Quantitative Macro
Problem Set 4

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1. Value function iteration

The bellman equation for the given optimization problem is following:

$$V_t(a, y) = \max_{c,a'} u(c) + \beta E_t V_{t+1}(a', y')$$

Substituting for consumption and expectations we obtain:

$$V_t(a, y) = \max_{wy+(1+r)a \geq a' \geq -A} u(wy + (1 + r)a - a') + \beta \sum_{y'} \pi(y'|y) V_{t+1}(a', y')$$

Taking the FOC with respect to a'.

$$-u_c(wy + (1 + r)a - a') + \beta \sum_{y'} \pi(y'|y) V'_{t+1}(a', y') = 0$$

Now, let us take the derivative with respect to a using the envelope condition.

$$V'_t(a, y) = u_c(wy + (1 + r)a - a')(1 + r)$$

Combining both equations:

$$u_c(c) = \beta \sum_{y'} \pi(y'|y) u_c(c')(1 + r')$$

Finally, the following stochastic Euler equation for consumption is obtained:.

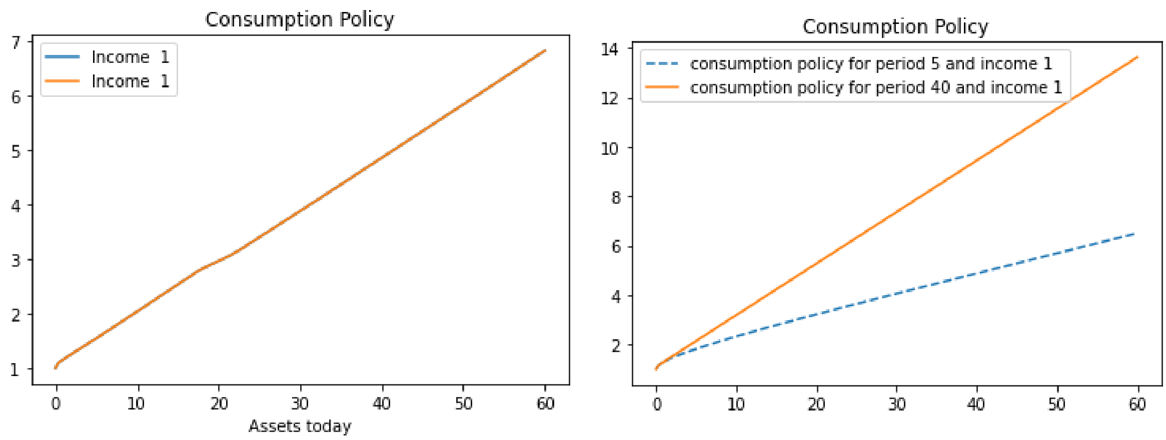
$$u_c(c) = \beta(1 + r) \sum_{y'} \pi(y'|y) u_c(c')$$

2. Partial Equilibrium

In the codes I provide solutions for infinite time horizon with continuous and discrete methods and for finite (45 periods) time horizon. In addition, the subroutine for simulating paths is added. The results below base on the continuous Value Function Iteration for the infinite time horizon. The simulation of time paths in the following results has the size of 2000 households.

2.1 With certainty

Let us set $\gamma = 0$, $\sigma_y = 0$, $\sigma = 2$ and $\bar{c} = 100$. Hence, there is no uncertainty of income here (both income levels are the same and equal to 1). Here, the utility function is CRRA. Using the code the following results are obtained:



(a) Infinite time horizon

(b) Finite time horizon

Figure 1: Consumption policies

The consumption policies for different levels of income overlap, it is a good sign, because there are two indistinguishable levels. In this case one policy should be obtained. For the infinite time horizon the policy is an almost linear function of assets. For finite horizon model it is similar and differs by time index. In period 5 the optimal policy is to consume less, compared to period 40.

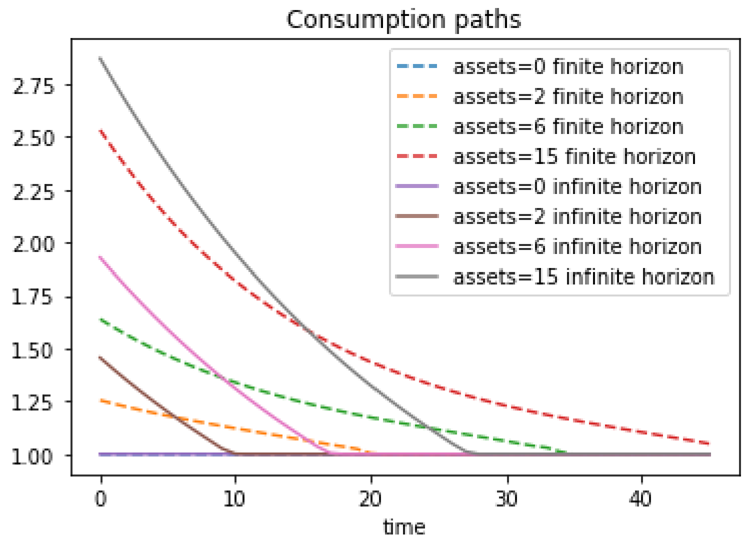


Figure 2: Consumption paths

Let us look at the consumption paths for 45 periods. One may notice that for the finite time horizon consumption is smoothed. However, for a given parametrization ($\rho > r$) the agent prefers to consume much more today (discounts the future heavily), than to save and consume in the future for the infinite time horizon model. Generally, the differences between the consumption path starting in different states are really small in the next parts of my homework, so I use there different income levels as substitutes.

