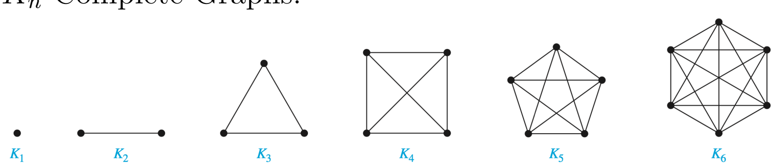
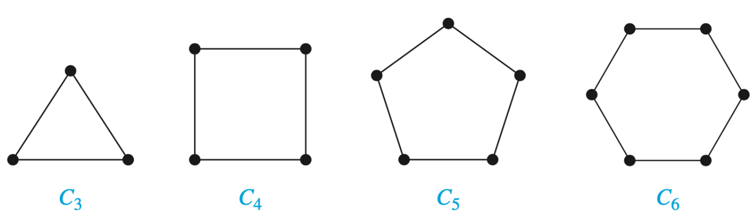
EECS 203: Discrete Mathematics
Winter 2024
FOF Worksheet 8

1 Graphs

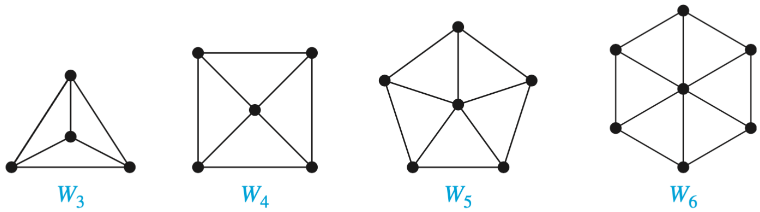
- **Graph, $G = (V, E)$:** A graph $G = (V, E)$ consists of V, a set of vertices, and E, a set of edges. Each edge has either one or two vertices associated with it, where there is only one if the edge is a loop.
- **Simple Graph:** A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices.
- **Directed Graph:** A graph where each edge is associated with an ordered pair of vertices (a directed edge), (u, v), and the edge is said to start at u and end at v.
- **Multigraph:** Edges can have multiplicity. Ex: $\{u, v\}$ and $\{u, v\}$ undirected, (u, v) and (u, v) directed
- **Loops:** An edge that connects a vertex to itself.
- **Adjacent Vertices:** Two vertices are adjacent if there is an edge that connects them. This edge is said to connect the two vertices.
- **Degree, $deg(v)$:** In an undirected graph, the degree of a vertex, v, is the number of edges attached to v (except that a loop at a vertex contributes twice to the degree of that vertex).
- **Special Simple Undirected Graphs:**
  - K_n Complete Graphs:



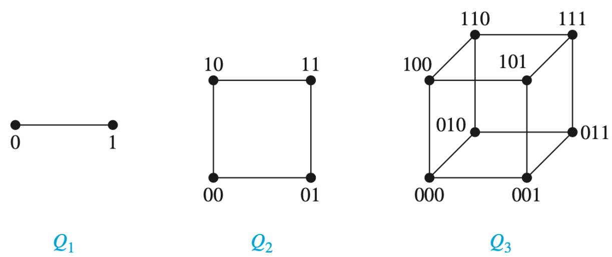
  - C_n Cycles:



– W_n Wheels:



– Q_n Hypercubes:



