Quantitative Macro
Problem Set 2

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Question 1. Computing Transitions in a Representative Agent Economy

a), b)

Let us start with computing as much analytically as it is possible. We may notice that the labour choice is constant over time and there is no stochasticity in the problem, hence we may omit expectation operator and subscript of variable h. Rewriting the program let us combine three constraints into one:

$$\max_{c_t, i_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

s.t

$$k_t^{1-\theta}(zh)^\theta - c_t + k_t(1-\delta) - k_{t+1} = 0 \tag{2}$$

Then we set up a Lagrangian :

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t [k_t^{1-\theta}(zh)^\theta - c_t + k_t(1-\delta) - k_{t+1}]$$

Now, let us calculate the FOCs, given that the

$$u(c_t) = ln(c_t) :$$

$$\frac{\partial \mathcal{L}}{\partial c_t} : \frac{\beta^t}{c_t} = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} : \lambda_{t+1} = \lambda_t [(zh)^\theta (1-\theta) k_{t+1}^{-\theta} + 1 - \delta]$$

Substituting for λ_t, we obtain an Euler equation:

$$\frac{c_{t+1}}{c_t} = \beta[(zh)^\theta(1-\theta)k_{t+1}^{-\theta} + 1 - \delta] \tag{3}$$

In the steady state the capital low of motion is: $i = k - (1-\delta)k$ also $\frac{k}{y} = 4$ $\frac{i}{y} = 4$, thus

$$i = \delta k, \frac{k}{i} = 16, => \delta = \frac{1}{16}$$

Rewriting the production function:

$$\frac{k}{4} = k^{1-\theta}(zh)^\theta, => k = 4^{\frac{1}{\theta}}(zh) \tag{4}$$

One can notice that the k (and also y, i, c) in the steady state is linear in z. Hence, let us choose z such that the production function is equal to one in the first steady state, this way it is easier and more visible to observe the relative changes in output. Without fixing y (or k, c, i), we would not have a fully determined unique steady state, so let us keep up the assumption of $y = 1$ and consider this specific steady state through the whole exercise. Then we get k, i and y from the given conditions and to obtain z we use equation 3 or 4. To get all the parameters of the model we have to obtain β substituting k from 4 equation to equation 3.

$$\frac{1}{\beta} = [(1-\theta)(zh)^\theta(zh)^{-\theta}4^{\frac{-\theta}{\theta}} + 1 - \delta]$$

$$\frac{1}{\beta} = \frac{0.33}{4} + 1 - \frac{1}{16}, => \frac{1}{\beta} = 1.02$$

When we have all parameters let us calculate steady steady for case a) and b). The results are in the table 1.

Variable	Steady State 1	Steady State 2
z	1.629676	3.259352
y	1.000000	2.000000
k	4.000000	8.000000
i	0.250000	0.500000
c	0.750000	1.500000
β	0.980392	0.980392
θ	0.670000	0.670000
h	0.310000	0.310000
δ	0.062500	0.062500

Table 1: Steady state variables

c) Transition path

Here let us bother with the transition path after a productivity shock. Let us assume that in the $t = 0$ the economy is in the first steady state. And then in period $t = 1$ the shock arrives

We have 4 variables which values are unknown: y, k, i and c. When we know them in period t we may use the four following equations to determine them in period $t + 1$.

$$k_t + 1 = i_t + (1 - \delta)k_t \tag{5}$$

$$c_{t+1} = c_t\beta[(zh)^\theta(1 - \theta)k_{t+1}^{-\theta} + 1 - \delta] \tag{6}$$

$$y_{t+1} = k_{t+1}^{1-\theta}(zh)^\theta \tag{7}$$

$$i_{t+1} = y_{t+1} - c_{t+1} \tag{8}$$

At the beginning we obtain capital by standard capital law of motion (equation 5), then we use it to compute consumption and output (equations 6 and 7). And from them we get investment.

Let us have a brief look at the graphs.

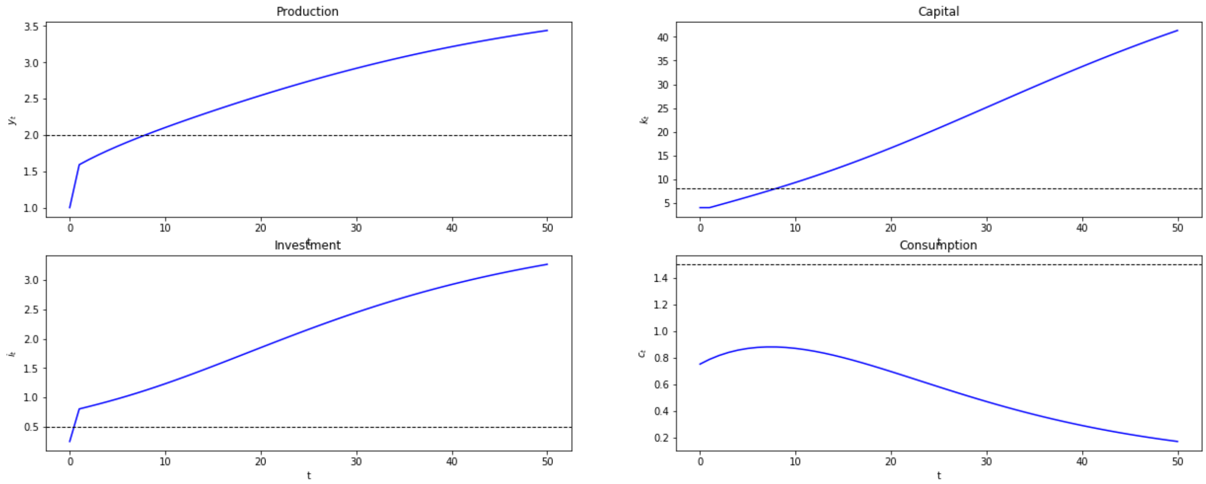

Figure 1: Transition path to the new steady state - wrong one

The horizontal dashed line represent the new steady state values of variables. One can notice that definitely something goes wrong, the system does not converge to the steady state. From the Macroeconomics course we should know the reason. After economy is hit by the shock we must adjust consumption (or investments complementary) to remain on the transition path to the new steady state. Using fsolve function from Python's scipy library I am going to find c such that the variables converge to the second steady state. To do it I define an objective function that is equal to the sum squared difference between value of the variable in the $t = 50$ and steady state value divided by the steady state value (the division is like assigning quasi-weights, in the steady state output is 16 times bigger than investment, so I want to consider it).