2.2 With Uncertainty

Let us set $\gamma = 0$, $\sigma_y = 0.1$. Now let us consider the case with the uncertainty of labour income. We have two levels: 0.9 and 1.1. Let us start with comparing policy functions between two different utility functions.

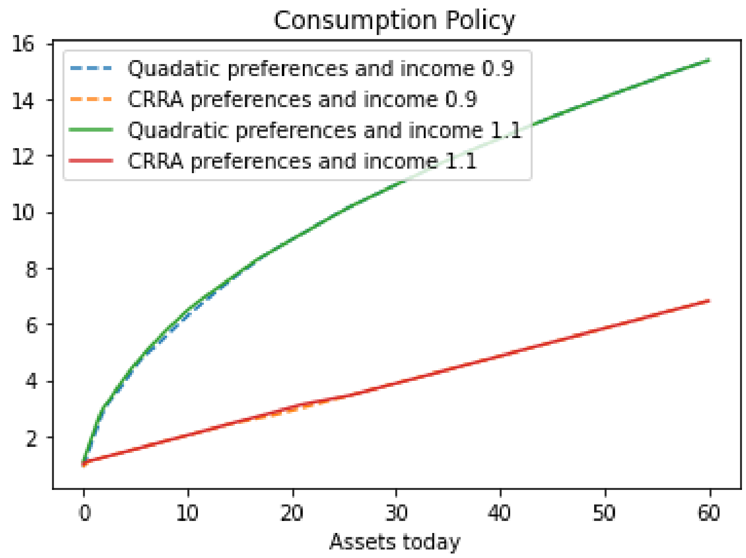


Figure 3: Consumption policies comparison between utilities

Consumption function for the CRRA utility is similar to those previously obtained, but takes slightly smaller values. The presence of uncertainty reduces the optimal consumption. Speaking of quadratic utility function, it has a concave shape and attains much higher values than CRRA one. The differences for policies between two states of income is negligible, however one may notice that the functions for lower income state are slightly below.

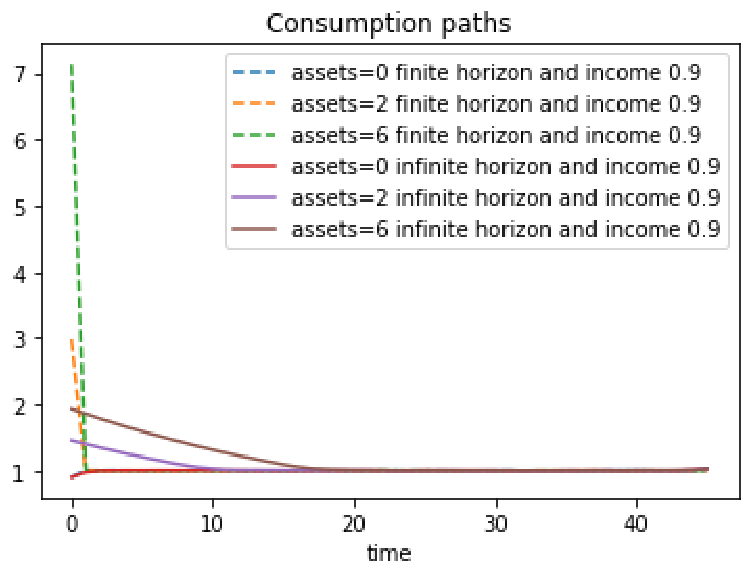


Figure 4: Consumption paths for quadratic utility

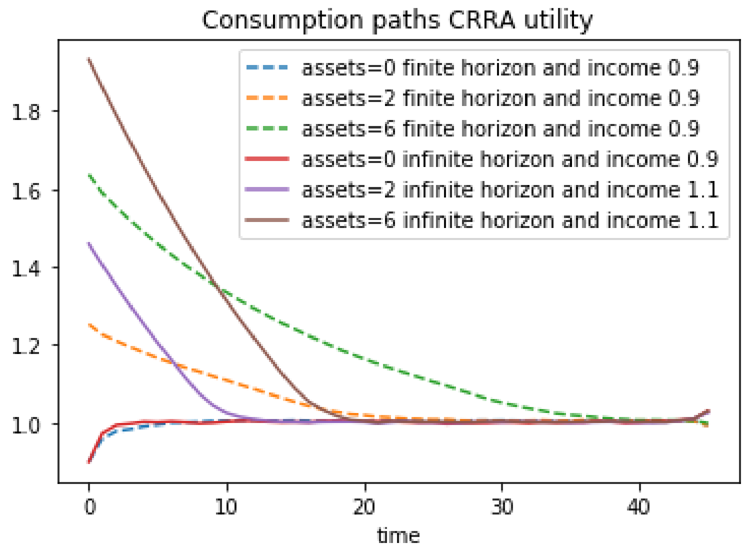


Figure 5: Consumption paths for CRRA utility

For the quadratic utilities, the smoothing is much shorter for the finite time horizon, whereas for the CRRA utility the consumption is not so smooth and attains lower levels.

2.3 Changing risk aversion, $\sigma = 5$

Here let us play with the risk aversion and see what are the implications of a higher risk perception.

