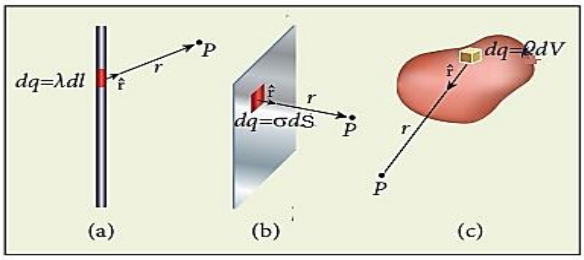
Introduction to Electromagnetism

Introduction: Electromagnetic theory can be thought of as generalization of circuit theory. There are certain situations that can be handled exclusively in terms of field theory. In electromagnetic theory, the quantities involved can be categorized as source quantities and field quantities. Source of electromagnetic field is electric charges: either at rest or in motion. However an electromagnetic field may cause a redistribution of charges that in turn change the field and hence the separation of cause and effect is not always visible.

Charge Distributions: In principle, the smallest unit of electric charge that can be isolated is the charge of a single electron, which is $\cong -1.60\times10^{-19}$ C. This is very small, and we rarely deal with electrons one at a time, so it is usually more convenient to describe charge as a quantity that is continuous over some region of space. In particular, it is convenient to describe charge as being distributed in one of three ways: along a curve, over a surface, or within a volume.



Line, surface and volume charge distribution

Line Charge Distribution: Imagine that charge is distributed along a curve C through space. Let Δq be the total charge along a short segment of the curve, and let Δl be the length of this segment. The line charge density λ at any point along the curve is defined as

$$\lambda = lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl}$$

which has units of C/m. We may then define λ to be a function of position along the curve, parameterized by l; e.g., λ(l). Then, the total charge q along the curve is

$$q = \oint \lambda(l)\, dl$$

Surface Charge Distribution: Imagine that charge is distributed over a surface. Let Δq be the total charge on a small patch on this surface, and let ΔS be the area of this patch. The surface charge density σ at any point on the surface is defined as

$$\sigma = lim_{\Delta S \to 0} \frac{\Delta q}{\Delta S} = \frac{dq}{dS}$$

which has units of C/m2. Let us define σ to be a function of position on this surface. Then the total charge over a surface S is

$$q = \int \sigma\, dS$$

Volume Charge Distribution: Imagine that charge is distributed over a volume. Let Δq be the total charge in a small cell within this volume, and let ΔV be the volume of this cell. The volume charge density ρ at any point in the volume is defined as

$$\rho = lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV}$$

which has units of C/m3. Since ρ is a function of position within this volume, the total charge within a volume V is

$$q = \int \rho \, dV$$

Gradient: Gradient of a scalar filed is a vector quantity, whose magnitude gives the maximum space rate variation of that scalar quantity at that point. It tends to point in the direction of greatest change of scalar field. Mathematically it can be represented as

$$grad\ A = \frac{\partial A}{\partial x}\hat{\imath} + \frac{\partial A}{\partial y}\hat{\jmath} + \frac{\partial A}{\partial z}\hat{k} = \nabla A$$

$Here\ \nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$

Physical significance: Physical significance: Gradient of potential field gives the electric field intensity i.e.

$$\vec{E} = \frac{\partial V}{\partial x}$$

Divergence: The divergence of a vector field at a point is a scalar quantity of magnitude equal to the flux of that vector field diverging out per unit volume. It tells us about the presence of source and sink within the volume under consideration. It is mathematically represented as

$$div\, \vec{A} = \nabla . \vec{A}$$

$where\ \vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

$$\therefore\ div\, \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Physical Significance: If div ($\vec{\boldsymbol{A}}$) = 0; then no source or sink is present in the volume under consideration it is also said as solenoid vector function. When div ($\vec{\boldsymbol{A}}$) >0 it indicates the presence of source and in case div ($\vec{\boldsymbol{A}}$) < 0 it indicates the presence of sink.

Curl: Curl of a vector field at point is a vector quantity whose magnitude is equal to the maximum value of line integral of that vector per unit area along the boundary of a small elementary area around that point. It is given as

$$curl\vec{A} = \nabla \times \vec{A}$$

$$curl\ \boldsymbol{A} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$$

Physical Significance: Curl of a vector field tells about the rotation of the vector field. If for a vector A value of curl A=$\boldsymbol{\nabla}$ ×A = 0 then the vector is said to be irrotational.

