W24

EECS 203 Discussion 8b

Graphs (Paths & Cycles), Counting Intro

Admin Notes

- **Homework/Groupwork 8**
  - **Due Apr. 4th (This upcoming Thursday)**
- **Exam 2**
  - Grades should be released early next week

Colorability

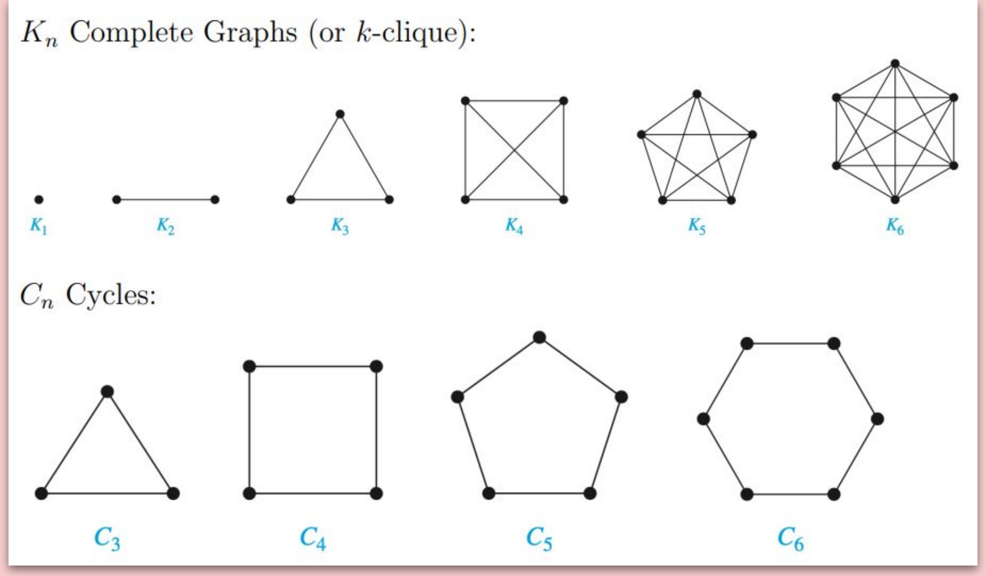
Bipartite Graphs/Colorability

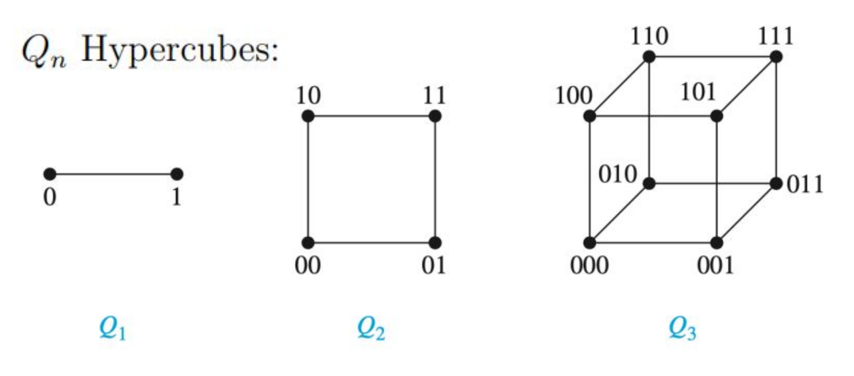
- **Bipartite Graph:** a simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V1 and V2 such that <u>every edge in the graph connects a vertex in V1 and a vertex in V2</u>. The pair **(V1, V2)** is called a **bipartition** of the vertex set V.

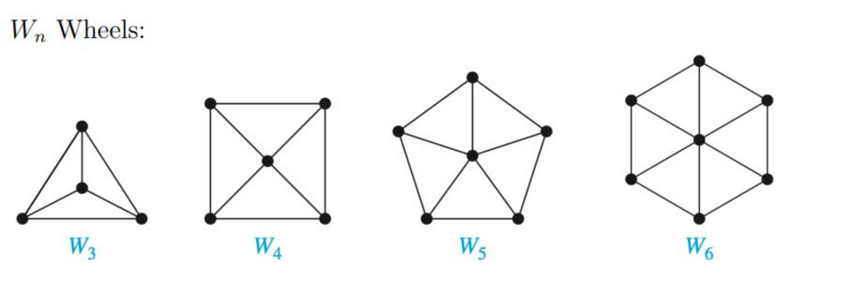
- **Bipartite Theorem (3 Equivalent Statements):** The following statements are equivalent:
  - G is **bipartite.**
  - G is **2-colorable.**
    (There is a function $f : V \Rightarrow \{red, blue\}$ such that $\mathbf{u, v \in E \Rightarrow f(u) \neq f(v)}$)
  - G **does not contain odd cycle** (C_{2k+1}) subgraphs.

