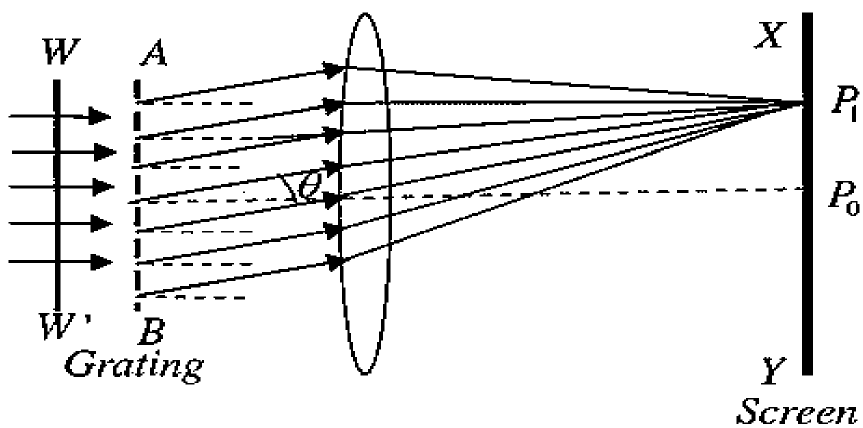
Diffraction due to N-Slits (Grating)

An arrangement consisting of large number of parallel slits of the same width and separated by equal opaque spaces is known as Diffraction grating.

Gratings are constructed by ruling equidistant parallel lines on a transparent material such as glass, with a fine diamond point. The ruled lines are opaque to light while the space between any two lines is transparent to light and acts as a slit. This is known as plane transmission grating. When the spacing between the lines is of the order of the wavelength of light, then an appreciable deviation of the light is produced.

Theory: A section of a plane transmission grating AB placed perpendicular to the plane of the paper is as shown in the figure.



Let 'e' be the width of each slit and'd' the width of each opaque space. Then (e+d) is known as grating element and XY is the screen. Suppose a parallel beam of monochromatic light of wavelength 'λ ' be incident normally on the grating. By Huygen's principle, each of the slit sends secondary wavelets in all directions. Now, the secondary wavelets travelling in the direction of incident light will focus at a point Po on the screen. This point Po will be a central maximum.

Now consider the secondary waves travelling in a direction inclined at an angle 'θ ' with the incident light will reach point P_1 in different phases. As a result dark and bright bands on both sides of central maximum are obtained.

The intensity at point P_1 may be considered by applying the theory of Fraunhofer diffraction at a single slit. The wavelets proceeding from all points in a slit along their direction are equivalent to a single wave of amplitude $A\frac{sin\alpha}{\alpha}$ starting from the middle point of the slit, Where $\alpha = \frac{\pi}{\lambda}esin\theta$

If there are N slits, then we have N diffracted waves. The path difference between two consecutive slits is $(e+d)sin\theta$. Therefore, the phase difference

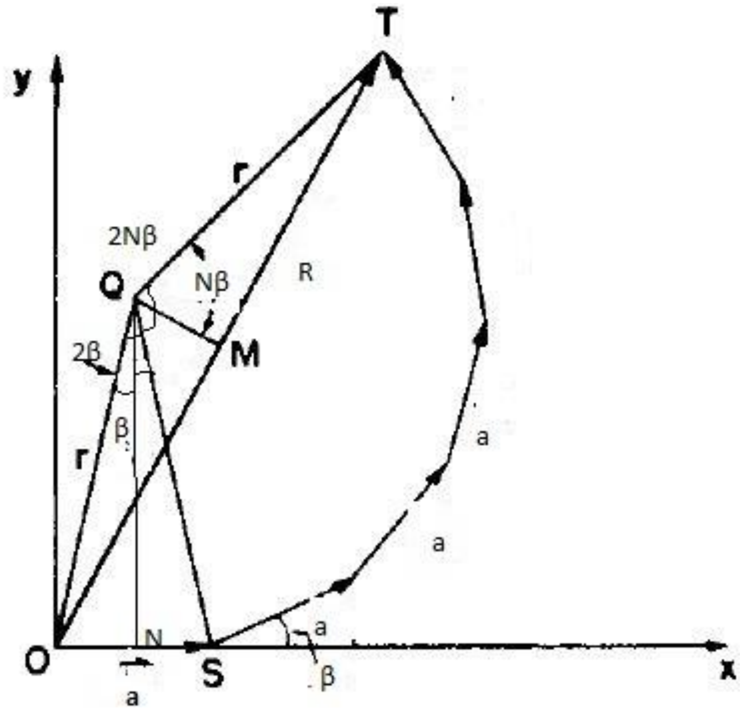
$$\delta = \frac{2\pi}{\lambda}(e+d)sin\theta = 2\beta \quad ....(1)$$

