EECS 203 Discussion 5a

Mathematical Induction & Exam 1 Review

Upcoming Exam

- **Exam 1** is on **Monday, February 19th** from **7:00 - 9:00 PM**!
- **Exam Review Sessions**
  - **Sat, February 17th, 1-4 PM** in CHRYS 220
    - **Topics:** Propositional Logic + Predicates and Quantifiers
  - **Sun, February 18th, 1-4 PM** in CHRYS 220
    - **Topics:** Proof Methods + Sets
- If you have a time conflict, contact the course staff **ASAP**!
- Practice exam questions have been released on Canvas!
  - They can be found on via **Files -> Practice Exams -> Exam 1**
  - See pinned Piazza post @448 for practice exam walkthrough videos

Upcoming Homework

- Homework/Groupwork 5 will be due **Mar. 7th – AFTER SPRING BREAK**
    - **Don't forget to match pages!**
    - Please note as soon as you press submit you've successfully submitted by the deadline. **You can still match pages** with no rush without adding to your submission time.
- Groupwork
    - Groupwork can be done alone, but the problems tend to be more difficult, and the goal is for you to puzzle them out with others!
    - Your discussion section is a great place to find a group!
    - There is also a pinned Piazza thread for searching for homework groups.

Mathematical Induction

Mathematical Induction

We want to show some statement **P(n)** is true for all integers **$n \geq c$**.

- **Base Case**
  - First, show that the statement **P(c)** is true for some initial value **c**.
- **Inductive Step**
  - Next, show that if **P(k)** is true for an arbitrary integer **$k \geq c$**, then **P(k+1)** is also true.
  - In other words, we want to prove the implication **$P(k) \rightarrow P(k+1)$**.
  - Since k is arbitrary, we start this step by assuming that **P(k)** is true.
  - When you assume **P(k)**, it's called the **inductive hypothesis**.
- **That's it!**
  - You've proven that **$\forall (n \geq c)\ P(n)$**, as desired.
  - Since **P(c)** is true and **P(k)** implies **P(k+1)**, we therefore have:
    $P(c) \rightarrow P(c+1) \rightarrow P(c+2) \rightarrow P(c+3) \rightarrow P(c+4) \ldots$

Problem 1

1. Bandar's Blunder ⋆

Bandar writes a proof for the following statement:

$$n! > n^2 \text{ for all } n \geq 4.$$

His proof is incorrect, and it's your task to help him identify his mistake!

Proof:

Inductive step:

Let k be arbitrary. Assume $P(k) : k! > k^2$. We need to show $P(k+1) : (k+1)! > (k+1)^2$

$$\begin{aligned}
(k+1)! &= (k+1) \cdot k! \\
&> (k+1) \cdot k^2 && \text{(By the Inductive Hypothesis)} \\
&= (k+1)(k \cdot k) \\
&\geq (k+1)(2 \cdot k) && \text{(Because } k \geq 2\text{)} \\
&= (k+1)(k+k) \\
&\geq (k+1)(k+1) && \text{(Because } k \geq 1\text{)} \\
&= (k+1)^2
\end{aligned}$$

This proves $(k+1)! > (k+1)^2$.

Base Case:

Prove $P(0) : 0! > 0^2$, $0! = 1 > 0^2 = 0$

Thus by mathematical induction, $n! > n^2$ for all $n \geq 0$.

What is wrong with Bandar's proof?

Solution

Solution: The key idea here is that although we have a valid base case, and a valid inductive step, they don't work together. In particular, the inductive step requires $k \geq 4$, but our base case only shows that $k = 0$ is valid (and in fact, $k = 1, k = 2,$ and $k = 3$ are false). A valid proof could have used the same inductive step with a base case of $n = 4$.

Some possible explanations:

- The base case and inductive step are individually valid, but the base case can't be used with the inductive step.
- The base case doesn't help prove the statement is true for $n = 4$, and this case can't be proved with the inductive step.
- The inductive step doesn't work with the given base case.

Problem 2

2. Sum Mathematical Induction

Using induction, prove that for all integers $n \geq 1$:

$$\sum_{r=1}^{n} (r+1) \cdot 2^{r-1} = n \cdot 2^{n}$$

Solution

Solution:

Inductive Step:

Let k be an arbitrary integer that is greater or equal to 1.

Assume $P(k) : \sum_{r=1}^{k}(r+1)\cdot 2^{r-1} = k \cdot 2^k$.

