第二次习题课参考解答(复合函数链式法则、高阶偏导数、方向导数)

1.（1）设 f 可微，且 $z = x^3 f\left(xy, \dfrac{y}{x}\right)$，求 $\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$.

解：$dz = f \cdot 3x^2 dx + x^3 df = 3x^2 f dx + x^3\left[f_1' d(xy) + f_2' d\left(\dfrac{y}{x}\right)\right]$

$$= 3x^2 f dx + x^3\left[f_1'(xdy + ydx + f_2' \frac{xdy - ydx}{x^2}\right]$$

$$= \left(3x^2 f + x^3 y f_1' - xy f_2'\right)dx + \left(x^4 f_1' + x^2 f_2'\right)dy$$

由一阶微分的形式不变性，

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \left(3x^2 f + x^3 y f_1' - xy f_2'\right)dx + \left(x^4 f_1' + x^2 f_2'\right)dy$$

故 $\dfrac{\partial z}{\partial x} = \left(3x^2 f + x^3 y f_1' - xy f_2'\right)$，$\dfrac{\partial z}{\partial y} = \left(x^4 f_1' + x^2 f_2'\right)$.

其中符号 f_1'，f_2' 分别表示函数 $f(x,y)$ 分别对第一个中间变量和第二个中间变量求偏导。

（2）设 $z = f(x^2 y, \dfrac{y}{x})$，其中 $f \in C^2$，求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial^2 z}{\partial y \partial x}$.

解：$\dfrac{\partial z}{\partial x} = 2xy f_1' - \dfrac{y}{x^2} f_2'$;

$$\frac{\partial^2 z}{\partial y \partial x} = 2xy(f_{11}'' x^2 + \frac{1}{x} f_{12}'') + 2x f_1' - \frac{y}{x^2}(x^2 f_{21}'' + \frac{1}{x} f_{22}'') - \frac{1}{x^2} f_2'$$

$$= 2x^3 y f_{11}'' + y f_{12}'' - \frac{y}{x^3} f_{22}'' + 2x f_1' - \frac{1}{x^2} f_2'.$$

2. 设 $g(x) = f\left(x, \phi\left(x^2, x^2\right)\right)$，其中函数 f 和 ϕ 的二阶偏导数连续，求 $g''(x)$.

解：由 $g(x) = f\left(x, \phi\left(x^2, x^2\right)\right)$ 两边对 x 求导，得

$$g'(x) = f_x'\left(x, \phi\left(x^2, x^2\right)\right) + 2 f_\phi'\left(x, \phi\left(x^2, x^2\right)\right)\left(\phi_1'\left(x^2, x^2\right) + \phi_2'\left(x^2, x^2\right)\right)x,$$

两边再对 x 求导，得

$$g''(x) = f_{xx}'' + 4 f_{x\phi}''(\phi_1' + \phi_2')x + 4 f_{\phi\phi}''(\phi_1' + \phi_2')^2 x^2 + 4 f_\phi'(\phi_{11}'' + 2\phi_{12}'' + \phi_{22}'')x^2 + 2 f_\phi'(\phi_1' + \phi_2'),$$

其中符号 ϕ_1'，ϕ_2' 分别表示 ϕ 对其第一个中间变量和第二个中间变量求偏导。

3. 设 $z=z(x,y)$ 二阶连续可微，并且满足方程 $A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}=0$，其中 A,B,C 都是非零常数。若令 $\begin{cases}u=x+\alpha y\\ v=x+\beta y,\end{cases}$ 试确定 α,β 为何值时原方程可转化为 $\frac{\partial^2 z}{\partial u\partial v}=0$.

解：因为 $z=z(x,y)$ 二阶连续可微，因此二阶混合偏导与求偏导顺序无关。将 x,y 看成自变量，u,v 看成中间变量，利用链式法则得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v},\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v},$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}\right)=\alpha^2\frac{\partial^2 z}{\partial u^2}+2\alpha\beta\frac{\partial^2 z}{\partial u\partial v}+\beta^2\frac{\partial^2 z}{\partial v^2},$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}\right)=\alpha\frac{\partial^2 z}{\partial u^2}+(\alpha+\beta)\frac{\partial^2 z}{\partial u\partial v}+\beta\frac{\partial^2 z}{\partial v^2},$$

