1. Problem Description

The port has $|\mathcal{J}|$ berths and $|\mathcal{K}|$ tugboats. During the scheduling period $\mathcal{T} = 1, 2, \ldots, T$, there are $|\mathcal{I}|$ ships applying for berthing. Each ship needs to be allocated a berth, berthing time, and inbound/outbound tugboat services to minimize total scheduling costs. Some ships may be allowed to remain unassigned.

2. Symbol Definitions

2.1 Sets and Indices

- $\mathcal{I} = 1, \ldots, m$: Ship set, index i
- $\mathcal{J} = 1, \ldots, n$: Berth set, index j
- $\mathcal{K} = 1, \ldots, K$: Tugboat set, index k
- $\mathcal{T} = 1, \ldots, T$: Time set, index t

2.2 Parameters

Ship-berth matching related:

- S_i: Size class of ship i
- C_j: Capacity class of berth j

Time related:

- ETA_i: Expected arrival time period of ship i
- D_i: Berthing operation duration of ship i (number of consecutive time periods)
- τ_i^{in}: Inbound tugboat service duration for ship i
- τ_i^{out}: Outbound tugboat service duration for ship i
- ρ^{in}: Preparation time after tugboat completes inbound service
- ρ^{out}: Preparation time after tugboat completes outbound service
- Δ_i^{early}: Maximum number of early time periods allowed for ship i
- Δ_i^{late}: Maximum number of late time periods allowed for ship i

Cost related:

- α_i: Priority weight of ship i
- β_i: Unit waiting cost of ship i
- γ_i: Unit cost of JIT deviation for ship i
- c_k: Unit time period usage cost of tugboat k

Horsepower related:

- P_k: Horsepower of tugboat k
- P_i^{req}: Minimum tugboat horsepower required by ship i

System parameters:

- H_{max}: Maximum number of tugboats allowed for a single service
- ϵ_{time}: Maximum time deviation allowed for timing constraints
- M: Big number parameter (used for penalty or logical relaxation)
- $\lambda_1, \lambda_2, \lambda_3, \lambda_4$: Weighting coefficients in the objective function

2.3 Decision Variables

- $x_{ijt} \in \{0, 1\}$: Takes 1 if ship i starts berthing at berth j in time period t and $C_j \geq S_i$
- $y_{ikt}^{in} \in \{0, 1\}$: Takes 1 if tugboat k starts inbound service for ship i in time period t

- $y_{ikt}^{out} \in \{0,1\}$: Takes 1 if tugboat k starts outbound service for ship i in time period t
- $z_{it}^{in} \in \{0,1\}$: Takes 1 if ship i starts inbound tugboat service in time period t (auxiliary variable)
- $z_{it}^{out} \in \{0,1\}$: Takes 1 if ship i starts outbound tugboat service in time period t (auxiliary variable)
- $u_i^{early} \geq 0$: Early time of ship i relative to ETA
- $u_i^{late} \geq 0$: Late time of ship i relative to ETA

3. Mathematical Model

3.1 Objective Function

$$\min Z = \lambda_1 Z_1 + \lambda_2 Z_2 + \lambda_3 Z_3 + \lambda_4 Z_4$$

Unserved penalty:

$$Z_1 = \sum_{i \in \mathcal{I}} M \cdot \alpha_i \left(1 - \sum_{j \in \mathcal{J}} \sum_{t \in \mathcal{T}} x_{ijt}\right)$$

Total time in port cost:

$$Z_2 = \sum_{i \in \mathcal{I}} \alpha_i \beta_i \left[\sum_{t \in \mathcal{T}} (t + \tau_i^{out}) z_{it}^{out} - \sum_{t \in \mathcal{T}} t \cdot z_{it}^{in}\right]$$

ETA deviation cost:

$$Z_3 = \sum_{i \in \mathcal{I}} \alpha_i \gamma_i \left(u_i^{early} + u_i^{late}\right)$$

Where u_i^{early} and u_i^{late} are linearization auxiliary variables for ETA deviation.

