W24

EECS 203 Discussion 7

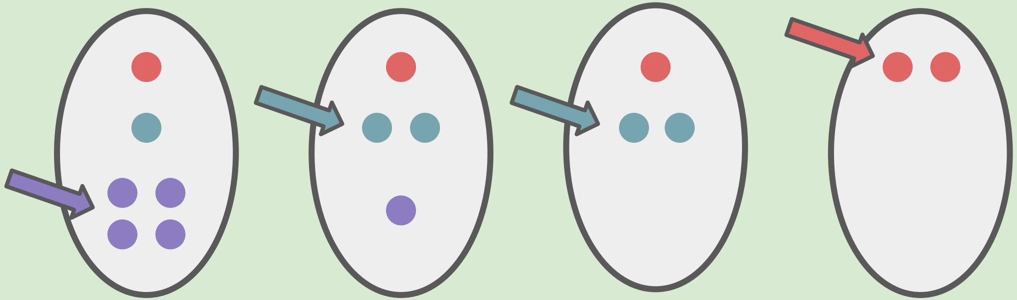
Countability and Pigeonhole Principle

Admin Notes:

- Homework/Groupwork 7 will be due **Mar. 21th**
  - **Don’t forget to match pages!**
  - Please note as soon as you press submit you’ve successfully submitted by the deadline. **You can still match pages** with no rush without adding to your submission time.

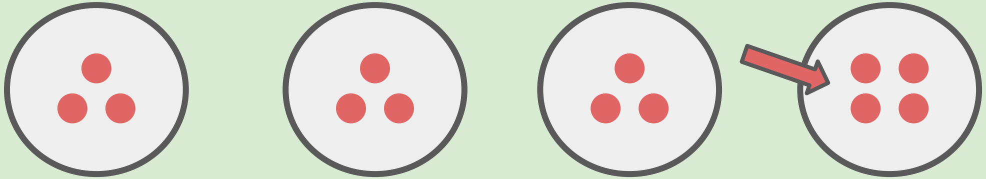
Pigeonhole Principle

- **Pigeonhole Principle:** If we put **k+1 objects** into **k boxes**, then at least one box contains **2 or more** objects.



*__Examples__ of putting 5 objects into 4 bins

- **Generalized Pigeonhole Principle:** If we put **N objects** into **k boxes**, then at least one box contains **ceil(N/k) or more** objects.



*__Example__ of putting 13 objects into 4 bins
ceil(13/4) = ceil(3.25) = 4

Problem 1

1. Pigeonhole Principle $\star$

How many distinct numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

Problem 2

2. More Pigeonhole Principle

(a) Undergraduate students at a college belong to one of four groups depending on the year in which they are expected to graduate. Each student must choose one of 21 different majors. How many students are needed to assure that there are two students expected to graduate in the same year who have the same major?

(b) What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Problem 3

3. Even More Pigeonhole Principle ⋆

Sophia has a bowl of 15 red, 15 blue, and 15 orange pieces of candy. Without looking, Sophia grabs a handful of pieces.

(a) What is the smallest number of pieces of candy Sophia has to grab to make sure she has at least 4 of the same color?

(b) What is the smallest number of pieces of candy Sophia has to grab to make sure she has 3 orange candies?

Problem 4

4. Pigeonhole Principle Is All You Need

A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Review: Functions

- **Function *f: A* → *B*:** associates each element of set A to <u>exactly one</u> element in set B
  - **Domain: *A***
  - **Codomain: *B***
- **Onto Function *f: A* → *B*:** all elements in B are mapped to by f

$$\forall b \in B\ \exists a \in A\ [\ f(a) = b\ ]$$

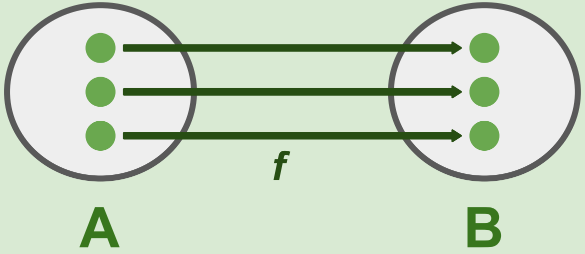
- **One-to-One Function *f: A* → *B*:** no two elements of A map to the same output in B

$$\forall a,b \in A\ [\ f(a) = f(b) \rightarrow a = b\ ]$$

- **Bijective Function:** onto and one-to-one (also called a one-to-one correspondence)

