Quantum Algorithm for Supersingular Isogeny Problem

Against Post-Quantum Crypto Protocol

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Elliptic curves

Definition

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If $char(k) \neq 2, 3$, every elliptic curve in affine space $\mathbb{A}^2$can be writen in the form:

$$y^2 = x^3 + ax + b$$

with$a, b \in k$
It can also be written in projective coordinates in projective space $\mathbb{P}^2$:

$$y^2 z = x^3 + axz^2 + bz^3$$

$(0 : 1 : 0)$ denotes the infinite point.

Group Law

Theorem

For $P, Q \in E(k)$, the line $\overline{PQ}$ intersects E in a rational point, because E is a cubic curve, $\#(\overline{PQ} \cap E(k)) = 3$.

We define a group operation $+$, for$P, Q, R \in E(k)$,
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Any elliptic curve over $\mathbb{C}$ is isomorphic by $\wp$-function to a torus $\mathbb{C}/L$, so the additive structure is natural.

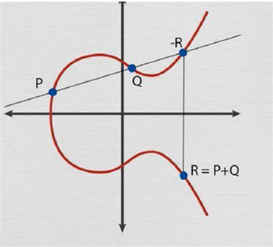

Figure 1: Group Law

Diffie-Hellman

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- Diffie-Hellman performs better than RSA. 256-bit Diffie-Hellman's security is better than 2048-bit RSA. (Its construction is more complicated than RSA)

However, it can be attacked by quantum computer in $poly(log(p))$ time using Shor's algorithm.

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For curves E, E' over k, an isogeny:

$$\psi : E(\bar{k}) \longrightarrow E'(\bar{k})$$

$\deg(\psi)$:=$|\ker(\psi)|$.
Every isogeny has its dual isogeny:

$$\hat{\psi} : E' \longrightarrow E, \ s.t.$$

$$\mathrm{Id}_E = \hat{\psi} \circ \psi : E \longrightarrow E$$

Over finite field

(Hesse Theorem) The points of elliptic curves E over finite field $\mathbb{F}_q$, char($\mathbb{F}_q$)=p:

$$\#E(\mathbb{F}_q) = q - 1 + t, \text{ where } |t| \leq 2\sqrt{q}$$

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$$\#E(\mathbb{F}_q) = q - 1 + t, \text{ where } |t| \leq 2\sqrt{q}$$

- E is ordinary if $p \not| t \;\Leftrightarrow \#E[p] = p$
- E is supersingular if $p|t \;\Leftrightarrow E[p] = \{\mathbf{0}_E\}$

Ordinary curves only isogeneous to ordinary curves, so do supersingular curves.

$$E, E' \text{ over } \mathbb{F}_q \text{ are isogeneous } \Leftrightarrow \#E(\mathbb{F}_q) = \#E'(\mathbb{F}_q)$$

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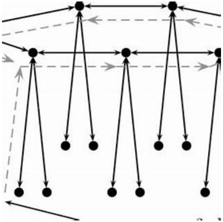
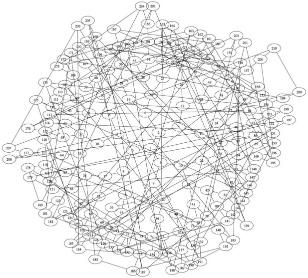


(a) ordinary



(b) supersingular

Figure 2: Isogeny graph

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Expander graph

Expander graph can be used to construct pseudorandom sequence due to its mixing property, which means random walk on it can converge to uniform distribution rapidly in just $O(log(n))$ steps.

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Hence, searching and finding path in an expander is a hard problem!

Ramanujan graph

Supersingular isogeny graph is the optimal expander Ramanujan Graph.
We assume the location has no regulation.
Because j-invariants of supersingular curves all lie in $\mathbb{F}_{p^2}$, we can consider $G_l(\mathbb{F}_{p^2})$. The graph is completely connected, so constructing isogeny can be reduced to finding path on the graph.

Ramanujan graph

Denote the set of all j-invariants in the graph as S_{p^2}, the j-invariants in $\mathbb{F}_p$ as S_p

$$\#S_{p^2} = \lfloor \frac{p}{12} \rfloor + \begin{cases} 0 & \text{if } p \equiv 1 \quad mod\ 12 \\ 1 & \text{if } p \equiv 5, 7 \quad mod\ 12 \\ 2 & \text{if } p \equiv 11 \quad mod\ 12 \end{cases}$$

$$\#S_p = \begin{cases} \frac{h(-4p)}{2} & \text{if } p \equiv 1\ mod\ 4 \\ h(-p) & \text{if } p \equiv 7\ mod\ 8 \\ 2h(-p) & \text{if } p \equiv 3\ mod\ 8 \end{cases}$$

where $h(d)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, $h(d) \in \tilde{O}(\sqrt{d})$

Quantum toolkits

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- It can solve isogeny problem whose endomorphism ring can be embedded to a imaginary quadratic field in subexponential time, while classical case is exponential time. e.g. ordinary isogeny problem and $\mathrm{End}_{\mathbb{F}_p}(E)$, E is defined over $\mathbb{F}_{p^2}$.

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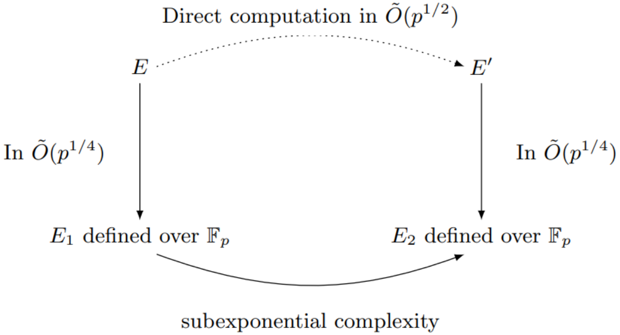
- For general searching problem on a graph, quantum walk can provide quadratic speedup.

Existing algorithm

- One algorithm to constructing isogeny is to construct isogeny to the curves whose j-invariant in $\mathbb{F}_p$, making use of the $\mathrm{End}_{\mathbb{F}_p}(E)$. Its time complexity is $\tilde{O}(p^{1/4})$
- To construct isogeny between E_1, E_2, we can implement quantum "meet in the middle", its complexity is $\tilde{O}(p^{1/6})$.



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Can we make improvement?

To make improvement to the existing algorithm, we need to make full use of the property of the isogeny problem instead of just taking the problem as an unordered searching problem. Then we need to exploring the detailed information.

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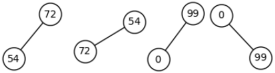
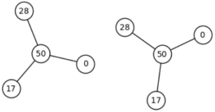
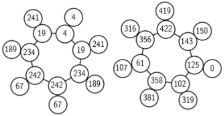
Considering the 2-isogeny graph, the distribution of S_p depends on p.



(a) $p \equiv 1 \bmod 4$



(b) $p \equiv 3 \bmod 8$



(c) $p \equiv 7 \bmod 8$

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The relationship can be got from localizing the endomorphism ring and considering the isomorphism classes of lattice.
We begin to consider isogeny problem in this view instead of isogeny graph.
I think it will provide more information about the isogeny relationship.

Thank you!