1.1 Graphs Intro

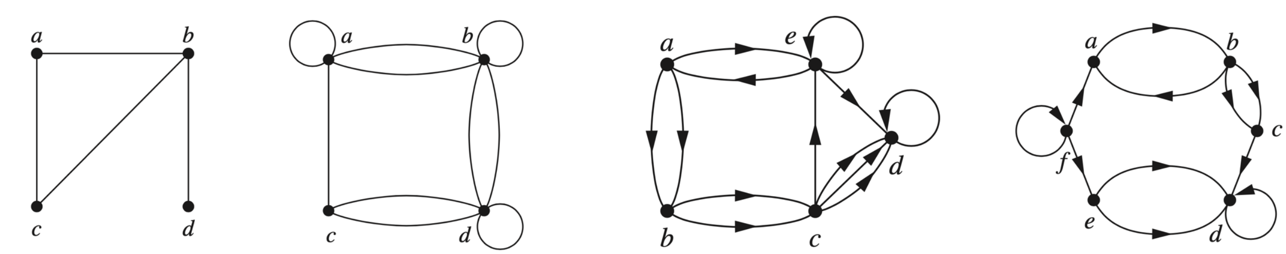
For the following graphs:

a) Identify whether the graph has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops.

b) For each undirected graph, identify whether or not it is simple. If it is not simple, find a set of edges to remove to make it simple.

c) Find $deg(b)$ or if the graph is directed, find $deg^-(b)$ and $deg^+(b)$.

d) Write out its degree sequence. For this part, treat the directed graphs as if they were undirected.



Solution:

a) undirected, no multiple edges, no loops

undirected, multiple edges, loops

directed, multiple edges (there are multiple going in the same direction between two vertices), loops

directed, multiple edges, loops

NOTE: The directed edges (u, v) and (v, u) are distinct edges and therefore aren't counted as multiple edges. However if we have two directed edges (u, v), (u, v) (same vertices, same direction), that IS a multigraph.

b) simple
not simple
not simple
not simple

c) $deg(b) = 3$
$deg(b) = 6$
$deg^-(b) = 2$, $deg^+(b) = 2$
$deg^-(b) = 1$, $deg^+(b) = 3$

d) 3, 2, 2, 1
6, 6, 5, 3
6, 6, 6, 4, 4
5, 4, 4, 3, 3, 3

1.2 The Handshake Theorem

The Handshake Theorem: Let $G = (V, E)$ be an undirected graph with m edges. Then:

$$2m = \sum_{v \in V} deg(v)$$

Handshake Theorem Equivalent for Directed Graphs:

$$|E| = \sum_{v \in V} deg^+(v) = \sum_{v \in V} deg^-(v)$$

Corollary of Handshake Theorem: Every graph has an even number of vertices with odd degree.

1.2.1 Edges and Vertices

Suppose a graph has 21 edges, and 3 vertices of degree 4. All other vertices have degree 2. How many vertices are in the graph?

Solution: We know that the total degree of everything in the graph must be $21 \times 2 = 42$ (by Handshake Theorem), and that the vertices of degree 4 contribute a total of 12 to this count. Thus, there are 30 degrees left in the graph. This means that there must be $30/2 = 15$ vertices with degree 2. This gives a total number of 18 vertices in the graph.

1.3 Trees

Acyclic graph: An acyclic graph is a graph having no cyclic subgraphs. **Tree:** A tree is a connected, acyclic graph. **Tree Theorems (2):**

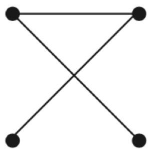
- If $T = (V, E)$ and $u, v \in V$, there is a unique simple path from u to v.
- Every tree on n vertices contains $(n - 1)$ edges.

Spanning Tree: Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.

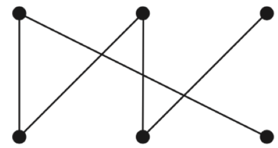
1.3.1 Trees

Which of the following graphs are trees? If it is not a tree, are you able to construct a spanning tree of the graph?