由 $0=A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}$ 得到

$$\left(A+2B\alpha+C\alpha^2\right)\frac{\partial^2 z}{\partial u^2}+2\left(A+B(\alpha+\beta)+C\alpha\beta\right)\frac{\partial^2 z}{\partial u\partial v}+\left(A+2B\beta+C\beta^2\right)\frac{\partial^2 z}{\partial v^2}=0\ \cdots(*)$$

故只要选取 α,β 使得

$$\begin{cases}A+2B\alpha+C\alpha^2=0\\ A+2B\beta+C\beta^2=0,\end{cases}$$

即得 $\frac{\partial^2 z}{\partial u\partial v}=0$. 这样问题转化为方程 $A+2Bt+Ct^2=0$ 有两不同实根，即要求 $B^2-AC>0$. 取 $\alpha=\frac{-B+\sqrt{B^2-AC}}{C}$, $\beta=\frac{-B-\sqrt{B^2-AC}}{C}$. 将其代入方程(*)，可知 $\frac{\partial^2 z}{\partial u\partial v}$ 的系数不为零，从而 $\frac{\partial^2 z}{\partial u\partial v}=0$.

4. 设 $u(x,y)\in C^2$，又 $\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$，$u(x,2x)=x$，$u_x'(x,2x)=x^2$，求 $u_{xx}''(x,2x)$，$u_{xy}''(x,2x)$，$u_{yy}''(x,2x)$.

解: 因为 $\frac{\partial u}{\partial x}(x,2x)=x^2$，两边对 x 求导，得

$$\frac{\partial^2 u}{\partial x^2}(x,2x)+\frac{\partial^2 u}{\partial y\partial x}(x,2x)\cdot 2=2x. \qquad (1)$$

由 $u(x,2x)=x$，两边对 x 求导，得 $\frac{\partial u}{\partial x}(x,2x)+\frac{\partial u}{\partial y}(x,2x)\cdot 2=1$，

所以，$\frac{\partial u}{\partial y}(x,2x)=\frac{1-x^2}{2}$. 此式两边再对 x 求导，得

$$\frac{\partial^2 u}{\partial x\partial y}(x,2x)+\frac{\partial^2 u}{\partial y^2}(x,2x)\cdot 2=-x. \qquad (2)$$

由已知，

$$\frac{\partial^2 u}{\partial x^2}(x,2x)-\frac{\partial^2 u}{\partial y^2}(x,2x)=0, \qquad (3)$$

因为 $u(x,y)\in C^2$， 因此 $\frac{\partial^2 u}{\partial x\partial y}(x,2x)=\frac{\partial^2 u}{\partial y\partial x}(x,2x)$.

联立 (1), (2), (3)解得:

$$\frac{\partial^2 u}{\partial x^2}(x,2x)=\frac{\partial^2 u}{\partial y^2}(x,2x)=-\frac{4}{3}x,\quad \frac{\partial^2 u}{\partial x\partial y}(x,2x)=\frac{5}{3}x.$$

5. 设 $z(x,y)$是定义在矩形区域 $D=\{(x,y)|0\le x\le a, 0\le y\le b\}$上的可微函数。证明:

（1） $z(x,y)=f(y)\Leftrightarrow \forall(x,y)\in D, \frac{\partial z}{\partial x}\equiv 0$;

（2） $z(x,y)=f(y)+g(x)\Leftrightarrow \forall(x,y)\in D, \frac{\partial^2 z}{\partial x\partial y}\equiv 0$.

证明: (1) “$\Rightarrow$”显然.

“$\Leftarrow$” 任取 $x_0\in[0,a]$. 任意固定 $y\in[0,a]$，关于 x 的一元函数 $z(x,y)$ 在以 x 与 x_0 为端点的区间上应用微分中值定理，则存在 ξ 使得

$$z(x,y)-z(x_0,y)=\frac{\partial z}{\partial x}(\xi,y)(x-x_0)=0,$$

这样 $z(x,y)=z(x_0,y)$，故 $z(x,y)=f(y)$ 与 x 无关.

(2) “$\Rightarrow$” 显然。

“$\Leftarrow$” 因为 $\dfrac{\partial^2 z}{\partial x\partial y}\equiv 0,\ \dfrac{\partial z}{\partial y}=h(y)$ 与 x 无关. 故

$$z(x,y)=\int h(y)dy+g(x)=f(y)+g(x).$$

6. 计算下列各题:

(1) 已知 $z=\left(\dfrac{y}{x}\right)^{\frac{x}{y}}$，求 $\left.\dfrac{\partial z}{\partial x}\right|_{(1,2)}$.

