W24

EECS 203 Discussion 7

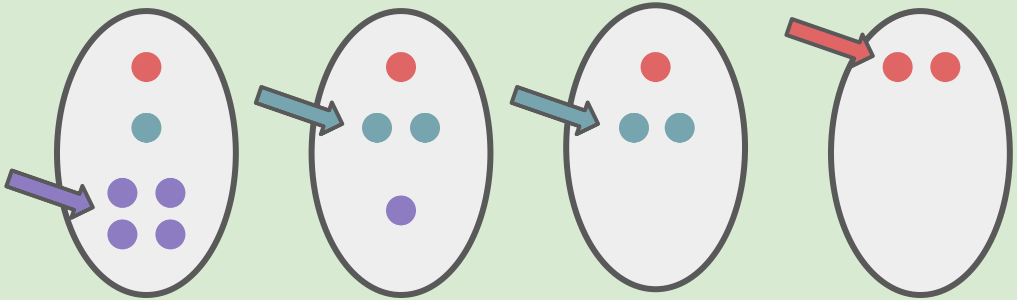
Countability and Pigeonhole Principle

Admin Notes:

- Homework/Groupwork 7 will be due **Mar. 21th**
    - **Don’t forget to match pages!**
    - Please note as soon as you press submit you’ve successfully submitted by the deadline. **You can still match pages** with no rush without adding to your submission time.

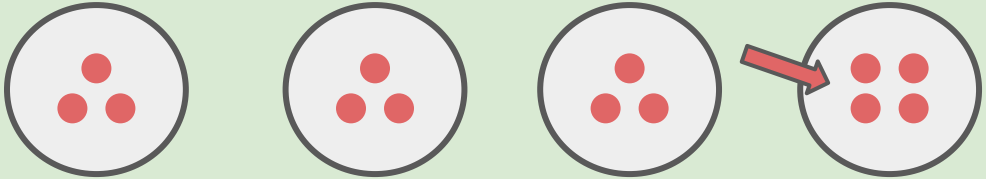
Pigeonhole Principle

- **Pigeonhole Principle:** If we put **k+1 objects** into **k boxes**, then at least one box contains **2 or more** objects.



*__Examples__ of putting 5 objects into 4 bins

- **Generalized Pigeonhole Principle:** If we put **N objects** into **k boxes**, then at least one box contains **ceil(N/k) or more** objects.



*__Example__ of putting 13 objects into 4 bins
ceil(13/4) = ceil(3.25) = 4

Problem 1

1. Pigeonhole Principle $\star$

How many distinct numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

Solution

1. Pigeonhole Principle ⋆

How many distinct numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

We can group these into pairs that add up to 16 of: $(1, 15), (3, 13), (5, 11), (7, 9)$. Notably, no other pair of numbers sum to 16. Therefore, if we pick 5 numbers, it is guaranteed that we pick both numbers from at least one of these pairs.

Problem 2

2. More Pigeonhole Principle

(a) Undergraduate students at a college belong to one of four groups depending on the year in which they are expected to graduate. Each student must choose one of 21 different majors. How many students are needed to assure that there are two students expected to graduate in the same year who have the same major?

(b) What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Solution

2. More Pigeonhole Principle

(a) Undergraduate students at a college belong to one of four groups depending on the year in which they are expected to graduate. Each student must choose one of 21 different majors. How many students are needed to assure that there are two students expected to graduate in the same year who have the same major?

(b) What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

(a) $(4 \cdot 21) + 1 = 85$

(b) $(50 \cdot 99) + 1 = 4951$

Problem 3

3. Even More Pigeonhole Principle ⋆

Sophia has a bowl of 15 red, 15 blue, and 15 orange pieces of candy. Without looking, Sophia grabs a handful of pieces.

(a) What is the smallest number of pieces of candy Sophia has to grab to make sure she has at least 4 of the same color?

(b) What is the smallest number of pieces of candy Sophia has to grab to make sure she has 3 orange candies?

Solution

3. Even More Pigeonhole Principle $\star$

Sophia has a bowl of 15 red, 15 blue, and 15 orange pieces of candy. Without looking, Sophia grabs a handful of pieces.

(a) What is the smallest number of pieces of candy Sophia has to grab to make sure she has at least 4 of the same color?

(b) What is the smallest number of pieces of candy Sophia has to grab to make sure she has 3 orange candies?