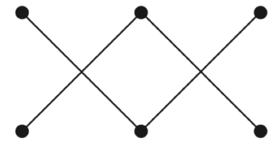
a)



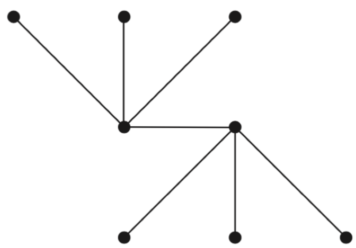
b)



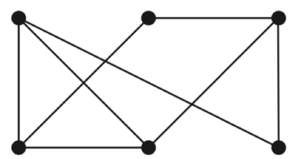
c)



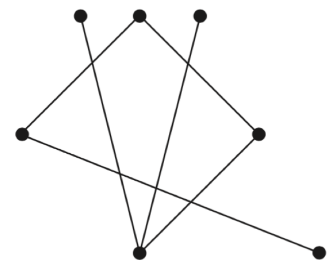
d)



e)



f)



Solution:

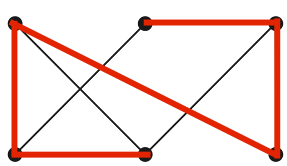
a) Tree

b) Tree

c) Not a tree (not a connected graph). You cannot create a spanning tree from this graph because it is not connected and thus, there is no way to remove edges and get a tree.

d) Tree

e) Not a tree (contains cyclic subgraphs). Yes, you can construct a spanning tree of this graph.
EX:



f) Tree

1.4 Isomorphic Graphs

Isomorphism: Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijection $f : V_1 \to V_2$ such that

$$\forall u, v \big[\{u, v\} \in E_1 \leftrightarrow \{f(u), f(v)\} \in E_2\big]$$

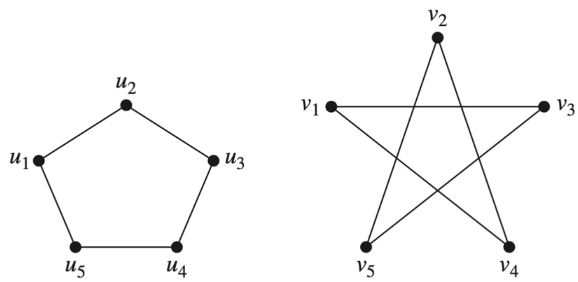
Graph Invariant: A graph invariant is a property preserved by isomorphism of graphs.

- Examples: number of vertices, number of edges, degrees of vertices, existence of subgraphs, path properties

1.4.1 Isomorphic Graphs

Determine whether each given pair of graphs is isomorphic. Exhibit an isomorphism or provide an argument that none exists.

39.



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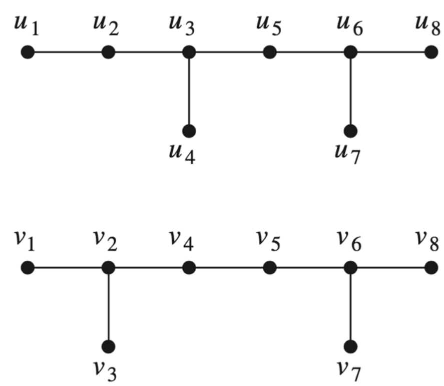

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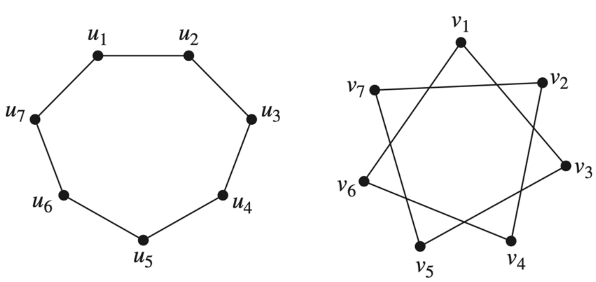

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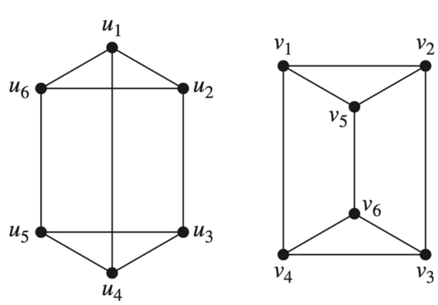