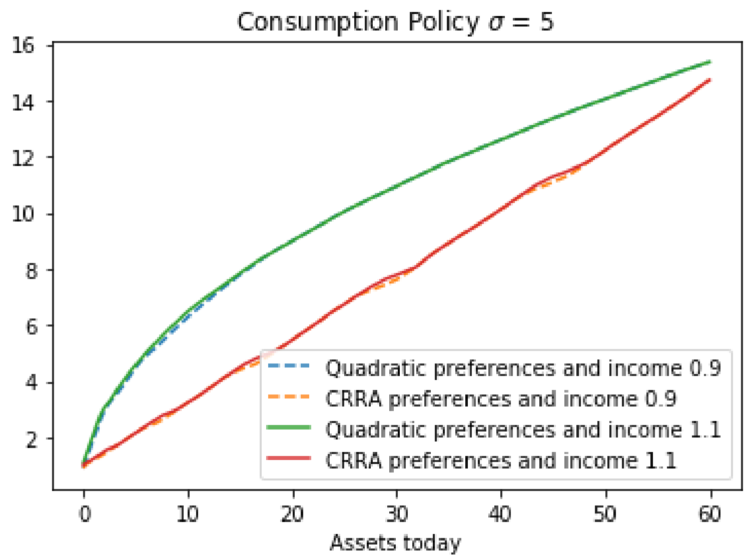


Figure 6: Consumption policies comparison between utilities

For the quadratic utilities there are no changes in the consumption policy, however the CRRA utility produces slightly different result. Namely the optimal level of consumption should be higher than previously and similar to the quadratic utility.

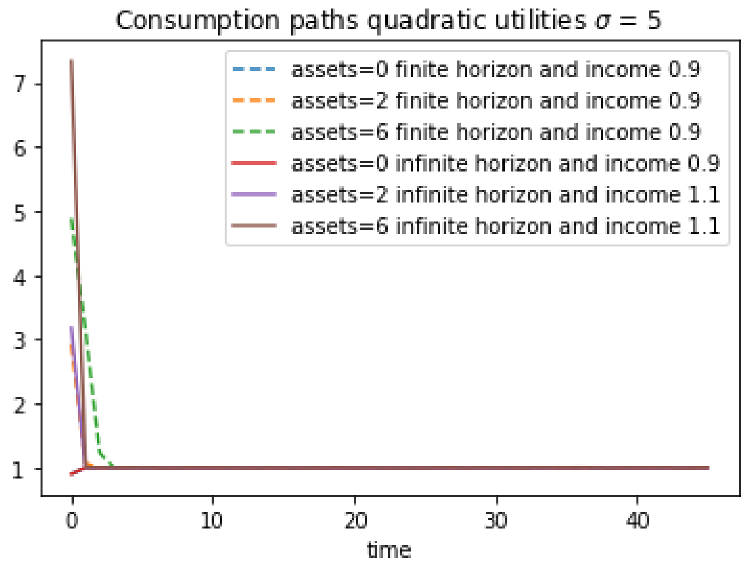


Figure 7: Consumption paths for quadratic utility

In the case of quadratic utility nothing has changed.

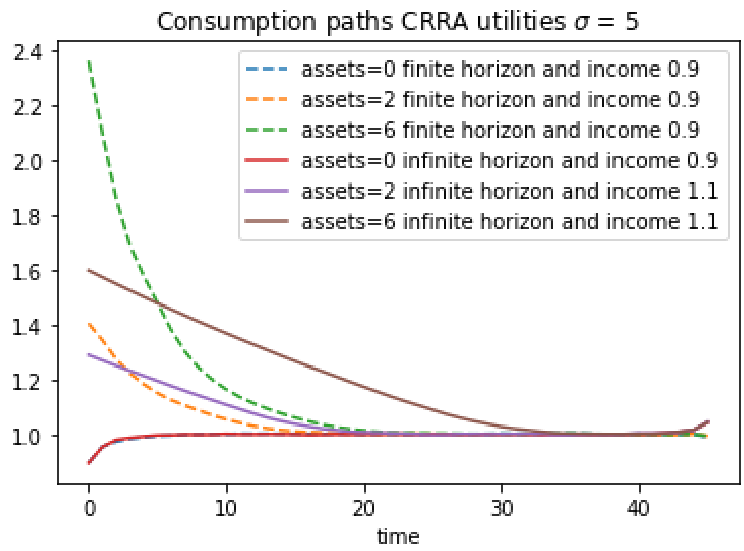


Figure 8: Consumption paths for CRRA utility

For the CRRA utility we see more of a smoothing motive, but surprisingly it is stronger for the infinite time horizon. The results here may be bit counterintuitive, higher consumption level in the consumption function more smoothing in the infinite time horizon. It could be that there is an issue in the code, or for a given parametrization with a relatively strong discounting those results are obtained.

2.4 Changing risk aversion, $\sigma = 20$

Now let us increase the risk aversion parameter even higher.

