EECS 203: Discrete Mathematics
Winter 2024
FoF Worksheet 7

1 Pigeonhole Principle

- **Pigeonhole Principle:** If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects
- **Generalized Pigeonhole Principle:** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

1.1 Multiple choice pigeons

What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

a) 100

b) 101

c) 4951

d) 5001

Solution: By the Pigeonhole Principle, if there are 4951 students (50 states multiplied by 99 students from each state, plus one), then there must be at least 100 students from the same state.

1.2 Pigeonhole Principle

How many distinct numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

Solution: We can group these into pairs that add up to 16 of: $(1, 15), (3, 13), (5, 11), (7, 9)$. Notably, no other pair of numbers sum to 16. Therefore, if we pick 5 numbers, it is guaranteed that we pick both numbers from at least one of these pairs.

1.3 More Pigeonhole Principle

Undergraduate students at a college belong to one of four groups depending on the year in which they are expected to graduate. Each student must choose one of 21 different majors. How many students are needed to assure that there are two students expected to graduate in the same year who have the same major?

Solution: $(4 \cdot 21) + 1 = 85$

1.4 Even More Pigeonhole Principle

Sophia has a bowl of 15 red, 15 blue, and 15 orange pieces of candy. Without looking, Sophia grabs a handful of pieces.

(a) What is the smallest number of pieces of candy Sophia has to grab to make sure she has at least 4 of the same color?

(b) What is the smallest number of pieces of candy Sophia has to grab to make sure she has 3 orange candies?

Solution:

(a) 10. Consider colors as boxes, and candies as pigeons. By pigeonhole principle, we have $\left\lceil \frac{N}{3} \right\rceil = 4$ where N is the number of pieces we have to grab to make this work. The smallest number N that works is 10.

(b) 33. This is not actually pigeonhole. We specifically need to have 3 orange candies. The only way to make sure this happens is to grab all 15 red, 15 blue, and then the next 3 we grab have to be orange.

1.5 Pigeonhole Principle

A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Solution: Let $K(x)$ be the number of other computers that computer x is connected to. The possible values for $K(x)$ are 1, 2, 3, 4, 5. Since there are 6 computers, the pigeonhole principle guarantees that at least two of the values $K(x)$ are the same, which is what we wanted to prove.

2 Functions: Countability

- **Countably Infinite:** A set is countably infinite if it has the same cardinality as the natural numbers. This can be proven for a set by finding a one-to-one correspondence between it and the natural numbers. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are common examples of sets that are countably infinite.
- **Uncountably Infinite:** A set is said to be uncountably infinite if its cardinality is larger than that of the set of all natural numbers. $\mathbb{R}$ is one example of a set that is uncountably infinite.

2.1 Countability Multiple Choice

Which of the following sets is countably infinite?

a) Set of real numbers $\mathbb{R}$

b) Set of integers $\mathbb{Z}$

c) Set of complex numbers

d) Set of irrational numbers

Solution: B) Set of integers $\mathbb{Z}$
The set of integers $\mathbb{Z}$ can be put into one-to-one correspondence with the set of natural numbers $\mathbb{N}$, thus making it countably infinite.

2.2 Multiple Answer Multiple Choice

Which of the following statements about countably infinite sets is true?

a) Countably infinite sets have a smaller cardinality than uncountably infinite sets.

b) Countably infinite sets cannot be enumerated in a systematic way.

c) Countably infinite sets have a one-to-one correspondence with the set of natural numbers.

d) Countably infinite sets include the set of real numbers $\mathbb{R}$.

Solution: A and C
Countably infinite sets have a one-to-one correspondence with the set of natural numbers. A set is countably infinite if it has a one-to-one correspondence with the set of natural numbers $\mathbb{N}$, meaning its elements can be enumerated in a systematic way.

2.3 Different Infinities

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) the integers greater than 10

b) the integers with absolute value less than 1,000,000

c) the real numbers between 0 and 2

d) the set $A \times \mathbb{Z}$ where $A = \{2, 3\}$

Solution: For each part, we will denote the set in question as S.
a) This is countably infinite. The function $f : \mathbb{Z}^+ \to S, f(x) = x + 10$ is a one-to-one correspondence between S and $\mathbb{Z}^+$.
b) This is finite. It is the integers in the range
$\{-999,999, -999,998, ..., -1, 0, 1, ..., 999,998, 999,999\}$
c) This is uncountable. We proved in lecture that $(0, 1)$ is uncountable, which is a subset of this set.
d) This is countably infinite. The function
$g : \mathbb{Z}^+ \to A \times \mathbb{Z}$,
$g(x) =$

$(2, \frac{x-4}{4})$ if $x \equiv 0 \mod 4$
$(3, \frac{x-1}{4})$ if $x \equiv 1 \mod 4$
$(2, \frac{-(x+2)}{4})$ if $x \equiv 2 \mod 4$
$(3, \frac{-(x+1)}{4})$ if $x \equiv 3 \mod 4$

is a one-to-one correspondence between $A \times \mathbb{Z}$ and $\mathbb{Z}^+$.