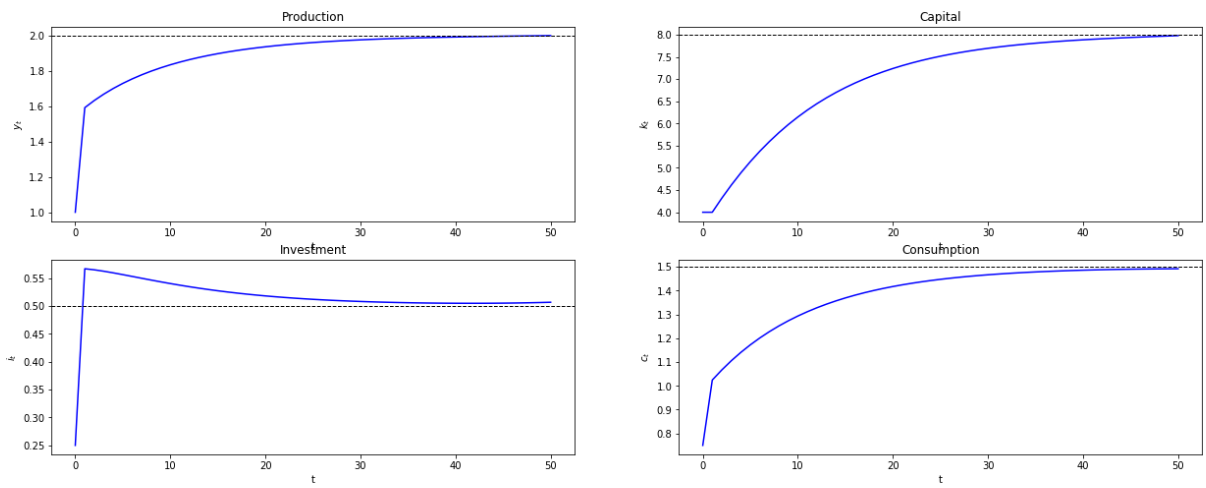

Figure 2: Transition path to the new steady state - good one

The value of 'adjusted' consumption in the first period was calculated to be equal to 1.20439. Here we can notice convergence in all variables. The production follows a sharp rise in the first period and then converges steadily. The growth of capital is more stable over the time. Consumption follows a similar pattern as output by with a lower increase in the first period. Investments are a but different, in the first period it rise above steady state value, and then slowly declines.

d) Unexpected shocks

Now we have a two shocks, first at t=1 and than at t=10, when value of z decreases to the primary value. Following a similar steps as above we need to adjust consumption changes in both shocks such that we remain on the transition path. Using a similar method we obtain the $c_{10} = 0.977578$. Let us look at the graph.

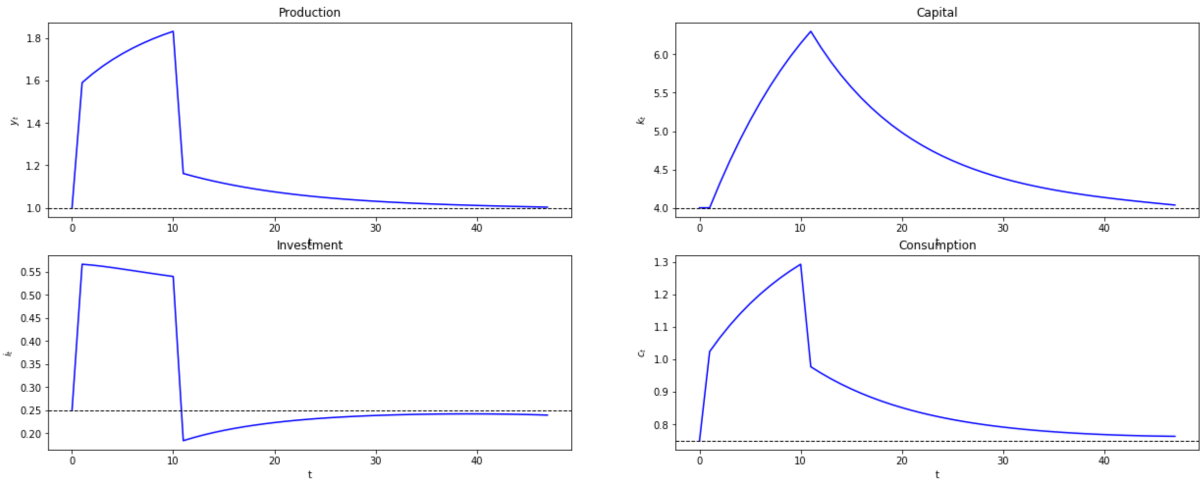

Figure 3: Transition path to the new steady state - two shocks

Until period $t = 10$ everything follows the same pattern as before, then the shock arrives. Consumption and output fall sharply to the value above steady state and then slightly decline. Investment fall even below the steady state value and recovers over the time. Whereas, the capital declines steadily.

Question 2: COVID-19 lockdown model

Let us start by rewriting the maximization problem stated during the lecture.

$$\max_{H_f, H_{nf}} Y(H_f, H_{nf}) - \kappa_f H_f - \kappa_{nf} H_{nf} - \omega D \tag{9}$$

s.t.

$$H_f + H_{nf} \leq N \tag{10}$$

To solve it I am going to use scipy.optimize function with Trust-Region Constrained Algorithm. Because it looks for a minimum of a function I will multiply obcective function by minus one. Using equations 1, 2, 3, 4 from lecture notes to plug in variables we obtain:

$$\min_{H_f, H_{nf}} -[[A_f H_f^{\frac{\rho-1}{\rho}} + A_{nf} c(TW) H_{nf}^{\frac{\rho-1}{\rho}}]^{\frac{\rho}{\rho-1}} - \kappa_f H_f - \kappa_{nf} H_{nf} - \omega(1-\gamma)\beta(HC) H_f^2 \frac{i_0}{N}] \tag{11}$$

Let us see the results.