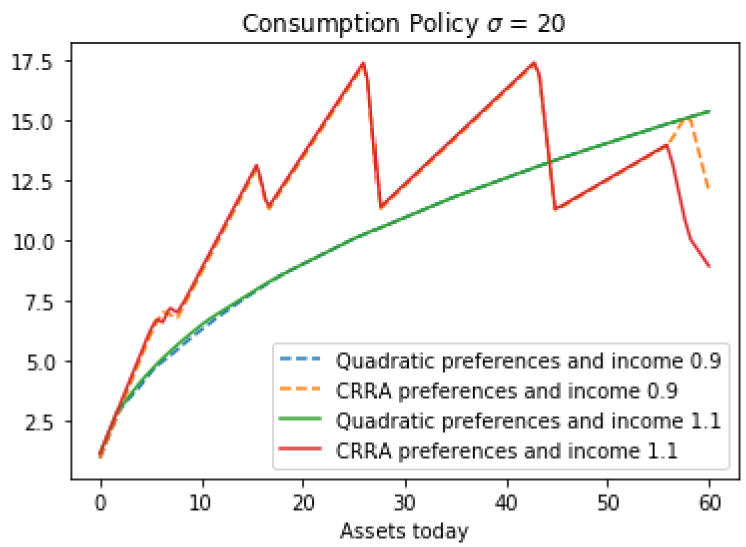


Figure 9: Consumption policies comparison between utilities

One may notice, that may code fails here slightly, and provides an odd policy for CRRA utility. But, after a closer look one may say that it resembles moderately the quadratic preferences.

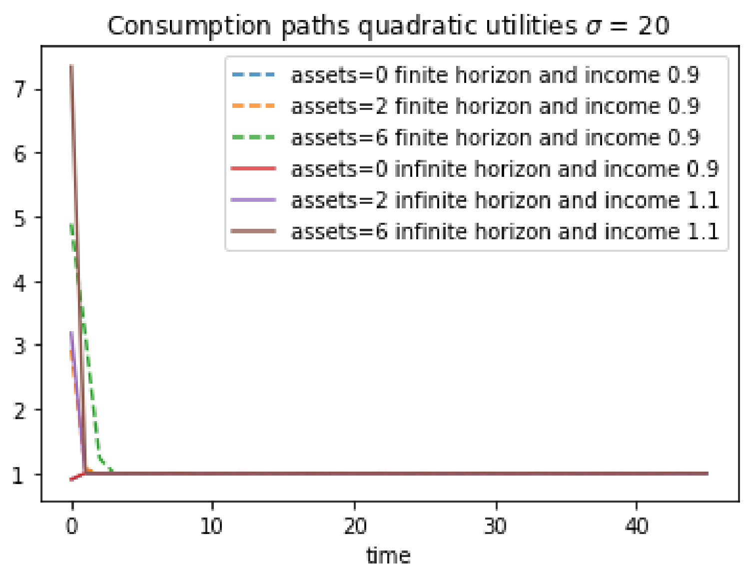


Figure 10: Consumption paths

As previously, nothing has changed here, because nothing had been changed.

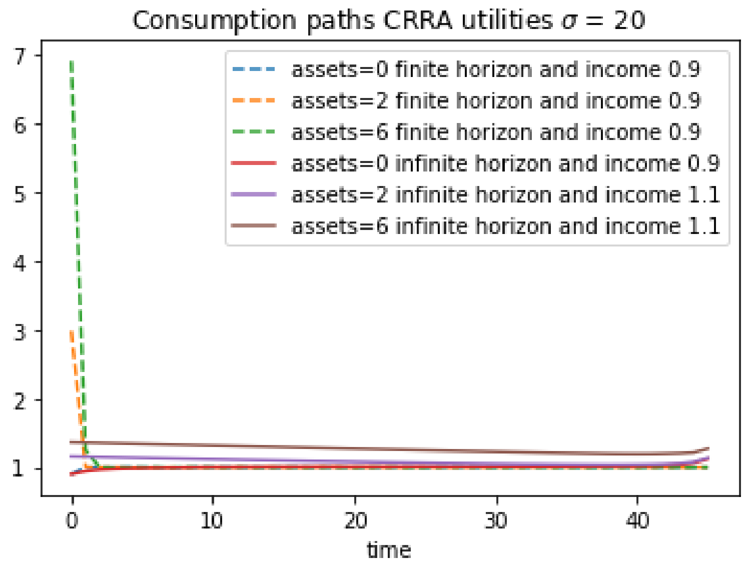


Figure 11: Consumption paths

Now, slightly different results are obtained. One may notice that for a finite horizon, almost no smoothing is observed. However, for the infinite horizon, the paths are parallel, hence there is kind of smoothing consumption enforced by the high coefficent of risk aversion.

2.5 More severe income shock

Now let us increse the severity of income shocks, by increasing σ_y to 0.5, there are two levels of labour income: 0.5 and 1.5. I keep all parameters as in subsection 2.2. In this part I obtained slightly unusual results, the policies are the same, probably there were some minor problems in the code.

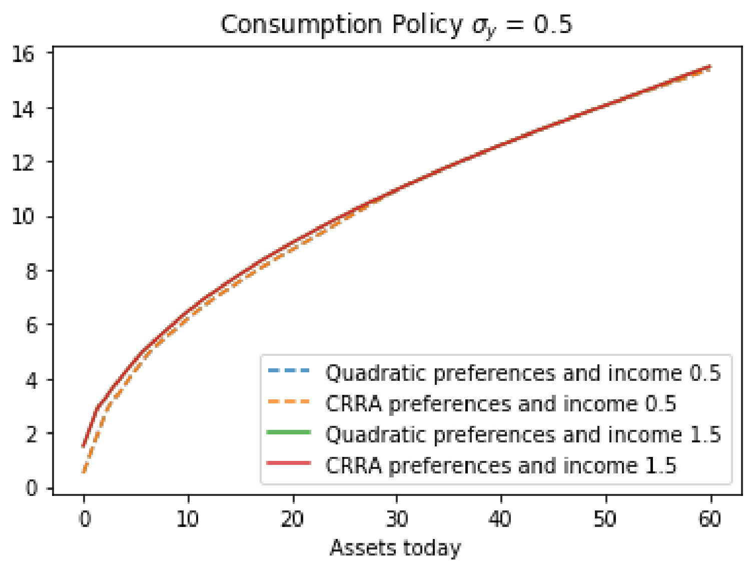


Figure 12: Consumption policies comparison between utilities

Here emerges one noticeable issue, the difference between policy function is bigger between the states, but still this difference not relatively small.

