Groupwork 10 Problems

1. Circular Reasoning [15 points]

Suppose we select $2n$ distinct points independently and uniformly at random on the border of a circle, and label them p_1 through p_{2n} counter-clockwise (i.e. point p_2 is counter-clockwise from point p_1).

(a) In the case where $n = 2$, we have four distinct points on the circle. If we select two of these points uniformly at random and draw a line segment between them, then draw a line segment between the remaining two points, what is the probability that these line segments intersect?

Hint: Consider the different cases corresponding to the point p_1 is paired with.

(b) Suppose we repeat the procedure in (a) where we select two points at random and draw a line segment between them. We'll call this line segment ℓ_1. We repeat this again with the $2n-2$ remaining points, creating a line segment ℓ_2, etc., until we have drawn n line segments: $\ell_1, \ldots, \ell_n$. After this procedure is completed, what is the expected number of intersections? Your answer should be in terms of n.

Hint: Create an indicator random variable for each possible intersection and apply linearity of expectation.

Note: The number of intersections is the number of pairs (ℓ_i, ℓ_j) of distinct line segments where ℓ_i and ℓ_j intersect.

Solution:

(a) consider the different cases corresponding to the point p_1 is paired with. there are 3 cases:

(i) p_1 is paired with p_3 and p_2 is paired with p_4 intersect

(ii) p_1 is paired with p_4 and p_2 is paired with p_3 no intersect

(iii) p_1 is paired with p_2 and p_3 is paired with p_4 no intersect

so the probability is $\frac{1}{3}$

(b) Let X_{ij} be the indicator random variable for the event that line segments ℓ_i and ℓ_j intersect. There are 2n points in total, since the starting point is p_1, the number of points that ℓ_1 can intersect is $2n - 3$. The probability is uniform, so it is $\frac{2n-3}{2n-1}$. So the expected number of intersections is $\binom{2n}{2} \cdot \frac{2n-3}{2n-1}$

2. Open or Closed [20 points]

Online Bayseian Inference is a process where we repeatedly apply Bayes rule to update our beliefs over time. Suppose we have a sensor that determines whether a door is open or closed. If the door is open, the sensor reads it as open with probability 0.9. If the door is closed, the sensor reads it as closed with probability 0.7. Suppose the door starts in an unknown position, and has equal probability of being open or closed.

(a) After one reading that the door is closed, what is the probability that the door is actually closed?

(b) Before the second reading, we believe that the door is closed with the probability found in part (a) (that is, we consider the probability that the door is closed to be the probability that we found the door is closed given our first reading). Suppose we make another reading that the door is closed. Now what is the probability that the door is closed?

(c) On the third reading, the sensor reads that the door is open. What is the probability that the door is actually closed, using the answer from part (b) as our initial probability for the door being closed?

Solution:

(a) We are given that the door is closed, and we want to find the probability that the door is actually closed. Let C be the event that the door is closed, and D be the event that the sensor reads the door as closed. We want to find $P(C|D)$. By Bayes' rule, we have

$$\begin{aligned} P(C|D) &= \frac{P(D|C)P(C)}{P(D)} \\ &= \frac{P(D|C)P(C)}{P(D|C)P(C) + P(D|C^c)P(C^c)} \\ &= \frac{0.7 \cdot 0.5}{0.7 \cdot 0.5 + 0.1 \cdot 0.5} \\ &= \frac{0.35}{0.35 + 0.05} \\ &= \frac{0.35}{0.4} \\ &= 0.875. \end{aligned}$$

So the probability that the door is actually closed is 0.875.