EECS 203: Discrete Mathematics
Winter 2024
Discussion 5b Notes

1 Weak Induction

1.1 Conceptual Understanding

Base Case: The part of the inductive proof which directly proves the predicate for the *first* value in the domain (generally n_0). The base case does not rely on $P(k)$ for any other value of k. Often this will be $P(0)$ or $P(1)$

Inductive Hypothesis: The assumption we make at the beginning of the inductive step. The inductive hypothesis assumes that the predicate holds for some *arbitrary* member of the domain

Inductive Step: The proof which shows that the predicate holds for the "next" value in the domain. The inductive step should make use of the inductive hypothesis.

Weak Induction: Weak Induction is a proof method used to prove a predicate $P(n)$ holds for "all" $n \geq n_0$. Often "all" n is $\mathbb{N}$ or $\mathbb{Z}^+$, but the desired domain of n varies by problem.

1.2 Inductive Conclusions

Suppose that $P(n)$ is an unknown predicate. Determine for which positive integers n the statement $P(n)$ must be true, and justify your answer, if

a) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then P(n + 2) is true.

b) $P(1)$ and $P(2)$ are true, and for all positive integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

c) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(2n)$ is true.

d) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.

Solution:

a) The inductive step here allows us to conclude that $P(3)$, $P(5)$, ... are all true, but we can conclude nothing about $P(2)$, $P(4)$, ...

b) $P(n)$ is true for all positive integers n, using **strong induction**.

c) The inductive step here enables us to conclude that $P(2)$, $P(4)$, $P(8)$, $P(16)$,... are all true, but we can conclude nothing about $P(n)$ when n is not a power of 2.

d) This is **weak induction**; we can conclude that $P(n)$ is true for all positive integers n.

1.3 Fill in the blanks

Prove that $2^n > n^2$ for all integers n greater than 4.

Define $P(n)$: $\underline{\quad 2^n > n^2 \quad}$
Inductive Step: Let $\underline{\ k > 4}$ **be arbitrary**. Assume $\underline{\quad P(k) \quad}$.
We will show $\underline{\quad P(k+1)}$**; that is,** $\underline{2^{k+1} > (k+1)^2 \quad}$.

$$2^{k+1} = \underline{\qquad\qquad\qquad}$$

$$= (k+1)^2$$

Solution:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k^2 \\ &= k^2 + k \cdot k \\ &> k^2 + 4k \\ &= k^2 + 2k + 2k \\ &\geq k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

Base Case: $P(\underline{\quad}$ **5** $\underline{\quad})$

Solution:

$$2^5 = 32 > 25 = 5^2$$

Conclusion: By weak induction, **for all integers** $n > 4$**,** $P(n)$ **holds**.

1.3.1 Fractions

Prove that, whenever n is a positive integer,

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot \ldots \cdot 2n}.$$

Solution: Let $P(n)$ be the predicate given above, so

$$P(n) \text{ means } \frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 3 \cdots 2n}.$$

We assume the inductive hypothesis $P(k)$, which is $\frac{1}{2k} \leq \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)}{2 \cdot 4 \cdot \ldots \cdot 2k}$ and want to prove the similar inequality $P(k+1)$.
We proceed as follows, using the trick of writing $\frac{1}{2(k+1)}$ in terms of $\frac{1}{2k}$ so that we can invoke the inductive hypothesis:

$$\begin{aligned}
\frac{1}{2(k+1)} &= \frac{1}{2k} \cdot \frac{2k}{2(k+1)} \\
&\leq \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)}{2 \cdot 4 \cdot \ldots \cdot 2k} \cdot \frac{2k}{2(k+1)} && \text{(by IH)} \\
&= \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1) \cdot 2k}{2 \cdot 4 \cdot \ldots \cdot 2(k+1)} \\
&\leq \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1) \cdot (2k+1)}{2 \cdot 4 \cdot \ldots \cdot 2(k+1)}
\end{aligned}$$

Our base case of $P(1)$ is true because $LHS = \frac{1}{2 \cdot 1} = \frac{1}{2} \leq \frac{1}{2} = \frac{2(1)-1}{2 \cdot 1} = RHS$.

