Entanglement Area Law

Quantum Complexity and Theoretical Physics

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Last week we discussed about local Hamiltonians and the complexity in finding the low-energy spectrum for them. The concept of k-local Hamiltonian has captured the "locality" exhibit in realistic quantum many-body system, yet physical systems are actually more constrained beyond locality. In this lecture we will explore one of the most interesting consequences from these properties – the area law for entanglement entropy.

1 Introduction to Area Law

1.1 Basic Concepts

Physical systems generally live in spaces of certain dimensions, with degrees of freedom (D.O.F) associated with spatial sites (discrete or continuum). The "locality" in this case not only means that local Hamiltonian involves only small number of D.O.F, but also requires that they are spatially close to each other.

- Examples that happens in nature: grids, glassy networks
- Examples that cannot happens in nature: complete graphs, expander graph, …

For macroscopic (or most mesoscopic) system with size $N \sim 10^{23}$, it is impossible to know the details of the system. Luckily, we only care about macroscopic measurables which do not relies on microscopic details of the system. In particular, we can take the $N \to \infty$ limit, called the **thermodynamic limit**, in theoretical study. In most cases, the thermodynamic limit allows one to ignore thermal fluctuations and boundary effect. The thermodynamic limit also allows system with analytic dynamics exhibit non-analytic behaviors in its macroscopic observables, which are phase transitions we see in experiments.

1.2 Entanglement Entropy vs. Locality

In studying low-energy states of local Hamiltonian, the entanglement properties are good indicators of how "complicated" these states are. In particular, one can choose some bipartitions (A, B) of the system and calculate the entanglement entropy:

$$E(A, B) \coloneqq S(\rho_A) = S(\rho_B)$$

For physical system under the thermodynamic limit, one case we are interested in is that A is a local region with $|A| \to \infty$ under the limit. For the most "general" quantum states one can imagine, it is expected to see that:

$$E(A, B) \sim |A|,$$

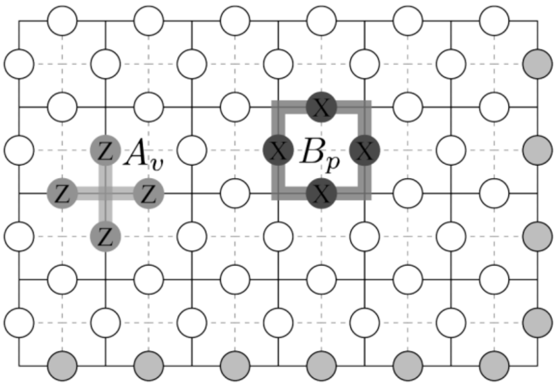
since any sites in A could have entanglement with some sites outside. However, it has been found that for a large class of physical systems, the entanglement entropy over a ground state behaves as:

$$E(A,B) \sim |\partial A|,$$

that is, the entanglement entropy is roughly proportional to the surface area of the region A. Entanglement entropy that scale as surface area are said to obey the **area law**.

The area law shows that for many systems, the locality in interaction also implies some locality the entanglement of the ground states. In particular, entanglement between A and the rest part B is strong only near the boundary, where sites can directly feel the influence from another part.

1.3 Some Examples



Example 1. (toric code) The toric code system is a system of 2D square lattice on a torus. The D.O.F are qubits sitting on edges. We interpret state $|1\rangle$ as connected while $|0\rangle$ as disconnected. There are two groups of operators needed to construct the Hamiltonian:

1. The vertex operator: $A_v = \prod_{j \in \text{edge}(v)} Z_j$, the eigenvalues of A_v is 1 for even number of edges connecting the vertex and -1 for odd.

2. The plaquette operator: $B_p = \prod_{j \in \text{plaquette}(p)} X_j$, B_p flips all the edges around a plaquette while keeping the parity of the numbers of edges connected to every vertex.

