EECS 203: Discrete Mathematics
Winter 2024
Homework 10

Due **Thursday, April 18**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $8+2$ Total Points: $100+35$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Color Conundrum [14 points]

Each day Donovan Edwards either wears a T-shirt or a tank top. On a given day, there is a 70% chance he wears a T-shirt and a 30% chance he wears a tank top. If he wears a T-shirt, he randomly picks one of 4 pink T-shirts, 3 blue T-shirts, and 2 black T-shirts (he is equally likely to pick any particular shirt). If he wears a tank top, he randomly picks one of 2 pink tank tops, 3 white tank tops, or 2 blue tank tops.

(a) What is the probability that he is wearing pink or white on a given day?

(b) Given that Donovan is wearing pink or white on a given day, what is the probability that he is wearing a T-shirt?

You **do not** need to simplify your answers.

Solution:

(a) Let E be the event that his shirt is pink or white, S be the event that he wears a T-shirt, and T be the event that he wears a tank top. Then by the law of total probability

$$\begin{aligned} P(E) &= P(E \cap S) + P(E \cap T) && (S \cap T = \emptyset) \\ &= P(E \mid S)P(S) + P(E \mid T)P(T). \end{aligned}$$

The probably that his shirt is pink or white given he wears a T-shirt is $\frac{4}{9}$, and the probability that it is pink or white given he wears a tank top is $\frac{5}{7}$. So the total probability is

$$P(E) = \frac{4}{9}(0.7) + \frac{5}{7}(0.3) = \frac{331}{630} \approx 0.5253968.$$

(b) The quantity we're looking for is $P(S \mid E)$. By Bayes' theorem,

$$P(S \mid E) = \frac{P(E \mid S)P(S)}{P(E)}.$$

We know from the problem statement that $P(S) = 0.7$. From part (a), we know $P(E \mid S) = \frac{4}{9}$ and $P(E) = \frac{331}{630}$. So plugging this in we get

$$P(S \mid E) = \frac{\frac{4}{9} \cdot 0.7}{\frac{331}{630}} = \frac{196}{331} \approx 0.5921450.$$

Draft Grading Guidelines [14 points]

Part a:

+2 identifies that law of total probability can be used
+2 correct probability for $P(E \mid S)$
+2 correct probability for $P(E \mid T)$
+2 correct final answer

Part b:

+2 identifies that Bayes' theorem can be used
+2 substitutes in correct values
+2 correct final answer

2. Bayes' ×3 [8 points]

Suppose that E, F_1, F_2, and F_3 are events from a sample space S. Furthermore, suppose that F_1, F_2, and F_3 are each mutually exclusive, and that their union is S. Find $P(F_2 \mid E)$ if

$$P(E \mid F_2) = \frac{3}{8} \qquad P(F_1) = \frac{1}{6}$$
$$P(E \mid F_3) = \frac{1}{2} \qquad P(F_2) = \frac{1}{2}$$
$$P(E \mid F_1) = \frac{2}{7} \qquad P(F_3) = \frac{1}{3}$$

Express your final answer as a **single, fully-simplified** number.

Solution:

By the generalized version of Bayes' theorem,

$$\begin{aligned} P(F_2 \mid E) &= \frac{P(E \mid F_2)P(F_2)}{P(E \mid F_1)P(F_1) + P(E \mid F_2)P(F_2) + P(E \mid F_3)P(F_3)} \\ &= \frac{\frac{3}{8} \cdot \frac{1}{2}}{\frac{2}{7} \cdot \frac{1}{6} + \frac{3}{8} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3}} \\ &= \frac{7}{15}. \end{aligned}$$

Draft Grading Guidelines [8 points]

+4 select correct form of Bayes' theorem
+4 correct final answer

3. There Snow Way I'm Running In This [12 points]

Ishaan likes to run, but he hates running in the snow. If it snows, the probability of Ishaan going for a run is $\frac{1}{10}$. If it doesn't snow, the probability of Ishaan running is $\frac{8}{10}$. If Ishaan goes for run, then the probability that it snowed is $\frac{1}{9}$. What is the probability that it snows?

Express your final answer as a **single, fully-simplified** number.