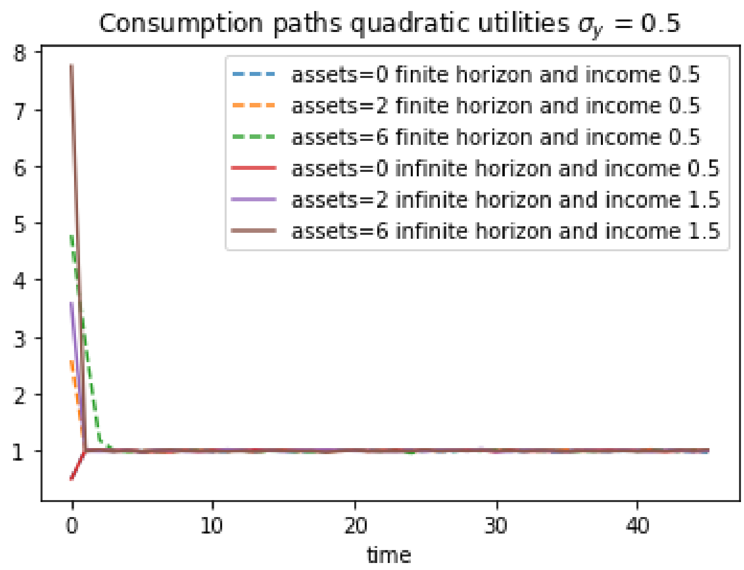


Figure 13: Consumption paths for quadratic utility

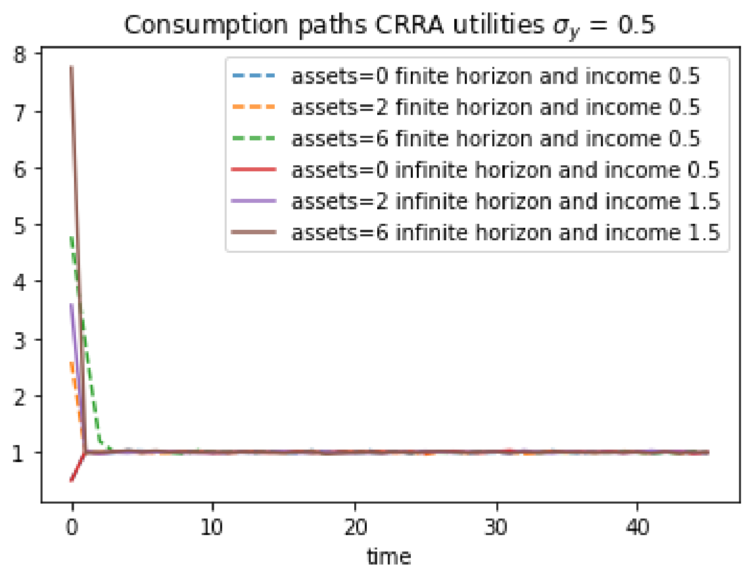


Figure 14: Consumption paths for CRRA utility

As mentioned before, the paths obtained are the same. Comparing to the previous results we may notice lack of smoothing and very fast drop to the long-run consumption value.

2.6 More persistent income shocks

Keeping the high severity of the income shocks, the persistence is increased here.

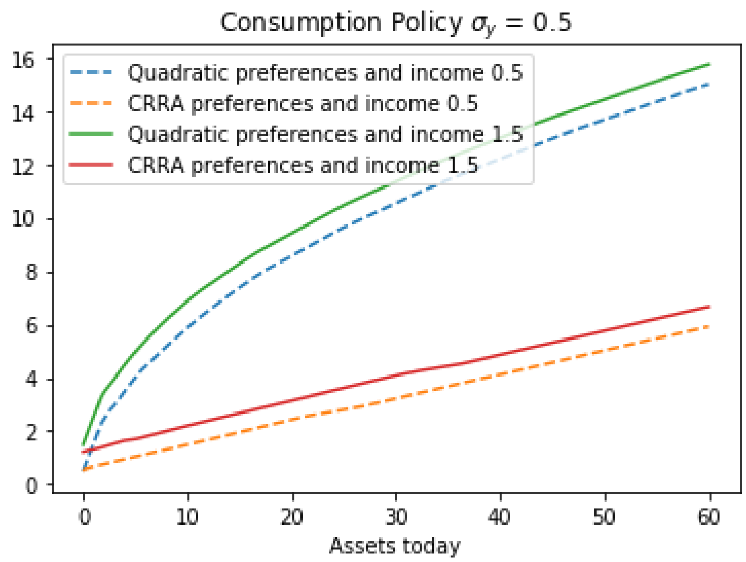


Figure 15: Consumption policies comparison between utilities

The result is similar to the 2, 2 subsection. However, in this case the differences between the decision rules for different states are the most visible (it is the result of increasing both severity and persistence of shock) .

