PRACTICE Exam 2 Solutions
EECS 203, Winter 2024

Name (ALL CAPS): ______________________________

Uniqname (ALL CAPS): ______________________________

8-Digit UMID: ______________________________

Instructions

- When you receive this packet, fill in your name, Uniqname, and UMID above.
- Once the exam begins, make sure you have problems 1-18 in this booklet.
- Write your UMID in the blank at the top of every other page.
- No one may leave within the last 10 minutes of the exam.
- After you complete the exam, sign the Honor Code below. If you finish when time is called, your proctor will give you time to sign the Honor Code.
- Do not detach the scratch paper at the end of the packet.
- Do not discuss the exam until solutions have been released!

Materials

- No electronics allowed, including calculators.
- You may use one 8.5″ by 11″ note sheet, front and back, created by you.
- You may not use any other sources of information.

Honor Code

This exam is administered under the College of Engineering Honor Code. Your signature endorses the pledge below. We will not grade your exam without your signature.

I have neither given nor received unauthorized aid on this examination, nor have I concealed any violations of the Honor Code. I further agree not to discuss any aspect of this examination in any way, shape, or form until the solutions have been published.

Signature: ______________________________

Part A: Single Answer Multiple Choice

Instructions

- There are 6 questions in this section.
- Shade **only one** circle corresponding to your answer choice.
- If you shade more than one circle, your answer will be marked as incorrect.

Example. ⓐ ⓑ ● ⓓ ⓔ

Make sure to SHADE A BUBBLE next to the question title, as shown above.

Problem 1. (4 points) ⓐ ⓑ ⓒ ⓓ ⓔ

Whats the minimum number of initial conditions necessary to define the following recurrence?

$$P(n) = P(n-2) + 2^{P(n-4)} + sin(n-5)$$

(a) 2

(b) 3

(c) 4

(d) 5

(e) 6

Solution

(c). Since our recurrence requires us knowing the value of P(n-4) to find the value of P(n), we need at least 4 initial conditions. Note: We don't need 5 initial conditions since we can find the value of $sin(n-5)$ for any n without needing to know a single step in the recurrence.

Problem 2. (4 points) ⓐ ⓑ ⓒ ⓓ ⓔ

Consider a set $A \subseteq \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$. What is the minimum cardinality of A that guarantees there exist two distinct elements a and b in A such that $ab = 100$?

(a) 5

(b) 6

(c) 7

(d) 8

(e) 9

Solution

(b)
We can use the Pigeonhole Principle. We can model the holes as the pairs of numbers that multiply to 100. $\{1, 100\}, \{2, 50\}, \{4, 25\}, \{5, 20\}$. However, we need any potentially selected number to have a hole for it, so we need a hole for 10 as well. We don't want to allow anything else with it, though, because it can't multiply by any *other* number to make 100. There are 5 holes, so $5 + 1 = 6$ pigeons is the lowest number needed to ensure there is a hole with 2 pigeons in it from the selected 8 numbers. The hole with 2 pigeons in it can't be the hole with 10 because that hole is only set up to fit 1 pigeon, so it must be one of the others, and for any of the other 4 holes, if the two numbers in it are both selected, they multiply to 100.

Problem 3. (4 points)

a b c d e

Suppose $a \equiv 4 \pmod{13}$ and $b \equiv 7 \pmod{13}$. Compute $(a^5 + ab^2) \bmod 13$.

(a) 1

(b) 4

(c) 7

(d) 9

(e) 11

Solution

(c)11

$$\begin{aligned}
c &\equiv a^5 + ab^2 \\
&\equiv 4^5 + (4)(7^2) \\
&\equiv (4^2)^2(4) + (4)(49) \\
&\equiv (16^2)(4) + 4(10) \\
&\equiv (3^2)(4) + 40 \\
&\equiv (9)(4) + 1 \\
&\equiv (36) + 1 \\
&\equiv 37 \equiv 11 \pmod 3
\end{aligned}$$

Problem 4. (4 points)

a b c d e

All 25 of the EECS 203 IAs are participating in a donut-eating contest sponsored by the GSIs. Anyone who ate the most donuts won the contest. If 520 donuts were eaten during the contest, what is the minimum number of donuts a winner could have eaten?
Note: There could be multiple winners.

(a) 20

(b) 21

(c) 22

(d) 23

(e) 24

Solution

(b) Here we use the generalized Pigeonhole Principle, which states that if there are N pigeons and k pigeonholes, then there are at least $\lceil \frac{N}{k} \rceil$ pigeons together in some pigeonhole. In this problem, we have 520 donuts and 25 IAs, and each donut must have been eaten by exactly one IA, so there are 520 "pigeons" and 25 "pigeonholes". Plugging these numbers into the Generalized Pigeonhole Principle, we get that there must have been at least $\lceil \frac{520}{25} \rceil = \lceil 20.8 \rceil = 21$ donuts that were eaten by the same person so a winner must have eaten at least 21 donuts.

Problem 5. (4 points)

ⓐ ⓑ ⓒ ⓓ ⓔ

How many initial conditions are needed for the following recurrence?

$$a_n = a_{n-1} + a_{n-4} + 2^n$$

(a) 1
(b) 2
(c) 3
(d) 4
(e) Not enough information

Solution

(d) This recurrence reaches back 4 terms, so we need 4 initial conditions. If we had fewer than that, a_{n-4} would be undefined for the first term after the initial conditions.

Problem 6. (4 points)

ⓐ ⓑ ⓒ ⓓ ⓔ

Let f and g be functions from $\mathbb{R} \to \mathbb{R}$ with $f(x) = x^2 - 4$ and $g(x) = 3x - 2$. Find $(f \circ g)(x)$.

