Portfolio Optimization

Define the vector of returns μ as

$$\mu = \begin{pmatrix} 0.08 \\ 0.10 \\ 0.10 \\ 0.14 \end{pmatrix}$$

Define the standard deviation matrix S as

$$S = \begin{pmatrix} 0.12 & 0 & 0 & 0 \\ 0 & 0.12 & 0 & 0 \\ 0 & 0 & 0.15 & 0 \\ 0 & 0 & 0 & 0.20 \end{pmatrix}$$

The covariance matrix Σ is given by

$$\Sigma = SRS$$

Define the weight vector w as

$$w = \begin{pmatrix} w_A \\ w_B \\ w_C \\ w_D \end{pmatrix}$$

Our optimization problem can be formulated as

$$\min_{w} \frac{1}{2} w^T SRSw$$

Subject to

$$\mu^T w = m$$
$$1^T w = 1$$

where the two constraints are respectively the return constraint and the budget equation.

Next, form the Lagrange Function: with two Lagrange Multipliers, λ and γ:

$$L(w, \lambda, \gamma) = \frac{1}{2} w^T SRSw + \lambda(m - \mu^T w) + \gamma(1 - 1^T w)$$

And solve for the first order condition:

$$\frac{\partial L}{\partial w}(w, \lambda, \gamma) = w^T \Sigma - \lambda \mu^T - \gamma 1^T = 0$$
$$\frac{\partial L}{\partial \lambda}(w, \lambda, \gamma) = m - \mu^T w = 0$$
$$\frac{\partial L}{\partial \gamma}(w, \lambda, \gamma) = 1 - 1^T w = 0$$

Then get the optimal weight vector $\boldsymbol{w}^*$

$$w^* = (SRS)^{-1}(\lambda\mu + \gamma 1)$$

where

$$\begin{cases} \lambda = \dfrac{Am - B}{AC - B^2} \\ \gamma = \dfrac{C - Bm}{AC - B^2} \end{cases}$$

and

$$\begin{cases} A = 1^T(SRS)^{-1}1 \\ B = \mu^T(SRS)^{-1}1 \\ C = \mu^T(SRS)^{-1}\mu \end{cases}$$

Note:

$$1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

- Case a:

$$R = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.3 \\ 0.2 & 1 & 0.7 & 0.4 \\ 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.4 & 0.9 & 1 \end{pmatrix}$$

$$\Sigma = SRS = \begin{pmatrix} 0.0144 & 0.00288 & 0.009 & 0.0072 \\ 0.00288 & 0.0144 & 0.0126 & 0.0096 \\ 0.009 & 0.0126 & 0.0225 & 0.027 \\ 0.0072 & 0.0096 & 0.027 & 0.04 \end{pmatrix}$$

$$(SRS)^{-1} = \begin{pmatrix} 442.83413 & 634.05797 & -1336.55394 & 670.28985 \\ 634.05797 & 1207.72946 & -2342.99516 & 1177.53623 \\ -1336.55394 & -2342.99516 & 4882.44766 & -2492.75362 \\ 670.28985 & 1177.53623 & -2492.75362 & 1304.34782 \end{pmatrix}$$

$$\begin{cases} A = 1^T (SRS)^{-1} 1 = 456.52173 \\ B = \mu^T (SRS)^{-1} 1 = 63.81642 \\ C = \mu^T (SRS)^{-1} \mu = 9.38969 \end{cases}$$

for m = 10% return constraint

$$\begin{cases} \lambda = \dfrac{Am - B}{AC - B^2} = -0.08485 \\ \gamma = \dfrac{C - Bm}{AC - B^2} = 0.01405 \end{cases}$$

$$w^* = (SRS)^{-1}(\lambda\mu + \gamma 1) = \begin{pmatrix} 0.76167 \\ 0.84320 \\ -0.98577 \\ 0.38019 \end{pmatrix}$$

- Case b:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Sigma = SRS = \begin{pmatrix} 0.0144 & 0 & 0 & 0 \\ 0 & 0.0144 & 0 & 0 \\ 0 & 0 & 0.0225 & 0 \\ 0 & 0 & 0 & 0.04 \end{pmatrix}$$

$$(SRS)^{-1} = \begin{pmatrix} 69.44444 & 0 & 0 & 0 \\ 0 & 69.44444 & 0 & 0 \\ 0 & 0 & 44.44444 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix}$$

$$\begin{cases} A = 1^T(SRS)^{-1}1 = 208.33332 \\ B = \mu^T(SRS)^{-1}1 = 20.44444 \\ C = \mu^T(SRS)^{-1}\mu = 2.07333 \end{cases}$$

for m = 10% return constraint

$$\begin{cases} \lambda = \dfrac{Am - B}{AC - B^2} = 0.02784 \\ \gamma = \dfrac{C - Bm}{AC - B^2} = 0.00206 \end{cases}$$

$$w^* = (SRS)^{-1}(\lambda\mu + \gamma 1) = \begin{pmatrix} 0.29772 \\ 0.33638 \\ 0.21528 \\ 0.14894 \end{pmatrix}$$

- Case c:

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\Sigma = SRS = \begin{pmatrix} 0.0144 & 0.0144 & 0.018 & 0.024 \\ 0.0144 & 0.0144 & 0.018 & 0.024 \\ 0.018 & 0.018 & 0.0225 & 0.03 \\ 0.024 & 0.024 & 0.03 & 0.04 \end{pmatrix}$$

$(SRS)^{-1} \longrightarrow$ Matrix is Singular

Design a portfolio producing any arbitrary level of return m with 0 volatility, and solve a system of 3 equations with n unknown variables, the weights:

$$\sigma^T w = 0$$
$$\mu^T w = m$$
$$1^T w = 1$$

where σ is the vector of standard deviation.

Since I have more unknown variables than constraints, I will have several possible portfolios satisfying this system.