Hence the intensity in a direction 'θ ' can be found by finding the resultant amplitude of N vibrations each of amplitude $A\frac{sin\alpha}{\alpha}$ and a phase difference of '2β '

Since in the previous case

$$a = A\frac{sin\alpha}{\alpha}; n = N; \delta = 2\beta$$

Substituting these in equation $R = a\frac{sin(n\delta/2)}{sin(\delta/2)}$



In ΔOQM

OM= OQ Sin Nβ…………(a)

And δ In ΔOQN

ON= OQ Sin β…………..(b)

From equation (a) and (b)

$$OM = ON\frac{\sin N\beta}{\sin \beta}$$

OR

$$R = a\frac{\sin N\beta}{\sin \beta}$$

Where a= resultant amplitude due to single slit diffraction

Therefore

$$a = A\frac{\sin \alpha}{\alpha}$$

The resultant amplitude on screen at P becomes

$$R = \left(A\frac{sin\alpha}{\alpha}\right)\frac{sinN\beta}{sin\beta} \text{..........(2)}$$

Thus Intensity at P_1 will be

$$I = R^2 = \left(A\frac{\sin\alpha}{\alpha}\right)^2 \frac{sin^2 N\beta}{sin^2\beta} \dots\dots\dots\dots\dots\dots (3)$$

The factor

$$\left(A\frac{sin\alpha}{\alpha}\right)^2$$

gives the distribution of Intensity due to a single slit while the factor

$$\frac{sin^2 N\beta}{sin^2\beta}$$

gives the distribution of intensity as a combined effect of all the slits

Intensity Distribution:

Case (i): Principal maxima: The eqn (2) will take a maximum value if

$$sin\beta = 0$$

$$\beta = \pm n\pi; n = 0, 1, 2, 3.....$$

$$\frac{\pi}{\lambda}(e+d)sin\theta = \pm n\pi$$

$$\boldsymbol{(e+d)sin\theta = \pm n\lambda} \text{.......(3)}$$

n = 0 corresponds to zero order maximum. For n = 1,2,3,… we obtain first, second, third,… principal maxima respectively. The ± sign indicates that there are two principal maxima of the same order lying on either side of zero order maximum.

Case(ii): Minima Positions: The eqn (2) takes minimum value if $sinN\beta = 0$ but $sin\beta \neq 0$

$$\therefore N\beta = \pm m\pi$$

$$N\frac{\pi}{\lambda}(e+d)sin\theta = \pm m\pi$$

$$N(e+d)sin\theta = \pm m\lambda \text{...........(4)}$$

Where m has all integral values except m = 0, N, 2N, …, nN, because for these values $sin\beta$ becomes zero and we get principal maxima. Thus, m = 1, 2, 3, …, (N-1). Hence

$$N(e+d)sin\theta = \pm m\lambda$$

where $m = 1, 2, 3....(N-1), (N+1), ...(2N-1)....$

gives the minima positions which are adjacent to the principal maxima.

Case(iii): Secondary maxima: As there are (N-1) minima between two adjacent principal maxima there must be (N-2) other maxima between two principal maxima. These are known as secondary maxima. To find their positions

$$\frac{\mathrm{dI}}{\mathrm{d}\beta} = 0$$

$$\frac{dI}{d\beta} = \left(A\frac{\sin\alpha}{\alpha}\right)^2 2\left(\frac{\sin N\beta}{\sin\beta}\right)\left[\frac{N\cos N\beta\sin\beta - \sin N\beta\cos\beta}{sin^2\beta}\right] = 0$$

$$\because \frac{sin\alpha}{\alpha} \neq 0; sinN\beta \neq 0$$

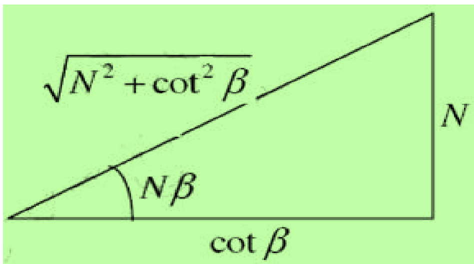
Only

$$[NcosN\beta sin\beta - SinN\beta cos\beta] = 0$$

$$\boldsymbol{Ntan\beta = tanN\beta} \text{............(5)}$$

The roots of the above equation other than those for which $\beta = \pm n\pi$ give the positions of secondary maxima The eqn (5) can be written as

$$tanN\beta = \frac{N}{cot\beta}$$



From the triangle we have

$$sinN\beta = \frac{N}{\sqrt{N^2 + cot^2\beta}}$$

$$\therefore \frac{sin^2N\beta}{sin^2\beta} = \frac{N^2}{(N^2 + cot^2\beta)sin^2\beta}$$

$$\therefore \frac{sin^2N\beta}{sin^2\beta} = \frac{N^2}{N^2sin^2\beta + (1 - sin^2\beta)}$$