解: 令 $u=\dfrac{y}{x},\ v=\dfrac{x}{y}$，则 $z=u^v$. 所以

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=vu^{v-1}\left(-\frac{y}{x^2}\right)+\frac{1}{y}u^v\ln u.$$

因为 $u(1,2)=2,\ v(1,2)=\dfrac{1}{2}$，因此 $\left.\dfrac{\partial z}{\partial x}\right|_{(1,2)}=\dfrac{\ln 2-1}{\sqrt{2}}$.

(2) 设 $f(u,v)\in C^2$ 且 $z=f(e^{x+y},xy)$. 求 $\dfrac{\partial z}{\partial x},\ \dfrac{\partial^2 z}{\partial x\partial y}$.

解: 令 $u=e^{x+y},\ v=xy$，则 $z=f(u,v)$. 由复合函数的链式法则，

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=e^{x+y}\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}.$$

$$\frac{\partial^2 z}{\partial x\partial y}=e^{x+y}\frac{\partial f}{\partial u}+e^{x+y}(\frac{\partial^2 f}{\partial u^2}e^{x+y}+x\frac{\partial^2 f}{\partial v\partial u})+\frac{\partial f}{\partial v}+y(\frac{\partial^2 f}{\partial u\partial v}e^{x+y}+x\frac{\partial^2 f}{\partial v^2})$$

$$=e^{x+y}\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}+e^{x+y}(x+y)\frac{\partial^2 f}{\partial u\partial v}+\frac{\partial^2 f}{\partial u^2}e^{2(x+y)}+yx\frac{\partial^2 f}{\partial v^2}.$$

(3) 设函数 f 二阶可导，函数 g 一阶可导。令

$$z(x,y)=f(x+y)+f(x-y)+\int_{x-y}^{x+y}g(t)dt.\ 求\ \frac{\partial^2 z}{\partial x^2},\ \frac{\partial^2 z}{\partial y^2},\ \frac{\partial^2 z}{\partial x\partial y}.$$

解: 由复合函数求导法则及变限积分求导，可得

$$\frac{\partial z}{\partial x} = f'(x+y) + f'(x-y) + g(x+y) - g(x-y),$$

$$\frac{\partial z}{\partial y} = f'(x+y) - f'(x-y) + g(x+y) + g(x-y),$$

所以$$\frac{\partial^2 z}{\partial x^2} = f''(x+y) + f''(x-y) + g'(x+y) - g'(x-y),$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+y) + f''(x-y) + g'(x+y) - g'(x-y),$$

$$\frac{\partial^2 z}{\partial x \partial y} = f''(x+y) - f''(x-y) + g'(x+y) + g'(x-y).$$

(4) 设 $f(x,y) = \begin{cases} e^{-\frac{1}{x^2+y^2}}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0. \end{cases}$ 求 $f''_{xx}(0,0)$ 与 $f''_{xy}(0,0)$.

解: $f'_x(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0) - f(0,0)}{x} = 0,$

对 $\forall (x,y) \neq (0,0)$, $f'_x(x,y) = \dfrac{2x}{(x^2+y^2)^2} e^{-\frac{1}{x^2+y^2}}$,

因此 $f''_{xx}(0,0) = \lim\limits_{x\to 0} \dfrac{f'_x(x,0) - f'_x(0,0)}{x} = \lim\limits_{x\to 0} \dfrac{2e^{-\frac{1}{x^2}}}{x^4} = 0$;

$f''_{xy}(0,0) = \lim\limits_{y\to 0} \dfrac{f'_x(0,y) - f'_x(0,0)}{y} = 0.$

(5) 设 $f(x,y) = e^x \sin y$, 求 $f^{(n+m)}_{x^n y^m}(0,0)$.

解: 因为 $f^{(n+m)}_{x^n y^m}(x,y) = e^x \sin(y + \dfrac{m}{2}\pi)$, 因此

$$f^{(n+m)}_{x^n y^m}(0,0) = \sin\frac{m}{2}\pi = \begin{cases} 0, & m = 2k, \\ (-1)^{k-1}, & m = 2k-1, \end{cases}$$ 其中 $k \in \mathbb{N}^+$.

7. 证明: 函数 $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ 在 $(0,0)$ 点不连续, 但在该点存在任意阶偏导数 $\dfrac{\partial^n f}{\partial x^n}(0,0)$ 和 $\dfrac{\partial^n f}{\partial y^n}(0,0)$.

