EECS 203: Discrete Mathematics
Winter 2024
Homework 6

Due **Thursday, Mar. 14th**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $7 + 2$ Total Points: $100 + 30$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Mod Warm-up [12 points]

Find the integer a such that

(a) $a \equiv 58 \pmod{18}$ and $0 \leq a \leq 17$;

(b) $a \equiv -142 \pmod{7}$ and $0 \leq a \leq 6$;

(c) $a \equiv 17 \pmod{29}$ and $-14 \leq a \leq 14$;

(d) $a \equiv -11 \pmod{21}$ and $110 \leq a \leq 130$.

Show your intermediate steps or briefly explain your process to justify your work.

Solution:

a) $58 \pmod{18} = 4$, so $a = 4$.
This is because $58 = 18 \cdot 3 + 4$.
b) $-142 \pmod{7} = -2$, so $a = -2 + 7 = 5$.
This is because $-142 = 7 \cdot (-21) + 5$.
c) $17 \pmod{29} = 17$, so $a = 17 - 29 = -12$.
This is because $17 = 29 \cdot 0 + 17$.
To make $-14 \leq a \leq 14$, we subtract 29 from 17.
d) $-11 \pmod{21} = -11$, so $a = -11 + 21 \cdot 6 = 115$.
This is because $-11 = 21 \cdot (-1) + 10$.
To make $110 \leq a \leq 130$, we add $21 \cdot 6$ to -11.

2. Multiple Modular Madness [14 points]

For each of the questions below, answer "always," "sometimes," or "never," then explain your answer. Your explanation should justify why you chose the answer you did, but does not have to be a rigorous proof.

Hint: Recall that if $a \equiv b \pmod{m}$ then there exists an integer k such that $a = b + mk$.

(a) Suppose $a \equiv 2 \pmod{21}$. When is $a \equiv 2 \pmod{7}$?

(b) Suppose $b \equiv 2 \pmod{7}$. When is $b \equiv 2 \pmod{21}$?

(c) Suppose $c \equiv 5 \pmod{8}$. When is $c \equiv 4 \pmod{16}$?

(d) Suppose $d \equiv 3 \pmod{21}$. When is $d \equiv 0 \pmod{6}$?

Solution:

a) Always. If $a \equiv 2 \pmod{21}$, then $a = 2 + 21k$ for some arbitary integer k. Then $a = 2 + 7(3k)$, so $a \equiv 2 \pmod 7$.
b) Sometimes. If $b \equiv 2 \pmod 7$, then $b = 2 + 7k$ for some arbitary integer k. Then $b = 2 + 21(\frac{1}{3}k)$, so b is not necessarily congruent to 2 modulo 21.
c) Never. If $c \equiv 5 \pmod 8$, then $c = 5 + 8k$ for some arbitary integer k. Suppose $c \equiv 4 \pmod{16}$, then $c = 4 + 16j$ for some other arbitary integer j.
Then $5 + 8k = 4 + 16j$, so $8k + 1 = 16j$. This implies that $16k$ is odd, which is a contradiction.
Therefore, c is never congruent to 4 modulo 16.
d) Sometimes. If $d \equiv 3 \pmod{21}$, then $d = 3 + 21k$ for some arbitary integer k.
Suppose $d \equiv 0 \pmod 6$, then $d = 6j$ for some arbitary integer j.
Then $3 + 21k = 6j$, so $7k = 2j - 1$. This holds as long as $2j - 1$ is a multiple of 7, and would be false when $j = k = 0$.

3. How Low Can You Go? [12 points]

Suppose $a \equiv 3 \pmod{10}$ and $b \equiv 8 \pmod{10}$. In each part, find c such that $0 \le c \le 9$ and

(a) $c \equiv 14a^2 - b^3 \pmod{10}$

(b) $c \equiv b^{15} - 99 \pmod{10}$

(c) $c \equiv a^{97} \pmod{10}$

Show your work! You should be doing the arithmetic/making substitutions **without using a calculator**. Your work must **not** include numbers above 100.

Solution:

Since $a \equiv 3 \pmod{10}$, $a = 3 + 10k$ for some integer k. Similarly, $b = 8 + 10l$ for some integer l.
a) $c \equiv 14(3 + 10k)^2 - (8 + 10l)^3 \pmod{10}$.
$\equiv 14(3)^2 - 8^3 \pmod{10}$.
$\equiv 14(9) - 8^2 \cdot 8 \pmod{10}$.
$\equiv 4 \cdot 9 - 4 \cdot 8 \pmod{10}$.
$\equiv 36 - 32 \pmod{10}$.
$\equiv 4 \pmod{10}$.
So $c = 4$.
b) $c \equiv (8 + 10l)^{15} - 99 \pmod{10}$.
$\equiv 8^{15} - 99 \pmod{10}$.