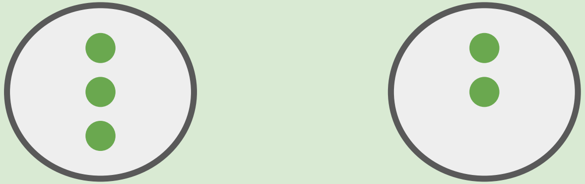
What do function properties tell us about the set cardinalities?

- **Onto Function *f* : A → B:** $\forall b \in B \; \exists a \in A \; [\, f(a) = b \,]$

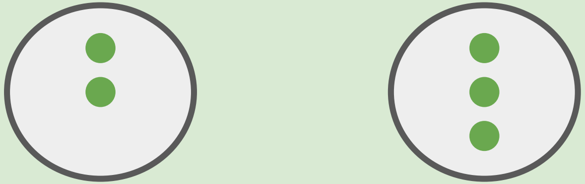


*$|A| = |B|$

  - Is it possible that $|A| > |B|$?

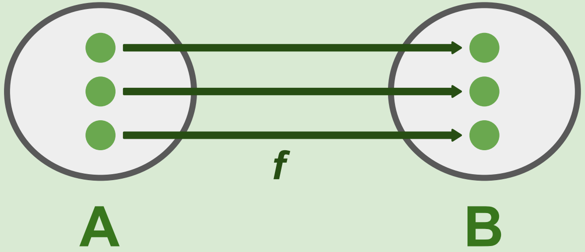


  - Is it possible that $|B| > |A|$?



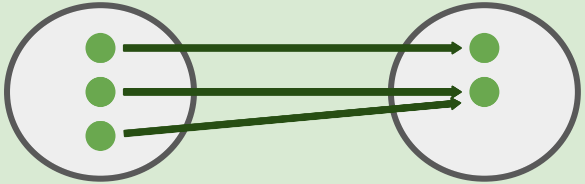
What do these properties tell us about the set cardinalities?

- **Onto Function f : A → B:** $\forall b \in B\ \exists a \in A\ [\ f(a) = b\ ]$



*$|A| = |B|$

  - Is it possible that $|A| > |B|$? **Yes!**



  - Is it possible that $|B| > |A|$? **No**

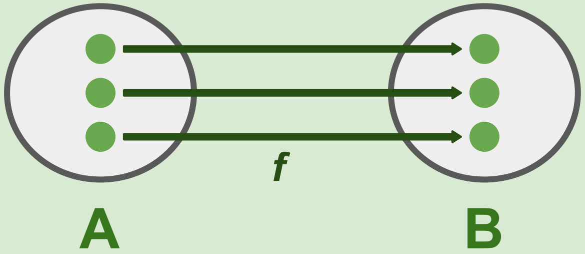
(can’t be a function and onto in this case)

*Thus, if we have an onto function from A to B,

$$|A| \geq |B|$$

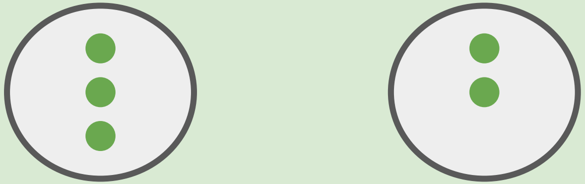
What do these properties tell us about the set cardinalities?