Special Graphs

You only need to know **complete graphs** and **cycles**. (The others will be defined later.)







Problem 1

1. Bipartite Intro

Draw each of the following graphs. Which of these graphs are bipartite?

a) C_6, a cycle with 6 nodes

b) W_4, a wheel with a center node and 4 spoke nodes

c) Q_3, a graph representing a 3-dimensional cube, with nodes on each corner

Solution

1. Bipartite Intro

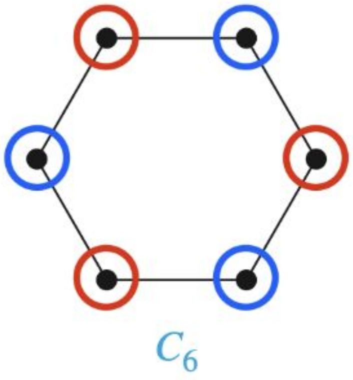
Draw each of the following graphs. Which of these graphs are bipartite?

a) C_6, a cycle with 6 nodes

b) W_4, a wheel with a center node and 4 spoke nodes

c) Q_3, a graph representing a 3-dimensional cube, with nodes on each corner

a) C_6 is bipartite. See the following bipartition:



Solution

1. Bipartite Intro

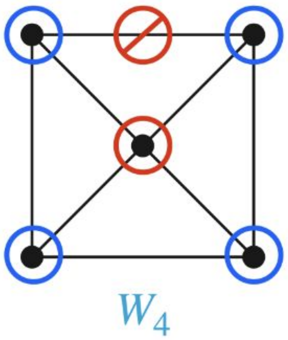
Draw each of the following graphs. Which of these graphs are bipartite?

a) C_6, a cycle with 6 nodes

b) W_4, a wheel with a center node and 4 spoke nodes

c) Q_3, a graph representing a 3-dimensional cube, with nodes on each corner

b) W_4 is not bipartite. The middle vertex is connected to everything other vertex, and thus must be its own set. However, we can see connections between the elements of the other set.



Solution

1. Bipartite Intro

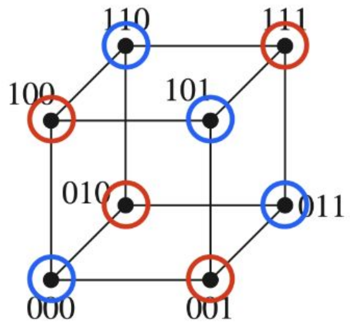
Draw each of the following graphs. Which of these graphs are bipartite?

a) C_6, a cycle with 6 nodes

b) W_4, a wheel with a center node and 4 spoke nodes

c) Q_3, a graph representing a 3-dimensional cube, with nodes on each corner

c) Q_3 is bipartite. See the following bipartition:



Q_3

Problem 2

2. Bipartite Conclusion

For which values of n are these graphs bipartite? Explain your answer.

a) K_n

b) C_n

c) W_n

d) Q_n

Solution

2. Bipartite Conclusion

For which values of n are these graphs bipartite? Explain your answer.

a) K_n

b) C_n

c) W_n

d) Q_n

a) K_1 and K_2 are bipartite. There is a triangle in K_n for $n > 2$, so they are not bipartite.

b) C_n is defined for $n > 3$. If n is even, then C_n is bipartite, since we can take one part to be every other vertex. If n is odd, then C_n is not bipartite.

c) Every wheel $n > 1$ contains triangles, so no W_n for any $n > 1$ is bipartite. Note that $W_1 = K_2$, which is bipartite.

d) Q_n is bipartite for all $n \geq 1$, since we can divide the vertices into these two classes: those bit strings with an odd number of 1's, and those bit strings with an even number of 1's.