We want to show $P(k+1) : \sum_{r=1}^{k+1}(r+1)\cdot 2^{r-1} = (k+1)\cdot 2^{k+1}$

$$\begin{aligned}
&\sum_{r=1}^{k+1}(r+1)\cdot 2^{r-1} \\
&= [\sum_{r=1}^{k}(r+1)\cdot 2^{r-1}] + (k+1+1)\cdot 2^{k+1-1} \\
&= [\sum_{r=1}^{k}(r+1)\cdot 2^{r-1}] + (k+2)\cdot 2^{k} \\
&= [k\cdot 2^k] + (k+2)\cdot 2^k \text{ (by Inductive Hypothesis)} \\
&= k\cdot 2^k + k2^k + 2^{k+1} \\
&= 2k\cdot 2^k + 2^{k+1} \\
&= k\cdot 2^{k+1} + (1)\cdot 2^{k+1} \\
&= (k+1)\cdot 2^{k+1}
\end{aligned}$$

Therefore, $P(k+1)$ is true.

(base case on next slide)

Solution

Base Case:

Prove $P(1) : \sum_{r=1}^{1}(r+1) \cdot 2^{r-1} = 1 \cdot 2^1$. $LHS = (1+1) \cdot (2)^0 = 2, RHS = (1) \cdot (2)^1 = 2,$ so $LHS = RHS$. Therefore, $P(1)$ is true.

Therefore we have shown by mathematical induction that for all integers $n \geq 1$,
$\sum_{r=1}^{n}(r+1) \cdot 2^{r-1} = n \cdot 2^n$

Exam 1 Review

Tautology, Contradiction, Satisfiability (Discussion 1b)

- **Tautology:** A compound proposition that is **always true** regardless of its input values

- **Contradiction:** A compound proposition that is **always false** regardless of its input values

- **Satisfiable:** A compound proposition is satisfiable if it **can be true** (there is at least one set of inputs that makes the proposition true)

Problem 3

3. REVIEW: Satisfiability ⋆

Determine whether each of these compound propositions is satisfiable.

(a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$

(b) $(p \to q) \wedge (p \to \neg q) \wedge (\neg p \to q) \wedge (\neg p \to \neg q)$

Solution

Solution:

(a) Satisfiable. The expression is satisfied when p is False and q is False. You could draw up a truth table to help you think through the possible combinations of truth values for p and q.

(b) Unsatisfiable (ie a contradiction)

p	q	$p \to q$	$p \to \neg q$	$\neg p \to q$	$\neg p \to \neg q$	$(p \to q) \wedge (p \to \neg q) \wedge (\neg p \to q) \wedge (\neg p \to \neg q)$
T	T	T	F	T	T	F
T	F	F	T	T	T	F
F	T	T	T	T	F	F
F	F	T	T	F	T	F

Since all boolean assignments of p and q result in the expression being False, this is compound proposition is unsatisfiable.

Solution

Alternate Solutions:

- Using Equivalence Laws:

$$
\begin{aligned}
&(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q) \\
&\qquad \equiv (\neg p \lor q) \land (\neg p \lor \neg q) \land (p \lor q) \land (p \lor \neg q) \\
&\qquad \equiv (\neg p \lor (q \land \neg q)) \land (p \lor q) \land (p \lor \neg q) \\
&\qquad \equiv \neg p \land (p \lor q) \land (p \lor \neg q) \\
&\qquad \equiv \neg p \land (p \lor (q \land \neg q)) \\
&\qquad = \neg p \land p \\
&\qquad = F
\end{aligned}
$$

- Verbal Argument: In order to show that this statement is not satisfiable, we will consider every possible assignment of p and q and show that in every case, the statement is false. When p is true and q is true, $p \to \neg q$ is false so the whole statement is false. When p is true and q is false, $p \to q$ is false, so the whole statement is false. When p is false and q is true, $\neg p \to \neg q$ is false, so the whole statement is false. When p is false and q is false, $\neg p \to q$ is false, so the whole statement is false. Therefore, in every possible assignment of p and q, the statement is false, which means that the statement is not satisfiable.

Quantifiers (Discussion 2)

- **Nested Quantifiers:** A nested quantifier is a quantifier that involves the use of two or more quantifiers to quantify a compound proposition P(x,y). In nested quantifiers, order matters…
    - **P(x,y):** some statement about x and y
    - **Example:** $\forall x \exists y\, P(x,y)$ is different from $\exists y \forall x\, P(x,y)$
        - **$\forall x \exists y\, P(x,y)$:** "For all x, there exists y such that…"
        - **$\exists y \forall x\, P(x,y)$:** "There exists y such that for all x…"

Problem 4

4. REVIEW: Nested Quantifier Translations

Let P(x, y) be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

a) $\exists x \exists y P(x, y)$

b) $\exists x \forall y P(x, y)$

c) $\forall x \exists y P(x, y)$

d) $\exists y \forall x P(x, y)$

e) $\forall y \exists x P(x, y)$

f) $\forall x \forall y P(x, y)$

Solution

Solution:

a) There is a student in your class who has taken a computer science course [at your school].

b) There is a student in your class who has taken every computer science course.

c) Every student in your class has taken at least one computer science course.

d) There is a computer science course that every student in your class has taken.

e) Every computer science course has been taken by at least one student in your class.

f) Every student in your class has taken every computer science course.

