EECS 203: Discrete Mathematics
Winter 2024
Discussion 5b Notes

1 Weak Induction

1.1 Conceptual Understanding

Base Case: The part of the inductive proof which directly proves the predicate for the *first* value in the domain (generally n_0). The base case does not rely on $P(k)$ for any other value of k. Often this will be $P(0)$ or $P(1)$

Inductive Hypothesis: The assumption we make at the beginning of the inductive step. The inductive hypothesis assumes that the predicate holds for some *arbitrary* member of the domain

Inductive Step: The proof which shows that the predicate holds for the "next" value in the domain. The inductive step should make use of the inductive hypothesis.

Weak Induction: Weak Induction is a proof method used to prove a predicate $P(n)$ holds for "all" $n \geq n_0$. Often "all" n is $\mathbb{N}$ or $\mathbb{Z}^+$, but the desired domain of n varies by problem.

1.2 Inductive Conclusions

Suppose that $P(n)$ is an unknown predicate. Determine for which positive integers n the statement $P(n)$ must be true, and justify your answer, if

a) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then P(n + 2) is true.

b) $P(1)$ and $P(2)$ are true, and for all positive integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

c) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(2n)$ is true.

d) $P(1)$ is true, and for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.

1.3 Fill in the blanks

Prove that $2^n > n^2$ for all integers n greater than 4.

Define $P(n)$: __________
Inductive Step: Let ____________________. Assume __________.
We will show ____.

$$2^{k+1} = \underline{\hspace{4cm}}$$

$$= (k+1)^2$$

Base Case: $P($__________$)$

Conclusion: By weak induction, ____.

1.3.1 Fractions

Prove that, whenever n is a positive integer,

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot \ldots \cdot 2n}.$$

1.4 Weak Induction - Sets Edition

Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

2 Strong Induction

2.1 Definitions

Strong Induction: Similar to weak induction, but the inductive step relies on ALL previous cases rather than just the case that came before it. So, you must show:
$\forall k[(P(c) \wedge P(c+1) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)]$, where $k \geq$ some starting case.

You must also prove bases cases. There is often more than one base case for strong induction depending on what information you need to do your inductive step. The number of base cases is the largest c for which you need to assume P(k-c) in order to prove P(k). For example, if your proof of P(k) uses the assumptions P(k-2) and P(k-5), then you would need 5 base cases.

1. Faulty Induction

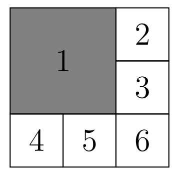
Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

Basis Step: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all non-negative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k+1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

2. Squares Strong Induction

Prove that a square can be subdivided into any number of squares $n \geq 6$. Note that sub-squares don't need to be the same size. For example, here's how you would subdivide a square into 6 squares.



3. Jigsaw Puzzle Induction

A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n-1$ moves are required to assemble a puzzle with n pieces.

4. Forming Discussion Groups 1$\star$

Tom is trying to do a group activity in his next discussion session. He wants to form groups of size 5 or 6.

(a) Show Tom that if there are 23 students attending his discussion, he will be able to split the students into groups of 5 or 6.

(b) In fact, there is some cutoff $p \in \mathbb{N}$ where $\forall n \geq p$, n students can be split into groups of 5 or 6. Find the smallest possible value of p.

(c) Now prove to Tom that if at least p students attends his discussion, he can successfully split the students in to groups of 5 or 6.

3 Recurrence Relations

3.1 Definitions

Recurrence Relation: A recurrence relation is an equation that defines each term of a sequence as a function of previous term(s).

5. Lobster Recurrence

A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years. Find a recurrence relation for $L(n)$, where $L(n)$ is the numbers of lobsters caught in year n, under the assumption for this model.

6. Stair Climbing

(a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.

(b) What are the initial conditions?