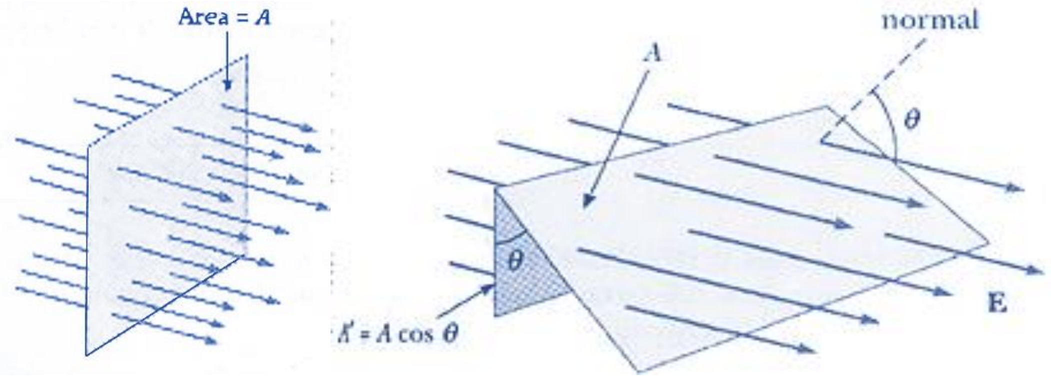
Electric Flux: Air is blowing in through a window. How much air comes through the window depends upon the speed of the air, the direction of the air, and the area of the window. We call this air that comes through the window the "air flux". In similar manner electric flux is the total number of lines of force passing through a surface.

"The total number of lines of force passing through the unit area of a surface held perpendicularly."

We will define the electric flux $\boldsymbol{\phi}$ *for an electric field that is perpendicular to an area as*

$$\phi = E.S$$

If the electric field E is not perpendicular to the area, we will have to modify this to account for that.



Think about the "Electric flux" of field lines passing through surface at an angle θ . The "effective area" is S cosθ.With this in mind, we will make a general definition of the electric flux as

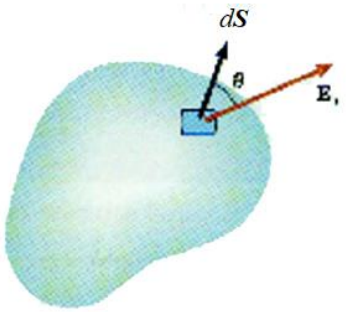
$$\phi = EScos\,\theta$$

Remembering the "dot product" or the "scalar product", we can also write this as

$$\phi = E.S$$

where E is the electric field and S is a vector equal to the area S and in a direction perpendicular to that area.

"The dot product of electric field intensity E and the vector area S is called electric flux."



$$\boldsymbol{S} = \hat{n}S$$

where $\hat{n}$ is a unit vector pointing perpendicular to the area. In that case, we could also write the electric flux across an area as

Then the electric flux through that small area is dϕ and

$$d\phi = E\,dScos\,\theta$$

or $\quad d\phi = \boldsymbol{E}.\,\hat{n}dS$

To find the flux through all of a closed surface, we need to sum up all these contributions of $d\phi$ over the entire surface,

$$\phi = \int_S E.\hat{n}dS$$

Divergence Theorem:

The divergence theorem states that the surface integral of the normal component of a vector point function 'F' over a closed surface 'S' is equal to the volume integral of the divergence of F taken over the volume 'V' enclosed by the surface S. Thus, the divergence theorem is symbolically denoted as:

$$\iint \vec{F}.d\vec{S} = \iiint div\,(\vec{F})dV$$

Stokes' Theorem

The Stoke's theorem states that "the surface integral of the curl of a function over a surface bounded by a closed surface is equal to the line integral of the particular vector function around that surface."

$$\iint (\nabla \times \vec{F}).d\vec{S} = \oint \vec{F}.\vec{dl}$$

Divergence of electrostatics field (Gauss's law in Differential Form):

From Gauss's Law of electrostatics

$$\oint \vec{E}.\vec{dS} = \frac{1}{\epsilon_o}q \ldots\ldots\ldots\ldots\ldots..(1)$$

Using divergence Theorem (Relates volume integral of divergence of a vector field to surface integral of the vector field)

$$\oint \vec{E}.\vec{dS} = \oint(\vec{\nabla}.\vec{E})dV \ldots \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

From Equation (1) and (2)

$$\oint_V \ (\vec{\nabla}.\vec{E})dV = \frac{q}{\epsilon_o}\ldots\ldots\ldots\ldots\ldots.(3)$$

Let a charge q be distributed over a volume V of the closed surface S and p be the charge density; then

$$\rho = \frac{dq}{dV}$$

$$q = \oint dq = \oint \rho dV$$

Substituting the value of net charge in terms of charge density, equation (3) becomes

$$\oint_V \ (\vec{\nabla}.\vec{E})dV = \frac{1}{\epsilon_o}\oint_V \ \rho dV$$

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_o}$$

$$div\,\vec{E} = \frac{\rho}{\epsilon_o}$$

Equation (4) and (5) represent Gauss Law in differential form.

Differential form of Gauss law states that "the divergence of electric field E at any point in space is equal to $1/\varepsilon_0$ times the volume charge density, ρ, at that point".

The displacement vector

$$\vec{D} = \epsilon_o \vec{E}$$

Then equation (4) and (5) Becomes

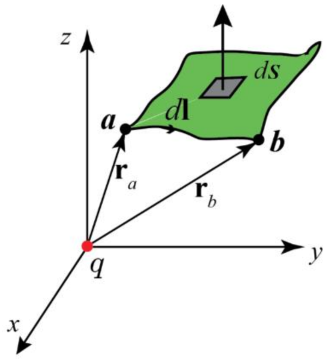
$$\nabla . \vec{D} = \rho$$

$$div\, \vec{D} = \rho$$

Differential form of Gauss law states that "the divergence of displacement vector $\vec{D}$ at any point in space is equal to volume charge density, ρ, at that point".

