W24

EECS 203 Discussion 8a

Graph Theory

Admin Notes

- **Homework/Groupwork 8 will be due Apr. 4th!**
- **Exam 2** is on **Wednesday, March 27th** from **7:00 - 9:00 PM**!
- **Exam Review Sessions**
  - **Sat, March 23rd, 2-5 PM** in **BBB 1670**
    - **Topics:** Induction, Strong Induction
  - **Sun, March 24th, 2-5 PM** in **BBB 1670**
    - **Topics:** Functions, Mod, PHP, Countability
- If you have a time conflict, contact the course staff **ASAP**!
- Practice exam questions have been released on Canvas!
  - They can be found on via **Files -> Practice Exams -> Exam 2**
  - See pinned Piazza post **@901** for practice exam walkthrough videos

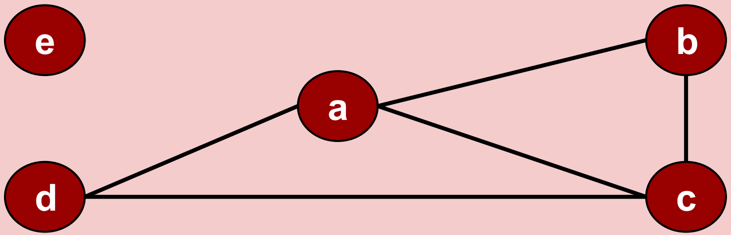
Introduction to Graphs

What is a graph?

- **Graph:** A graph **G = (V, E)** consists of **V** a set of **vertices** and **E** a set of **edges**.
- **(Simple) Graph:** a graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices
  - When we say graph we are referring to this type of graph. We will clarify when referring to some other type of graph like a multigraph.

Example:

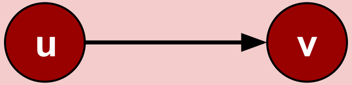
G = (V, E) where **V = {a,b,c,d,e}** and **E = { {a,b}, {a,c}, {a,d}, {b,c}, {c,d} }**



Note: Edge pairs listed as sets: **{}**
This is because there is no order to the vertices, since the edges are undirected.

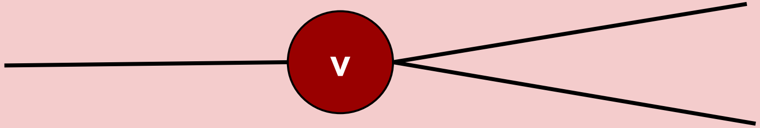
Graph Definitions

- **Directed Graph:** a graph where each edge is associated with an <u>ordered pair</u> of vertices **(u,v)** and the edge is to start at u and end at **v**.



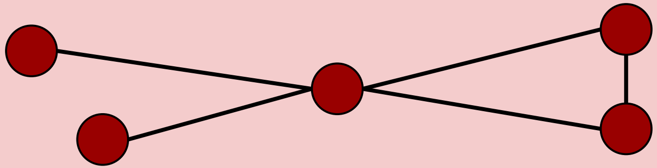
- **Adjacent Vertices:** Two vertices are adjacent if there is an edge that connects them.

- **Degree of a Vertex *deg(v)*:** In an undirected graph, the degree of a vertex, v, is the number of edges attached to v. (In the example below, ***deg(v) = 3***)



Graph Definitions

- **Degree Sequence:** The sequence of degrees of the vertices of a graph in non-increasing order. Ex: **(4,2,2,1,1)**



- **Neighborhood of a Vertex *N(v)*:** The set of all adjacent (or neighbor) vertices of that vertex. For a set of vertices A, the neighborhood ***N(A)***, is the set of all neighbor vertices to any vertex within the set A.

The Handshake Theorem

- **The Handshake Theorem:** Let G = (V, E) be an undirected graph with ***m* edges**. Then:

$$2m = \sum_{v \in V} deg(v)$$

In other words, the sum of the degrees in a graph is equal to two times the edges.

- **Corollary of Handshake Theorem:** Every graph has an even number of vertices with odd degrees.

