MPS

Today we will discuss something about MPS.

Basic definition of Matrix Product State(MPS)

Matrix product states(MPS) provide a kind of powerful trial states which can represent ground state of 1D quantum system faithfully. Given a 1D quantum system with N sites, at each site, the local state can be described by vectors in a d dimensional Hilbert space $\mathcal{H}_d$. Hence, the total Hilbert space of the system is $\mathcal{H} = \bigoplus_{i=1}^{N} \mathcal{H}_d$. A general state can be expressed as

$$|\psi\rangle = \sum_{i_1 i_2 \cdots i_N} a_{i_1 i_2 \cdots i_N} |i_1 i_2 \cdots i_N\rangle$$

Regarding $a_{i_1 i_2 \cdots i_N}$ as an $d \times d^{N-1}$ matrix $a_{i_1, i_2 \cdots i_N}$, we can do SVD

$$a_{i_1, i_2 \cdots i_N} = U_{i_1 a_1} D_{a_1 a_2} V^{\dagger}_{a_2, i_2 \cdots i_N}$$

Denote $U_{i_1 a_1}$ as $A^{i_1}_{a_1}$, and we can do SVD continuously like $D_{a_1 a_2} V^{\dagger}_{a_2, i_2 \cdots i_N} = a'_{a_1 i_2, i_3 \cdots i_N} = U'_{a_1 i_2, a_2} D'_{a_2 a_3} V'^{\dagger}_{a_3, i_3 \cdots i_N}$. In a similiar way, we denote $U'_{a_1 i_2, a_2} = A^{i_2}_{a_1 a_2}$. Repeating this procedure, we will obtain a compact representation for the wavefunction.

$$|\psi\rangle = \sum_{i_1 i_2 \cdots i_N} A^{i_1} A^{i_2} \cdots A^{i_N} |i_1 i_2 \cdots i_N\rangle$$

This decomposition can be regarded as a kind of chain decomposition. For PBC system, we shall need a kind of ring decomposition, in the form

$$|\psi\rangle = \sum_{i_1 i_2 \cdots i_N} Tr(A^{i_1} A^{i_2} \cdots A^{i_N}) |i_1 i_2 \cdots i_N\rangle$$

MPS form provides a construction way from local properties to global wavefunction.

Properties of MPS

norm and physical observables

The norm of MPS can be calculated as

$$\begin{aligned}
\langle \psi | \psi \rangle &= \sum_{i_1, \cdots, i_2} Tr((A^{i_1})^* (A^{i_2})^* \cdots (A^{i_n})^*) Tr(A^{i_1} A^{i_2} \cdots A^{i_n}) \\
&= \sum_{i_1, i_2, \cdots, i_n} (A^{i_1}_{a_1 a_2} (A^{i_1}_{b_1 b_2})^*) (A^{i_2}_{a_2 a_3} (A^{i_2}_{b_2 b_3})^*) \cdots (A^{i_n}_{a_n a_1} (A^{i_n}_{b_n b_1})^*) \\
&= Tr(E^n)
\end{aligned}$$

Here E is called transfer matrix(or double tensor) which is a useful quantity when dealing with MPS.

$$E_{a_1 b_1, a_2 b_2} = \sum_i A^i_{a_1 a_2} (A^i_{b_1 b_2})^*$$

If we combine a_1 with b_1 as row index and combine a_2 with b_2 as column index, we can write E in a matrix form,

$$E = \sum_i A^i \otimes (A^i)^*$$

and can be represented as in Fig.

The expectation value of local observable $\hat{O}_i$ is given by

$$\langle \hat{O}_l \rangle = \frac{\langle \psi | \hat{O}_l | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{Tr(E^{l-1} \sum_{i,j} (O_l^{ij} A^i \otimes A^j) E^{n-l})}{Tr(E^n)} = \frac{Tr(E^{l-1} E[\hat{O}_l] E^{n-l})}{Tr(E^n)}$$

Finite correlation length

Consider two sites i, j, the correlation function between two local operators $\hat{O}_i$ and $\hat{O}_j$ is

$$\langle \hat{O}_i \hat{O}_j \rangle - \langle \hat{O}_i \rangle \langle \hat{O}_j \rangle = \frac{Tr(E^{i-1} E[\hat{O}_i] E^{j-i} E[\hat{O}_j] E^{N-j})}{Tr(E^N)} - \frac{Tr(E^{i-1} E[\hat{O}_i] E^{N-i}) Tr(E^{j-1} E[\hat{O}_j] E^{N-j})}{Tr^2(E^N)}$$

