EECS 203: Discrete Mathematics
Winter 2024
FoF Discussion 4 Notes

1 Definitions

- **Types of Proofs:**
  - **Direct Proof:** Prove that if some proposition p is true, then another proposition q is true "directly". Start by assuming that p is true, then make some deductions and eventually arrive at the conclusion that q must be true.

    $$p \to q$$

  - **Proof by Contraposition:** Prove that "if p is true, then q is true" by proving that if q is false, then p is false (since these are logically equivalent).

    $$\neg q \to \neg p$$

  - **Proof by Contradiction:** Prove p is true by assuming it is false, and arriving at a contradiction, i.e. a conclusion that we know is false.
    When using a proof by contradiction to prove "if p is true then q is true", we assume that p is true and that q is false, and derive a contradiction. This shows us that if p is true, then q is true.

    $$\neg(p \to q) \equiv (p \wedge \neg q) \to F \to \neg(p \wedge \neg q) \equiv (p \to q)$$

    A simpler way to view this: Assume p is true and show that

    $$\neg q \to \mathrm{F} \to q$$

  - **Proof by Cases:** Prove by considering all possibilities, or all categories of possibilities (i.e., cases), and showing that in each of those cases, the proposition you're trying to prove is true.

- **Set:** A set is an unordered collection of distinct objects

- **Universe:** In set theory, a universe is a collection that contains all the entities one wishes to consider in a given situation.

- **Set Operations:**
  - **Union** $S \cup T$: The set containing those elements that are in S or T
    $S \cup T = \{x \mid x \in S \vee x \in T\}$
  - **Intersection** $S \cap T$: The set containing those elements that are in S and T
    $S \cap T = \{x \mid x \in S \wedge x \in T\}$
  - **Complement** $\overline{S}$: The set containing those elements that are in the universe U but not in S.
    $\overline{S} = \{x \mid x \in U \wedge x \notin S\}$
  - **Minus** $S - T$: The set containing those elements that are in S but not in T
    $S - T = \{x \mid x \in S \wedge x \notin T\}$

- **Inclusion−Exclusion Principle:** The inclusion-exclusion principle states the the size of the union of two sets is equal to the sum or their sizes minus the size of their intersection:
  $$|A \cup B| = |A| + |B| - |A \cap B|$$

- **Subset:** The set A is a subset of B if and only if every element of A is also an element of B. Denoted $A \subseteq B$. Note that A and B may be the same set.
  $A \subseteq B$ iff $\forall x\, [x \in A \to x \in B]$

- **Proper Subset:** The set A is a proper subset of B if and only if A is a subset of B and $A \neq B$. That is, A is a subset of B and there is at least one element of B that is not in A. Denoted $A \subsetneq B$.
  $A \subsetneq B$ iff $\forall x\, [x \in A \to x \in B] \ \wedge\ A \neq B$

- **Disjoint:** The sets A and B are disjoint if and only if they do not share any elements.

- **Power Set:** The power set of a set S is the set of all subsets of S. $P(S)$ denotes the power set of S.
  $P(S) = \{T \mid T \subseteq S\}$

- **Cardinality:** The number of elements in a set. The cardinality of a set S is denoted by $|S|$.

- **Cartesian Product:** $A \times B$ is the set of all ordered pairs of elements (a, b) where a $\in A$ and $b \in B$.
  $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

- **Empty Set:** The empty set, denoted $\emptyset$ or $\{\}$, is the unique set having no elements.

2 Exercises

1. Associativity of minimum

Use proof by cases to show that $min(a, min(b, c)) = min(min(a, b), c)$ whenever a, b, c are real numbers.
Let a, b and c be arbitrary real numbers.
Case 1: a is the smallest ($a < b$ and $a < c$)

Case 2: b is the smallest

Case 3: c is the smallest

Since it is true in all the cases, we have thus shown through proof by cases that $min(a, min(b, c)) = min(min(a, b), c)$.

Solution: Case 1: a is the smallest ($a < b$ and $a < c$)
In this case since a < b and c a < min(b, c) and thus min(a, min(b, c)) = a. In addition, min(a, b) = a, and min(a, c) = a so min(min(a, b), c) = a. Thus min(a, min(b, c)) = min(min(a, b), c) when a is the smallest number.