Problem

1. Graphs Intro

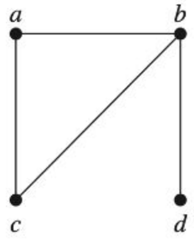
For the following graphs:

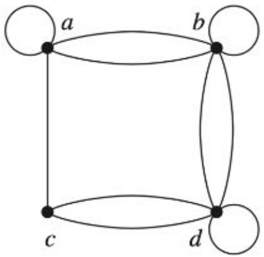
a) Identify whether the graph has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops.

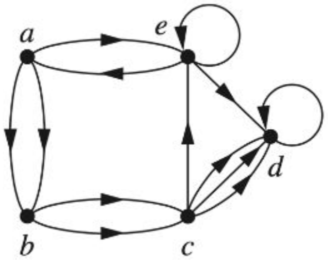
b) For each undirected graph, identify whether or not it is simple. If it is not simple, find a set of edges to remove to make it simple.

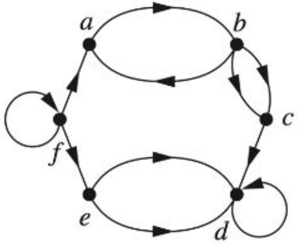
c) Find $deg(b)$ or if the graph is directed, find $deg^-(b)$ and $deg^+(b)$.

d) Write out its degree sequence. For this part, treat the directed graphs as if they were undirected.









Solution

1. Graphs Intro

For the following graphs:

a) Identify whether the graph has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops.

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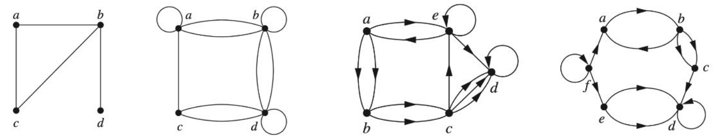


Solution:

a) undirected, no multiple edges, no loops

undirected, multiple edges, loops

directed, multiple edges (there are multiple going in the same direction between two vertices), loops

directed, multiple edges, loops

NOTE: The directed edges (u, v) and (v, u) are distinct edges and therefore aren't counted as multiple edges. However if we have two directed edges (u, v), (u, v) (same vertices, same direction), that IS a multigraph.

b) simple
not simple
not simple
not simple

c) $deg(b) = 3$
$deg(b) = 6$
$deg^-(b) = 2$, $deg^+(b) = 2$
$deg^-(b) = 1$, $deg^+(b) = 3$

d) 3, 2, 2, 1
6, 6, 5, 3
6, 6, 6, 4, 4
5, 4, 4, 3, 3, 3

Problem

2. Edges and Vertices

Suppose a graph has 21 edges, and 3 vertices of degree 4. All other vertices have degree 2. How many vertices are in the graph?

Solution

2. Edges and Vertices

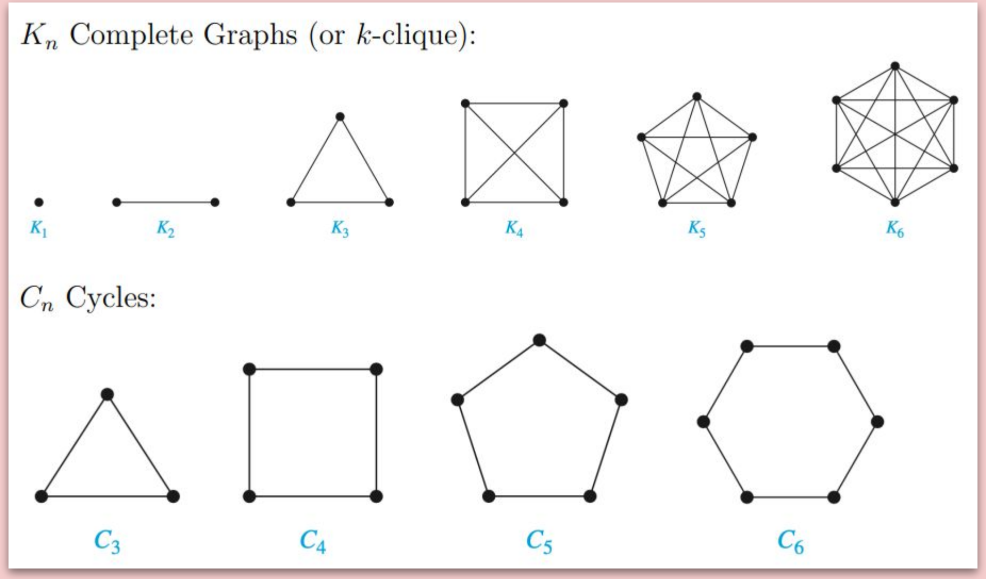
Suppose a graph has 21 edges, and 3 vertices of degree 4. All other vertices have degree 2. How many vertices are in the graph?

