EECS 203: Discrete Mathematics
Winter 2024
FOF Discussion 5a Notes

1 Induction

- **Mathematical Induction**: Mathematical Induction is a proof method used to prove a predicate $P(n)$ holds for "all" $n \geq n_0$. Often "all" n is $\mathbb{N}$ or $\mathbb{Z}^+$, but the desired domain of n varies by problem. Mathematical induction consists of a base case and an inductive step, which proves: $[P(n_0) \ \wedge \ \forall k \geq n_0(P(k) \implies P(k+1))] \implies \forall n \geq n_0, P(n)$
- **Induction Steps**:
  - **Base Case**: The part of the inductive proof which directly proves the predicate for the *first* value in the domain (generally n_0). The base case does not rely on $P(k)$ for any other value of k. Often this will be $P(0)$ or $P(1)$
  - **Inductive Hypothesis**: The assumption we make at the beginning of the inductive step. The inductive hypothesis assumes that the predicate holds for some *arbitrary* member of the domain
  - **Inductive Step**: The proof which shows that the predicate holds for the "next" value in the domain. The inductive step should make use of the inductive hypothesis.
- **Exponent Product Rule**: $b^n \cdot b^m = b^{n+m}$. Often useful in induction proofs involving exponents

1.1 Equality

Prove by induction that the following equality is true for all positive integers n.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

Let $P(n)$ be ______________________________ = _______________.

Inductive Step: We assume that $P(k)$ is true for an arbitrary positive integer k such that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\frac{1}{k\cdot(k+1)}=\frac{k}{k+1}$. It must be shown that $\underline{P(\quad)}$ follows from this assumption.

Consider the LHS of P(k+1):

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k\cdot(k+1)}+\underline{\qquad\qquad\qquad}=\underline{\qquad\qquad\quad}+\frac{1}{(k+1)\cdot((k+1)+1)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)\cdot(k+2)}$$

$$=\frac{\qquad\qquad}{(k+1)(k+2)}+\frac{1}{\qquad\qquad\qquad}$$

$$=\underline{\qquad\qquad\qquad\qquad}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\underline{\qquad\qquad\qquad\qquad}$$

$$=\frac{k+1}{(k+1)+1}$$

$=$ RHS of P(k+1)

This shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. Note that the equality in line 1 is true by _________________________.

Base Case:
Our base case of $P(1)$ is true since ______________________________.

Therefore, since $P(1)$ and $\forall k[P(k)\rightarrow P(k+1)]$ are both true, then by mathematical induction, the claim is proven.

Solution: Let $P(n)$ be $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$.

Inductive Step:

We assume that $P(k)$ is true for an arbitrary positive integer k such that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\frac{1}{k\cdot(k+1)}=\frac{k}{k+1}$. It must be shown that $P(k+1)$ follows from this assumption.

$$\begin{aligned}
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k\cdot(k+1)}+\frac{1}{(k+1)\cdot((k+1)+1)} &= \frac{k}{k+1}+\frac{1}{(k+1)\cdot((k+1)+1)}\\
&= \frac{k}{k+1}+\frac{1}{(k+1)\cdot(k+2)}\\
&= \frac{k(k+2)}{(k+1)(k+2)}+\frac{1}{(k+1)\cdot(k+2)}\\
&= \frac{k^2+2k+1}{(k+1)(k+2)}\\
&= \frac{(k+1)^2}{(k+1)(k+2)}\\
&= \frac{k+1}{k+2}\\
&= \frac{k+1}{(k+1)+1}
\end{aligned}$$

This shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. Note that the equality in line 1 is true by the inductive hypothesis.

Base Case:
Our base case of $P(1)$ is true since $LHS=\frac{1}{1\cdot 2}=\frac{1}{2}=\frac{1}{1+1}=RHS$.

Therefore, since $P(1)$ and $\forall k P(k)\rightarrow P(k+1)$ are both true, then by mathematical induction, the claim is proven.