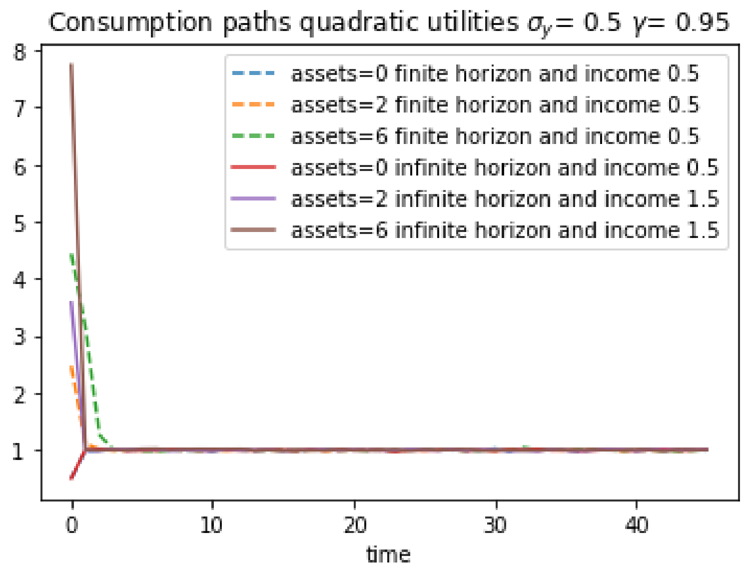


Figure 16: Consumption paths for quadratic utility

The result is very similar to the previous sub-points, consumption path is flat after a few periods.

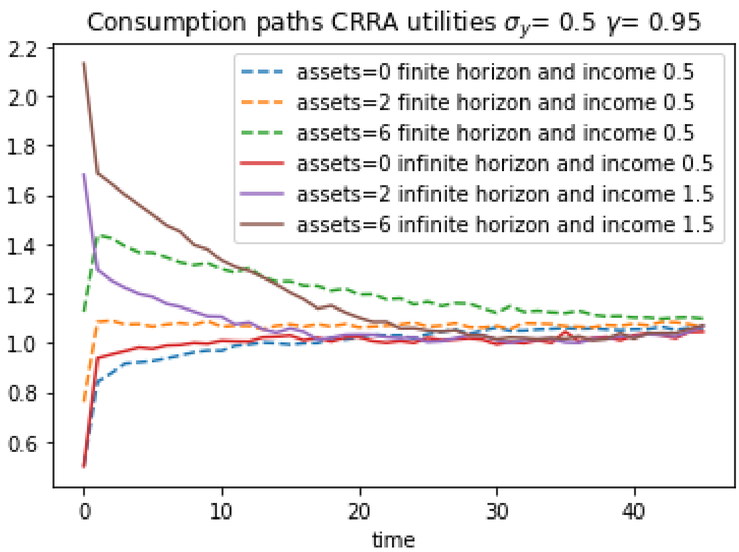


Figure 17: Consumption paths for CRRA utility

However, here we may see slightly different result. In the presence of the severe risk, agents seem to smooth consumption. In addition one may notice a plethora of small perturbations in the paths.

3. General Equilibrium

3.1 The simple ABHI model

For this exercise let us set the following parameters: $\beta = \frac{1}{\rho}$ where $\rho = 0.06$, utility function is CRRA with risk aversion parameter equal to 2, $\sigma_y = 0.3$ and $\gamma = 0.5$ and capital share, $\alpha = 0.33$. In addition the households cannot borrow.

Following the steps from the algorithm, I start with an initial guess of interest rate of 0.04. Then, I modify my previous codes such that I get a function which solves households problem given the interest rate. I solve the problem with a continuous function for the infinite time horizon (same as in the Kruger, Mitman, Perri handbook), linear splines are used here with 30 knots for asset level. Next step is to compute invariant distribution. To do this, Monte Carlo simulation for 250 periods for 8000 households is performed, each household starts with zero assets. Then, I compute the expected value of assets and calculate the distance function : $d(r) = K(r) - Ea(r)$. When the distance is above the tolerance level (epsilon) then the interest rate is updated using following equation.

$$r_{s+1} = r_s + \omega d(r)$$

Where $\omega = 0.0004$, when distance is positive for r we should increase it in the next step to decrease $K(r)$ and increase $Ea(r)$.

Finally, the interest rate of 0.05968 was obtained. The longest component of iterations were the Monte Carlo. For the model, the Aiyagari's graph was created.