We can do decomposition $E = \sum_i \lambda_i \hat{P}_i$, in the thermodynamic limit($N \to \infty$), we only keep the part corresponding to the largest eigenvalue in terms like E^N, we denote the projector as $\hat{P}_1$. Then, we have

$$\begin{aligned}
\langle \hat{O}_i \hat{O}_j \rangle - \langle \hat{O}_i \rangle \langle \hat{O}_j \rangle &= \frac{Tr(\hat{P}_1 E[\hat{O}_i] \sum_k (\lambda_k \hat{P}_k)^{j-i} E[\hat{O}_j])}{Tr(\hat{P}_1)} - \frac{Tr(\hat{P}_1 E[\hat{O}_i]) Tr(\hat{P}_1 E[\hat{O}_j])}{Tr^2(\hat{P}_1)} \\
&= \frac{Tr(\hat{P}_1 E[\hat{O}_i] \hat{P}_1 E[\hat{O}_j])}{Tr(\hat{P}_1)} - \frac{Tr(\hat{P}_1 E[\hat{O}_i]) Tr(\hat{P}_1 E[\hat{O}_j])}{Tr^2(\hat{P}_1)} + \frac{Tr(\hat{P}_1 E[\hat{O}_i] \sum_{k \neq 1} (\lambda_k \hat{P}_k)^{j-i} E[\hat{O}_j])}{Tr(\hat{P}_1)}
\end{aligned}$$

For convenience, we have set the largest eigenvalue of E to be 1.

If $\hat{P}_1$ is rank-1, then in the above expression, the first term minus the second term equals to 0. And the third term decays as λ^{j-i}, here $\lambda < 1$ thus the correlator decays to zero exponentially. On the contrary, if $rank(\hat{P}_1) > 1$, then in the above expression, the first term does not equal to the second term. As a result, besides the exponentially decaying term, there is also constant correlation not decaying which corresponding to infinite correlation length.

So **a MPS has finite correlation length if and only if the largest eigenvalue of its transfer matrix is non-degenerate**. **Moreover, we can only get expentially decaying or constant correlators for any MPS. It is not possible to obtain power law decaying correlation from MPS with finite bond dimension D. And this is the reason MPS can not simulate critical system successfully.**

Low entanglement

Suppose we split the whole system into two parts L and R, then like we discussed in the basic definition of MPS, we can use SVD to decompose the coefficients. For state like $|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} a_{i_1, i_2, \cdots, i_n} |i_1 i_2 \cdots i_n\rangle$, we can have

$$a_{i_1 i_2 \cdots i_n} = a_{i_1 i_2 \ldots i_l, i_{l+1} \ldots i_n} = U_{i_1 i_2 \ldots i_l, i} \lambda_i V^\dagger_{i, i_{l+1} \ldots i_n}$$

which means we can write the state in a form like $|\psi\rangle = \sum_i \lambda_i |L_i\rangle \otimes |R_i\rangle$. This decomposition is called Schmit decomposition and λ_i is called Schmit weight.

The $\alpha-$th Renyi entropy is defined as

$$S_\alpha = \frac{1}{1-\alpha} \ln Tr(\hat{\rho}^\alpha)$$

So we see the entanglement entropy of subsystem L(or R) is determined by the Schmit weights. Directly, when we set $\alpha = 0$, we obtain

$$S_0 = \ln(rank(\hat{\rho})) = \ln(\#\lambda_i)$$

This entropy is also called Hartley entropy, and one can prove $S_0 \geq S_\alpha$, for any $\alpha \geq 0$.

Obviously this quantity is upper bound by the bond dimension of a MPS. As a result, we have the minimal cut rule: the entanglement entropy between any segment and rest of the chain is upper bounded by $S_{cut} \ln D$

$$S_\alpha \leq S_{cut} \ln D$$

here S_{cut} denotes the minimal number of bonds that the curve which split the two parts passes through and D denotes the bond dimension.

When D is a constant, the entanglement entropy will be $O(1)$. In that case, the MPS satisfies the area law in 1D and is a kind of low entanglement state.

