EECS 203: Discrete Mathematics
Winter 2024
Homework 4

Due **Thursday, Feb. 15th**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $8 + 2$ Total Points: $100 + 20$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Even Just One [12 points]

Prove that if $n^3 + 4$ is even or $3n + 3$ is odd, then n is even.

Solution:

Contraposition of original is if n is odd, then $n^3 + 4$ is odd and $3n + 3$ is even.
Prove by contrapositive:

Assume n is odd	premise
Let $n = 2k + 1$, k is an arbitary integer	definition of odd
$n^3 + 4 = (2k + 1)^3 + 4 = 8k^3 + 12k^2 + 6k + 4 + 1$	
$= 2(4k^3 + 6k^2 + 3k + 2) + 1$	substitution
Let arbitary integer $j = 4k^3 + 6k^2 + 3k + 2$, $n^3 + 4 = 2j + 1$	
Therefore, $n^3 + 4$ is also odd	definition of odd
Similarly, $3n + 3 = 3(2k + 1) + 3$	
$= 6k + 6 = 2(k + 3)$	substitution
Therefore, $3n + 3$ is even	definition of even
Thus, the original proposition is true by contraposition.	

2. Odd^2 [20 points]

Prove the following for all integers x and y:

(a) If $x + y$ is even, then (x is even and y is even) or (x is odd and y is odd).

(b) Using your answer from part (a), show that if $(x - y)^2$ is odd, then $x + y$ is odd.

Solution:

a) Proof by contrapositive with cases
First, the contraposition is:
(x is odd $\vee$ y is odd)$\wedge$ (x is even $\vee$ y is even) $\rightarrow$ $x + y$ is odd.

Assume arbitary integers x and y	premise
Case 1: x is even, y is even	
$x = 2k$, $y = 2j$, k, j be arbitary integers	definition of even
$x + y = 2k + 2j = 2(k + j)$	substitution
Thus, $x + y$ is even.	definition of even
Case 2: x is odd, y is odd	
$x = 2k + 1$,$y = 2j + 1$	definition of odd
$x + y = 2(k + j + 1)$	substitution
Thus, $x + y$ is even.	definition of even
Case 3: x is even, y is odd	
$x = 2k$	
$y = 2j + 1$, k, j be arbitary integers	definition of even / odd
$x + y = 2(k + j) + 1$	substitution
Thus, $x + y$ is odd, the contrapositive holds.	definition of odd
Case 4: x is odd, y is even	
This would be the same as Case 3 but with x and y swapped.	

Therefore, the original proposition holds by proof by contrapositive with cases.

b) Similarly, proof by contrapositive with cases:

The contrapositive is ($x + y$ is even $\rightarrow (x - y)^2$ is even

a) concludes that (x is even $\wedge y$ is even) or (x is odd $\wedge y$ is odd)	premise
Thus, there are two cases to consider	
Case 1:x is even $\wedge y$ is even	
Let $x = 2k, y = 2j, k, j$ be arbitary integers	definition of even
$(x - y)^2 = (2k - 2j)^2 = (2(k - j))^2 = 2(2k^2 - 4kj + 2j^2)$	substitution
Let integer l be $2k^2 - 4kj + 2j^2$	
$(x - y)^2 = 2l$	substitution
Thus, $(x - y)^2$ is even.	definition of even
Case 2:x is odd $\wedge y$ is odd	
Similar to Case 1,	
Let $x = 2k + 1, y = 2j + 1, k, j$ be arbitary integers	definition of odd
$(x - y)^2 = (2k + 1 - 2j - 1)^2 = (2k - 2j)^2 =$	
$(2(k - j))^2 = 2(2k^2 - 4kj + 2j^2)$	substitution
Thus, $(x - y)^2$ is even.	definition of even

Therefore, the original proposition holds by proof by contrapositive with cases.

3. Do you ∃xist...? [8 points]

Prove or disprove the following: There exist integers x and y so that $20x + 4y = 1$.

Solution:

Disproof by contradiction:

Assume $\exists x \exists y$such that $20x + 4y = 1 (x, y \in \mathbb{Z})$	premise
$20x + 4y = 2(10x + 2y)$	factor
Let arbitary integer $k = 10x + 2y$, $20x + 4y = 2k$	substitution
Thus, $20x + 4y$ is even	definition of even
$1 = 2 \times 0 + 1$, thus 1 is odd	definition of odd
Thus $even = odd$, which clearly contradicts	
Therefore, disproved the original proposition by contradiction.	

