W24

EECS 203 Discussion 2

Predicates, Quantifiers, and Intro to Proofs

Important Forms

- Two beginning-of-semester surveys on Canvas
  - **FCI BoT** Survey and **Better Belonging in Computer Science (BBCS) Entry** Survey
  - **Due:** Friday, Feb. 2nd @11:59pm
- Exam Date Confirmation Survey
  - **Due:** Friday, Feb. 2nd @11:59pm
  - Please fill this out, even if you don't have an exam conflict!
- They are each worth a few points, so make sure to fill them out!

Upcoming Homework

- Homework/Groupwork 2 will be due **Feb. 1st**
    - **Don’t forget to match pages!**
    - Please note as soon as you press submit you’ve successfully submitted by the deadline. **You can still match pages** with no rush without adding to your submission time.
- Groupwork
    - Groupwork can be done alone, but the problems tend to be more difficult, and the goal is for you to puzzle them out with others!
    - Your discussion section is a great place to find a group!
    - There is also a pinned Piazza thread for searching for homework groups.

Predicates & Quantifiers

Predicates & Quantifiers

- **Predicate:** A sentence or mathematical expression whose truth value depends on a parameter, and becomes a proposition when the parameter is specified.
  - **Example:** **$x > 10$** predicate that depends on parameter x

- **Universal Quantifier:** Denoted by **∀** and translated as **"for all"**, it specifies that the following propositional function is true for all possible parameters in the domain.
  - **Example:** Let x be a positive integer. $\forall x\ [x > 0]$

- **Existential Quantifier:** Denoted by **∃** and translated as **"there exists"**, it specifies that the following propositional function is true for at least one parameter in the domain.
  - **Example:** Let x be an integer. $\exists x\ [x = 3]$

Quantifiers Continued

- **Nested Quantifiers:** A nested quantifier is a quantifier that involves the use of two or more quantifiers to quantify a compound proposition P(x,y). In nested quantifiers, order matters…
  - **P(x,y):** some statement about x and y
  - **Example:** $\forall x \exists y\ P(x,y)$ is different from $\exists y \forall x\ P(x,y)$
    - **$\forall x \exists y\ P(x,y)$:** "For all x, there exists y such that…"
    - **$\exists y \forall x\ P(x,y)$:** "There exists y such that for all x…"

- **De Morgan's Laws for Quantifiers:**
  - $\neg \forall x\ P(x) \equiv \exists x\ \neg P(x)$
  - $\neg \exists x\ P(x) \equiv \forall x\ \neg P(x)$

Problem 1

1. Quantifiers and Negations $\star$

Find the negation of each of these propositions. Simplify so that your answers do not include the negation symbol.

a) $\exists x[-4 < x \leq 1]$

b) $\forall z \exists x \exists y[x^3 + y^3 = z^3]$

Solution

1. Quantifiers and Negations ⋆

Find the negation of each of these propositions. Simplify so that your answers do not include the negation symbol.

a) $\exists x[-4 < x \leq 1]$

b) $\forall z \exists x \exists y[x^3 + y^3 = z^3]$

a)
$$\begin{aligned} &\neg(\exists x(-4 < x \leq 1)) \\ &\equiv \forall x(\neg(-4 < x \leq 1)) \\ &\equiv \forall x((x \leq -4) \vee (x > 1)) \end{aligned}$$

b)
$$\begin{aligned} &\neg(\forall z \exists x \exists y(x^3 + y^3 = z^3)) \\ &\equiv \exists z \neg(\exists x \exists y(x^3 + y^3 = z^3)) \\ &\equiv \exists z \forall x \neg(\exists y(x^3 + y^3 = z^3)) \\ &\equiv \exists z \forall x \forall y(\neg(x^3 + y^3 = z^3)) \\ &\exists z \forall x \forall y(x^3 + y^3 \neq z^3) \end{aligned}$$

Problem 2

2. Quantified Statement Counterexamples

Find a counterexample, if possible, to these quantified statements, where the domain for all variables is integers.

a) $\forall x \exists y (x = 1/y)$

b) $\forall x \exists y (y^2 - x < 100)$

c) $\forall x \forall y (x^2 \neq y^3)$

Solution

2. Quantified Statement Counterexamples

Find a counterexample, if possible, to these quantified statements, where the domain for all variables is integers.

a) $\forall x \exists y (x = 1/y)$

b) $\forall x \exists y (y^2 - x < 100)$

c) $\forall x \forall y (x^2 \neq y^3)$

a) Consider $x = 2$, then there is no y among the integers such that $2 = 1/y$, since the only solution of this equation is $y = 1/2$.

b) Consider $x = -200$. The statement claims there exists a y such that $y^2 + 200 < 100$. This would require our y^2 to be negative, which is not possible in the domain of integers.

c) Consider $x = y = 0$. $x^2 = 0$ and $y^3 = 0$, so $x^2 = y^3$.