The Hamiltonian of the toric code is the sum over all these operators:

$$H = -\sum_v A_v - \sum_p B_p$$

Since all A_v and B_p commute with each other, toric code has a frustration-free Hamiltonian, and the eigenvectors of H can be determined from the simultaneous eigenvectors of all these operators. For the ground state, lowering A_v requires that all vertices are connected even numbers of edges, or equivalently, the ground state is a superposition of close loops. At the same time, lowering B_p

requires that the ground state be symmetric over string configurations that are equivalent under all B_p's actions. Notice that B_p actions correspond to local deformations, so the only choices left is the number of locally inequivalent loop configurations, which is just loops winding around the torus in either direction. Thus, the ground state is four-fold degenerate (either loop may exist or not).

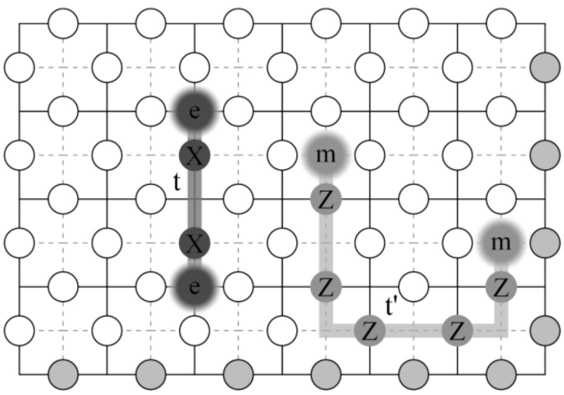
The entanglement entropy of such ground state is easy to compute. For any subregion A and exterior B. We denote the set of vertices that have connection to both edges in A and in B as S. Given any loop configuration, we can record the parity of the numbers of edges in A that is connected to a given vertex in S as $q: S \to \{0,1\}$, which must be the same for B since the total number must be even. On the other hand, when B is absent, any such $\{q\}$ with $\sum q$ even can act as boundary condition for constructing configurations in A. Consequently, one among the four ground states $|\psi\rangle$ can be naturally decomposed as:

$$|\psi\rangle \propto \sum_{\substack{q:S\to\{0,1\} \\ \sum q \text{ even}}} |\psi_q^A\rangle \otimes |\psi_q^B\rangle,$$

where $|\psi_q^A\rangle$ and $|\psi_q^B\rangle$ are evenly superposed by all configurations in A and B that satisfy the boundary condition given by q (plus the topological constrain for $|\psi\rangle$). Since there is no overlap configuration between $|\psi_q^A\rangle$ (same for $|\psi_q^B\rangle$) with different q, all $|\psi_q^A\rangle$ are orthogonal to each other and there is no repetition in the summation above, so $|\psi\rangle$ does have the same coefficient for each loop configurations. Consequently, by tracing out region B we get an equal-weight mixing of all $|\psi_q^A\rangle$, yielding the entropy:

$$S = \log 2^{|S|-1} = (|S| - 1)\log 2.$$

For large region A, we can safely interpret $|S|$ as the perimeter of region A, proving that the entanglement entropy of the ground state does satisfy area law.



For future purposes, we also discuss two physical properties in toric code system. The first one about the low-energy excitation states. Since the system is frustration free, excitation states are also eigenstates of all A_v and B_p, but with different eigenvalues. The lowest excitation states can be achieved by applying string operators:

$$\sigma_v(t) = \prod_{j \in t} X_j, \quad \sigma_p(t') = \prod_{j \in t'} Z_j,$$

where t is an open path from vertex to vertex and t' is an open path from plaquette to plaquette. The states created by these operators are the same if the two paths are locally equivalent. All these lowest excited states have energy 2 above the ground state. Excited states of higher energy can be obtained by successively apply these string operators.

The toric code system is an example of so-called **gapped system**, which is defined as follow:

Definition 1. (gapped system) A system with Hamiltonian H_N is gapped iff as $N \to \infty$, there exist $m \sim O(1)$ many lowest energy states with energy separation $\varepsilon \to 0$ among themselves, and the energy separation between them and all other states is lower bounded by $\Delta \sim O(1)$. Otherwise the system is called gapless.