Curl of the Electric Field:

Electric field due to a single point charge q is:



$$\boldsymbol{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

We need to find the curl of it

$$\nabla \times \boldsymbol{E}(r) = \frac{q}{4\pi\varepsilon_0}\nabla \times \left(\frac{1}{r^2}\hat{r}\right) \dots \dots (1)$$

Taking the surface integral of equation (1)

$$\int \nabla \times \boldsymbol{E}(r).d\boldsymbol{S} = \frac{q}{4\pi\varepsilon_0}\int \nabla \times \left(\frac{1}{r^2}\hat{r}\right).d\boldsymbol{S} \dots \dots (2)$$

Applying Stokes's theorem on RHS of equation (2)

$$\int \nabla \times \boldsymbol{E}(r).d\boldsymbol{S} = \frac{q}{4\pi\varepsilon_0}\oint \left(\frac{1}{r^2}\hat{r}\right).d\boldsymbol{l}$$

Since dl =dr $\hat{r}$

$$\int \nabla \times \boldsymbol{E}(r).d\boldsymbol{S} = \frac{q}{4\pi\varepsilon_0}\oint \left(\frac{1}{r^2}\hat{r}\right).dr\hat{r}$$

$$\int \nabla \times \boldsymbol{E}(r).d\boldsymbol{S} = \frac{q}{4\pi\varepsilon_0}\oint \frac{1}{r^2}.dr$$

$$\int \nabla \times \boldsymbol{E}(r).d\boldsymbol{S} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_{r_a}^{r_b}$$

If $\mathrm{r_a} = \mathrm{r_b}$

$$\int \nabla \times \boldsymbol{E}(r).d\boldsymbol{S} = 0$$

$$\int \nabla \times \boldsymbol{E}(r).d\boldsymbol{S} = 0$$

$$\therefore \quad \nabla \times \boldsymbol{E}(r) = 0$$

Poisson's and Laplace Equation:

This equation is used to solve the boundary value problem. When electrostatics conditions at some boundaries are given and we have to find the E and V throughout the region, these problems are known as boundary value problem.

From Gauss's Law

$$div\,\vec{E} = \frac{\rho}{\epsilon_o}$$

$$div\,\vec{D} = \rho$$

$$\because\ \vec{D} = \epsilon\vec{E} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore \vec{\nabla}(\epsilon\vec{E}) = \rho$$

We also know that

$$E = -\nabla V$$

$$\therefore \quad \overrightarrow{\nabla.\,(}- \epsilon\nabla V) = \rho$$

$$\nabla^2 V = -\frac{\rho}{\epsilon} \ldots\ldots\ldots\ldots\ldots. (3)$$

Equation (3) is known as Poisson's equation

Laplace Equation:

For a charge free region $\rho = 0$

Then

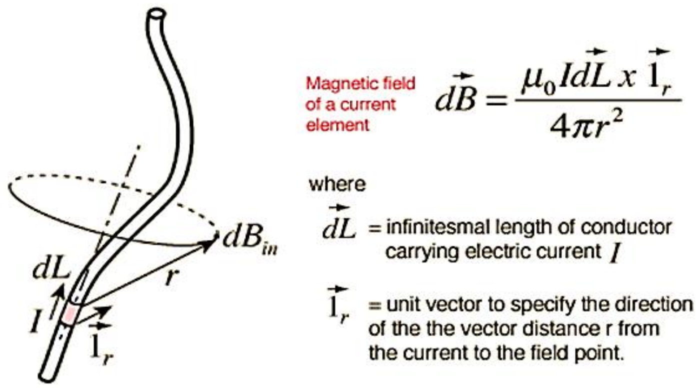
$$\nabla^2 V = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (4)$$

Equation (4) is known as Laplace equation.

Poisson equation is for inhomogeneous medium and Laplace equation is for in homogeneous medium with $\rho = 0$.

Bio –Savart Law:

The Biot-Savart Law relates magnetic fields to the currents which are their sources. In a similar manner, Coulomb's law relates electric fields to the point charges which are their sources. Finding the magnetic field resulting from a current distribution involves the vector product, and is inherently a calculus problem when the distance from the current to the field point is continuously changing.



