W24

EECS 203 Discussion 4

Proof by Cases, Intro to Sets

Upcoming Exam

- **Exam 1** is on **Monday, February 19th** from **7:00 - 9:00 PM**!
- If you have a time conflict, contact the course staff **ASAP**!
- Practice exam questions have been released on Canvas!
    - They can be found on via **Files -> Practice Exams -> Exam 1**

Upcoming Homework

- Homework/Groupwork 4 will be due **Feb. 15th**
  - **Don't forget to match pages!**
  - Please note as soon as you press submit you've successfully submitted by the deadline. **You can still match pages** with no rush without adding to your submission time.
- Groupwork
  - Groupwork can be done alone, but the problems tend to be more difficult, and the goal is for you to puzzle them out with others!
  - Your discussion section is a great place to find a group!
  - There is also a pinned Piazza thread for searching for homework groups.

Proof Methods Overview

Making a Valid Argument (Writing a Proof)

- **Argument/Proof:** An **argument** for a statement S is a sequence of statements ending with S. S is called the **conclusion**. An argument starts with some beginning statements you assume are true, called the **premises**.
- **Valid Argument/Proof:** An argument is **valid** if every statement after the premises is implied ($\rightarrow$) by the some combination of the statements before it.
    - Whenever the premises are true, the conclusion must be true.

- Today we will be discussing word-style proofs

Proof Methods

- **Direct Proof:** Proves $p \rightarrow q$ by showing

  $p \rightarrow \text{stuff} \rightarrow q$

- **Proof by Contraposition:** Proves $p \rightarrow q$ by showing

  $\neg q \rightarrow \text{stuff} \rightarrow \neg p$

- **Proof by Contradiction:** Proves $p \rightarrow q$ by showing

  $(p \land \neg q) \rightarrow F \rightarrow \neg(p \land \neg q) \equiv \neg p \lor q \equiv p \rightarrow q$

- **Proof by Cases:** Proves $p \rightarrow q$ by showing

  $p \rightarrow p1 \lor p2 \lor \ldots \lor pn \rightarrow q$

Some Methods of Proving $p \rightarrow q$:

- **Direct Proof:**
  Proves $p \rightarrow q$ by showing **$p \rightarrow$ stuff $\rightarrow q$**

- **Proof by Contraposition:**
  Proves $p \rightarrow q$ by showing **$\neg q \rightarrow$ stuff $\rightarrow \neg p$**
  (Once you show $\neg q \rightarrow \neg p$, you can immediately conclude $p \rightarrow q$ by contraposition)

- **Proof by Contradiction:**
  Assume p and $\neg q$ are true. Derive a contradiction (F), by arriving at a mathematically incorrect statement (ex: $0 = 2$) or two statements that contradict each other ($x = y$ and $x \neq y$). Therefore, $p \rightarrow q$.
  $(p \wedge \neg q) \rightarrow F \rightarrow \neg(p \wedge \neg q) \equiv \neg p \vee q \equiv p \rightarrow q$

- **Proof by Cases:**
  Break p into cases and show that each case implies q (in which case $p \rightarrow q$).
  $p \rightarrow p_1 \vee p_2 \vee \ldots \vee p_n \rightarrow q$

