Probability Distributions

Discrete Probability Distribution

Binomial and Bernoulli Distribution

Hypergeometric Distribution

Poisson Distribution

Bernoulli trials

An experiment that results in one of only **two possible outcomes**, which could be labeled as success or a failure, is called a Bernoulli trial.

If **n=1** in a binomial distribution, you will have the Bernoulli Distribution

Example

If you let X be the number of tails in the experiment of flipping a coin, what are the possible values of X?

In tossing a die, you may define the random variable Y=1, if the outcome is a prime number and Y=0, if the outcome is not a prime number. Then Y is a Bernoulli random variable.

If a student randomly selected from a statistics class, the random variable W would be defined to have the value W=1, if the selected student is a female and the value W=0, if the selected student is a male. As defined, W is a Bernoulli random variable.

Bernoulli Distribution

A Bernoulli random variable X,

denoted by $\boldsymbol{X \sim \mathrm{Ber}(p) \equiv \mathrm{Bin}(n=1,p)}$,

has the probability mass function

$$p(x) = \begin{cases} p^x(1-p)^{1-x}, & x = 0,1 \\ 0, & \text{otherwise} \end{cases}$$

It has

mean $\mu_X = p$ and

variance $\sigma_X^2 = p(1-p)$.

Note: the probability of "success" is p , the probability of "failure" is $1-p$.

Example of Bernoulli Distribution

1. Suppose there is an experiment where you flip a coin that is fair. This means that the probability of getting heads is $1/2$. If the outcome of the flip is heads then you will win.

a) Define the random variable X

b) Find the pmf of X.

c) What is the value of $P(X = 1)$

d) What are the mean and variance of X

Example of Bernoulli Distribution

2. Suppose there is an experiment where you flip a coin that is not fair, the probability of getting heads is $3/4$. If the outcome of the flip is heads then you will win.

a) Define the random variable X
b) Find the pmf of X.
c) What is the value of $P(X = 1)$
d) What are the mean and variance of X

Example of Bernoulli Distribution

3. Suppose there is an experiment where you flip a coin that is not fair, the probability of getting heads is $3/4$. If the outcome of the flip is heads then you will lose.

a) Define the random variable X.
b) Find the pmf of X.
c) What is the value of $P(X = 1)$
d) What are the mean and variance of X

Binomial Distribution

A Binomial experiment consists of n independent and identical Bernoulli trials and observe the number of successes.

Binomial experiment has the following characteristics:

1. It consists of n repeated trials
2. The trials are independent from one another
3. Each of these trials is a Bernoulli trial.
4. The probability of success(p) for each trial is a constant and the probability of failure (q=1-p) for each trial is also a constant.

Examples of Binomial trials

1. Observing the number of tails occurring in 20 tosses of a coin,
2. Counting the number of times of getting a sum of seven out of 10 tosses of a pair of dice
3. Counting the number of defective chips in ten electronic chips randomly selected from a large shipment of electronic chips that has a defect rate of 10%.
4. Monitoring the number of patients who will recover from a certain disease a group of 15 such patients if a recovery rate of 80% is known, are some examples of binomial experiments.

Binomial Distribution

The discrete random variable X, which signifies the **number of successes** in a binomial experiment, has the binomial distribution,

denoted by $\boldsymbol{X} \sim \mathbf{Bin}(\boldsymbol{n}, \boldsymbol{p})$**,**

and probability mass function

$$p(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0,1,2,\dots,n, \\ 0, & \text{otherwise.} \end{cases}$$

The binomial random variable X has

mean $\mu_X = np$ and

variance $\sigma_X^2 = np(1-p)$.

A Binomial experiment consists of $\boldsymbol{n}$ **independent** and identical Bernoulli trials and observe the number of successes.

Example 4. Harris, a basketball fanatic, has a shooting average of 60%. He attempts to shoot the ball 12 times.

a. What is the probability that he will make exactly eight shots?

b. What is the probability that he will make more than one shot?

c. What is the probability that he will make between three to seven shots?

d. On the average, how many times will Harris make a shot?

Example 5. A machine used to manufacture a popular toy robot produces 3% defective robots. Lynne, the owner of a small toy shop, placed an order with the toy manufacturer for 15 of these toy robots to be delivered to her toy shop. She plans to inspect the toy robots upon delivery and will return all of them to the manufacturer if she finds at least one defective toy.

a. Let X be the number of defective robots. Give the pmf of X.
b. Find the probability that Lynne will accept and purchase all the 15 toy robots.

