EECS 203: Discrete Mathematics
Winter 2024
Homework 3

Due **Thursday, Feb. 8**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $7 + 1$ Total Points: $100 + 30$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

1. On the Contrary [12 points]

Let n be an integer. Prove that if $4 \mid (n^2 - 1)$, then n is odd using

(a) a proof by contraposition, and

(b) a proof by contradiction.

Then,

(c) compare your answers to parts (a) and (b). What is different? What is the same?

Solution:

Let p be $4 \mid (n^2 - 1)$, q be n is odd.
The original proposition can therefore be expressed as $\forall n(p \rightarrow q)$.

a) Proof:

To prove by contraposition,	
the contrapositive is $\forall n(\neg q \rightarrow \neg p)$.	contraposition
Assume n is an even integer.	
$n = 2k$, k is a random integer.	definition of even
$n^2 - 1 \equiv 4k^2 - 1$	substitution
It does not divide 4.	definition of divide
Thus, the original proposition is true by contraposition.	

b) Proof:

To prove by contradiction,	
the negation is $\exists n(p \land \neg q)$.	Implication Breakout Rule
Assume n is an even integer, $n^2 - 1$ divides 4.	
$n = 2k$, k is a random integer.	definition of even
$n^2 - 1 \equiv 4k^2 - 1$	substitution
It does not divide 4.	definition of divide
Thus, the original proposition is true by contradiction.	

c)
The math is the same, but the logic is different (premise is different due to the difference between contradiction and contrapostion).

2. An Even-Numbered Question about Even Numbers [16 points]

Prove or disprove the following statements:

(a) For all integers x, if x is even, then x^2 is even.

(b) For all integers x, if x^2 is even, then x is even.

(c) For all integers x, if x is even, then $2x$ is even.

(d) For all integers x, if $2x$ is even, then x is even.

Solution:

a) Proof:

Assume x is a random even integer.	premise
Since x is even,	
$x = 2k$, k is a random integer.	definition of even
$x^2 = 4k^2 = 2(2k^2)$.	subsitution
Since $2k^2$ is also an integer,	
let $2j = 2k^2$, j also a random integer.	
Thus, x^2 is also even.	definition of even

b) Proof:

To prove by contrapositive, assume x is a random odd integer.	premise
Since x is odd,	
let $x = 2k + 1$, k is a random integer	definition of odd
$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$	substitution
Since $2k^2 + 2k$ is also an integer,	
let $2j = 2k^2 + 2k$, j also a random integer.	
Thus, x^2 is, $2j + 1$, also odd.	definition of odd
Thus the original proposition is true by contraposition.	

c) Proof:

Assume x is a random even integer.	premise
Since x is even,	
$x = 2k$, k is a random integer.	definition of even
$2x = 2(2k)$	substitution
Since k is also an integer,	
let $j = 2k$, j also a random integer.	
Thus, $2x$ is also even.	definition of even
Thus the original proposition is true by direct proof.	

d) Disproof:

To disprove by contrapositive, assume x is odd.	premise
Since x is odd,	
let $x = 2k + 1$, k is a random integer	definition of even
$2x = 4k + 2 = 2(2k + 1)$	substitution
Since k is also an integer,	
let $j = 2k + 1$, j also a random integer.	
Thus, x, is $2j$, even.	definition of even
Thus the original proposition is false.	contraposition is false

3. Even Stevens [16 points]

Prove or disprove the following statement: "There is a finite amount of even numbers."

Solution:

Disproof:
Assume there is a set of finite amount of even numbers $n = a, b, c, d,$
with a domain of even numbers
and d being the largest possible even number premise
Let e be a random even number that is twice d.
Since e is $2d$, an even number greater than the assumed d,
the proposition is false.
Thus, disproved there is a set of finite amount of even numbers.

4. Pay it Forward (Or Don't, It's Up To You) [12 points]

Consider a centipede game, where there are two players: Ka-chun and Zyaire. The game starts by Ka-chun's decision of take or wait.

- If Ka-chun takes, Ka-chun earns $1 while Zyaire earns nothing, and the game ends.
- If Ka-chun waits, then Zyaire can choose between take or wait. If Zyaire takes, Zyaire earns $2 while Ka-chun earns nothing and the game ends. If Zyaire waits it becomes Ka-chun's turn to choose again.
- If they keep waiting the reward grows by $1 each round, until Zyaire's choice of taking $20 or waiting, when the game will end no matter what.

Both of Ka-chun and Zyaire want to maximize their rewards, and behave as perfect logicians.

(a) Suppose Ka-chun and Zyaire made it to round 20. What happens in round 20?

(b) Using your answer to (a), what would happen if they made it to round 19?

(c) Building off of parts (a) and (b), argue that Ka-chun should take $1 in the very first round.

Solution:

a) Zyaire takes $20 and wins.
This is because the game ends after 20 rounds, if Z does not take, Z wins nothing when

the game is over.
b) Ka-Chun takes \$19 and wins.
This is because if Z does not take before the 19th round, it is the only chance for K to win.
c) Ka-Chun has to take the \$1 in order to win something for sure, otherwise Z will win it all.

5. Proofs to the Max [12 points]

Prove that for all real numbers a, b, and c, if $\max\{a^2(b-c), -a\}$ is non-negative, then $a \leq 0$ or $b \geq c$.

Note: You can use the following facts in your proof:

- If x and y are positive, then $x \cdot y$ is positive.
- If x is positive and y is negative, then $x \cdot y$ is negative.
- If x and y are negative, then $x \cdot y$ is positive.