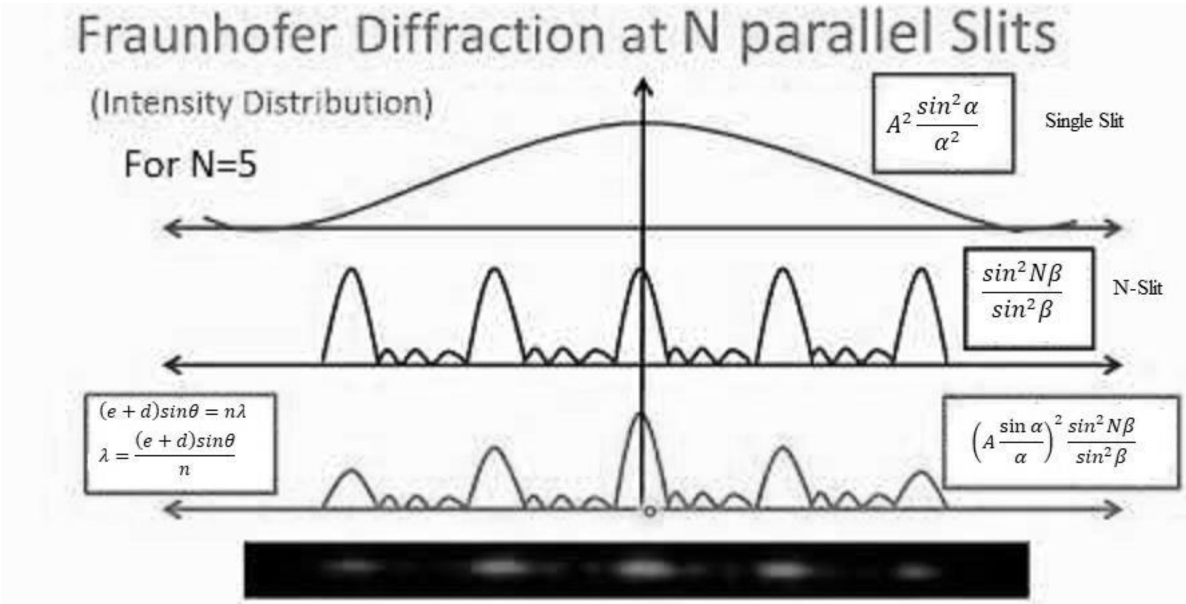
$$\therefore \frac{sin^2N\beta}{sin^2\beta} = \frac{N^2}{1 + (N^2 - 1)sin^2\beta}$$

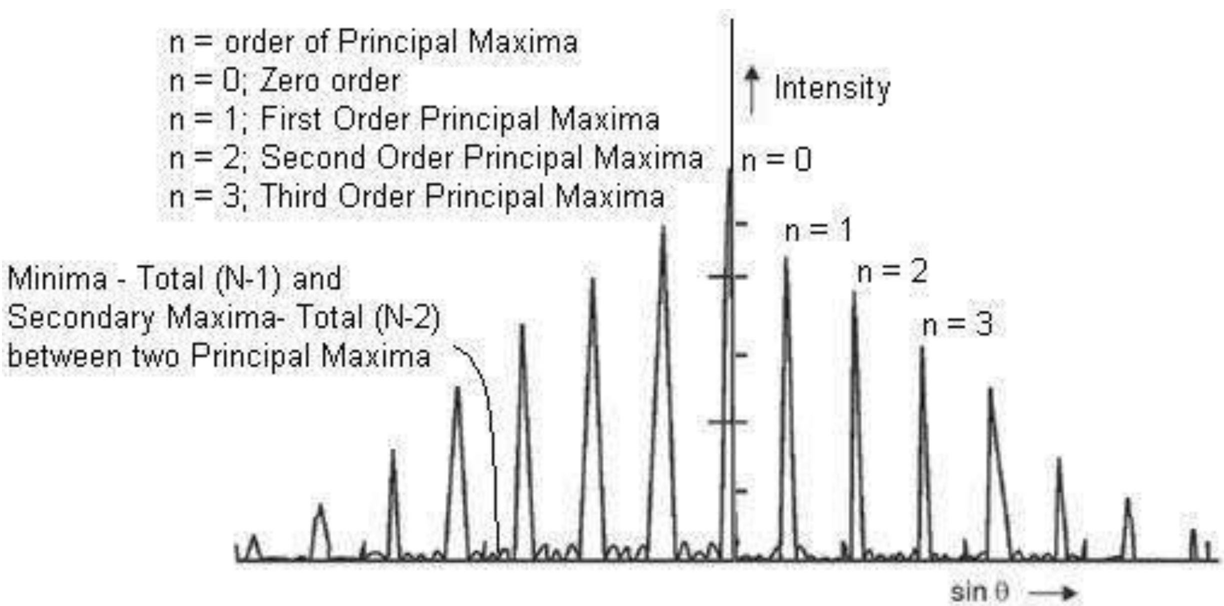
$$I = I_o\left(\frac{sin\alpha}{\alpha}\right)^2 \frac{N^2}{1 + (N^2 - 1)sin^2\beta} \backslash$$