Δ is called the gap of the system, while m is called the ground state degeneracy. For the toric code, we have $\Delta = 2$ and $m = 4$ for arbitrarily large N.

The second one is the **correlation functions** defined as:

$$C(\mathcal{O}_1, \mathcal{O}_2) = \langle \mathcal{O}_1 \mathcal{O}_2 \rangle - \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle,$$

where $\mathcal{O}_1$, $\mathcal{O}_2$ are local operators acting on finite regions R_1, R_2 separated by a distance r, and $\langle \cdot \rangle$ denotes expectation values in the ground state (zero temperature) or the thermal state with finite temperature. For the toric code ground state, the correlation functions are always zero unless the two regions are in contact, so the ground state has zero correlation length.

Example 2. (quasi-free system) The quasi-free systems are vibrational modes on lattices within lowest order approximations (harmonic approximation). The general Hamiltonian for such systems is given by:

$$H = \frac{1}{2} \sum_{i,j} \left(p_i P_{i,j} p_j + x_i M_{i,j} x_j \right).$$

The x_i and p_i are Hermitian operators satisfying canonical commutation / anti-commutation relation (CCR / CAR):

$$x_i p_j \mp x_j p_i = \mathrm{i} \delta_{ij}.$$

Depending on whether the system is bosonic or fermionic, the corresponding single-site Hilbert space is then $L^2(\mathbb{R})$ for bosons or $\mathbb{C}^2$ for fermions, and $P_{i,j}$, $M_{i,j}$ are real, symmetric and non-negative matrices for bosons or real, antisymmetric for fermions.

Here we take bosonic quasi-free system as an example. The procedure of solving such system is to simultaneously diagonalize both $P_{i,j}$, $M_{i,j}$ by changing variables in the (x, p) space while keeping CCRs. On the other hand, without solving the normal modes explicitly, the vacuum wave function can be expressed as:

$$\langle x_1 x_2 \cdots x_N | \Omega \rangle = \sqrt{\frac{\det \Gamma}{\pi^{N/2}}} \exp\left(-\frac{1}{2} \sum_{i,j} \Gamma_{i,j} x_i x_j \right), \quad \Gamma = M^{1/2} \left(M^{1/2} P M^{1/2} \right)^{-1/2} M^{1/2}$$

This form can be used to compute entanglement entropy between a subset A of the sites and the rest system B formally as (see [3])

$$E(A,B) = \mathrm{tr}\left(\frac{C+1}{2} \log \frac{C+1}{2} - \frac{C-1}{2} \log \frac{C-1}{2} \right),$$

$$C := \left(1 - \Gamma_A^{-1/2} \Gamma_{AB} \Gamma_B^{-1} \Gamma_{BA} \Gamma_A^{-1/2} \right)^{-1/2}$$

For the simplest case, we consider a system on a 1-dimensional loop of chain of length N with $P = \mathbb{I}$ and

$$M_{i,j} = a\delta_{i,j} - b\left(\delta_{i,j+1} + \delta_{i+1,j} \right).$$

Obviously H is now a 3-local Hamiltonian with translational symmetry over i. The non-negativity of M requires that $a \geq 2|b|$. The normal frequencies of this quasi-free system are given by:

$$\omega_k^2 = a - 2b \cos \frac{2k\pi}{N}, \qquad -N \leq k < N$$

while the spectrum of H is given by:

$$E = \sum_{k=-N}^{N-1} \left(n_k + \frac{1}{2} \right) \omega_k, \qquad n_k \in \mathbb{N}$$

Even in this simple case, exact values for $E(A,B)$ are hard to obtain and numerical method is needed. Nonetheless, for symmetric bisection ($|A| = N/2$), a clear upper bound has been derived in [4]:

$$E(A,B) \leq \frac{1}{4} \log \frac{a + 2|b|}{a - 2|b|} = \frac{1}{2} \log \frac{\|M\|}{\Delta}$$

The upper limit is a constant, which is expected from area law for an 1D system.