Proof Methods (Discussion 2)

- **Direct Proof:**
  Proves $p \rightarrow q$ by showing **$p \rightarrow$ stuff $\rightarrow q$**

Even and Odd (Discussion 2)

- **Even:** An integer **x** is even iff there exists an integer k such that **x = 2k**
- **Odd:** An integer **x** is odd iff there exists an integer k such that **x = 2k + 1**

Problem 5

5. REVIEW: Direct Proof

Use a direct proof to show that the product of two odd numbers is odd.

Solution

Solution: Using a Direct Proof,

Let a and b be arbitrary odd integers. Then, a and b can be written as $a = 2m + 1$ and $b = 2n + 1$ for some integers n and m. Looking at their product, we have

$$\begin{aligned} ab &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

Since $ab = 2k + 1$, where k is the integer $2mn + m + n$, then by definition ab is odd.

Proof Methods (Discussion 3)

- **Direct Proof:**
  Proves $p \rightarrow q$ by showing **$p \rightarrow$ stuff $\rightarrow q$**
- **Proof by Contradiction:**
  Proves p by showing **$\neg p \rightarrow F$**
  To prove $p \rightarrow q$, assume the negation: **$\neg(p \rightarrow q) \equiv \neg(\neg p \lor q) \equiv p \land \neg q$**
  "Seeking contradiction, assume that…"

Problem 6

6. REVIEW: Proof by Contradiction $\star$

Prove that for all integers n, if $n^2 + 2$ is even, then n is even using a proof by contradiction.

Note: When using proof by contradiction to prove p → q, there are multiple places where one could introduce the assumption that is "seeking contradiction":

1. "Seeking contradiction, assume the negation of the entire claim, including negating the quantifier…"
2. "Let x be an arbitrary element of the domain. Seeing contradiction, assume p and not(q). [ie negate the if-then] …"
3. "Let x be an arbitrary element of the domain. Assume p [ie begin direct proof of if p then q]. Seeking contradiction, assume not(q). …"

Solution

Solution: Let n be an arbitrary integer. For the sake of contradiction, assume $n^2 + 2$ is even and n is odd.
(Note that we could have also assumed the negation of the entire statement: "Assume that there exists some n such that $n^2 + 2$ is even and n is odd".)

- Since n is odd, we can say $n = 2k + 1$ for some integer k.
- This means $n^2 + 2 = (2k + 1)^2 + 2$.
$= 4k^2 + 4k + 1 + 2$
$= 2(2k^2 + 2k + 1) + 1$
$= 2j + 1$, where j is an integer equal to $2k^2 + 2k + 1$
- Thus from the definition of an odd number, $n^2 + 2$ is odd. This contradicts our earlier assumption that $n^2 + 2$ is even.

Therefore, using proof by contradiction, we have showed that for all integers n, if n is odd, then $n^2 + 2$ is odd.

Proof Methods (Discussion 3)

- **Direct Proof:**
  Proves $p \rightarrow q$ by showing $\mathbf{p \rightarrow stuff \rightarrow q}$
- **Proof by Contradiction:**
  Proves p by showing $\mathbf{\neg p \rightarrow F}$
  To prove $p \rightarrow q$, assume the negation: $\mathbf{\neg(p \rightarrow q) \equiv \neg(\neg p \lor q) \equiv p \land \neg q}$
  "Seeking contradiction, assume that…"
- **Proof by Contrapositive:**
  Proves $p \rightarrow q$ by showing $\mathbf{\neg q \rightarrow stuff \rightarrow \neg p}$

Problem 7

7. REVIEW: Proof by Contrapositive ⋆

Prove that for all integers x and y, if xy^2 is even, then x is even or y is even.

Solution

Solution:

We will prove the statement via proof by contrapositive. Let x and y be arbitary integers. Because we are using proof by contrapositive, we want to assume x is odd and y is odd and eventually conclude that xy^2 is odd. First, we will assume x is odd and y is odd. Since x and y are odd, $x = 2k+1$ and $y = 2n+1$ where k and n are integers. Therefore, $xy^2 = (2k+1)(2n+1)^2 = (2k+1)(4n^2+4n+1) = 8kn^2+8kn+2k+4n^2+4n+1 = 2(4kn^2+4kn+k+2n^2+2n)+1 = 2j+1$ where j is an integer and $j = 4kn^2+4kn+k+2n^2+2n$. Therefore, xy^2 is odd. Thus, we have shown via proof by contrapositive that for all integers x and y, if xy^2 is even, then x is even or y is even.