Problem 3

3. Quantifier Translations ⋆

Let $P(x)$ be "x is perfect"; let $F(x)$ be "x is your friend"; and let the domain of quantifiers be all people. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.

b) Not everyone is perfect.

c) All your friends are perfect.

d) At least one of your friends is perfect

e) Everyone is your friend and is perfect.

f) Not everybody is your friend or someone is not perfect.

Solution

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f) Not everybody is your friend or someone is not perfect.

Solution: Let $P(x)$ be "x is perfect"; let $F(x)$ be "x is your friend"; and let the domain (universe of discourse) be all people.

a) This means that everyone has the property of being not perfect: $\forall x \neg P(x)$. Alternatively, we can write this as $\neg \exists x P(x)$, which says that there does not exist a person who is perfect.

b) This is just the negation of "Everyone is perfect": $\neg \forall x P(x)$. Alternatively we could write $\exists x \neg P(x)$ (i.e. there exists someone who is not perfect).

c) If someone is your friend, then that person is perfect: $\forall x (F(x) \rightarrow P(x))$. Note the use of conditional statement with universal quantifiers.

Solution

3. Quantifier Translations ⋆

Let $P(x)$ be "x is perfect"; let $F(x)$ be "x is your friend"; and let the domain of quantifiers be all people. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.

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d) At least one of your friends is perfect

e) Everyone is your friend and is perfect.

f) Not everybody is your friend or someone is not perfect.

d) $\exists x(F(x) \land P(x))$. Note the use of conjunction ($\land$) with existential quantifiers to restrict the domain. (Note this allows the possibility that more than one of your friends are perfect.)

e) The expression is $\forall x(F(x) \land P(x))$. Note that here we did use a conjunction with the universal quantifier because $\forall$ distributes over $\land$. We can also split up the expression into two quantified statements and write $(\forall x F(x)) \land (\forall x P(x))$.

f) The correct expression is $(\neg \forall x F(x)) \lor (\exists x \neg P(x))$.

Intro to Direct Proofs

Introduction to Proofs

A **proof** (or *argument*) for a statement **S** is a sequence of statements ending with **S** (the **conclusion**).

A proof starts with some beginning statements you assume are true, called the **premises**.

A proof is **valid** if every statement after the premises is implied ($\rightarrow$) by the some combination of the statements before it.

Then, whenever the premises are true, the conclusion **must be true**.

Proof Methods

- **Direct Proof:** Proves $p \rightarrow q$ by showing: **$p \rightarrow$ stuff $\rightarrow q$**
- **Proof by Contraposition:** *next week!*
- **Proof by Contradiction:** *next week!*
- **Proof by Cases:** *next week!*

Some Useful Definitions

***Note:** **iff** stands for if and only if ($\leftrightarrow$)

- **Even:** An integer **x** is even iff there exists an integer k such that **x = 2k**
    - **Mod Definition:** An integer x is even iff **$x \equiv 0 \pmod{2}$**
- **Odd:** An integer **x** is odd iff there exists an integer k such that **x = 2k + 1**
    - **Mod Definition:** An integer x is odd iff **$x \equiv 1 \pmod{2}$**

Problem 4

4. Odd Proof

Prove or disprove: The product of two odd numbers is odd.

Solution

4. Odd Proof

Prove or disprove: The product of two odd numbers is odd.

Solution: We will prove this statement.
Let x and y each be **arbitrary** odd integers (which are not necessarily equal). By definition, x and y can be written as $x = 2m + 1$ and $y = 2n + 1$ for some integers m and n. Looking at their sum, we have

$$\begin{aligned} xy &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

Since $xy = 2k + 1$, where k is the integer $2mn + m + n$, then by definition, xy is odd. Therefore, this relation holds for all odd x and y, and we have shown, via a direct proof, that the product of two odd numbers is odd.

Problem 5

5. Even Proof

Prove (using a direct proof) that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even.