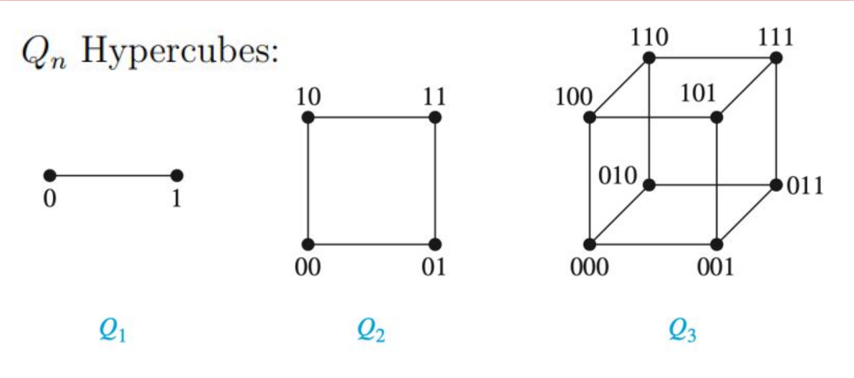
Solution: We know that the total degree of everything in the graph must be $21 \times 2 = 42$ (by Handshake Theorem), and that the vertices of degree 4 contribute a total of 12 to this count. Thus, there are 30 degrees left in the graph. This means that there must be $30/2 = 15$ vertices with degree 2. This gives a total number of 18 vertices in the graph.

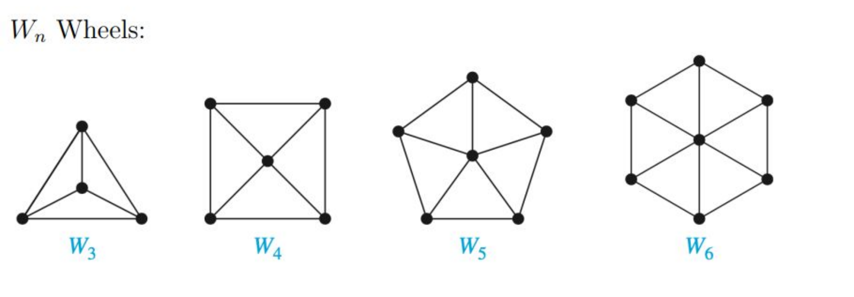
Special Graphs & Colorability

Special Graphs

You only need to know **complete graphs** and **cycles**. (The others will be defined later.)







Bipartite Graphs/Colorability

- **Bipartite Graph:** a simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V1 and V2 such that **every edge in the graph connects a vertex in V1 and a vertex in V2**. The pair **(V1, V2)** is called a **bipartition** of the vertex set V.

- **Bipartite Theorem (3 Equivalent Statements):** The following statements are equivalent…
  - G is **bipartite**.
  - G is **2-colorable**.
    (There is a function f : V ⇒ {red, blue} such that **u, v ∈ E ⇒ f(u) ≠ f(v)**)
  - G **does not contain odd cycle** (C2k+1) subgraphs.

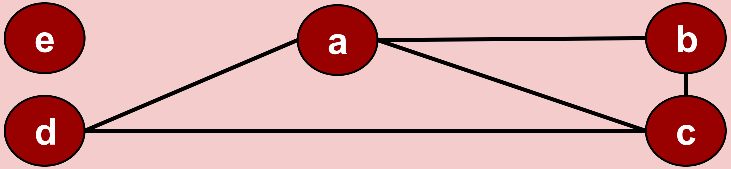
Graph Connectivity & Trees

Graph Connectivity

- **Path:** a path **$(u_0, u_1, \dots u_k)$** is a sequence of vertices in which consecutive vertices in the sequence are adjacent in the graph (connected by an edge).
    - Note parentheses **()** because a path DOES indicate an order
- **Simple Path:** a path that <u>does not repeat any vertices</u>

