EECS 203: Discrete Mathematics
Winter 2024
FoF Discussion 2 Notes

1 Predicates

Predicate: A sentence or mathematical expression whose truth value depends on a parameter, and becomes a proposition when the parameter is specified. For example, "$x > 10$" is a predicate that depends on the parameter x.

Universal quantifier: Denoted by $\forall$ and read as "for all", it specifies that the following propositional function is true for all possible parameters in the domain.

Existential quantifier: Denoted by $\exists$ and read as "there exists", it specifies that the following propositional function is true for at least one of the possible parameters in the domain.

1.1 Instantiation and Quantification

1.1.1 Predicates into Propositions

Let $Q(x)$ be the statement "$x + 1 > 2x$", where the domain of x is all integers. For each of the following parts, write the full proposition and decide if it is true or false.

a) $Q(0)$

b) $Q(-1)$

c) $Q(1)$

d) $\exists x\, Q(x)$

e) $\forall x\, Q(x)$

f) $\exists x \neg Q(x)$

g) $\forall x \neg Q(x)$

1.1.2 Rewriting in English

Translate these statements into English where the predicate $C(x, y)$ is "x is the color y", where the domain for x is all objects and the domain for y is all possible colors, respectively.

a) $\exists x C(x, \text{yellow})$

b) $\forall x (C(x, \text{black}) \vee C(x, \text{white}))$

c) $\forall x (C(x, \text{black})) \vee \forall x (C(x, \text{white}))$

d) $\exists x C(x, \text{yellow}) \vee \exists x C(x, \text{black})$

e) $\exists x \neg C(x, \text{purple})$

f) $\forall x \neg C(x, \text{white})$

1.1.3 Rewriting in Logic

Let $P(x)$ be "x is perfect"; let $F(x)$ be "x is your friend"; and let the domain of quantifiers be all people. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.
b) Not everyone is perfect.
c) All your friends are perfect.
d) At least one of your friends is perfect
e) Everyone is your friend and is perfect.
f) Not everybody is your friend or someone is not perfect.

1.2 Nested Quantifiers

Nested Quantifiers: There's nothing stopping us from using multiple quantifiers at a time. When one quantifier is inside another, we can call it *nested*. By default, we should never assume that we can rearrange things; in particular, rearranged quantifiers can change the meaning of a statement. For example, $\forall x \exists y\, P(x, y)$ is different from $\exists y \forall x\, P(x, y)$.

1.2.1 Rewriting in English

Let P(x, y) be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

a) $\exists x \exists y P(x, y)$

b) $\exists x \forall y P(x, y)$

c) $\forall x \exists y P(x, y)$

d) $\exists y \forall x P(x, y)$

e) $\forall y \exists x P(x, y)$

f) $\forall x \forall y P(x, y)$

1.2.2 Rewriting in Logic

Let $K(x, y)$ be the statement "x knows y". Translate each of the following English statements into logical expressions using predicates, quantifiers, and logical connectives

a) Everybody knows everybody

b) Somebody knows somebody

c) Nobody knows anybody

d) Nobody knows everybody

e) Everybody knows somebody

f) There is somebody who everybody knows

1.3 Quantifiers and Negation

Find the negation of each of these propositions. Simplify so that your answers do not include the negation symbol.

a) $\exists x[-4 < x \leq 1]$
b) $\forall z \exists x \exists y[x^3 + y^3 = z^3]$

1.4 Quantified Statement Counterexamples

Find a counterexample, if possible, to these quantified statements, where the domain for all variables is integers.

a) $\forall x \exists y(x = 1/y)$
b) $\forall x \exists y(y^2 - x < 100)$
c) $\forall x \forall y(x^2 \neq y^3)$

1.5 Domain Restrictions

1.5.1 Math

Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators.

a) The product of two negative real numbers is positive.

b) The difference of a real number and itself is zero.

c) Every positive real number has exactly two square roots.

d) A negative real number does not have a square root that is a real number.

2 Introduction to Proofs

2.1 Even and Odd

Integer: A positive or negative whole number (including 0)

Even: An integer x is even if there exists an integer k with $x = 2k$.

Odd: An integer x is odd if there exists an integer k with $x = 2k + 1$.

2.1.1 Even Examples

Come up with three examples of even numbers. Prove that they are even.

2.1.2 Odd Examples

Come up with three examples of odd numbers. Prove that they are odd.

2.1.3 Odd Proof

Prove that the sum of an even and an odd integer is always odd.

2.1.4 Even Proof

Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even.

2.2 Divisibility

Divisibility: Given integers n and a, we say that n *divides* a, written $n \mid a$, when there exists an integer k so that $nk = a$.

2.2.1 Divides Proof

Prove that if n is odd, then $4 \mid (n^2 - 1)$.

2.2.2 Divides Proof 2

Prove that if $a \mid c$ and $b \mid d$, then $ab \mid cd$ where a, b, c, and d are all integers.

2.3 Disproofs

Disproof: To disprove a statement means to prove the negation of that statement.

$$\text{Disprove } P(x) \equiv \text{Prove } \neg P(x)$$

Note that if the statement you are trying to disprove is a for all statement, all you need to disprove it is a singular counterexample (since $\neg \forall x P(x) \equiv \exists x \neg P(x)$).

Rational Numbers: A number is rational if it can be written as the ratio of two integers: $\frac{p}{q}$.

Prime Numbers A prime number is a number greater than 1 whose only factors are 1 and itself.

Composite Numbers: A composite number is a number which has at least one factor other than 1 and itself (ie not a prime number). Note that 1 is neither prime nor composite.

2.3.1 Two Sides of the Same Coin

Disprove each of the following statements.

a) For all real numbers x and y, if they sum to zero, one of them is negative and the other is positive.

b) For all nonzero rational numbers x and y, if they are multiplicative inverses, $x \neq y$. (Two numbers are multiplicative inverses if their product is 1.)

2.4 Connecting Logic and Proofs

2.4.1 Negation Station

For each of the following statements, write the statement's negation. Then, determine which is true: the original statement or the negated statement? (You do not need a rigorous proof.)

Reminder: Two numbers, x and y, are **multiplicative inverses** if $xy = 1$.

a. For all real numbers x and y, if $x + y = 0$, then one of them is negative and the other is positive.
b. For all nonzero rational numbers x and y, if they are multiplicative inverses, then $x \neq y$.
c. Each non-zero rational number has a rational multiplicative inverse.
d. Each non-zero integer has an integer multiplicative inverse.

2.4.2 Quantifier Proofs

Building on the last question, prove or disprove each of the following statements.
(If you find it helpful to translate the statements to logical connectives and symbols first, you can, but it's not required that you; you can just work with the English statements directly.)

a. For all nonzero rational numbers x and y, if they are multiplicative inverses, then $x \neq y$.

b. Each non-zero rational number has a multiplicative inverse that is also a rational number.