1.2 Bandar's Blunder $\star$

Bandar writes a proof for the following statement:

$$n! > n^2 \text{ for all } n \geq 4.$$

His proof is incorrect, and it's your task to help him identify his mistake!

Proof:

Inductive step:
Let k be an arbitrary integer ≥ 4.
Assume $P(k) : k! > k^2$. We need to show $P(k+1) : (k+1)! > (k+1)^2$

$$\begin{aligned}
(k+1)! &= (k+1) \cdot k! \\
&> (k+1) \cdot k^2 && \text{(By the Inductive Hypothesis)} \\
&= (k+1)(k \cdot k) \\
&\geq (k+1)(2 \cdot k) && \text{(Because } k \geq 2\text{)} \\
&= (k+1)(k+k) \\
&\geq (k+1)(k+1) && \text{(Because } k \geq 1\text{)} \\
&= (k+1)^2
\end{aligned}$$

This proves $(k+1)! > (k+1)^2$.

Base Case:
Prove $P(0) : 0! > 0^2$, $0! = 1 > 0^2 = 0$

Thus by mathematical induction, $n! > n^2$ for all $n \geq 0$.

What is wrong with Bandar's proof?

Solution: The key idea here is that although we have a valid base case, and a valid inductive step, they don't work together. In particular, the inductive step requires $k \geq 4$, but our base case only shows that $k = 0$ is valid (and in fact, $k = 1, k = 2,$ and $k = 3$ are false). A valid proof could have used the same inductive step with a base case of $n = 4$.

Some possible explanations:

- The base case and inductive step are individually valid, but the base case can't be used with the inductive step.

- The base case doesn't help prove the statement is true for $n = 4$, and this case can't be proved with the inductive step.
- The inductive step doesn't work with the given base case.

1.3 Check Your Understanding

a) If I wanted to show by induction that 3 divides $n^3 + 2n$ whenever n is a positive integer. What would I need to show?

b) Prove by induction that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Hint : $(a + b)^3 = a^3 + 3ab(a + b) + b^3$

Solution:

a) You would need to show that $3 \mid \left[1^3 + 2(1)\right]$ (the base case). Then you would need to show that IF $3 \mid \left[k^3 + 2(k)\right]$ for some $k \geq 1$ THEN $3 \mid \left[(k + 1)^3 + 2(k + 1)\right]$ (the inductive step).

b)

Inductive step: Assume that 3 divides $k^3 + 2k$ for some $k \geq 1$. Then, prove 3 divides $(k + 1)^3 + 2(k + 1)$.

$$\begin{aligned}(k + 1)^3 + 2(k + 1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= k^3 + 2k + 3k^2 + 3k + 3 \\ &= (k^3 + 2k) + 3(k^2 + k + 1)\end{aligned}$$

3 divides $3(k^2+k+1)$. By the inductive hypothesis, 3 divides k^3+2k. Thus, 3 divides $(k + 1)^3 + 2(k + 1)$.

Alternate Solution for Inductive Step: Assume that 3 divides $k^3 + 2k$. Then, prove 3 divides $(k + 1)^3 + 2(k + 1)$.

$$\begin{aligned}3|k^3 + 2k &\rightarrow 3|k^3 + 2k + 3(k^2 + k + 1) \\ &\rightarrow 3|k^3 + 2k + 3k^2 + 3k + 3 \\ &\rightarrow 3|k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &\rightarrow 3|(k + 1)^3 + 2(k + 1)\end{aligned}$$

Base case: $n = 1$. $n^3 + 2n = 1 + 2 = 3$. Since 3 is divisible by 3, the base case holds.

By induction, we have proven that for every positive integer n, 3 divides $n^3 + 2n$.

1.4 Sum Mathematical Induction

Using induction, prove that for all integers $n \geq 1$:

$$\sum_{r=1}^{n}(r+1) \cdot 2^{r-1} = n \cdot 2^n$$

Solution:

Inductive Step:

Let k be an arbitrary integer that is greater or equal to 1.