Since intensity of principal maxima is proportional to N^2,

$$\frac{\text{Intensity of secondry maxima}}{\text{Intensity of Principal maxima}} = \frac{\frac{N^2}{1+(N^2-1)\sin^2\beta}}{N^2} = \frac{1}{1+(N^2-1)\sin^2\beta}$$

Hence if the value of N is larger, then the secondary maxima will be weaker and becomes negligible when N becomes infinity.





Formation of Spectra with Grating:

The principle maxima in a grating are formed in direction given by

$$(e+d)sin\theta = n\lambda$$

Where

(e + d) $\Rightarrow$ the grating element,

'n'$\Rightarrow$ the order of the maxima and

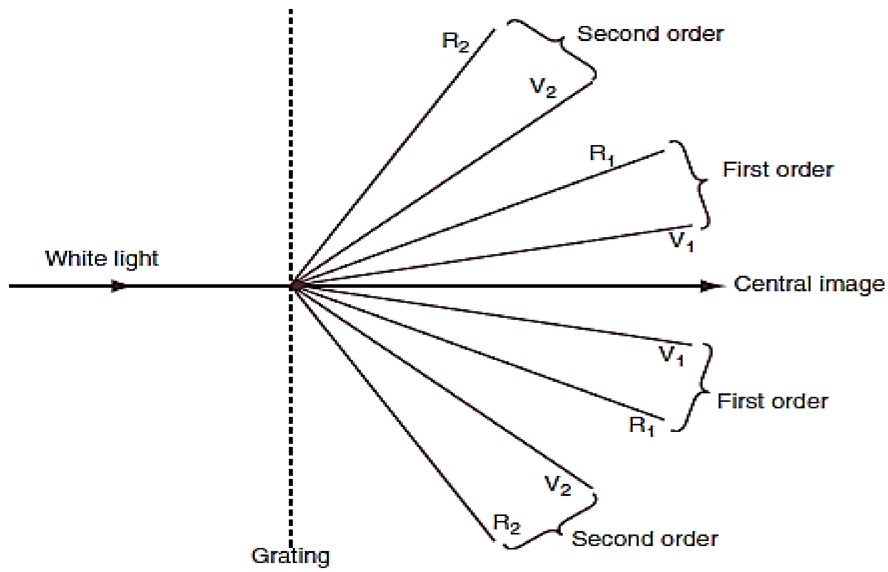
$\lambda \Rightarrow$the wavelength of the incident light.

- For a given wavelength λ the angle of diffraction θ is different for principal maxima of different orders.
- For white light and for a particular order n, the light of different wavelengths will be diffracted in different directions.

The longer the wavelength, greater is the angle of diffraction. So in each order, we will get the spectra having as many lines as the wavelength in the light source.

At centre (n = 0, zero order) $\theta = 0$ gives the maxima of all wavelengths. So here different wavelengths coincide to form the central image of the same colour as that the light source.

Similarly the principal maxima of all wavelengths corresponding to n = 1 will form the first order spectrum, the principal maxima of all wavelengths corresponding to n = 2, will form the second order spectrum and so on.



From this we conclude that

Important characteristics of grating spectra:

1. Spectra of different orders are situated symmetrically on both sides of zero order.
2. Spectral lines are almost straight and quite sharp.
3. Spectral colours are in the order from Violet to Red.
4. Spectral lines are more dispersed as we go to higher orders.
5. Most of the incident intensity goes to zero order and rest is distributed among the other orders.

