F23

EECS 203 Discussion 10

Bayes Rule, Random Variables, and Expected Value

Admin Notes:

Homework and Check-in 10 are due **Thursday, April 18th.**

Bayes’ Rule

Review: Conditional Probability and Independence

- **Conditional Probability:** The probability of E_1 given E_2, denoted $P(E_1 \mid E_2)$, is:

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

- **Independence:** Events E and F are independent if and only if:

$$P(E \cap F) = P(E) * P(F)$$

  Note: This is not the same thing as two events being mutually exclusive!

- **Independence via Conditional Probability:** Events E and F are also independent if: $P(E) = P(E \mid F)$.

Bayes' Theorem

- **Bayes' Theorem:** Suppose that E and F are events from a sample space S such that $p(E) \neq 0$ and $p(F) \neq 0$. Then

$$P(F|E) = \frac{p(E|F)p(F)}{P(E)} = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})} = \frac{p(E \cap F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}$$

Which version you use depends on what you are given, but they are all equivalent.

Problem 1

1. Conditional Probability

A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0? (We assume that 0 bits and 1 bits are equally likely.)

Solution

1. Conditional Probability

A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0? (We assume that 0 bits and 1 bits are equally likely.)

Let E be the event that a bit string of length four contains at least two consecutive 0s, and let F be the event that the first bit of a bit string of length four is a 0. The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 0, equals

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

Because $E \cap F = 0000, 0001, 0010, 0011, 0100$, we see that $p(E \cap F) = 5/16$. Because there are eight bit strings of length four that start with a 0, we have $p(F) = 8/16 = 1/2$. Consequently,

$$P(E|F) = \frac{5/16}{1/2} = 5/8.$$

Problem 2

2. Bayes' Theorem ⋆

An electronics company is planning to introduce a new camera phone. The company commissions a marketing report for each new product that predicts either the success or the failure of the product. Of new products introduced by the company, 60% have been successes. Furthermore, 70% of their successful products were predicted to be successes, while 40% of failed products were predicted to be successes. Find the probability that this new camera phone will be successful if its success has been predicted.

Solution

2. Bayes' Theorem ⋆

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Let S be the event "Product is a success", and let A be the event that "the product is predicted to be successful." We are given $P(S) = 0.6, P(A|S) = 0.7$, and $P(A|\bar{S}) = 0.4$. We want to find $P(S|A)$, the probability that the product is a success given that the it was predicted to be a success.

Problem 3

3. Conditional Probability ⋆

Jakub has created an app that classifies images as either being a Hot Dog or Not a Hot Dog, and he needs your help for some analysis.

Suppose that 4% of the images in a data set are images of hot dogs. Furthermore, suppose that when Jakub's app classifies an image, 97% of the hot dog images are classified correctly (as hot dogs), and 2% of the images that are not hot dogs are classified incorrectly (as hot dogs). What is the probability that:

a) an image classified as a hot dog is really a hot dog?

b) an image classified as a hot dog is **not** a hot dog?

c) an image classified as "not a hot dog" is a hot dog?

d) an image classified as "not a hot dog" is **not** a hot dog?

Solution

3. Conditional Probability ⋆

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Let A be the event that a randomly chosen image in the dataset is a hot dog image. Also, let T be the event that a randomly chosen image classification instance comes out positive (the app identifies the image as a hot dog). Based on the problem, $p(A) = 0.04, p(\overline{A}) = 0.96, p(T|A) = 0.97, p(T|\overline{A}) = 0.02$. From these, we can also say that $p(\overline{T}|A) = 0.03$ and $p(\overline{T}|\overline{A}) = 0.98$.

a) This is asking for $p(A|T)$.

$$p(A|T) = \frac{p(T|A)p(A)}{p(T|A)p(A) + p(T|\overline{A})p(\overline{A})} = \frac{(0.97)(0.04)}{(0.97)(0.04) + (0.02)(0.96)} \approx 0.669$$

Solution

3. Conditional Probability $\star$

Jakub has created an app that classifies images as either being a Hot Dog or Not a Hot Dog, and he needs your help for some analysis.

Suppose that 4% of the images in a data set are images of hot dogs. Furthermore, suppose that when Jakub's app classifies an image, 97% of the hot dog images are classified correctly (as hot dogs), and 2% of the images that are not hot dogs are classified incorrectly (as hot dogs). What is the probability that:

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b) This is asking for $p(\overline{A}|T)$.
Since this is the probability of the complementary event of (a), we can say $p(\overline{A}|T) = 1 - p(A|T) \approx 1 - 0.669 = 0.331$.

c) This is asking for $p(A|\overline{T})$.

$$p(A|\overline{T}) = \frac{p(\overline{T}|A)p(A)}{p(\overline{T}|A)p(A) + p(\overline{T}|\overline{A})p(\overline{A})} = \frac{(0.03)(0.04)}{(0.03)(0.04) + (0.98)(0.96)} \approx 0.001$$

d) This is asking for $p(\overline{A}|\overline{T})$.
Since this is the probability of the complementary event of (c), we can say $p(\overline{A}|\overline{T}) = 1 - p(A|\overline{T}) \approx 1 - 0.001 = 0.999$.