Problem 3

3. Bipartite Graphs

Does there exist a bipartite graph with degree sequence 3,3,3,3,3,3,3,3,3,3,5,6,9 (in other words, a graph with ten nodes of degree 3, one of degree 5, one of degree 6, and one of degree 9)? If not, explain why.

Solution

3. Bipartite Graphs

Does there exist a bipartite graph with degree sequence 3,3,3,3,3,3,3,3,3,3,5,6,9 (in other words, a graph with ten nodes of degree 3, one of degree 5, one of degree 6, and one of degree 9)? If not, explain why.

Since the graph is bipartite, let A, B be the two partitions. Then we know the sum of the degrees of the vertices in A must be equal to half of the sum of all the degrees.

The sum of all the degrees is 50. The the sum of the degrees of the vertices in A is 25. But no subset of the above vertices sum of 25. (If 5 is not in the subset, then it sums to a multiple of 3. If 5 is there, then subtracting 5 should sum to a multiple of 3. In this case neither 25 nor 20 are multiples of 3).

Hence, there is a contradiction, and thus no such graph exists.

Graph Connectivity

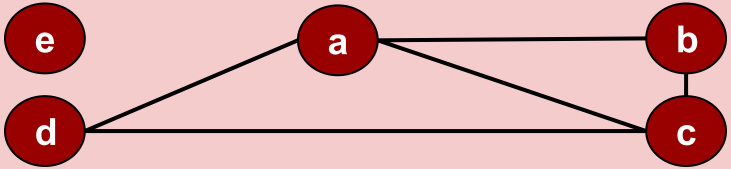
Graph Connectivity

- **Path:** a path **$(u_0, u_1, \dots u_k)$** is a sequence of vertices in which consecutive vertices in the sequence are adjacent in the graph (connected by an edge).
  - Note parentheses **()** because a path DOES indicate an order

- **Simple Path:** a path that <u>does not repeat any vertices</u>

Graph Connectivity

- **Connected Vertices:** Two vertices u and v are connected if there is a path from u to v: **(u, ..., v)**
  - Note that vertices *don't have to be adjacent* to be connected
    - **Ex from pic below:** **(d,b)** are connected but not adjacent
- **Connected Component:** A nonempty set of vertices in which every pair of vertices in the set is connected. **Example below: 2 connected components**



- **Connected Graph:** a graph G in which there is a path connecting any two vertices u, v ∈ G. In other words, there is only one connected component in the graph. Example above is NOT a connected graph.

Special Types of Graph Paths

- **Euler Path:** A Euler (pronounced “oiler”) path is a path that uses **every edge** of a graph exactly once. An Euler path can start and end at the same vertex OR at different vertices.

- **Euler Circuit:** An Euler path that **starts and ends at the same vertex**. Sometimes, this is also referred to as an **Euler cycle**, but note that an Euler circuit is not necessarily an actual cycle, since it can visit the same vertex multiple times, as long as it doesn’t repeat an edge.

- **Euler’s Theorem:** A connected graph (or multigraph) has an Euler cycle **if and only if** every vertex has even degree.

Special Types of Graph Paths

- **Hamiltonian Path:** A Hamiltonian path (or Hamilton path) is a path between two vertices of a graph that visits **every vertex** in the graph exactly once.

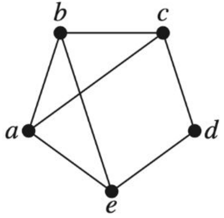
- **Hamiltonian Cycle:** If a Hamiltonian path exists whose endpoints are adjacent, then the resulting graph cycle (**starting and ending at same vertex**) is called a Hamiltonian cycle (or Hamilton cycle).

Problem 4

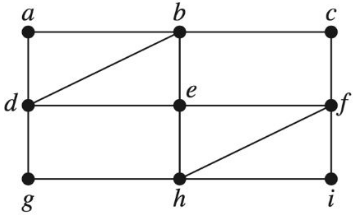
4. Euler Paths and Circuits

For each of the following graphs: Determine whether the graph has an Euler circuit. Construct such a circuit if one exists. If no Euler circuit exists, determine whether the graph has an Euler path. Construct such a path if one exists.

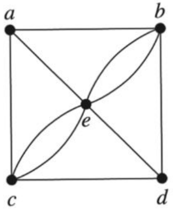
1.



