EECS 203 Exam 2 Review

Day 1

Today’s Review Topics

- Modular Arithmetic
- Induction
    - Weak Induction
    - Strong Induction

Divisibility and Modular Arithmetic

Divisibility Recap

- Divisibility: a | b iff $\exists c\ (b = ac)$ 0|3? 3|0?
- Prime Number p>1: p is only divisible by 1 and itself

Two types of “mods”

- **a ≡ b (mod m)** is a predicate involving three numbers. Sometimes we leave out the parens; **≡** is the important part
- **a mod m** is the remainder after dividing a by m. This is always an integer between 0 and m–1. (a%m in C++)

Modular

- We can write b = na + r (n is some int and 0<=r<a)
- a ≡ b (mod m) “a and b have same remainder upon division by m”?
- More about modular arithmetic:

- Suppose a ≡ b (mod m) and c ≡ d (mod m).
- **Claim:** a+c ≡ b+d (mod m) (***Addition works!***)
- **Claim:** a–c ≡ b–d (mod m) (***Subtraction works!***)
- **Claim:** ac ≡ bd (mod m) (***Multiplication works!***)
    - Simplify the bases of exponents and constant addition/multiplication terms
    - Split exponents using exponent rules

Mods Question 1

Let $x \equiv 3 \pmod{12}$, $y \equiv 11 \pmod{21}$, and $z \equiv 3 \pmod 4$. Which of the following statements must be true?

(a) $x + y \equiv 2 \pmod 3$

(b) $x + z \equiv 3 \pmod 4$

(c) $x - y \equiv -8 \pmod{12}$

(d) $x \cdot y \equiv 12 \pmod{21}$

(e) $x \cdot z \equiv 1 \pmod 4$

Mods Question 2

Find c with $0 \leq c < 11$ such that $c \equiv 14^6 + 22^{203} \pmod{11}$

Induction

Cheat Sheet

//	Induction	Strong Induction
Define Predicate	Let P(n) be the statement...	Let P(n) be the statement...
Basis Step	Form your base case $P(x)$ (it can be more than one)	Form your base case(s) $P(x), P(x+1), \ldots$ (usually more than one)
Inductive Hypothesis	$P(x)$ is true	$P(j)$ is true for all j such that smallest base case$\leq j \leq k$
Inductive Step	$P(x) \rightarrow P(x+1)$	$P(i) \land P(i+1) \land \cdots P(k) \rightarrow P(k+1)$ i =smallest base case $P(j) \rightarrow P(k+1)$, base$\leq j \leq k$

Induction Recap

- Two types of Induction
  - Weak Induction
  - Strong Induction
- Base Case(s), Inductive Hypothesis, Inductive Step
- “Mathematical ladder”

Weak Induction

Weak Induction

1. Show that the expression/statement is true for the base case (often in the form of n = 0 or n = 1).
2. Assume that the expression is true for some arbitrary element k in the domain appropriate for the problem.*
3. Show that the statement is true for P(k+1) when P(k) is true. (i.e P(k) -> P(k+1))

* The domain is often $\mathbf{Z}^{+}$, but it may be different.

Induction 1

Prove that the following equality holds for all positive integers n:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

“Inequality” Induction

- Hardest part is substituting using inequality
- Manipulate the expressions to reduce them to desired form
- Consider things like “is the product greater than the sum?”

Induction 2

Prove using induction that $1 + (3^0 + 3^1 + 3^2 + \cdots + 3^{n-1}) < 3^n$ for all $n \geq 1$.

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Induction 3

Prove using induction that

$$n^2 + n < 2^n, \qquad \text{for all integers } n \geq 5.$$

Every inequality in your proof should be justified by one of the following:

- The inductive hypothesis (IH)
- $k^i < k^j$ when $i < j$ because $k > 1$ (e.g., $k^2 < k^4$)
- $c \leq k$ when $c \leq 5$ because $k \geq 5$ (e.g., $3 \leq k$)

Induction 4

Prove that for all $n \geq 1$, the sum of the squares of the first 2n positive integers is given by the formula

$$1^2 + 2^2 + 3^2 + ... + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$$

Strong Induction

Strong Induction

- Similar to Weak Induction
- Major Differences
  - Possibly multiple base cases
  - Assumes all previous steps to be true
- Still has the same format as weak induction

Strong Induction 1

Prove that every integer $n \geq 12$ can be written as $n = 4a + 5b$ for some non-negative integer a, b using strong induction.

Strong Induction 2

Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. [Hint: For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. Note that when $(k + 1)$ is even, $(k + 1)/2$ is an integer.]

Have a great rest of the weekend!