Consider a conductor through which a current I flows and let small elemental length dl at a source point. We want to calculate magnetic field at a point which is distance r from source point. θ be the angle between r and dl. Then from biot-savart law the magnetic field due to current carrying conductor is

The magnetic field directly proportional to the magnitude of current

$$dB \propto I$$

The magnetic field directly proportional to the length of element dl

$$dB \propto dl$$

The magnetic field directly proportional to the sine of angle between r and dl

$$dB \propto sin\theta$$

The magnetic field inversely proportional to c square of the distance between source point and field point

$$dB \propto \frac{1}{r^2}$$

Now from above equations we get,

$$dB \propto \frac{Idlsin\theta}{r^2}$$

$$dB = \frac{kIdlsin\theta}{r^2}$$

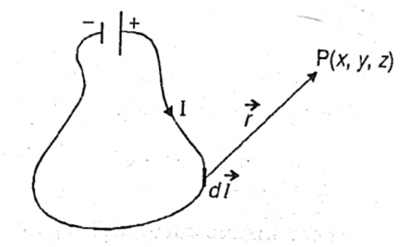
$$in\ SI\ unit\ k = \frac{\mu_\circ}{4\pi}$$

Hence total magnetic field due to current carrying conductor is

$$B = \int dB = \int \frac{\mu_\circ}{4\pi}\frac{Idlsin\theta}{r^2} = \frac{\mu_\circ}{4\pi}\int \frac{Idlsin\theta}{r^2} Tesla$$

Divergence of Magnetic Field:

We know, the magnetic field produced by a current element ***Idl* vector** at a point P (x,y,z) whose distance from the current element r is given by



$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}sin\theta}{r^2}$$

Therefore, the magnetic field at P due to the whole current loop is given by

$$\oint d\vec{B} = \oint \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \vec{r}}{r^3}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\oint \frac{Id\vec{l} \times \vec{r}}{r^3} \dots\dots\dots\dots(1)$$

Taking divergence both sides, we get

$$\vec{\nabla}.\vec{B} = \frac{\mu_0}{4\pi} I \oint \vec{\nabla}.\left(\frac{d\vec{l} \times \vec{r}}{r^3}\right) \dots \dots \dots \dots (2)$$

$$Using \;\; \vec{\nabla}.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{\nabla} \times \vec{A}) - \vec{A}.(\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla}\vec{B} = \frac{\mu_0}{4\pi} I \left[\oint \left(\frac{\vec{r}}{r^3}\right).(\vec{\nabla} \times \vec{dl}) - d\vec{l}.\left(\vec{\nabla} \times \frac{\vec{r}}{r^3}\right)\right] \dots \dots \dots \dots (3)$$

Since dl is not the function of x, y, and z

$$\text{Therefore } \nabla \times \overrightarrow{\mathrm{dl}} = 0 \dots \dots (4)$$

$$\vec{\nabla} \times \frac{\vec{r}}{r^3} = -\vec{\nabla} \times \vec{\nabla}\left(\frac{1}{r}\right)$$

We know curl of gradient is zero.

$$\vec{\nabla} \times \frac{\vec{r}}{r^3} = -\vec{\nabla} \times \vec{\nabla}\left(\frac{1}{r}\right) = 0 \dots \dots \dots (5)$$

Using eqns. (4) and (5), eqn. (3) becomes

$$\vec{\nabla}.\vec{B} = 0$$

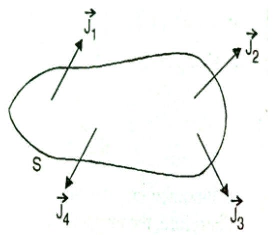
Thus, divergence of B vector is zero.

Any vector whose divergence is zero is known as a solenoidal vector. Thus, magnetic field vector B vector is a solenoidal vector.

This is the proof of Divergence of magnetic field.

Curl of Magnetic Field

Let us consider a region of space in which currents are flowing, the current density J vector varies from point to point but is time-independent.



The total steady current I is given by

$$I = \iint_S \vec{j}.d\vec{S} \dots \dots \dots (1)$$

where J vector is the current density of an element dS vector of the surface S bounded by the closed path.

$$\oint \vec{B}.d\vec{l} = \mu_0 I \; (Amperes \; circuital \; law)$$

$$\oint \vec{B}.d\,\vec{l} = \mu_0 \iint_S \vec{J}.d\vec{S} \; (using \; eqn. 1) \dots \dots \dots (2)$$

But according to Stokes theorem, the closed line integral of the B vector is equal to the surface integral of its curl.

$$\oint \vec{B}.d\vec{l} = \iint_S (\vec{\nabla} \times \vec{B}).d\vec{S} \ldots\ldots\ldots (3)$$

From (2) and (3)

$$\iint_S (\vec{\nabla} \times \vec{B}).d\vec{S} = \mu_0 \iint_S \vec{J}.d\vec{S}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$
$$Curl\ \vec{B} = \mu_0 \vec{J} \ldots\ldots\ldots (4)$$

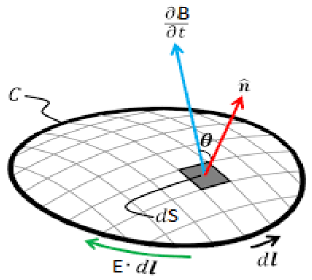
Which is the differential form of ampere's law or curl of a magnetic field.

Curl of the Electric Field (Digression): (Faraday's Law)

According to Faraday's law, an electromotive force (emf) is set up in a circuit when the magnetic flux linking the circuit is changed in some manner; the magnitude of this emf is proportional to the time rate of change of flux linkages with the circuit and the direction of the induced emf is such as tends to oppose the cause.