(a) $3x^2 - 14$
(b) $3x^2 - x$
(c) $x^2 + 3x - 6$
(d) $3x^3 - 2x^2 - 12x + 8$
(e) $9x^2 - 12x$

Solution

(e)

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= (g(x))^2 - 4 \\ &= (3x-2)^2 - 4 \\ &= 9x^2 - 12x + 4 - 4 \\ &= 9x^2 - 12x\end{aligned}$$

Note that (a) is $(g \circ f)(x)$, (c) is $f(x) + g(x)$, and (d) is $f(x) \cdot g(x)$.

Part B: Multiple Answer Multiple Choice

Instructions

- There are 4 questions in this section.
- Shade whichever boxes you believe are correct. **This could be all answers, no answers, or anything in between.**
- If there are no correct answers, leave all the boxes blank.

Example	a b c d e

Make sure to SHADE 0 OR MORE BOXES next to the question title, as shown above.

Problem 7. (4 points)	a b c d e

Let P be a predicate over $\mathbb{N}$ that is not always true, and such that $\forall n, P(n+1) \to P(n)$. Which of the following **must** be true?

(a) If $P(0)$, then $P(203)$.

(b) If $P(203)$, then $P(0)$.

(c) There is some $c \in \mathbb{N}$ such that $P(c)$ is true but $P(c+1)$ is false.

(d) There is some $c \in \mathbb{N}$ such that $P(n)$ is true if and only if $n < c$.

(e) None of the above

Solution

b, d

(a) We're given that P is not always true, and the truth value of $P(0)$ doesn't tell us anything about the successive predicates. Then, $P(203)$ is not necessarily true.

(b) This follows from the same ideas as induction:
we can see that $P(203) \to P(202) \to \cdots \to P(0)$.

(c) Consider the case where $P(n)$ is always false.

(d) There must be a largest value x such that $P(x)$ is true, as otherwise, P would be always true. Thanks to the given implies statement, every value less than $x+1$ must be true. In the case that P is always false (so there is no largest x that makes $P(x)$ true because there is no value that makes it true), we can use $c = 0$.

(e) Since (b) and (d) are correct, this answer choice is incorrect.

Problem 8. (4 points)

a b c d e

- Let S be the set of University of Michigan students.
- Let M be the set of all college majors formally offered by the University of Michigan.
- Consider the function $f: S \to M$ that maps each student to their declared major.

Assume each student has declared exactly one major. Select each of the following that **must** be true.

(a) If $|S| > |M|$, then there exists a major declared by multiple students.

(b) If $|S| > |M|$, then each major has at least one student.

(c) If $|S| < |M|$, then no two students have declared the same major.

(d) If $|S| < |M|$, then some major has no student declared.

(e) f has an inverse, $f^{-1}: M \to S$.

Solution

The correct choices are **(a), (d)**.

(a) In this scenario, each student must declare exactly one major and $|S| > |M|$. With this restriction, if there wasn't a major declared by multiple students (e.g. each student declared a unique major), then there must be at least as many majors as students. However, $|M| < |S|$, so there must be at least 1 major declared by multiple students. Therefore, this is true.

(b) In this scenario, $|S| > |M|$, and each student needs to have declared exactly one major. It's possible that there could be some majors with multiple students declared while others have none. This answer choice assumes that every major must have at least one student, which may not be the case, since multiple students can declare the same major (ex. every student declares CSE as their major). Therefore, this is false.

(c) In this scenario, $|S| < |M|$ and each student needs to have declared exactly 1 major. It's possible for each student to declare a unique major (since it would satisfy the constraint of every student having declared exactly 1 major), but there is also nothing stopping two or more students from declaring the same major. Therefore, since there is no restriction constraining multiple students from declaring the same major, this is false.

(d) In this scenario, $|S| < |M|$ and every student has to declare exactly 1 major. It is possible for each student to declare a unique major, which would mean $|S| = |M|$. Since $|M| > |S|$, we know that even in the most favorable case, where each student declares a unique major, there must exist a major that doesn't have a student declared. Therefore, this is true.

(e) For a function to have an inverse, it must be a bijection which means it must be both 1-1 and onto. In this scenario, all we know about $f\colon S \to M$ is that each student has declared a major. However, it could be the case that multiple students declared the same major, which means the function won't be 1-1 since the mapping isn't unique. Also, there could be majors without any students declaring them, meaning the function wouldn't be onto. So, the function wouldn't be bijective. Therefore, this is false.

Problem 9. (4 points)

a b c d e

Let $x \equiv 4 \pmod 5$ and $y \equiv 7 \pmod{10}$. Which of the following statements **must** be true?

(a) $xy \equiv 3 \pmod 5$

(b) $x - y \equiv 3 \pmod 5$

(c) $x + y \equiv 1 \pmod 5$

(d) $x + y \equiv 1 \pmod{10}$

(e) $xy \equiv 8 \pmod{10}$

Solution

(a), (c)

The key to solving this question is recognizing that $y \pmod 5$ can be computed whereas $x \pmod{10}$ cannot. Since $y \equiv 7 \pmod{10}$,

$$y = 10k + 7 \text{ where k is an integer} \tag{1}$$
$$y = 5(2k + 1) + 2 \tag{2}$$
$$y \equiv 2 \pmod 5 \tag{3}$$

x on the other hand is $4 + 5k$ which cannot be rewritten as $4 + 10(int)$ without knowing the value of k. Hence x can be either $4 \pmod{10}$ or $9 \pmod{10}$.

(a) $xy \equiv 4 \cdot 2 \equiv 8 \equiv 3 \pmod 5$

(b) $x - y \equiv 4 - 2 \equiv 2 \pmod 5$

(c) $x + y \equiv 4 + 2 \equiv 6 \equiv 1 \pmod 5$

(d) Since we don't know what $x \pmod{10}$ is, this value cannot be computed

(e) Similarly, this cannot be computed either

Problem 10. (4 points)

a b c d e

Which of the following sets are **countably infinite**?