Graph Connectivity

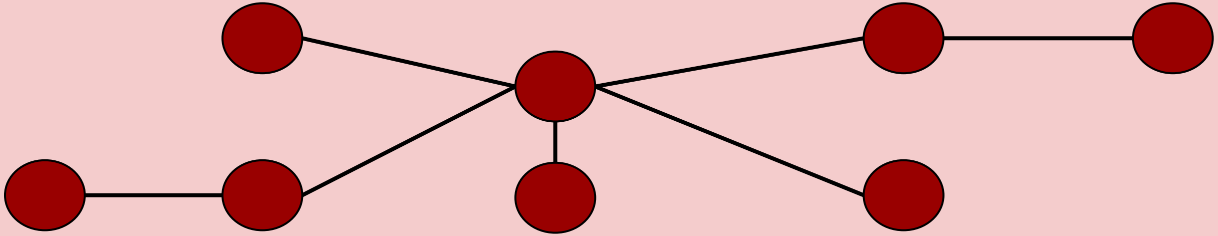
- **Connected Vertices:** Two vertices u and v are connected if there is a path from u to v: **(u, ..., v)**
  - Note that vertices *don't have to be adjacent* to be connected
    - **Ex from pic below:** **(d,b)** are connected but not adjacent
- **Connected Component:** A nonempty set of vertices in which every pair of vertices in the set is connected. **Example below: 2 connected components**



- **Connected Graph:** a graph G in which there is a path connecting any two vertices u, v ∈ G. In other words, there is only one connected component in the graph. Example above is NOT a connected graph.

Cycles and Trees

- **Subgraph:** $\mathbf{H = (V_H, E_H)}$ is a subgraph of $G = (V_G, E_G)$ iff $\mathbf{V_H \subseteq V_G}$ and $\mathbf{E_H \subseteq E_G}$
- **Cyclic Graph:** a graph containing <u>at least one cycle</u>
- **Acyclic Graph:** a graph having <u>no cyclic subgraphs</u>
- **Tree:** a <u>connected, acyclic</u> graph $\mathbf{T = (V, E)}$



- **Tree Theorems (2):**
  - If $T = (V, E)$ and $u,v \in V$, there is a <u>unique simple path</u> from u to v
  - Every tree on n vertices contains <u>n-1 edges</u>

Special Types of Graph Paths

- **Euler Path:** A Euler (pronounced "oiler") path is a path that uses **every edge** of a graph exactly once. An Euler path can start and end at the same vertex OR at different vertices.

- **Euler Circuit:** An Euler path that **starts and ends at the same vertex**. Sometimes, this is also referred to as an **Euler cycle**, but note that an Euler circuit is not necessarily an actual cycle, since it can visit the same vertex multiple times, as long as it doesn't repeat an edge.

- **Euler's Theorem:** A connected graph (or multigraph) has an Euler cycle **if and only if** every vertex has even degree.

Special Types of Graph Paths

- **Hamiltonian Path:** A Hamiltonian path (or Hamilton path) is a path between two vertices of a graph that visits **every vertex** in the graph exactly once.

- **Hamiltonian Cycle:** If a Hamiltonian path exists whose endpoints are adjacent, then the resulting graph cycle (**starting and ending at same vertex**) is called a Hamiltonian cycle (or Hamilton cycle).

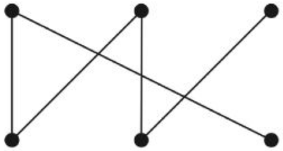
Problem

3. Trees

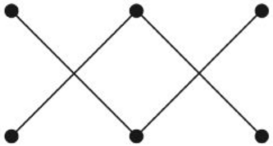
Which of the following graphs are trees? If it is not a tree, are you able to construct a spanning tree of the graph?

a)

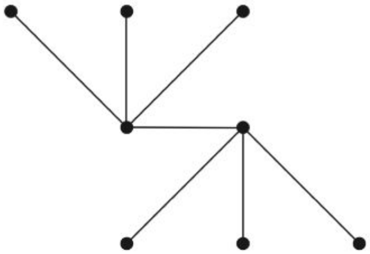
b)



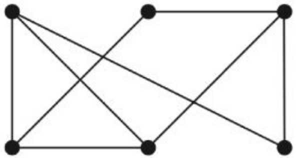
c)