Case 2: b is the smallest
In this case since b < c and a min(b, c) = b and min(a, b) = b. Thus min(a, min(b, c)) = a. In addition, min(a, b) = b, and min(b, c) = b so min(min(a, b), c) = b. Thus min(a, min(b, c)) = min(min(a, b), c) when b is the smallest number.

Case 3: c is the smallest
In this case since c < a and b min(b, c) = c and min(a, c) = c. Thus min(a, min(b, c)) = a. We also know that c must be less than min(a, b) (because it is the smallest) so min(min(a, b), c) = c. Thus min(a, min(b, c)) = min(min(a, b), c) when c is the smallest number.

Thus we have shown through proof by cases that min(a, min(b, c)) = min(min(a, b), c)

2. Proof by Cases/Contradiction ⋆

Prove that there is no rational solution to the equation $x^3 + x + 1 = 0$. **Hint:** Use the fact that 0 is an even number.
You can use the following lemmas without proving:

- Odd $\times$ Even = Even
- Odd $\times$ Odd = Odd
- Even $\times$ Even = Even
- Odd + Even = Odd
- Odd + Odd = Even
- Even + Even = Even

Solution:
Suppose there is. Let a solution be $\frac{a}{b}$, with a, b in reduced form.
Then we know that $\frac{a^3}{b^3} + \frac{a}{b} + 1 = 0 \iff a^3 + ab^2 + b^3 = 0$.
Since the RHS is even, LHS should be even as well.

Case 1: a, b both odd.
Then we have LHS = odd + odd $\times$ odd + odd = odd.
Case 2: a is odd, b is even.
Then we have LHS = odd + even + even = odd.
Case 3: a is even, b is odd.
(note that WLOG does not apply here since a, b are not symmetric; there is a term ab^2).
Then we have LHS = even + even + odd = odd.
Case 4: a, b are both even.
This cannot occur since a, b is in reduced form.

Each case results in LHS being odd which is a contradiction if LHS = 0. Thus we have proved by contradiction that the equation $x^3 + x + 1$ has no solution in $\mathbb{Q}$.

3. Prime Proof $\star$

Show that for any prime number p, $p^2 + 11$ is composite (not prime). Recall that a prime p is defined to be a natural number ≥ 2 such that p and 1 are the only factors that divide p.

Solution:
We can consider two cases: either p is even, or it is odd.

- Case 1: Consider the even primes, which is just $p = 2$. $p^2 + 11 = 15$, and $15 = 5 \cdot 3$ is composite.
- Case 2: Now we consider the odd primes, or any prime greater than 2. Since p is odd, we have $p = 2k + 1$ for some integer $k > 1$. Then

  $$p^2 + 11 = (2k + 1)^2 + 11 = 4k^2 + 4k + 12 = 2(2k^2 + 2k + 6).$$

  Hence, p^2+11 can be factored into 2 and $2k^2+2k+6$, therefore p^2+11 is composite.

We have exhausted all non-overlapping cases and proved that for all primes p, $p^2 + 11$ is composite.

4. Proving the Triangle Inequality

Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).

Solution:
This is a proof by cases. There are 4 cases to consider:

- x and y are both nonnegative
- x and y are both negative
- $x \geq 0$, $y < 0$, $x \geq -y$
- $x \geq 0$, $y < 0$, $x < -y$

Since x and y play symmetric roles (you can switch the values of x and y without impacting the validity of the triangle inequality), we can assume without loss of generality (WLOG) for the last two cases that $x \geq 0$ and $y < 0$.

- Case 1: If x and y are both nonnegative, then $|x| + |y| = x + y = |x + y|$.
- Case 2: If x and y are both negative, then $|x| + |y| = (-x) + (-y) = -(x + y) = |x + y|$.
- Case 3: If $x \geq 0$ and $y < 0$ and $x + y \geq 0$, then $|x| + |y| = x + (-y)$ is some number greater than x. $|x + y|$ is some positive number less than x since y is negative. Thus, $|x| + |y| \geq x \geq |x + y|$.
- Case 4: If $x \geq 0$ and $y < 0$ and $x + y < 0$, then $|x| + |y| = x + (-y)$ is some number greater than $-y$. $|x + y| = -(x + y) = (-x) + (-y)$ which is some positive number less than or equal to $-y$, since $-x$ is nonpositive. Thus, we have $|x| + |y| \geq -y \geq |x + y|$.

We have now proved for all cases that the triangle inequality is valid. This example is purposely lengthy to show in full detail a proof by cases.

