EECS 203: Discrete Mathematics
Winter 2024
Discussion 3 Notes

1 Proof Styles

1.1 Proofs by Contraposition

Proof by Contraposition: Prove a conditional (in the form "if p, then q") by proving that proving that if q is false, then p must also be false. This is done by assuming the negation of q and concluding the negation of p

1.1.1 Proof by Contraposition $\star$

Prove that if $n^2 + 2$ is even, then n is even using a proof by contrapositive.

Solution: We will prove the contrapositive, that is: If n is odd, then $n^2 + 2$ is odd.

- Assume n is odd. Then we can write it as $n = 2k + 1$ for some integer k.
- This means $n^2 + 2 = (2k + 1)^2 + 2$.
  $= 4k^2 + 4k + 1 + 2$
  $= 2(2k^2 + 2k + 1) + 1$
  $= 2j + 1$, where j is an integer equal to $2k^2 + 2k + 1$
- Thus from the definition of an odd number, $n^2 + 2$ is odd.

This completes the proof of the contrapositive, and thus the original statement.

1.1.2 Proof by Contraposition II

Prove by contrapositive that if $a^2 + a + 2 \geq b^2 + b + 2$, then $a \geq b$, where a and b are positive integers.
You may use without proving:

1. $c < d$ and $e < f \rightarrow (c + e) < (d + f)$
2. $c < d$ and $e < f \rightarrow ce < df$, where c, d, e, f are positive integers

Solution:

- The contrapositive of the above statement is - If $a < b$, then $a^2 + a + 2 < b^2 + b + 2$.
- Assume $a < b$, where a and b are positive integers.
- $a^2 < b^2$ (Using lemma 2)
- $a^2 + a < b^2 + b$ (Using lemma 1)
- $a^2 + a + 2 < b^2 + b + 2$ (Adding 2 to both sides does not change the inequality)

Hence the contrapositive is true. This proves that if $a^2 + a + 2 \geq b^2 + b + 2$, then $a \geq b$

1.2 Proofs by Contradiction

Prove p is true by assuming $\neg p$, and arriving at a contradiction, i.e. a conclusion that we know is false.

When using a proof by contradiction to prove "if p is true then q is true", we assume that p is true and that q is false, and derive a contradiction. This shows us that if p is true, then q is true.

$$\neg(p \to q) \equiv (p \wedge \neg q) \to F \to \neg(p \wedge \neg q) \equiv (p \to q)$$

A simpler way to view this: Assume p is true and show that

$$(\neg q \to \mathrm{F}) \to q$$

1.2.1 Contraposition vs Contradiction

Show that for an integer n: if $n^3 + 5$ is odd, then n is even, using

a) a proof by contraposition.

b) a proof by contradiction.

Note: The algebra in either case is the same. You don't need to rewrite the algebra for part (b), just reformat your proof from (a) into a proof by contradiction.

Solution:

a) We will prove the contrapositive of the proposition, which is: "if n is odd, then n^3+5 is even".
Since n is odd, n can be written as $2k + 1$, where k is some integer. Then,

$$\begin{aligned} n^3 + 5 &= (2k + 1)^3 + 5 \\ &= (8k^3 + 12k^2 + 6k + 1) + 5 \\ &= 8k^3 + 12k^2 + 6k + 6 \\ &= 2(4k^3 + 6k^2 + 3k + 3) \end{aligned}$$

So $n^3 + 5 = 2m$, where m is the integer $4k^3 + 6k^2 + 3k + 3$. Because $n^3 + 5$ is two times some integer, we can say that $n^3 + 5$ is even.

b) We will use a proof by contradiction. Let $n^3 + 5$ be odd. *Seeking a contradiction*, assume that n is odd. Since n is odd, it can be written as $2k + 1$, where k is some integer. So

$$\begin{aligned}
n^3 + 5 &= (2k + 1)^3 + 5 \\
&= (8k^3 + 12k^2 + 6k + 1) + 5 \\
&= 8k^3 + 12k^2 + 6k + 6 \\
&= 2(4k^3 + 6k^2 + 3k + 3)
\end{aligned}$$

Since $n^3 + 5 = 2m$, for an integer m ($m = 4k^3 + 6k^2 + 3k + 3$), then $n^3 + 5$ is even. Since the premise was that $n^3 + 5$ is odd, this completes the contradiction. Therefore, our assumption that n is odd must be false, leading to the conclusion that n is even.