Gauge freedom

A point need to note is that there are gauge freedom in a MPS. We can always insert an arbitrary unitary $\hat{M}$ into a MPS, like

$$|\psi\rangle = \sum_{i_1 i_2 \ldots i_n} Tr(A^{i_1} A^{i_2} \cdots A^{i_n}) \, |i_1 i_2 \cdots i_n\rangle = \sum_{i_1 i_2 \ldots i_n} Tr(M A^{i_1} M^{-1} M A^{i_2} M^{-1} \cdots M A^{i_n} M^{-1}) \, |i_1 \cdots i_n\rangle$$

we can pose some conditions to partially fix the gauge. Two usual choice are left-normalized($\sum_\sigma (A^\sigma)^\dagger A^\sigma = I$) and right-normalized($\sum_\sigma B^\sigma (B^\sigma)^\dagger = I$).

Example

We illustrate two examples. One is GHZ state, another is AKLT state.

▼ Example 1: GHZ state

For an N-spin system, GHZ state is

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$$

To cast this state to MPS form, we can choose matrix at each site as

$$A^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; A^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Another always talked example wihch can be constructed in MPS form is the ground state of AKLT model.

▼ Example 2: ground state of AKLT model

AKLT model is proposed by Affleck, Kennedy, Lieb and Tasaki. For an 1D spin-1 chain, consider projectors $\hat{P}_{i,i+1}$ which project neighbor spins to the subspace where total spin is 2. It can be constructed as

$$\hat{P}_{i,i+1} = \frac{1}{2}\hat{S}_i \cdot \hat{S}_{i+1} + \frac{1}{6}(\hat{S}_i \cdot \hat{S}_{i+1})^2 + \frac{1}{3}$$

AKLT model considers the Hamiltonian which is the sum of all $\hat{P}_{i,i+1}$

$$\hat{H} = \sum_i \hat{P}_{i,i+1} = \sum_i \frac{1}{2}\hat{S}_i \cdot \hat{S}_{i+1} + \frac{1}{6}(\hat{S}_i \cdot \hat{S}_{i+1})^2 + \frac{1}{3}$$

Then the ground state is the common null space of $\hat{P}_{i,i+1}$. What surprises is this state can be constructed in MPS form. A spin-1 can be regarded as a triplet composed by a pair of spin-1/2. So we consider $2N$ virtual spin-1/2, and project pairs $\{(2n+1, 2n+2) | 1 \le n \le N-1\}$ to triplet subspace(total spin equals 1).

$$\begin{aligned} |1,-1\rangle &= |\downarrow\downarrow\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) \\ |1,1\rangle &= |\uparrow\uparrow\rangle \end{aligned}$$

For the bonds which connect pair $\{(2n, 2n+1)|1 \leq n \leq N-1\}$, we project every pair of spin-1/2 to singlet subspace(total spin equals 0).

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

It can be showed that in such construction, for every spin-1 pair $(\hat{S}_i, \hat{S}_{i+1})$(virtual spin $(2i-1, 2i, 2i+1, 2i+2)$), they are ruled out the subspace where total spin is 2. Hence this state is in the common null space of $\hat{P}_{i,i+1}$, so it is the ground state of $\hat{H}$. We use PBC here. First, project pairs $(2i, 2i+1)$ to singlet space. The corresponding state is

$$\sum_{\mathbf{a},\mathbf{b}} M_{a_1 b_1} M_{a_2 b_2} \cdots M_{a_{N-1} b_{N-1}} M_{a_N b_0} \, |b_0 a_1 b_1 \cdots a_{N-1} b_{N-1} a_N\rangle$$

where

$$M = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Then project pairs $(2i, 2i+1)$ to triplet space. The projector is

$$\hat{P} = \sum_{\sigma} \sum_{\mathbf{a},\mathbf{b}} P^{\sigma_1}_{b_0 a_1} P^{\sigma_2}_{b_1 a_2} \cdots P^{\sigma_N}_{b_{N-1} a_N} \, |\sigma\rangle \langle b_0 a_1 b_1 \cdots a_{N-1} b_{N-1} a_N|$$

where

$$P^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; P^{0} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \; P^{+1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Finally, the AKLT state can be written as

$$\begin{aligned} |AKLT\rangle &= \sum_{\sigma} \sum_{\mathbf{a},\mathbf{b}} P^{\sigma_1}_{b_0 a_1} M_{a_1 b_1} P^{\sigma_2}_{b_1 a_2} M_{a_2 b_2} \cdots P^{\sigma_N}_{b_{N-1} a_N} M_{a_N b_0} \, |\sigma\rangle \\ &= \sum_{\sigma} Tr(A^{\sigma_1} A^{\sigma_2} \cdots A^{\sigma_N}) \, |\sigma\rangle \end{aligned}$$

where

$$A^{-1} = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \; A^{0} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \; A^{+1} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

