EECS 203 Exam 2 Review

Day 2

Today’s Review Topics

- Functions
  - Properties
  - Compositions
- Countability
  - Schroder-Bernstein
- Recurrence Relations
- Pigeonhole Principle

Functions & Countability

(Functions) Definitions

- **Function** $f : A \to B$: A function f is a relation between two sets, say A and B that associates each element of set A to exactly one element from the set B. The set A and set B are respectively called the domain and codomain of f. The range of f is the set of all elements in the codomain which are mapped to by an element in the domain.

1. Each "x"-value is mapped to something
2. Each "x"-value maps to exactly one thing

- **Onto**: A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$, there is an element $a \in A$ with $f(a) = b$. A function f is called surjective if it is onto.

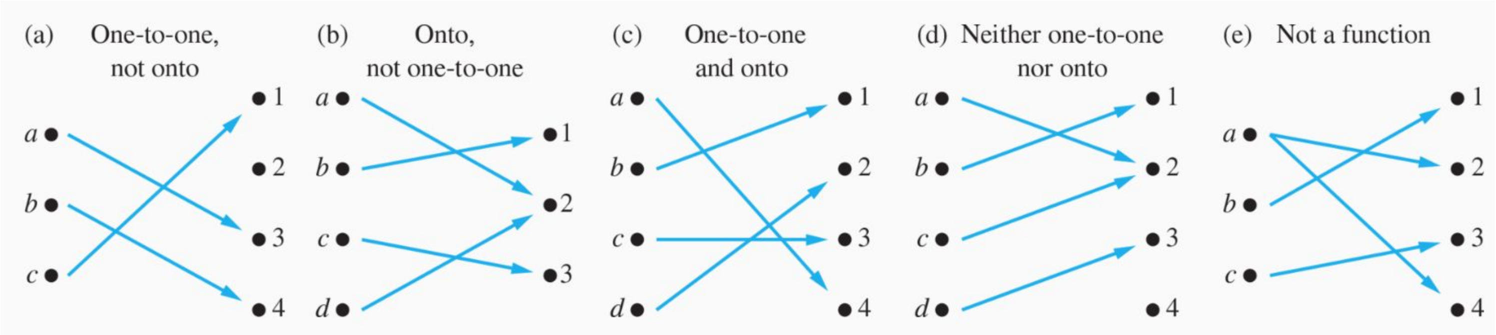
$$\forall b \in B, \exists a \in A\ [f(a) = b]$$

- **One-to-One**: A function f is said to be one-to-one, or an injection, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be injective if it is one-to-one.

$$\forall a_1, a_2\ [(f(a_1) = f(a_2)) \rightarrow (a_1 = a_2)]$$

- **Bijection**: A function f is called a bijection (or one-to-one correspondence) if it is both one-to-one and onto.

Some Pictures



Function Property Proof Templates

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

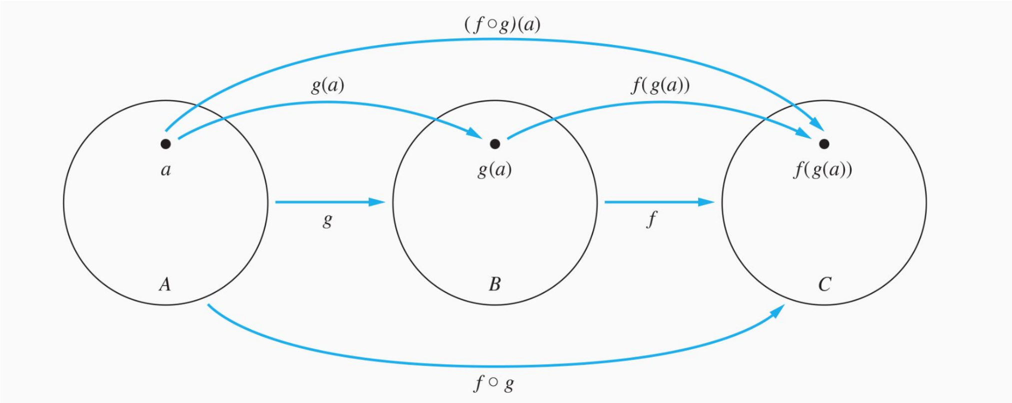
To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

More (Functions) Definitions

- f^{-1}: Let f be a bijection from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Bijective == Invertible

- $f \circ g$: Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.