Solution

5. Even Proof

Prove (using a direct proof) that if $m+n$ and $n+p$ are even integers, where m, n, and p are integers, then $m+p$ is even.

Solution: Using a Direct Proof,

- Let $m+n$ and $n+p$ be **arbitrary** even integers, $m+n=2a$ and $n+p=2b$, for some integers a and b.
- Since $m+n=2a$, we have that $m=2a-n$
- Likewise, since $n+p=2b$, we have that $p=2b-n$
- Therefore, $m+p=(2a-n)+(2b-n)=2(a+b-n)$
- Since a, b and n are all integers, $(a+b-n)$ is an integer
- Since $m+p$ is equal to two times an integer, $m+p$ is even

Therefore, the statement "if $m+n$ and $n+p$ are even integers, where m, n, and p are integers, then $m+p$ is even" holds for all even $m+n$ and $n+p$.

Disproof

To **disprove** a statement means to **prove the negation** of that statement:

Disprove $P(x) \equiv$ Prove $\neg P(x)$

Note that if the statement you are trying to disprove is a for-all statement, all you need to disprove it is a singular counter example since $\boldsymbol{\neg \forall x P(x) \equiv \exists x \neg P(x)}$.

Example: Disprove it's raining today ≡ **Prove** it's not raining today 🌞

Example: Disprove $P \rightarrow Q \equiv$ **Prove** $\neg(P \rightarrow Q) \equiv \neg(\neg P \vee Q) \equiv (P \wedge \neg Q)$

More Useful Definitions

*Note: **iff** stands for if and only if ($\leftrightarrow$)

- **Rational:** A number **x** is rational iff it can be written as the quotient of two integers. **x = p/q**

- **Irrational:** Not rational–cannot be written as the quotient of two integers

- **Prime:** A prime number **p** is a number greater than 1 whose only factors are 1 and itself. $\forall x\ [x|p \rightarrow (x=1 \lor x=p)]$

- **Composite:** A whole number **p** is composite if it has at least one divisor other than 1 and itself. $\exists x\ [x \neq 1 \land x \neq p \land x|p]$

Problem 6

6. Negation Station

For each of the following statements, write the statement's negation. Then, determine which is true: the original statement or the negated statement? (You do not need a rigorous proof.)

Reminder: Two numbers, x and y, are **multiplicative inverses** if $xy = 1$.

a. For all real numbers x and y, if $x + y = 0$, then one of them is negative and the other is positive.

b. For all nonzero rational numbers x and y, if they are multiplicative inverses, then $x \neq y$.

c. Each non-zero rational number has a rational multiplicative inverse.

d. Each non-zero integer has an integer multiplicative inverse.

Solution

6. Negation Station

For each of the following statements, write the statement's negation. Then, determine which is true: the original statement or the negated statement? (You do not need a rigorous proof.)

Reminder: Two numbers, x and y, are **multiplicative inverses** if $xy = 1$.

a. For all real numbers x and y, if $x + y = 0$, then one of them is negative and the other is positive.

b. For all nonzero rational numbers x and y, if they are multiplicative inverses, then $x \neq y$.

c. Each non-zero rational number has a rational multiplicative inverse.

d. Each non-zero integer has an integer multiplicative inverse.

Solution:

a. "There exist real numbers x and y such that $x + y = 0$ and they do not have opposite signs."

The **negated** statement is true. (Consider $x = y = 0$.)

b. "There exist non-zero rational numbers x and y such that they are multiplicative inverses and $x = y$."

The **negated** statement is true. (Consider $x = y = 1$.)

c. "There exists a non-zero rational number that does not have a rational multiplicative inverse."

The **original** statement is true. (Any non-zero rational number $x = a/b$ has a multiplicative inverse of b/a, which is also rational, where a and b are both integers.)

d. "There exists a non-zero integer that does not have an integer multiplicative inverse."

The **negated** statement is true. (Consider $x = 2$, whose multiplicative inverse is $1/2$, which is not an integer.)

Proving “For All” and “There Exists” Statements

Claim: **For all x, P(x).**

- Start with an arbitrary domain element
- End with the statement inside the “for all”

Sample Language:

Let x be an **arbitrary** domain element

… (make some deductions) …

Thus, P(x).

Therefore, P(x) holds for all x in the domain.

