EECS 203: Discrete Mathematics
Winter 2024
FOF Discussion 5a Notes

1 Induction

- **Mathematical Induction**: Mathematical Induction is a proof method used to prove a predicate $P(n)$ holds for "all" $n \geq n_0$. Often "all" n is $\mathbb{N}$ or $\mathbb{Z}^+$, but the desired domain of n varies by problem. Mathematical induction consists of a base case and an inductive step, which proves: $[P(n_0) \land \forall k \geq n_0(P(k) \implies P(k+1))] \implies \forall n \geq n_0, P(n)$
- **Induction Steps**:
  - **Base Case**: The part of the inductive proof which directly proves the predicate for the *first* value in the domain (generally n_0). The base case does not rely on $P(k)$ for any other value of k. Often this will be $P(0)$ or $P(1)$
  - **Inductive Hypothesis**: The assumption we make at the beginning of the inductive step. The inductive hypothesis assumes that the predicate holds for some *arbitrary* member of the domain
  - **Inductive Step**: The proof which shows that the predicate holds for the "next" value in the domain. The inductive step should make use of the inductive hypothesis.
- **Exponent Product Rule**: $b^n \cdot b^m = b^{n+m}$. Often useful in induction proofs involving exponents

1.1 Equality

Prove by induction that the following equality is true for all positive integers n.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

Let $P(n)$ be ______________________________ = ________________.

Inductive Step: We assume that $P(k)$ is true for an arbitrary positive integer k such that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\frac{1}{k\cdot(k+1)}=\frac{k}{k+1}$. It must be shown that $\underline{P(\quad)}$ follows from this assumption.

Consider the LHS of P(k+1):

$$\begin{aligned}
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k\cdot(k+1)}+\underline{\hspace{4cm}} &= \underline{\hspace{3cm}}+\frac{1}{(k+1)\cdot((k+1)+1)}\\
&= \frac{k}{k+1}+\frac{1}{(k+1)\cdot(k+2)}\\
&= \frac{\underline{\hspace{3cm}}}{(k+1)(k+2)}+\frac{1}{\underline{\hspace{3.5cm}}}\\
&= \underline{\hspace{5cm}}\\
&= \frac{(k+1)^2}{(k+1)(k+2)}\\
&= \underline{\hspace{5cm}}\\
&= \frac{k+1}{(k+1)+1}\\
&= \text{RHS of P(k+1)}
\end{aligned}$$

This shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. Note that the equality in line 1 is true by __________________________.

Base Case:
Our base case of $P(1)$ is true since ______________________________.

Therefore, since $P(1)$ and $\forall k[P(k)\rightarrow P(k+1)]$ are both true, then by mathematical induction, the claim is proven.

1.2 Bandar's Blunder $\star$

Bandar writes a proof for the following statement:

$$n! > n^2 \text{ for all } n \geq 4.$$

His proof is incorrect, and it's your task to help him identify his mistake!

Proof:

Inductive step:
Let k be an arbitrary integer ≥ 4.
Assume $P(k) : k! > k^2$. We need to show $P(k+1) : (k+1)! > (k+1)^2$

$$\begin{aligned}
(k+1)! &= (k+1) \cdot k! \\
&> (k+1) \cdot k^2 && \text{(By the Inductive Hypothesis)} \\
&= (k+1)(k \cdot k) \\
&\geq (k+1)(2 \cdot k) && \text{(Because } k \geq 2\text{)} \\
&= (k+1)(k+k) \\
&\geq (k+1)(k+1) && \text{(Because } k \geq 1\text{)} \\
&= (k+1)^2
\end{aligned}$$

This proves $(k+1)! > (k+1)^2$.

Base Case:
Prove $P(0) : 0! > 0^2$, $0! = 1 > 0^2 = 0$

Thus by mathematical induction, $n! > n^2$ for all $n \geq 0$.

What is wrong with Bandar's proof?

1.3 Check Your Understanding

a) If I wanted to show by induction that 3 divides $n^3 + 2n$ whenever n is a positive integer. What would I need to show?

b) Prove by induction that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Hint : $(a + b)^3 = a^3 + 3ab(a + b) + b^3$

1.4 Sum Mathematical Induction

Using induction, prove that for all integers $n \geq 1$:

$$\sum_{r=1}^{n} (r + 1) \cdot 2^{r-1} = n \cdot 2^n$$

2 Exam Review

2.1 Satisfiability $\star$

Determine whether each of these compound propositions is satisfiable.

(a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$

(b) $(p \rightarrow q) \wedge (p \rightarrow \neg q) \wedge (\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$

2.2 Nested Quantifier Translations

Let P(x, y) be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

a) $\exists x \exists y P(x, y)$

b) $\exists x \forall y P(x, y)$

c) $\forall x \exists y P(x, y)$

d) $\exists y \forall x P(x, y)$

e) $\forall y \exists x P(x, y)$

f) $\forall x \forall y P(x, y)$

2.3 English to Logic Translation

Define the following propositions:

- p: the user enters a valid password
- q: access is granted
- r: the user has paid the subscription fee

Express the following using p, q, r and logical operators

a) the user has paid the subscription fee, but does not enter a valid password

b) access is granted whenever the user has paid the subscriptoin fee and enters a valid password

c) access is denied if the user has not paid the subscription fee

d) if the user has not entered a valid password but has paid the subscription fee, then access is granted

2.4 Proof Practice

2.4.1 Proof I

Prove that the product of two odd numbers is odd.

2.4.2 Proof II

Prove that for all integers n, if $n^2 + 2$ is even, then n is even.

2.4.3 Proof III

Prove that for all integers x and y, if xy^2 is even, then x is even or y is even.

2.4.4 Proof IV

Prove or Disprove that for all integers n, $n^2 + n$ is even.

2.4.5 Proof V

Prove or Disprove that for all integers a and b, $\frac{a}{b}$ is a rational number.

2.5 Set Equality

Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$ by showing that either side is a subset of the other.

2.6 More Power Sets $\star$

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$

b) $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

2.7 Inclusion−Exclusion Principle:

The inclusion-exclusion principle states the the size of the union of two sets is equal to the sum or their sizes minus the size of their intersection:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

A similar principle can be applied to obtain the following formula, for the cardinality of the union of three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

2.8 Three sets

Suppose there is a group of 120 U of M students. Here's what you know:

- There are 31 in Engineering.
- There are 65 in LSA.
- There are 44 in Ross.
- There are 20 that are not in any of these 3 schools.
- There are 15 in Engineering and Ross.
- There are 17 in Engineering and LSA.
- There are 18 in LSA and Ross.

How many are in all 3 schools?