Maximum number of orders formed by a Grating:

The principal maxima in a grating satisfy the condition

$$(e+d)sin\theta = n\lambda$$

Or $$n = \frac{(e+d)sin\theta}{\lambda}$$

The maximum angle of diffraction is 90^o, hence the maximum possible order is given by

$$n_{max} = \frac{(e+d)sin90^o}{\lambda} = \frac{(e+d)}{\lambda}$$

Ex: Consider a grating having grating element which is less than twice the wavelength of the incident light, then $(e+d) < 2\lambda$

$$\therefore n_{max} < \frac{2\lambda}{\lambda} < 2$$

i.e., only the first order is possible.

Absent spectrum with a diffraction grating:

Sometimes it happens that the first order spectrum is clearly visible, second order is not visible and third order is again visible, i.e., the second order is absent, and so on. This happens when for a given angle of diffraction θ , the path difference between the diffracted rays from the two extreme ends of one slit is equal to an integral multiple of λ. Suppose the path difference is λ , then each slit can be considered to be made up of two halves, the path difference between the secondary waves from the corresponding points in the two halves will be l / 2 . Now they will cancel one another resulting zero intensity.

We know that, in case of a grating the principal maxima are obtained in the directions given by

$$(e+d)sin\theta = n\lambda$$

Also, in case of a single slit, the minima are obtained in the directions given by

$$esin\theta = m\lambda$$

If both the conditions are satisfied simultaneously, a particular maximum of order n will be missing in the grating spectrum. Dividing above equations we have

$$\boldsymbol{\frac{(e+d)}{e} = \frac{n}{m}}$$

which is the condition of absent spectra.

If the width of the ruling is equal to the width of the slit,

$$\because e = d \therefore \frac{(e+e)}{e} = 2 \Rightarrow n = 2m$$

the second order spectrum will be missed.

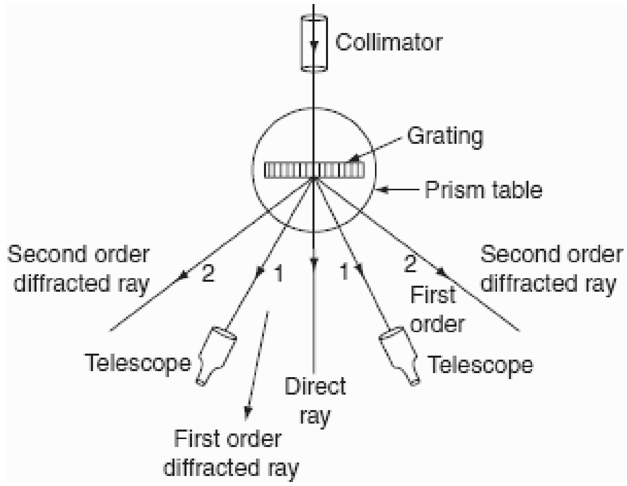
Determination of Wavelength Using Grating

In laboratory, the grating spectrum can be obtained by using a spectrometer.

Adjustments: Before performing the experiment, the following adjustments are made.

(1) The spectrometer is adjusted for parallel rays by Schuster’s method.

(2) The grating is adjusted for normal incidence.



Measurement of θ : When a white light is incident on the grating normally, the beam gets dispersed and in each order of the spectrum we can observe constituent wavelengths (i.e., VIBGYOR)

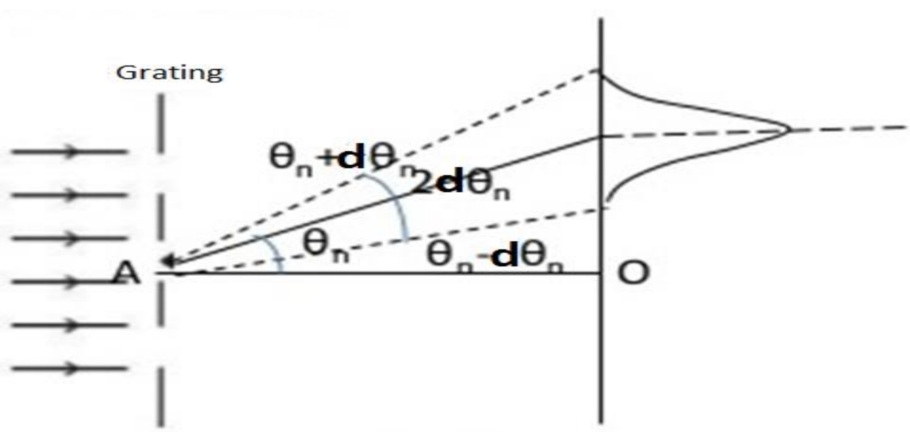
1. The telescope is now turned to get the first order spectrum in the field of view on left.
2. The cross-wire is adjusted on the line whose wavelength is to be determined (say on RED line)
3. Now, the readings of the two verniers are noted.
4. The telescope is then turned to the right side to receive the first order spectrum and repeat steps (2) & (3).
5. The difference between readings of the same vernier gives twice the angle of diffraction q for that line in first order.
6. By substituting the values of θ , (e + d) and n in $(e+d)sin\theta = n\lambda$ we can determine the wavelength of light.