Assume $P(k) : \sum_{r=1}^{k}(r+1) \cdot 2^{r-1} = k \cdot 2^k$.

We want to show $P(k+1) : \sum_{r=1}^{k+1}(r+1) \cdot 2^{r-1} = (k+1) \cdot 2^{k+1}$

$$\begin{aligned}
&\sum_{r=1}^{k+1}(r+1) \cdot 2^{r-1} \\
&= [\sum_{r=1}^{k}(r+1) \cdot 2^{r-1}] + (k+1+1) \cdot 2^{k+1-1} \\
&= [\sum_{r=1}^{k}(r+1) \cdot 2^{r-1}] + (k+2) \cdot 2^{k} \\
&= [k \cdot 2^k] + (k+2) \cdot 2^k \text{ (by Inductive Hypothesis)} \\
&= k \cdot 2^k + k2^k + 2^{k+1} \\
&= 2k \cdot 2^k + 2^{k+1} \\
&= k \cdot 2^{k+1} + (1) \cdot 2^{k+1} \\
&= (k+1) \cdot 2^{k+1}
\end{aligned}$$

Therefore, $P(k+1)$ is true.

Base Case:

Prove $P(1) : \sum_{r=1}^{1}(r+1) \cdot 2^{r-1} = 1 \cdot 2^1$. $LHS = (1+1) \cdot (2)^0 = 2, RHS = (1) \cdot (2)^1 = 2,$

so $LHS = RHS$. Therefore, $P(1)$ is true.

Therefore we have shown by mathematical induction that for all integers $n \geq 1$,
$\sum_{r=1}^{n}(r+1) \cdot 2^{r-1} = n \cdot 2^n$

2 Exam Review

2.1 Satisfiability $\star$

Determine whether each of these compound propositions is satisfiable.

(a) $(p \lor \neg q) \land (\neg p \lor q) \land (\neg p \lor \neg q)$

(b) $(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q)$

Solution:

(a) Satisfiable. The expression is satisfied when p is False and q is False. You could draw up a truth table to help you think through the possible combinations of truth values for p and q.

(b) Unsatisfiable (ie a contradiction)

p	q	$p \to q$	$p \to \neg q$	$\neg p \to q$	$\neg p \to \neg q$	$(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q)$
T	T	T	F	T	T	F
T	F	F	T	T	T	F
F	T	T	T	T	F	F
F	F	T	T	F	T	F

Since all boolean assignments of p and q result in the expression being False, this is compound proposition is unsatisfiable.

Alternate Solutions:

- Using Equivalence Laws:

$$\begin{aligned}
&(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q) \\
&\qquad \equiv (\neg p \lor q) \land (\neg p \lor \neg q) \land (p \lor q) \land (p \lor \neg q) \\
&\qquad \equiv (\neg p \lor (q \land \neg q)) \land (p \lor q) \land (p \lor \neg q) \\
&\qquad \equiv \neg p \land (p \lor q) \land (p \lor \neg q) \\
&\qquad \equiv \neg p \land (p \lor (q \land \neg q)) \\
&\qquad = \neg p \land p \\
&\qquad = F
\end{aligned}$$

- Verbal Argument: In order to show that this statement is not satisfiable, we will consider every possible assignment of p and q and show that in every case, the statement is false. When p is true and q is true, $p \to \neg q$ is false so the whole statement is false. When p is true and q is false, $p \to q$ is false, so the whole statement is false. When p is false and q is true, $\neg p \to \neg q$ is false, so the whole statement is false. When p is false and q is false, $\neg p \to q$ is false, so the whole statement is false. Therefore, in every possible assignment of p and q, the statement is false, which means that the statement is not satisfiable.

