W24

EECS 203 Discussion 4

Proof by Cases, Intro to Sets

Upcoming Exam

- **Exam 1** is on **Monday, February 19th** from **7:00 - 9:00 PM**!
- If you have a time conflict, contact the course staff **ASAP**!
- Practice exam questions have been released on Canvas!
  - They can be found on via **Files -> Practice Exams -> Exam 1**

Upcoming Homework

- Homework/Groupwork 4 will be due **Feb. 15th**
    - **Don’t forget to match pages!**
    - Please note as soon as you press submit you’ve successfully submitted by the deadline. **You can still match pages** with no rush without adding to your submission time.
- Groupwork
    - Groupwork can be done alone, but the problems tend to be more difficult, and the goal is for you to puzzle them out with others!
    - Your discussion section is a great place to find a group!
    - There is also a pinned Piazza thread for searching for homework groups.

Proof Methods Overview

Making a Valid Argument (Writing a Proof)

- **Argument/Proof:** An **argument** for a statement S is a sequence of statements ending with S. S is called the **conclusion**. An argument starts with some beginning statements you assume are true, called the **premises**.
- **Valid Argument/Proof:** An argument is **valid** if every statement after the premises is implied ($\rightarrow$) by the some combination of the statements before it.
  - Whenever the premises are true, the conclusion must be true.

- Today we will be discussing word-style proofs

Proof Methods

- **Direct Proof:** Proves $p \to q$ by showing

  $\mathbf{p \to stuff \to q}$

- **Proof by Contraposition:** Proves $p \to q$ by showing

  $\mathbf{\neg q \to stuff \to \neg p}$

- **Proof by Contradiction:** Proves $p \to q$ by showing

  $\mathbf{(p \land \neg q) \to F \to \neg(p \land \neg q) \equiv \neg p \lor q \equiv p \to q}$

- **Proof by Cases:** Proves $p \to q$ by showing

  $\mathbf{p \to p1 \lor p2 \lor \ldots \lor pn \to q}$

Some Methods of Proving $p \rightarrow q$:

- **Direct Proof:**
  Proves $p \rightarrow q$ by showing **$p \rightarrow$ stuff $\rightarrow q$**

- **Proof by Contraposition:**
  Proves $p \rightarrow q$ by showing **$\neg q \rightarrow$ stuff $\rightarrow \neg p$**
  (Once you show $\neg q \rightarrow \neg p$, you can immediately conclude $p \rightarrow q$ by contraposition)

- **Proof by Contradiction:**
  Assume p and $\neg q$ are true. Derive a contradiction (F), by arriving at a mathematically incorrect statement (ex: $0 = 2$) or two statements that contradict each other ($x = y$ and $x \neq y$). Therefore, $p \rightarrow q$.
  $(p \wedge \neg q) \rightarrow F \rightarrow \neg(p \wedge \neg q) \equiv \neg p \vee q \equiv p \rightarrow q$

- **Proof by Cases:**
  Break p into cases and show that each case implies q (in which case $p \rightarrow q$).
  $p \rightarrow p_1 \vee p_2 \vee \ldots \vee p_n \rightarrow q$

Proof by Cases

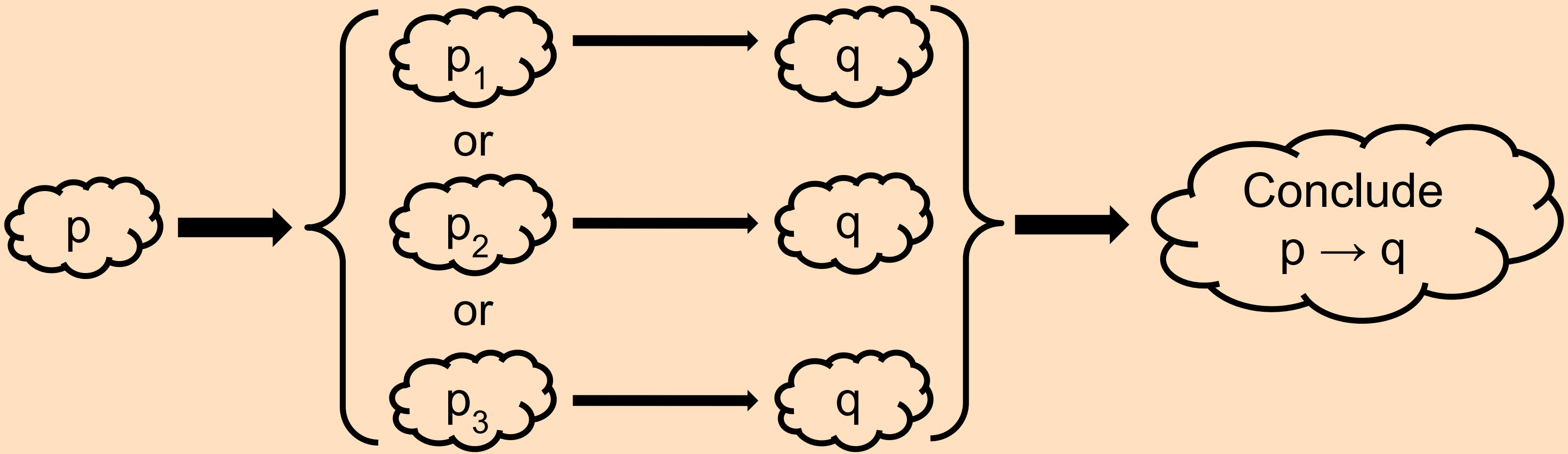
Proof by Cases

Break **p** into cases and show that each case implies **q** (in which case **p** → **q**).

$$\mathbf{p} \rightarrow \mathbf{p_1} \vee \mathbf{p_2} \vee \ldots \vee \mathbf{p_n} \rightarrow \mathbf{q}$$

$\mathbf{p_1} \vee \mathbf{p_2} \vee \ldots \vee \mathbf{p_n}$ should cover all possible cases for **p**.