Expectation

Random Variables and Expected Value

- **Random Variable:** A random variable is a function from the sample space of an experiment to the set of real numbers. That is, a random variable assigns a real number to each possible outcome.

- **Expected Value:** The average value of values that a random variable can take on, where each possible value is weighted by its respective probability. It can be found using:

$$E(X) = \sum_{s \in S} p(s) \times X(S) \quad \text{or} \quad E(X) = \sum_{r \in X(S)} p(X = r) \times r$$

- **Linearity of Expectations:** Expected values are linear. This tells us that the expected value of the sum of some random variables is equal to the sum of their individual expected values.

$$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$$

Indicator Random Variables

- **Indicator Random Variable:** Let A be an event. Then I_A, the indicator random variable of A, equals 1 if A occurs and equals 0 otherwise. The expectation of the indicator random variable of A equals the probability of A, that is, $E(I_A) = p(A)$.

Problem 4

4. Random Variables and Expected Value

Suppose that a coin is flipped three times. Let X be a random variable representing the total number of heads across the three flips.

(a) What are the possible values of X and which outcome(s) are associated with each value of X?

(b) Find $E(X)$?

Solution

4. Random Variables and Expected Value

Suppose that a coin is flipped three times. Let X be a random variable representing the total number of heads across the three flips.

(a) What are the possible values of X and which outcome(s) are associated with each value of X?

(b) Find $E(X)$?

(a) $X(HHH) = 3$
$X(HHT) = X(HTH) = X(THH) = 2$
$X(TTH) = X(THT) = X(HTT) = 1$
$X(TTT) = 0$

(b) Because the coin is fair and the flips are independent, the probability of each outcome is $1/8$. Consequently,

$$\begin{aligned}
E(X) &= \sum_{r \in range(X)} p(X=r) \cdot r \\
&= p(X=3) \cdot 3 + p(X=2) \cdot 2 + p(X=1) \cdot 1 + p(X=0) \cdot 0 \\
&= \frac{1}{8} \cdot 3 + \frac{3}{8} \cdot 2 + \frac{3}{8} \cdot 1 + \frac{1}{8} \cdot 0 \\
&= \frac{3}{8} + \frac{6}{8} + \frac{3}{8} \\
&= \frac{3}{2}
\end{aligned}$$

Binomial & Geometric Distribution

Bernoulli Trials and the Binomial Distribution

For a set number of trials, examining number of successes.

- **Bernoulli Trials:** Each performance of an experiment with two possible outcomes is called a Bernoulli trial.
  - We typically label these outcomes success and failure
  - If $p = Pr(success)$ and $q = Pr(failure)$, then $\mathbf{p + q = 1}$
- **Binomial Distribution:** We call the probability distribution of the number of successes in a sequence of *n* Bernoulli trials the Binomial Distribution
  - The **probability of exactly *k* successes** in *n* independent Bernoulli trials, with $p = Pr(success)$ and $q = Pr(failure) = 1 - p$, is

$$Pr(numSuccesses = k) = C(n,k)\ (p^k)\ (q^{n-k})$$

  - The **expected number of successes** in *n* independent Bernoulli trials where $p = Pr(success)$, is

$$E(numSuccesses) = np$$

The Geometric Distribution

For examining number of trials to get a success.

- **Bernoulli Trials:** A random variable X has a geometric distribution with parameter p if

$$Pr(X = k) = (1 - p)^{k-1}p$$

  for k = 1,2,3, ...

  This is saying that the first $k-1$ trials were failures (so probability $1-p$) and the k^{th} trial was a success (so probability p). So the **parameter *p*** stands for the probability of success and the **random variable X** is representing some number of trials it took to get a success.

  - The **expected value of X** (ie the expected number of trials to get a success) is

$$E(X) = 1/p$$

Problem 5

5. Bernoulli Trials and the Binomial Distribution (I) ⋆

A biased program is generating a 10-bit string. For any single bit, that the probability that the program generates a 0 is 0.9, and each of the 10 bits are generated independently.

(a) What is the probability that the bitstring will contain exactly 8 0's?