The above expression diverges when the gap Δ approach zero (this happens in systems such as phonons). By detailed analysis one can show that the divergence is due to the $k = 0$ mode (for $b > 0$) which becomes a zero-mode in the $\Delta \to 0$ limit. By removing its contribution $E(A,B)$ approaches constant value as $\Delta \to 0$.

The correlation function behaves as:

$$C(x_i, x_j) = (M^{-1})_{i,j} = \frac{1}{N} \sum_{k=0}^{N-1} \frac{\mathrm{e}^{\frac{2\pi \mathrm{i} k(i-j)}{N}}}{\omega_k^2} \approx \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{\mathrm{i}l(i-j)}}{l^2 + \Delta^2} \mathrm{d}l = \frac{\mathrm{e}^{-\Delta|i-j|}}{2\Delta}, \qquad (1 \ll |i-j| \ll N)$$

We see that correlations decrease exponentially with characteristic length Δ^{-1}, which is called the

correlation length ξ of the system. (In case when $\Delta = 0$, the correlation function becomes ill defined as a consequence of Mermin-Wagner theorem.)

By studying the continuum limit of the quasi-free system in different dimensions, we can see the typical behavior for correlation functions in gapped and gapless system:

1. For gapped system: $C(r) \sim \mathrm{e}^{-r/\xi}$.

2. For gapless system: $C(r) \sim \frac{1}{r^{\alpha}}$ $(\alpha > 0)$

The two examples shown here gives us the following observation:

Observation: the property of being gapped, having finite correlation length may be related to the entanglement area law.

2 Conditions for Area Law and its Correction

2.1 When do Area Law Hold and When Not?

We have observed that systems satisfying the entanglement area law are gapped and having finite correlation length. To begin with, we first claim that gapped systems always have finite correlation length:

Theorem 1. (M. B. Hastings, 2004) The ground state of a gapped system of any dimension has finite correlation length [5].

To understand this result, we should first know that all perturbation travels within some limited speed v for any locally-interacting system. This is captured in so-called Lieb-Robinson Bound stating that commutators of spatially separated observables at different time is bounded as [6]:

$$\| [A(t), B(0)] \| \leq \|A\| \, \|B\| O[\exp(-\alpha L_{AB})], \qquad t = L_{AB}/v$$

for $v \sim k \|H_i\|$ in system with k-local Hamiltonian.

At the same time, the uncertainty principle shows that the typical time for energy fluctuation is given by:

$$\Delta E \cdot \Delta t \backsim 1.$$

Taking $\Delta E = \Delta$ to be the gap of the system, then the typical time is around Δ^{-1}. During the time, this quantum fluctuation can propagate at speed v, creating a correlation of length $v\Delta^{-1}$.

It should also be mentioned that the inverse is not true in general: for example, in a typical glassy system, there exists a range of energy in which states are localized and do not contribute to long range behavior (Anderson localization). Such energy range can effectively serve as the energy gap, producing a finite correlation length. Also, thermal fluctuations at finite temperature T can also constitutes a length cut-off in correlation.

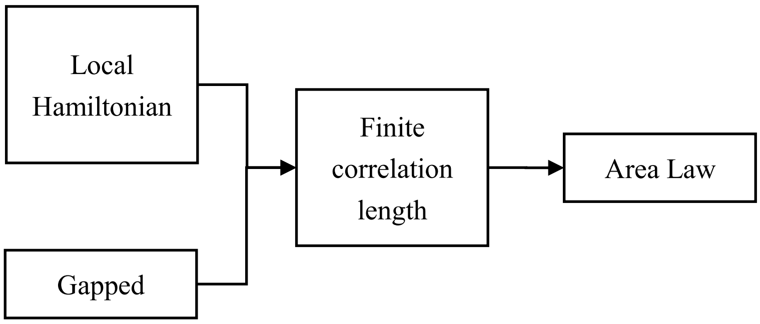
For 1D system, it has been proved that the area law can be deduced from either energy gap in

Hamiltonian or exponential decay in any correlation functions.