The same procedure from step (1) through (6) is repeated for second order and even in higher orders.

Width of the principal Maxima

The angular width of principal maxima of n^{th} *order is defined as the angular separation between the first two minima lying adjacent to principal maxima on either side.*

If θ_n *is position of nth order principal maxima and* $\theta_n+d\theta_n$ *and* $\theta_n-d\theta_n$ *are the direction of first outer and inner sided minima adjacent to* n^{th} *maxima, then the value of angular width will be* $2d\theta_n$ *.*



The direction of nth principal maxima and minima are given as follows

$$(\mathrm{e}+\mathrm{d}) \sin \theta_{\mathrm{n}}=\mathrm{n} \lambda \ldots \ldots \ldots \ldots(1)$$

$$\mathrm{N}(\mathrm{e}+\mathrm{d}) \sin \theta_{\mathrm{n}}=\mathrm{m} \lambda \ldots \ldots \ldots \ldots(2)$$

For the first order outer and inner sided minima adjacent to the nth maxima, θ_n should be replacing with (θ_n+$d\theta_n$) and m=(nN+1). Then from equation (2), we get

$$N(e+d) \sin(\theta_n \pm d\theta_n)=(nN \pm 1)\lambda \ldots \ldots \ldots \ldots(3)$$

$$(e+d) \sin(\theta_n \pm d\theta_n)=\frac{(nN \pm 1)}{N}\lambda \ldots \ldots \ldots \ldots(4)$$

On dividing equation (4) by equation (1)

$$\frac{\sin(\theta_n+d\theta_n)}{\sin\theta_n}=\frac{(nN \pm 1)}{N}$$

$$\boldsymbol{\frac{\sin\theta_n \cos d\theta_n \pm \cos\theta_n \sin d\theta_n}{\sin\theta_n}=\left(1 \pm \frac{1}{nN}\right)}$$

$$\boldsymbol{\cos d\theta_n \pm \frac{\cos\theta_n \sin d\theta_n}{\sin\theta_n}=\left(1 \pm \frac{1}{nN}\right)}$$

If dθn is very small then cos dθn=1, sin dθn=dθn

$$1 \pm \frac{\cos\theta_n d\theta_n}{\sin\theta_n}=\left(1 \pm \frac{1}{nN}\right)$$

$$\cot\theta_n d\theta_n=\frac{1}{nN} \Longrightarrow d\theta_n=\frac{\tan\theta_n}{nN}$$

Width of principal maxima

$$\boldsymbol{2d\theta_n=\frac{2\tan\theta_n}{nN}}$$

Resolving Power:

Rayleigh Criterion of Resolution:

Statement: Two sources are resolvable by an optical instrument when the central maximum of one diffraction pattern falls over the first minimum of the other diffraction pattern and vice versa.

For example:

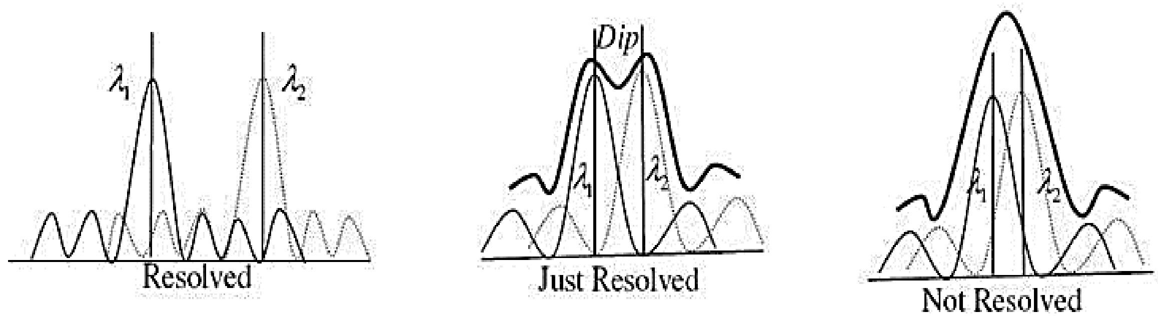
Let us consider the resolution of two wavelengths and by a grating. When the difference in wavelengths is smaller and such that the central maximum of the wavelength coincides with the first minimum of the other as shown in figure, then the resultant intensity curve is as shown by the thick curve. The curve shows a distinct dip in the middle of two central maxima. Thus the two wavelengths can be distinguished from one another and according to Rayleigh they are said to be **"Just Resolved".**