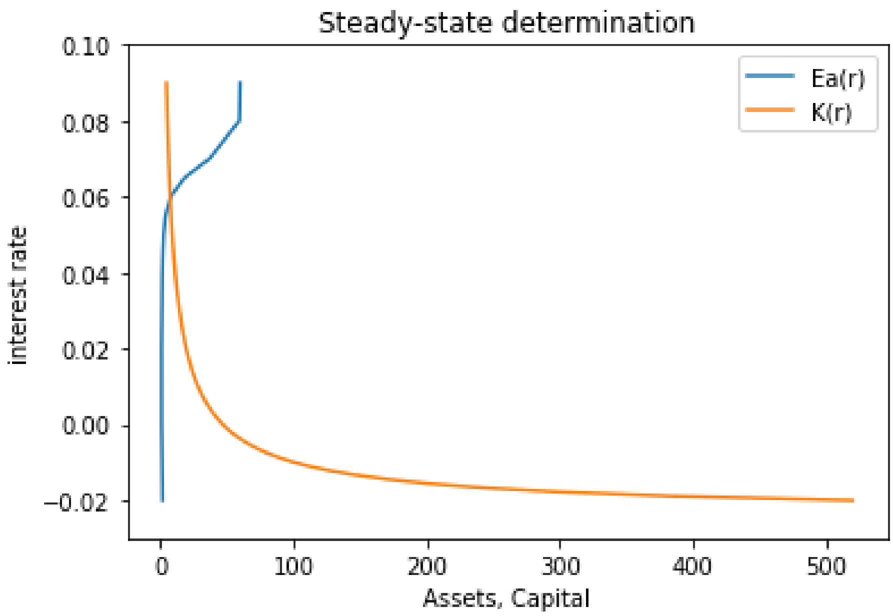


Figure 18: Aiyagari's graph

The variables distribution looks at follows at the graphs. Here, we consider an income process with only two states, hence the comparison of them from the obtained data is almost meaningless.

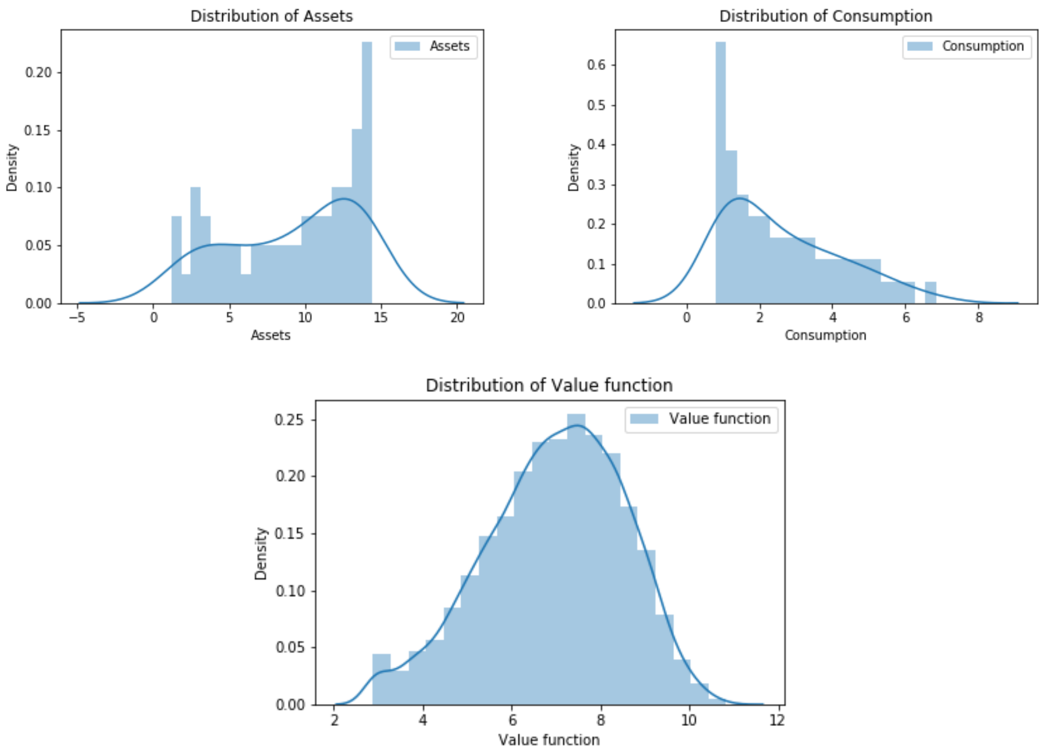


Figure 19: Distributions for the simple model

In the simple model the inequalities are much lower than in the data. The biggest discrepancy is in the assets, with gini coefficient around 0.31 and top one-percent having the 2.69% of it. One may notice of the graph that there are many agents with relatively high level of assets $(13 - 14)$ and the second group is for agents with low asset level $(3 - 4)$. The consumption is relatively equally distributed, similar to value function. If we take the assets as a proxy of Net Worth, then it might be said that the results are qualitatively the same, but the magnitude is much lower.

	Assets	Consumption	Value function
Mean	7.611672	1.754502	6.957550
Q1	5.516863	15.258637	13.300290
Q2	12.866568	18.065894	17.748076
Q3	18.981030	19.828693	20.346474
Q4	25.640621	21.755698	22.723996
Q5	36.994918	25.091077	25.881164
90-95	9.418555	6.310966	6.531730
95-99	8.741149	5.417833	5.502785
Top 1%	2.695594	1.505842	1.475475
Gini	0.319410	0.099026	0.127230

Table 1: Distributions for the simple ABHI model

Table 1: Means and Marginal Distributions in 2006

	Variable						
	Earn.	Disp Y		Cons. Exp		Net Worth	
Source	PSID	PSID	CPS	PSID	CE	PSID	SCF(2007)
Mean (2006$)	54,349	64,834	60,032	42,787	47,563	324,951	538,265
% Share by:							
Q1	3.6	4.5	4.4	5.6	6.5	-0.9	-0.2
Q2	9.9	9.9	10.5	10.7	11.4	0.8	1.2
Q3	15.3	15.3	15.9	15.6	16.4	4.4	4.6
Q4	22.7	22.8	23.1	22.4	23.3	13.0	11.9
Q5	48.5	47.5	46.0	45.6	42.4	82.7	82.5
90 – 95	10.9	10.8	10.1	10.3	10.2	13.7	11.1
95 – 99	13.1	12.8	12.8	11.3	11.1	22.8	25.3
Top 1%	8.0	8.0	7.2	8.2	5.1	30.9	33.5
Gini	0.43	0.42	0.40	0.40	0.36	0.77	0.78
Sample Size	6,232	6,232	54,518	6,232	4,908	6,232	2,910

Figure 20: Table from Krueger, Mitman and Perri.

