Pigeon Hole:
Hilbert's hotel paradox: make sure the function is bijective

Absolute functions are not one-to-one

Impl breakout/contrapos: $p \to q \equiv \neg p \vee q \equiv \neg q \to \neg p$

The function is continuous and always increasing->one-to-one.

Strong induction: splitting something
Weak induction: everything else basically

For all functions: (if there is a point where the function is undefined, it is not a function)
∀x∃y[f(x)=y] is true
∃x∈A ∀y∈B [f(x)=y] is false (this is saying there are multiple out to the same in)
For **surjectivity (onto)**, we ask: "Does every output have at least one input?"∀y∃x [f(x)=y]

For **injectivity (one-to-one)**, we ask: "Does every input have exactly one unique output?"∀x∃y[f(x)=f(y)->x=y]

Composition functions: if comp is bi, outside is onto and inside is one-to-one

Def. of mod: a=b+mk

If the total number of pigeons is even, there won't be leftovers, but highly likely it's odd and the middle number gets left out(still gets its own hole)-> Pigeons that don't work are still in holes

Injective (1-1) and Surjective (Onto) Proofs

Red: assumption

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

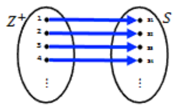
To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Set Comparisons

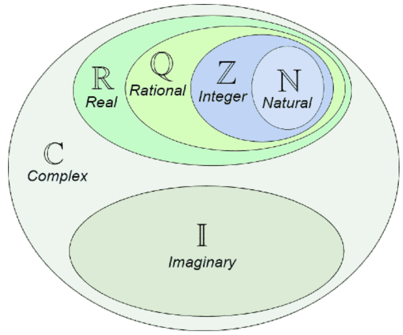
Definition: A set S is **countable** iff $|S| \le |\mathbb{Z}^+|$
(otherwise it's *uncountable*)

$|\mathbb{Z}^+| = \aleph_0$ = "aleph null"

One way to think about countability:
A set is countable if you can list its elements, one after the other, and any given element would be listed within finitely many steps



Some Countable Sets	Some Uncountable Sets
$\mathbb{Z}^+$	$\mathbb{R}$
$\mathbb{N}, \mathbb{Z}$	$[0,1]$
$\mathbb{Q}$	$\mathbb{R} - \mathbb{Q}$
$\mathbb{Z} \times \mathbb{Z}$	$P(\mathbb{Z})$
and many more!	and many more!



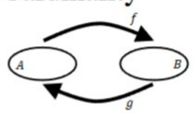
Proving onto

Let $b \in \mathbb{R}^+$ be arbitrary. Let $a = \frac{2}{b}$. Note $b > 0$ so $a \in \mathbb{R}^+$.

$$\begin{aligned} f(a) &= f(2/b) \\ &= \left|\frac{2}{2/b}\right| \\ &= |b| \\ &= b \qquad\qquad b > 0 \end{aligned}$$

One-to-one, Onto, and Cardinality

If $f: A \to B$ is one-to-one,
then $|A| \le |B|$
If $g: B \to A$ is one-to-one,
then $|A| \ge |B|$

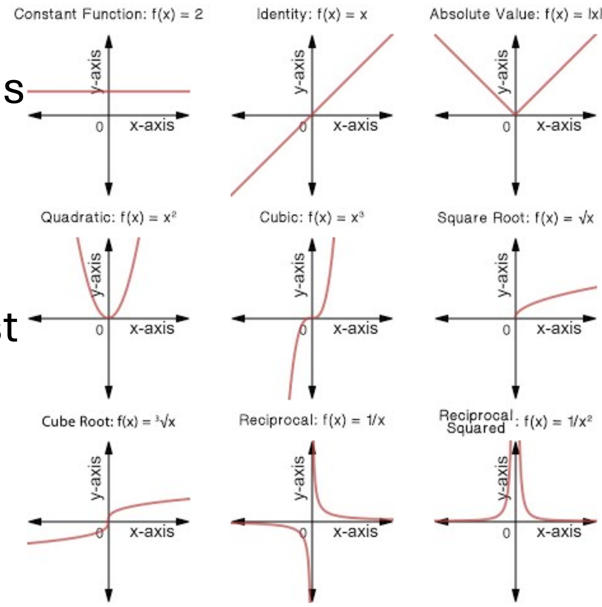


Schroeder-Bernstein Theorem:
If $|A| \le |B|$ and $|B| \le |A|$, then $|A| = |B|$

Ways to show $\|A\| \le \|B\|$	Ways to show $\|B\| \le \|A\|$
$f: A \to B$ is one-to-one	$g: B \to A$ is one-to-one