If the difference in wavelengths is such that their principal maxima are separately visible, then there is a distinct point of zero intensity in between the two wavelengths. Hence according to Rayleigh they are said to be **"Resolved".**

When the difference in wavelengths is so small that the central maxima corresponding to two wavelengths come still closer as shown in figure, then the resultant intensity curve is quite smooth without any dip. This curve is as if there is only one wavelength somewhat bigger and stronger.

Hence according to Rayleigh the two wavelengths are **"Not Resolved".**

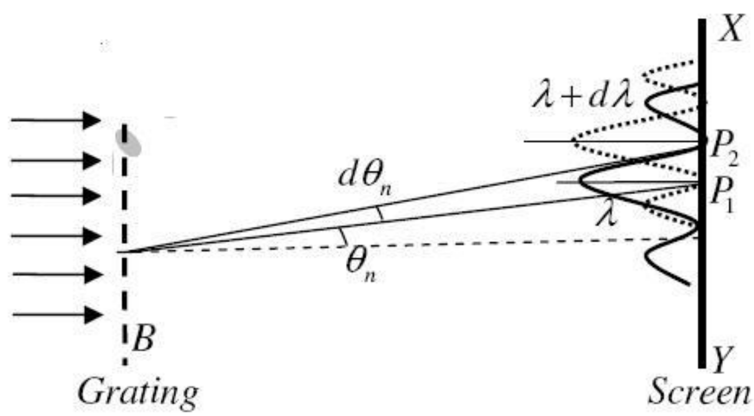
Thus the two spectral lines can be resolved only up to a certain limit expressed by Rayleigh Criterion.



Resolving Power of Grating:

It is defined as the capacity of a grating to form separate diffraction maxima of two wavelengths which are very close to each other

It is measured by $\frac{\lambda}{d\lambda}$ where $d\lambda$ is the smallest difference in two wavelengths which are just resolvable by grating and λ is the wavelength of either of them or mean wavelength.



Let AB represent the surface of a plane transmission grating having grating element (e+d) and N total number of slits. Let a beam of light having two wavelengths λ and $\lambda + d\lambda$ is normally incident on the grating. Let P_1 is nth primary maximum of a spectral line of wavelength λ at an angle of diffraction θ and P_2 is the nth primary maximum of wavelength $\lambda + d\lambda$ at diffracting angle $\theta + d\theta$

According to Rayleigh criterion, the two wavelengths will be resolved if the principal maximum $\lambda + d\lambda$ of nth order in a direction $\theta + d\theta$ falls over the first minimum of nth order in the same direction $\theta + d\theta$. Let us consider the first minimum of λ of nth order in the direction $\theta + d\theta$ as below.

The principal maximum of λ in the θ direction is given by

$$(e + d)sin\theta = n\lambda \text{.......... (1)}$$

The equation of minima is $N(e+d)sin\theta = m\lambda$ where m takes all integers except 0, N, 2N, …, nN, because for these values of m, the condition for maxima is satisfied. Thus first minimum adjacent to nth principal maximum in the direction $\theta + d\theta$ can be obtained by substituting the value of 'm' as (nN+1). Therefore, the first minimum in the direction of $\theta + d\theta$ is given by

$$N(e+d)sin(\theta + d\theta) = (nN+1)\lambda$$

$$(e+d)sin(\theta + d\theta) = (n + \frac{1}{N})\lambda \quad \text{.. (2)}$$

The principal maximum of $\lambda + d\lambda$ in direction $\theta + d\theta$ is given by

$$(e+d)sin(\theta + d\theta) = n(\lambda + d\lambda) \quad \text{.......(3)}$$

Dividing eqn(2) by eqn(3), we get

$$(n + \frac{1}{N})\lambda = n(\lambda + d\lambda)$$

$$n\lambda + \frac{\lambda}{N} = n\lambda + nd\lambda$$

$$\frac{\lambda}{N} = nd\lambda$$

$$\frac{\lambda}{d\lambda} = nN$$

$$\boldsymbol{R.P. = \frac{\lambda}{d\lambda} = nN} \quad \text{.............(4)}$$

Thus the resolving power is directly proportional to

(i) The order of the spectrum 'n'

(ii) The total number of lines on the grating 'N'

Diffraction of X-ray

Soon after the discovery of X-rays, Schuster pointed out that X-rays behave as electromagnetic waves of wavelength much shorter than that of visible light. Attempts were made to measure the wave length of X-rays by means of diffraction gratings which proved unsuccessful, as the grating failed to disperse X-rays on account of their very small wavelength. Obviously, diffraction effects can only be observed if the spacing between the lines ruled on the grating is of the order of magnitude of wavelength of the wave used. Thus, in order to diffract X-rays, grating with much finer rulings, having distance between rulings comparable to the wave length of X-rays are required. It is impossible to construct a grating of such fine dimensions artificially.