Solution:

Proof:
To prove by contradiction,
assume the negation $\max\{a^2(b-c), -a\} \geq 0 \wedge (a \geq 0 \wedge b \leq c)$ premise
Since $a \geq 0$, $-a \leq 0$
Since $\max\{a^2(b-c), -a\} \geq 0 \wedge -a \leq 0$, $a^2(b-c) \geq 0$
Since $a^2 > 0 \wedge b - c < 0$, $a^2(b-c) < 0$ fact 2
This contradicts with $a^2(b-c) \geq 0$
Thus, the original proposition is true by contradiction.

6. Let's All Be Rational [16 points]

Show that these statements about a real number x are equivalent to each other:

(i) x is rational
(ii) $\frac{x}{2}$ is rational
(iii) $3x - 1$ is rational.

Hint: One way to prove statements (i), (ii) and (iii) are equivalent is by proving (i) $\rightarrow$ (ii), (ii) $\rightarrow$ (iii), and (iii) $\rightarrow$ (i).

Solution:

Proof: $(i) \rightarrow (ii)$

Assume $x \in \mathbb{Q}$	premise
$x = \frac{a}{b}, a \in \mathbb{Z} \wedge b \in \mathbb{Z} \wedge b \neq 0$	definition of rational
$\frac{x}{2} = \frac{a}{2b}$, which is also an $\frac{integer}{integer}$.	subsitution
Thus, $x \in \mathbb{Q} \rightarrow \frac{x}{2} \in \mathbb{Q}$.	definition of rational

Proof: $(ii) \rightarrow (iii)$

Assume $\frac{x}{2} \in \mathbb{Q}$	premise
$\frac{x}{2} = \frac{a}{b}, a \in \mathbb{Z} \wedge b \in \mathbb{Z} \wedge b \neq 0$	definition of rational
$\frac{x}{=}\frac{2a}{b}, 3x - 1 = \frac{3\dot{2}a}{b} - 1 = \frac{3a-b}{b}$, which is also an $\frac{integer}{integer}$.	subsitution
Thus, $\frac{x}{2} \in \mathbb{Q} \rightarrow 3x - 1 \in \mathbb{Q}$.	definition of rational

Proof: $(iii) \rightarrow (i)$

Assume $3x - 1 \in \mathbb{Q}$	premise
$3x - 1 = \frac{a}{b}, a \in \mathbb{Z} \wedge b \in \mathbb{Z} \wedge b \neq 0$	definition of rational
$x = \frac{1+\frac{a}{b}}{3}$, which is also an $\frac{integer}{integer}$.	subsitution
Thus, $3x - 1 \in \mathbb{Q} \rightarrow x \in \mathbb{Q}$.	definition of rational

Thus, the three statements are equivlent to each other.

7. Irrational Pr00f [16 points]

Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

Solution:

Proof:

To prove by contradiction, assume the negation.

Let $x \times y = z, x \notin \mathbb{Q}, y \in \mathbb{Q}, z \in \mathbb{Q}$.	premise
Since y and z are rational, $y = \frac{a}{b}, z = \frac{c}{d}$,	
$a \in \mathbb{Z}, b \in \mathbb{Z}, c \in \mathbb{Z}, d \in \mathbb{Z}$	definion of rational
$x \times \frac{a}{b} = \frac{c}{d}$	substitution

$x = \frac{cb}{ad}$, which is also $\frac{integer}{interger}$, therefore $x \in \mathbb{Q}$.

It contradicts with the premise $x \notin \mathbb{Q}$.

Thus, the original proposition holds.

Grading of Groupwork 2

Using the solutions and Grading Guidelines, grade your Groupwork 2 Problems:

- Use the table below to grade your past groupwork submission and calculate scores.
- While grading, mark up your past submission. Include this with the table when you submit your grading.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- See "All About Groupwork" on Canvas for more detailed guidance, and what to do if you change groups.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	Total:
Problem 1												/20
Problem 2												/20
Total:												/40

Groupwork 3 Problems

1. ∀re These ∃quiv♢lent? [30 points]

Let $P(x)$ and $Q(x)$ be arbitrary predicates.

(a) Prove or disprove that for any domain of x, $\forall x(P(x) \leftrightarrow Q(x))$ must be logically equivalent to $\forall x P(x) \leftrightarrow \forall x Q(x)$.

(b) Prove or disprove that for any domain of x, $\exists x(P(x) \leftrightarrow Q(x))$ must be logically equivalent to $\exists x P(x) \leftrightarrow \exists x Q(x)$.

(c) Let $\diamondsuit x$ mean that "there exists **at most one** x." Prove or disprove that for any domain of x, $\diamondsuit x(P(x) \leftrightarrow Q(x))$ must be logically equivalent to $\diamondsuit x P(x) \leftrightarrow \diamondsuit x Q(x)$.

Solution:

a) Disproof:
To disprove $\forall x(P(x) \leftrightarrow Q(x)) \equiv \forall x P(x) \leftrightarrow \forall x Q(x)$, $P(x)$ has to be sometimes true but not always, ar
Let $P(x)$ be $2 \mid x$ and $Q(x)$ be $3 \mid x$.
The left proposition will be false because there exists an x that will only satisfy either proposition but
The right, however, will be always true because not all x satisfies both propositions.
Thus, $F \not\equiv T$, therefore it is disproved.