W24

# EECS 203 Discussion 5a

## Mathematical Induction & Exam 1 Review

# Upcoming Exam

- **Exam 1** is on **Monday, February 19th** from **7:00 - 9:00 PM**!
- **Exam Review Sessions**
  - **Sat, February 17th, 1-4 PM** in CHRYS 220
    - **Topics:** Propositional Logic + Predicates and Quantifiers
  - **Sun, February 18th, 1-4 PM** in CHRYS 220
    - **Topics:** Proof Methods + Sets
- If you have a time conflict, contact the course staff **ASAP**!
- Practice exam questions have been released on Canvas!
  - They can be found on via **Files -> Practice Exams -> Exam 1**
  - See pinned Piazza post @448 for practice exam walkthrough videos

# Upcoming Homework

- Homework/Groupwork 5 will be due **Mar. 7th – AFTER SPRING BREAK**
  - **Don’t forget to match pages!**
  - Please note as soon as you press submit you’ve successfully submitted by the deadline. **You can still match pages** with no rush without adding to your submission time.
- Groupwork
  - Groupwork can be done alone, but the problems tend to be more difficult, and the goal is for you to puzzle them out with others!
  - Your discussion section is a great place to find a group!
  - There is also a pinned Piazza thread for searching for homework groups.

# Mathematical Induction

# Mathematical Induction

We want to show some statement **P(n)** is true for all integers **$n \geq c$**.

- **Base Case**
  - First, show that the statement **P(c)** is true for some initial value **c**.
- **Inductive Step**
  - Next, show that if **P(k)** is true for an arbitrary integer **$k \geq c$**, then **P(k+1)** is also true.
  - In other words, we want to prove the implication **$P(k) \rightarrow P(k+1)$**.
  - Since k is arbitrary, we start this step by assuming that **P(k)** is true.
  - When you assume **P(k)**, it's called the **inductive hypothesis**.
- **That's it!**
  - You've proven that **$\forall (n \geq c)\ P(n)$**, as desired.
  - Since **P(c)** is true and **P(k)** implies **P(k+1)**, we therefore have:
    **$P(c) \rightarrow P(c+1) \rightarrow P(c+2) \rightarrow P(c+3) \rightarrow P(c+4) \ldots$**

# Problem 1

## 1. Bandar's Blunder ⋆

Bandar writes a proof for the following statement:

$$n! > n^2 \text{ for all } n \geq 4.$$

His proof is incorrect, and it's your task to help him identify his mistake!

**Proof:**

**Inductive step:**

Let $k$ be arbitrary. Assume $P(k) : k! > k^2$. We need to show $P(k+1) : (k+1)! > (k+1)^2$

$$\begin{aligned}
(k+1)! &= (k+1) \cdot k! \\
&> (k+1) \cdot k^2 && \text{(By the Inductive Hypothesis)} \\
&= (k+1)(k \cdot k) \\
&\geq (k+1)(2 \cdot k) && \text{(Because } k \geq 2\text{)} \\
&= (k+1)(k+k) \\
&\geq (k+1)(k+1) && \text{(Because } k \geq 1\text{)} \\
&= (k+1)^2
\end{aligned}$$

This proves $(k+1)! > (k+1)^2$.

**Base Case:**

Prove $P(0) : 0! > 0^2$, $0! = 1 > 0^2 = 0$

Thus by mathematical induction, $n! > n^2$ for all $n \geq 0$.

What is wrong with Bandar's proof?

# Problem 2

## 2. Sum Mathematical Induction

Using induction, prove that for all integers $n \geq 1$:

$$\sum_{r=1}^{n} (r+1) \cdot 2^{r-1} = n \cdot 2^{n}$$

# Exam 1 Review

# Tautology, Contradiction, Satisfiability (Discussion 1b)

- **Tautology:** A compound proposition that is **always true** regardless of its input values

- **Contradiction:** A compound proposition that is **always false** regardless of its input values

- **Satisfiable:** A compound proposition is satisfiable if it **can be true** (there is at least one set of inputs that makes the proposition true)

# Problem 3

## 3. REVIEW: Satisfiability ⋆

Determine whether each of these compound propositions is satisfiable.