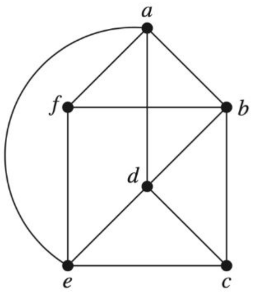
2.



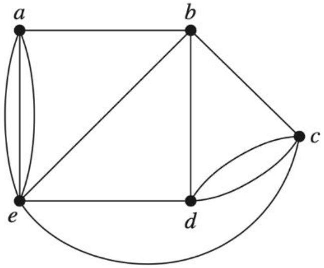
3.



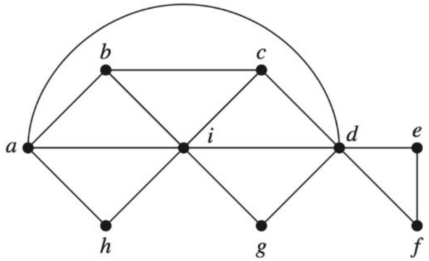
4.



5.



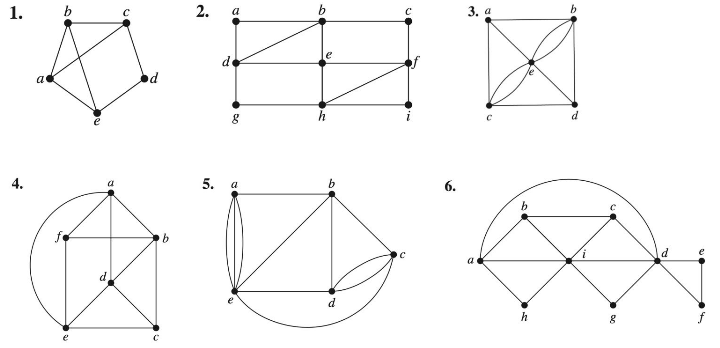
6.



Solution

4. Euler Paths and Circuits

For each of the following graphs: Determine whether the graph has an Euler circuit. Construct such a circuit if one exists. If no Euler circuit exists, determine whether the graph has an Euler path. Construct such a path if one exists.



1) Neither

2) Euler circuit: d,a,b,d,e,b,c,f,e,h,i,f,h,g,d

3) No Euler circuit; Euler path: a,e,c,e,b,e,d,b,a,c,d

4) No Euler circuit; Euler path: c,e,d,c,b,d,a,b,f,e,a,f

5) Euler circuit: a,b,c,d,c,e,d,b,e,a,e,a

6) No Euler circuit; Euler path: b,a,i,h,a,d,e,f,d,g,i,b,c,i,d,c

Intro to Counting

Counting Rules

- **Product Rule:** Suppose a procedure can be broken down into a sequence of k tasks, where you have to do each task to complete the procedure. If there are n_k ways to do the k-th task, then there are $\mathbf{n_1 \cdot n_2 \cdot n_3 \cdot \ ... \ \cdot n_k}$ ways to do the entire procedure.
- **Sum Rule:** Suppose there are k distinct, disjoint methods to complete a procedure such that k-th method can be done in n_k ways, then there are $\mathbf{n_1 + n_2 + n_3 + ... + n_k}$ ways to do exactly one of these tasks.
- **Division Rule**: If there are N ways to choose an object, and each object can be chosen in exactly k ways, there are $\boldsymbol{N/k}$ objects.

Counting Rules

- **Difference Rule**: Any process with *n* total choices, which has *k* extra choices that shouldn't have been counted, means ***n - k*** possible choices total

- **Inclusion Exclusion:** If a task can be done in either n_1 or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Problem 5

5. Product Rule

a. The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

b. How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

c. How many functions are there from a set with m elements to a set with n elements?

d. How many one-to-one functions are there from a set with m elements to one with n elements?

Solution

5. Product Rule

a. The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

b. How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

c. How many functions are there from a set with m elements to a set with n elements?

d. How many one-to-one functions are there from a set with m elements to one with n elements?

a. The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600.

b. There are 26 choices for each of the three uppercase English letters and 10 choices for each of the three digits. Hence, by the product rule there are a total of $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$ possible license plates.