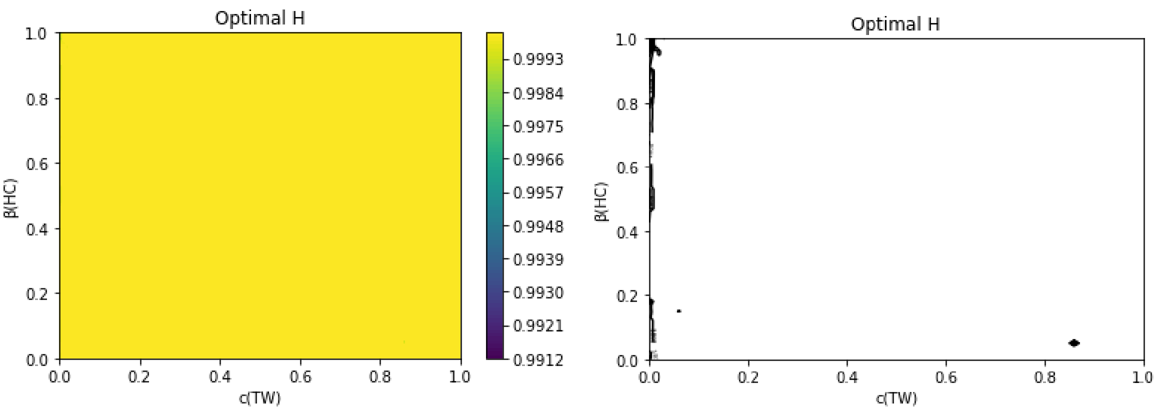

Figure 4: Contour plot of optimal H

Speaking of H (number of aggregated hours) contour plots provide information that for any combinations of β and c it is very high and close to 1.

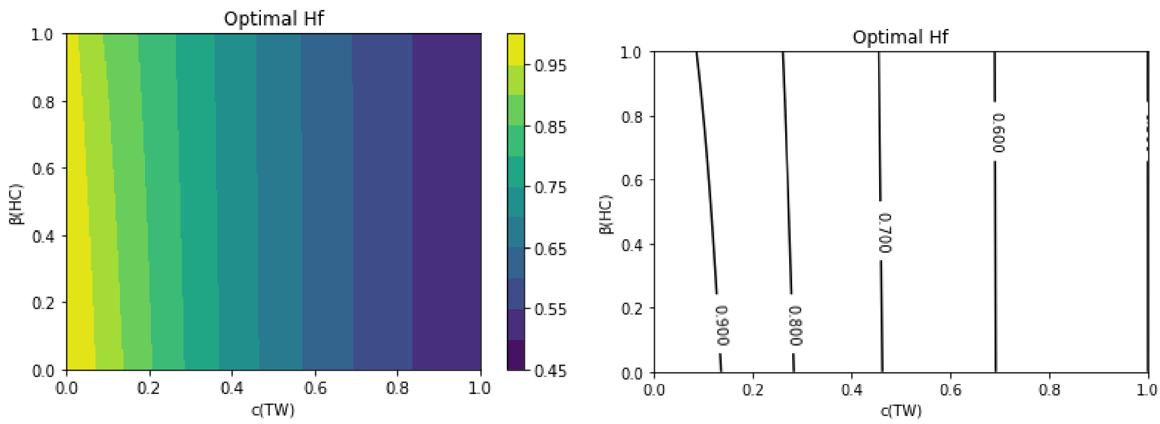

Figure 5: Contour plot of optimal H_f

For an optimal level of H_f it is the productivity of teleworking that has the biggest impact on it. In addition, it is noticeable that the contour lines are not completely straight, hence the influence of the conditional on human contact infection's rate is present in the results, especially for low teleworking abilities. The optimal level of people working at the workplace varies from 0.45 to almost 1.

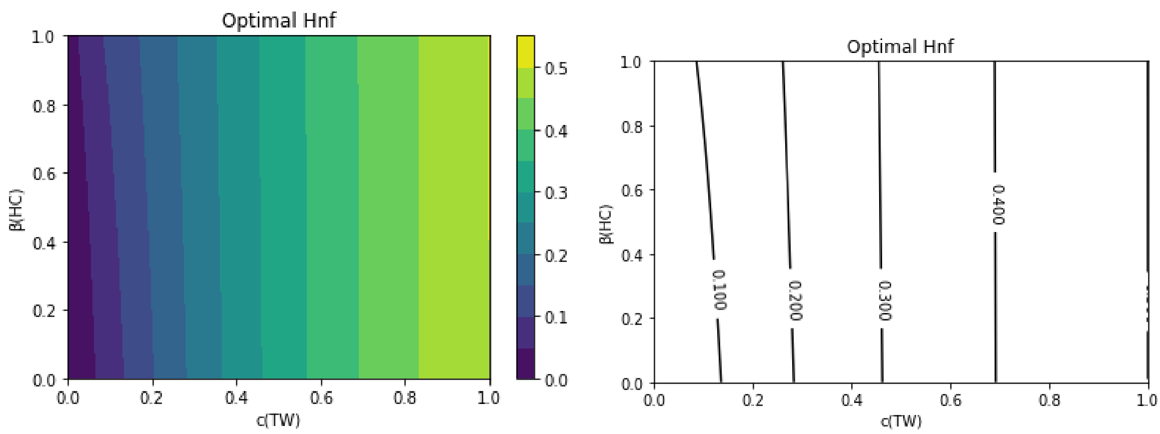

Figure 6: Contour plot of optimal H_{nf}

Results for H_{nf} looks similar to those previously stated but are inversed. For low productivity of teleworking the optimal level of people who do telework tends to zero, and for very high productivity it is almost reaching one-half.