Solution:

Let S be the event that it snowed, and R be the event that Ishaan runs. We are given $P(R \mid S) = \frac{1}{10}$, $P(R \mid \overline{S}) = \frac{8}{10}$, and $P(S \mid R) = \frac{1}{9}$. By Bayes' theorem, we know

$$P(S \mid R) = \frac{P(R \mid S)P(S)}{P(R \mid S)P(S) + P(R \mid \overline{S})P(\overline{S})}$$

which implies

$$P(S \mid R)\left(P(R \mid S)P(S) + P(R \mid \overline{S})P(\overline{S})\right) = P(R \mid S)P(S).$$

We can substitute $P(\overline{S}) = 1 - P(S)$, as well as the values we know to get

$$\frac{1}{9}\left(\frac{1}{10}P(S) + \frac{8}{10}(1 - P(S))\right) = \frac{1}{10}P(S).$$

Combining like terms and solving for $P(S)$, we get

$$P(S) = \frac{1}{2}.$$

Draft Grading Guidelines [12 points]

+2 correctly identifies $P(R \mid S)$
+2 correctly identifies $P(R \mid \overline{S})$
+2 correctly identifies $P(S \mid R)$
+2 attempts to solve for $P(S)$
+2 applies correct form of Bayes' theorem
+2 correct final answer

4. What did you expect? [12 points]

The EECS 203 staff is going on a road trip! The 36 staff members have decided to split up into 6 different cars with 9, 8, 6, 6, 4, 3 people in each of the respective cars.

(a) Suppose we pick a car uniformly at random, and consider X to be the random variable defined by the number of staff members in that car. What is the expected value of X?

(b) Now suppose we pick one of the staff members uniformly at random. Let Y be the random variable defined by the number of people in the car that staff member is in. What is the expected value of Y?

Express your final answers as **single, fully-simplified** numbers.

Solution:

(a) In this case each car is equally likely to be chosen with probability $\frac{1}{6}$. So

$$E(X) = \frac{1}{6}(9+8+6+6+4+3) = 6.$$

(b) With this set up it is no longer equally likely for each car to be chosen. So

$$\begin{aligned} E(Y) &= 3 \cdot P(Y=3) + 4 \cdot P(Y=4) + 6 \cdot P(Y=6) + 8 \cdot P(Y=8) + 9 \cdot P(Y=9) \\ &= 3 \cdot \frac{3}{36} + 4 \cdot \frac{4}{36} + 6 \cdot \frac{12}{36} + 8 \cdot \frac{8}{36} + 9 \cdot \frac{9}{36} \\ &= \frac{121}{18} \approx 6.722. \end{aligned}$$

Draft Grading Guidelines [12 points]

Part a:

+2 correct expected value
+2 correct justification

Part b:

+2 identifies the rooms have different probabilities of being selected
+2 selects correct expected value formula
+2 correct expected value
+2 correct justification

5. Zero-sum game...or is it? [12 points]

Your friend proposes to play the following game. You roll a fair, 6-sided dice twice and record the result. Let X be the random variable defined as twice the value of the first roll, minus three times the value of the second roll. For example, if you rolled 3 then 4, then X would equal $2 \cdot 3 - 3 \cdot 4 = -6$. You win X dollars if X is positive, but have to give your friend $|X|$ dollars if X is negative. If X is zero then you neither win nor lose money. How much money do you expect to win or lose?

Express your final answer as a **single, fully-simplified** number.

Solution:

Define random variables X_1 and X_2 as the number rolled on the first and second die respectively.

Then, $E(X_1) = E(X_2) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$.

By definition $X = 2X_1 - 3X_2$, so by linearity of expectation

$$E(X) = E(2X_1 - 3X_2) = 2E(X_1) - 3E(X_2) = 2 \cdot \frac{7}{2} - 3 \cdot \frac{7}{2} = -\frac{7}{2}.$$

Therefore, you would expect to lose 3.5 dollars.

Draft Grading Guidelines [12 points]

+3 defines random variables for each roll
+2 correct expected value of a particular roll
+2 correct definition of X in terms of previous variables
+2 applies linearity of expectation
+3 correct final answer

6. Rollie Pollie [15 points]

Rohit recently became super passionate about rolling dice. He decides to roll a single fair 6-sided die 100 times. What is the expected number of times he rolls a 5 followed by a 6?