(a) 10. Consider colors as boxes, and candies as pigeons. By pigeonhole principle, we have $\lceil \frac{N}{3} \rceil = 4$ where N is the number of pieces we have to grab to make this work. The smallest number N that works is 10.

(b) 33. This is not actually pigeonhole. We specifically need to have 3 orange candies. The only way to make sure this happens is to grab all 15 red, 15 blue, and then the next 3 we grab have to be orange.

Problem 4

4. Pigeonhole Principle Is All You Need

A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Solution

4. Pigeonhole Principle Is All You Need

A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Let $K(x)$ be the number of other computers that computer x is connected to. The possible values for $K(x)$ are 1, 2, 3, 4, 5. Since there are 6 computers, the pigeonhole principle guarantees that at least two of the values $K(x)$ are the same, which is what we wanted to prove.

Review: Functions

- **Function *f: A* → *B*:** associates each element of set A to <u>exactly one</u> element in set B
  - **Domain:** ***A***
  - **Codomain:** ***B***
- **Onto Function *f: A* → *B*:** all elements in B are mapped to by f

$$\forall b \in B\ \exists a \in A\ [\ f(a) = b\ ]$$

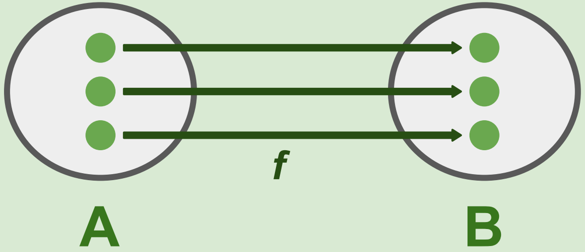
- **One-to-One Function *f: A* → *B*:** no two elements of A map to the same output in B

$$\forall a,b \in A\ [\ f(a) = f(b) \rightarrow a = b\ ]$$

- **Bijective Function:** onto and one-to-one (also called a one-to-one correspondence)

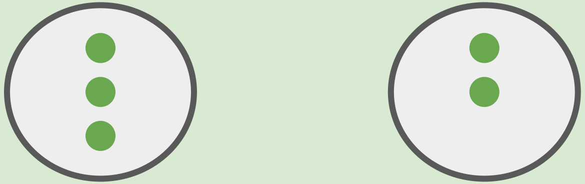
What do function properties tell us about the set cardinalities?

- **Onto Function f : A → B:** $\forall b \in B\ \exists a \in A\ [\ f(a) = b\ ]$

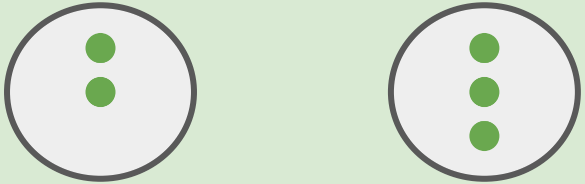


*$|A| = |B|$

  - Is it possible that $|A| > |B|$?

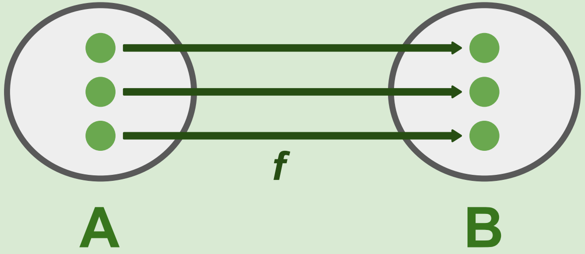


  - Is it possible that $|B| > |A|$?



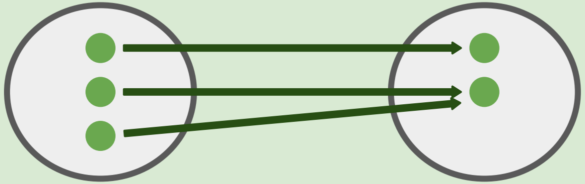
What do these properties tell us about the set cardinalities?