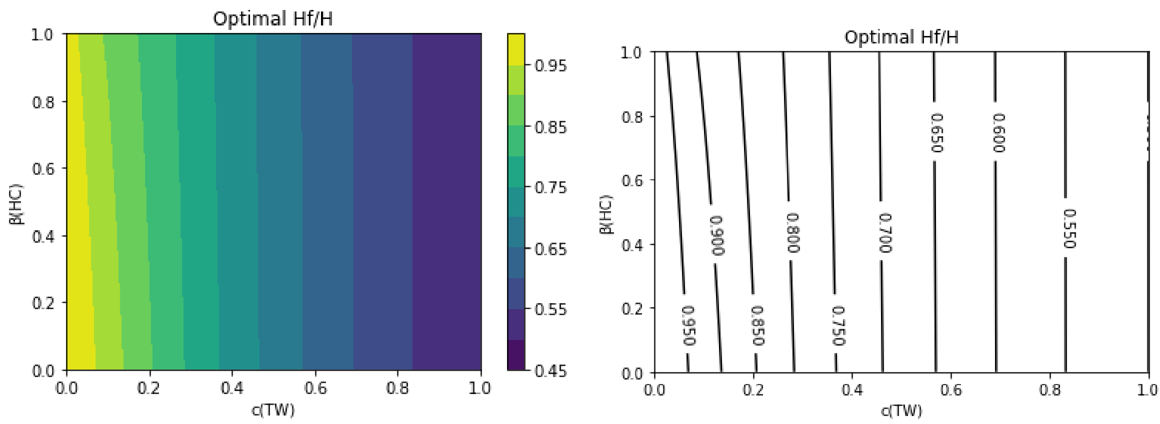

Figure 7: Contour plot of optimal $\frac{H_f}{H}$

Remembering that N was normalized to one and that for any combination of parameters H was almost equal to one, the results of the optimal share of people who work at the workplace are not surprising, because it resembles the results of the optimal value of people working at the workplace.

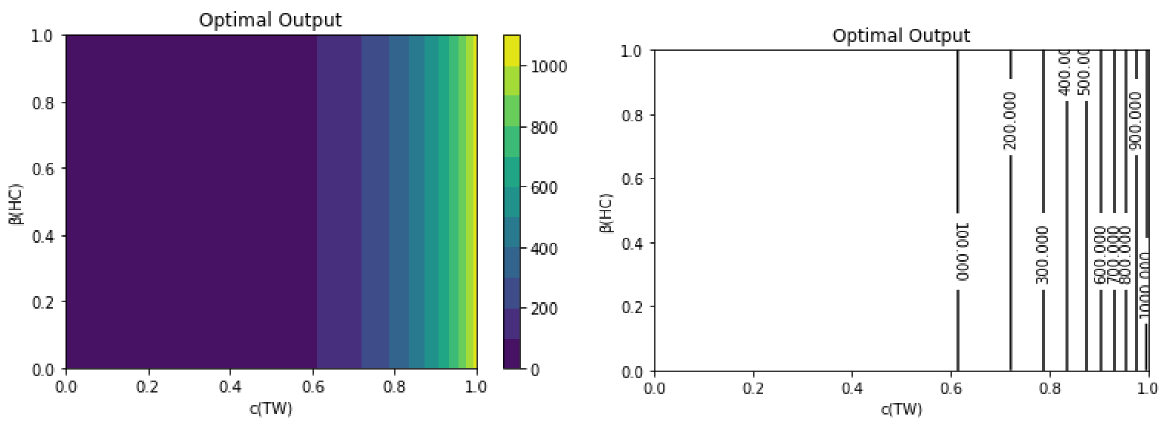

Figure 8: Contour plot of output

It is the value of productivity that for a given parameter sets determines the level of output for optimal allocations. The closer is the c(TW) to 1 the higher output is produced.

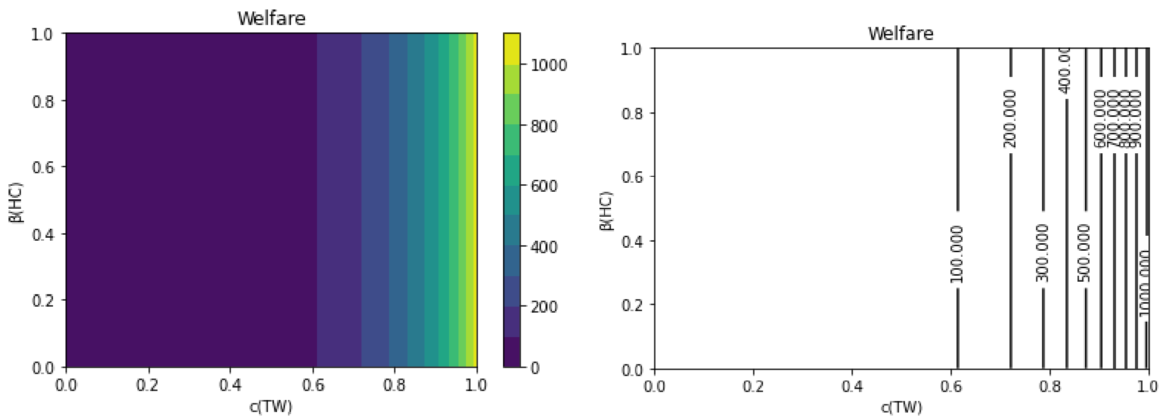

Figure 9: Contour plot of welfare

Looking at equation 11, we may notice that welfare is a production function minus some penalties that for given parameter sets and high output are not relatively big. Hence the pattern represented on the graph is identical as for the output. However, the values are a bit lower than for output.

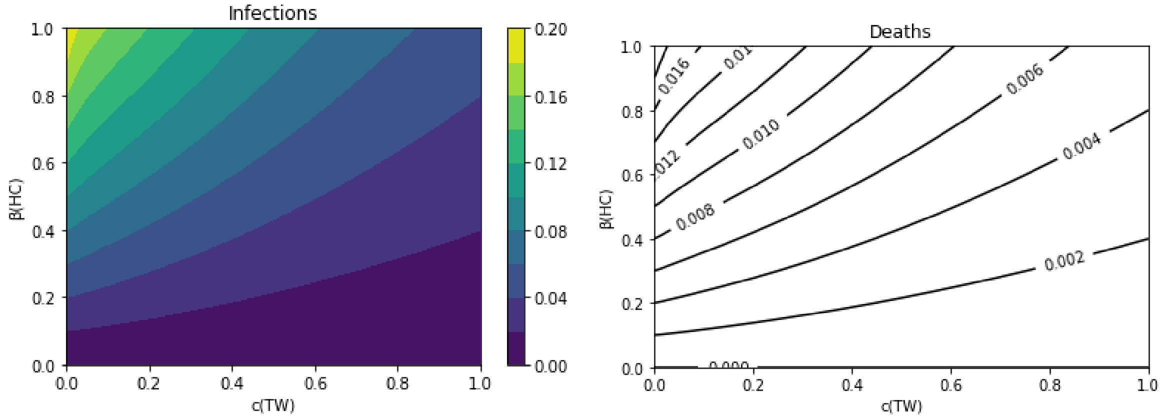

Figure 10: Contour plot of indecftions and deaths

Plots for infections and deaths are almost identical (deaths are multiplied infections by some constant). One may notice that the most vital for reducing the number of infections and deaths is the conditional (on human contact) probability of getting infected. Additionally high productivity of teleworking also provides optimal allocations that reduces the infections among the population.