Express your final answer as a **single, fully-simplified** number.

Solution:

Since we are looking for a 5 followed by a 6, we define an experiment/trial to be two dice rolls. For example, the dice roll sequence 1, 2, 3 would produce 2 trials (1,2) and (2,3).

Note that this means for 100 rolls we will have 99 trials.

Let I_k be an indicator random variable for the k-th trial, where $I_k = 1$ if the result of the k-th trial is (5,6), and is 0 otherwise. Let X be the random variable for the total number of times a 5 followed by a 6 appears in the sequence. By this construction

$$X = \sum_{k=1}^{99} I_k.$$

Moreover we know the probability that we roll a 5 then a 6 on any pair of consecutive rolls is $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$, so $I_k = \frac{1}{36}$ for every trial, k. By linearity of expectation, we have

$$E(X) = E\left(\sum_{k=1}^{99} I_k\right) = \sum_{k=1}^{99} E(I_k) = \sum_{k=1}^{99} \frac{1}{36} = \frac{99}{36} = \frac{11}{4}.$$

Grading Guidelines [15 points]

+3 correct probability of rolling a 5 followed by a 6
+4 correct number of trials (99)
+4 attempts to apply linearity of expectation
+4 correct final answer

7. Bernoulli trials, binomial distribution [15 points]

You roll a fair six-sided die 12 times. Find:

(a) The probability that exactly two rolls come up as a 6.

(b) The probability that exactly two rolls come up as a 6, given that the first four rolls each came up as 3.

(c) The probability that at least two rolls come up as a 6.

(d) The expected number of rolls that come up as 6.

You **do not** need to simplify your answers.

Solution:

(a) We apply the formula for a binomial distribution with $p = \frac{1}{6}$, $n = 12$, and $k = 2$. If X is the number of rolls that come up as a 6: $P(X = 2) = \binom{12}{2}(\frac{1}{6})^2(\frac{5}{6})^{10}$

(b) Since these events are independent, then the first four rolls being determined reduces to the previous problem, but with $n = 8$, so $P(X = 2) = \binom{8}{2}(\frac{1}{6})^2(\frac{5}{6})^6$

(c) We can find the probability that less than two rolls come up as 6, and subtract that from 1:

$$\begin{aligned}
P(X \geq 2) &= 1 - (P(X = 0) + P(X = 1)) \\
&= 1 - \left[\binom{12}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{12} + \binom{12}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{11}\right] \\
&= 1 - \left[\left(\frac{5}{6}\right)^{12} + 12\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11}\right]
\end{aligned}$$

(d) We can apply the formula for expected value of a binomial with $p = \frac{1}{6}$ and $n = 12$. Then $E(X) = np = 12(\frac{1}{6}) = 2$.

Draft Grading Guidelines [15 points]

Parts a, b, d:

+2 identifies correct parameters for binomial distribution
+1 correct answer

Part c:

+2 uses difference rule
+1 correct expression for $P(X = 0)$
+1 correct expression for $P(X = 1)$
+2 correct answer

8. Fastest Draw in the Midwest [12 points]

Suppose Grace has a standard deck of 52 cards. Grace expects she can draw all 52 cards in order (defined below) in 1300 draws. Yunsoo expects 1600 draws. Explain why Yunsoo is further away from the real expected value.

Note: The order of cards goes Ace, 2, 3, . . . , King and $\clubsuit, \diamondsuit, \heartsuit, \spadesuit$. If the next card in order is not drawn, then it is placed back into the deck at random. If the next card in order is drawn, then Grace sets it aside, removing it from the deck.

Note: The cards do not have to be selected consecutively. For example, $\underline{A\clubsuit}$, $3\diamondsuit$, $J\spadesuit$, $\underline{2\clubsuit}$ is a valid start, and there would only be 50 cards left in the deck at this point.

Solution:

We can use linearity of expectations to find the number of expected number of cards that Grace must draw.

The first card (A$\clubsuit$) has a probability of $\frac{1}{52}$ of being drawn, so the expected value is 52 draws. Since the card will not be replaced, the next card (2$\clubsuit$), has a probability of $\frac{1}{51}$ of being drawn and the expected value of 51 draws. Since the number of tries until Grace draws a particular card follows a geometric distribution, in expectation it takes her 52 tries to draw A$\clubsuit$, 51 tries to draw 2$\clubsuit$, etc.