Example 6. In a large consulting firm, one out of every five consultants is a graduate of statistics degree program. Shirlee, the Human Resources Department manager, wants to interview seven of the consultants. Let X be the number of consultants with a degree in statistics selected at random.

a. What is the probability that Shirlee will interview at least one consultant with a degree in statistics?
b. What is the probability that she will interview three or four consultants with a degree in statistics?
c. How many consultant with a degree in statistics will Shirlee get on the average?
d. What is the standard deviation of X?

Hypergeometric Experiments

The random variable X is defined as the **number of successes** in a random sample of size n taken without replacement (**dependent**) from a population of size N whose elements could be classified into one of **two** classes, which consists of *K successes* and *N-K failures.*

Examples of experiments

a. Counting the number of defective items in a random sample of five taken from a box of 20 such items consisting of three defective items and 17 nondefective items

b. Counting the number of male students included in a committee of four that were randomly selected from the class of 10 boys and 20 girls.

c. Observing how many of six numbers picked by a person for a raffle matched the six winning numbers from the numbers 1 to 49.

Hypergeometric Distribution

If the discrete random variable X is defined as **the number of successes** in a hypergeometric experiment,
denoted by $\boldsymbol{X} \sim \mathbf{Hyp}(\mathbf{N}, \mathbf{n}, \mathbf{K})$**,** where K successes and $N - K$ failures.

then it has the hypergeometric distribution with probability mass function

$$p(x) = \begin{cases} \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}, x = \max(0, n - N + K), \dots, \min(n, K); n = 1,2, \dots, N \\ \qquad 0, \qquad \textit{otherwise} \end{cases}$$

The mean and variance of X are given by $\mu_X = n\frac{K}{N}$ and $\sigma_X^2 = n\frac{K}{N}\left(1 - \frac{K}{N}\right)\left(\frac{N-n}{N-1}\right)$, respectively.

Example 7. Pocholo has four pairs of black socks and two pairs of white socks in his bedroom drawer. If he randomly picks two socks from his drawer,

a. Give the pmf of X, the number of black socks selected, and

b. Solve for the probability that he will get a pair of black socks.

$\boldsymbol{X} \sim \mathbf{Hyp}(\mathbf{N}, \mathbf{n}, \mathbf{K})$**,** where K successes and $N - K$ failures.

$$p(x) = \begin{cases} \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}, x = \max(0, n - N + K), \ldots, \min(n, K); n = 1,2, \ldots, N \\ \qquad\qquad 0, \qquad\qquad otherwise \end{cases}$$

Example 8. Among the ten officers of the Student Council, seven favor while three do not favor a proposal to change their school uniform. If four of the ten officers are selected at random and X is the number of randomly selected officers who favor the proposal, do the following.

a. Give the probability mass function of X

b. Give the probability that all the four randomly selected officers are in favor of the proposal.

$\boldsymbol{X} \sim \mathbf{Hyp}(\mathbf{N}, \mathbf{n}, \mathbf{K})$**,** where K successes and $N - K$ failures.

$$p(x) = \begin{cases} \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}, x = \max(0, n - N + K), \dots, \min(n, K); n = 1,2, \dots, N \\ 0, \qquad otherwise \end{cases}$$

Example 8. Among the ten officers of the Student Council, seven favor while three do not favor a proposal to change their school uniform. If four of the ten officers are selected at random and X is the number of randomly selected officers who favor the proposal, do the following.

c. Give the probability that at least one of the four randomly selected officers do not favor the proposal.

Example 8. Among the ten officers of the Student Council, seven favor while three do not favor a proposal to change their school uniform. If four of the ten officers are selected at random and X is the number of randomly selected officers who favor the proposal, do the following.

d. Find the expected value and variance of X

$$\mu_X = n\frac{K}{N} \text{ and } \sigma_X^2 = n\frac{K}{N}\left(1 - \frac{K}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Example 9.Yvette, the purchasing officer of a large manufacturing company, inspects ten items selected at random from a lot of 50 items that contain five defective items. Let X be the number of defective items inspected.

a. Find the expected number of defective items among the inspected items.

b. Find the variance of X.

$$\mu_X = n\frac{K}{N} \text{ and } \sigma_X^2 = n\frac{K}{N}\left(1-\frac{K}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Poisson distribution

A process that examines the number of times an event will occur over a specified time interval or region of space.