Inverse

What is the inverse of function f?

$$f : \mathbb{Z} \to \mathbb{Z} \quad \text{where} \quad f(x) = \begin{cases} x - 2 & : x \geq 5 \\ x + 1 & : x \leq 4 \end{cases}$$

(A)

$$f^{-1}(x) = \begin{cases} x - 2 & : x \geq 4 \\ x + 1 & : x \leq 5 \end{cases}$$

(B)

$$f^{-1}(x) = \begin{cases} x + 2 & : x \geq 5 \\ x - 1 & : x \leq 4 \end{cases}$$

(C)

$$f^{-1}(x) = \begin{cases} x + 2 & : x \geq 4 \\ x - 1 & : x \leq 5 \end{cases}$$

(D) f does not have an inverse

Define $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$: $f(x) = \log_{10}(x)$ and $g : \mathbb{R} \longrightarrow \mathbb{R}^+$: $g(x) = 3^x$. Which of the following can we conclude?

(a) f is one-to-one

(b) $f \circ g$ is a bijective function

(c) $g \circ f$ is a bijective function

(d) f is onto

(e) g is bijective

Practice

For each of the functions below determine whether it is i) one-to-one, ii) onto, iii) bijective. Prove your answers.

a) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3 + 1$

b) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = -3x^2 + 7$

Prove

Suppose that g is a function from A to B and f is a function from B to C.

c) If $f \circ g$ is a bijection, then g is onto if and only if f is one-to-one.

Proof

Given: $f \circ g$ is a bijection.

Want to show: g is onto if and only if f is one-to-one

Equivalent: g is onto ↔ f is one-to-one

Split into two implications: g is onto → f is one-to-one

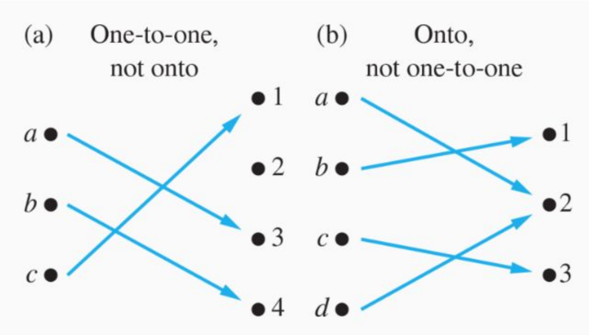
f is one-to-one → g is onto

Sufficient to prove both of them separately.

Countability

Function Properties & Cardinality

	$f : A \to B$	$f : B \to A$
f is **one-to-one**	$\lvert A\rvert \leq \lvert B\rvert$	$\lvert B\rvert \leq \lvert A\rvert$
f is **onto**	$\lvert A\rvert \geq \lvert B\rvert$	$\lvert B\rvert \geq \lvert A\rvert$



Let $A = \mathbb{R}$. For which of the following will $A - B$ be countably infinite?

(a) $B = (\mathbb{R} - \mathbb{Q}) \cup (\mathbb{Z} \times \mathbb{Z})$

(b) $B = \mathbb{Q}$

(c) $B = \mathbb{Q} \oplus \mathbb{R}^+$

(d) $B = \{x | x \in \mathbb{R} \wedge x \neq -x\}$

(e) $B = \{x | x \in \mathbb{R} \wedge \exists y (y \in \mathbb{Z} \wedge |x - y| < \frac{1}{2})\}$

Given the following, which of the statements must be **FALSE**?

$$|A| = 5,\ |B| = 8,\ |C| = 7,\ |D| = 4$$
$$f : A \to B,\ g : B \to C,\ h : C \to D$$

Scantron 10

(A) f can be one to one

(B) g can be onto

(C) h can be onto

(D) $h \circ g$ can be onto

(E) $g \circ f$ can be bijective

Problem 13. (4 points)

Which of the following functions can be used to show that $|\mathbb{Z}| = |\mathbb{N}|$? (Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.)

