EECS 203: Discrete Mathematics
Winter 2024
FoF Worksheet 7

1 Pigeonhole Principle

- **Pigeonhole Principle:** If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects
- **Generalized Pigeonhole Principle:** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

1.1 Multiple choice pigeons

What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

a) 100

b) 101

c) 4951

d) 5001

1.2 Pigeonhole Principle

How many distinct numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

1.3 More Pigeonhole Principle

(a) Undergraduate students at a college belong to one of four groups depending on the year in which they are expected to graduate. Each student must choose one of 21 different majors. How many students are needed to assure that there are two students expected to graduate in the same year who have the same major?

(b) What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

1.4 Even More Pigeonhole Principle

Sophia has a bowl of 15 red, 15 blue, and 15 orange pieces of candy. Without looking, Sophia grabs a handful of pieces.

(a) What is the smallest number of pieces of candy Sophia has to grab to make sure she has at least 4 of the same color?

(b) What is the smallest number of pieces of candy Sophia has to grab to make sure she has 3 orange candies?

1.5 Pigeonhole Principle

A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

2 Functions: Countability

- **Countably Infinite:** A set is countably infinite if it has the same cardinality as the natural numbers. This can be proven for a set by finding a one-to-one correspondence between it and the natural numbers. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are common examples of sets that are countably infinite.
- **Uncountably Infinite:** A set is said to be uncountably infinite if its cardinality is larger than that of the set of all natural numbers. $\mathbb{R}$ is one example of a set that is uncountably infinite.

2.1 Countability Multiple Choice

Which of the following sets is countably infinite?

a) Set of real numbers $\mathbb{R}$

b) Set of integers $\mathbb{Z}$

c) Set of complex numbers

d) Set of irrational numbers

2.2 More Countability Multiple Choice

Which of the following statements about countably infinite sets is true?

a) Countably infinite sets have a smaller cardinality than uncountably infinite sets.

b) Countably infinite sets cannot be enumerated in a systematic way.

c) Countably infinite sets have a one-to-one correspondence with the set of natural numbers.

d) Countably infinite sets include the set of real numbers $\mathbb{R}$.

2.3 Different Infinities

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) the integers greater than 10

b) the integers with absolute value less than 1,000,000

c) the real numbers between 0 and 2

d) the set $A \times \mathbb{Z}$ where $A = \{2, 3\}$

2.4 Different Infinities with Sets

Give an example of two uncountable sets A and B such that $A \cap B$ is

a) finite

b) countably infinite

c) uncountably infinite

2.5 Cardinality Proof

Show that $|(0,1)| \geq |\mathbb{Z}^+|$.

2.6 Countability

(a) Find a countably infinite subset A of $(0,1)$.

(b) Find a bijection between A and $A \cup \{0,1\}$

(c) Find an explicit one-to-one and onto mapping from the open interval $(0,1)$ to the closed interval $[0,1]$.

3 Schroder-Bernstein Theorem

Schroder-Bernstein Theorem: For two sets A and B, if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. Note that finding an onto function from A to B shows that $|A| \geq |B|$, and finding a one-to-one function from A to B shows that $|A| \leq |B|$.

1. Schroder-Bernstein Theorem

Show that $(0, 1)$ and $[0, 1]$ have the same cardinality.

4 Supplemental Problems

4.1 Generic pairwise sum

Let $n \geq 2$ be an integer, and suppose that we have selected $n+1$ different integers from the set $\{1, 2, 3, ..., 2n\}$. Prove that there will always be two among the selected integers whose sum is equal to $2n+1$.

4.2 Deck of Cards

a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are selected?

b) How many must be selected from a standard deck of 52 cards to guarantee that at least three hearts are selected?