EECS 203 Exam 1 Review

Day 2

Today’s Review Topics

- Proof Methods
  - Direct Proof
  - Proof by Contrapositive
  - Proof by Contradiction
  - Proof by Cases
- Sets

Proof Methods

Proofs Overview

- Direct Proof - Prove $p \rightarrow q$ by showing that **if p is true, then q must also be true.**
- Proof by Contraposition - Prove $p \rightarrow q$ by showing that **if not q, then not p.**
  - Assume not q and arrive at not p
- Proof by Contradiction
  - Prove p by **assuming ¬p and arriving at a contradiction, therefore proving p is true (can think of this as ¬-intro from natural deduction)**
  - Prove $p \rightarrow q$ by **assuming p and ¬q and arriving at a contradiction, therefore ¬(p and ¬q) is true which is equivalent to saying $p \rightarrow q$ is true**

Overview Cont.

- Proof by Cases
    - Prove that a predicate is true by **separating into all possible cases and showing that the predicate is true in each individual case.**
    - Proof by cases is similar to the idea of $\vee$ - elimination.

NOTE: Proof by Induction will not be covered in Exam 1

Proof Methods Table

$p \rightarrow q$	Assumptions	Want to Reach
Direct Proof	p	q
Proof By Contrapositive	$\neg q$	$\neg p$
Proof By Contradiction	$p \land \neg q$	F

Proving + Disproving Quantified Statements

	Prove	**Disprove**
$\forall x P(x)$	Show that **arbitrary** x **satisfies** P(x)	Find a **counterexample** x which does not satisfy P(x)
$\exists x P(x)$	Find an **example** x which satisfies P(x)	Show that an **arbitrary** x **does not satisfy** P(x)

NOTE: The above does not show proof by example. Proof by example is **never** valid.

WLOG

Without Loss of Generality (WLOG) – used when the same argument can be made for multiple cases

Example: Show that if x and y are integers and both $x \cdot y$ *and* $x + y$ are even, then both x and y are even.

Proof: Use a proof by contraposition. Suppose x and y are not both even. Then, one or both are odd. Without loss of generality, assume that x is odd. Then $x = 2m + 1$ for some integer k.

Case 1: y is even. Then $y = 2n$ for some integer n, so $x + y = (2m + 1) + 2n = 2(m + n) + 1$ is odd.

Case 2: y is odd. Then $y = 2n + 1$ for some integer n, so $x \cdot y = (2m + 1)(2n + 1) = 2(2m \cdot n + m + n) + 1$ is odd.

Prove that if n is an odd integer, then n^2 is odd.

Direct Proof Solution

Prove that if n is an odd integer, then n^2 is odd.

p = Odd(n)

q = Odd(n^2)

Direct Proof of $p \rightarrow q$:

1) Let n be odd; odd(n) $\rightarrow$ n = 2k + 1 for some arbitrary $k \in \mathbb{Z}$
2) $n^2 = (2k + 1)^2$
3) $= 4k^2 + 4k + 1$
4) $= 2(2k^2 + 2k) + 1$
5) Since this is of the form 2(some integer) + 1, then we conclude odd(n^2)
6) Therefore, since we started by assuming p and were able to conclude q, then $p \rightarrow q$.

Prove that if $a \cdot b < 0$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then $(a / b) < 0$.

Proof by Cases Solution

Prove that if $a \cdot b < 0$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then $a / b < 0$.

If $a \cdot b < 0$, then a and b must be of opposite signs and a, $b \neq 0$ (since then $a \cdot b = 0$)

Case 1: $a > 0$, $b < 0$

Then a / b would be + / - which would divide to become a negative number.

Case 2: $a < 0$, $b > 0$

Then a / b would be - / + which would divide to become a negative number.

In all (both) cases $a / b < 0$, therefore we have proven our implication that $a \cdot b < 0 \rightarrow a / b < 0$.

Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof by Contraposition Solution

Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

p : $n = ab$

q : $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

1) Start by assuming not q, that $a > \sqrt{n}$ and $b > \sqrt{n}$ (by De Morgan's Law)
2) Multiply the two inequalities (able to do this since left side is > right side for both inequalities)
3) $ab > \sqrt{n} \cdot \sqrt{n} \equiv ab > n$, so $ab \neq n$ (this is $\neg p$)
4) Therefore we have reached $\neg p$ and have shown that $\neg q \to \neg p$
5) By using proof by contraposition, we have now shown that $p \to q$

Prove that if $3n + 2$ is odd, then n is odd.

Proof by Contradiction Solution

Prove that if 3n + 2 is odd, then n is odd.

p = odd(3n + 2)

q = odd(n)

1) Assume odd(3n + 2), then assume ¬odd(n) ≡ even(n)
2) If n is even, then n = 2k, $k \in \mathbb{Z}$.
3) 3(2k) + 2 = 6k + 2 = 2(3k + 1)
4) This shows even(3n + 2) which is a contradiction with our first assumption, therefore our second assumption (¬odd(n)) must have been false and odd(n) must be true.
5) Therefore, $p \rightarrow q$ by proof by contradiction

Prove or Disprove: For all rational numbers x and y, x^y is also rational.

Prove/Disprove For-all Statement Solution

For all rational numbers x and y, x^y is also rational.

Recall that roots are applied to numbers raised to fractions. We can use this to our advantage in coming up with a counterexample.

Disproof by counterexample: let $x=2$, $y=\frac{1}{2}$

- x and y are both rational numbers.
- However, $x^y = 2^{1/2} = \sqrt{2}$, which is not rational.
- Thus, our counterexample disproves the statement.

Prove or Disprove: There exists an integer n such that

$4n^2 + 8n + 16$ is prime

Prove/Disprove There-exists Statement Solution

There exists an integer n such that $4n^2 + 8n + 16$ is prime.

Notice that $4n^2 + 8n + 16$ is divisible by 4, no matter what integer can be plugged in. Therefore, it cannot be prime, so we know to disprove this statement.

Disproof of an Exists Statement:

- Let x be an arbitrary integer.
- Then, we have the expression $y = 4x^2 + 8x + 16$ (abbreviate y for less writing)
- We can factor out a 4 to get $y = 4(x^2 + 2x + 4)$.
- $x^2 + 2x + 4$ is an integer, and because we have written y as 4 times (some integer), y is divisible by 4.

Prove/Disprove There-exists Statement Solution (cont.)

- Now we have 2 cases: $y = 4$ and $y \neq 4$
  1. $y = 4$: y is not prime, because it (4) has a factor of 2
  2. $y \neq 4$: y is not prime, because it is divisible by 4 (a factor that is not equal to y, as y is not 4)
- In all cases y is not prime. Therefore, there does not exist any integer such that $4n^2 + 8n + 16$ is prime.

Which of the following describe the proof method(s) used to show the following statement? Mark all that apply.

Statement: If x is rational and y is irrational, then $x + y$ is irrational.
Proof: Assume that x is rational, y is irrational, and $x + y$ is rational. Notice that $y = (x + y) - x$. Since both $x + y$ and x are rational, and the difference of two rational numbers is also rational, this means that y is rational. But we assumed y was irrational. So it must be the case that whenever x is rational and y is irrational, $x + y$ is irrational.

(a) Proof by contrapositive

(b) Proof by cases

(c) Proof by contradiction

(d) Direct Proof

(e) Exhaustive proof Proving all cases possible

Solution

C, we assume p and not q and arrive at a contradiction

Identify the mistakes in the following proof, multiple answers

We prove that $0 = 2$ as follows.

S1. We have $4x^2 = 4x^2$.

S2. Rewriting the left and right hand sides, we get $(-2x)^2 = (2x)^2$.

S3. Taking the square root, we get $-2x = 2x$.

S4. Adding $x^2 + 1$ on both sides gives $-2x + x^2 + 1 = 2x + x^2 + 1$.