1.4 Weak Induction - Sets Edition

Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

Solution:

Inductive Step:
Let k be an arbitrary integer greater than or equal to 2. Assume $P(k)$: a set with k elements has $k(k-1)/2$ two-element subsets. We want to show $P(k+1)$: a set with $k+1$ elements has $(k+1)k/2$ two-element subsets

Now we consider a set of $k+1$ elements. Take k elements of this set. We know that this set of elements has $k(k-1)/2$ two-element subsets (by our inductive hypothesis). Now if we add in the $(k+1)$th element, it adds k two-element subsets (($k+1$)th element + one of the k elements that we originally took). So a set of size $k+1$ has $k(k-1)/2+k$ two-element subsets.
$k(k-1)/2+k = k(k-1)/2+2k/2 = [k(k-1)+2k]/2 = k(k-1+2)/2 = (k+1)k/2$.
Therefore, $P(k+1)$ is true.

Base Case:
Prove $P(2)$: a set with 2 elements contains $(2 \cdot 1)/2$ two-element subsets. A set with 2 elements only has one two-element subset: the set itself.
$1 = 2/2 = 2(2-1)/2$ ✓

Thus, by weak induction, a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

2 Strong Induction

2.1 Definitions

Strong Induction: Similar to weak induction, but the inductive step relies on ALL previous cases rather than just the case that came before it. So, you must show:
$\forall k[(P(c) \wedge P(c+1) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)]$, where $k \geq$ some starting case.

You must also prove bases cases. There is often more than one base case for strong induction depending on what information you need to do your inductive step. The number of base cases is the largest c for which you need to assume P(k-c) in order to prove P(k). For example, if your proof of P(k) uses the assumptions P(k-2) and P(k-5), then you would need 5 base cases.

1. Faulty Induction

Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

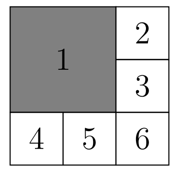
Basis Step: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all non-negative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k+1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

Solution: The proof is invalid for $k = 4$. We cannot increase the postage from 4 cents to 5 cents by either of the replacements indicated, because there is no 3-cent stamp present and there is only one 4-cent stamp present. There is also a minor flaw in the inductive step, because the condition that $j \geq 3$ is not mentioned.

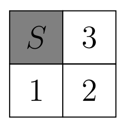
2. Squares Strong Induction

Prove that a square can be subdivided into any number of squares $n \geq 6$. Note that subsquares don't need to be the same size. For example, here's how you would subdivide a square into 6 squares.

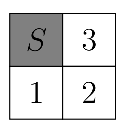


Solution:
We need to show that for any $n \geq 6$, a square can be subdivided into n subsquares.

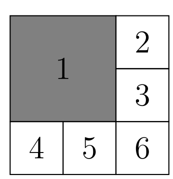
Notice that if we have a square S then we can create a bigger square by adding 3 squares of the same size as shown below. This guides our thinking for the inductive step.

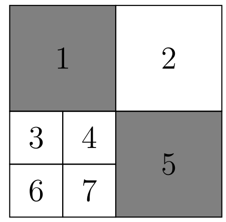


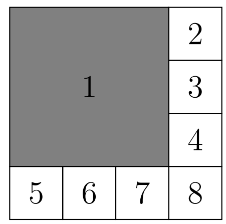
Inductive Step: Assume for $6 \leq i \leq k$ where $k \geq 8$, a square can be split into i subsquares. We want to show that a square can be split into $k+1$ subsquares. We know that we can split a square into $(k+1-3) = (k-2)$ subsquares, by our inductive hypothesis since $(k-2) \geq 6$. Thus, we can split a square into $(k+1)$ subsquares by taking the square of $(k-2)$ subsquares and adding 3 squares as we showed above:



Base Cases:
Since we showed that $P(k)$ implies $P(k+1)$ for all $k \geq 8$ and we want to show $P(c)$ for all $n \geq 6$, we need to bridge the gap by showing the base cases $P(6)$, $P(7)$, $P(8)$...