(a) $\mathbb{R} - \mathbb{Q}$

(b) $[0, 203] \cap \mathbb{Z}$

(c) $\mathbb{Q} \times \mathbb{Q}$

(d) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y = x^2\}$

(e) $\{(x, y) \in \mathbb{R} \times \mathbb{Z} \mid y = x^2\}$

Solution

(c), (d), (e)

(a) The set $\mathbb{R}$ is uncountable and the set $\mathbb{Q}$ is countably infinite. To construct the set $\mathbb{R} - \mathbb{Q}$, we remove all of the rational numbers from the set of real numbers so that we are left with only irrational numbers. The set of irrational numbers is still uncountably infinite. As a proof, assume for contradiction that the set of irrational numbers is countable. Therefore, $\mathbb{R} = \mathbb{Q} \cup$ irrational numbers. Therefore, $\mathbb{R}$ must be countable because $\mathbb{Q}$ and the set of irrational numbers are both countable. This is a contradiction because $\mathbb{R}$ is uncountably infinite. By proof by contradiction, the set of irrationals must be uncountably infinite. Therefore, $\mathbb{R} - \mathbb{Q}$ is not countably infinite.

(b) We know that there are a finite number of integers in $[0, 203]$. Therefore, $[0, 203] \cap \mathbb{Z}$ must be finite. Therefore, $[0, 203] \cap \mathbb{Z}$ is not countably infinite.

(c) We know that $\mathbb{Q}$ is countably infinite and we also know that the cartesian product of two countably infinite sets is still countably infinite. The proof for this is very similar to the argument that $\mathbb{Z} \times \mathbb{Z}$ is countably infinite. As a proof for this question, we start with the fact that $\mathbb{Q}$ is countably infinite. Therefore, $\mathbb{Q}$ is listable. We can then lay out all the elements of $\mathbb{Q} \times \mathbb{Q}$ in a table. Listing all of the rationals along the x axis and list all of the rationals along the y axis, we can then "zig-zag" through every cartesian pair of rational numbers. Therefore, since we are able to reach every cartesian product of rationals and there are infinitely many of them, we know that $\mathbb{Q} \times \mathbb{Q}$ is countably infinite.

(d) In this set, each value of $x \in \mathbb{Z}$ will have exactly one corresponding value of $y \in \mathbb{Z}$. Therefore, this set has the same cardinality as $\mathbb{Z}$ does, so it is countably infinite.

(e) In this set, each value of $y \in \mathbb{Z}$ will have at most two corresponding values of $x \in \mathbb{R}$. Therefore, this set is countable. Also, in this set, each value of $y \in \mathbb{Z}^+$ will have at least one corresponding value of $x \in \mathbb{R}$. Therefore, this set is infinite.

Part C: Short Answer

Instructions

- There are 5 questions in this section.
- Write your solution in the space provided.
- Show your work and include justification.

Problem 11: Onto Proof/Disproof (6 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5x^3 - 10$. Prove or disprove that f is onto.

● **Prove** ○ Disprove

Solution

Prove. We would like to show that $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}\ f(x) = y$. Here, let $x = \sqrt[3]{\frac{y+10}{5}}$. Note that this is a well-defined real number. Then $f(x) = f(\sqrt[3]{\frac{y+10}{5}}) = 5(\sqrt[3]{\frac{y+10}{5}})^3 - 10 = 5(\frac{y+10}{5}) - 10 = y + 10 - 10 = y$, so $f(x) = y$ and we are done. Thus, f is onto.

Common Mistakes

- Incorrectly quantifying either codomain or domain element
- Algebraic errors
- Forgetting to substitute the codomain element back into the function
- Assuming $f(a) = b$ from the start of the proof, whereas this is what you should be proving. This is a good start for scratch work, but not for your proof.

Grading Guidelines [6 points]

+1 Indicates proving that the function f is onto
+1 Selects an arbitrary codomain element
+2 Selects a correct corresponding domain element
+2 Shows that the selected domain element maps to the codomain element

Problem 12: Subset Proof (7 points)

Suppose you want to prove the following claim using strong induction:

Claim: $P(n)\ \ \forall n \geq 5$.

(a) Fill in the blanks to complete the inductive step below.
Note: Write all your answers in the Answer Packet. Nothing written below will be graded.

Inductive Step:
Let $k \geq$ ___.
Assume $P(j)$ is true for all _______ $\leq j \leq$ _______.
Since $P(k-4)$, ... [*specific deductions omitted*] ..., then $P(k)$.

(b) Given the inductive step in Part (a), for which values of n will $P(n)$ need to be proven using base cases?

Solution

(a) *Inductive Step:*
Let $k \geq \mathbf{9}$.
Assume $P(j)$ is true for all $\mathbf{5} \leq j \leq \mathbf{k-1}$.
Since $P(k-4)$, then $P(k)$.

For the inductive hypothesis, any range of j that includes $j = k-4$ and does not allow $j \geq k$ or $j < 5$ is correct and should get full credit.

(b) $n = 5, 6, 7, 8$. These values are not covered by the inductive step, but are part of the claim, and thus need to be proven with base cases.

Grading Guidelines:

(a) [+5 total]
[+2] Correct minimum value for k, such as **9** or max(base case) $+ 1$
[+3] Correct range for j: includes $k-4$ for all k's and excludes k.

[+1.5] Correct lower bound and includes k-4, but incorrect upper bound
[+1.5] Incorrect lower bound but correct upper bound

(b)
[+2] Completely correct: $n = 5, 6, 7, 8$
[+1] More or less than the number of base cases required for student's IH
[+1] Base cases does not start from 5, but right number of base cases.
[-0.1] If the number of base cases is more than 4, but is still appropriate for the answer in part a.