In a crystal, the atoms or molecules are arranged symmetrically in a three dimensional space. Any plane containing an arrangement of atoms is known as lattice plane or cleavage plane. The spacing between the atoms is of the order of 10^{-10} m, comparable to the wavelength of X-rays. It was suggested that the regular arrangement of atoms or molecules in the cleavage planes of a crystal might provide a grating element suitable to diffract X-rays. The crystal might serve as a three dimensional grating, whereas optical grating is a two dimensional one.

Bragg's law for X-ray diffraction:

W.L. Bragg and W.H. Bragg studied the diffraction of X-rays in detail and used a crystal of rock salt to diffract X-rays and succeeded in measuring the wavelength of X-rays.

Consider homogeneous X-rays of wave length λ incident on a crystal at a glancing angle θ. The incident rays AB and DE after reflection from the lattice planes Y and Z travel along BC and EF respectively as shown in Fig .

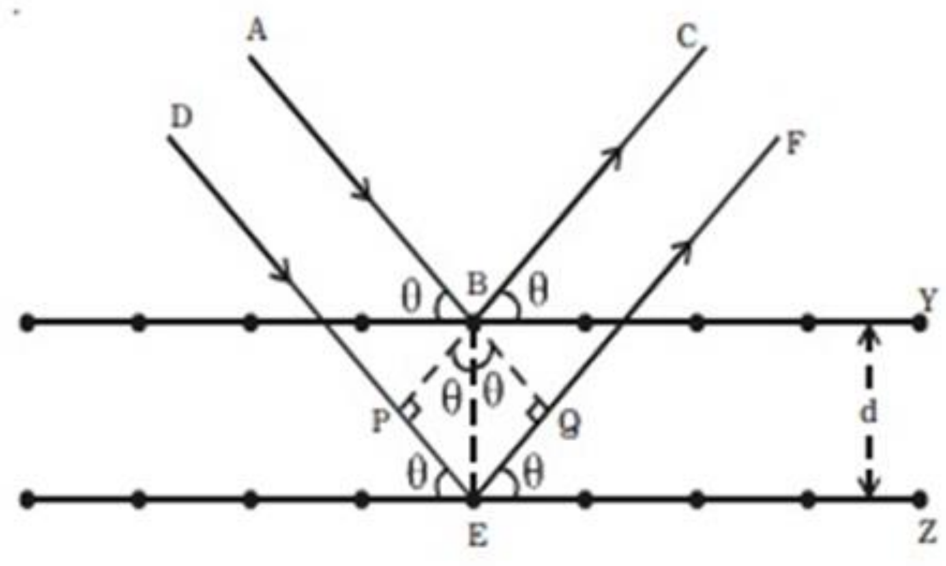


Fig Bragg's law

Let the crystal lattice spacing between the planes be d. BP and BQ are perpendiculars drawn from B on DE and EF respectively. Therefore, the path difference between the two waves ABC and DEF is equal to PE + EQ.

In the ΔPBE, $\sin\theta = PE/BE$ (or) $PE = BE \sin\theta = d \sin\theta$

In the ΔQBE, $\sin\theta = EQ/BE$ (or) $EQ = BE \sin\theta = d \sin\theta$

Path difference = $PE + EQ = d \sin\theta + d \sin\theta = 2d \sin\theta$

If this path difference $2d \sin\theta$ is equal to integral multiple of wavelength of X-ray i.e. $n\lambda$, then constructive interference will occur between the reflected beams and they will reinforce with each other. Therefore the intensity of the reflected beam is maximum.

$2d \sin\theta = n\lambda$

where, n = 1, 2, 3 etc.

This is known as Bragg's law.

Applications of Bragg's Law

There are numerous applications of Bragg's law in the field of science. Some common applications are given in the points below.

- *In the case of XRF (X-ray fluorescence spectroscopy) or WDS (Wavelength Dispersive Spectrometry), crystals of known d-spacing are used as analyzing crystals in the spectrometer.*
- *In XRD (X-ray diffraction) the inter-planar spacing or d-spacing of a crystal is used for characterization and identification purposes.*