Claim: **There exists an x such that P(x).**

- Name a specific domain element
- Show that the named value satisfies the claim

Sample Language:

Consider x = __ [specific domain element]

… show that P(x) holds for that value of x.

Problem 7

7. Quantifier Proofs

Building on the last question, prove or disprove each of the following statements.
(If you find it helpful to translate the statements to logical connectives and symbols first, you can, but it's not required that you; you can just work with the English statements directly.)

a. For all nonzero rational numbers x and y, if they are multiplicative inverses, then $x \neq y$.

b. Each non-zero rational number has a multiplicative inverse that is also a rational number.

Solution

7. Quantifier Proofs

Building on the last question, prove or disprove each of the following statements.
(If you find it helpful to translate the statements to logical connectives and symbols first, you can, but it's not required that you; you can just work with the English statements directly.)

a. For all nonzero rational numbers x and y, if they are multiplicative inverses, then $x \neq y$.

b. Each non-zero rational number has a multiplicative inverse that is also a rational number.

Solution:

a. **Disprove:**
We will prove the negation: "There exist non-zero rational numbers x and y such that they are multiplicative inverses and $x = y$."
Consider the non-zero rational numbers $x = 1$ and $y = 1$. They are multiplicative inverses since $xy = 1 \cdot 1 = 1$, and they are equal to each other, i.e., $x = y$.

Alternate disproof (for logical expression lovers):
Let x and y come from the domain of all non-zero rational numbers. We can restate the claim as:

$$\forall x \forall y [xy = 1 \rightarrow x \neq y]$$

We can disprove with a counterexample. Note that in symbolic notation, the negation of the original statement is

$$\exists x \exists y [xy = 1 \wedge x = y]$$

We will prove the negation is true. Consider $x = 1$ and $y = 1$. Then $xy = 1 \cdot 1 = 1$, and $x = y = 1$. Therefore, the negation of the original statement is true, which means the original statement must be false.

Solution

7. Quantifier Proofs

Building on the last question, prove or disprove each of the following statements.
(If you find it helpful to translate the statements to logical connectives and symbols first, you can, but it's not required that you; you can just work with the English statements directly.)

a. For all nonzero rational numbers x and y, if they are multiplicative inverses, then $x \neq y$.

b. Each non-zero rational number has a multiplicative inverse that is also a rational number.

b. **Prove:**
Let x be an arbitrary non-zero rational number. The multiplicative inverse of x is $1/x$ because $x \cdot 1/x = 1$. Since x is rational and non-zero, we have $x = a/b$ for some non-zero integer a and integer b. This gives us $1/x = b/a$, where a and b are both integers. So $1/x$, the multiplicative inverse of x, is rational.

Alternate proof (for logical expression lovers):
Let x and y come from the domain of all non-zero rational numbers. We will prove the given statement, which can be written in logic as:

$$\forall x \exists y [xy = 1]$$

- Take an arbitrary non-zero rational number x.
- By definition of rational numbers, $x = \frac{p}{q}$ for some integers p and q where $q \neq 0$.
- Since $x \neq 0$, $p \neq 0$.
- Let $y = \frac{q}{p}$. (We can do this since $p \neq 0$)
- y is a rational number by definition
- Since $q \neq 0$, $y \neq 0$.
- $xy = \frac{p}{q} \cdot \frac{q}{p} = 1$

Thus, for all non-zero rational numbers x there exists rational number y that is x's multiplicative inverse, i.e., there exists a rational y such that $xy = 1$.

More Useful Definitions

*Note: **iff** stands for if and only if ($\leftrightarrow$)

- **Divisibility:**
  - The statement **n|a** means "**n divides a**".
  - *In other words*, "**a is divisible by n**".
  - *In other words*:
    - **n|a** iff **$\exists$ k (nk = a)**, *where n, a, and k are integers.*

Problem 8

8. Divides Proof

Prove that if n is odd, then $4|(n^2 - 1)$.

Solution

8. Divides Proof

Prove that if n is odd, then $4|(n^2-1)$.

- Assume n is odd. So $n = 2k+1$ for some integer k.
- So $n^2 - 1 = (2k+1)^2 - 1 = (4k^2 + 4k + 1) - 1 = 4k^2 + 4k = 4(k^2 + k)$.
- Since k in an integer, $(k^2 + k)$ is also an integer.
- Thus $(n^2 - 1)$ is 4 times an integer, i.e., $4|(n^2 - 1)$.