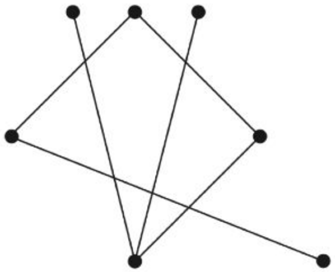
d)



e)



f)



Solution

3. Trees

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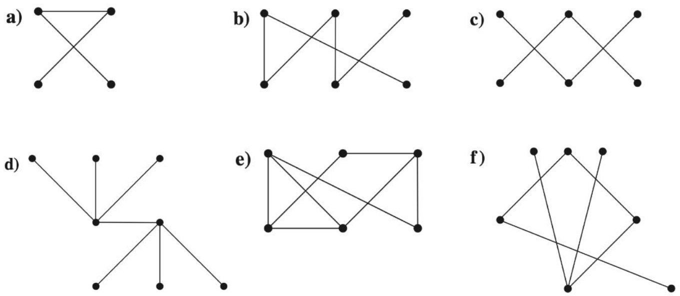


Solution:

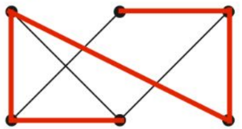
a) Tree

b) Tree

c) Not a tree (not a connected graph). You cannot create a spanning tree from this graph because it is not connected and thus, there is no way to remove edges and get a tree.

d) Tree

e) Not a tree (contains cyclic subgraphs). Yes, you can construct a spanning tree of this graph.
EX:



f) Tree

Graph Isomorphisms

Graph Isomorphisms

- **Graph Isomorphism:** Two simple graphs **G1 = (V1, E1)** and **G2 = (V2, E2)** are **isomorphic** if there exists a bijection **f : V1 → V2** such that:

$$\forall u, v\ [\ \{u, v\} \in E1 \leftrightarrow \{f(u), f(v)\} \in E2\ ].$$

This bijection is called an **isomorphism**.

- **Graph Invariant:** A graph invariant is a property preserved by isomorphism of graphs. If two graphs are isomorphic, then a graph invariant holds, but not necessarily vice-versa.
    - **What are some examples of graph invariants you can think of?**

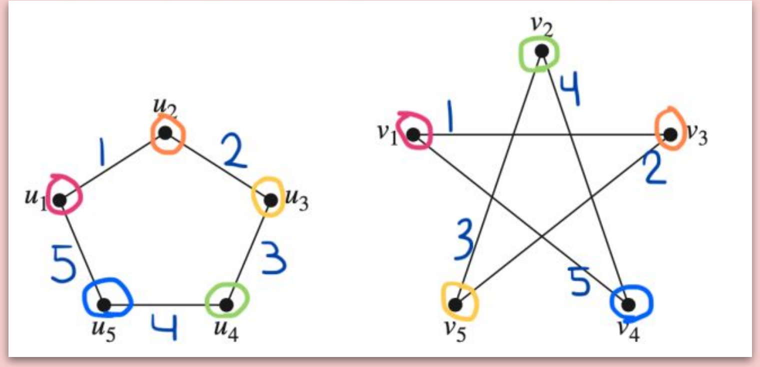
Graph Isomorphisms

- **Graph Isomorphism:** Two simple graphs **G1 = (V1, E1)** and **G2 = (V2, E2)** are **isomorphic** if there exists a bijection **f : V1 → V2** such that:

$$\forall u, v \; [\; \{u, v\} \in E1 \leftrightarrow \{f(u), f(v)\} \in E2 \;].$$

This bijection is called an **isomorphism**.

- **Graph Invariant:** A graph invariant is a property preserved by isomorphism of graphs. If two graphs are isomorphic, then a graph invariant holds, but not necessarily vice-versa.
  - Number of vertices
  - Number of edges
  - Degree sequence
  - Existence of subgraphs/path properties
  - Cyclic or acyclic
  - having paths of certain length



Proving/Disproving Graph Isomorphisms

- **To prove that two graphs are isomorphic:**
  - The **only** way to prove that two graphs are isomorphic is to **provide an example of an isomorphism**.
  - **An isomorphism** is a function from one set of vertices to the other such that $\forall u,v\ [\{u,v\} \in E1 \leftrightarrow \{f(u),f(v)\} \in E2]$, as defined on the previous slide.
    - It is not sufficient to simply list some consistent invariants.
    - The following statement is true: **IF two graphs are isomorphic, THEN the invariants are preserved**, but NOT the other way around **(the converse – not necessarily true)**.
    - Because of this, however, it is easier to disprove isomorphism.