3.2 Aiyagari model

Now let us consider Aiyagari's parametrization, $\alpha = 0.36$, $\beta = 0.96$, depreciation rate should be changed to 0.1, because he considered one period as a year, the risk aversion parameter could take 3 values $(1, 3, 5)$. Finally, he takes a seven-state Markov process as an approximation for the labour income process. Following the same steps as previously, the results below are obtained. I use the following combination of parameters to simulate the model and compare the answers with Aiyagari's, $\sigma_y = 0.4$, $\rho = 0.9$, $\sigma/\mu = 3$.

TABLE II

A. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.2$)

$\rho \backslash \mu$	1	3	5
0	4.1666/23.67	4.1456/23.71	4.0858/23.83
0.3	4.1365/23.73	4.0432/23.91	3.9054/24.19
0.6	4.0912/23.82	3.8767/24.25	3.5857/24.86
0.9	3.9305/24.14	3.2903/25.51	2.5260/27.36

B. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.4$)

$\rho \backslash \mu$	1	3	5
0	4.0649/23.87	3.7816/24.44	3.4177/25.22
0.3	3.9554/24.09	3.4188/25.22	2.8032/26.66
0.6	3.7567/24.50	2.7835/26.71	1.8070/29.37
0.9	3.3054/25.47	1.2894/31.00	−0.3456/37.63

Figure 21: Aiyagari's results

The interest rate of -0.0353 is obtained. The result is different, but Aiyagari for similar parametrization got also one negative result. Plausibly, there is some issue in the code.

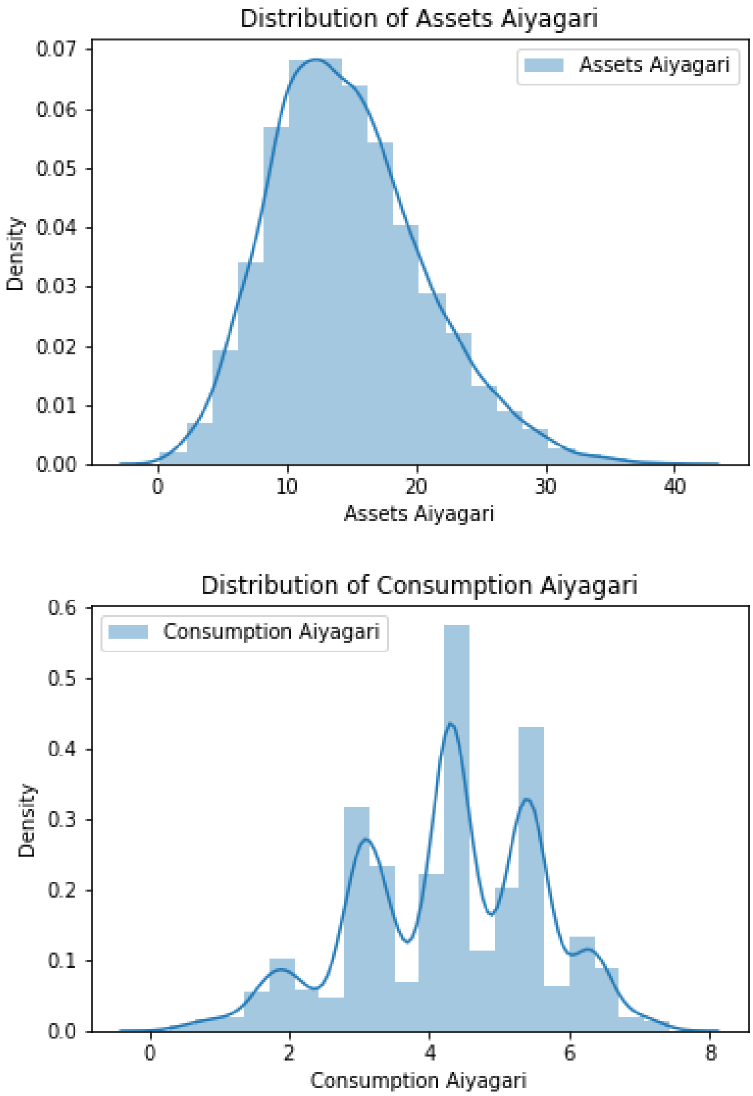


Figure 22: Distributions for the Aiyagari's model

Now, let us look at the distributions. Speaking of assets, there is less inequality than in the simple model and data. Distribution of consumption looks unusual, with some mass cumulated in few points. The gini coefficient is higher than previously, but still much below the real data values. The value function behaves really strange here.

Probably there are some minor issues in the code, that became visible when solving the Aiyagari model, that provide wrong answers.

	Assets	Consumption	Value function
Mean	14.684373	4.270337	10.860492
Q1	9.777909	11.094049	17.738117
Q2	15.172707	16.823416	20.048681
Q3	19.190864	20.293054	20.476904
Q4	23.667832	23.833905	20.748395
Q5	32.190689	27.955575	20.987903
90-95	8.168062	7.193838	5.260335
95-99	7.506836	6.079094	4.221428
Top 1%	2.254346	1.662461	1.061374
Gini	0.225834	0.172799	0.032231

Table 2: Distributions for the the Aiyagari model