- **One-to-One Function f : A → B:** $\forall a,b \in A\ [\ f(a) = f(b) \rightarrow a = b\ ]$

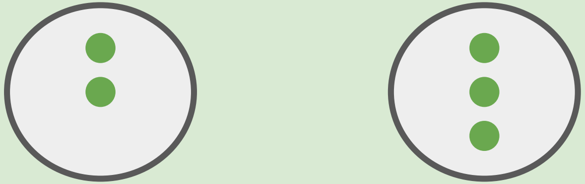


*$|A| = |B|$

  - Is it possible that $|A| > |B|$?

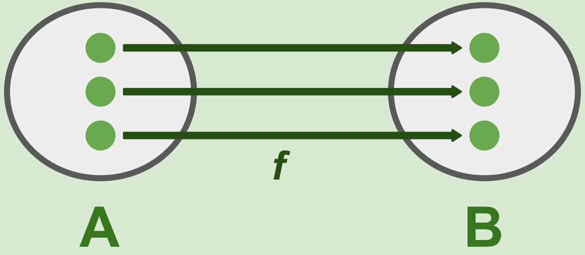


  - Is it possible that $|B| > |A|$?



What do these properties tell us about the set cardinalities?

- **One-to-One Function *f* : A → B:** ***∀ a,b ∈ A [f(a) = f(b) → a = b]***



*|*A*| = |*B*|

- Is it possible that |*A*| > |*B*|? **No**

(can’t be a function and 1-1 in this case)

- Is it possible that |*B*| > |*A*|? **Yes!**

(still a function, still 1-1)

*Thus, if we have an 1-1 function from A to B,

|*A*| ≤ |*B*|

Countably vs Uncountably Infinite

- **Countably Infinite:** A set is said to be countably infinite if it has the same cardinality as the natural numbers. One way to prove this is by finding a bijection between the set and the natural numbers. Examples:
  - **The natural numbers**
  - **The integers**
  - **The rational numbers**
- **Uncountably Infinite:** A set is said to be uncountably infinite if its cardinality is larger than the cardinality of the natural numbers. Examples:
  - **The real numbers**
  - **The irrational numbers**
  - **(0,1)**

Problem 5

5. Different Infinities $\star$

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

(a) The set of all integers greater than 10.

(b) The set of all integers with absolute value less than 1,000,000.

(c) The set of all real numbers between 0 and 2.

(d) The set $A \times \mathbb{Z}$, where $A = \{2, 3\}$.

Problem 6

6. Different Infinities with Sets

Give an example of two uncountable sets A and B such that $A \cap B$ is:

a) Finite.

b) Countably infinite.

c) Uncountably infinite.

Problem 7

7. Cardinality Proof ⋆

Show that $|(0,1)| \geq |\mathbb{Z}^+|$.

Schroder-Bernstein & Applications

- **Schroder-Bernstein Theorem:**
  If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

  - So using this theorem, injectivity, and surjectivity, how can we show that $|A| = |B|$ for sets A and B?

    1. Find a **bijection** $f: A \to B$ (or a bijection $g: B \to A$)
    2. Find **1-1** $f: A \to B$ and **1-1** $g: B \to A$
    3. Find **onto** $f: A \to B$ and **onto** $g: B \to A$

$\lvert A\rvert \leq \lvert B\rvert$	$\lvert A\rvert \geq \lvert B\rvert$
$f_1: A \to B$ is 1-1	$f_2: A \to B$ is onto
$g_1: B \to A$ is onto	$g_2: B \to A$ is 1-1

Problem 8

8. Schroder-Bernstein Theorem $\star$

Show that $(0,1)$ and $[0,1]$ have the same cardinality.

Problem 9

9. Countability

(a) Find a countably infinite subset A of $(0, 1)$.

(b) Find a bijection between A and $A \cup \{0, 1\}$.

(c) Find an explicit one-to-one and onto mapping from the open interval $(0, 1)$ to the closed interval $[0, 1]$.