Now let us consider changes in parameters ρ and ω. First let us analyse the results of increased $\rho = 1.5$

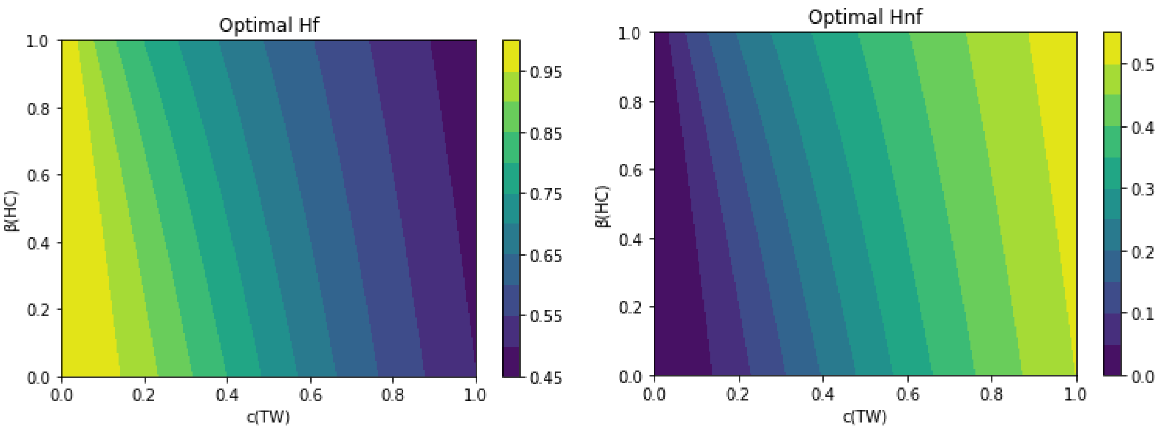

Figure 11: Contour plots of H_f and H_{nf}

First thing we may notice is the bigger influence of $\beta(HC)$ for the optimal values in our model (the contour lines are more curve now), but generally the pattern remains the same.

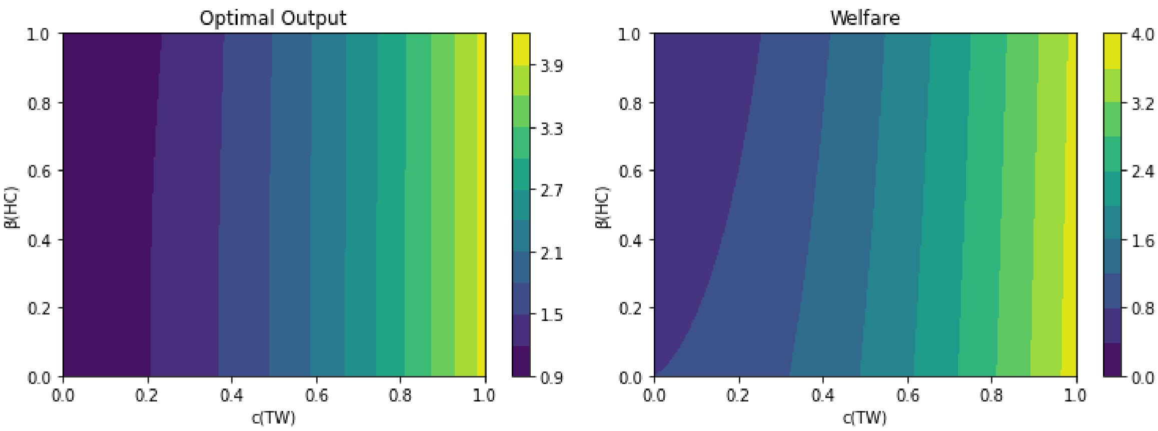

Figure 12: Contour plots of output and welfare

Here we may also notice the increased 'importance' of $\beta(HC)$, for higher values the output is lower for output and much lower for welfare. In addition the output is not so big now as for previous model settings, and that is the reason why the welfare differs more from output in terms of pattern and values. Welfare here increases also with the lower value of infection's rate - what was not so visible for previous settings.

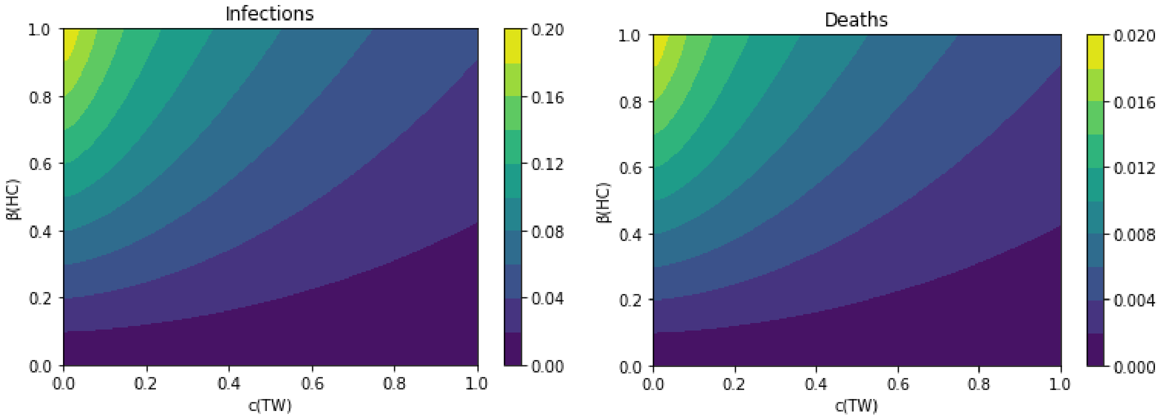

Figure 13: Contour plots of infections and deaths

An increase of elasticity of substitution decreases slightly number of infected people but the general pattern remains the same.