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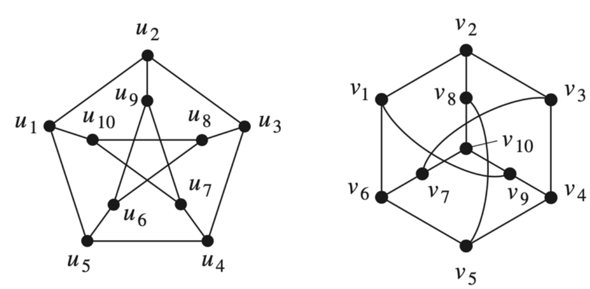
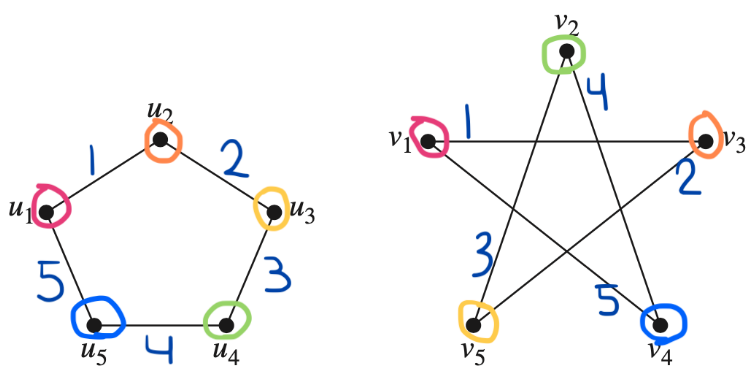


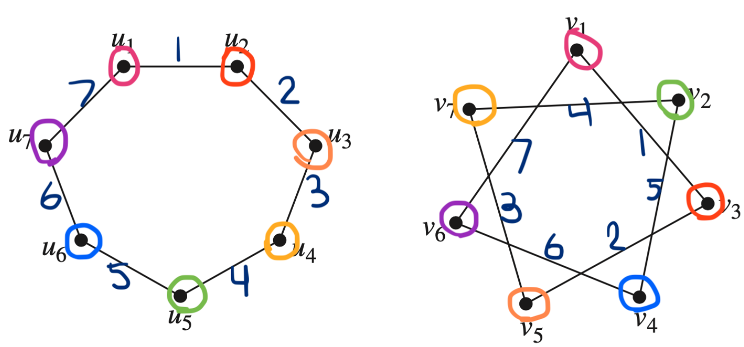
Solution: 39. Isomorphic
Let f be as follows:
$f(u_1) = v_1$
$f(u_2) = v_3$
$f(u_3) = v_5$
$f(u_4) = v_2$
$f(u_5) = v_4$
Then all of the edges corresond between the two graphs, ie $\{u_i, u_j\}$ is an edge in the first graph iff $\{f(u_i), f(u_j)\}$ is an edge in the second graph.

39.



41. Isomorphic
Let f be as follows:
$f(u_1) = v_1$
$f(u_2) = v_3$
$f(u_3) = v_5$
$f(u_4) = v_7$
$f(u_5) = v_2$
$f(u_6) = v_4$
$f(u_7) = v_6$

41.



43. Isomorphic
Let f be as follows:

$f(u_1) = v_5$
$f(u_2) = v_1$
$f(u_3) = v_4$
$f(u_4) = v_6$
$f(u_5) = v_3$
$f(u_6) = v_2$

2 Exam 2 Review

2.1 Induction

2.1.1 Inductive Conclusions

Suppose that $P(n)$ is an unknown predicate. Determine for which positive integers n the statement $P(n)$ must be true, and justify your answer, if

a) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then P(n + 2) is true.

b) $P(1)$ and $P(2)$ are true, and for all positive integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

c) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(2n)$ is true.

d) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.