Furthermore, we rescale these matrixs to make such matrix be left-normalized$(\sum_{\sigma} (A^{\sigma})^{\dagger} A^{\sigma} = I)$.

$$A^{-1} = \begin{pmatrix} 0 & 0 \\ -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}, \; A^{0} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \; A^{+1} = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 \end{pmatrix}$$

We now calculate the correlators of the AKLT state.

$$\begin{aligned} \langle \hat{S}^z_i \hat{S}^z_j \rangle &= \langle AKLT| \hat{S}^z_i \hat{S}^z_j |AKLT\rangle \\ &= \sum_{\sigma} \sum_{\sigma'_i \sigma'_j} Tr(A^{\sigma_1} A^{\sigma_2} \cdots A^{\sigma_N})^* Tr(A^{\sigma_1} \cdots A^{\sigma'_i} \cdots A^{\sigma'_j} \cdots A^{\sigma_N}) \, \langle \sigma_i | \hat{S}^z_i | \sigma'_i \rangle \, \langle \sigma_j | \hat{S}^z_j | \sigma'_j \rangle \\ &= \sum_{\sigma} \sum_{\sigma'_i \sigma'_j} Tr(((A^{\sigma_1})^* \otimes A^{\sigma_1})((A^{\sigma_2})^* \otimes A^{\sigma_2}) \cdots (S^z_{\sigma_i \sigma'_i} (A^{\sigma_i})^* \otimes A^{\sigma'_i})) \cdots (S^z_{\sigma_j \sigma'_j} (A^{\sigma_j})^* \otimes A^{\sigma'_j})) \cdots ((A^{\sigma_N})^* \otimes \\ &= Tr(E^{i-1} E_i E^{j-i-1} E_j E^{N-j}) \end{aligned}$$

where

$$E = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix},\ E_i = E_j = \begin{pmatrix} 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2}{3} & 0 & 0 & 0 \end{pmatrix}$$

After calculating, we obtain $\langle \hat{S}_i^z \hat{S}_j^z \rangle = \frac{4}{27}[(-\frac{1}{3})^{j-i} + (-\frac{1}{3})^{N-j+i}]$. In thermodynamic limit$(N \to \infty)$, $\langle \hat{S}_i^z \hat{S}_j^z \rangle \propto (-\frac{1}{3})^{j-i}$ corresponding to exponential decaying correlation.

In the construction of AKLT state, what we do is essentially doing projection from a large state space of virtual spin-1/2 pairs connected by singlet(maximally entanglement state) onto a smaller physical subspace(spin-1).

MPS and local gapped Hamiltonian

From above discussion, we find for an MBS under PBC, whatever segment C we choose, the rank of the reduced density matrix is less than or equal D^2, here D denotes the bond dimension. But the dimension of the Hilbert space corresponds to segment C is d^{L_c}. If we select a segment large enough so that $d^{L_c} > D^2$, then the kernel of the reduced density matrix $\hat{\rho}_c$ is not empty. We can construct projectors $\hat{P}_C$ onto the kernel of $\hat{\rho}_c$. As a result, for Hamiltonian

$$\hat{H} = \sum_C \hat{P}_C$$

we must have $\hat{H}\left|\psi\right\rangle = 0$. Since the Hamiltonian is non-negative, $\left|\psi\right\rangle$ must be a ground state of $\hat{H}$. $\hat{H}$ is called the parent Hamiltonian of $\left|\psi\right\rangle$.

We illustrate this process using the example of AKLT state. In the case of AKLT state, the bond dimension $D = 2$, and physics dimension $d = 3$. So we need to select at least two sites as the block C. In the block $(i, i+1)$, the corresponding matrix element is

$$A^{-1}A^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ A^{0}A^{-1} = \begin{pmatrix} 0 & 0 \\ -\frac{\sqrt{2}}{3} & 0 \end{pmatrix},\ A^{+1}A^{-1} = \begin{pmatrix} -\frac{2}{3} & 0 \\ 0 & 0 \end{pmatrix}$$
$$A^{-1}A^{0} = \begin{pmatrix} 0 & 0 \\ \frac{\sqrt{2}}{3} & 0 \end{pmatrix},\ A^{0}A^{0} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix},\ A^{+1}A^{0} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{3} \\ 0 & 0 \end{pmatrix}$$
$$A^{-1}A^{+1} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2}{3} \end{pmatrix},\ A^{0}A^{+1} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{3} \\ 0 & 0 \end{pmatrix},\ A^{+1}A^{+1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

