Groupwork 6 Problems

1. Multiple Multiples [12 points]

Let $a, b \in \mathbb{Z}$. Show that $7a - 8b$ is a multiple of 5 if and only if $19a - 21b$ is a multiple of 5.

Solution:

Let $a, b \in \mathbb{Z}$. We want to show that $7a - 8b$ is a multiple of 5 if and only if $19a - 21b$ is a multiple of 5.
In other words, we want to show that $7a - 8b \equiv 0 \pmod 5$ if and only if $19a - 21b \equiv 0 \pmod 5$.
i.e. in logic notation, we want to show that $(7a - 8b \equiv 0 \pmod 5) \iff (19a - 21b \equiv 0 \pmod 5)$.
We will show this by proving both directions of the biconditional.

LHS:
Assume $7a - 8b \equiv 0 \pmod 5$

$7a - 8b \pmod 5 \equiv 2a - 3b \pmod 5 \equiv 0 \pmod 5$	definition of mod
$2a - 3b = 5k$ for some $k \in \mathbb{Z}$	definition of mod
$2a = 5k + 3b$	algebra
$4a = 10k + 6b$	algebra
$19a - 21b \pmod 5 \equiv 4a - 1b \pmod 5$	definition of mod
$10k + 6b - 1b \pmod 5 \equiv 10k + 5b \pmod 5 \equiv 0 \pmod 5$	algebra

Therefore, $19a - 21b \equiv 0 \pmod 5$.

RHS:
Similarly, assume $19a - 21b \equiv 0 \pmod 5$

$19a - 21b \pmod 5 \equiv 4a - 1b \pmod 5 \equiv 0 \pmod 5$	definition of mod
$4a - 1b = 5k$ for some $k \in \mathbb{Z}$	definition of mod
$b = 4a - 5k$	algebra
$7a - 8b \pmod 5 \equiv 7a - 8b \pmod 5 \equiv 2a - 3b \pmod 5$	definition of mod
$2a - 3b \pmod 5 \equiv 2a - 3(4a - 5k) \pmod 5$	substitution
$2a - 3(4a - 5k) \pmod 5 \equiv 2a - 12a + 15k \pmod 5$	algebra
$2a - 12a + 15k \pmod 5 \equiv -10a + 15k \pmod 5$	algebra
$-10a + 15k \pmod 5 \equiv 0 \pmod 5$	algebra

Therefore, $7a - 8b \equiv 0 \pmod 5$.
Since we have shown both directions of the biconditional, we have shown that $7a - 8b$ is a multiple of 5 if and only if $19a - 21b$ is a multiple of 5.

2. Rapidly Rising [18 points]

For this problem, we will say a function $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ is "rapidly rising" if:

$$\forall x_1, x_2 \in \mathbb{Z}^+ \; [x_1 < x_2 \to 2f(x_1) < f(x_2)]$$

(a) Prove that $f(x) = 3^x$ is rapidly rising.

Hint: It may be easier to show $f(x_2) > 2f(x_1)$ than the other way around.

(b) Is a rapidly rising function always one-to-one? Is a one-to-one function from $\mathbb{Z}^+ \to \mathbb{Z}^+$ always rapidly rising? Is a one-to-one function (again from $\mathbb{Z}^+ \to \mathbb{Z}^+$) always strictly increasing? Briefly explain your answer; a formal proof is not necessary but is encouraged.

Note: $f\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(c) Prove that, for any rapidly rising function f, it must **not** be onto.

Solution:

a) Let $f(x) = 3^x$. We want to show that $f(x)$ is rapidly rising.
Let $x_1, x_2 \in \mathbb{Z}^+$ such that $x_1 < x_2$.
We want to show that $2f(x_1) < f(x_2)$.
We have $f(x_1) = 3^{x_1}$ and $f(x_2) = 3^{x_2}$.
We want to show that $2 \cdot 3^{x_1} < 3^{x_2}$.
Since for $f(x)$, the base is 3, thus for every increase in x, the value of $f(x)$ is multiplied by 3.
Thus, no matter what the value of x_1 is, 3^{x_1} will always be less than 3^{x_2} for any $x_2 > x_1$.
Therefore, $2 \cdot 3^{x_1} < 3^{x_2}$.
Since we have shown that $2f(x_1) < f(x_2)$, we have shown that $f(x) = 3^x$ is rapidly rising.
b) A rapidly rising function is not always one-to-one.
A one-to-one function from $\mathbb{Z}^+ \to \mathbb{Z}^+$ is not always rapidly rising.
A one-to-one function from $\mathbb{Z}^+ \to \mathbb{Z}^+$ is not always strictly increasing.
A rapidly rising function is not always one-to-one because a rapidly rising function only guarantees that $2f(x_1) < f(x_2)$ for $x_1 < x_2$. It does not guarantee that $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$.
A one-to-one function from $\mathbb{Z}^+ \to \mathbb{Z}^+$ is not always rapidly rising because a one-to-one function only guarantees that $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$. It does not guarantee that $2f(x_1) < f(x_2)$ for $x_1 < x_2$.
A one-to-one function from $\mathbb{Z}^+ \to \mathbb{Z}^+$ is not always strictly increasing because a one-to-one function only guarantees that $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$. It does not guarantee that $f(x_1) < f(x_2)$ for $x_1 < x_2$.

c) Let f be a rapidly rising function. We want to show that f must not be onto.
A function f is onto if for every $y \in \mathbb{Z}^+$, there exists an $x \in \mathbb{Z}^+$ such that $f(x) = y$.
We will show that f must not be onto by contradiction.
Assume for the sake of contradiction that f is onto.
Since f is onto, for every $y \in \mathbb{Z}^+$, there exists an $x \in \mathbb{Z}^+$ such that $f(x) = y$.
Let $x_1, x_2 \in \mathbb{Z}^+$ such that $x_1 < x_2$.
We have $f(x_1) < f(x_2)$ because f is rapidly rising.
Since f is onto, for every $y \in \mathbb{Z}^+$, there exists an $x \in \mathbb{Z}^+$ such that $f(x) = y$.
Let $y = f(x_2)$.
Since f is onto, there exists an $x \in \mathbb{Z}^+$ such that $f(x) = f(x_2)$.
Since f is one-to-one, $x = x_2$.
Since $x_1 < x_2$, $f(x_1) < f(x_2)$.
This is a contradiction because $f(x_1) < f(x_2)$ but $f(x_1) = f(x_2)$.
Therefore, f must not be onto.
Since we have shown that f must not be onto, we have shown that for any rapidly rising function f, it must not be onto.