4. What's Nunya? Nunya Products are Negative. [12 points]

Given any three real numbers, prove that the product of two of them will always be non-negative.

Solution:

Proof by cases:

Let $x, y, z \in \mathbb{R}$	premise
Case 1: $x \geq 0, y \geq 0, z \geq 0$	
$xy \geq 0$	
Case 2: $x \geq 0, y \geq 0, z \leq 0$	
$xy \geq 0$	
Case 3: $x \geq 0, y \leq 0, z \leq 0$	
$yz \geq 0$	
Case 4: $x \leq 0, y \leq 0, z \leq 0$	
$xy \geq 0$	
Case 5: $x \leq 0, y \leq 0, z \geq 0$	
$xy \geq 0$	
Case 6: $x \leq 0, y \geq 0, z \geq 0$	
$yz \geq 0$	
Case 7: $x \leq 0, y \geq 0, z \leq 0$	
$xz \geq 0$	
Case 8: $x \geq 0, y \leq 0, z \geq 0$	
$xz \geq 0$	

WLOG, the product of two of the three real numbers will always be non-negative.

5. Element or Subset? [8 points]

Let $A = \{1, 2, \text{“a”}\}$. State whether each statement is true or false. Give a brief explanation if false (you do not need to justify why a statement is true).

(a) “a” $\in A$

(b) “a” $\subseteq A$

(c) $\{1, 2\} \in A$

(d) $\{1, 2\} \subseteq A$

Solution:

a) T
b) F This is because for “a” to be a set, it needs brackets
c) F This is because for $\{1, 2\}$ is not an element in the set
d) T

6. Ready, $\{s, e, t\}$, go! [12 points]

Let $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{4, 5\}$. Compute the following, where complements are taken within S. Show intermediate steps as part of your justification.

(a) $\mathcal{P}\left((A \cap B) \cap \overline{C})\right)$

(b) $\mathcal{P}\left((\overline{C} - B) \cap A\right)$

(c) $\{A \times B\} \cap \{S \times B\}$

(d) $(A \times B) \cap (S \times B)$

Solution:

a) $A \cap B = \{2\}, \overline{C} = \{1, 2, 3\}$
$(A \cap B) \cap \overline{C} = \{2\}$
$\mathcal{P}\left((A \cap B) \cap \overline{C})\right) = \{\emptyset, 2\}$

b)$\overline{C} - B = \{1\}$
$(\overline{C} - B) \cap A = \{1\}$
$\mathcal{P}\left((\overline{C} - B) \cap A\right) = \{\emptyset, 1\}$

c) $A \times B = \{(1,2),(1,3),(2,2),(2,3)\}$
$S \times B = \{(1,2),(1,3),(2,2),(2,3),(3,2),(3,3),(4,2),(4,3),(5,2),(5,3))\}$
$\{A \times B\} \cap \{S \times B\} = \{\{(1,2),(1,3),(2,2),(2,3)\}\}$
$\cap\{\{(1,2),(1,3),(2,2),(2,3),(3,2),(3,3),(4,2),(4,3),(5,2),(5,3))\}\}$
$= \{\emptyset\}$

d)$(A \times B) \cap (S \times B) = \{(1,2),(1,3),(2,2),(2,3)\}$

7. Subset Proofs [16 points]

Prove that if A and B are sets, then $A \cup (A \cap B) = A$ by proving each side is a subset of the other. This set identity is known as an absorption law. Your answer should be a word proof, and not use any set equivalence laws.

Solution:

Proof:
We will show that these two sets are equal by showing that each is a subset of the other.
Suppose $x \in A \cup (A \cap B)$. Then $x \in A \vee x \in A \cap B$ by the definition of union.
In the former case, we have $x \in A$, and in the latter case we have $x \in A \wedge x \in B$ by the definition of intersection.
Thus in any event, $x \in A$, so we have proved that the left-hand side is a subset of the right-hand side.
Conversely, let $x \in A$. Then by the definition of union, $x \in A \vee (A \cap B)$ as well.
Thus we have shown that the right-hand side is a subset of the left-hand side.
Therefore, proved.
will revise later

8. IceCream-Exclusion [12 points]

Out of the 40 EECS 203 staff members, 21 like vanilla ice cream, 18 like chocolate ice cream, and 24 like strawberry ice cream. In addition, 13 like both strawberry and vanilla, and 7 like chocolate and vanilla.