To find projectors which nullify the reduced density matrix of the block is equivalent to find the null space of the below 4×9 matrix

$$\begin{pmatrix} 0 & 0 & -\frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{3} & 0 & -\frac{\sqrt{2}}{3} & 0 \\ 0 & -\frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & -\frac{2}{3} & 0 & 0 \end{pmatrix}$$

And the corresponding null vector is

$$\begin{aligned} \left|\phi_1\right\rangle &= \left|-1\,-1\right\rangle \\ \left|\phi_2\right\rangle &= \left|11\right\rangle \\ \left|\phi_3\right\rangle &= \frac{1}{\sqrt{2}}(\left|0\,-1\right\rangle + \left|-10\right\rangle) \\ \left|\phi_4\right\rangle &= \frac{1}{\sqrt{2}}(\left|10\right\rangle + \left|01\right\rangle) \\ \left|\phi_5\right\rangle &= \frac{1}{\sqrt{6}}(\left|1\,-1\right\rangle + 2\left|00\right\rangle + \left|-11\right\rangle) \end{aligned}$$

We suprisingly find the null vectors corresponds to the $S=2$ representation of two spin-1. So we find the parent Hamiltonian of AKLT state,

$$\hat{H}_{AKLT}=\sum_i \hat{h}_{i,i+1}=\sum_i \hat{P}^{S=2}_{i,i+1}$$

Now we see, given a MPS, we can always construct a Hamiltonian $\hat{H}$ for which the MPS serves as a ground state of $\hat{H}$. However, is the MPS we have the unique ground state of the constructed Hamiltonian $\hat{H}$? Or say, is the Hamiltonian gapped? The answer is not always. Only when MPS has the property called “injectivity”, the constructed Hamiltonian will be gapped and the MPS will be the unique ground state.

Injectivity

The matrices A^i in the MPS form can be decomposed to a block diagonal form, like

$$A^i=\begin{pmatrix} A^i_1 & & \\ & A^i_2 & \\ & & \cdots \end{pmatrix}$$

Then the matrix product state $|\psi\rangle$ corresponding to A^i can also be decomposed to a superposition of $|\psi_k\rangle$, which corresponds to A^i_k and is orthogonal with each other

$$|\psi\rangle=\sum_k |\psi_k\rangle$$

For each block, the transer matrix is $E_k=\sum_i A^i_k\otimes (A^i_k)^*$. If the largest eigenvalue of E_k is not degenerate, then the corresponding $|\psi_k\rangle$ is short range correlated.

If the cannonical form of an MPS contains only one block and the transfer matrix has only one eigenvalue with largest magnitude, then the MPS is said to be “injective”. In that case, injective MPS has only one component which is short range correlated.

Parent Hamiltonian

Return to our initial problem about the uniqueness of ground state of parent Hamiltonian, one can prove that if the MPS is injective, then the projectors $\hat{P}_C$ impose strong enough constraints such that the constructed parent Hamiltonian has a unique ground state and a finite gap. Notice that, injectivity means the MPS is short range correlated(the largest magnitude eigenvalue of transfer matrix is not degenerate). So we can say, **a parent Hamiltonian can be constructed for a finite dimensional matrix product state with finite correlation length, such that the matrix product state is the unique gapped state of the parent Hamiltonian.** This result relates the injective MPS which has finite correlation length to gapped Hamiltonian.

Wave function renormaliation on MPS

We have saw that injective MPS can be the unique ground state of a gapped Hamiltonian. A natural quesition raised, is there any intrinsic distinguish between these MPS, or say do all the injective MPS belong to the same phase? In the zero temperature, which determines properties of system is the ground state, so the phase of a quantum system is determined by properties of the ground state. We consider a gapped system $\hat{H}(\lambda)$, when we tune the parameters from λ_i to λ_f, if there is no cossover between the ground state and the first excited state, then we can relate the ground state of $\hat{H}(\lambda_i)$ and $\hat{H}(\lambda_f)$ using abiabatic evolution and also there is not phase transition. On the contrary, if there is energy crossover, we can not relate the ground state of $\hat{H}(\lambda_i)$ and $\hat{H}(\lambda_f)$ using abiabatic evolution and there exists phase transition. For **gapped system**, a quantum phase transition can happen only when energy gap closes. Thus we say, if two gapped states $|\phi(0)\rangle$ and $|\phi(1)\rangle$ are in the same phase, then we can always find a family of Hamiltonian $\hat{H}(g)$, such that the energy gap for $\hat{H}(g)$ are finite for all g in $[0,1]$, and $|\phi(0)\rangle$ and $|\phi(1)\rangle$ are ground states of $\hat{H}(0)$ and $\hat{H}(1)$, respectively. That is to say,