Characteristics:

1. The occurrences (Poisson events) within a specified time or region are independent of occurrences in the other time or region.
2. Number of occurrences is proportional to length time or size or region.
3. Probability that at least two events will occur simultaneously is negligible

Examples

- Monitoring the number of days classes are suspended in one school year due to typhoons
- Counting the number of airplane disasters in one year
- Counting the number of defects found in one unit of a high quality laptop brand

Poisson Distribution

A random variable X defined as the **number of occurrences** of a Poisson event **within a specified time interval or size of a region** has the Poisson distribution,

denoted by $\boldsymbol{X \sim \mathbf{Poi}(\lambda)}$**,**

and the following probability mass function:

$$p(x) = \begin{cases} \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}, & x = 0,1,2,\ldots, \text{ where } e \approx 2.71828 \\ 0, & \text{otherwise,} \end{cases}$$

where λ is the mean rate of occurrence of the Poisson event within the specified time interval or size of the region.

mean $\mu_X = \lambda$ and variance $\sigma_X^2 = \lambda$.

Law of small numbers

- Poisson distribution is also referred to as the law of small numbers because Poisson events occur rarely. It is commonly used to model rare phenomena.

Example 10. Mario hired an encoder for the manuscript of the school play he is writing. The encoder makes an average of three errors per page. Assume that the number of errors per page has the Poisson distribution. If a page is randomly selected from the manuscript, find the probability that

a. No error will be found
b. At least one error will be found
c. Exactly one error will be found

Example 10. Mario hired an encoder for the manuscript of the school play he is writing. The encoder makes an average of three errors per page. Assume that the number of errors per page has the Poisson distribution. If a page is randomly selected from the manuscript, find the probability that

d. 4 errors will be found in two pages

λ is the mean rate of occurrence

Example 10. Mario hired an encoder for the manuscript of the school play he is writing. The encoder makes an average of three errors per page. Assume that the number of errors per page has the Poisson distribution. If a page is randomly selected from the manuscript, find the probability that

e. At least 2 errors will be found in the next three pages

λ is the <u>mean rate of occurrence</u>

Example 10. Mario hired an encoder for the manuscript of the school play he is writing. The encoder makes an average of three errors per page. Assume that the number of errors per page has the Poisson distribution. If a page is randomly selected from the manuscript, find the probability that

e. Less than 2 errors will be found in the next two pages

λ is the mean rate of occurrence

Example 11. According to the Philippine Atmospheric, Geophysical, and Astronomical Administration (PAGASA), an average of 20 typhoons enter the Philippine area of responsibility (PAR) every year. Assuming that the number of typhoons that enter the country every year has the Poisson distribution, find the probability that

a. Exactly 15 typhoons will enter the country next year.

Example 11. According to the Philippine Atmospheric, Geophysical, and Astronomical Administration (PAGASA), an average of 20 typhoons enter the Philippine area of responsibility (PAR) every year. Assuming that the number of typhoons that enter the country every year has the Poisson distribution, find the probability that

b. Between 12 to 14 typhoons will enter the country next year

Example 11. According to the Philippine Atmospheric, Geophysical, and Astronomical Administration (PAGASA), an average of 20 typhoons enter the Philippine area of responsibility (PAR) every year. Assuming that the number of typhoons that enter the country every year has the Poisson distribution, find the probability that

c. At least one typhoon will enter the country next month.

Example 12. Reggie and Michelle agreed to spend one Sunday baking chocolate chip cookies for their friends. Reggie was in charge of preparing the cookie dough, while Michelle took care of cutting the dough into portions and sprinkling mini chocolate chips on them.

a. Suppose that the number of mini chocolate chips on each cookie has the Poisson distribution with an average of three, what proportion of the cookies will have more than five mini chocolate chips each?

Example 12. Reggie and Michelle agreed to spend one Sunday baking chocolate chip cookies for their friends. Reggie was in charge of preparing the cookie dough, while Michelle took care of cutting the dough into portions and sprinkling mini chocolate chips on them.

b. If the cookies were placed in boxes of 15 cookies, what is the probability that exactly two of the 15 cookies in a randomly selected box contain more than five mini chocolate chips?