With this pattern, it takes

$$52 + 51 + \cdots + 2 + 1 = \sum_{k=1}^{52} k = \frac{52 \cdot 53}{2} = 1378$$

draws in expectation. So Grace's guess of 1300 draws was closer than Yunsoo's guess of 1600.

Draft Grading Guidelines [12 points]

+3 identifies the use of a geometric distribution
+3 correct expectation of a particular trial
+3 applies linearity of expectation
+3 correct final answer

Grading of Groupwork 9

Using the solutions and Grading Guidelines, grade your Groupwork 9 Problems:

- Use the table below to grade your past groupwork submission and calculate scores.
- While grading, mark up your past submission. Include this with the table when you submit your grading.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- See "All About Groupwork" on Canvas for more detailed guidance, and what to do if you change groups.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 1												/15
Problem 2												/15
Total:												/30

Groupwork 10 Problems

1. Circular Reasoning [15 points]

Suppose we select $2n$ distinct points independently and uniformly at random on the border of a circle, and label them p_1 through p_{2n} counter-clockwise (i.e. point p_2 is counter-clockwise from point p_1).

(a) In the case where $n = 2$, we have four distinct points on the circle. If we select two of these points uniformly at random and draw a line segment between them, then draw a line segment between the remaining two points, what is the probability that these line segments intersect?

Hint: Consider the different cases corresponding to the point p_1 is paired with.

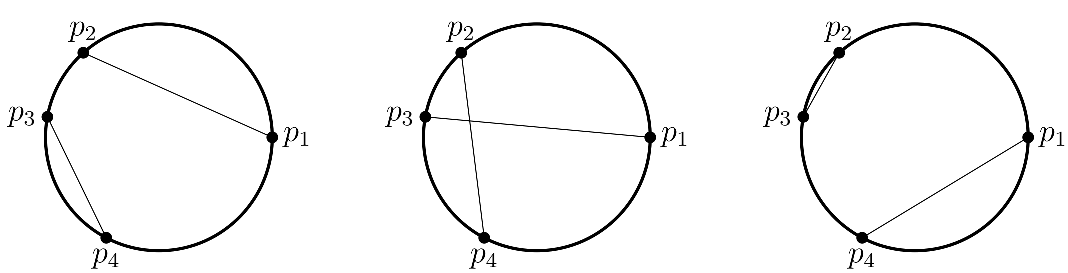
(b) Suppose we repeat the procedure in (a) where we select two points at random and draw a line segment between them. We'll call this line segment ℓ_1. We repeat this again with the $2n-2$ remaining points, creating a line segment ℓ_2, etc., until we have drawn n line segments: $\ell_1, \ldots, \ell_n$. After this procedure is completed, what is the expected number of intersections? Your answer should be in terms of n.

Hint: Create an indicator random variable for each possible intersection and apply linearity of expectation.

Note: The number of intersections is the number of pairs (ℓ_i, ℓ_j) of distinct line segments where ℓ_i and ℓ_j intersect.

Solution:

(a) We have two line segments, ℓ_1 and ℓ_2. Without loss of generality, let p_1 be an endpoint of ℓ_1. Then it is equally likely that any of p_2, p_3, or p_4 is the other endpoint. Define ℓ_2 using the remaining points. Note that ℓ_1 and ℓ_2 will only intersect when ℓ_1 is bounded by p_1 and p_3. Hence, the probability of intersection is $\frac{1}{3}$.



(b) We first label our lines $\ell_1, \ldots, \ell_n$. Let X be the random variable that counts intersections between these lines. Now, define indicator random variables X_{ij} for all $1 \leq i < j \leq n$ so that $X_{ij} = 1$ provided that ℓ_i and ℓ_j intersect, and $X_{ij} = 0$ otherwise. By part (a), we know that $E(X_{ij}) = \frac{1}{3}$. Moreover, since there are $\binom{n}{2}$ possible intersections between n distinguishable line segments, we can use linearity of expectation to say that

$$\begin{aligned} E(X) &= E\left(\sum_{1\leq i<j\leq n} X_{ij}\right) \\ &= \sum_{1\leq i<j\leq n} E(X_{ij}) \\ &= \binom{n}{2} \cdot \frac{1}{3} \\ &= \frac{n!}{6(n-2)!} \\ &= \frac{n(n-1)}{6}. \end{aligned}$$

Grading Guidelines [15 points]

Note: Students are not required to simplify their answers to receive full credit.