证明：令 $y=kx$，则 $\lim\limits_{(x,y)\to(0,0)} f(x,kx)=\dfrac{k}{1+k^2}$ 依赖于 k 的值，因此 $f(x,y)$ 在 $(0,0)$ 的极限不存在，从而在该点不连续。但

$$f_x'(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=0.$$

当 $(x,y)\neq(0,0)$ 时，$f_x'(x,y)=\dfrac{y(y^2-x^2)}{(y^2+x^2)^2}$，因此

$$f_{xx}''(0,0)=\lim_{x\to 0}\frac{f_x'(x,0)-f_x'(0,0)}{x}=0.$$

当 $(x,y)\neq(0,0)$ 时，$f_{xx}''(x,y)=\dfrac{2xy(x^2-y^2)}{(y^2+x^2)^3}$ 且

$$\frac{\partial^3 f}{\partial x^3}(0,0)=\lim_{x\to 0}\frac{f_{xx}''(x,0)-f_{xx}''(0,0)}{x}=0;$$

当 $(x,y)\neq(0,0)$ 时，假设 $\dfrac{\partial^{n-2} f}{\partial x^{n-2}}(x,y)=\dfrac{yp(x,y)}{(y^2+x^2)^{n-1}}$ 且 $\dfrac{\partial^{n-2} f}{\partial x^{n-2}}(0,0)=0$，其中 $p(x,y)$ 是关于 x, y 的多项式. 则

$$\frac{\partial^{n-1} f}{\partial x^{n-1}}(x,y)=\frac{y[p_x'(x,y)(y^2+x^2)-2nxp(x,y)]}{(y^2+x^2)^n}$$

且 $\dfrac{\partial^{n-1} f}{\partial x^{n-1}}(0,0)=\lim\limits_{x\to 0}\dfrac{\dfrac{\partial^{n-2} f}{\partial x^{n-2}}(x,0)-\dfrac{\partial^{n-2} f}{\partial x^{n-2}}(0,0)}{x}=0$，从而

$$\frac{\partial^{n} f}{\partial x^{n}}(0,0)=\lim_{x\to 0}\frac{\dfrac{\partial^{n-1} f}{\partial x^{n-1}}(x,0)-\dfrac{\partial^{n-1} f}{\partial x^{n-1}}(0,0)}{x}=0.$$

由于 $f(x,y)$ 的表达式中 x, y 的地位对称，同理可得 $\dfrac{\partial^n f}{\partial y^n}(0,0)=0$. 证毕

8. 设 $u(x,y)$ 有二阶偏导数，无零点。证明：$u(x,y)$ 满足方程 $u\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial u}{\partial y}$ 当且仅当 $u(x,y)=f(x)g(y)$.

证明：充分性显然,下证必要性. 假设 $u\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial u}{\partial y}$. 由于 $u(x,y)$ 没有零点，因此有

$$\frac{u\dfrac{\partial^2 u}{\partial x\partial y}-\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}}{u^2}=0,$$

即 $\dfrac{\partial\left(\dfrac{1}{u}\dfrac{\partial u}{\partial y}\right)}{\partial x}=0.$ 故 $\dfrac{1}{u}\dfrac{\partial u}{\partial y}=r(y)$，所以 $\dfrac{\partial(\ln u)}{\partial y}=r(y)$；两端关于 y 求不定积分，得

$\ln u=t(x)+I(y)$，从而 $u(x,y)=f(x)g(y)$. 证毕

9. 设 $f(x,y)\in C^2(\mathbb{R}^2)$ 满足 Laplace 方程 $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0.$

证明：$u(x,y)=f\left(\dfrac{x}{x^2+y^2},\dfrac{y}{x^2+y^2}\right)$也满足 Laplace 方程.