2.2 Nested Quantifier Translations

Let P(x, y) be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

a) $\exists x \exists y P(x, y)$

b) $\exists x \forall y P(x, y)$

c) $\forall x \exists y P(x, y)$

d) $\exists y \forall x P(x, y)$

e) $\forall y \exists x P(x, y)$

f) $\forall x \forall y P(x, y)$

Solution:

a) There is a student in your class who has taken a computer science course [at your school].

b) There is a student in your class who has taken every computer science course.

c) Every student in your class has taken at least one computer science course.

d) There is a computer science course that every student in your class has taken.

e) Every computer science course has been taken by at least one student in your class.

f) Every student in your class has taken every computer science course.

2.3 English to Logic Translation

Define the following propositions:

- p: the user enters a valid password
- q: access is granted
- r: the user has paid the subscription fee

Express the following using p, q, r and logical operators

a) the user has paid the subscription fee, but does not enter a valid password

b) access is granted whenever the user has paid the subscriptoin fee and enters a valid password

c) access is denied if the user has not paid the subscription fee

d) if the user has not entered a valid password but has paid the subscription fee, then access is granted

Solution:

a) $r \land \neg p$

b) $(r \land p) \implies q$

c) $\neg r \implies \neg q$

d) $(\neg p \land r) \implies q$

2.4 Proof Practice

2.4.1 Proof I

Prove that the product of two odd numbers is odd.

Solution: Using a Direct Proof,
Let a and b be arbitrary odd integers. Then, a and b can be written as $a = 2m + 1$ and $b = 2n + 1$ for some integers n and m. Looking at their product, we have

$$\begin{aligned} ab &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

Since $ab = 2k + 1$, where k is the integer $2mn + m + n$, then by definition ab is odd.

2.4.2 Proof II

Prove that for all integers n, if $n^2 + 2$ is even, then n is even.

Solution: Let n is an arbitrary integer. For the sake of contradiction, assume $n^2 + 2$ is even and n is odd.
(Note that we could have also assumed the negation of the entire statement: "Assume that there exists some n such that $n^2 + 2$ is even and n is odd".)

- Since n is odd, we can say $n = 2k + 1$ for some integer k.
- This means $n^2 + 2 = (2k + 1)^2 + 2$.
  $= 4k^2 + 4k + 1 + 2$
  $= 2(2k^2 + 2k + 1) + 1$
  $= 2j + 1$, where j is an integer equal to $2k^2 + 2k + 1$
- Thus from the definition of an odd number, $n^2 + 2$ is odd. This contradicts our earlier assumption that $n^2 + 2$ is even.

Therefore, using proof by contradiction, we have showed that for all integers n, if n is odd, then $n^2 + 2$ is odd.

2.4.3 Proof III

Prove that for all integers x and y, if xy^2 is even, then x is even or y is even.

Solution: We will prove the statement via proof by contrapositive. Let x and y be arbitary integers. Because we are using proof by contrapositive, we want to assume x is odd and y is odd and eventually conclude that xy^2 is odd. First, we will assume x is odd and y is odd. Since x and y are odd, $x = 2k + 1$ and $y = 2n + 1$ where k and n are integers. Therefore, $xy^2 = (2k + 1)(2n + 1)^2 = (2k + 1)(4n^2 + 4n + 1) = 8kn^2 + 8kn + 2k + 4n^2 + 4n + 1 = 2(4kn^2 + 4kn + k + 2n^2 + 2n) + 1 = 2j + 1$ where j is an integer and $j = 4kn^2 + 4kn + k + 2n^2 + 2n$. Therefore, xy^2 is odd. Thus, we have shown via proof by contrapositive that for all integers x and y, if xy^2 is even, then x is even or y is even.

2.4.4 Proof IV

Prove or Disprove that for all integers n, $n^2 + n$ is even.

Solution: We prove the statement via proof by cases. Let x be an arbitrary integer.

- **Case 1:** x is even
  Since x is even, $x = 2k$ where k is an integer. Therefore, $x^2 + x = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k) = 2j$ where j is some integer. Therefore, $x^2 + x$ is even.