S5. By algebra, this can be written as $(x - 1)^2 = (x + 1)^2$.

S6. Taking the square root, we get $x - 1 = x + 1$.

S7. Subtracting $x - 1$ on both sides, we get $x - 1 - (x - 1) = x + 1 - (x - 1)$, i.e., $0 = 2$.

Solution: The mistake was made in steps 3 and 6. $a^2 = b^2$ does not imply that $a = b$ and so $(-2x)^2 = (2x)^2$ does not imply that $-2x = 2x$. Similarly, $(x-1)^2 = (x+1)^2$ does not imply that $x - 1 = x + 1$. The problem only asks to select the step where first error is made, therefore the correct answer is S3.

5 Minute Break

https://paveldogreat.github.io/WebGL-Fluid-Simulation/

Sets and Set Proofs

Overview/Definitions

Set: An unordered collection of distinct objects

Subset ($\subseteq$): A set *A* is considered to be a **subset** of *B* if every element in *A* is also in *B* (Note that, with this definition, *A* is a subset of itself)

Proper Subset ($\subsetneq$): A set *A* is considered to be a **proper subset** of *B* if *A* is a subset of *B*, and *B* contains at least one element not in *A*.

Power set (P(S)): A set containing all of the subsets of S as **elements** in the set.

Inclusion-Exclusion Principle: $|A \cup B| = |A| + |B| - |A \cap B|$

Sets Question 1

Which of the following are valid subsets of the set S where S = $\{1, \{2\}, \varnothing\}$? Select all that apply.

A. $\varnothing$
B. $\{\varnothing\}$
C. 1
D. $\{1\}$
E. $\{2\}$

Sets Answer 1

Which of the following are valid subsets of the set S where S = {1, {2}, ∅}? Select all that apply.

A. ∅ ✅
B. {∅} ✅
C. 1 ❌
D. {1} ✅
E. {2} ❌

Sets Solution 1

Answer: A , B and D

S = {1, {2}, ∅} Of the answer choices, only ∅ , {∅} and {1} appear as answers so A and D are correct.

So we have 1 is an element so {1} would be a subset. Not 1

So we have ∅ is an element so {∅} would be a subset

∅ is a subset of everything

{2} is an element so {{2}} would be a subset not {2}

More definitions and Sets Question 2

Cardinality: The number of elements in a set, denoted |A|

Note that power sets of sets with n elements are of cardinality 2^n

Cartesian Product: A x B is the set of all pairs of elements from A and B, i.e. (a,b) where a ∈ A and b ∈ B. Note that |A x B| = |A| * |B|

What is the cardinality of {E,E,C,S} X {2,0,3}?

Sets Solution 2

{E, E, C, S} has cardinality 3, as does {2, 0, 3}. Note this is because the cardinality is the number of *unique* elements in a set.

We know that $|A \times B| = |A| * |B|$, so $|\{E, E, C, S\} \times \{2, 0, 3\}| = |\{E, E, C, S\}| * |\{2, 0, 3\}| = 3 * 3 = 9$.

Sets Question 3

Prove that if C ⊆ comp(A - B), then A ∩ C ⊆ B. Note that comp() is the complement of the set.

Solution:

To prove this implication, we will assume the premise and try to derive the conclusion. We therefore assume that $C \subseteq \overline{(A-B)}$. This means if $x \in C$ then $x \notin A - B$. That is, if $x \in C$, then either $x \notin A$ or $x \in B$.

We want to show that $A \cap C \subseteq B$. Take any $x \in A \cap C$. Then $x \in A$ and $x \in C$. We know from above that if $x \in C$, then either $x \notin A$ or $x \in B$. But it cannot be the case that $x \notin A$ as we already know that $x \in A$. The only possibility, then, is that $x \in B$. We have shown that for every $x \in A \cap C$, we have $x \in B$. We conclude that $A \cap C \subseteq B$.

Good luck studying!