Thus we've shown by strong induction that for any $n \geq 6$, a square can be subdivided into n subsquares.

3. Jigsaw Puzzle Induction

A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n-1$ moves are required to assemble a puzzle with n pieces.

Solution: Let $P(n)$ be the statement that exactly $n-1$ moves are required to assemble a puzzle with n pieces. Now $P(1)$ is trivially true. Consider a puzzle with k pieces for some $k \geq 1$, and assume that $P(j)$ is true for all $0 < j < k$ where $j \in \mathbb{Z}^+$. The final move must be the joining of two blocks, of size i and $k-i$ for an arbitrary integer i such that $1 \leq i \leq k-1$. (Note that we can view "adding" a piece to a block as joining a the block to another block of size 1.) By the inductive hypothesis, it required $i-1$ moves to construct the one block, and $k-i-1$ moves to construct the other. Therefore $1+(i-1)+(k-i-1) = k-1$ moves are required in all, so $P(k)$ is true.

Thus, by strong induction, $P(n)$ is true for all $n \geq 1$.

4. Forming Discussion Groups 1⋆

Tom is trying to do a group activity in his next discussion session. He wants to form groups of size 5 or 6.

(a) Show Tom that if there are 23 students attending his discussion, he will be able to split the students into groups of 5 or 6.

(b) In fact, there is some cutoff $p \in \mathbb{N}$ where $\forall n \geq p$, n students can be split into groups of 5 or 6. Find the smallest possible value of p.

(c) Now prove to Tom that if at least p students attends his discussion, he can successfully split the students in to groups of 5 or 6.

Solution:

(a) $23 = 5 + 6 + 6 + 6$

(b) The smallest value of k is 20

(c) Proof by strong induction
Let $P(n)$ be the statement "n students can be split into groups of 5 or 6".

Inductive step: Assume that $P(j)$ is true for all $20 \leq j \leq k$, for some $k \geq 24$. Want to show that $k+1$ students can be split into groups of 5 or 6.
Since $k \geq 24$, $k - 4 \geq 20$, thus, by our IH, $k - 4$ students can be split into groups of 5 or 6. Thus, we can create an other group of 5 from the remaining 5 students. In other words, if we can create a groups of 5 and b groups of 6 with $k - 4$ students, we will be able to create $a + 1$ groups of 5 and b groups of 6 with $k + 1$ students.

Base cases:

- $n = 20 : 20 = 5 + 5 + 5 + 5$
- $n = 21 : 21 = 5 + 5 + 5 + 6$
- $n = 22 : 22 = 5 + 5 + 6 + 6$
- $n = 23 : 23 = 5 + 6 + 6 + 6$
- $n = 24 : 24 = 6 + 6 + 6 + 6$

3 Recurrence Relations

3.1 Definitions

Recurrence Relation: A recurrence relation is an equation that defines each term of a sequence as a function of previous term(s).

5. Lobster Recurrence

A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years. Find a recurrence relation for $L(n)$, where $L(n)$ is the numbers of lobsters caught in year n, under the assumption for this model.

Solution: The number of lobsters caught in the previous year is $L(n-1)$ and the number 2 years prior is $L(n-2)$. The average of the two is therefore $\frac{L(n-1)+L(n-2)}{2}$. Thus, the recurrence relation is, $L(n) = \frac{L(n-1)+L(n-2)}{2}$

6. Stair Climbing

(a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.

(b) What are the initial conditions?

Solution:

(a) Let $a(n)$ be the number of ways to climb n stairs. In order to climb n stairs, a person must either start with a step of one stair and then climb $n-1$ stairs (and this can be done in $a(n-1)$ ways), start with a step of two stairs and then climb $n-2$ stairs (and this can be done in $a(n-2)$ ways), or start with a step of three stairs and then climb $n-3$ stairs (and this can be done in $a(n-3)$ ways). From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3 : a(n) = a(n-1) + a(n-2) + a(n-3)$.

(b) $a(0) = 1, a(1) = 1, a(2) = 2$. There is one way to climb up zero stairs (do nothing), one way to go up one stair (one step), and two ways to go up two stairs (one step twice or two step once).