- **Case 2:** x is odd
  Since x is odd, $x = 2k + 1$ where k is an integer. Therefore, $x^2 + x = (2k + 1)^2 + (2k + 1) = (4k^2 + 4k + 1) + (2k + 1) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1) = 2j$ where j is some integer. Therefore, $x^2 + x$ is even.

For all cases of x, we have shown that $x^2 + x$ is even. Therefore, we have shown that for all integers n, $n^2 + n$ is even.

2.4.5 Proof V

Prove or Disprove that for all integers a and b, $\frac{a}{b}$ is a rational number.

Solution:
We will disprove this statement. Consider the case, $a = 1$ and $b = 0$. In this case, $\frac{a}{b}$ is not a rational number because $b = 0$.

2.5 Set Equality

Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$ by showing that either side is a subset of the other.

Solution: We will show that these two sets are equal by showing that each is a subset of the other.

LHS $\subseteq$ RHS:
Suppose $x \in (A - B) - C$
By the definition of set difference, we have $x \in (A - B)$ but $x \notin C$.
Applying the definition of set difference again, we have $x \in A$ but $x \notin B$ and $x \notin C$.
Thus as $x \in A$ and $x \notin C$, then we have $x \in (A - C)$.
We also know that $x \notin B$, as $B - C$ is a subset of B, then we also have that $x \notin B - C$.

Therefore we can conclude that $x \in (A - C) - (B - C)$ and that $(A - B) - C \subseteq (A - C) - (B - C)$.

RHS $\subseteq$ LHS:
Suppose that $x \in (A - C) - (B - C)$.
Thus by the definition of set difference, we have $x \in (A - C)$ and $x \notin (B - C)$.
Thus we know that $x \in A$ and $x \notin C$.

We also know that $x \notin B$, because if $x \in B$, then $x \in (B - C)$ which does not satisfy our given that $x \notin (B - C)$.
We now have $x \in A$ and $x \notin B$ and $x \notin C$.
By the definition of set difference, we have $x \in (A - B)$ and $x \notin C$ which then leads to $x \in (A - B) - C$.
Thus $(A - C) - (B - C) \subseteq (A - B) - C$.

Therefore, as we have shown both sides are subsets of each other, we can conclude that $(A - B) - C = (A - C) - (B - C)$

2.6 More Power Sets $\star$

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$

b) $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Solution:

a) The power set of every set includes at least the empty set, so the power set cannot be empty. Thus $\emptyset$ is not the power set of any set.

b) This is the power set of $\{a\}$.

c) We know that the power set a set of size n has 2^n elements, but this set has three elements. Since 3 is not a power of 2, this set cannot be the power set of any set. Set cardinality aside, the set $\{\emptyset, a\}$ may come to mind, but $P(\{\emptyset, a\}) = \{\emptyset, \{\emptyset\}, \{a\}, \{\emptyset, a\}\}$.

d) This is the power set of $\{a, b\}$.

2.7 Inclusion−Exclusion Principle:

The inclusion-exclusion principle states the the size of the union of two sets is equal to the sum or their sizes minus the size of their intersection:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

A similar principle can be applied to obtain the following formula, for the cardinality of the union of three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

2.8 Three sets

Suppose there is a group of 120 U of M students. Here's what you know:

- There are 31 in Engineering.
- There are 65 in LSA.
- There are 44 in Ross.
- There are 20 that are not in any of these 3 schools.
- There are 15 in Engineering and Ross.
- There are 17 in Engineering and LSA.
- There are 18 in LSA and Ross.

How many are in all 3 schools?

Solution: We see that the 20 people that are not in any of the schools do not contribute to the count of any other number, and so we can simply look at the $120 - 20 = 100$ students that are in the schools that we are looking at.

By the inclusion-exclusion principle, we have to add the number of people in the 3 individual categories independently, then subtract the pairwise totals, and add back the number of those in all 3 categories. This should yield our original number of students.

$$31 + 65 + 44 - 15 - 17 - 18 + x = 100$$

Doing some algebra, we see that $x = 10$. Thus there are 10 people in all three schools.