Alternate Solution:

$$a + b \leq |a| + b \leq |a| + |b|$$
$$-(a + b) = -a - b \leq |-a| - b \leq |-a| + |-b| = |a| + |b|$$

Since $|a + b| = a + b$ or $-(a + b)$, and both cases are $\leq |a| + |b|$, we have proven the triangle inequality.

5. Set Exploration $\star$

a) What is $|\emptyset|$?

b) Let $A = \{1,2,3\}$, $B = \{\emptyset\}$, $C = \{\emptyset, \{\emptyset\}\}$, $D = \{4,5\}$, and $E = \{\emptyset, 5\}$.

i. Is $\emptyset \in A$?

ii. Is $\emptyset \subseteq A$?

iii. Is $\emptyset \in B$?

iv. Is $\emptyset \subseteq B$?

v. Is $\emptyset \in C$?

vi. Is $\emptyset \subseteq C$?

vii. What is $A \cap D$?

viii. What is $B \cap C$?

ix. What is $B \cap E$?

x. What is $|B|$, $|C|$, $|E|$?

c) Let A and C be the sets defined above.

i. What is $P(A)$?

ii. What is $P(C)$?

iii. Find a formula for the size of the power set of S, $|P(S)|$, in terms of $|S|$.

iv. What is $C \times A$?

v. What is A^2? ($A^2 = A \times A$)

vi. Find a formula for the size of the Cartesian product of A and B, $|A \times B|$ in terms of $|A|$ and $|B|$.

Solution:

a) $|\emptyset| = 0$

b) i. No, $\emptyset$ is not an element of A, you would see it in A if it was.

ii. Yes, $\emptyset$ is a subset of all sets. All elements of $\emptyset$ (none) are elements of A. $\{\} \subseteq \{1, 2, 3\}$

iii. Yes, $\emptyset \in \{\emptyset\}$

iv. Yes, $\emptyset$ is a subset of all sets.

v. Yes, $\emptyset \in \{\emptyset, \{\emptyset\}\}$

vi. Yes, $\emptyset$ is a subset of all sets.

vii. $A \cap D = \emptyset$

viii. $B \cap C = \{\emptyset\}$

ix. $B \cap E = \{\emptyset\}$

x. $|B| = 1$, $|C| = 2$, $|E| = 2$

c) i. $\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\Big\}$

ii. $\mathcal{P}(C) = \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\Big\}$

iii. $|\mathcal{P}(S)| = 2^{|S|}$

iv. $C \times A = \Big\{(\emptyset, 1), (\emptyset, 2), (\emptyset, 3), (\{\emptyset\}, 1), (\{\emptyset\}, 2), (\{\emptyset\}, 3)\Big\}$

v. $A^2 = A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

vi. $|A \times B| = |A| \times |B|$

6. Double Subset Equality ⋆

Prove the set equivalence: $A - (B \cap C) = (A - B) \cup (A - C)$

Solution:
First, let's show $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.
Let x be an arbitrary element of the domain.
Assume $x \in A - (B \cap C)$

- $x \in A \wedge x \in \overline{(B \cap C)}$
- $x \in A \wedge (x \notin B \vee x \notin C)$ (using DeMorgan's Law)

- $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$ (using the distributive property)
- $(x \in A - B) \vee (x \in A - C)$
- $x \in (A - B) \vee (A - C)$

Therefore, $A - (B \cap C) \subseteq (A - B) \cup (A - C)$

Now we will show $(A - B) \cup (A - C) \subseteq A - (B \cap C)$
Let x be an arbitrary element of the domain.
Assume $x \in (A - B) \cup (A - C)$

Case 1: $x \in A - B$

- $x \in A \wedge x \notin B$
- $x \in A$
- $x \notin B$
- $x \notin B \vee x \notin C$
(**Note:** we can add whatever we want with an or statement, since we know the first half is always true!)
- $x \in A \wedge (x \notin B \vee x \notin C)$

Case 2: $x \in A - C$

- $x \in A \wedge x \notin C$
- $x \in A$
- $x \notin C$
- $x \notin B \vee x \notin C$
- $x \in A \wedge (x \notin B \vee x \notin C)$

Now we need to use the conclusions of our cases:

- In both cases, we have $x \in A \wedge (x \notin B \vee x \notin C)$
- $x \in A \cap (\overline{B} \cup \overline{C})$
- $x \in A \cap \overline{(B \cap C)}$ (using DeMorgan's Law)
- $x \in A - (B \cap C)$