证明：因为 $\dfrac{\partial u}{\partial x}=f_1'\dfrac{y^2-x^2}{(y^2+x^2)^2}-f_2'\dfrac{2xy}{(y^2+x^2)^2}$，$\dfrac{\partial u}{\partial y}=f_1'\dfrac{-2xy}{(y^2+x^2)^2}+f_2'\dfrac{x^2-y^2}{(y^2+x^2)^2}$，

因此

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=f_{11}''\frac{(y^2-x^2)^2}{(y^2+x^2)^4}-f_{12}''\frac{4xy(y^2-x^2)}{(y^2+x^2)^4}+f_{22}''\frac{4y^2x^2}{(y^2+x^2)^4}\\&\quad+f_1'\frac{2x(y^2+x^2)(x^2-3y^2)}{(y^2+x^2)^4}+f_2'\frac{2y(y^2+x^2)(3x^2-y^2)}{(y^2+x^2)^4},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2}&=f_{11}''\frac{4y^2x^2}{(y^2+x^2)^4}-f_{12}''\frac{4xy(x^2-y^2)}{(y^2+x^2)^4}+f_{22}''\frac{(y^2-x^2)^2}{(y^2+x^2)^4}\\&\quad+f_1'\frac{2x(y^2+x^2)(3y^2-x^2)}{(y^2+x^2)^4}+f_2'\frac{2y(y^2+x^2)(y^2-3x^2)}{(y^2+x^2)^4},\end{aligned}$$

由于 $f_{11}''+f_{22}''=0$，故 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=(f_{11}''+f_{22}'')\dfrac{(y^2-x^2)^2}{(y^2+x^2)^4}+(f_{11}''+f_{22}'')\dfrac{4y^2x^2}{(y^2+x^2)^4}=0.$证毕

10. 设 n 为整数，若对任意的 $t>0$，$f(tx,ty)=t^n f(x,y)$，则称 f 是 n 次齐次函数。证明：

可微函数 $f(x,y)$ 是零次齐次函数的充要条件是 $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=0.$

证明：先证必要性。设可微函数 $f(x,y)$ 是零次齐次函数，即

$$f(tx,ty)=f(x,y)\ \ (\forall t>0). \qquad (4)$$

若 f 在坐标原点处有定义，则由 f 的连续性可知 $f(x,y)=f(0,0),\ (\forall(x,y))$. 结论显

然成立。

现在假设 f 在坐标原点处没有定义。则由复合函数的链式法则，方程(4)两边分别对 t 求导，得 $x\frac{\partial f}{\partial x}(tx,ty)+y\frac{\partial f}{\partial y}(tx,ty)=0$. 令 $t=1$，即得

$$x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=0.$$

必要性得证。

下证充分性。设 $f(x,y)$ 满足 $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0.$ 令 $x=r\cos\theta,\ y=r\sin\theta.$ 则

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\cdot\cos\theta+\frac{\partial f}{\partial y}\cdot\sin\theta=\frac{1}{r}(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y})=0.$$

上式说明 f 在极坐标系中只是 $\theta=\arctan\frac{y}{x}$ 的函数，这等价于只是 $\frac{y}{x}$ 的函数。可记 $f(x,y)=\phi(\frac{y}{x})$. 显然 ϕ 是零次齐次函数。

充分性证法二：任取 $(x,y)\in\mathbb{R}^2$，并令 $\vec{r}=(x,y)$. 因为 $xf'_x+yf'_y=0,$ 因此

$$\left.\frac{\partial f}{\partial\vec{r}}\right|_{(x,y)}=\frac{1}{\|\vec{r}\|}(xf'_x+yf'_y)=0,$$

即 f 沿着任意方向的方向导数都等于零，从而 f 沿着任意方向的函数值不变。故在极坐标系中，由原点出发的任一射线上函数值相等。所以在极坐标系中 f 只是 θ 的函数。证毕

11. 设 $f(x,y)$ 在 $P_0(x_0,y_0)$ 可微。已知 $\vec{v}=\vec{i}-\vec{j}$， $\vec{u}=-\vec{i}+2\vec{j}$，且 $\left.\frac{\partial f}{\partial\vec{v}}\right|_{P_0}=2$， $\left.\frac{\partial f}{\partial\vec{u}}\right|_{P_0}=1$，求 $f(x,y)$ 在 $P_0(x_0,y_0)$ 的微分。

解：因为 $\vec{v}=\vec{i}-\vec{j}=(1,-1)$， $\vec{u}=-\vec{i}+2\vec{j}=(-1,2)$，且 $f(x,y)$ 在 $P_0(x_0,y_0)$ 可微，因此

$$2=\left.\frac{\partial f}{\partial\vec{v}}\right|_{P_0}=(f'_x(P_0),f'_y(P_0))\cdot(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})=\frac{1}{\sqrt{2}}(f'_x(P_0)-f'_y(P_0)),$$

$$1=\left.\frac{\partial f}{\partial\vec{u}}\right|_{P_0}=(f'_x(P_0),f'_y(P_0))\cdot(-\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})=\frac{1}{\sqrt{5}}(-f'_x(P_0)+2f'_y(P_0)),$$