Proof by Cases

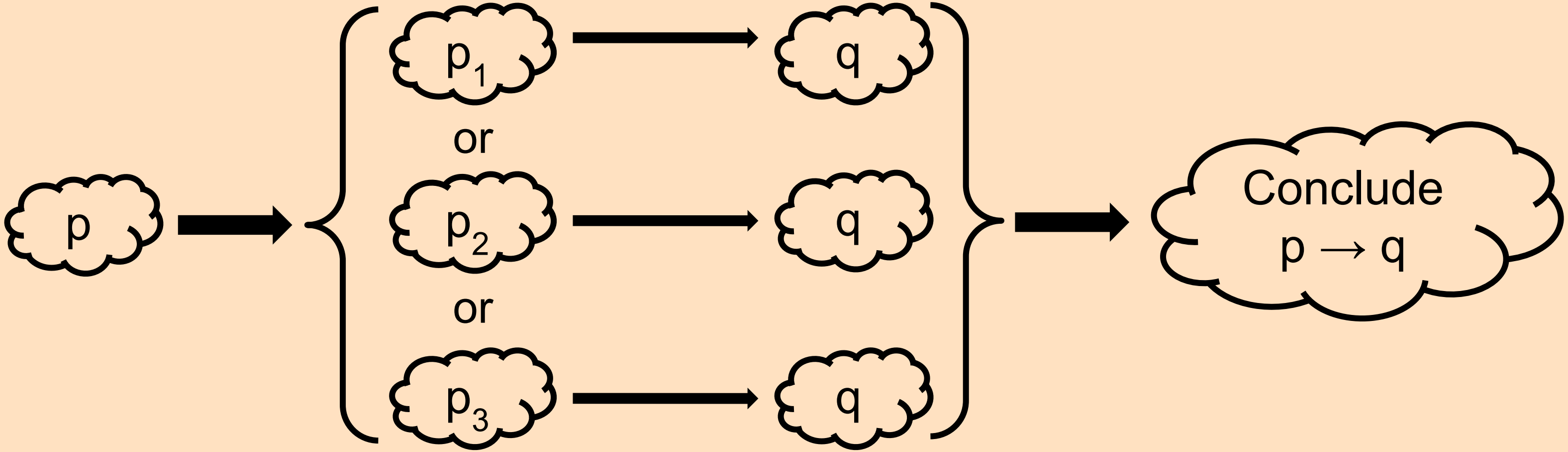
Proof by Cases

Break **p** into cases and show that each case implies **q** (in which case **p** → **q**).

$$p \to p_1 \lor p_2 \lor \ldots \lor p_n \to q$$

$p_1 \lor p_2 \lor \ldots \lor p_n$ should cover all possible cases for **p**.

- We break our statement into all possible cases
- We show that each case leads to the conclusion we want



Problem 1

1. Proof by Cases/Contradiction $\star$

Prove that there is no rational solution to the equation $x^3 + x + 1 = 0$. **Hint:** Use the fact that 0 is an even number.
You can use the following lemmas without proving:

- Odd $\times$ Even $=$ Even
- Odd $\times$ Odd $=$ Odd
- Even $\times$ Even $=$ Even
- Odd $+$ Even $=$ Odd
- Odd $+$ Odd $=$ Even
- Even $+$ Even $=$ Even

Solution

1. Proof by Cases/Contradiction $\star$

Prove that there is no rational solution to the equation $x^3 + x + 1 = 0$. **Hint:** Use the fact that 0 is an even number.

Seeking contradiction, suppose there is a rational solution. Let a solution be $\frac{a}{b}$, with a, b in reduced form.
Then we know that $\frac{a^3}{b^3} + \frac{a}{b} + 1 = 0 \iff a^3 + ab^2 + b^3 = 0$.
Since the RHS is even, LHS should be even as well.

Case 1: a, b both odd.
Then we have LHS = odd + odd + odd = odd.
Case 2: a is odd, b is even.
Then we have LHS = odd + even + even = odd.
Case 3: a is even, b is odd.
(note that WLOG does not apply here since a, b are not symmetric; there is a term ab^2).
Then we have LHS = even + even + odd = odd.
Case 4: a, b are both even.
This cannot occur since a, b is in reduced form.

Each case results in LHS being odd which is a contradiction if LHS = 0. Thus we have proved by contradiction that the equation $x^3 + x + 1$ has no solution in $\mathbb{Q}$.

- Odd $\times$ Even = Even
- Odd $\times$ Odd = Odd
- Even $\times$ Even = Even
- Odd + Even = Odd
- Odd + Odd = Even
- Even + Even = Even

Problem 2

2. Prime Proof $\star$

Show that for any prime number p, $p^2 + 11$ is composite (not prime). Recall that a prime p is defined to be a natural number ≥ 2 such that p and 1 are the only factors that divide p.

Solution

2. Prime Proof $\star$

Show that for any prime number p, $p^2 + 11$ is composite (not prime). Recall that a prime p is defined to be a natural number ≥ 2 such that p and 1 are the only factors that divide p.

We can consider two cases: either p is even, or it is odd.

- Case 1: Consider the even primes, which is just $p = 2$. $p^2 + 11 = 15$, and $15 = 5 \cdot 3$ is composite.
- Case 2: Now we consider the odd primes, or any prime greater than 2. Since p is odd, we have $p = 2k + 1$ for some integer $k > 1$. Then

  $$p^2 + 11 = (2k + 1)^2 + 11 = 4k^2 + 4k + 12 = 2(2k^2 + 2k + 6).$$

  Hence, p^2+11 can be factored into 2 and $2k^2+2k+6$, therefore p^2+11 is composite.