Proof Methods (Discussion 4)

- **Direct Proof:**
  Proves $p \rightarrow q$ by showing $\mathbf{p \rightarrow stuff \rightarrow q}$
- **Proof by Contradiction:**
  Proves p by showing $\mathbf{\neg p \rightarrow F}$
  To prove $p \rightarrow q$, assume the negation: $\mathbf{\neg(p \rightarrow q) \equiv \neg(\neg p \lor q) \equiv p \land \neg q}$
  “Seeking contradiction, assume that…”
- **Proof by Contrapositive:**
  Proves $p \rightarrow q$ by showing $\mathbf{\neg q \rightarrow stuff \rightarrow \neg p}$
- **Proof by Cases:**
  Break p into cases and show that each case implies q (in which case $p \rightarrow q$).
  Make sure to prove q for every possible case!
  $\mathbf{p \rightarrow p_1 \lor p_2 \lor \ldots \lor p_n \rightarrow q}$

Problem 8

8. REVIEW: Proof by Cases/Disproofs ⋆

a) Prove or Disprove that for all integers n, $n^2 + n$ is even

b) Prove or Disprove that for all integers a and b, $\frac{a}{b}$ is a rational number.

Solution

a) We prove the statement via proof by cases. Let x be an arbitrary integer.

- **Case 1:** x is even
  Since x is even, $x = 2k$ where k is an integer. Therefore, $x^2 + x = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k) = 2j$ where j is some integer. Therefore, $x^2 + x$ is even.
- **Case 2:** x is odd
  Since x is odd, $x = 2k+1$ where k is an integer. Therefore, $x^2 + x = (2k+1)^2 + (2k+1) = (4k^2+4k+1)+(2k+1) = 4k^2+6k+2 = 2(2k^2+3k+1) = 2j$ where j is some integer. Therefore, $x^2 + x$ is even.

For all cases of x, we have shown that $x^2 + x$ is even. Therefore, we have shown that for all integers n, $n^2 + n$ is even.

b) We will disprove this statement. Consider the case, $a = 1$ and $b = 0$. In this case, $\frac{a}{b}$ is not a rational number because $b = 0$.

Problem 9

9. REVIEW: Sets $\star$

Let our domain U be the set of the 26 lowercase letters in the English alphabet. Let $A = \{i, a, n\}, B = \{s, h, u, b\}, C = \{i, s, a, b, e, l\}$. Compute the following, where complements are taken within U. Write your answers in list notation.

Hint: For parts (b) and (c), simplifying the expressions using set identities may make the sets quicker to compute.

Solution

Solution:

(a) The set union of A and B is $\{i, a, n, s, h, u, b\}$. Recall that the set minus removes the elements in C that are also in $A \cup B$, so it removes $\{i, s, a, b\}$, which leaves us with $(A \cup B) - C = \{n, h, u\}$.

(b)

$$
\begin{aligned}
&\overline{\overline{B \cup C} \cup A} \\
&= \overline{\overline{B \cup C}} \cap \overline{A} && \text{DeMorgan's Law} \\
&= (B \cup C) \cap \overline{A} && \text{Complementation Law} \\
&= (B \cup C) - A && \text{Definition of Set Minus}
\end{aligned}
$$

The set union of B and C is $\{s, h, u, b, i, a, e, l\}$. Recall that the set minus removes the elements of A that are also in $B \cup C$, so it removes $\{i, a\}$, which leaves us with $\overline{\overline{B \cup C} \cup A} = \{s, h, u, b, e, l\}$.

Solution

(c) $(A \times B) \cap (A \times C) = A \times (B \cap C)$ by the Distributive Property for Cartesian Product, which we proved in Groupwork 3 Problem 3. The set intersection of B and C is $\{s, b\}$, so $A \times (B \cap C) = \{(i, s), (i, b), (a, s), (a, b), (n, s), (n, b)\}$.

Alternate Solution: An alternate solution would be to calculate $(A \times B)$ and $(A \times C)$ and manually calculate their intersection.

$(A \times B) = \{(i, s), (i, h), (i, u), (i, b), (a, s), (a, h), (a, u), (a, b), (n, s), (n, h), (n, u), (n, b)\}$

$(A \times C) = \{(i, i), (i, s), (i, a), (i, b), (i, e), (i, l), (a, i), (a, s), (a, a), (a, b), (a, e), (a, l), (n, n$
$(n, s), (n, a), (n, b), (n, e), (n, l)\}$.

$(A \times B) \cap (A \times C) = \{(i, s), (i, b), (a, s), (a, b), (n, s), (n, b)\}$.