(b) What is the expected number of of 1's in bitstring?

Solution

5. Bernoulli Trials and the Binomial Distribution (I) $\star$

A biased program is generating a 10-bit string. For any single bit, that the probability that the program generates a 0 is 0.9, and each of the 10 bits are generated independently.

(a) What is the probability that the bitstring will contain exactly 8 0's?

(b) What is the expected number of of 1's in bitstring?

(a) This can be viewed as 10 Bernoulli Trails, where "success" is generating a 0 bit, so $p = 0.9$. The probability of exactly 8 successes in 10 trials is $\binom{10}{8}p^8(1-p)^2 = \binom{10}{8}(0.9)^8(0.1)^2$.

(b) By linearity of expectation, $E(\textit{number of } 1s) = 10 \cdot P(\textit{single bit is } 1) = 10(0.1) = 1$.

Problem 6

6. Bernoulli Trials and the Binomial Distribution (II)

A group of six people play the game of "odd person out" to determine who will buy refreshments. Each person flips a fair coin. If there is a person whose outcome is not the same as that of any other member of the group, this person has to buy the refreshments.

(a) What is the probability that there is an odd person out after the coins are flipped once?

(b) Suppose the group continues to play the game until there is an odd person out. In each iteration of the game, every person flips their coin and then the group checks for an odd person out. What is the expected number of iterations until there is finally an odd person out?

Solution

6. Bernoulli Trials and the Binomial Distribution (II)

A group of six people play the game of "odd person out" to determine who will buy refreshments. Each person flips a fair coin. If there is a person whose outcome is not the same as that of any other member of the group, this person has to buy the refreshments.

(a) What is the probability that there is an odd person out after the coins are flipped once?

(b) Suppose the group continues to play the game until there is an odd person out. In each iteration of the game, every person flips their coin and then the group checks for an odd person out. What is the expected number of iterations until there is finally an odd person out?

(a) There are two ways that there could be a odd person out after the coins are flipped: all but one person flipped heads (i.e., exactly 1 tail), or all but one person flipped tails (i.e., exactly 1 head). So, the probability of an odd person out is:

$$\begin{aligned}
p(\textit{odd out}) &= p(\textit{exactly } 1 \textit{ tail}) + p(\textit{exactly } 1 \textit{ head}) \\
&= \binom{6}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^5 + \binom{6}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^5 \\
&= 2 \cdot 6 \cdot \left(\frac{1}{2}\right)^6 \\
&= \frac{12}{64} \\
&= \frac{3}{16}
\end{aligned}$$

(b) This is a "waiting time" experiment, which is represented with a Geometric Random Variable. Let X be a random variable representing the number of iterations until the first time there is an odd person out. Each iteration of the game is a Bernoulli trial with probability of success $p = 3/16$ (calculated in Part (a)). We repeat the Bernoulli trials until the first success, which means X has a geometric distribution, and $E(X) = 1/p = 16/3$.

Problem 7

7. Expected Value ⋆

Rebecca is creating a new 15-digit passcode using only the digits 1, 2, and 3. She chooses each digit uniformly at random from that set of 3 digits, $\{1, 2, 3\}$. What is the expected number of times that the sequence 3113 appears in her passcode? For example, it appears 4 times in the passcode

$$\overline{311\underline{3}}\underline{11\overline{3}}\overline{11}32\underline{3113}.$$

Solution

7. Expected Value ⋆

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311311311323113.

First, we look at how many possible locations there are for 3113 in the passcode. Since 3113 is 4 characters long and the passcode is 15 characters long, then the possible starting positions for 3113 are positions 1, 2, 3, ... , 12. (If 3113 are the last 4 digits of the passcode, the starting position for that sequence is position 12.)

Let I_k be an indicator random variable relating to starting position k, where $I_k = 1$ if there is a 3113 starting at position k and $I_k = 0$ otherwise. Let p be the probability that there is a 3113 starting at position k. Then, for any starting position, $p = (1/3)^4$. Then $E(I_k) = 1 \cdot P(I_k = 1) + 0 \cdot P(I_k = 1) = 1 \cdot (1/3)^4 + 0 = (1/3)^4$

Let X be a random variable representing the number of times 3113 appears in the passcode. Using the indicator random variable, defined above, and linearity of expectation, we have:

$$
\begin{aligned}
E(X) &= E(I_1 + I_2 + \cdots + I_{12}) \\
&= E(I_1) + E(I_2) + \cdots + E(I_{12}) \\
&= \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^4 + ... + \left(\frac{1}{3}\right)^4 \\
&= 12\left(\frac{1}{3}\right)^4 \\
&= \frac{4}{27}
\end{aligned}
$$