Proving/Disproving Graph Isomorphisms

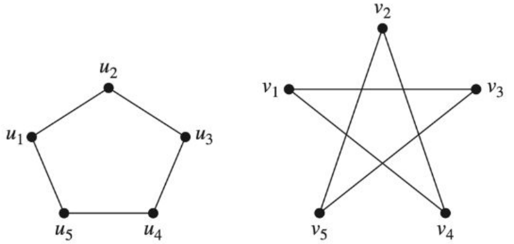
- **To prove that two graphs are NOT isomorphic:**
  - If you are trying to disprove that two graphs are isomorphic, you are trying to prove that there does not exist an isomorphism between them.
  - Thus, if a graph invariant is not the same in two graphs, they are NOT isomorphic.
  - As such, it is sufficient to simply **list or describe an invariant** that is different between the two graphs.

Problem

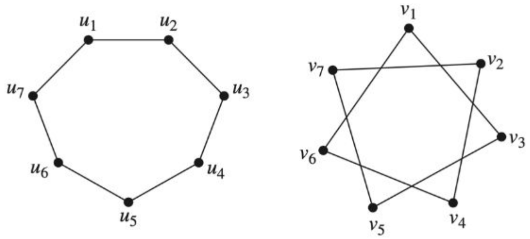
4. Isomorphic Graphs

Determine whether each given pair of graphs is isomorphic. Exhibit an isomorphism or provide an argument that none exists.

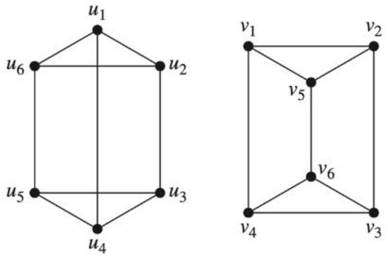
39.



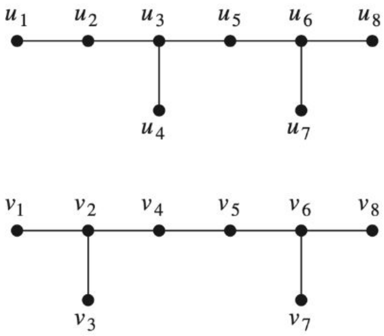
41.



43.



45.



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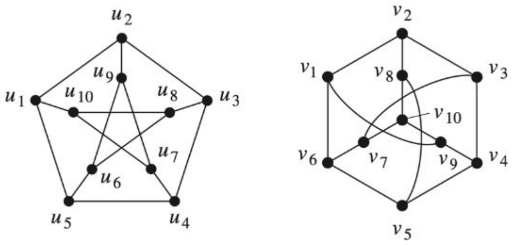

Solution

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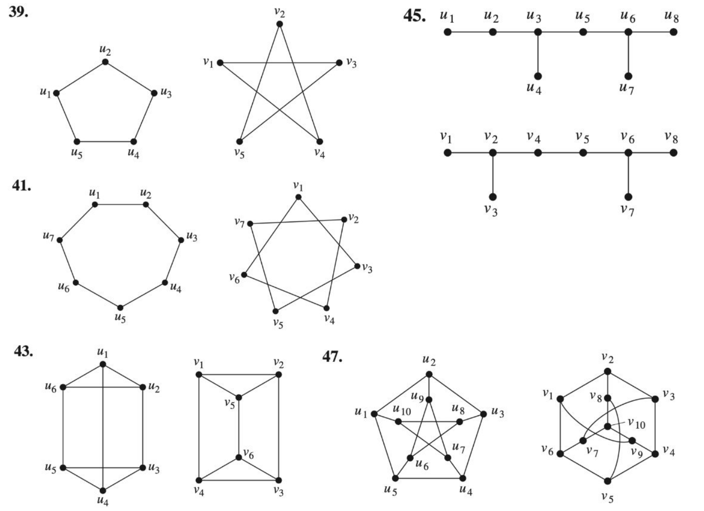
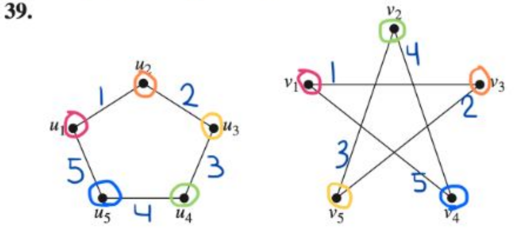
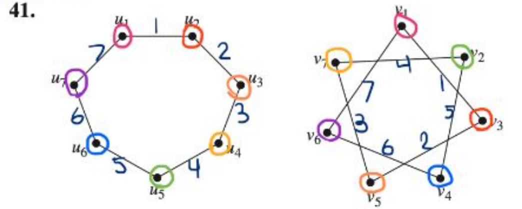