Common mistakes:

1. $k \geq 5$
   P(5), P(6), P(7) and P(8) cannot be proven in the inductive step since, in order to prove for example P(5), you need to use $P(k-4) = P(1)$. However, we are not given in the question that P(1) is true.
2. $5 \leq j \leq k$
   Since we are trying to prove P(k), we cannot asssume that P(k) is true.
3. Base cases n = 1, 2, 3, 4
   We are not sure that P(1) ... P(4) is true. We only have to prove $P(5)$ and up
4. Base cases n = 5, 6, 7, ...
   If the answer ends in ellipses, even though it starts at 5, no points were given (even if you start from 5), since this defeats the point of induction.
5. Note that we did accept answers that had a tighter range than 5 to $k-1$. As long as your range included $k-4$ and doesn't go below 5 or above $k-1$, you can get full credit.

Problem 13: Weak Induction Proof (7 points)

Prove by weak induction that for all integers $n \geq 1$, $\quad 1 + 3 + 5 + ... + (2n-1) = n^2$.

Proof:

Solution

Let $P(n)$ be defined as the predicate that $1 + 3 + 5 + ... + (2n - 1) = n^2$.
Base Case: We prove that $P(1)$ holds. $1 = 1^2$ so the base case is met.
Inductive Step: Let k be an arbitrary integer ≥ 1. Assume $P(k)$; that is $1+3+5+...+(2k-1) = k^2$.
We want to show $P(k+1)$. We begin with the left hand side:

$$\begin{aligned} &1 + 3 + 5 + ... + (2k-1) + (2k+1) && \text{LHS} \\ &= k^2 + 2k + 1 && \text{By the inductive hypothesis} \\ &= (k+1)^2 && \text{By factoring, yielding RHS} \end{aligned}$$

Thus $P(k+1)$ holds. We have completed the base case and the inductive step, so we have shown by weak induction that for all integers $n \geq 1$, $\quad 1+3+5+...+(2n-1) = n^2$.

Alternate inductive step: Let k be an arbitrary integer ≥ 1. Assume $P(k)$; that is $1 + 3 + 5 + ... + (2k - 1) = k^2$.
We want to show $P(k+1)$. We begin with the right hand side:

$$\begin{aligned} &(k+1)^2 && \text{RHS} \\ &= k^2 + 2k + 1 && \text{Expanding the square} \\ &= 1 + 3 + 5 + ... + (2k-1) + (2k+1) && \text{By the inductive hypothesis, yielding LHS} \end{aligned}$$

Thus $P(k+1)$ holds.
Common mistakes:

- Forgetting to include the base case

- Working from both the LHS and RHS, or assuming $P(k+1)$ and concluding a true statement. We did not generally take points off for most errors of this form, though this is not a valid proof.
- Assuming $P(k)$ for all integers $k \geq 1$. This is assuming what you want to prove and involves circular reasoning.
- Not specifying the domain when you assume $P(k)$. I.E. k is an arbitrary integer greater ≥ 1.
- Attempting to use P(k) in algebra. P(k) is a predicate that contains the truth value of $1 + 3 + 5 + ... + (2k - 1) = k^2$ (an equality), so you can not plug in P(k) for either side of the equation.

Grading Guidelines [7 points]

+.5 Defines $P(n)$ or writes it correctly in-line in the proof.
+1 Specifies that k is an arbitrary positive integer.
+1 Assumes $P(k)$
+1 Starts from either the LHS or RHS of $P(k+1)$ and attempts to show equality to the other side.
+1 Correctly applies the induction hypothesis
+1.5 Correctly reaches the opposite side (either by factoring or expanding).
+1 Correct base case and work.

Problem 14: Cardinality Proof (7 points)

Let $A = [0, 1]$ and $B = [-2, -1] \cup [1, 2]$.

(a) Provide a function f that shows that $|A| \leq |B|$

(b) Provide a function g that shows that $|A| \geq |B|$

For each function you define, make sure to state its domain and codomain, and state which relevant property/properties your function has (e.g., one-to-one or onto).
You do **not** need to prove that your functions have the property/properties you indicate.

Proof:

(a) $|A| \leq |B|$

f: __________ $\rightarrow$ __________ $f(x) =$

f is ______________________________ (State property/properties)

(b) $|A| \geq |B|$

g: __________ $\rightarrow$ __________ $g(x) =$

g is ______________________________ (State property/properties)

Solution

Using one-to-one functions:
We can show that $|A| \leq |B|$ by defining a one-to-one $f : A \to B$ as $f(x) = x + 1$.
We can show that $|A| \geq |B|$ by defining a one-to-one $g : B \to A$ as $g(x) = \frac{x+2}{4}$.

Using onto functions:
We can show that $|A| \leq |B|$ by defining an onto $f : B \to A$ as $f(x) = |x| - 1$.
We can show that $|A| \geq |B|$ by defining an onto $g : A \to B$ as $g(x) =$

$$\begin{cases} 4x - 2 & x \leq \frac{1}{4} \vee x \geq \frac{3}{4} \\ 1 & \frac{1}{4} < x < \frac{3}{4} \end{cases}$$

Note: there are many other functions that can be used
3.5 points per part:

Grading Guidelines [7 points]

+1 Correct combination of domain, codomain, and function property that gives the desired inequality

+1.25 The given function is well defined on the entire domain and the range of the function is a subset of the codomain where the codomain and domain are either A or B.

+1.25 The function has the listed property over A or B.

Common Mistakes

- Listing a domain or codomain besides A or B. One example of this is $f : \mathbb{Z} \to \mathbb{Z}$. This example would show that $|\mathbb{Z}| \leq |\mathbb{Z}|$, which is not what the question was asking for.
- Writing a proper subset of B as the domain or codomain instead of B itself. One example of this is a one-to-one function $f : [0, 1] \to [-2, -1]$ instead of a one-to-one function $f : [0, 1] \to [-2, -1] \cup [1, 2]$. This example would show that $|[0, 1]| \leq |[-2, -1]|$, which is not what the question was asking for. (For the purposes of grading, this was considered acceptable if the answer both noted what relevant set their codomain was a subset of and stated that their codomain was a subset.)
- Mixing up whether the function should be one-to-one or onto.
- Mixing up square bracket notation with curly brace notation. For example, this could incorrectly lead to thinking that $B = \{-2, -1, 1, 2\}$.
- Writing a function such that B is the listed codomain but an element of $(-1, 1)$ is in the range. This results in the function's range not being a subset of the listed codomain.