Solution:

a) The inductive step here allows us to conclude that $P(3)$, $P(5)$, ... are all true, but we can conclude nothing about $P(2)$, $P(4)$, ...

b) $P(n)$ is true for all positive integers n, using **strong induction**.

c) The inductive step here enables us to conclude that $P(2)$, $P(4)$, $P(8)$, $P(16)$,... are all true, but we can conclude nothing about $P(n)$ when n is not a power of 2.

d) This is **mathematical induction**; we can conclude that $P(n)$ is true for all positive integers n.

2.1.2 Equality Weak Induction

Use induction to prove the summation of cubes formula. That is, for $n \geq 1$ prove:

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Solution: Let $P(n)$ be the statement $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$
Base Step: n = 1
$1^3 + 2^3 + \cdots + n^3 = 1^3 = 1$ and $\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$
Thus, $P(1)$ is true.

Inductive Hypothesis: Assume that $P(k)$ is true

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$$

Inductive Step: Prove that $P(k+1)$ is true given the assumption.

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= (k+1)^2 \cdot \frac{k^2}{4} + (k+1)^2(k+1) \\
&= (k+1)^2(\frac{k^2}{4} + (k+1)) \\
&= (k+1)^2(\frac{k^2+4k+4}{4}) \\
&= (k+1)^2(\frac{(k+2)^2}{4}) \\
&= \frac{(k+1)^2((k+1)+1)^2}{4}
\end{aligned}$$

Therefore, using mathematical induction, we have proven that $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for all $n \geq 1$.

2.1.3 Strong Induction

Prove that every positive integer is either a power of 2, or can be written as the sum of distinct powers of 2.

Solution: Define $P(n)$ to be the predicate: n can be written as the sum of distinct powers of two.

Inductive Step: Let $k > 0$ Assume $P(j)$ is true for all $j < k$. We can split this into cases:

1. If k is a power of two, so $k = 2^m$, for some $m \in \mathbb{Z}^+$. We have written k as the sum of one (distinct) power of two, so $P(k)$ is true.

2. Otherwise, We can write k as the sum of the largest power of two smaller than k, j_1, and some other positive integer, $j_2 = n - j_1$.

   Lemma: $j_2 < j_1$. Assume for contradiction that $j_2 \geq j_1$. Then $k \geq 2j_1$. Because j_1 is a power of two, $2j_1$ is also a power of two, and is smaller than k, so this contradicts the fact that j_1 is the largest power of two smaller than k.

   Because $P(j_2)$ is true, j_2 can be written as the sum of distinct powers of two (all of which must be less than, and therefore not equal to, j_1). So k must be able to be written as the sum of distinct powers of two.

We have that if $P(j)$ is true for all $j < k$, then $P(k)$ is true.

Base Case: $P(1)$ is true because 1 can be written as 2^0.
We can then conclude that every positive integer is either a power of 2 or can be written as the sum of distinct powers of 2 by strong induction.

2.2 Recurrence

2.2.1 Initial Conditions

What is the minimum number of initial conditions required for the following recurrence relation?