We have exhausted all non-overlapping cases and proved that for all primes p, $p^2 + 11$ is composite.

Problem 3

3. Proving the Triangle Inequality

Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).

Solution

3. Proving the Triangle Inequality

Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).

This is a proof by cases. There are 4 cases to consider:

- x and y are both nonnegative
- x and y are both negative
- $x \geq 0$, $y < 0$, $x \geq -y$
- $x \geq 0$, $y < 0$, $x < -y$

Since x and y play symmetric roles (you can switch the values of x and y without impacting the validity of the triangle inequality), we can assume without loss of generality (WLOG) for the last two cases that $x \geq 0$ and $y < 0$.

- Case 1: If x and y are both nonnegative, then $|x| + |y| = x + y = |x + y|$.
- Case 2: If x and y are both negative, then $|x| + |y| = (-x) + (-y) = -(x + y) = |x + y|$.
- Case 3: If $x \geq 0$ and $y < 0$ and $x + y \geq 0$, then $|x| + |y| = x + (-y)$ is some number greater than x. $|x + y|$ is some positive number less than x since y is negative. Thus, $|x| + |y| \geq x \geq |x + y|$.
- Case 4: If $x \geq 0$ and $y < 0$ and $x + y < 0$, then $|x| + |y| = x + (-y)$ is some number greater than $-y$. $|x + y| = -(x + y) = (-x) + (-y)$ which is some positive number less than or equal to $-y$, since $-x$ is nonpositive. Thus, we have $|x| + |y| \geq -y \geq |x + y|$.

We have now proved for all cases that the triangle inequality is valid. This example is purposely lengthy to show in full detail a proof by cases.

Alternate Solution

3. Proving the Triangle Inequality

Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).

$$a + b \leq |a| + b \leq |a| + |b|$$

$$-(a + b) = -a - b \leq |-a| - b \leq |-a| + |-b| = |a| + |b|$$

Since $|a + b| = a + b$ or $-(a + b)$, and both cases are $\leq |a| + |b|$, we have proven the triangle inequality.

Intro to Sets

Set Terminology

- **Set:** A set is an unordered collection of distinct objects
- **Universe:** In set theory, a universe is a collection that contains all the entities one wishes to consider in a given situation.
- **Union $S \cup T$:** The set containing the elements that are in *S* or *T:*

  $$S \cup T = \{x \mid x \in S \lor x \in T\}$$

- **Intersection $S \cap T$:** The set containing the elements that are in *S* and *T:*

  $$S \cap T = \{x \mid x \in S \land x \in T\}$$

- **Complement $\bar{A}$ of A:** The set containing the elements that are in the universe *U* but not in *A*:

  $$\bar{A} = \{x \mid x \in U \land x \notin A\}$$

- **Minus $S - T$:** The set containing the elements that are in *S* but not in *T:*

  $$S - T = \{x \mid x \in S \land x \notin T\}$$

Set Terminology

- **Subset:** The set A is a subset of B if and only if every element of A is also an element of B. Denoted $A \subseteq B$. Note that A and B may be the same set.

  $$A \subseteq B \text{ iff } \forall x\, [x \in A \rightarrow x \in B]$$

- **Proper Subset:** The set A is a proper subset of B if and only if A is a subset of B and $A \neq B$. That is, A is a subset of B and there is at least one element of B that is not in A.

  $$A \subsetneq B.\ A \subsetneq B \text{ iff } \forall x\, [x \in A \rightarrow x \in B] \wedge (A \neq B)$$

- **Disjoint:** The sets A and B are disjoint if and only if they do not share any elements

- **Empty Set:** The empty set, denoted $\varnothing$ or **{}**, is the unique set having no elements.

Set Terminology

- **Inclusion−Exclusion Principle:** The inclusion-exclusion principle states that the size of the union of two sets is equal to the sum or their sizes minus the size of their intersection:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- **Power Set:** The power set of a set S is the set of all subsets of S. $P(S)$ denotes the power set of S:

$$P(S) = \{T \mid T \subseteq S\}$$

- **Cardinality:** The number of elements in a set. The cardinality of a set S is denoted by $|S|$.