Tugboat usage cost:

$$Z_4 = \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} \sum_{t \in \mathcal{T}} c_k (\tau_i^{in} y_{ikt}^{in} + \tau_i^{out} y_{ikt}^{out})$$

3.2 Constraints

Each ship assigned at most once:

$$\sum_{j \in \mathcal{J}} \sum_{t \in \mathcal{T}} x_{ijt} \leq 1, \quad \forall i \in \mathcal{I} \tag{1}$$

Ship-berth matching constraint:

$$x_{ijt} = 0, \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, t \in \mathcal{T} : C_j < S_i \tag{2}$$

Inbound tugboat and berth coupling:

$$\sum_{t \in \mathcal{T}} z_{it}^{in} = \sum_{j \in \mathcal{J}} \sum_{t \in \mathcal{T}} x_{ijt}, \quad \forall i \in \mathcal{I} \tag{3}$$

Outbound tugboat and berth coupling:

$$\sum_{t \in \mathcal{T}} z_{it}^{out} = \sum_{j \in \mathcal{J}} \sum_{t \in \mathcal{T}} x_{ijt}, \quad \forall i \in \mathcal{I} \tag{4}$$

Tugboat horsepower constraint:

$$\sum_{k \in \mathcal{K}} P_k y_{ikt}^{in} \geq P_i^{req} z_{it}^{in}, \quad \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{5}$$

$$\sum_{k \in \mathcal{K}} P_k y_{ikt}^{out} \geq P_i^{req} z_{it}^{out}, \quad \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{6}$$

Tugboat quantity limit:

$$\sum_{k\in\mathcal{K}} y_{ikt}^{in} \leq H_{max} \cdot z_{it}^{in}, \quad \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{7}$$

$$\sum_{k\in\mathcal{K}} y_{ikt}^{out} \leq H_{max} \cdot z_{it}^{out}, \quad \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{8}$$

Auxiliary variable definition:

$$z_{it}^{in} \leq \sum_{k\in\mathcal{K}} y_{ikt}^{in}, \quad \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{9}$$

$$z_{it}^{out} \leq \sum_{k\in\mathcal{K}} y_{ikt}^{out}, \quad \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{10}$$

Berth capacity constraint:

$$\sum_{j\in\mathcal{J}} \sum_{\tau=\max(1,t-D_i+1)}^{t} x_{ij\tau} \leq 1, \quad \forall j \in \mathcal{J}, t \in \mathcal{T} \tag{11}$$

Tugboat capacity constraint (considering preparation time):

$$\sum_{i\in\mathcal{I}} \left(\sum_{\tau=\max(1,t-\tau_i^{in}-\rho^{in}+1)}^{t} y_{ik\tau}^{in} + \sum_{\tau=\max(1,t-\tau_i^{out}-\rho^{out}+1)}^{t} y_{ik\tau}^{out} \right) \leq 1, \quad \forall k \in \mathcal{K}, t \in \mathcal{T} \tag{12}$$

Tugboat service boundary constraint:

$$y_{ikt}^{in} = 0, \quad \forall i \in \mathcal{I}, k \in \mathcal{K}, t < ETA_i - \Delta_i^{early} \text{ or } t > ETA_i + \Delta_i^{late} \tag{13}$$

Inbound timing constraint:

$$0 \leq \sum_{j\in\mathcal{J}} \sum_{t\in\mathcal{T}} t \cdot x_{ijt} - \sum_{t\in\mathcal{T}} (t + \tau_i^{in}) \cdot z_{it}^{in} \leq \epsilon_{time} \cdot \sum_{j\in\mathcal{J}} \sum_{t\in\mathcal{T}} x_{ijt}, \quad \forall i \in \mathcal{I} \tag{14}$$

Outbound timing constraint:

$$0 \leq \sum_{t\in\mathcal{T}} t \cdot z_{it}^{out} - \sum_{j\in\mathcal{J}} \sum_{t\in\mathcal{T}} (t + D_i) \cdot x_{ijt} \leq \epsilon_{time} \cdot \sum_{j\in\mathcal{J}} \sum_{t\in\mathcal{T}} x_{ijt}, \quad \forall i \in \mathcal{I} \tag{15}$$

ETA deviation linearization constraint:

$$\sum_{t\in\mathcal{T}} t \cdot z_{it}^{in} = ETA_i + u_i^{late} - u_i^{early}, \quad \forall i \in \mathcal{I} \tag{16}$$

Variable domain constraints:

$$\begin{gathered} x_{ijt}, y_{ikt}^{in}, y_{ikt}^{out}, z_{it}^{in}, z_{it}^{out} \in 0, 1, \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathcal{K}, t \in \mathcal{T} \\ u_i^{early}, u_i^{late} \geq 0, \quad \forall i \in \mathcal{I} \end{gathered} \tag{17}$$