If Φ_B be the magnetic flux linked with circuit at any instant and e be the induced emf then

$$e = -\left(\frac{d\emptyset_B}{dt}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$



Consider that magnetic field produced by a stationary magnet or current carrying coil. Suppose there is a closed circuit C of any shape which enclosed a surface S in the field. Let B be the magnetic flux density in the neighborhood of the circuit. The magnetic flux through a small area dS will be B.dS . Now the flux through the entire circuit is

$$\emptyset_B = \int_S B.dS \ldots\ldots\ldots\ldots (2)$$

When magnetic flux is changed, an electric field is induced around the circuit. The line integral of the electric field gives the induced emf in the closed circuit. Thus,

$$e = \oint E.dl \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (3)$$

Substituting the value of e and Φ_B in equation (1)

$$\oint E.dl = -\frac{d}{dt}\oint B.dS \dots\dots\dots\dots\dots.(4)$$

Further by stokes theorem, we have

$$\oint E.dl = \oint(\vec{\nabla} \times E)dS \dots\dots\dots\dots\dots\dots\dots(5)$$

From equation (4) and (5)

$$\oint(\vec{\nabla} \times E)dS = -\frac{\partial}{\partial t}\oint B.dS$$

$$\oint(\vec{\nabla} \times E)dS = -\oint\frac{\partial B}{\partial t}.dS$$

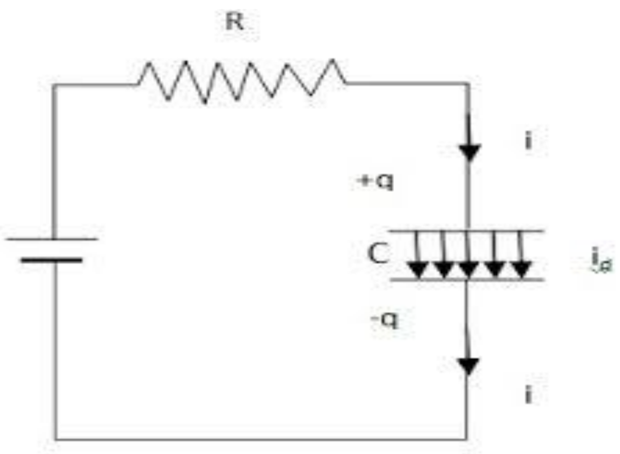
It follows that

$$\vec{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$curl\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Displacement Current:

We consider a parallel combination of a resistance and a capacitor. If we apply voltage V to such a circuit current, current flowing through resistance is given by



$$I_R = \frac{V}{R}$$

In this case there is actual charge motion and so it is called conduction current.

The current flowing through the capacitor is given by

$$I_c = \frac{dQ}{dt}$$

But $Q = CV$

$$\therefore I_c = C\frac{dV}{dt}$$

I_c does not imply actual motion of charges. The current flows through the capacitor only when the voltage is changing. The current flowing out from one plate is equal to the current flowing into other plate is known as displacement current I_d

The displacement current per unit area is called displacement current density J_d.

$$\therefore \quad I_d = I_c = C\frac{dV}{dt}$$

$$\because \quad C = \frac{\epsilon A}{d}$$

$$\therefore\ I_d = \frac{\epsilon A}{d}\frac{dV}{dt}$$

$$I_d = \frac{\epsilon A}{d}\frac{d.\,d\vec{E}}{dt}$$

$$I_d = \epsilon\, A\frac{d\vec{E}}{dt}$$

But $\vec{D} = \epsilon\vec{E}$ (Displacement vector)

$$I_d = A\frac{d\vec{D}}{dt}$$

Current density

$$J_d = \frac{I_d}{A} = \frac{d\vec{D}}{dt}$$

The total current density in vector form is given by

$$\vec{J} = \overrightarrow{J_R} + \overrightarrow{J_d}$$

Maxwell's Equations for Electromagnetism:

Formulation in SI units

Name	Integral equations	Differential equations	Meaning
Gauss's law	$\oiint_{\partial\Omega} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\iiint_\Omega \rho\,\mathrm{d}V$	$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$	The electric field leaving a volume is proportional to the charge inside.
Gauss's law for magnetism	$\oiint_{\partial\Omega} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0$	$\nabla\cdot\mathbf{B} = 0$	There are no magnetic monopoles; the total magnetic flux piercing a closed surface is zero.
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_\Sigma \mathbf{B}\cdot\mathrm{d}\mathbf{S}$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$	The voltage accumulated around a closed circuit is proportional to the time rate of change of the magnetic flux it encloses.
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{B}\cdot \mathrm{d}\boldsymbol{\ell} = \mu_0\iint_\Sigma \mathbf{J}\cdot\mathrm{d}\mathbf{S} + \mu_0\varepsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\iint_\Sigma \mathbf{E}\cdot\mathrm{d}\mathbf{S}$	$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)$	Electric currents and changes in electric fields are proportional to the magnetic field circulating about the area they pierce.