- **Cartesian Product:** $A \times B$ is the set of all ordered pairs of elements (a, b) where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

Problem 4

4. Set Exploration ⋆

a) What is $|\emptyset|$?

b) Let $A = \{1, 2, 3\}$, $B = \{\emptyset\}$, $C = \{\emptyset, \{\emptyset\}\}$, $D = \{4, 5\}$, and $E = \{\emptyset, 5\}$.

i. Is $\emptyset \in A$?

ii. Is $\emptyset \subseteq A$?

iii. Is $\emptyset \in B$?

iv. Is $\emptyset \subseteq B$?

v. Is $\emptyset \in C$?

vi. Is $\emptyset \subseteq C$?

vii. What is $A \cap D$?

viii. What is $B \cap C$?

ix. What is $B \cap E$?

x. What is $|B|$, $|C|$, $|E|$?

c) Let A and C be the sets defined above.

i. What is $P(A)$?

ii. What is $P(C)$?

iii. Find a formula for the size of the power set of S, $|P(S)|$, in terms of $|S|$.

iv. What is $C \times A$?

v. What is A^2? $(A^2 = A \times A)$

vi. Find a formula for the size of the Cartesian product of A and B, $|A \times B|$ in terms of $|A|$ and $|B|$.

Solution

a) $|\emptyset| = 0$

b)
i. No, $\emptyset$ is not an element of A, you would see it in A if it was.
ii. Yes, $\emptyset$ is a subset of all sets. All elements of $\emptyset$ (none) are elements of A: $\{\} \subseteq \{1, 2, 3\}$
iii. Yes, $\emptyset \in \{\emptyset\}$
iv. Yes, $\emptyset$ is a subset of all sets.
v. Yes, $\emptyset \in \{\emptyset, \{\emptyset\}\}$
vi. Yes, $\emptyset$ is a subset of all sets.
vii. $A \cap D = \emptyset$
viii. $B \cap C = \{\emptyset\}$
ix. $B \cap E = \{\emptyset\}$
x. $|B| = 1$, $|C| = 2$, $|E| = 2$

c)
i. $\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\Big\}$
ii. $\mathcal{P}(C) = \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\Big\}$
iii. $|\mathcal{P}(S)| = 2^{|S|}$
iv. $C \times A = \Big\{(\emptyset, 1), (\emptyset, 2), (\emptyset, 3), (\{\emptyset\}, 1), (\{\emptyset\}, 2), (\{\emptyset\}, 3)\Big\}$
v. $A^2 = A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
vi. $|A \times B| = |A| \times |B|$

4. Set Exploration ⋆

a) What is $|\emptyset|$?

b) Let $A = \{1, 2, 3\}$, $B = \{\emptyset\}$, $C = \{\emptyset, \{\emptyset\}\}$, $D = \{4, 5\}$, and $E = \{\emptyset, 5\}$.

i. Is $\emptyset \in A$?
ii. Is $\emptyset \subseteq A$?
iii. Is $\emptyset \in B$?
iv. Is $\emptyset \subseteq B$?
v. Is $\emptyset \in C$?
vi. Is $\emptyset \subseteq C$?
vii. What is $A \cap D$?
viii. What is $B \cap C$?
ix. What is $B \cap E$?
x. What is $|B|$, $|C|$, $|E|$?

c) Let A and C be the sets defined above.

i. What is $P(A)$?
ii. What is $P(C)$?
iii. Find a formula for the size of the power set of S, $|P(S)|$, in terms of $|S|$.
iv. What is $C \times A$?
v. What is A^2? $(A^2 = A \times A)$
vi. Find a formula for the size of the Cartesian product of A and B, $|A \times B|$ in terms of $|A|$ and $|B|$.

Problem 5

5. Double Subset Equality $\star$

Prove the set equivalence: $A - (B \cap C) = (A - B) \cup (A - C)$

Solution

5. Double Subset Equality $\star$

Prove the set equivalence: $A - (B \cap C) = (A - B) \cup (A - C)$

First, let's show $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.
Let x be an arbitrary element of the domain.
Assume $x \in A - (B \cap C)$

- $x \in A \wedge x \in \overline{(B \cap C)}$
- $x \in A \wedge (x \notin B \vee x \notin C)$ (using DeMorgan's Law)
- $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$ (using the distributive property)
- $(x \in A - B) \vee (x \in A - C)$
- $x \in (A - B) \vee (A - C)$

Therefore, $A - (B \cap C) \subseteq (A - B) \cup (A - C)$

Now we will show $(A - B) \cup (A - C) \subseteq A - (B \cap C)$
Let x be an arbitrary element of the domain.
Assume $x \in (A - B) \cup (A - C)$