- We break our statement into all possible cases
- We show that each case leads to the conclusion we want



Problem 1

1. Proof by Cases/Contradiction $\star$

Prove that there is no rational solution to the equation $x^3 + x + 1 = 0$. **Hint:** Use the fact that 0 is an even number.
You can use the following lemmas without proving:

- Odd $\times$ Even $=$ Even
- Odd $\times$ Odd $=$ Odd
- Even $\times$ Even $=$ Even
- Odd $+$ Even $=$ Odd
- Odd $+$ Odd $=$ Even
- Even $+$ Even $=$ Even

Problem 2

2. Prime Proof $\star$

Show that for any prime number p, $p^2 + 11$ is composite (not prime). Recall that a prime p is defined to be a natural number ≥ 2 such that p and 1 are the only factors that divide p.

Problem 3

3. Proving the Triangle Inequality

Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).

Intro to Sets

Set Terminology

- **Set:** A set is an unordered collection of distinct objects
- **Universe:** In set theory, a universe is a collection that contains all the entities one wishes to consider in a given situation.
- **Union $S \cup T$:** The set containing the elements that are in S or T:

  $$S \cup T = \{x \mid x \in S \vee x \in T\}$$

- **Intersection $S \cap T$:** The set containing the elements that are in S and T:

  $$S \cap T = \{x \mid x \in S \wedge x \in T\}$$

- **Complement $\bar{A}$ of A:** The set containing the elements that are in the universe U but not in A:

  $$\bar{A} = \{x \mid x \in U \wedge x \notin A\}$$

- **Minus $S - T$:** The set containing the elements that are in S but not in T:

  $$S - T = \{x \mid x \in S \wedge x \notin T\}$$

Set Terminology

- **Subset:** The set A is a subset of B if and only if every element of A is also an element of B. Denoted $A \subseteq B$. Note that A and B may be the same set.

  $$A \subseteq B \text{ iff } \forall x\, [x \in A \rightarrow x \in B]$$

- **Proper Subset:** The set A is a proper subset of B if and only if A is a subset of B and $A \neq B$. That is, A is a subset of B and there is at least one element of B that is not in A.

  $$A \subsetneq B.\ A \subsetneq B \text{ iff } \forall x\, [x \in A \rightarrow x \in B] \land (A \neq B)$$

- **Disjoint:** The sets A and B are disjoint if and only if they do not share any elements

- **Empty Set:** The empty set, denoted $\varnothing$ or **{}**, is the unique set having no elements.

Set Terminology

- **Inclusion−Exclusion Principle:** The inclusion-exclusion principle states the the size of the union of two sets is equal to the sum or their sizes minus the size of their intersection:

  $$|A \cup B| = |A| + |B| - |A \cap B|$$

- **Power Set:** The power set of a set S is the set of all subsets of S. $P(S)$ denotes the power set of S:

  $$P(S) = \{T \mid T \subseteq S\}$$

- **Cardinality:** The number of elements in a set. The cardinality of a set S is denoted by $|S|$.

- **Cartesian Product:** $A \times B$ is the set of all ordered pairs of elements (a, b) where $a \in A$ and $b \in B$:

  $$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

Problem 4

4. Set Exploration ⋆

a) What is $|\emptyset|$?

b) Let $A = \{1, 2, 3\}$, $B = \{\emptyset\}$, $C = \{\emptyset, \{\emptyset\}\}$, $D = \{4, 5\}$, and $E = \{\emptyset, 5\}$.

i. Is $\emptyset \in A$?
ii. Is $\emptyset \subseteq A$?
iii. Is $\emptyset \in B$?
iv. Is $\emptyset \subseteq B$?
v. Is $\emptyset \in C$?
vi. Is $\emptyset \subseteq C$?
vii. What is $A \cap D$?
viii. What is $B \cap C$?
ix. What is $B \cap E$?
x. What is $|B|$, $|C|$, $|E|$?

c) Let A and C be the sets defined above.

i. What is $P(A)$?
ii. What is $P(C)$?
iii. Find a formula for the size of the power set of S, $|P(S)|$, in terms of $|S|$.
iv. What is $C \times A$?
v. What is A^2? ($A^2 = A \times A$)
vi. Find a formula for the size of the Cartesian product of A and B, $|A \times B|$ in terms of $|A|$ and $|B|$.

Problem 5

5. Double Subset Equality $\star$

Prove the set equivalence: $A - (B \cap C) = (A - B) \cup (A - C)$

Problem 6

6. Subset Proofs

Let A, B, and C be sets. Prove that

a) $(A \cap B \cap C) \subseteq (A \cap B)$

b) $(A - B) - C \subseteq A - C$

Problem 7

7. Power Sets

Can you conclude that $A = B$ if A and B are two sets with the same power set?

Problem 8

8. More Power Sets $\star$

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$

b) $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Problem 9

9. Power Set of a Cartesian Product

Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.