由此解出 $f'_x(P_0)=4\sqrt{2}+\sqrt{5},\ f'_y(P_0))=2\sqrt{2}+\sqrt{5}.$ 所以 $f(x,y)$ 在 $P_0(x_0,y_0)$ 的微分

$$df(P_0)=(4\sqrt{2}+\sqrt{5})dx+(2\sqrt{2}+\sqrt{5})dy.$$

12. 设 $f(x,y)$ 为可微函数，$\vec{l}_1$，$\vec{l}_2$ 是 $\mathbb{R}^2$ 上的一组线性无关的向量。试证：$f(x,y)$ 在任一点 $P(x,y)$ 沿任意向量 $\vec{l}$ 的方向导数 $f'_{\vec{l}}(P)$ 必定能用 $f'_{\vec{l}_1}(P)$ 与 $f'_{\vec{l}_2}(P)$ 线性表示。

证明：令 $\vec{l}_1=(\cos\alpha_1,\cos\beta_1)$，$\vec{l}_2=(\cos\alpha_2,\cos\beta_2)$.

因为 $f(x,y)$ 可微，故

$$\begin{cases} f'_{\vec{l}_1}(P)=f'_x(P)\cos\alpha_1+f'_y(P)\cos\beta_1=d_1 \\ f'_{\vec{l}_2}(P)=f'_x(P)\cos\alpha_2+f'_y(P)\cos\beta_2=d_2. \end{cases}$$

由于 $\vec{l}_1$，$\vec{l}_2$ 线性无关，因此由上式解出 $\begin{pmatrix} f'_x(P) \\ f'_y(P) \end{pmatrix}=\begin{pmatrix} \cos\alpha_1 & \cos\beta_1 \\ \cos\alpha_2 & \cos\beta_2 \end{pmatrix}^{-1}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$.

于是，对任意的向量 $\vec{l}=(\cos\alpha,\cos\beta)$，

$$\begin{aligned} f'_{\vec{l}}(P)&=f'_x(P)\cos\alpha+f'_y(P)\cos\beta=\left(\cos\alpha,\quad \cos\beta\right)\begin{pmatrix} f'_x(P) \\ f'_y(P) \end{pmatrix} \\ &=\left(\cos\alpha,\quad \cos\beta\right)\begin{pmatrix} \cos\alpha_1 & \cos\beta_1 \\ \cos\alpha_2 & \cos\beta_2 \end{pmatrix}^{-1}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \\ &=\left(a,\ b\right)\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \end{aligned}$$

其中 $\left(a,\ b\right)=\left(\cos\alpha,\quad \cos\beta\right)\begin{pmatrix} \cos\alpha_1 & \cos\beta_1 \\ \cos\alpha_2 & \cos\beta_2 \end{pmatrix}^{-1}$.

13. 设 $f(x,y)=x^2-xy+y^2$，$P_0(1,1)$. 试求 $\left.\dfrac{\partial f}{\partial \vec{l}}\right|_{P_0}$，并问：在怎样的方向 $\vec{l}$ 上，方向导数 $\left.\dfrac{\partial f}{\partial \vec{l}}\right|_{P_0}$ 分别有最大值、最小值和零值。

解：因为 $f(x,y)$ 可微，且 $f'_x(P_0)=(2x-y)|_{(1,1)}=1$，$f'_y(P_0)=(2y-x)|_{(1,1)}=1$，

因此对任意的单位向量 $\vec{l}=(\cos\alpha,\cos\beta)$，$\left.\dfrac{\partial f}{\partial \vec{l}}\right|_{P_0}=\cos\alpha+\cos\beta.$

当 $\vec{l}=(1,1)$ 是梯度方向时，$\left.\dfrac{\partial f}{\partial \vec{l}}\right|_{P_0}=\sqrt{2}$ 达到最大；

当 $\vec{l}=(-1,-1)$ 时，$\left.\dfrac{\partial f}{\partial \vec{l}}\right|_{P_0}=-\sqrt{2}$ 达到最小；

当 $\vec{l}=(1,-1)$ 或 $\vec{l}=(-1,1)$ 时，即 $\alpha=\dfrac{3\pi}{4}$ 或 $\dfrac{7\pi}{4}$ 时，$\left.\dfrac{\partial f}{\partial \vec{l}}\right|_{P_0}=0$.