Problem 15: Pigeonhole (7 points)

(a) Name the smallest integer n such that you can guarantee that if you select n distinct integers, there must be two distinct integers you selected, a and b, such that

$$a \equiv \pm b \pmod{10}.$$

You do not need to justify your answer or show work.

(b) Use the Pigeonhole Principle to prove that your value of n satisfies the claim.
Make sure to include what your pigeons and holes are, and how many of each you have.

(a) $n =$ ______________

(b)

Solution

(a) $n = 7$

(b) Our pigeons are distinct integers. To construct our holes, we first consider congruence classes of integers modulo 10. That is, the possible remainders of integers when divided by 10: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. However, since we want $a \equiv \pm b \pmod{10}$, not just $a \equiv b \pmod{10}$, some of these classes can be combined. This leaves us with 6 holes corresponding to $(0), (1, 9), (2, 8), (3, 7), (4, 6), (5)$. Now, if we pick 7 distinct integers (our pigeons), the pigeonhole principle guarantees that at least two of them will be in the same hole since $\lceil \frac{7}{6} \rceil = 2$. That is, we will have chosen an a and b such that $a \equiv \pm b \pmod{10}$.

Grading Guidelines [7 points]

+2 Correct numerical answer for n
+1.5 Constructs the remainders of the integers mod 10.
+2 Correctly groups the remainders into the holes $(0), (1, 9), (2, 8), (3, 7), (4, 6), (5)$
+1.5 Correctly applies pigeonhole principle to the claimed number of holes
+0 Blank

Partial Credit

+1 Partial credit for grouping the remainders into 5 of these holes $(0), (1, 9), (2, 8), (3, 7), (4, 6), (5)$

Common Mistakes

- Constructing the holes as each remainder modulo 10, so the integers equivalent to 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 under modulo 10. This gives 10 holes. While it is true that if two integers a and b are in the same hole then $a \equiv b \pmod{10}$ and thus $a \equiv \pm b \pmod{10}$, this is not necessarily the smallest number of holes that we need to guarantee that $a \equiv \pm b \pmod{10}$. For example, consider $a = 11$ and $b = 29$. Then a and b would go into the holes represented by the remainders 1 and 9, respectively. So a and b are not in the same hole. However, it is still the case that $a \equiv \pm b \pmod{10}$ since $a \equiv 1 \pmod{10}$ and $b \equiv -1 \pmod{10}$. Thus, we can reduce the number of holes by combining the two holes represented by 1 and 9 into a single hole. Per the solution, similar logic follows for some of the other holes.

- Attempting a worst case analysis instead of directly applying the pigeonhole principle. Consider the "worst case" is a good way to get intuition for the principle, but it in itself is *not* the Pigeonhole Principle and is not sufficient to prove it. For example, if we have 6 holes, it is not sufficient to say "if we have 7 pigeons, in the worst case we have 1 of 6 pigeons in each hole, and then the 7th pigeon must share a hole with one of the pigeons". To use this reasoning, we would need to more precisely define the "worst case" and also consider other cases. Hence, we instead recommend directly applying the Pigeonhole Principle to conclude "since we have more pigeons than holes, by the Pigeonhole Principle there are at least two pigeons in the same hole."

- Excluding the holes for 0 modulo 10 and 5 modulo 10. To apply the Pigeonhole Principle, we must have that every pigeon goes into one pigeonhole. Thus if our holes are only the pairs of remainders $(1, 9), (2, 8), (3, 7), (4, 6)$ modulo 10, then the integers that are equivalent to 0 or 5 modulo 10 can not be placed in a hole.

Part D: Free Response

Instructions

- There are 3 questions in this section.
- Write your solution in the space provided.
- Write down your answer with care: answers that are unreadable (such as too faint or too messy) will not be graded. If you have multiple answers, you must indicate which one you want graded.
- **Show your work** and include justification.

Problem 16: Composition Proof (10 points)

Let $f : A \to B$ and $g : B \to C$. Prove:

If f is onto and g is **not** one-to-one, then $g \circ f$ is **not** one-to-one.

Hint: It could be helpful to write out what it means for g to be **not** one-to-one. Do the same for $g \circ f$.

Solution

Primary solution (direct proof)

Assume f is onto and g is not one-to-one.

Since g is not one-to-one, we can find $b_1, b_2 \in B$ such that $b_1 \neq b_2$ but $g(b_1) = g(b_2)$.

Then since f is onto, we can find $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$.

We know $b_1 \neq b_2$, so $f(a_1) \neq f(a_2)$, so $a_1 \neq a_2$, because f is a function. To see why this works, it helps to take the contrapositive: $a_1 = a_2 \to f(a_1) = f(a_2)$.

We now have that $g(f(a_1)) = g(b_1) = g(b_2) = g(f(a_2))$, so $(g \circ f)(a_1) = (g \circ f)(a_2)$. We also have that $a_1 \neq a_2$. Therefore, by definition $g \circ f$ is not one-to-one.

Alternate solution (contradiction)

Assume f is onto, g is not one-to-one, and $g \circ f$ *is* one-to-one. Take arbitrary $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$. We will try to prove that $b_1 = b_2$. Because f is onto, there are $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Knowing this, $(g \circ f)(a_1) = g(b_1)$ and $(g \circ f)(a_2) = g(b_2)$. Since $g(b_1) = g(b_2)$, we can say $(g \circ f)(a_1) = (g \circ f)(a_2)$. Since $g \circ f$ is one-to-one, $a_1 = a_2$. Since $a_1 = a_2$, $f(a_1) = f(a_2)$ by definition of function.