Theorem 2. (M. B. Hastings, 2007) The ground state of a gapped 1D system satisfies the area law with an upper-bound $E(A,B) \leq O\left(e^{c/\Delta}\right)$ [7]. The bound has been improved to $E(A,B) \leq O(\Delta^{-1})$ in later work.

Theorem 3. (M. Horodecki, 2014) For any quantum state of a 1D system satisfying exponential decay for all correlation functions, there is an upper-bound on its entanglement entropy $E(A,B) \leq O\left(\xi^{c\xi}\right)$ [8]. Notice that the state needs not to be a ground state of some local Hamiltonian.

To summarize the three theorems above, in 1D we have proved the following relation:



For higher dimension, whether area law still holds for general physical states are still an open question. Nonetheless, for quasi-free systems, there is a simple result for higher-dimensional quasi-free system with finite correlation length in [9]:

Theorem 4. (M. Cramer, 2006) If the ground state of a quasi-free system of arbitrary dimension D has exponentially decaying correlation functions, the entanglement entropy satisfies the area law.

Since Theorem 1 applies in any dimension, this also implies that gapped system with local Hamiltonian also satisfied the area law.

Beyond quasi-free systems, there is no rigorous results besides some specially constructed models or states (such as the toric code).

2.2 Area Law in Gapless System

When the system becomes gapless, depending on how fast the gap vanish as $N \to \infty$, the area law receive modification from these low-energy states. In particular, the gapless limit for many physical systems, such as quasi-free systems, are supposed to have long-range conformal symmetry dues to the argument from the renormalization group theory, which suggest that critical points are RG fixed point with scale invariance. Thus, powerful methods provided by conformal field theory (CFT) can help determine the entanglement properties of such systems.

The conformal symmetry is most useful in $(1+1)$-dimensional spacetime, making it best for solving 1D system. By utilizing holomorphic transformations, one can transform the problem of calculating the entanglement entropy of the ground state into calculating the path integral for a finite temperature, giving a general result [10]:

$$E(A,B) = \frac{c}{3}\log\frac{L}{\epsilon}.$$

Here L represent the length of a single interval region A, and ϵ is an ultraviolet cut-off, which is in the same order as lattice spacing. c is the central charge of the CFT, which is a defining property for different CFT ($c = 1$ for free single boson and $c = 1/2$ for free Majorana fermion). The expression above suggests that area law is violated mildly by a logarithmic correction in these gapless systems.

Once we go beyond 1D, the area law for gapless quasi-free system behaves differently depending on whether the system is bosonic or fermionic. In short, for gapless bosonic quasi-free system the area law is satisfied, while for gapless fermionic quasi-free system the area law is violated by a logarithmic factor [11]. In this sense, critical fermionic systems are more entangled than critical bosonic systems.

It should be noticed that violation of area law can be worse than a logarithmic factor in a general gapless model. This mainly depends on how fast states accumulates around the ground state as $N \to \infty$. Nonetheless, physical gapless systems are usually obtained as special limits of some gapped systems, so it is expected that logarithmic correction is enough for most realistic models.

3 Implication from the Area Law

The area law has strong implication on what a realistic quantum system should looks like. For computational purposes, the area law implies that the ground state of most quantum system is not "too complicated" compared to a general quantum state. In fact, it has been known (at least in 1D) that quantum states satisfying the area law can be well represented by matrix product states (MPS), which allows many powerful numerical algorithms to perform well.

Another motivation for studying the area law is its resemblance to the Bekenstein-Hawking entropy of black hole, which is proportional to the area of the event horizon A:

$$S_{\mathrm{BH}} = \frac{A}{4\ell_{\mathrm{p}}^2}$$

Since conversion from normal matters into black hole is invertible, it is suggested by the second law of thermodynamics that any system with surface area A (for the enclosing sphere at least) should not have entropy larger than S_{BH}, leading to the famous principle of holography.

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