(a) How many staff members like all three ice cream flavors if 9 staff members like both strawberry and chocolate ice cream, assuming everyone likes at least one type of ice cream?

(b) How many staff members don't like any of the ice cream flavors if 14 staff members like both strawberry and chocolate ice cream and 3 staff members like all three ice cream flavors?

Solution:

Let V, C, S be the population that likes the types of ice cream

a) According to the inclusion-exclusion principle, $|V \cup C \cup S| = |V| + |C| + |S| - |V \cap C| - |V \cap S| - |C \cap S| + |V \cap C \cap S|$
$= 40 - 21 - 18 - 24 + 13 + 7 + 9 = 6$

b) Similarly, $|\overline{V} \cap \overline{C} \cap \overline{S}| = 40 - |V \cup C \cup S| = 40 - 21 - 18 - 24 + 13 + 7 + 14 - 3 = 8$

Grading of Groupwork 3

Using the solutions and Grading Guidelines, grade your Groupwork 3 Problems:

- Use the table below to grade your past groupwork submission and calculate scores.
- While grading, mark up your past submission. Include this with the table when you submit your grading.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- See "All About Groupwork" on Canvas for more detailed guidance, and what to do if you change groups.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 1	+9	+6	+2	+2								19/30
Total:												19/30

Groupwork 3 Problems

1. ∀re These ∃quiv♢lent? [30 points]

Let $P(x)$ and $Q(x)$ be arbitrary predicates.

(a) Prove or disprove that for any domain of x, $\forall x(P(x) \leftrightarrow Q(x))$ must be logically equivalent to $\forall x P(x) \leftrightarrow \forall x Q(x)$.

(b) Prove or disprove that for any domain of x, $\exists x(P(x) \leftrightarrow Q(x))$ must be logically equivalent to $\exists x P(x) \leftrightarrow \exists x Q(x)$.

(c) Let $\diamondsuit x$ mean that "there exists **at most one** x." Prove or disprove that for any domain of x, $\diamondsuit x(P(x) \leftrightarrow Q(x))$ must be logically equivalent to $\diamondsuit x P(x) \leftrightarrow \diamondsuit x Q(x)$.

Solution:

a) Disproof:
To disprove $\forall x(P(x) \leftrightarrow Q(x)) \equiv \forall x P(x) \leftrightarrow \forall x Q(x)$,
$P(x)$ and $Q(x)$ has to be sometimes, but not always, true.
Let $P(x)$ be $2 \mid x$ and $Q(x)$ be $3 \mid x$.
The left proposition will be false.
It is because there exists an x such that either proposition but not both is true.
(e.g. $x = 4$ or $x = 9$ will make it false).
The right, however, will be always true because not all x satisfies both propositions.
Thus, $F \not\equiv T$, it is disproved.
b) Disproof:
Same idea as a),
except for that in the right proposition, x does not have to be the same.
c) Disproof:
Same as a) and b),
but with counter examples showing that the proposition does not stand.

Groupwork 4 Problems

1. Mostly Rational [12 points]

Show that if r is an irrational number, there is a unique integer n such that the distance between r and n is *strictly* less than $\frac{1}{2}$.

Solution:

Proof by contradiction:

Assume $r \notin \mathbb{Q}, \exists n_1 \exists n_2 [|r - n_1| < \frac{1}{2} \wedge |r - n_2| < \frac{1}{2}](n_1, n_2 \in \mathbb{Z})$	premise
$|n_1 - n_2| \leq |r - n_1| + |r - n_2|$	triangle inequality
$|n_1 - n_2| \leq \frac{1}{2} + \frac{1}{2} = 1$	substitution
Since $n_1, n_2 \in \mathbb{Z}$,	
they cannot have a distance < 1,	
this contradicts with the premise	contradiction

Therefore, the original proposition holds by proof by contradiction.

2. Set in Stone [8 points]

Prove using set identities that

$$(A \cap C) - (B \cap A) = (C - B) \cap A$$

for any three sets A, B and C.

Solution:

Let $x \in (A \cap C) - (B \cap A)$	premise
$\equiv x \in (A \cap C) \wedge x \notin (B \cap A)$	definition of minus
$\equiv x \in A \wedge x \in C \wedge (x \notin B \vee x \notin A)$	DeMorgan's Laws
$\equiv (x \in A \wedge x \in C \wedge x \notin B) \vee (\equiv x \in A \wedge x \in C \wedge x \notin A)$	distributive