To prove $|A| = |B|$:
- Give a bijection $A \to B$, or a bijection $B \to A$, OR
- **** Easiest **** Give a **one-to-one** function $A \to B$ **and** a **one-to-one** function $B \to A$

Recurrence: nothing is also one way
If there are no restrictions on repeating, you can repeat



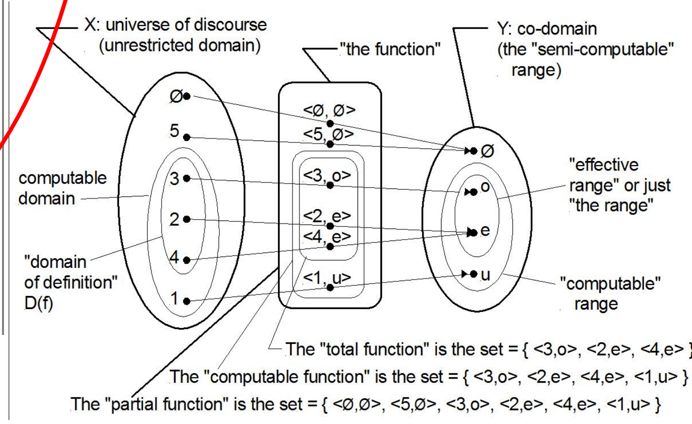
Last largest base case

Guide for Induction Proofs

- Restate the claim you are trying to prove
- **Base case**: Prove the claim holds for the "first" value of n
  - Prove $P(n_0)$ is true
- **Inductive Step**: Prove that $P(k) \to P(k+1)$ for an arbitrary integer k in the desired range.
  - Let k be an arbitrary integer with $k \ge n_0$
  - Assume $P(k)$
  - Show that $P(k+1)$ holds

Equivalently: Show $P(k-1) \to P(k)$

- Conclusion: explain that you've proven the desired claim.



Guide for **Strong** Induction Proofs

- Restate the claim you are trying to prove

Equivalently: Show
$[\,P(n_0) \wedge P(n_0+1) \wedge \cdots \wedge P(k)\,] \to P(k+1)$

- **Inductive Step**: Prove that for an arbitrary integer k in the desired range,
  $[\,P(n_0) \wedge P(n_0+1) \wedge \cdots \wedge P(k-1)\,] \to P(k)$
  - Let k be an arbitrary integer with $k \ge$ ____ ← value depends on the proof
  - Assume $P(j)$ is true for all $n_0 \le j \le k-1$
  - Show that $P(k)$ holds
- **Base case(s)**: Prove the claim holds for the "first" **value(s)** of n
  - Prove $P(n_0)$ is true
  - May also need to prove $P(n_0+1)$ and more, depending on the inductive step
- Conclusion: explain that you've proven the desired claim.

Function	Domain	Codomain	Injective (One-to-One)	Surjective (Onto)	Bijective
$f(x) = x^2$	$\mathbb{R}$	$\mathbb{R}$	No	No	No
$f(x) = x^2$	$\mathbb{R}^+$	$\mathbb{R}^+$	Yes	Yes	Yes
$f(x) = x^2 + 1$	$\mathbb{R}$	$\mathbb{R}$	No	No	No
$f(x) = x^2 + 1$	$\mathbb{R}$	$\mathbb{R}^+$	No	Yes	No
$f(x) = 31 - x$	$\mathbb{R}$	$\mathbb{R}$	Yes	Yes	Yes
$f(x) = 203x$	$\mathbb{R}$	$\mathbb{R}$	Yes	Yes	Yes
$f(x) = 203^x$	$\mathbb{R}$	$\mathbb{R}^+$	Yes	Yes	Yes
$f(x) = 2x + 1$	$\mathbb{Z}$	$\mathbb{Z}$	Yes	No	No
$f(x) = \lfloor x \rfloor$	$\mathbb{R}$	$\mathbb{Z}$	No	Yes	No
$f(x) = 3/(1-x)$	$\mathbb{R} - \{1\}$	$\mathbb{R}$	Yes	No	No
$f(x) = x$	$\mathbb{R}$	$\mathbb{R}$	Yes	Yes	Yes
$f(x) = \log_{10}(x)$	$\mathbb{R}^+$	$\mathbb{R}$	Yes	Yes	Yes
$f(x) = \ln(x)$	$\mathbb{R}^+$	$\mathbb{R}$	Yes	Yes	Yes
$f(x) = \sin(x)$	$\mathbb{R}$	$\mathbb{R}$	No	No	No
$f(x) = \sin(x)$	$[0, \pi]$	$[-1, 1]$	Yes	Yes	Yes
$f(x) = \sin(x)x^2$	$\mathbb{R}$	$\mathbb{R}$	No	No	No