$\equiv (8^3)^5 - 99 \pmod{10}$.
$\equiv (8^2 \cdot 8)^5 - 99 \pmod{10}$.
$\equiv (4 \cdot 8)^5 - 99 \pmod{10}$.
$\equiv 32^5 - 99 \pmod{10}$.
$\equiv 2^5 - 99 \pmod{10}$.
$\equiv 32 - 99 \pmod{10}$.
$\equiv 2 - 9 \pmod{10}$.
$\equiv -7 \pmod{10}$.
So $c = -7 + 10 = 3$.
c) $c \equiv (3 + 10k)^{97} \pmod{10}$.
$\equiv 3^{100} \cdot 3^{-3} \pmod{10}$.
$\equiv (3^5)^{20} \cdot 3^{-3} \pmod{10}$.
$\equiv (3^3 \cdot 3^2)^{20} \cdot 3^{-3} \pmod{10}$.
$\equiv (27 \cdot 9)^{20} \cdot 3^{-3} \pmod{10}$.
$\equiv (7 \cdot 9)^{20} \cdot 3^{-3} \pmod{10}$.
$\equiv 3^{20} \cdot 3^{-3} \pmod{10}$.
$\equiv (3^5)^4 \cdot 3^{-3} \pmod{10}$.
$\equiv 3^4 \cdot 3^{-3} \pmod{10}$.
$\equiv 3$.
So $c = 3$.

4. Be There or Be Square [16 points]

Prove that if n is an odd integer, then $n^2 \equiv 1 \pmod{8}$.

Note: You **cannot** use the fact that all integers are equivalent to one of 0-7 (mod 8) without proof.

Solution:

Proof:
Proof by cases.

Assume n is an arbitrary odd integer.	assumption
$n = 2k + 1$ for some integer k, $k \in \mathbb{Z}$	Definition of odd
$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$.	substitution
Case 1: k is even.	
$k = 2m$ for some integer m, $m \in \mathbb{Z}$.	Definition of even
$n^2 = 4(4m^2 + 2m) + 1 = 16m^2 + 8m + 1$.	substitution
$n^2 \equiv 1 \pmod 8$.	Definition of congruence
Case 2: k is odd.	
$k = 2m + 1$ for some integer m, $m \in \mathbb{Z}$.	Definition of odd
$n^2 = 4(4m^2 + 6m + 2) + 1 = 16m^2 + 24m + 9$.	substitution
$n^2 \equiv 9 \pmod 8 \equiv 1 \pmod 8$.	Definition of congruence
WLOG, all cases have been exhausted.	
Therefore, $n^2 \equiv 1 \pmod 8$.	

5. Functions and Fakers [16 points]

Determine if each of the examples below are functions or not.

- If it is not a function, explain why not.
- If it is a function, state whether or not it is bijective, and briefly justify your answer.

All domains and codomains are given as intended.

(a) $f\colon \mathbb{R}^\times \to \mathbb{R}^\times$ such that $f(x) = x^{-1}$.

Note: The set $\mathbb{R}^\times$ is the set $\mathbb{R} - \{0\}$. Additionally, recall that $x^{-1} = \frac{1}{x}$.

(b) $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(x) = y$ iff $y \leq x$.

(c) h: **U-M Courses** $\to \{\text{EECS}, \text{MATH}\}$ which maps each class to its department.

(d) k: **U-M Courses** $\to \mathbb{N}$ which maps each class to its course number

For example, h(EECS 203) = EECS and k(EECS 203) = 203.

Note: For the purpose of parts (c) and (d), two courses are considered "equal" if and only if they have the same department and course number. In particular, cross-listed courses are treated as distinct elements of **U-M Courses**.

Solution:

a) f is a bijective function. It is a function because for every $x \in \mathbb{R}^\times$, there is a unique $x^{-1} \in \mathbb{R}^\times$. It is bijective because it is both 1-1 and onto. This is because $\forall x \in \mathbb{R}^\times \exists y \in \mathbb{R}^\times$

such that $f(x) = f(y) \to x = y$ and $\forall y \in \mathbb{R}^{\times} \exists x \in \mathbb{R}^{\times}$ such that $f(x) = y$.
b) g is not a function. It is not a function because for every $x \in \mathbb{R}$, there is not a unique $y \in \mathbb{R}$ such that $g(x) = y$. Consider $x = 0$, then $g(0) = y$ for all $y \leq 0$.
c) h is a function but not bijective. It is a function because for every class in **U-M Courses**, there is a unique department(and cross-listed courses are considered distinctive). It is not bijective because it is onto but not 1-1. This is because each class has its own department, but two different classes within the same department will map to the same department.
d) k is a function but not bijective. It is a function because for every class in **U-M Courses**, there is a unique course number. It is not bijective because it is not onto or 1-1. This is because two different classes within different department can have the same course number, and some natural numbers do not get mapped as course number usually starts at 100.