2.4 Different Infinities with Sets

Give an example of two uncountable sets A and B such that $A \cap B$ is

a) finite

b) countably infinite

c) uncountably infinite

Solution: There are a lot of possible answers, but here are a few:

a) $A = [0, 1)$ and $B = (-1, 0]$. $A \cap B = \{0\}$

b) $A = \mathbb{R}^+$ and $B = \mathbb{R}^- \cup \mathbb{Z}^+$. $A \cap B = \mathbb{Z}^+$

c) $A = [0, 2]$ and $B = [1, 3]$. $A \cap B = [1, 2]$

2.5 Cardinality Proof

Show that $|(0, 1)| \geq |\mathbb{Z}^+|$.

Solution: We can show that $|(0, 1)| \geq |\mathbb{Z}^+|$ is true through the existence of an onto function from $(0, 1)$ to $\mathbb{Z}^+$, or a one-to-one function from $\mathbb{Z}^+$ to $(0, 1)$.

Onto example function ($(0, 1)$ to $\mathbb{Z}^+$): $f(x) = \lceil \frac{1}{x} \rceil - 1$
One-to-one example function ($\mathbb{Z}^+$ to $(0, 1)$): $g(x) = \frac{1}{x+1}$

Note: as the set $(0, 1)$ is exclusive, you cannot write $f(x) = \lceil \frac{1}{x} \rceil$ as this does not map to 1. Similarly, you cannot write $g(x) = \frac{1}{x}$; this is not a function, as 1 has no mapping in the codomain.

2.6 Countability

(a) Find a countably infinite subset A of $(0, 1)$.

(b) Find a bijection between A and $A \cup \{0, 1\}$

(c) Find an explicit one-to-one and onto mapping from the open interval $(0, 1)$ to the closed interval $[0, 1]$.

Solution:

(a) $\{\frac{1}{n}\}$ where n is an integer greater or equal to 2.

(b) Map $\frac{1}{2}$ to 0, $\frac{1}{3}$ to 1, and for all $n > 3$, map $\frac{1}{n}$ to $\frac{1}{n-2}$

(c) Map every element to itself, except those in A. Map those in A according to the mapping came up with in part b.

3 Schroder-Bernstein Theorem

Schroder-Bernstein Theorem: For two sets A and B, if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. Note that finding an onto function from A to B shows that $|A| \geq |B|$, and finding a one-to-one function from A to B shows that $|A| \leq |B|$.

1. Schroder-Bernstein Theorem

Show that $(0,1)$ and $[0,1]$ have the same cardinality.

Solution: By Schroder-Bernstein theorem, it suffices to find one-to-one functions $f : (0,1) \to [0,1]$ and $g : [0,1] \to (0,1)$. Let $f(x) = x$ and $g(x) = (x+1)/3$.

4 Supplemental Problems

4.1 Generic pairwise sum

Let $n \geq 2$ be an integer, and suppose that we have selected $n+1$ different integers from the set $\{1, 2, 3, ..., 2n\}$. Prove that there will always be two among the selected integers whose sum is equal to $2n+1$.

Solution: Consider the n pairs $\{1, 2n\}, \{2, 2n-1\}, \{3, 2n-2\}, \ldots, \{n, n+1\}$. Each of our $n+1$ selected integers is a member of exactly one of these pairs, and so by the Pigeonhole Principle there we must have selected two numbers from the same pair. By construction of the pair, these two numbers sum to $2n+1$.

4.2 Deck of Cards

a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are selected?

b) How many must be selected from a standard deck of 52 cards to guarantee that at least three hearts are selected?

Solution: a) Suppose there are four boxes, one for each suit, and as cards are selected they are placed in the box reserved for cards of that suit. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least one box containing at least $\lceil N/4 \rceil$ cards. Consequently, we know that at least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2(4) + 1 = 9$, so nine cards suffice. Note that if eight cards are selected, it is possible to have two cards of each suit, so more than eight cards are needed. Consequently, nine cards must be selected to guarantee that at least three cards of one suit are chosen. One good way to think about this is to note that after the eighth card is chosen, there is no way to avoid having a third card of some suit.

b) We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three hearts, not just three cards of one suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next three cards will be all hearts, so we may need to select 42 cards to get three hearts.