Solution:
39. Isomorphic
Let f be as follows:
$f(u_1) = v_1$
$f(u_2) = v_3$
$f(u_3) = v_5$
$f(u_4) = v_2$
$f(u_5) = v_4$
Then all of the edges corresond between the two graphs, ie $\{u_i, u_j\}$ is an edge in the first graph iff $\{f(u_i), f(u_j)\}$ is an edge in the second graph.



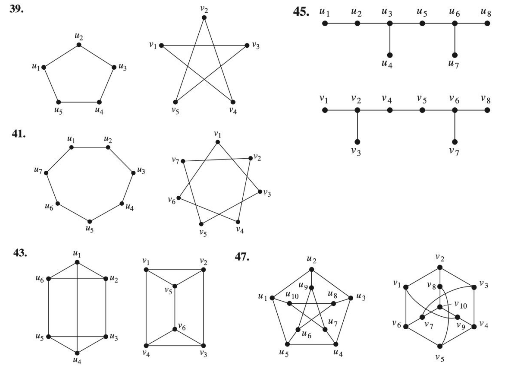
41. Isomorphic
Let f be as follows:
$f(u_1) = v_1$
$f(u_2) = v_3$
$f(u_3) = v_5$
$f(u_4) = v_7$
$f(u_5) = v_2$
$f(u_6) = v_4$
$f(u_7) = v_6$



Solution

4. Isomorphic Graphs

Determine whether each given pair of graphs is isomorphic. Exhibit an isomorphism or provide an argument that none exists.

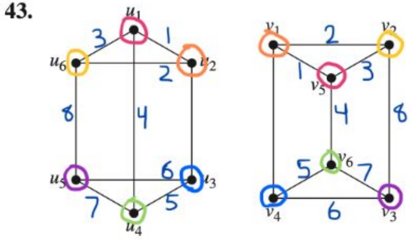


43. Isomorphic

Let f be as follows:

$f(u_1) = v_5$
$f(u_2) = v_1$
$f(u_3) = v_4$
$f(u_4) = v_6$
$f(u_5) = v_3$
$f(u_6) = v_2$

See how with this assignment all of the edges correspond:



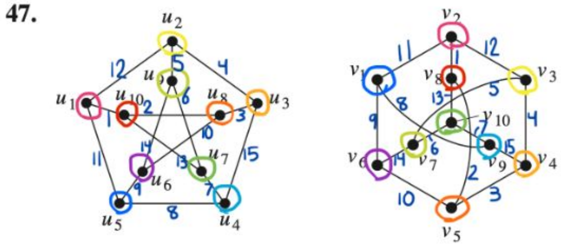
45. Not Isomorphic

It's not possible to create a bijection between the vertices in graph one and graph two and have the edges correspond correctly.

47. Isomorphic

Let f be as follows:

$f(u_1) = v_2$
$f(u_2) = v_3$
$f(u_3) = v_4$
$f(u_4) = v_9$
$f(u_5) = v_1$
$f(u_6) = v_6$
$f(u_7) = v_{10}$
$f(u_8) = v_5$
$f(u_9) = v_7$
$f(u_{10}) = v_8$



Exam Review

Problem

5. Inductive Conclusions

Suppose that $P(n)$ is an unknown predicate. Determine for which positive integers n the statement $P(n)$ must be true, and justify your answer, if

a) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $\mathrm{P(n + 2)}$ is true.

b) $P(1)$ and $P(2)$ are true, and for all positive integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

c) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(2n)$ is true.

d) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.

