Groupwork 7 Problems

1. Get to the Point [10 points]

Consider an arbitrary set A. We say a function $f : A \to A$ has a *fixed point* iff there exists $a \in A$ such that $f(a) = a$.

Consider the notation $f^{(n)}$ to mean $\underbrace{f \circ \cdots \circ f}_{n \text{ times}}$, where $n \in \mathbb{Z}^+$. Essentially, n copies of f are composed together.

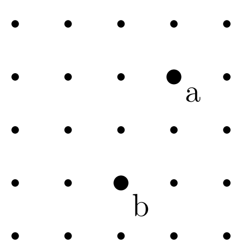
Prove by **induction** that if f is a function with a fixed point, then for all positive integers n, $f^{(n)}$ has a fixed point.

Solution:

Let P(n) be the statement that $f^{(n)}$ has a fixed point.
Inductive step: Assume P(k) is true for some $k \in \mathbb{Z}^+$.
Want to show: $P(k) \implies P(k+1)$ for all $k \in \mathbb{Z}^+$.
i.e. $\exists a \in A$ such that $f^{(k)}(a) = a \implies \exists b \in A$ such that $f^{(k+1)}(b) = b$.
$f^{(k+1)}(b) = f(f^{(k)}(b))$.
By inductive hypothesis, $f^{(k)}(b) = b$.
Then $f(f^{(k)}(b)) = f(b) = b$.
So b is a fixed point of $f^{(k+1)}$.
Thus, $P(k) \implies P(k+1)$ for all $k \in \mathbb{Z}^+$.
Base case: $n = 1$. Then $f^{(1)} = f$ has a fixed point by assumption.

2. Going Off the Grid [8 points]

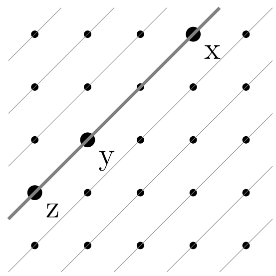
In a grid, we say that a point a *dominates* a point b iff a lies strictly above and to the right of b. For example, in the picture below, a dominates b.



Prove using the Pigeonhole Principle that if we choose $4n - 1$ points from an $n \times n$ grid $(n \geq 4)$, there must be three chosen points x, y, z such that x dominates y and y dominates

z. Make sure to state what your pigeons are and what your holes are, as well as how many of each you have.

Hint: If x, y, z lie on the same increasing diagonal as shown in the picture below, then x dominates y and y dominates z.



Solution:

Pigeons: the $4n - 1$ points chosen from the grid.
Holes: the $n - 1$ increasing diagonals of the grid.
Each point lies on exactly one increasing diagonal.
By the Pigeonhole Principle, at least one increasing diagonal contains at least $\lceil \frac{4n-1}{n-1} \rceil = 4$ points.
By the hint, there must be three chosen points x, y, z such that x dominates y and y dominates z. Thus, if we choose $4n - 1$ points from an $n \times n$ grid ($n \geq 4$), there must be three chosen points x, y, z such that x dominates y and y dominates z.