- **Onto Function *f* : A → B:** $\forall b \in B\ \exists a \in A\ [\ f(a) = b\ ]$



*$|A| = |B|$

  - Is it possible that $|A| > |B|$? **Yes!**



  - Is it possible that $|B| > |A|$? **No**

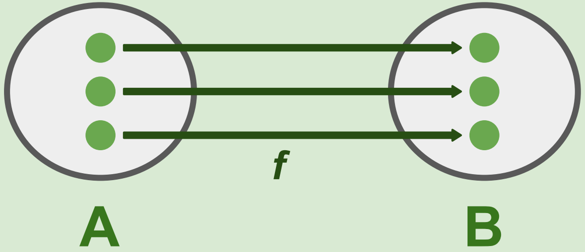
(can’t be a function and onto in this case)

*Thus, if we have an onto function from A to B,

$$|A| \geq |B|$$

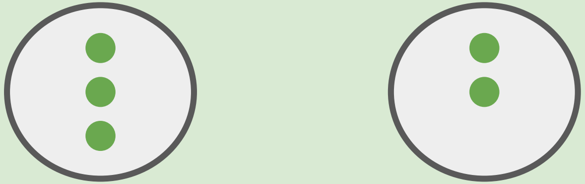
What do these properties tell us about the set cardinalities?

- **One-to-One Function *f* : A → B:** $\forall a,b \in A\ [\ f(a) = f(b) \rightarrow a = b\ ]$

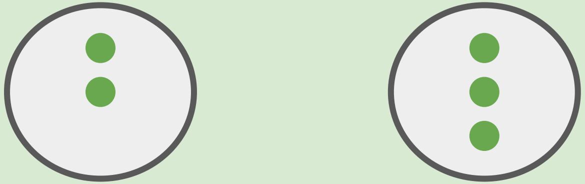


$^*|A| = |B|$

  - Is it possible that $|A| > |B|$?

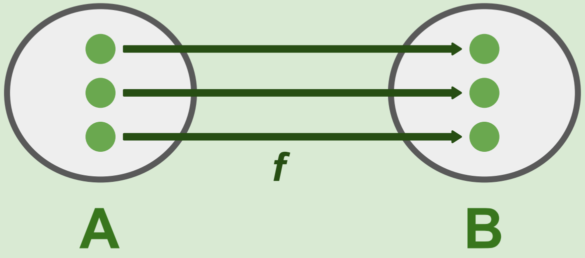


  - Is it possible that $|B| > |A|$?



What do these properties tell us about the set cardinalities?

- **One-to-One Function $f : A \rightarrow B$:** $\forall a,b \in A\ [\ f(a) = f(b) \rightarrow a = b\ ]$



*$|A| = |B|$

*Thus, if we have an 1-1 function from A to B,

$|A| \leq |B|$

- Is it possible that $|A| > |B|$? **No**

(can’t be a function and 1-1 in this case)

- Is it possible that $|B| > |A|$? **Yes!**

(still a function, still 1-1)

Countably vs Uncountably Infinite

- **Countably Infinite:** A set is said to be countably infinite if it has the same cardinality as the natural numbers. One way to prove this is by finding a bijection between the set and the natural numbers. Examples:
  - **The natural numbers**
  - **The integers**
  - **The rational numbers**
- **Uncountably Infinite:** A set is said to be uncountably infinite if its cardinality is larger than the cardinality of the natural numbers. Examples:
  - **The real numbers**
  - **The irrational numbers**
  - **(0,1)**

Problem 5

5. Different Infinities $\star$

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

(a) The set of all integers greater than 10.

(b) The set of all integers with absolute value less than 1,000,000.

(c) The set of all real numbers between 0 and 2.

(d) The set $A \times \mathbb{Z}$, where $A = \{2, 3\}$.

Solution

5. Different Infinities ⋆

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

(a) The set of all integers greater than 10.

(b) The set of all integers with absolute value less than 1,000,000.

(c) The set of all real numbers between 0 and 2.

(d) The set $A \times \mathbb{Z}$, where $A = \{2, 3\}$.

For each part, we will denote the set in question as S.

(a) This is countably infinite. The function $f : \mathbb{Z}^+ \to S, f(x) = x + 10$ is a one-to-one correspondence between S and $\mathbb{Z}^+$.

(b) As defined, S is the set is the integers in the following range:

$$\{-999,999, -999,998, \ldots, -1, 0, 1, \ldots, 999,998, 999,999\}.$$

This set is finite, with size 1,999.

(c) This is uncountable. We proved in lecture that $(0, 1)$ is uncountable, and $(0, 1)$ is a subset of $(0, 2)$.

(d) This is countably infinite. Consider the function $g : \mathbb{Z}^+ \to A \times \mathbb{Z}$, where

$$g(x) = \begin{cases} \left(2, \frac{x-4}{4}\right) & \text{if } x \equiv 0 \mod 4, \\ \left(3, \frac{x-1}{4}\right) & \text{if } x \equiv 1 \mod 4, \\ \left(2, \frac{-(x+2)}{4}\right) & \text{if } x \equiv 2 \mod 4, \\ \left(3, \frac{-(x+1)}{4}\right) & \text{if } x \equiv 3 \mod 4. \end{cases}$$

Then g is a one-to-one correspondence between $A \times \mathbb{Z}$ and $\mathbb{Z}^+$, and thus we have shown that $A \times \mathbb{Z}$ is countably infinite.