14. 设a,b是实数，函数$z=2+ax^2+by^2$在点$(3,4)$处的方向导数中，沿$\boldsymbol{l}=-3\mathbf{i}-4\mathbf{j}$的方向导数最大，最大值为 10，求$a,b$.

解：由于函数可微，因此函数沿着梯度方向的方向导数达到最大，又

$$\left.\frac{\partial z}{\partial x}\right|_{(3,4)}=6a,\quad \left.\frac{\partial z}{\partial y}\right|_{(3,4)}=8b\,,$$

且方向导数的最大值为 10，故$\frac{1}{5}(-3,-4)=(\frac{6a}{10},\frac{8b}{10})$. 从而$\begin{cases}\frac{6a}{10}=-\frac{3}{5}\\ \frac{8b}{10}=-\frac{4}{5},\end{cases}$ 所以$\begin{cases}a=-1\\ b=-1.\end{cases}$

15. 设$f(x,y)\in C^2(\mathbb{R}^2)$满足$f_x'(x,y)=f_y'(x,y)$，且$f(x,0)>0.$

试证明：对任意的$(x,y)\in\mathbb{R}^2$，有$f(x,y)>0.$

证明：令$\vec{l}=(1,-1)$. 则对任意的$(x,y)\in\mathbb{R}^2$，因为$f_x'(x,y)=f_y'(x,y)$，

所以$\left.\frac{\partial f}{\partial \vec{l}}\right|_{(x,y)}=0$，即函数$f(x,y)$在任意一点沿方向$\vec{l}=(1,-1)$的方向导数为零，故函数$f(x,y)$在该方向$\vec{l}=(1,-1)$上是常数，即在直线$x+y=c$上$f(x,y)$是常数。

对任意的点$(x,y)\in\mathbb{R}^2$，总存在直线$L:x+y=c$使得$(x,y)\in L$，所以

$f(x,y)=f(c,0)>0$. 证毕

16. 设$f(x,y)$在区域$D\subset\mathbb{R}^2$上具有连续的偏导数，$L:\begin{cases}x=x(t)\\ y=y(t)\end{cases}$（$a\le t\le b$）是$D$中的一段曲线，$L$的端点为$A,\ B$. 假设$x,\ y\in C^1[a,b]$且$x'(t)^2+y'(t)^2\neq 0\ \ (\forall t\in[a,b])$.

证明: 若$f(A)=f(B)$, 则存在点$P_0(x_0,y_0)\in L$使得$f_{\vec{l}}'(P_0)=0$，其中$\vec{l}$是曲线L在P_0的单位切向量。

证明: 令$g(t)=f(x(t),y(t)),\ a\le t\le b.$ 不妨设$A,\ B$分别对应着$t=a,\ t=b.$ 则由条件可知$g(t)$可导，且$g(a)=g(b).$ 由洛尔定理，存在$\mu\in(a,b)$使得$g'(\mu)=0.$ 故

$$g'(\mu)=f_x'(x(\mu),y(\mu))x'(\mu)+f_y'(x(\mu),y(\mu))y'(\mu)=0\,.$$

取 $x_0 = x(\mu),\ y_0 = y(\mu)$. 则 $P_0(x_0, y_0) \in L$. 令 $\vec{l} = \dfrac{(x'(\mu), y'(\mu))}{\sqrt{x'(\mu)^2 + y'(\mu)^2}}$，所以

$$\left.\frac{\partial f}{\partial \vec{l}}\right|_{P_0} = f_x'(x(\mu), y(\mu))\frac{x'(\mu)}{\sqrt{x'(\mu)^2 + y'(\mu)^2}} + f_y'(x(\mu), y(\mu))\frac{y'(\mu)}{\sqrt{x'(\mu)^2 + y'(\mu)^2}} = 0.$$ 证毕

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以下供学有余力的同学参考。

17. 若 $f_x'(x,y),\ f_y'(x,y)$ 在 $P_0(x_0,y_0)$ 的邻域内存在，且在 $P_0(x_0,y_0)$ 点可微，则 $f_{xy}''(P_0) = f_{yx}''(P_0)$.