Now let us consider $\rho = 1.05$

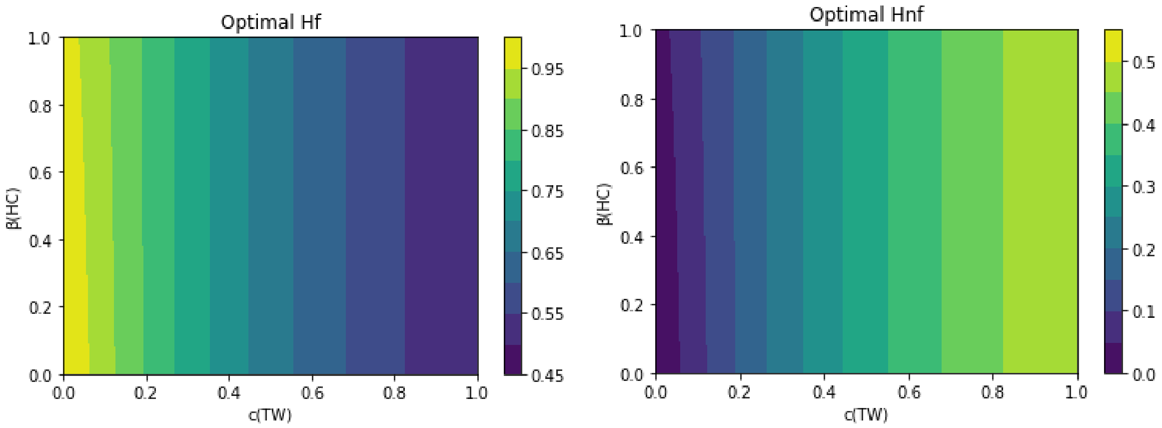

Figure 14: Contour plots of H_f and H_{nf}

Generally we see similar pattern as in the baseline case (figure 6 and 7).

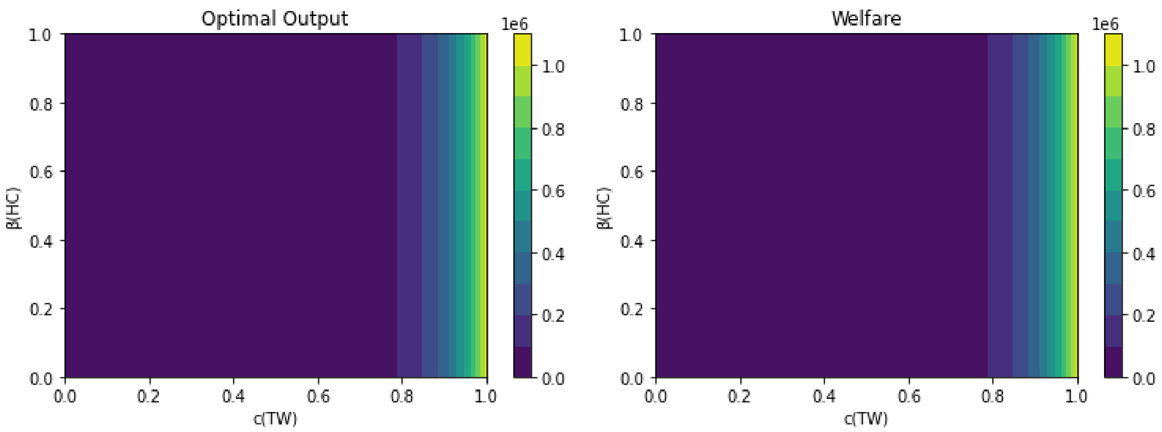

Figure 15: Contour plots of output and welfare

When increasing ρ the value of production function goes up and also the welfare, which for relatively big outputs is mainly determined by it.

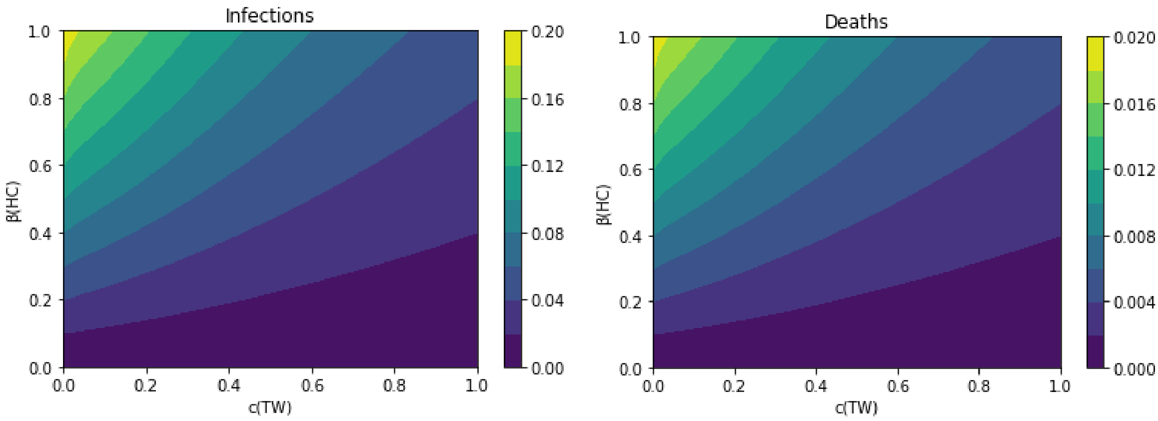

Figure 16: Contour plots of infections and deaths

A decrease of elasticity of substitution decreases very slightly number of infected people but the general pattern remains the same.