(a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$

(b) $(p \to q) \wedge (p \to \neg q) \wedge (\neg p \to q) \wedge (\neg p \to \neg q)$

# Quantifiers (Discussion 2)

- **Nested Quantifiers:** A nested quantifier is a quantifier that involves the use of two or more quantifiers to quantify a compound proposition P(x,y). In nested quantifiers, order matters…
  - **P(x,y):** some statement about x and y
  - **Example:** $\forall x \exists y \, P(x,y)$ is different from $\exists y \forall x \, P(x,y)$
    - **$\forall x \exists y$ P(x,y):** "For all x, there exists y such that…"
    - **$\exists y \forall x$ P(x,y):** "There exists y such that for all x…"

# Problem 4

## 4. REVIEW: Nested Quantifier Translations

Let P($x$, $y$) be the statement "Student $x$ has taken class $y$," where the domain for $x$ consists of all students in your class and for $y$ consists of all computer science courses at your school. Express each of these quantifications in English.

a) $\exists x \exists y P(x, y)$

b) $\exists x \forall y P(x, y)$

c) $\forall x \exists y P(x, y)$

d) $\exists y \forall x P(x, y)$

e) $\forall y \exists x P(x, y)$

f) $\forall x \forall y P(x, y)$

# Proof Methods (Discussion 2)

- **Direct Proof:**

  Proves $p \rightarrow q$ by showing **$p \rightarrow$ stuff $\rightarrow q$**

# Even and Odd (Discussion 2)

- **Even:** An integer **$x$** is even iff there exists an integer k such that **$x = 2k$**
- **Odd:** An integer **$x$** is odd iff there exists an integer k such that **$x = 2k + 1$**

# Problem 5

## 5. REVIEW: Direct Proof

Use a direct proof to show that the product of two odd numbers is odd.

# Proof Methods (Discussion 3)

- **Direct Proof:**
  Proves $p \to q$ by showing $\mathbf{p \to stuff \to q}$
- **Proof by Contradiction:**
  Proves p by showing $\mathbf{\neg p \to F}$
  To prove $p \to q$, assume the negation: $\mathbf{\neg(p \to q) \equiv \neg(\neg p \lor q) \equiv p \land \neg q}$
  **“Seeking contradiction, assume that…”**

# Problem 6

**6. REVIEW: Proof by Contradiction ⋆**

Prove that for all integers $n$, if $n^2 + 2$ is even, then $n$ is even using a proof by contradiction.

Note: When using proof by contradiction to prove p → q, there are multiple places where one could introduce the assumption that is "seeking contradiction":

1. "Seeking contradiction, assume the negation of the entire claim, including negating the quantifier…"
2. "Let x be an arbitrary element of the domain. Seeing contradiction, assume p and not(q). [ie negate the if-then] …"
3. "Let x be an arbitrary element of the domain. Assume p [ie begin direct proof of if p then q]. Seeking contradiction, assume not(q). …"

# Proof Methods (Discussion 3)

- **Direct Proof:**
  Proves $p \rightarrow q$ by showing $\mathbf{p \rightarrow stuff \rightarrow q}$
- **Proof by Contradiction:**
  Proves p by showing $\mathbf{\neg p \rightarrow F}$
  To prove $p \rightarrow q$, assume the negation: $\mathbf{\neg(p \rightarrow q) \equiv \neg(\neg p \lor q) \equiv p \land \neg q}$
  **"Seeking contradiction, assume that…"**
- **Proof by Contrapositive:**
  Proves $p \rightarrow q$ by showing $\mathbf{\neg q \rightarrow stuff \rightarrow \neg p}$

# Problem 7

## 7. REVIEW: Proof by Contrapositive ⋆

Prove that for all integers $x$ and $y$, if $xy^2$ is even, then $x$ is even or $y$ is even.

# Proof Methods (Discussion 4)

- **Direct Proof:**
  Proves $p \rightarrow q$ by showing $\mathbf{p \rightarrow stuff \rightarrow q}$
- **Proof by Contradiction:**
  Proves p by showing $\mathbf{\neg p \rightarrow F}$
  To prove $p \rightarrow q$, assume the negation: $\mathbf{\neg(p \rightarrow q) \equiv \neg(\neg p \lor q) \equiv p \land \neg q}$
  **"Seeking contradiction, assume that…"**
- **Proof by Contrapositive:**
  Proves $p \rightarrow q$ by showing $\mathbf{\neg q \rightarrow stuff \rightarrow \neg p}$
- **Proof by Cases:**
  Break p into cases and show that each case implies q (in which case $p \rightarrow q$).
  Make sure to prove q for every possible case!
  $\mathbf{p \rightarrow p_1 \lor p_2 \lor \ldots \lor p_n \rightarrow q}$

# Problem 8

## 8. REVIEW: Proof by Cases/Disproofs $\star$

a) Prove or Disprove that for all integers $n$, $n^2 + n$ is even

b) Prove or Disprove that for all integers $a$ and $b$, $\frac{a}{b}$ is a rational number.

# Problem 9

## 9. REVIEW: Sets ⋆

Let our domain $U$ be the set of the 26 lowercase letters in the English alphabet. Let $A = \{i, a, n\}, B = \{s, h, u, b\}, C = \{i, s, a, b, e, l\}$. Compute the following, where complements are taken within $U$. Write your answers in list notation.

*Hint:* For parts (b) and (c), simplifying the expressions using set identities may make the sets quicker to compute.