1.3 Choosing Proof Style

A number is considered **rational** if and only if it can be written as the ratio of two integers: $\frac{p}{q}$ where $q \neq 0$.

1.3.1 Proof Practice

Prove or disprove that the sum of a rational number and an irrational number must be irrational.

Solution: We prove the statement via proof by contradiction. Let $\frac{a}{b}$ be a rational number with a and b as integers and $b \neq 0$. Let x be an irrational number. Seeking contradiction assume that the sum $x + \frac{a}{b}$ is rational. Then we can write $x + \frac{a}{b} = \frac{p}{q}$ for some integers p and q with $q \neq 0$. This gives $x = \frac{p}{q} - \frac{a}{b} = \frac{pb-aq}{bq}$. Note that both the numerator and the denominator are integers and that $bq \neq 0$ since b and q were both nonzero. Therefore, x is, by definition, a rational number, which is a contradiction since x was assumed to be irrational. Hence, it must be that the sum of a rational number and an irrational number is irrational.

1.3.2 Odd Proof III

Prove that for all integers a and b, if a divides b and $a + b$ is odd, then a is odd.

Solution: Proof by Contradiction

- We are supposed to prove: $[(a \text{ divides } b) \wedge (a + b \text{ is odd})] \rightarrow a$ is odd
- Seeking contradiction, assume the negation of the above statement: $\neg\,[[a \text{ divides } b \wedge a + b \text{ is odd}] \rightarrow a \text{ is odd}]$, which is $(a \text{ divides } b) \wedge (a + b \text{ is odd}) \wedge (a \text{ is even})$.
- Since a is even, $a = 2k$ for some integer k.
- Since a divides b we have $b = m \cdot a$.
- So, $a + b$ becomes $2k + m(a) = 2k + m(2k) = 2(k + km) = 2p$, where p is an integer equal to $k + km$
- Thus $a + b = 2p$ and is even. However, we had originally assumed that $a + b$ is odd. This leads to our **contradiction**.
- Hence the assumption in the second bullet point is false, and $[(a \text{ divides } b) \wedge (a + b \text{ is odd})] \rightarrow a$ is odd

1.3.3 Proofs

(a) Prove or disprove: For all nonzero rational numbers x and y, x^y is rational

(b) Prove or disprove: For all even integers x and all positive integers y, x^y is even.

(c) Prove or disprove: For all real numbers x and y, if x^y is irrational, then x or y is not a positive integer

Solution:

1. This is false. Let $x = 2$ and $y = \frac{1}{2}$. Then $x^y = \sqrt{2}$ which is irrational.

2. This is true. Let $x = 2k$, where k is some integer.
   Now, let's substitute this into x^y and rearrange it:
   $x^y = (2k)^y = (2k) \cdot (2k)^{y-1} = 2 \cdot (k(2k)^{y-1})$
   Since y is a positive integer and k is an integer, $k(2k)^{y-1}$ is an integer, since the integers are closed on multiplication. Therefore, we have written x^y in the form of the definition of even (2 times some integer).
   Therefore, for all even x and positive integer$y$, x^y is always even.

3. This is true. Let's look at the contrapositive: "For all real numbers x and y, if x and y are both positive integers, x^y is rational."
   Since $y > 0$, x^y is x multiplied by itself y times - and thus x^y is an integer. As we know all integers are rational, x^y must be rational. Thus, we have proven the contrapositive, and the original statement must therefore be true as well.