Now let us consider $\omega = 40$

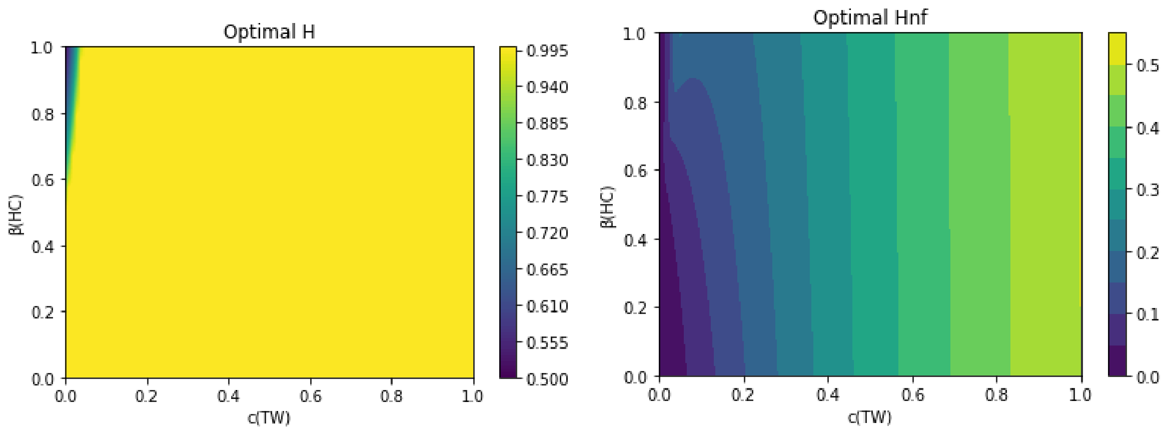

Figure 17: Contour plots of H and H_{nf}

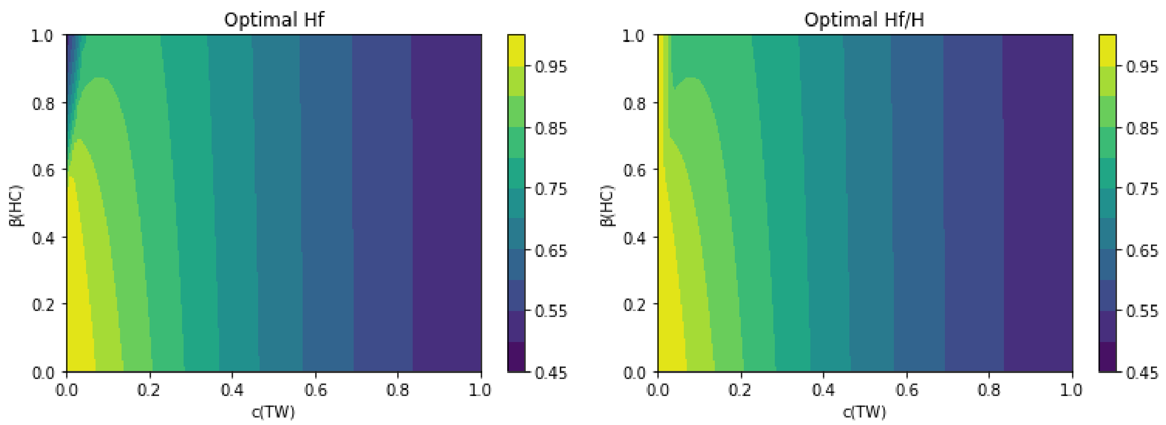

Figure 18: Contour plots of Hf and $\frac{H_f}{H}$

Now, one can notice something unusual comparing to previous examples - top left corner. For low teleworking productivity and high possibility of getting infected the optimal choice is not to work for the entire population. It seams reasonable, keeping in mind that we assigned a higher weight for deaths. Apart from that region the other figures are same as for the baseline model.

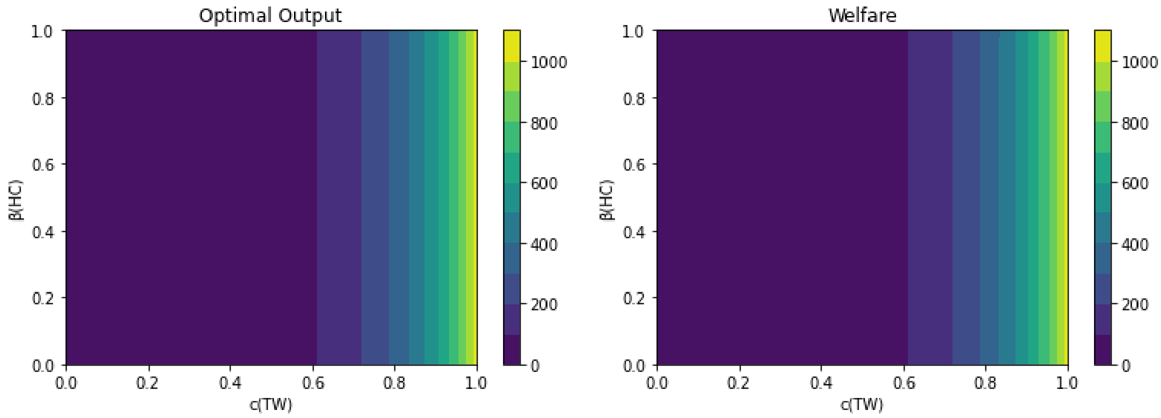

Figure 19: Contour plots of output and welfare

When increasing ω the value of production neither goes up nor down and the welfare behaves in the same way.

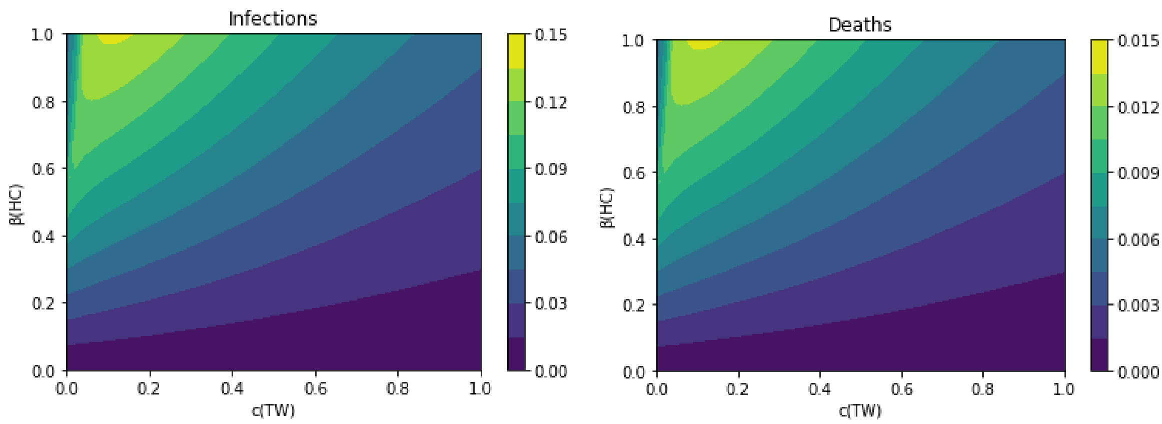

Figure 20: Contour plots of infections and deaths

Analysing these graphs there are two things to bear in mind. The first is this previously mentioned unusual behavior, that leads here to the lower number of infections and deaths. Second is that the maximal values reach about three-quarter of the previously presented results.