$$R(n) = R(n-4) + R(n-3) + 2log(n)$$

(a) 1

(b) 2

(c) 3

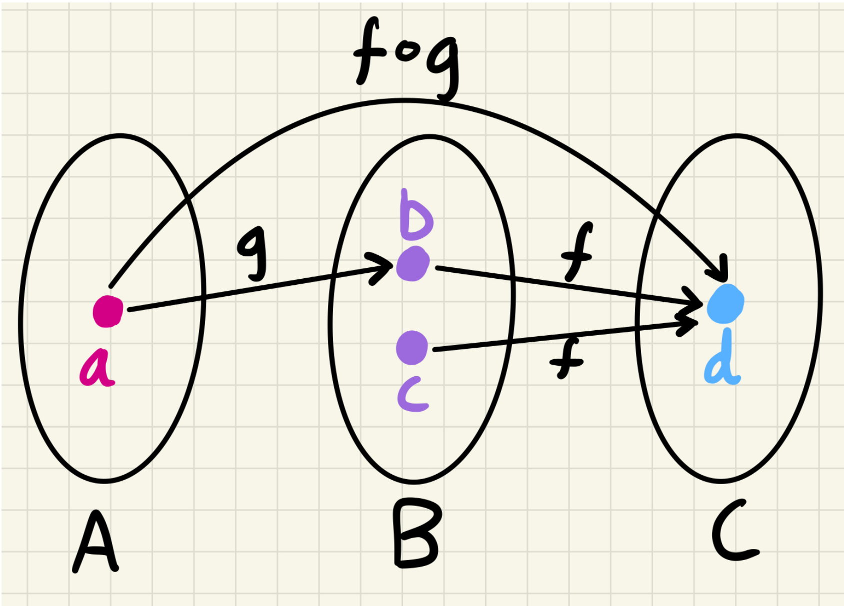
(d) 4

Solution: (d), 4 conditions are needed

2.3 Functions

If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

Solution: No. For example, suppose that $A = \{a\}, B = \{b, c\}$, and $C = \{d\}$. Let $g(a) = b, f(b) = d$, and $f(c) = d$. Then f and $f \circ g$ are onto, but g is not.



2.4 Countability

Which of the following sets are uncountably infinite? (select all that apply)

(a) $\mathbb{R}^-$

(b) $\mathbb{R} \cap \mathbb{Z}$

(c) $\mathbb{Q} - \mathbb{Z}$

(d) $\mathbb{R} \cup \mathbb{N}$

Solution: (a), (d)

(a) The real numbers are uncountably infinite

(b) This is just $\mathbb{Z}$, so it is countably infinite

(c) This leaves non-integer rationals, which is countably infinite

(d) This is $\mathbb{R}$, which is uncountably infinite

2.5 Schröder–Bernstein

Use the Schröder–Bernstein theorem to show $|[0,1)| = |(-2,0) \cup (0,2)|$.

Solution: We want to show that $|[0,1)| \leq |(-2,0)\cup(0,2)|$ and $|[0,1)| \geq |(-2,0)\cup(0,2)|$.

- First, we show $|[0,1)| \leq |(-2,0)\cup(0,2)|$. To show this, we can provide a one-to-one function from $[0,1)$ to $(-2,0) \cup (0,2)$ or an onto function from $(-2,0) \cup (0,2)$ to $[0,1)$. There are multiple valid, solutions, but a possible one-to-one function is as follows:
- $f : [0,1) \to (-2,0) \cup (0,2)$ defined as $f(x) = x + 1$.
  - First, note that this function is well-defined: for any $x \in [0,1)$, $0 \leq x < 1$. Thus $1 \leq x + 1 < 2$ so $f(x) \in (0,2)$, and thus $f(x) \in (-2,0) \cup (0,2)$.
  - This function is also one-to-one: Let x, y be arbitrary elements of $[0,1)$, and suppose $f(x) = f(y)$. Then $x + 1 = y + 1$, so $x = y$.
  - Thus we have shown that there exists a one-to-one function from $[0,1)$ to $(-2,0) \cup (0,2)$, so we conclude $|[0,1)| \leq |(-2,0) \cup (0,2)|$.
- Now, we must show the opposite direction, that is: $|[0,1)| \geq |(-2,0) \cup (0,2)|$. To do this, we find a function $g : (-2,0) \cup (0,2) \to [0,1)$ that is one-to-one.
- One possible function g could be $g(x) = \frac{x+2}{4}$.
  - This function is well defined. Consider an arbitrary $x \in (-2,0) \cup (0,2)$. Then observe that $x \in (-2,2)$, and $-2 < x < 2$. Thus $0 < \frac{x+2}{4} < 1$, so $g(x) \in [0,1)$.
  - Finally, we show this function is one-to-one. Let x, y be arbitrary elements of $(-2,0) \cup (0,2)$, and suppose $f(x) = f(y)$. Then $\frac{x+2}{4} = \frac{y+2}{4}$, so $x = y$ by algebra.