Solution

5. Inductive Conclusions

Suppose that $P(n)$ is an unknown predicate. Determine for which positive integers n the statement $P(n)$ must be true, and justify your answer, if

a) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then P(n + 2) is true.

b) $P(1)$ and $P(2)$ are true, and for all positive integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

c) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(2n)$ is true.

d) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.

Solution:

a) The inductive step here allows us to conclude that $P(3)$, $P(5)$, ... are all true, but we can conclude nothing about $P(2)$, $P(4)$, ...

b) $P(n)$ is true for all positive integers n, using **strong induction**.

c) The inductive step here enables us to conclude that $P(2)$, $P(4)$, $P(8)$, $P(16)$,... are all true, but we can conclude nothing about $P(n)$ when n is not a power of 2.

d) This is **mathematical induction**; we can conclude that $P(n)$ is true for all positive integers n.

Problem

6. Strong Induction

Prove that every positive integer is either a power of 2, or can be written as the sum of distinct powers of 2.

Solution

6. Strong Induction

Prove that every positive integer is either a power of 2, or can be written as the sum of distinct powers of 2.

Solution: Let $P(n)$ be true if and only if n can be written as the sum of distinct powers of two.

Inductive Step: Assume $P(k)$ is true for all $k < n$. We can split this into cases:

1. If n is a power of two, so $n = 2^j$, for some $j \in \mathbb{Z}^+$, then $P(n)$ is true trivially.
2. Otherwise, We can write n as the sum of the largest power of two smaller than n, k_1, and some other positive integer, $k_2 = n - k_1$.

   Lemma: We know that $k_2 < k_1$. Assume $k_2 \geq k_1$. Then there $n \geq 2k_1$. Because k_1 is a power of two, $2k_1$ is also a power of two, and is smaller than n, so this contradicts the fact that k_1 is the largest power of two smaller than n.

   Because $P(k_2)$, k_2 can be written as the sum of distinct powers of two (all of which must be less than, and therefore not equal to, k_1). So n must be able to be written as the sum of distinct powers of two.

We have proven using strong induction that if $P(k)$ is true for all $k < n$, then $P(n)$ is true.

Base Case: $P(1)$ is true because 1 can be written as 2^0.

Problem

7. Mod

Let $a \equiv 38 \pmod{15}$, $b \equiv 2 \pmod{15}$, and $c \equiv 3 \pmod{5}$. Compute the following if possible:

1. $d \equiv a^{24} \pmod{15}$
2. $e \equiv a^3 b^7 + b^{13} \pmod{15}$
3. $g \equiv a + c \pmod{15}$
4. $h \equiv a + c \pmod{5}$

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Solution:

1. $d \equiv 38^{24} \equiv 8^{24} \equiv (8^2)^{12} \equiv 64^{12} \equiv 4^{12} \equiv 16^6 \equiv 1^6 \equiv 1 \pmod{15}$
2. $e \equiv 8^3 2^7 + 2^{13} \pmod{15} \equiv (64 \cdot 8)(16 \cdot 2^3) + 2^{13} \equiv (4 \cdot 8)(8) + 2^{13} \equiv 16 + (2^7 \cdot 2^6) \equiv 1 + (8 \cdot 4) \equiv 1 + 2 \equiv 3 \pmod{15}$
3. $a = 15k_1 + 38$ and $c = 5k_2 + 3$, so $a + c = 15k_1 + 5k_2 + 41$. We cannot factor 15 out of the term with variables k_2, so this is impossible to compute.
4. $a = 15k_1 + 38$ and $c = 5k_2 + 3$, so $a + c = 15k_1 + 5k_2 + 41 = 5(3k_1 + k_2 + 8) + 1$. $h \equiv 1 \pmod{5}$.

Problem

8. Composition and Onto

If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

Solution

8. Composition and Onto

If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

Solution: No. For example, suppose that $A = \{a\}, B = \{b, c\}$, and $C = \{d\}$. Let $g(a) = b, f(b) = d$, and $f(c) = d$. Then f and $f \circ g$ are onto, but g is not.