$$|\phi(0)\rangle \ |\phi(1)\rangle \text{ , iff } |\phi(1)\rangle = T[e^{-i\int_0^1 dg \hat{H}(g)}]\,|\phi(1)\rangle$$

When the Hamiltonian $\hat{H}(g)$ is local, the unitary evolution is also local. Thus, states related by a local unitary evolution (which is equivalent to local quantum circuits) belong to the same phase. Short range entanglement can always be erased by local unitary

evolution, so whether two states are in different phase depends on the existence of long range entanglement.

Now we want to investigate the existence of long range entanglement in MPS. To see this, first we need to remove short range entanglement between near sites. This can be done by the wave function renormalization. Starting from the transfer matrix $E = \sum_i A^i \otimes (A^i)^*$, we consider two sites, the corresponding transfer matrix is

$$E^2_{a_i b_i, a_{i+2} b_{i+2}} = \sum_{a_{i+1}, b_{i+1}} E_{a_i b_i, a_{i+1} b_{i+1}} E_{a_{i+1} b_{i+1}, a_{i+2} b_{i+2}}$$

if we combine index a_i with a_{i+2} and b_i with b_{i+2}, then E^2 is a non-negtive Hermitian matrix, we can do eigenvalue decompose $E^2_{a_i b_i, a_{i+2} b_{i+2}} = \sum_k \lambda_k V^k_{a_i a_{i+2}} (V^k_{b_i b_{i+2}})^*$, then we can set matrix for the two-site block to preserve E^2 (it is obviously, that if two MPS have the same tansfer matrix then the two states can be mapped to each other by unitary transformations on each physical index),

$$\tilde{A}^k_{a_i a_{i+2}} = \sqrt{\lambda_k} V^k_{a_i a_{i+2}}$$

Now we have, $E^2 = \sum_k \tilde{A}^k \otimes (\tilde{A}^k)^*$. That is to say, now E^2 is independent of the local entanglement between the two sites, and we can erase them by unitary transformation $(A^i A^j)_{ab} = (\tilde{A}^k)_{ab}$, here $k = (i, j)$.

This finishes one step of the wavefunction renormalization. Continue this process, since we are dealing with injective MPS, the transfer matrix has only one eigenvalue with the largest magnitude(for convenience, we set to 1). So we have

$$E_{\alpha\gamma, \beta\chi} = \Lambda^l_{\alpha\gamma} (\Lambda^r_{\beta\gamma})^* + O(|\lambda|)$$

here $|\lambda| < 1$. So the fixed point of $E^{(\infty)}$ is

$$E^{(\infty)}_{\alpha\gamma, \beta\chi} = \Lambda^l_{\alpha\gamma} (\Lambda^r_{\beta\gamma})^*$$

Λ^l and Λ^r can also be decomposed

$$\Lambda^l_{\alpha\gamma} = \sum_i \lambda^i v^i_\alpha (v^i_\gamma)^*,\ \Lambda^r_{\beta\chi} = \sum_i \eta^j u^j_\beta (u^j_\chi)^*$$

So we get the fixed points of $(A^{(\infty)})^{i,j}_{\alpha,\beta} = \sqrt{\lambda^i \eta^j} v^i_\alpha u^j_\beta$, the corresponding state is

$$|\phi^{(\infty)}\rangle = \bigotimes_k (\sum_{i,j} \sqrt{\lambda_i \eta_j} \sum_\alpha v^i_\alpha u^j_\beta \, |ij\rangle)_{k,k+1}$$

We can also implement a local unitary transformation to disentanglize the sites $(k, k+1)$. Thus, we obtain a suprising result, all injective MPS can be transformed to product state using local unitary transformation. And hence, all gapped 1D bosonic systems belong to the same phase if no symmetry is required.