Solution

5. Product Rule

a. The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

b. How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

c. How many functions are there from a set with m elements to a set with n elements?

d. How many one-to-one functions are there from a set with m elements to one with n elements?

c. A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain. Hence, by the product rule there are $n \cdot n \cdots n = n^m$ functions from a set with m elements to one with n elements. For example, there are $5^3 = 125$ different functions from a set with three elements to a set with five elements.

d. First note that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements. Now let $m \leq n$. Suppose the elements in the domain are $a_1, a_2, ..., a_m$. There are n ways to choose the value of the function at a_1. Because the function is one-to-one, the value of the function at a_2 can be picked in $n-1$ ways (because the value used for a_1 cannot be used again). In general, the value of the function at ak can be chosen in $n-k+1$ ways. By the product rule, there are $n(n-1)(n-2)\cdots(n-m+1)$ one-to-one functions from a set with m elements to one with n elements.

Problem 6

6. Sum Rule

a. A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

b. A wired equivalent privacy (WEP) key for a wireless fi- delity (WiFi) network is a string of either 10, 26, or 58 hexadecimal digits. How many different WEP keys are there?

Solution

6. Sum Rule

a. A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

b. A wired equivalent privacy (WEP) key for a wireless fi- delity (WiFi) network is a string of either 10, 26, or 58 hexadecimal digits. How many different WEP keys are there?

a. The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

b. $16^{10} + 16^{26} + 16^{58}$

Problem 7

7. Inclusion Exclusion

a. How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

b. A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Solution

7. Inclusion Exclusion

a. How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

b. A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

a. Use inclusion-exclusion:
ways that start with 1 or end with 00 = #ways that start with 1 + # ways to end with 00 - # ways to do both (start with 1 and end with 00).

Let S = set of 8-bit strings that start with 1.
Let E = set of 8-bit strings that end with 00.
We are looking for $|S \cup E| = |S| + |E| - |S \cap E|$.

$|S|$ = #ways that start with 1:
We can construct a bit string of length eight that begins with a 1 in $2^7 = 128$ ways. This follows by the product rule, because the first bit can be chosen in only one way and each of the other seven bits can be chosen in two ways.

$|E|$ = #ways that end with 00:
Similarly, we can construct a bit string of length eight ending with the two bits 00, in $2^6 = 64$ ways. This follows by the product rule, because each of the first six bits can be chosen in two ways and the last two bits can be chosen in only one way.

$|S \cap E|$ = #ways that start with 1 and end with 00:
Some of the bitstrings that start with 1 also end with 00. For these strings, there are only 5 bits to choose (bits 2-6). There are $2^5 = 32$ ways to construct such a string. This follows by the product rule, because the first bit can be chosen in only one way, each of the second through the sixth bits can be chosen in two ways, and the last two bits can be chosen in one way.

$|S \cup E|$: Using inclusion-exclusion, we can find the number of bit strings of length eight that begin with a 1 **or** end with a 00:
$S \cup E| = |S| + |E| - |S \cap E| = 128 + 64 - 32 = 160$.

Solution

7. Inclusion Exclusion

a. How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

b. A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

b. To find the number of these applicants who majored neither in computer science nor in business, we can subtract the number of students who majored either in computer science or in business (or both) from the total number of applicants. Let A_1 be the set of students who majored in computer science and A_2 the set of students who majored in business. Then $A_1 \cup A_2$ is the set of students who majored in computer science or business (or both), and $A_1 \cap A_2$ is the set of students who majored both in computer science and in business. By the subtraction rule the number of students who majored either in computer science or in business (or both) equals:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316$$

We conclude that $350 - 316 = 34$ of the applicants majored neither in computer science nor in business.

Problem 8

8. Division Rule

How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

Solution

8. Division Rule

How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

We arbitrarily select a seat at the table and label it seat 1. We number the rest of the seats in numerical order, proceeding clockwise around the table. Note that are four ways to select the person for seat 1, three ways to select the person for seat 2, two ways to select the person for seat 3, and one way to select the person for seat 4. Thus, there are $4! = 24$ ways to order the given four people for these seats. However, each of the four choices for seat 1 leads to the same arrangement, as we distinguish two arrangements only when one of the people has a different immediate left or immediate right neighbor. Because there are four ways to choose the person for seat 1, by the division rule there are $24/4 = 6$ different seating arrangements of four people around the circular table.