W24

EECS 203 Discussion 5b

Strong Induction & Recurrence Relations

Admin Notes:

- Homework/Groupwork 5 will be due **Mar. 7th – AFTER SPRING BREAK**
  - **Don't forget to match pages!**
  - Please note as soon as you press submit you've successfully submitted by the deadline. **You can still match pages** with no rush without adding to your submission time.

- Exam 1:
  - **Grades will release before the end of the day! (Friday, Feb. 23)**

Weak Induction

Recall Mathematical (Weak) Induction:

We want to show some statement **P(n)** is true for all integers **n ≥ c**.

- **Base Case**
  - First, show that the statement **P(c)** is true for some initial value **c**.
- **Inductive Step**
  - Next, show that if **P(k)** is true for an arbitrary integer **k ≥ c**, then **P(k+1)** is also true.
  - In other words, we want to prove the implication **P(k)** → **P(k+1)**.
  - Since k is arbitrary, we start this step by assuming that **P(k)** is true.
  - When you assume **P(k)**, it's called the **inductive hypothesis**.
- **That's it!**
  - You've proven that **∀(n ≥ c) P(n)**, as desired.
  - Since **P(c)** is true and **P(k)** implies **P(k+1)**, we therefore have:
    P(c) → **P(c+1)** → **P(c+2)** → **P(c+3)** → **P(c+4) . . .**

Problem 1

1. Mathematical Induction - Sets Edition

Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

Strong Induction

Strong Induction

As before, we want to show some statement **P(n)** is true for all integers **$n \geq c$**.

- **Inductive Step**
  - Show that if P(j) is true for **$c \leq j \leq k$** , then **P(k+1)** is true
    P(c), … , P(k) → P(k+1)

- **Base Case**
  - Show P(c) and any other base cases that are needed…
    P(c), P(c+1), … , P(s)

- **Now, you've shown $\forall n \geq c$ P(n)** because **P(c),...,P(s)** are true, and:

$$\left.\begin{matrix}P(c)\\ P(c+1)\\ \vdots\\ P(s)\end{matrix}\right\} \longrightarrow P(s+1) \longrightarrow P(s+2)\ \ldots$$

Problem 2

2. Faulty Induction

Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

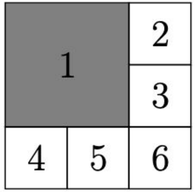
Base Case: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all non-negative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k+1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

Problem 3

3. Squares Strong Induction⋆

Prove that a square can be subdivided into any number of squares $n \geq 6$. Note that sub-squares don't need to be the same size. For example, here's how you would subdivide a square into 6 squares.



Problem 4

4. Jigsaw Puzzle Induction

A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n-1$ moves are required to assemble a puzzle with n pieces.

Problem 5

5. Forming Discussion Groups 1⋆

Tom is trying to do a group activity in his next discussion session. He wants to form groups of size 5 or 6.

(a) Show Tom that if there are 23 students attending his discussion, he will be able to split the students into groups of 5 or 6.

(b) In fact, there is some cutoff $p \in \mathbb{N}$ where $\forall n \geq p$, n students can be split into groups of 5 or 6. Find the smallest possible value of p.

(c) Now prove to Tom that if at least p students attends his discussion, he can successfully split the students in to groups of 5 or 6.

Recurrence Relations

Recurrence Relation: an equation that defines a sequence based on a rule that gives the next term as a function of previous terms.

Example:

A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years. Find a recurrence relation for *L(n)*, where *L(n)* is the number of lobsters caught in year *n*.

$$L(n) = \big(L(n-1) + L(n-2)\big) / 2$$

Recurrence Relations

Recurrence Relation: an equation that defines a sequence based on a rule that gives the next term as a function of previous terms.

Example:

- If we're searching an ordered list of length *n* for a particular number, how many total comparisons will we need to make?

$$S_n = S_{n/2} + 1$$

- We do this by checking the middle of the list each time, recursively narrowing the range we're looking at to half of the previous iteration.

Problem 6

6. Forming Discussion Groups 2$\star$

In the previous question, we proved that Tom can split a total of n students into groups of 5 or 6 when $n \geq 20$ using induction.

(a) Give a recurrence relation for the minimum number of groups, $G(n)$ that needs to be formed for a class of n students to be split into groups of 5 or 6.

(b) What are the initial conditions?

Problem 7

7. Lobster Recurrence

A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years. Find a recurrence relation for $L(n)$, where $L(n)$ is the numbers of lobsters caught in year n, under the assumption for this model.

Problem 8

8. Stair Climbing

(a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.

(b) What are the initial conditions?