Therefore, $(A - B) \cup (A - C) \subseteq A - (B \cap C)$

Since each side is a subset of the other, we can say $(A - B) \cup (A - C) = A - (B \cap C)$

7. Subset Proofs

Let A, B, and C be sets. Prove that

a) $(A \cap B \cap C) \subseteq (A \cap B)$

b) $(A - B) - C \subseteq A - C$

Solution:

a) Consider an arbitrary $x \in (A \cap B \cap C)$
- By the definition of intersection, we have $(x \in A) \wedge (x \in B) \wedge (x \in C)$
- So we have $(x \in A) \wedge (x \in B)$ (like $\wedge-$ elim in ND)
- Thus we have, $x \in (A \cap B)$

Therefore, $(A \cap B \cap C) \subseteq (A \cap B)$ by definition.

b) Consider an arbitrary $x \in (A - B) - C$
- By definition of set difference, we know that $(x \in A - B) \wedge (x \notin C)$
- Since $x \in A - B$, we know that $(x \in A) \wedge (x \notin B)$
- Thus, $(x \in A) \wedge (x \notin B) \wedge (x \notin C)$
- Then (like $\wedge$-elim), we have $(x \in A) \wedge (x \notin C)$
- Finally, by definition of set minus, we have $x \in A - C$

Therefore, we have shown that $(A - B) - C \subseteq A - C$

8. Power Sets

Can you conclude that $A = B$ if A and B are two sets with the same power set?

Solution: The union of all the sets in the power set of a set X must be exactly X . In other words, we can recover X from its power set, uniquely. Therefore the answer is yes.

We can also show this is true through a proof by contraposition. Let A and B be sets and assume $A \neq B$. Because A and B are not equal, WLOG there exists an element $x \in A$ such that $x \notin B$. Therefore we conclude that $\{x\} \in \mathcal{P}(A)$, and that $\{x\} \notin \mathcal{P}(B)$, therefore $\mathcal{P}(A) \neq \mathcal{P}(B)$. Thus by the contrapositive, the original statement is true.

9. More Power Sets $\star$

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$

b) $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Solution:

a) The power set of every set includes at least the empty set, so the power set cannot be empty. Thus $\emptyset$ is not the power set of any set.

b) This is the power set of $\{a\}$.

c) We know that the power set a set of size n has 2^n elements, but this set has three elements. Since 3 is not a power of 2, this set cannot be the power set of any set. Set cardinality aside, the set $\{\emptyset, a\}$ may come to mind, but $P(\{\emptyset, a\}) = \{\emptyset, \{\emptyset\}, \{a\}, \{\emptyset, a\}\}$.

d) This is the power set of $\{a, b\}$.

10. Power Set of a Cartesian Product

Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

Solution: This is not true. The simplest counterexample is to let $A = B = \emptyset$ (or any two disjoint sets). Then $A \times B = \emptyset$ and $\mathcal{P}(A \times B) = \{\emptyset\}$, whereas $\mathcal{P}(A) = \mathcal{P}(B) = \{\emptyset\}$ and $\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset)\}$. Thus, $\mathcal{P}(A \times B) \neq \mathcal{P}(A) \times \mathcal{P}(B)$

Inclusion−Exclusion Principle:

The inclusion-exclusion principle states the the size of the union of two sets is equal to the sum or their sizes minus the size of their intersection:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

11. Expanding to 3 sets

Come up with the inclusion exclusion principle for the union of 3 sets: $|A \cup B \cup C|$.

Solution:
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

12. Three sets

Suppose there is a group of 120 U of M students. Here's what you know:

- There are 31 in Engineering.
- There are 65 in LSA.
- There are 44 in Ross.
- There are 20 that are not in any of these 3 schools.
- There are 15 in Engineering and Ross.
- There are 17 in Engineering and LSA.
- There are 18 in LSA and Ross.

How many are in all 3 schools?

Solution: We see that the 20 people that are not in any of the schools do not contribute to the count of any other number, and so we can simply look at the $120 - 20 = 100$ students that are in the schools that we are looking at.

By the inclusion-exclusion principle, we have to add the number of people in the 3 individual categories independently, then subtract the pairwise totals, and add back the number of those in all 3 categories. This should yield our original number of students.

$$31 + 65 + 44 - 15 - 17 - 18 + x = 100$$

Doing some algebra, we see that $x = 10$. Thus there are 10 people in all three schools.