- Thus we have shown that there exists a one-to-one function from $(-2,0)\cup(0,2)$ to $[0,1)$, so we conclude $|[0,1)| \geq |(-2,0)\cup(0,2)|$.

- Thus by Schröder–Bernstein, we conclude $|[0,1)| = |(-2,0)\cup(0,2)|$

2.6 Mod

Let $a \equiv 38 \pmod{15}$, $b \equiv 2 \pmod{15}$, and $c \equiv 3 \pmod{5}$. Compute the following if possible:

a) $d \equiv a^{24} \pmod{15}$

b) $e \equiv a^3 b^7 + b^{13} \pmod{15}$

c) $g \equiv a + c \pmod{15}$

d) $h \equiv a + c \pmod{5}$

Solution:

1. $d \equiv 38^{24} \equiv 8^{24} \equiv (8^2)^{12} \equiv 64^{12} \equiv 4^{12} \equiv 16^6 \equiv 1^6 \equiv 1 \pmod{15}$
2. $e \equiv 8^3 2^7 + 2^{13} \pmod{15} \equiv (64 \cdot 8)(16 \cdot 2^3) + 2^{13} \equiv (4 \cdot 8)(8) + 2^{13} \equiv 16 + (2^7 \cdot 2^6) \equiv 1 + (8 \cdot 4) \equiv 1 + 2 \equiv 3 \pmod{15}$
3. $a = 15k_1 + 38$ and $c = 5k_2 + 3$, so $a + c = 15k_1 + 5k_2 + 41$. We cannot factor 15 out of the term with variables k_2, so this is impossible to compute.
4. $a = 15k_1 + 38$ and $c = 5k_2 + 3$, so $a + c = 15k_1 + 5k_2 + 41 = 5(3k_1 + k_2 + 8) + 1$. $h \equiv 1 \pmod{5}$.

2.7 Pigeonhole Practice

a) How many integers do we need to select to guarantee one of them is divisible by 7?

b) How many integers do we need to select to guarantee that we'll have a pair whose difference is divisible by 7?

Solution:

a) This can never be guaranteed. For example, we could have the set $\{1 + 7m | m \in \mathbb{Z}\}$ - in words, all the integers congruent to 1 (mod 7). None of the integers is divisible by 7, but this is an infinite set.

b) 8
Let our pigeon holes be the possible values of our integer mod 7, or the range $\{0, 1, 2, 3, 4, 5, 6\}$. Note that there are 7 pigeonholes. After selecting 8 integers, there are more pigeons than holes, so two of the selected integers are guaranteed to be the same remainder when divided by 7. Thus, by properties of mod, their difference is divisible by 7.

3 Additional Graph Definitions

- **In-Degree,** $deg^-(v)$**:** n a directed graph, the in-degree of a vertex, v, is the number of edges with v as their terminal vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)
- **Out-Degree,** $deg^+(v)$**:** In a directed graph, the out-degree of a vertex, v, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)
- **Neighborhood,** $N(v)$**:** The neighborhood of a vertex, v, is the set of all adjacent (or neighbor) vertices of that vertex and is denoted as $N(v)$. For a set of vertices, A, the neighborhood of A, denoted $N(A)$, is the set of all neighbor vertices to any vertex within the set A.
- **In-Neighborhood,** $N^-(u)$**:** The in-neighborhood of a vertex, u, is the set of all vertices v adjacent to u such that the edge between them is directed towards u.

$$N^-(u) = \{v | (v, u) \in E\}$$

- **Out-Neighborhood,** $N^+(u)$**:** The out-neighborhood of a vertex, u, is the set of all vertices v adjacent to u such that the edge between them is directed towards v (away from u).

$$N^-(u) = \{v | (u, v) \in E\}$$

- **Degree Sequence:** The degree sequence of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order.