Problem 6

6. Different Infinities with Sets

Give an example of two uncountable sets A and B such that $A \cap B$ is:

a) Finite.

b) Countably infinite.

c) Uncountably infinite.

Solution

6. Different Infinities with Sets

Give an example of two uncountable sets A and B such that $A \cap B$ is:

a) Finite.

b) Countably infinite.

c) Uncountably infinite.

There are a lot of possible answers, but here are a few:

a) $A = [0, 1)$ and $B = (-1, 0]$. $A \cap B = \{0\}$

b) $A = \mathbb{R}^+$ and $B = \mathbb{R}^- \cup \mathbb{Z}^+$. $A \cap B = \mathbb{Z}^+$

c) $A = [0, 2]$ and $B = [1, 3]$. $A \cap B = [1, 2]$

Problem 7

7. Cardinality Proof $\star$

Show that $|(0,1)| \geq |\mathbb{Z}^+|$.

Solution

7. Cardinality Proof $\star$

Show that $|(0,1)| \geq |\mathbb{Z}^+|$.

We can show that $|(0,1)| \geq |\mathbb{Z}^+|$ is true through the existence of an onto function from $(0,1)$ to $\mathbb{Z}^+$, or a one-to-one function from $\mathbb{Z}^+$ to $(0,1)$.

Onto example function ($(0,1)$ to $\mathbb{Z}^+$): $f(x) = \lceil \frac{1}{x} \rceil - 1$
One-to-one example function ($\mathbb{Z}^+$ to $(0,1)$): $g(x) = \frac{1}{x+1}$

Note: as the set $(0,1)$ is exclusive, you cannot write $f(x) = \lceil \frac{1}{x} \rceil$ as this does not map to 1. Similarly, you cannot write $g(x) = \frac{1}{x}$; this is not a function, as 1 has no mapping in the codomain.

Schroder-Bernstein & Applications

- **Schroder-Bernstein Theorem:**
  If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

  - So using this theorem, injectivity, and surjectivity, how can we show that $|A| = |B|$ for sets A and B?

    1. Find a **bijection** $f: A \to B$ (or a bijection $g: B \to A$)
    2. Find **1-1** $f: A \to B$ and **1-1** $g: B \to A$
    3. Find **onto** $f: A \to B$ and **onto** $g: B \to A$

$\|A\| \leq \|B\|$	$\|A\| \geq \|B\|$
$f_1: A \to B$ is 1-1	$f_2: A \to B$ is onto
$g_1: B \to A$ is onto	$g_2: B \to A$ is 1-1

Problem 8

8. Schroder-Bernstein Theorem $\star$

Show that $(0,1)$ and $[0,1]$ have the same cardinality.

Solution

8. Schroder-Bernstein Theorem $\star$

Show that $(0,1)$ and $[0,1]$ have the same cardinality.

By Schroder-Bernstein theorem, it suffices to find the following one-to-one functions: $f:(0,1)\to[0,1]$, and $g:[0,1]\to(0,1)$. As an example, let $f(x)=x$ and $g(x)=(x+1)/3$. This concludes the proof.

Problem 9

9. Countability

(a) Find a countably infinite subset A of $(0, 1)$.

(b) Find a bijection between A and $A \cup \{0, 1\}$.

(c) Find an explicit one-to-one and onto mapping from the open interval $(0, 1)$ to the closed interval $[0, 1]$.

Solution

9. Countability

(a) Find a countably infinite subset A of $(0, 1)$.

(b) Find a bijection between A and $A \cup \{0, 1\}$.

(c) Find an explicit one-to-one and onto mapping from the open interval $(0, 1)$ to the closed interval $[0, 1]$.

(a) Consider the set $\{\frac{1}{n}\}$, where n is an integer greater or equal to 2.

(b) Consider a function which maps $\frac{1}{2}$ to 0, $\frac{1}{3}$ to 1, and for all $n > 3$, maps $\frac{1}{n}$ to $\frac{1}{n-2}$.

(c) Consider a function which maps every element to itself, except for the elements in A, and maps the elements in A according to the mapping we used in part (b).