Thus, $b_1 = b_2$, proving g is one-to-one. But we assumed g was not one-to-one, so this is a contradiction.

Grading Guidelines [10 points]

Primary Solution: Direct Proof

+1 Assumes f is onto and g is not one-to-one (with no extraneous assumptions)
+1.5 Using that g is not one-to-one, defines variables $b_1 \neq b_2 \in B$ such that $g(b_1) = g(b_2) \in C$
+1 b_1 and b_2 are specific (not arbitrary)
+1.5 Using that f is onto, defines variables $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$
+1 a_1 and a_2 are specific (not arbitrary)
+1 Applies the fact that f is a function to find that $a_1 \neq a_2$
+1.5 Computes $(g \circ f)(a_1) = g(f(a_1)) = g(b_1)$, likewise for a_2 and b$_2$
+1.5 Concludes $g \circ f$ is not one-to-one, because $(g \circ f)(a_1) = (g \circ f)(a_2)$ and $a_1 \neq a_2$

Alternative Solution: Proof by Contradiction

+1 Assumes for the sake of contradiction that f is onto, g is not one-to-one, and $g \circ f$ is one-to-one (with no extraneous assumptions)
+1.5 Introduces arbitrary $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$
+1 Using that f is onto, defines variables $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$
+1 a_1 and a_2 are specific (not arbitrary)
+1.5 Computes that $(g \circ f)(a_1) = (g \circ f)(a_2)$
+2 Applies the fact that $g \circ f$ is one-to-one to find that $a_1 = a_2$
+1 Applies the fact that f is a function to find that $b_1 = b_2$
+1 Reach a contradiction that g is simultaneously one-to-one and not one-to-one

Note: Some proofs by contradiction were graded by the first rubric. For example, proofs that assume $g \circ f$ is one-to-one and show a contradiction by concluding $g \circ f$ is not one-to-one end up being very similar to direct proofs.

In general, proofs intending to end by showing $g \circ f$ is not one-to-one were graded by the first rubric, and proofs intending to end by showing a contradiction with the assumption that g is not one-to-one were graded by the second rubric.

Common Mistakes

- Many students made mistakes in stating and applying the assumption g is **not** one-to-one. These generally took the form of incorrect negations or defining variables as arbitrary when they should have been specific.

  Because g is not one-to-one, we know that $\exists b_1, b_2 \in B\ [g(b_1) = g(b_2) \wedge b_1 \neq b_2]$. To properly use this fact, we must consider specific b_1, b_2 that meet this condition. Common incorrect statements of this assumption included:

  - $\forall b_1, b_2 \in B,\ [g(b_1) = g(b_2) \wedge b_1 \neq b_2]$

This is missing an application of DeMorgan's for quantifiers. Note that this claim is trivially impossible to satisfy when B is nonempty because you can just set $b_1 = b_2$

- $\forall b_1, b_2 \in B,\ [g(b_1) = g(b_2) \to b_1 \neq b_2]$
- $\forall b_1, b_2 \in B,\ [g(b_1) \neq g(b_2) \to b_1 = b_2]$
- $\forall b_1, b_2 \in B,\ [g(b_1) = g(b_2)]$
- $\exists b_1, b_2 \in B,\ [g(b_1) = g(b_2) \to b_1 \neq b_2]$
- $\exists b_1, b_2 \in B,\ [g(b_1) = g(b_2)]$
- $\exists b_1, b_2 \in B,\ [g(b_1) \neq g(b_2)]$

Some students considered arbitrary $b_1, b_2 \in B$, and then retroactively attempted to apply the definition of not one-to-one. This is invalid as g being not one-to-one does **not** mean that $g(b_1) = g(b_2) \to b_1 \neq b_2$ for any $b_1, b_2 \in B$.

The same mistakes were present when considering the conclusion statement $g \circ f$ is not one-to-one.

- Relatedly, many students considered arbitrary variables $a_1, a_2 \in A$ and attempted to use the definition of onto to state that $f(a_1) = b_1$ and $f(a_2) = b_2$ for preexisting elements $b_1, b_2 \in B$. This is an incorrect application of the definition of onto. f being onto only gives information about the existence of such elements, not that the preexisting values a_1, a_2 when used as inputs to f, are equal to b_1 and b_2.
- Several students wrote that because f is onto, $\forall a \in A$ and $b \in B$, $f(a) = b$. Or, $\forall a \in A\ \exists b \in B\ [f(a) = b]$.
- Many students used the same variable name in multiple definitions, and then assumed that these variables were the same without justification. For example:

  "Let $b_1 \in B$ be arbitrary. Because f is onto, there is $a_1 \in A$ such that $f(a_1) = b_1$. Because g is one-to-one, there are $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$ and $b_1 \neq b_2$."

  Although it's obscured by the notation, there are two different b_1's here, and they each have different properties by construction. For example, the b_1 in the first sentence is arbitrary, and the b_1 in the third sentence is specific.
- Many students misinterpreted the intended conclusion, that $g \circ f$ is **not** one-to-one, as the claim that $\exists a_1, a_2 \in A\ [(g \circ f)(a_1) = (g \circ f)(a_2) \wedge f(a_1) \neq f(a_2)]$. These solutions ignored that $a_1 \neq a_2$ is required to reach the conclusion that $g \circ f$ is not one-to-one.
- Many students made type errors when applying the functions f and g, or discussing their inputs and outputs. For the expression $f(x)$ to be valid, we must have that $x \in A$; similarly, if $f(x) = y$, then we **must** have that $y \in B$, because these sets are respectively the domain and codomain of f. For the same reasons, $f \circ g$ is not well-defined.
- Some students embedded A, B, and/or C in some standard set, such as $\mathbb{R}$.