Case 1: $x \in A - B$

- $x \in A \wedge x \notin B$
- $x \in A$
- $x \notin B$
- $x \notin B \vee x \notin C$
  (**Note:** we can add whatever we want with an or statement, since we know the first half is always true!)
- $x \in A \wedge (x \notin B \vee x \notin C)$

Case 2: $x \in A - C$

- $x \in A \wedge x \notin C$
- $x \in A$
- $x \notin C$
- $x \notin B \vee x \notin C$
- $x \in A \wedge (x \notin B \vee x \notin C)$

Now we need to use the conclusions of our cases:

- In both cases, we have $x \in A \wedge (x \notin B \vee x \notin C)$
- $x \in A \cap (\overline{B} \cup \overline{C})$
- $x \in A \cap \overline{(B \cap C)}$ (using DeMorgan's Law)
- $x \in A - (B \cap C)$

Therefore, $(A - B) \cup (A - C) \subseteq A - (B \cap C)$

Since each side is a subset of the other, we can say $(A - B) \cup (A - C) = A - (B \cap C)$

Problem 6

6. Subset Proofs

Let A, B, and C be sets. Prove that

a) $(A \cap B \cap C) \subseteq (A \cap B)$

b) $(A - B) - C \subseteq A - C$

Solution

6. Subset Proofs

Let A, B, and C be sets. Prove that

a) $(A \cap B \cap C) \subseteq (A \cap B)$

b) $(A - B) - C \subseteq A - C$

a) Consider an arbitrary $x \in (A \cap B \cap C)$
- By the definition of intersection, we have $(x \in A) \wedge (x \in B) \wedge (x \in C)$
- So we have $(x \in A) \wedge (x \in B)$
- Thus we have, $x \in (A \cap B)$

Therefore, $(A \cap B \cap C) \subseteq (A \cap B)$ by definition.

b) Consider an arbitrary $x \in (A - B) - C$
- By definition of set difference, we know that $(x \in A - B) \wedge (x \notin C)$
- Since $x \in A - B$, we know that $(x \in A) \wedge (x \notin B)$
- Thus, $(x \in A) \wedge (x \notin B) \wedge (x \notin C)$
- Then, we have $(x \in A) \wedge (x \notin C)$
- Finally, by definition of set minus, we have $x \in A - C$

Therefore, we have shown that $(A - B) - C \subseteq A - C$

Problem 7

7. Power Sets

Can you conclude that $A = B$ if A and B are two sets with the same power set?

Solution

7. Power Sets

Can you conclude that $A = B$ if A and B are two sets with the same power set?

The union of all the sets in the power set of a set X must be exactly X . In other words, we can recover X from its power set, uniquely. Therefore the answer is yes.

We can also show this is true through a proof by contraposition. Let A and B be sets and assume $A \neq B$. Because A and B are not equal, WLOG there exists an element $x \in A$ such that $x \notin B$. Therefore we conclude that $\{x\} \in \mathcal{P}(A)$, and that $\{x\} \notin \mathcal{P}(B)$, therefore $\mathcal{P}(A) \neq \mathcal{P}(B)$. Thus by the contrapositive, the original statement is true.

Problem 8

8. More Power Sets ⋆

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$

b) $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Solution

8. More Power Sets ⋆

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$

b) $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

a) The power set of every set includes at least the empty set, so the power set cannot be empty. Thus $\emptyset$ is not the power set of any set.

b) This is the power set of $\{a\}$.

c) We know that the power set a set of size n has 2^n elements, but this set has three elements. Since 3 is not a power of 2, this set cannot be the power set of any set. Set cardinality aside, the set $\{\emptyset, a\}$ may come to mind, but $P(\{\emptyset, a\}) = \{\emptyset, \{\emptyset\}, \{a\}, \{\emptyset, a\}\}$.

d) This is the power set of $\{a, b\}$.

Problem 9

9. Power Set of a Cartesian Product

Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

Solution

9. Power Set of a Cartesian Product

Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

Solution: This is not true. The simplest counterexample is to let $A = B = \emptyset$ (or any two disjoint sets). Then $A \times B = \emptyset$ and $\mathcal{P}(A \times B) = \{\emptyset\}$, whereas $\mathcal{P}(A) = \mathcal{P}(B) = \{\emptyset\}$ and $\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset)\}$. Thus, $\mathcal{P}(A \times B) \neq \mathcal{P}(A) \times \mathcal{P}(B)$