W24

# EECS 203 Discussion 3

## Proof by Contrapositive & Contradiction

# Important Forms

- Two beginning-of-semester surveys on Canvas
  - **FCI BoT** Survey and **Better Belonging in Computer Science (BBCS) Entry** Survey
  - **Due:** Friday, Feb. 2nd @11:59pm
- Exam Date Confirmation Survey
  - **Due:** Friday, Feb. 2nd @11:59pm
  - Please fill this out, even if you don’t have an exam conflict!
- They are each worth a few points, so make sure to fill them out!

# Upcoming Homework

- Homework/Groupwork 3 will be due **Feb. 8th**
  - **Don’t forget to match pages!**
  - Please note as soon as you press submit you’ve successfully submitted by the deadline. **You can still match pages** with no rush without adding to your submission time.
- Groupwork
  - Groupwork can be done alone, but the problems tend to be more difficult, and the goal is for you to puzzle them out with others!
  - Your discussion section is a great place to find a group!
  - There is also a pinned Piazza thread for searching for homework groups.

# Proof Techniques

# Proof Methods

- **Direct Proof:** Proves $p \rightarrow q$ by showing

  **$p \rightarrow$ stuff $\rightarrow q$**

- **Proof by Contraposition:** Proves $p \rightarrow q$ by showing

  **$\neg q \rightarrow$ stuff $\rightarrow \neg p$**

- **Proof by Contradiction:** Proves p by showing

  **$\neg p \rightarrow F$**

- **Proof by Cases:** next week

# Some Methods of Proving $p \rightarrow q$

- **Direct Proof:**
  Proves $p \rightarrow q$ by showing **$p \rightarrow$ stuff $\rightarrow q$**
- **Proof by Contraposition:**
  Proves $p \rightarrow q$ by showing **$\neg q \rightarrow$ stuff $\rightarrow \neg p$**
  (Knowing $\neg q \rightarrow \neg p$ enables concluding $p \rightarrow q$ because $\neg q \rightarrow \neg p \equiv p \rightarrow q$)
- **Proof by Contradiction:**
  Proves p by showing **$\neg p \rightarrow F$**
  To prove $p \rightarrow q$, assume the negation: **$\neg(p \rightarrow q) \equiv \neg(\neg p \lor q) \equiv p \land \neg q$**
  Derive a contradiction (F) from this assumption by arriving at a mathematically incorrect statement (ex: $0 = 2$) or two statements that contradict each other ($x = y$ and $x \neq y$). Therefore, $p \rightarrow q$.

# Problem 1

## 1. Proof by Contraposition $\star$

Prove that if $n^2 + 2$ is even, then $n$ is even using a proof by contrapositive.

# Solution

## 1. Proof by Contraposition ⋆

Prove that if $n^2 + 2$ is even, then $n$ is even using a proof by contrapositive.

**Solution:** We will prove the contrapositive, that is: If $n$ is odd, then $n^2 + 2$ is odd.

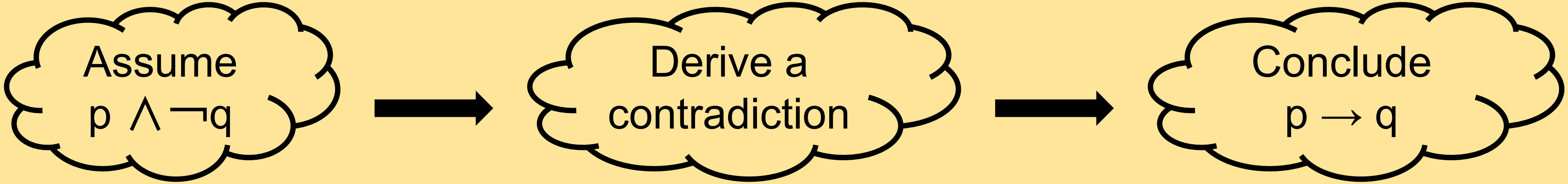
- Assume $n$ is odd. Then we can write it as $n = 2k + 1$ for some integer k.
- This means $n^2 + 2 = (2k + 1)^2 + 2$.
  $= 4k^2 + 4k + 1 + 2$
  $= 2(2k^2 + 2k + 1) + 1$
  $= 2j + 1$, where $j$ is an integer equal to $2k^2 + 2k + 1$
- Thus from the definition of an odd number, $n^2 + 2$ is odd.

This completes the proof of the contrapositive, and thus the original statement.

# Proof by Contradiction

# Proof by Contradiction

- When trying to prove p implies q, assume p is true and q is false. Derive a **contradiction**, (something that is always false, **ex:** $0 = 2$, **ex:** $x = y$ and $x \neq y$). Therefore, $p \rightarrow q$.
  - We assume the negation of what we want to prove
  - We arrive at something false
  - Therefore the negation of the thing we assumed must be true (ie the thing we wanted to prove)




$\neg(p \rightarrow q) \equiv (p \land \neg q)$ ⟶ $F$ ⟶ $\neg(p \land \neg q) \equiv \neg p \lor q \equiv p \rightarrow q$

# Problem 2

## 2. Contraposition vs Contradiction ⋆

Show that for an integer $n$: if $n^3 + 5$ is odd, then $n$ is even, using

a) a proof by contraposition.

b) a proof by contradiction.

**Note:** The algebra in either case is the same. You don't need to rewrite the algebra for part (b), just reformat your proof from (a) into a proof by contradiction.

# Solution

**2. Contraposition vs Contradiction** $\star$

Show that for an integer $n$: if $n^3+5$ is odd, then $n$ is even, using

a) a proof by contraposition.

b) a proof by contradiction.

**Note:** The algebra in either case is the same. You don't need to rewrite the algebra for part (b), just reformat your proof from (a) into a proof by contradiction.