- Several students attempted to solve the problem via a specific example, e.g. drawing a diagram or defining f and g as some algebraic functions over the reals.
- Many students forgot to explicitly state assumptions at the beginning of the proof. As a note, even if a submission didn't explicitly state their assumptions, as long as it was clear in their proof that they had the correct assumptions, the point was still given.
- Many students cited f being onto for the reason why $a_1 \neq a_2$. Given $f(a_1) \neq f(a_2)$, we know that $a_1 \neq a_2$ because a well-defined function (regardless of onto-ness) will never output two different outputs for the same input.
- Several students attempted to prove the claim using cardinality-related arguments. This question cannot be solved using cardinalities of the sets A, B, and C.
- There were a few function/set terminology errors, like saying "two domains map to the same codomain" rather than "two domain elements map to the same codomain element" in the definition of not one-to-one.
- Many students used counterintuitive variable names, e.g. defining elements $a \in B$ and $b \in A$. This is not incorrect, but as a result, these submissions often confused their variables, for example plugging a into f when $a \in B$, and reached incorrect conclusions. Clear notation can help you avoid these mistakes!

Problem 17: Brad is Blessing (10 points)

Brad is U of M's favorite dog and he spreads smiles by doing 1 of his favorite 4 activities every day:

- Dog therapy, Celebrity photo op, Fetching tennis balls, or Resting

He has two restrictions:

- He will never Rest 2 days in a row.
- He always plays Fetch the day after a Celebrity photo op.

(a) Find a recurrence relation representing the number of ways Brad can spend n days.

Recurrence Relation: ____

Justify your answer by showing your work below

(b) Find the initial conditions for your recurrence relation in part (a). For full credit, you must provide the fewest initial conditions possible to satisfy your recurrence.
List your initial conditions below. Show your work.

Solution

Recurrence: $a_n = 2a_{n-1} + 3a_{n-2} + a_{n-3}$
Base Cases: $a_0 = 1, a_1 = 3, a_2 = 9$

Forward solution:
On day 1, Brad can perform any of the 4 actions.

Case 1: Dog Therapy
There are no further requirements, so on day 2, he can again do anything. This means the number of ways to select activities for the remaining $n-1$ days is a_{n-1}.

Case 2: Fetch
Similarly, there are no further requirements, so there are a_{n-1} ways to select the remaining days.

Case 3: Celebrity Photo Op
In this case, Brad will play fetch on day 2. After that, though, we are back to being able to do anything, so there are a_{n-2} ways to select for the remaining $n-2$ days.

Case 4: Rest
In this case, Brad will not be able rest on day 2, so we can't just use use a_{n-1}. Instead, we need subcases for the other options:
Case 4a: Rest, Dog Therapy. a_{n-2}
Case 4b: Rest, Fetch. a_{n-2}
Case 4c: Rest, Celebrity Photo Op, Fetch. a_{n-3}
For these subcases, reference the corresponding case above.

Each case is a separate option, so we add them together and combine like terms to get the final reccurence relation: $a_n = 2a_{n-1} + 3a_{n-2} + a_{n-3}$

Backward solution:
On day n, Brad cannot do a celebrity photo opt as he would have to fetch the following day, but there is no following day. However, he can perform any of the remaining activities. It is noteworthy that due to the special case that a sequence cannot end on a celebrity photo opt, we will not be working back to an unconstrained state but rather, an identical state compared to day n. In other words we will work backwards, until he performed dog therapy, played fetch, or rested on the previous day.

Case 1: Dog Therapy
There are no restrictions on dog therapy, therefore on the previous day, he could've performed dog therapy, played fetch, or rested (and not celebrity photo opt as it needs to be followed by fetch). We have worked back to our initial conditions for day $n-1$ and therefore this case produces a_{n-1}.

Case 2: Fetch
This case is identical to dog therapy with the notable exception that a celebrity photo opt could've also happened on day $n-1$. However, this couldn't have been a terminal state of a previous sequence so we must reason about the day before it.
Case 2a: Day $n-1$ Brad could've performed dog therapy, played fetch, or rested on the previous day identically to our terminal state, a_{n-1}.

Case 2b: Day $n-1$ Brad hosted a celebrity photo opt. Previous to that day, on day $n-2$, he could've performed dog therapy, played fetch, or rested. Therefore this subcase yields a_{n-2}.

Case 3: Rest
In this case, Brad will not be able to do a celebrity photo opt or rest on the previous days. This leaves dog therapy and fetch.
Case 3a: Day $n-1$ Brad performed dog therapy. This case is identical to Case 1, however, this case works backwards for day $n-1$ instead of day n. Therefore we get a_{n-2} for this subcase.
Case 3b. Day $n-1$ Brad played fetch. This case mirrors that of case 2 but again, we start from day $n-1$ instead of day n. Therefore we get $a_{n-2} + a_{n-3}$ from this subcase.

For those that prefer a visual solution:
`Exams/Exam 2/images/Exam2_Question16_VisualSol.jpg`

Each case is a separate option, so we add them together and combine like terms to get the final recurrence relation: $a_n = 2a_{n-1} + 3a_{n-2} + a_{n-3}$

Part (b)
Our recurrence goes back to a_{n-3}, so we must have 3 initial conditions. $a_0 = 1$ because there is 1 way to do 0 activities across 0 days. $a_1 = 3$ because Brad can fetch, rest, or do dog therapy the first day. However, he cannot do a celebrity photo op because that would violate the restriction that it must be followed by fetch. For a_3, we can count using cases based on the first activity. If Brad played fetch or did dog therapy, he can do any of the activities on day 2 besides celebrity photo op for the same reason as above, accounting for 6 total possible combinations. If Brad rested on day 1, he cannot rest on day 2 but can do fetch or dog therapy, accounting for 2 possible combinations. We also need to consider the case where Brad did celebrity photo op on day 1 and fetch on day 2. This gives us $a_2 = 6 + 1 + 2 = 9$.