6. Fantastic Functions [18 points]

For each of the functions below, determine whether it is (i) one-to-one, (ii) onto. Prove your answers.

(a) $f\colon \mathbb{R} \to \mathbb{R} - \mathbb{R}^-,\ f(x) = e^{2x+1}$.

(b) $g\colon \mathbb{R} - \left\{-\frac{2}{5}\right\} \to \mathbb{R} - \left\{\frac{3}{5}\right\},\ g(x) = \frac{3x-1}{5x+2}$.

(c) $h\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z},\ h(m,n) = |m| - |n|$

Solution:

a) f is one-to-one but not onto
i) Proof one-to-one:
Assume $x_1, x_2 \in \mathbb{R}$. assumption
$e^{2x_1+1} = e^{2x_2+1}$.
$2x_1 + 1 = 2x_2 + 1$.
$x_1 = x_2$.
Therefore, f is one-to-one.
ii) Proof: not onto
Seeking contradiction, assume $y \in \mathbb{R} - \mathbb{R}^-$. assumption
$e^{2x+1} = y$.
$2x + 1 = \ln(y)$.
$x = \frac{\ln(y)-1}{2}$.
Since $\ln(y)$ is undefined for $y \leq 0$, x is undefined for $y \leq 0$.
Therefore, f is not onto.

b) g is one-to-one and onto.
i) Proof one-to-one:
Assume $x_1, x_2 \in \mathbb{R} - \left\{-\frac{2}{5}\right\}$. assumption
$\frac{3x_1-1}{5x_1+2} = \frac{3x_2-1}{5x_2+2}$.
$3x_1(5x_2+2) - 5x_1(3x_2-1) = 3x_2(5x_1+2) - 5x_2(3x_1-1)$.
$15x_1x_2 + 6x_1 - 15x_1x_2 + 5x_1 = 15x_2x_1 + 6x_2 - 15x_2x_1 + 5x_2$.
$x_1 = x_2$.
Therefore, g is one-to-one.
ii) Proof onto:
Assume $y \in \mathbb{R} - \left\{\frac{3}{5}\right\}$. assumption
$\frac{3x-1}{5x+2} = y$.
$3x - 1 = y(5x+2)$.
$3x - 1 = 5xy + 2y$.
$3x - 5xy = 2y + 1$.
$x = \frac{2y+1}{3-5y}$.
Therefore, g is onto.
c) h is onto but not one-to-one.
i)
Proof not one-to-one:
It is possible to have $h(m_1, n_1) = h(m_2, n_2)$ for $m_1 \neq m_2$ and $n_1 \neq n_2$. For example, $h(1,0) = h(0,1) = 1$.
Therefore, h is not one-to-one.
ii)
Proof onto:
To get negative values, set $m = 0$. To get positive values, set $n = 0$. To get 0, set $m = n$.
All cases have been exhausted.
Therefore, h is onto.

7. Comp$\circ$sition(Functions) [12 points]

For each of the following pairs of functions f and g, find $f \circ g$ and $g \circ f$. Make sure to include the domain and codomain of each composed function you give. If either can't be computed, explain why.

(a) $f\colon \mathbb{N} \to \mathbb{Z}^+,\ f(x) = x + 1$
$g\colon \mathbb{Z}^+ \to \mathbb{N},\ g(x) = x^2 - 1$

(b) $f\colon \mathbb{Z} \to \mathbb{R},\ f(x) = \left(\frac{3}{2}x + 3\right)^3$
$g\colon \mathbb{R} \to \mathbb{R}_{\geq 0},\ g(x) = |x|$

Note: $\mathbb{R}_{\geq 0}$ is the set of real numbers greater than or equal to 0.

Solution:

For composite functions to be computable, the domain of $f \circ g$ is the codomain of g and the codomain of $f \circ g$ is the codomain of f.
The domain of $g \circ f$ is the codomain of f and the codomain of $g \circ f$ is the codomain of g.
a) $f \circ g$, and $g \circ f$ are computable.
$f \circ g \colon \mathbb{N} \to \mathbb{Z}^+,\ f \circ g(x) = (x^2 - 1) + 1 = x^2.$
$g \circ f \colon \mathbb{Z}^+ \to \mathbb{N},\ g \circ f(x) = (x+1)^2 - 1 = x^2 + 2x.$
b) $f \circ g$ is not computable, but $g \circ f$ is computable.
$f \circ g \colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ is not computable because $f(x)$ is not defined for non-integers. i.e. when $g = 1.2$.
$g \circ f \colon \mathbb{R} \to \mathbb{R}_{\geq 0},\ g \circ f(x) = \left| \left(\frac{3}{2}x + 3\right)^3 \right|.$