11$

(b) $f(x) = x\text{log}x$

(c) $f(x) = 2^x$

(d) $f(x) = x^2 + 1000$

(e) $f(x) = x^4/2$

(f) $f(x) = \lfloor x \rfloor \cdot \lceil x \rceil$

1.2.3 Algorithms

Give the tightest big-O estimate for the number of operations (where an operation is arithmetic, a comparison, or an assignment) used in each of the following algorithms:

(a)

procedure $findMax(a_1, a_2, ..., a_N$: real numbers)
  $max := 0$
  for $i := 1$ to N
    if $a_i > max$
      $max = a_i$
  return max

(b)

procedure $sumOddIndices(a_1, a_2, ...a_N$: real numbers)
  $i := 1$
  $oddIndexSum := 0$
  while $i \leq N$
    $oddIndexSum := oddIndexSum + a_i$
    $i := i + 2$
  return $oddIndexSum$

(c)

procedure $findMinPowerAboveN(N$: positive integer)
  $i := 1$
  while $i \leq N$
    $i := i * 2$
  return i

(d)

procedure $findMaxDifference(a_1, a_2, ..., a_N$: real numbers)
  $maxDiff := 0$
  for $i := 1$ to N
    for $j := 1$ to N
      if $a_i - a_j > maxDiff$
        $maxDiff := a_i - a_j$
  return $maxDiff$

(e)

procedure $countElementsGreaterThanMean(a_1, a_2, ..., a_N$: real numbers)
  $sum := 0$
  $numGreaterThanMean := 0$
  for $i := 1$ to N
    $sum := sum + a_i$
  $mean := sum/N$
  for $j := 1$ to N
    if $a_j > mean$
      $numGreaterThanMean := numGreaterThanMean + 1$
  return $numGreaterThanMean$

1.2.4 Master Theorem

Consider the function f such that

$$f(n) = 2f(\frac{n}{4}) + n, f(1) = 2$$

a) Find $f(16)$.

b) Use the master theorem to find the tightest big-O estimate of f

2 Exam Review

2.1 Distributing Objects into Bins

For each of the following identify whether the objects/bins are indistinguishable or distinguishable. Then solve the problem.

(a) How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

(b) How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

(c) How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

2.2 Counting

How many ways are there for a horse race with three horses to finish if ties are possible? [Note: Two or three horses may tie.]

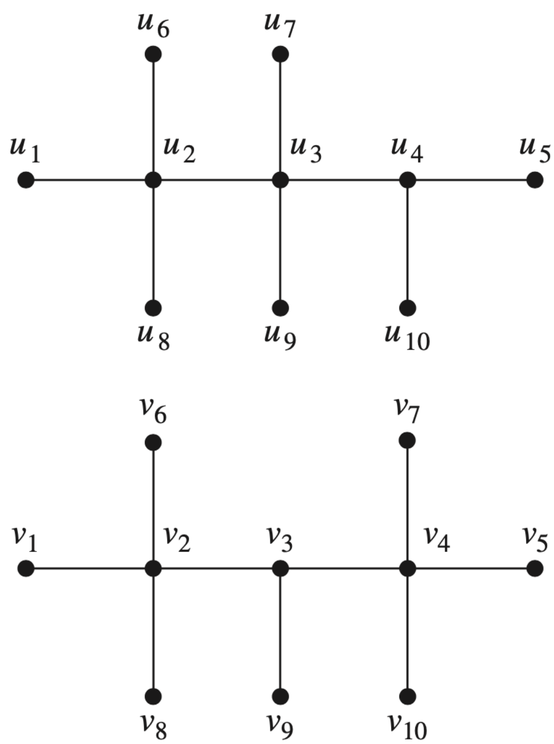
2.3 Poker Hands

(a) Find the probability that a hand of five cards in poker contains at least 2 Aces.

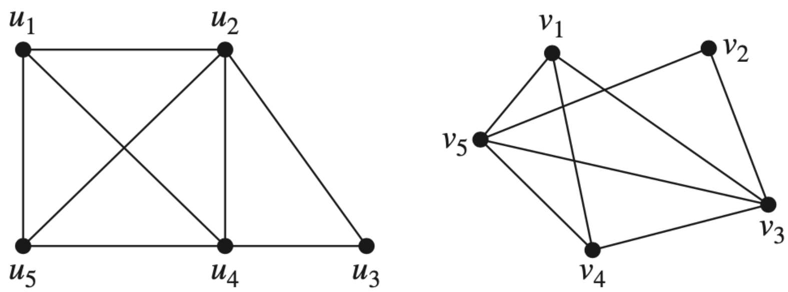
(b) Find the probability a hand of five cards in poker has exactly one of every face card(Jack, Queen, King).

2.4 Graph Isomorphisms

Determine whether or not the following graphs are isomorphic and thoroughly justify your answers.



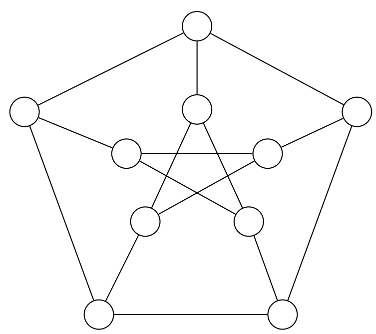
(a)



(b)

2.5 Coloring

Show that the following graph is 3-colorable and that this is the smallest number of colors needed to color it. Do so by explaining why it is not two colorable and then giving a three coloring.



2.6 Exam Scores

The final exam of a discrete mathematics course consists of 50 true/false questions, each worth two points, and 25 multiple-choice questions, each worth four points. The probability that Linda answers a true/false question correctly is 0.9, and the probability that she answers a multiple-choice question correctly is 0.8. What is her expected score on the final?

2.7 Hat Check Problem

Each of n customers gives a hat to a hat-check person at a restaurant. The hat-check person gives the hats back to the customers in a random order. What is the expected number of customers who get back their own hat?

2.8 Predicting Success

An electronics company is planning to introduce a new camera phone. The company commissions a marketing report for each new product that predicts either the success or the failure of the product. Of new products introduced by the company, 60% have been successes. Furthermore, 70% of their successful products were predicted to be successes, while 40% of failed products were predicted to be successes. Find the probability that this new camera phone will be successful if its success has been predicted.