Only 3 base cases necessary, but if starting at 1, then $a_3 = 28$

Grading Guidelines [10 points]

Part a (Working Forwards):
+1 split into correct cases based on approach
+1 correct term for Dog Therapy case
+1 correct term for Fetch case
+1 correct term for Celebrity Photo Op case
+0.5 splits Rest case into subcases

- +1 correct term for Dog Therapy and Fetch subcase

- +1 correct term for Celebrity Photo Op subcase

+1 Adds cases together to get recurrence relation

Part a (Working Backwards):
+1 split into correct cases based on approach
+1 correct term for Dog Therapy case
+1.5 correct term for Fetch case
+0.5 splits Rest case into subcases

- +1 correct term for Dog Therapy subcase
- +1.5 correct term for Fetch subcase

+1 Adds cases together to get recurrence relation

Part b:
+1 correct number of initial conditions (based on final recurrence relation)
+1.5 correct values for initial conditions (-0.5 for each incorrect)

Common Mistakes

- **Going backwards and considering the option of ending on a celebrity photo op**. If Brad always plays fetch the day after a photo op, *he cannot have a photo op on the last day*, else he will not be able to play fetch the day after. These solutions were not awarded the "split into correct cases" rubric item for going backwards. There was therefore no viable solution for this problem if one considers the possibility of a celeb photo op on day n.
- Stemming from the previous mistake, many had the initial conditions of $a_0 = 1$, $a_1 = 4$, and $a_2 = 12$. Again, a celebrity photo op cannot be the last activity done by Brad, which eliminates one option from a_1 and three options from a_2.
- Associating Brad fetching as only possible after a celebrity photo op. Brad's option to fetch is **not restricted**, so it can be done at any time. This led to an a_{n-1} term missing and an a_{n-2} term missing within the context of resting.
- Going forwards and backwards at the same time. (i.e. stating that the solution is backwards and forward solving)
- Stating the sequence of [rest $\rightarrow$ photo op $\rightarrow$ fetch] (forwards)/ [photo op $\rightarrow$ fetch $\rightarrow$ rest] (backwards) at a_{n-2}, when it should be a_{n-3}
- Giving unnecessary coefficients to each particular case (i.e. considering the "dog therapy" case to be associated with the term $3a_{n-1}$ instead of a_{n-1})
- Including more base cases than the minimum needed (i.e. if the recurrence went as far back as a_{n-3}, including a_0, a_1, a_2, **and** a_3

Problem 18: Recurrence Inequality (10 points)

Consider the following recurrence with initial conditions $a_1 = 1$ and $a_2 = -1$:

$$a_n = -a_{n-1} + 6a_{n-2}$$

Use strong induction to prove: for all $n > 0$, $\quad |a_n| < 3^n$.

Note: You may use without proof that for all $x, y \in \mathbb{Z}$, $|x + y| \leq |x| + |y|$.

Solution

We use strong induction.

Inductive Step:

Let $k \geq 3$. Assume $|a_j| < 3^j$ for all $1 \leq j < k$ (take $k \geq 2$ and $1 \leq j \leq k$ if proving

$k+1$). Then

$$\begin{aligned}
|a_k| &= |-a_{k-1} + 6a_{k-2}| \\
&\leq |-a_{k-1}| + |6a_{k-2}| \\
&= |a_{k-1}| + 6|a_{k-2}| \\
&< 3^{k-1} + 6 \cdot 3^{k-2} && \text{(by IH)} \\
&= 3^{k-1} + 2 \cdot 3^{k-1} \\
&= 3 \cdot 3^{k-1} \\
&= 3^k.
\end{aligned}$$

Base Cases: $n = 1$ and $n = 2$

$$\begin{aligned}
|a_1| &= |1| = 1 < 3 = 3^1 \quad \text{and} \\
|a_2| &= |-1| = 1 < 9 = 3^2,
\end{aligned}$$

so the base cases hold.

Grading Guidelines [10 points]

+1.5 Correct inductive hypothesis to prove claim
+1.5 First productive use of inductive hypothesis
+1.0 Second productive use of inductive hypothesis
+1.5 The goal of the inductive step is shown either implicitly with work, or written out explicitly if work in Inductive Step is missing.
+1.0 Correct algebra to be able to apply IH
+1.0 Correct algebra to reach goal after applying IH
+1.5 Sufficient number and choice of base cases
+1.0 Correct work for each base case

Errors:
-0.5 Bounds error in inductive hypothesis – lower bound on k and/or j
-0.5 Missing absolute value in inductive hypothesis
-0.5 Minor algebraic error in the inductive step
-0.5 Missing absolute value in inductive step
-0.5 Minor mistake in base cases – missing absolute value, slightly incorrect choice, or extra base case
-1.0 Starts inductive step with proof goal and works to true statement

Common Mistakes

1. The most common error was probably setting incorrect bounds for k. The correct value is determined by the fact that we use two previous values of the predicate in the inductive step. If attempting to prove $P(k)$, then k must be at least 3, and if attempting to prove $P(k+1)$, then k must be at least 2.

2. Similarly, students often had an incorrect bound on j. Regardless of which proof was attempted for the inductive step, the lower bound on j should be 1.
3. There were a number of mistakes which are common in most inductive proofs – forgetting base cases, including extra base cases, and forgetting to set bounds on the variables in the inductive hypothesis.
4. Some students dropped the absolute value bars in various parts of the proof – most often the base cases.
5. Some students attempted to prove $a_n = -a_{n-1} + 6a_{n-2}$ rather than the predicate that $|a_n| < 3^n$. Similarly, and more commonly, students would say in the base cases that $a_1 = 1$ and $a_2 = -1$ without showing that the predicate holds – i.e. that $|1| < 3$ and $|-1| < 3^2$
6. Some students appeared to misunderstand the definition of the recurrence relation; for example, stating $a_k = a_{k-2} + a_2$ or $a_k = a_{k-2} + 2$. The latter often came with the sort of reasoning common in "stamps-style" problems.

This is a space for scratch work. DO NOT DETACH THIS PAPER FROM YOUR EXAM.

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