Maxwell's first equation or Gauss's law in electrostatics:

Statement: It states that the total electric flux ϕ_E passing through a closed hypothetical surface is equal to $1/\varepsilon_0$ times the net charge enclosed by the surface

Integral Form:

$$\phi_E = \oint \vec{E}.\overrightarrow{dS} = \frac{q}{\epsilon_0}$$

$$\oint \vec{D}.\vec{dS} = q$$

where $\vec{D}=\varepsilon_0\vec{E}$= Displacement vector

Let the charge be distributed over a volume V and ρ be the volume charge density. Therefore

$$q = \oint \rho dV$$

Therefore

$$\oint E.dS = \frac{1}{\epsilon_0}\oint \rho dV \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1a)$$

Or

$$\oint D.dS = \oint \rho dV \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1b)$$

Equation (1) is the integral form of Maxwell's first equation or Gauss's law in electrostatics.

Differential form:

Apply Gauss's Divergence theorem to change L.H.S. of equation (1) from surface integral to volume integral

That is

$$\oint E.dS = \oint (\nabla.E)dV$$

Substituting this equation in equation (1a), we get

$$\oint (\nabla.E)dV = \frac{1}{\epsilon_0}\oint \rho dV \ldots\ldots\ldots\ldots\ldots(2)$$

In equation (2) as two volume integrals are equal only if their integrands are equal
Thus,

$$\nabla.E = \frac{1}{\epsilon_0}\rho \ldots\ldots\ldots\ldots(3a)$$

Or

$$\nabla.D = \rho \ldots\ldots\ldots.(3b)$$

Equation (3) is the Differential form of Maxwell's first equation.

Maxwell's second equation or Gauss's law for Magnetism:

Statement: It states that the total magnetic flux ϕ_B emerging through a closed surface is zero.

$$i.e.\ \phi_B = 0$$

Integral Form:

We have

$$\phi_B = \oint \boldsymbol{B.dS} \ldots\ldots\ldots\ldots\ldots\ldots..(\mathbf{1})$$

The equation (1) is the Integral form of Maxwell's second equation.

This equation also proves that magnetic monopole does not exist.

Differential Form:

Apply Gauss's Divergence theorem to equation (1)

That is

$$\oint \mathrm{B.dS} = \oint (\nabla.\mathrm{B})\mathrm{dV}$$

As

$$\oint \mathrm{B.dS} = 0$$

$$\therefore \oint (\nabla.\mathrm{B})\mathrm{dV} = 0$$

Thus ,

$$\nabla, B = divB = 0,,,,,,,,,,,,,,,,,,(2)$$

The equation (2) is differential form of Maxwell's second equation.

Maxwell's Third equation

Statement:

.(a) It states that, whenever magnetic flux linked with a circuit changes then induced electromotive force (emf) is set up in the circuit. This induced emf lasts so long as the change in magnetic flux continues.

(b) The magnitude of induced emf is equal to the rate of change of magnetic flux linked with the circuit.

Therefore

$$e = -\frac{\partial \phi_B}{\partial t} \ldots\ldots\ldots\ldots\ldots (1)$$

Here negative sign is because of Lenz's law which states that the induced emf set up a current in such a direction that the magnetic effect produced by it opposes the cause producing it.

Where 'e' is induced emf

Integral Form:

Since

$$\phi_B = \oint B.dS \ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Also definition of emf states that emf is the closed line integral of the non-conservative electric field generated by the battery.

That is

$$e = \oint E.dl \ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

From equations (1) and (2), we get

$$\oint E.dl = -\frac{d}{dt}\oint B.dS = -\oint \frac{dB}{dt}.dS \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Equation (4) is the integral form of Maxwell's third Equation or Faraday's law of electromagnetic induction.

This is integral form of Maxwell's third equation

Differential form:

Apply Stoke's theorem to L.H.S. of equations (4) to change line integral to surface integral.

That is

$$\oint \mathrm{E.dl} = \oint (\nabla \times \mathrm{E}).\mathrm{dS}$$

By substituting above equation in equation (3), we get

$$\oint (\nabla \times \mathrm{E}).\mathrm{dS} = -\oint \left(\frac{\partial \mathrm{B}}{\mathrm{dt}}\right).\mathrm{dS}$$

As two surface integral are equal only when their integrands are equal.

Thus

$$(\nabla \times \mathrm{E}) = -\frac{\partial \mathrm{B}}{\partial \mathrm{t}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

Equation (5) is the Differential form of Maxwell's third equation.

Maxwell's Fourth Equation or Modified Ampere's Circuital Law

Integral Form:

According to Ampere's circuital law, i.e. the line integral of magnetic field (B) over closed path is equal to μ times the net current (I) flowing through the closed path.

That is

$$\oint B.dl = \mu_o I = \mu_o(I_C + I_d) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Here the first question arises, why there was need to modify Ampere's circuital Law?

The net current is sum of conduction current (I_c) and displacement current (I_d). In terms of conduction current density (J_c) and displacement current density (J_d), We can express I_c and I_d. We have

$$I_c = \oint J_c.dS \qquad and \quad I_d = \oint J_d.dS \ldots\ldots\ldots\ldots.(2)$$

Substituting these values in equation (1), we get

$$\oint B.dl = \mu_o\left(\oint J_c.dS + \oint J_d.dS\right) \ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

Normally conduction current density Jc is represented by J and J_d by$\left(\frac{\partial D}{\partial t}\right)$, So equation (3) can be written as

$$\oint B.dl = \mu_o\left(\oint J.dS + \oint \frac{\partial D}{\partial t}.dS\right)$$

$$\oint B.dl = \mu_o\left(\oint \left(J + \frac{\partial D}{\partial t}\right).dS\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

Since $H = \frac{B}{\mu_o}$ is magnetic field intensity

$$\oint H.dl = \oint \left(J + \frac{\partial D}{\partial t}\right).dS \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

This is integral form of Maxwell's equation.