**Solution:**

a) We will prove the contrapositive of the proposition, which is: "if $n$ is odd, then $n^3+5$ is even".
Since $n$ is odd, $n$ can be written as $2k+1$, where $k$ is some integer. Then,

$$\begin{aligned} n^3+5 &= (2k+1)^3+5 \\ &= (8k^3+12k^2+6k+1)+5 \\ &= 8k^3+12k^2+6k+6 \\ &= 2(4k^3+6k^2+3k+3) \end{aligned}$$

So $n^3+5=2m$, where $m$ is the integer $4k^3+6k^2+3k+3$. Because $n^3+5$ is two times some integer, we can say that $n^3+5$ is even.

b) We will use a proof by contradiction. Let $n^3+5$ be odd. *Seeking a contradiction*, assume that $n$ is odd. Since $n$ is odd, it can be written as $2k+1$, where $k$ is some integer. So

$$\begin{aligned} n^3+5 &= (2k+1)^3+5 \\ &= (8k^3+12k^2+6k+1)+5 \\ &= 8k^3+12k^2+6k+6 \\ &= 2(4k^3+6k^2+3k+3) \end{aligned}$$

Since $n^3+5=2m$, for an integer $m$ ($m=4k^3+6k^2+3k+3$), then $n^3+5$ is even. Since the premise was that $n^3+5$ is odd, this completes the contradiction. Therefore, our assumption that $n$ is odd must be false, leading to the conclusion that $n$ is even.

# Problem 3

## 3. Proof Practice

Prove or disprove that the sum of a rational number and an irrational number must be irrational.

# Solution

### 3. Proof Practice

Prove or disprove that the sum of a rational number and an irrational number must be irrational.

**Solution:**

We prove the statement via proof by contradiction. Let $\frac{a}{b}$ be a rational number with $a$ and $b$ as integers and $b \neq 0$. Let $x$ be an irrational number. We assume that the sum $x + \frac{a}{b}$ is rational. Then we can write $x + \frac{a}{b} = \frac{p}{q}$ for some integers $p$ and $q$ with $q \neq 0$. This gives $x = \frac{p}{q} - \frac{a}{b} = \frac{pb-aq}{bq}$. Note that both the numerator and the denominator are integers, and that $bq \neq 0$ since $b$ and $q$ were both nonzero. Therefore, $x$ is, by definition, a rational number, which is a contradiction since $x$ was assumed to be irrational. Hence, it must be that the sum of a rational number and an irrational number is irrational.

# Problem 4

## 4. Odd Proof III

Prove that for all integers $a$ and $b$, if $a$ divides $b$ and $a + b$ is odd, then $a$ is odd.

# Solution

## 4. Odd Proof III

Prove that for all integers $a$ and $b$, if $a$ divides $b$ and $a + b$ is odd, then $a$ is odd.

**Solution: Proof by Contradiction**

- We are supposed to prove: $[(a \text{ divides } b) \wedge (a + b \text{ is odd})] \rightarrow a$ is odd
- Seeking contradiction, assume the negation of the above statement: $\neg$ $[[a$ divides $b$ $\wedge$ $a + b$ is odd$] \rightarrow a$ is odd$]$, which is $(a$ divides $b) \wedge (a + b$ is odd$) \wedge (a$ is even$)$.
- Since $a$ is even, $a = 2k$ for some integer $k$.
- Since $a$ divides $b$ we have $b = m \cdot a$.
- So, $a + b$ becomes $2k + m(a) = 2k + m(2k) = 2(k + km) = 2p$, where $p$ is an integer equal to $k + km$
- Thus $a + b = 2p$ and is even. However, we had originally assumed that $a + b$ is odd. This leads to our **contradiction**.
- Hence the assumption in the second bullet point is false, and $[(a$ divides $b) \wedge (a + b$ is odd$)] \rightarrow a$ is odd

# Problem 5

## 5. Proofs ⋆

1. Prove or disprove: For all nonzero rational numbers $x$ and $y$, $x^y$ is rational
2. Prove or disprove: For all even integers $x$ and all positive integers $y$, $x^y$ is even.
3. Prove or disprove: For all real numbers $x$ and $y$, if $x^y$ is irrational, then $x$ is not a positive integer or $y$ is not a positive integer

# Solution

## 5. Proofs ⋆

1. Prove or disprove: For all nonzero rational numbers $x$ and $y$, $x^y$ is rational
2. Prove or disprove: For all even integers $x$ and all positive integers $y$, $x^y$ is even.
3. Prove or disprove: For all real numbers $x$ and $y$, if $x^y$ is irrational, then $x$ is not a positive integer or $y$ is not a positive integer

**Solution:**

1. This is false. Let $x = 2$ and $y = \frac{1}{2}$. Then $x^y = \sqrt{2}$ which is irrational.
2. This is true. Let $x = 2k$, where $k$ is some integer.
   Now, let's substitute this into $x^y$ and rearrange it:
   $x^y = (2k)^y = (2k) \cdot (2k)^{y-1} = 2 \cdot (k(2k)^{y-1})$
   Since $y$ is a positive integer and $k$ is an integer, $k(2k)^{y-1}$ is an integer, since the integers are closed on multiplication. Therefore, we have written $x^y$ in the form of the definition of even (2 times some integer).
   Therefore, for all even $x$ and positive integers $y$, $x^y$ is always even.
3. This is true. Let's look at the contrapositive: "For all real numbers $x$ and $y$, if $x$ and $y$ are both positive integers, $x^y$ is rational."
   Since $y > 0$, $x^y$ is $x$ multiplied by itself $y$ times - and thus $x^y$ is an integer. As we know all integers are rational, $x^y$ must be rational. Thus, we have proven the contrapositive, and the original statement must therefore be true as well.