Now let us consider $\omega = 10$

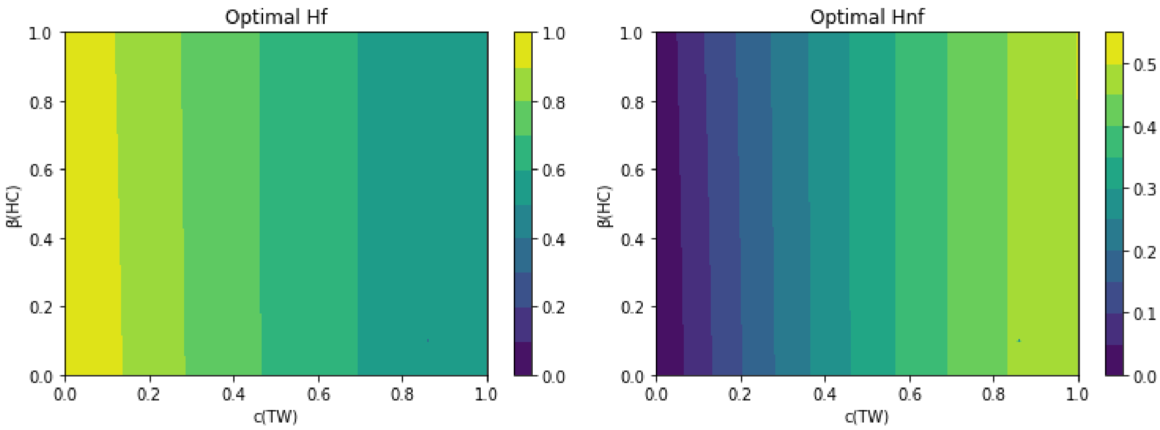

Figure 21: Contour plots of H_f and H_{nf}

Generally we see similar pattern as in the baseline case (figure 6 and 7), there are no essential changes.

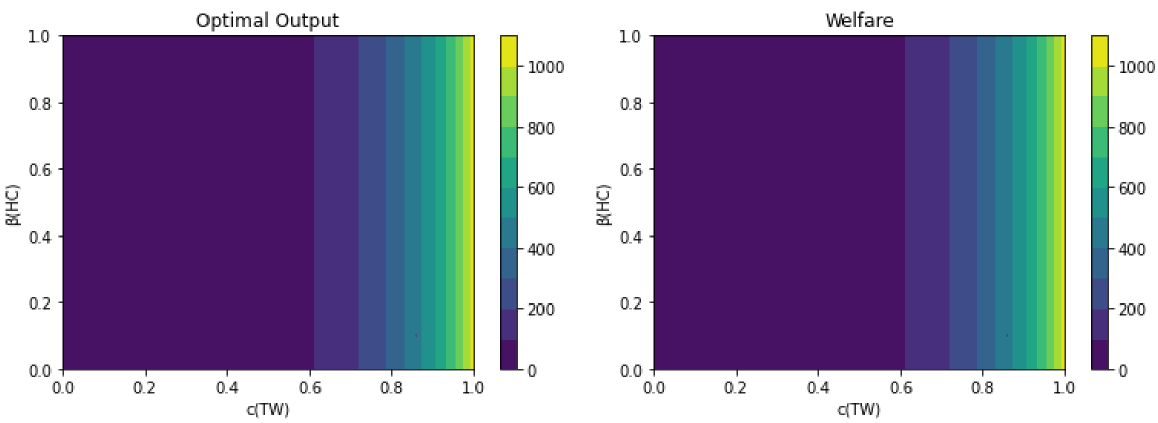

Figure 22: Contour plots of output and welfare

Also speaking of output and welfare the 50-percent decrease in weight attached to the deaths does not change anything.

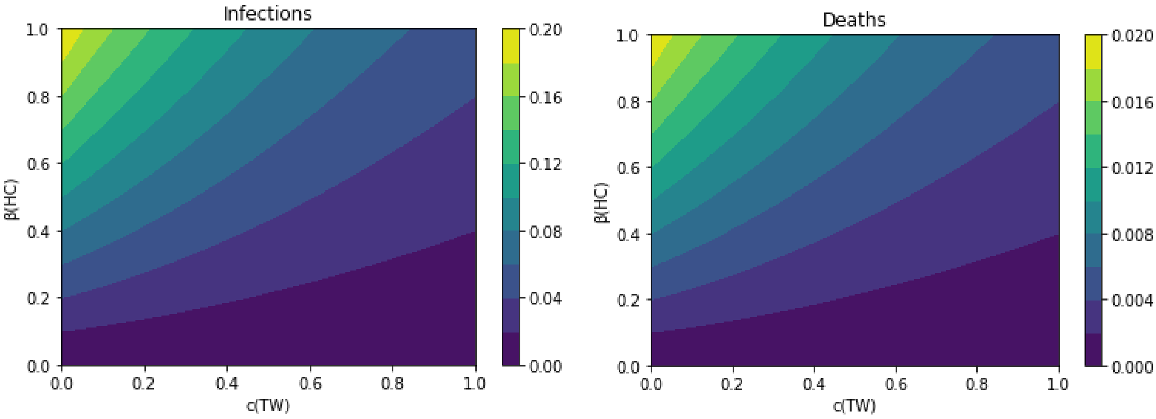

Figure 23: Contour plots of infections and deaths

Finally contour plots for infections and deaths does not present any deviation from the baseline settings.