Differential form:

According to Stoke's Law

$$\oint B.dl = \oint (\vec{\nabla} \times \mathrm{B}).\mathrm{dS} \ldots\ldots\ldots\ldots..(6)$$

From equations (4) and (6) we get,

$$\oint (\vec{\nabla} \times \mathrm{B}).\mathrm{dS} = \mu_o \left(\oint \left(J + \frac{\partial D}{\partial t} \right).dS \right)$$

Or

$$\left(\vec{\nabla} \times \mathrm{B}\right) = \mu_o \left(J + \frac{\partial D}{\partial t} \right) \ldots\ldots\ldots\ldots..(7)$$

$$\left(\vec{\nabla} \times \mathrm{H}\right) = \left(J + \frac{\partial D}{\partial t} \right) \ldots\ldots.(8)$$

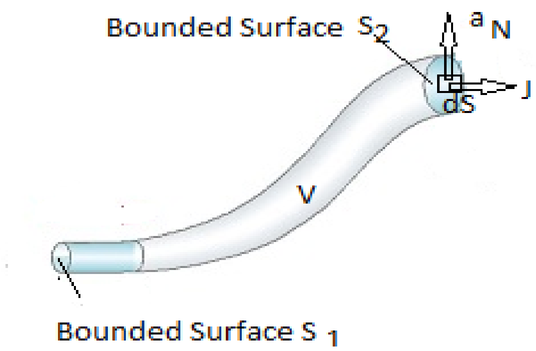
This is differential form of Maxwell's Equation

Equation of Continuity:

According to the law of conservation of charges. Consider and arbitrary volume V bounded by surface $S_1$and S_2. A net charge q exists within this region. If a net current I flows across the surface S_1 into the volume bounded by surface S_1 and S_2, then the change in the volume must increase at the equal to current.

On the other hand, if a net current I flows across the surface S_2, the charge in the volume bounded between S_1 and S_2 must decrease at the rate equals the current.

The current leaving the volume is the total outward flux of current density vector through the surface S_2



The Maxwell's fourth equation can be expressed as

$$\mathrm{curl}\,\vec{\mathrm{B}} = \mu_0 \mathrm{J} + \mu_0 \epsilon_0 \frac{\partial \vec{\mathrm{E}}}{\partial \mathrm{t}}$$

Taking divergence on both sides

$$\mathrm{div.curl}\vec{\mathrm{B}} = \mathrm{div.}\ \mu_0 \mathrm{J} + \mu_0 \epsilon_0 \frac{\partial \mathrm{div.}\vec{\mathrm{E}}}{\partial \mathrm{t}}$$

Now

$$\mathrm{div.curl}\vec{\mathrm{B}} = \vec{\nabla}.\left(\vec{\nabla} \times \vec{\mathrm{B}}\right) = 0$$

Then

$$0 = \ \mathrm{div.}\ \mu_0 \mathrm{J} + \mu_0 \epsilon_0 \frac{\partial \mathrm{div.}\vec{\mathrm{E}}}{\partial \mathrm{t}}$$

From Maxwell's first equation

$$\text{div}.\vec{E} = \frac{\rho}{\epsilon_0}$$

Therefore

$$0 = \text{ div. } \mu_0 J + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \frac{\rho}{\epsilon_0}$$

Therefore

$$0 = \text{div. } J + \frac{\partial \rho}{\partial t}$$

$$\boldsymbol{\vec{\nabla}.\mathbf{J} = -\frac{\partial \rho}{\partial t}}$$

Or

$$\boldsymbol{divJ + \frac{\partial \rho}{\partial t}}$$

This is Known Equation of continuity

Poynting vector and poynting theorem:

When electromagnetic wave travels in space, it carries energy and energy density is always associated with electric fields and magnetic fields.

The rate of energy travelled through per unit area i.e. the amount of energy flowing through per unit area in the perpendicular direction to the incident energy per unit time is called poynting vector. Mathematically poynting vector is represented as

$$\vec{P} = \vec{E} \times \vec{H} \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (1)$$

The direction of poynting vector is perpendicular to the plane containing $\vec{E}$ and $\vec{H}$. Poynting vector is also called as instantaneous energy flux density. Here rate of energy transfer $\vec{P}$ is perpendicular to both $\vec{E}$ and $\vec{H}$. Since it represents the rate of energy transfer per unit area, its unit is W/m^2

Poynting Theorem: Poynting theorem states that the net power flowing out of a given volume V is equal to the time rate of decrease of stored electromagnetic energy in that volume decreased by the conduction losses.

i.e.