Part a:

(i) +3 considers three cases, one for each possible pairing of p_1 (does not need to be explicit)
(ii) +3 concludes that the line segments intersect in only one case, thus the probability is $\frac{1}{3}$

Part b:

(iii) +2 correct setup of indicator variables
(iv) +2 recognizes that $P(X_{ij} = 1) = \frac{1}{3}$ by part (a)
(v) +3 recognizes that there are $\binom{n}{2}$ indicator RVs
(vi) +2 correct final answer

2. Open or Closed [20 points]

Online Bayesian Inference is a process where we repeatedly apply Bayes rule to update our beliefs over time. Suppose we have a sensor that determines whether a door is open or closed. If the door is open, the sensor reads it as open with probability 0.9. If the door is closed, the sensor reads it as closed with probability 0.7. Suppose the door starts in an unknown position, and has equal probability of being open or closed.

(a) After one reading that the door is closed, what is the probability that the door is actually closed?

(b) Before the second reading, we believe that the door is closed with the probability found in part (a) (that is, we consider the probability that the door is closed to be the probability that we found the door is closed given our first reading). Suppose we make another reading that the door is closed. Now what is the probability that the door is closed?

(c) On the third reading, the sensor reads that the door is open. What is the probability that the door is actually closed, using the answer from part (b) as our initial probability for the door being closed?

Solution:

(a) Let C be the event that the door is closed. At first, we believe $P(C) = 0.5$. Let R_1 be the event that the first reading is closed. We know $P(R_1 \mid C) = 0.7$ and $P(R_1 \mid \overline{C}) = 1 - 0.9 = 0.1$. Then by Bayes' rule:

$$\begin{aligned} P(C \mid R_1) &= \frac{P(R_1 \mid C)P(C)}{P(R_1 \mid C)P(C) + P(R_1 \mid \overline{C})P(\overline{C})} \\ &= \frac{0.7 \cdot 0.5}{0.7 \cdot 0.5 + 0.1 \cdot 0.5} \\ &= \frac{0.35}{0.4} \\ &= 0.875. \end{aligned}$$

(b) We repeat the same process but use the updated probabilities from part (a) as $P(C)$ and $P(\overline{C})$. This gives us:

$$\begin{aligned} P(C \mid R_2) &= \frac{P(R_2 \mid C)P(C)}{P(R_2 \mid C)P(C) + P(R_2 \mid \overline{C})P(\overline{C})} \\ &= \frac{0.7 \cdot 0.875}{0.7 \cdot 0.875 + 0.1 \cdot 0.125} \\ &= \frac{0.6125}{0.6125 + 0.0125} \\ &= 0.98. \end{aligned}$$

(c) We use a similar process, but different probabilities.

$$
\begin{aligned}
P(C \mid \overline{R_3}) &= \frac{P(\overline{R_3} \mid C)P(C)}{P(\overline{R_3} \mid C)P(C) + P(\overline{R_3} \mid \overline{C})P(\overline{C})} \\
&= \frac{0.3 \cdot 0.98}{0.3 \cdot 0.98 + 0.9 \cdot 0.02} \\
&= \frac{0.294}{0.294 + 0.018} \\
&\approx 0.94.
\end{aligned}
$$

Grading Guidelines [20 points]

Part a:

(i) +2 correctly identifies $P(C)$
(ii) +2 correctly identifies $P(R_1 \mid C)$
(iii) +2 correctly identifies $P(R_1 \mid \overline{C})$
(iv) +2 attempts to solve for $P(C \mid R_1)$
(v) +2 correct final answer

Part b:

(vi) +2 correctly uses value from (a) for $P(C)$
(vii) +2 correct final answer (based on answer from (a))

Part b:

(viii) +2 correctly uses value from (a) for $P(C)$
(ix) +2 correctly identifies $P(\overline{R_3} \mid C)$ and $P(\overline{R_3} \mid \overline{C})$
(x) +2 correct final answer (based on answer from (a))