证明：任意固定充分小的 $h \neq 0$，令 $\varphi(x) = f(x, y_0 + h) - f(x, y_0)$ 且

$$g(h) = f(x_0 + h, y_0 + h) - f(x_0 + h, y_0) - f(x_0, y_0 + h) + f(x_0, y_0).$$

因为 $f_x'(x,y)$ 在 $P_0(x_0,y_0)$ 的邻域内存在，故 $\varphi(x)$ 可导且

$$\varphi'(x) = f_x'(x, y_0 + h) - f_x'(x, y_0).$$

由一元函数的微分中值定理，存在 $0 < \theta_1 < 1$ 使得

$$\begin{aligned} g(h) &= \varphi(x_0 + h) - \varphi(x_0) = \varphi'(x_0 + \theta_1 h)h \\ &= \left(f_x'(x_0 + \theta_1 h, y_0 + h) - f_x'(x_0 + \theta_1 h, y_0) \right) h. \end{aligned}$$

又因为 $f_x'(x,y)$ 在 $P_0(x_0,y_0)$ 可微，故当 $h \to 0$ 时，

$$f_x'(x_0 + \theta_1 h, y_0 + h) = f_x'(x_0, y_0) + f_{xx}''(x_0, y_0)\theta_1 h + f_{xy}''(x_0, y_0)h + o(h),$$

且 $f_x'(x_0 + \theta_1 h, y_0) = f_x'(x_0, y_0) + f_{xx}''(x_0, y_0)\theta_1 h + o(h)$，从而

$$g(h) = f_{xy}''(x_0, y_0)h^2 + o(h^2) \tag{5}$$

任意固定充分小的 $h \neq 0$，令 $\psi(y) = f(x_0 + h, y) - f(x_0, y)$. 则 $\psi(y)$ 可导，且存在 $0 < \theta_2 < 1$ 使得

$$\begin{aligned} g(h) &= \psi(y_0 + h) - \psi(y_0) = \psi'(y_0 + \theta_2 h)h \\ &= \left(f_y'(x_0 + h, y_0 + \theta_2 h) - f_y'(x_0, y_0 + \theta_2 h) \right) h. \end{aligned}$$

类似地，由 $f_y'(x,y)$ 在 $P_0(x_0,y_0)$ 可微，得到

$$g(h) = f''_{yx}(x_0, y_0)h^2 + o(h^2) \quad (h \to 0) \tag{6}$$

由(5)式与(6)式，得 $f''_{xy}(P_0) = \lim\limits_{h\to 0}\dfrac{g(h)}{h^2} = f''_{yx}(P_0)$. 证毕

18. 设 $f'_x(x,y)$, $f'_y(x,y)$ 在 $P_0(x_0,y_0)$ 的邻域内存在, 且 f''_{xy} 在 $P_0(x_0,y_0)$ 连续, 则 $f''_{yx}(P_0)$ 存在且 $f''_{xy}(P_0) = f''_{yx}(P_0)$.

证明：对任意固定的充分小的 $\Delta y \neq 0$，令

$$\varphi(x) = f(x, y_0+\Delta y) - f(x, y_0).$$

则 $\varphi(x)$ 可导，且

$$\begin{aligned}\Delta &= \varphi(x_0+\Delta x) - \varphi(x_0)\\ &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0) - f(x_0, y_0+\Delta y) + f(x_0, y_0).\end{aligned}$$

由一元函数的微分中值定理，存在 $0<\theta_1<1$ 使得

$$\Delta = \varphi'(x_0+\theta_1\Delta x) = \left(f'_x(x_0+\theta_1\Delta x, y_0+\Delta y) - f'_x(x_0+\theta_1\Delta x, y_0)\right)\Delta x.$$

再一次应用一元函数的微分中值定理，存在 $0<\theta_2<1$ 使得

$$\Delta = f''_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)\Delta x\Delta y.$$

这样

$$\begin{aligned}&\frac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0)}{\Delta y} - \frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}\\ &= f''_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)\Delta x,\end{aligned}$$

从而两边对 $\Delta y \to 0$ 时取极限，得

$$f'_y(x_0+\Delta x, y_0) - f'_y(x_0, y_0) = f''_{xy}(x_0+\theta_1\Delta x, y_0)\Delta x.$$

故 $\lim\limits_{\Delta x\to 0}\dfrac{f'_y(x_0+\Delta x, y_0) - f'_y(x_0, y_0)}{\Delta x} = \lim\limits_{\Delta x\to 0} f''_{xy}(x_0+\theta_1\Delta x, y_0) = f''_{xy}(x_0, y_0)$.

所以 $f''_{yx}(x_0, y_0)$ 存在且 $f''_{yx}(x_0, y_0) = f''_{xy}(x_0, y_0)$. 证毕