(a) $f\colon \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \pmod 2 \\ -\frac{n+1}{2}, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

(b) $f\colon \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = n$$

(c) $f\colon \mathbb{Z} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} n, & \text{if } n \geq 0 \\ -n, & \text{if } n < 0 \end{cases}$$

(d) $f\colon \mathbb{Z} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} 2n, & \text{if } n \geq 0 \\ -2n-1, & \text{if } n < 0 \end{cases}$$

(e) $f\colon \mathbb{Z} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} 4n, & \text{if } n \geq 0 \\ -4n-1, & \text{if } n < 0 \end{cases}$$

Prove that $\mathbb{Z}^+$ has the same cardinality as $\mathbb{Z}$

(Hint: use Schroder-Bernstein)

Which of the following sets have the same cardinality?

(a) $\mathbb{Z}$ and $\mathbb{Q}^+$

(b) $\mathbb{Q}^- \cap \mathbb{R}^-$ and $[0, 203)$

(c) $\mathbb{Z}^- \cap \mathbb{R}+$ and $\mathbb{Z}^-$

(d) $\mathbb{N} \cup (4, 20)$ and the set of even integers

(e) $\mathbb{Z} \times \mathbb{Q}$ and $\mathbb{N}$

Break

Recurrence

Recurrence Recap

- A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses $\{a_n\}$ in terms of one or more of the previous terms of the sequence.

$$a_n = 8a_{n-1} - 16a_{n-2}$$

- Includes associated base cases
- Will be relevant when solving recursive problems
- For example, finding Fibonacci numbers

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3$$

with the initial values

$$F_1 \text{ and } F_2 = 1.$$

Warm up

You are trying to walk up a total of n steps up a stair, and you can walk up either 1 or 2 steps. Find the recurrence relation.

Finding Recurrence Relations

- Either think about the “first steps” or the “last steps”
  - For example, the stairs problem solution thought about the "last steps".
- Try to go forward from "first steps" or backwards from "last steps" as little as possible; use the recursive leap of faith whenever possible
- Base cases are usually the first cases that aren't covered by the equation (they're like edge cases)
- Every term after base cases should be able to be calculated from the recurrence

At a local candy shop, Milky Way, Twix, and Kit-Kat are each \$2 per bar and Reese's and Hershey's are each \$3 per bar.

a) Let a_n represent the number of ways to buy exactly $\$n$ worth of candy bars where the order in which candy bars are bought matters. Find a recurrence relation for a_n. Include an explanation for why this is the correct recurrence.

b) What are the initial conditions for the recurrence relation from part (a)? For full credit, you must use the fewest number of initial conditions necessary for your recurrence and must start at a_0. Also, please provide brief justification for each initial condition.

c) Compute a_6. Express your answer as a single number.

More practice with recurrence relations.

Suppose that Bob has 3 different hats, a red one, a blue one, and a yellow one, where he wears one hat per day. He has a couple of rules on how he distributes wearing them.

1) If he wears a red hat, he must have worn a blue hat on the day before.
2) If he wears a yellow hat, he cannot wear the yellow hat again the next day, unless he also wore a blue hat on the day before he wore the first yellow hat.

Find a recurrence relation to describe how many ways there are for Bob to wear his hats in a period of n days. For example, for n = 5, a valid way for Bob to wear his hats is BRBYY.

How many initial conditions are necessary to define the following recurrence?

$$P(n) = P(n-2) + 2^{P(n-4)} + log(n-5)$$

(a) 2

(b) 3

(c) 4

(d) 5

(e) 6

Pigeonhole Principle

Pigeonhole Principle Recap

- If there are *k+1* objects and *k* boxes, at least one box must contain 2 or more objects
- Generalized Version: If *N* objects are placed into *k* boxes, then there is at least one box containing at least *ceiling(N/k)* objects
- This does NOT guarantee which box will have 2 or more objects
- This does NOT guarantee the number of objects in the special box

Warm-Up

What's the minimum number of phone numbers in order to ensure that at least three people have the same area code where X can be any number from 0 to 9 (XXX-213-435)?

Pigeonhole Principle

- Think of the “worst case scenarios”
- If the question asks “what is the least number required to get x number of objects per box ?” consider “how many can I have that will allow for x - 1 per box?”

Practice Problem

If there are n people who can shake hands with one another (where $n > 1$), show that there is always a pair of people who will shake hands with the same number of people.

Have a great weekend!