Total power leaving the volume = (rate of decrease of stored electromagnetic energy) – (ohmic power dissipated due to motion of charge)

Proof: We shall **derive** the desired result by using Maxwell's equation

By Maxwell's equation

$$\vec{\nabla} \times H = J + \frac{\partial D}{\partial t} \dots \dots \dots \dots \dots \dots . (1)$$

Taking dot product with E on both sides, we get

$$E.\left(\vec{\nabla}\times H\right) = E.J + \left(E.\frac{\partial D}{\partial t}\right) \ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

We know that the vector identity

$$\vec{\nabla}.(E\times H) = H.\left(\vec{\nabla}\times E\right) - E.(\vec{\nabla}\times H)$$

Or

$$E.\left(\vec{\nabla}\times H\right) = H.\left(\vec{\nabla}\times E\right) - \vec{\nabla}.(E\times H)$$

Putting this value in equation (2), we get

$$H.\left(\vec{\nabla}\times E\right) - \vec{\nabla}.(E\times H) = E.J + \left(E.\frac{\partial D}{\partial t}\right) \ldots\ldots\ldots\ldots\ldots.(3)$$

Consider the following Maxwell's equation

$$\left(\vec{\nabla}\times E\right) = -\frac{\partial B}{\partial t} = -\mu\frac{\partial H}{\partial t}$$

Then equation written as

$$\therefore \quad H.\left(\vec{\nabla}\times E\right) = H.\left(-\mu\frac{\partial H}{\partial t}\right) = -\mu\left(H.\frac{\partial H}{\partial t}\right)$$

Substituting in equation (3), we get

$$-\mu\left(H.\frac{\partial H}{\partial t}\right) - \vec{\nabla}.(E\times H) = E.J + \left(E.\frac{\partial D}{\partial t}\right)$$

Since D=εE

$$-\mu\left(H.\frac{\partial H}{\partial t}\right) - \vec{\nabla}.(E\times H) = E.J + \epsilon\left(E.\frac{\partial E}{\partial t}\right) \ldots\ldots\ldots\ldots.(4)$$

Let us consider

$$\frac{\partial}{\partial t}(H.H) = H.\frac{\partial H}{\partial t} + H.\frac{\partial H}{\partial t} = 2H.\frac{\partial H}{\partial t}$$

Or

$$\frac{\partial}{\partial t}(H^2) = 2H.\frac{\partial H}{\partial t}$$

Or

$$H.\frac{\partial H}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(H^2)$$

Similarly

$$E.\frac{\partial E}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E)^2$$

Substituting these values in equation (4), we get

$$-\mu\left(H.\frac{\partial H}{\partial t}\right) - \vec{\nabla}.(E\times H) = E.J + \epsilon\left[\frac{1}{2}\frac{\partial}{\partial t}(E)^2\right]$$

Or

$$-\vec{\nabla}.(E\times H) = E.J + \frac{\partial}{\partial t}\left[\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right]$$

$$\vec{\nabla}.(E\times H) = -E.J - \frac{\partial}{\partial t}\left[\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5)$$

Integrating over volume V, we have

$$\int_V \vec{\nabla}.(E \times H)dV = -\int_V (E.J)dV - \frac{\partial}{\partial t}\int_V \left[\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right] dV \ldots\ldots\ldots\ldots\ldots..(6)$$

Using divergence theorem

$$\int_V \vec{\nabla}.(E \times H)dV = \int_S (E \times H)ds$$

Equation (6) becomes

$$\int_S (E \times H)ds = -\int_V (E.J)dV - \frac{\partial}{\partial t}\int_V \left[\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right] dV \ldots\ldots\ldots\ldots\ldots..(7)$$

This equation is an important equation which tells us about the flow of power in a wave. Let us discuss the term in this equation one by one:

(1) The term $\int_V (E.J)$ represents instantaneous power dissipated in volume V

In an electric field

$$E = \frac{Voltage}{Distance} = \frac{V}{d} \text{ and J(current density)} = \frac{I}{A}$$

Therefore

$$E.J = \frac{VI}{Ad} = Power\ loss\ in\ unit\ volume = I^2R$$

(2) - $\frac{\partial}{\partial t}$ represents the rate at which the stored energy is decreasing

$\left[\frac{\mu H^2}{2}\right]$ Represents the energy density due to magnetic field

$\left[\frac{\epsilon E^2}{2}\right]$ represents the energy density due to electric field

Therefore $\left[\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right]$ represents the total stored energy in volume V due to both electric and magnetic fields. Thus,

$$-\frac{\partial}{\partial t}\int_V \left[\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right] dV$$

Represents the rate of decrease of stored energy

(3) $\int_S (E \times H)ds$ gives the rate flow of energy outwards through the surface.

The product $(E \times H) = P$ is measure of rate of energy flow per unit area. P is known a pointing vector. The unit of P is Watt/m^2 and is in the direction of $(E \times H)$.therefore Poynting vector

$$\vec{P} = \vec{E} \times \vec{H} \text{ Watt/m}^2$$

Thus the integral of P or $(E \times H)$ over a closed surface represents the rate at which electromagnetic energy the closed surface. In this relation E and H represents the instantaneous values.