STATS 500 HW4

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Github repo: https://github.com/PKUniiiiice/STATS_500

Problem 1

(a)

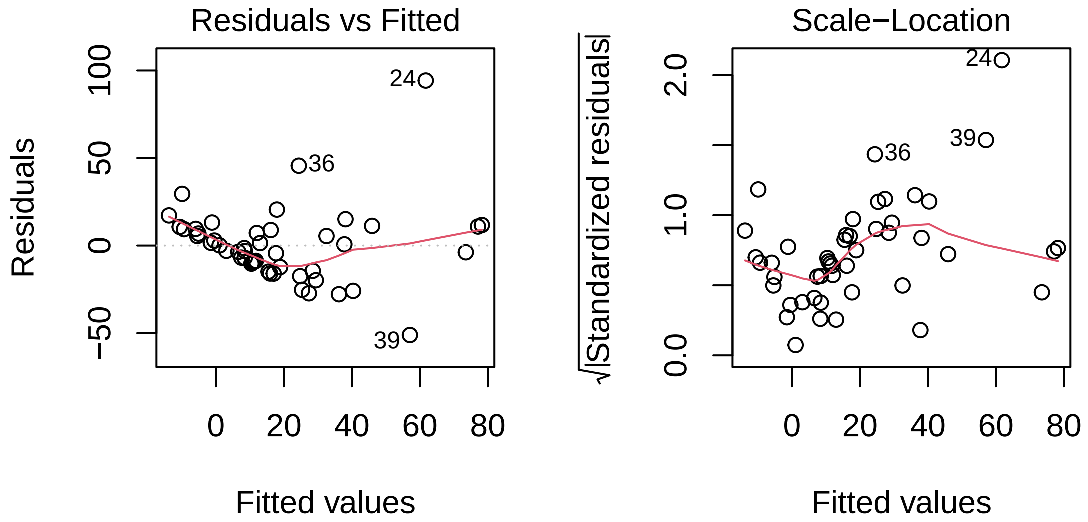
First, we perform diagnostics on the original model.

```
library(faraway)
data(teengamb)
# sex has already been encoded as 0,1, so we don't need to convert it to factor
m.ori <- lm(gamble ~ sex+status+income+verbal, data=teengamb)
#summary(m.ori)
```

- Check the constant variance assumption for the errors.

We can use both the `Residuals vs Fitted` and `Scale-Location` plots to check for variance issues. It's evident that, as x increases, the magnitude of residuals also increases. So we conclude that there is heteroscedasticity. Note that this violation may result in bias into all inferences. As a remedy, we choose to apply a transformation and proceed with the remaining diagnosis on the new model.

```
par(mfrow = c(1,2))
plot(m.ori, which=1)
plot(m.ori, which=3)
```

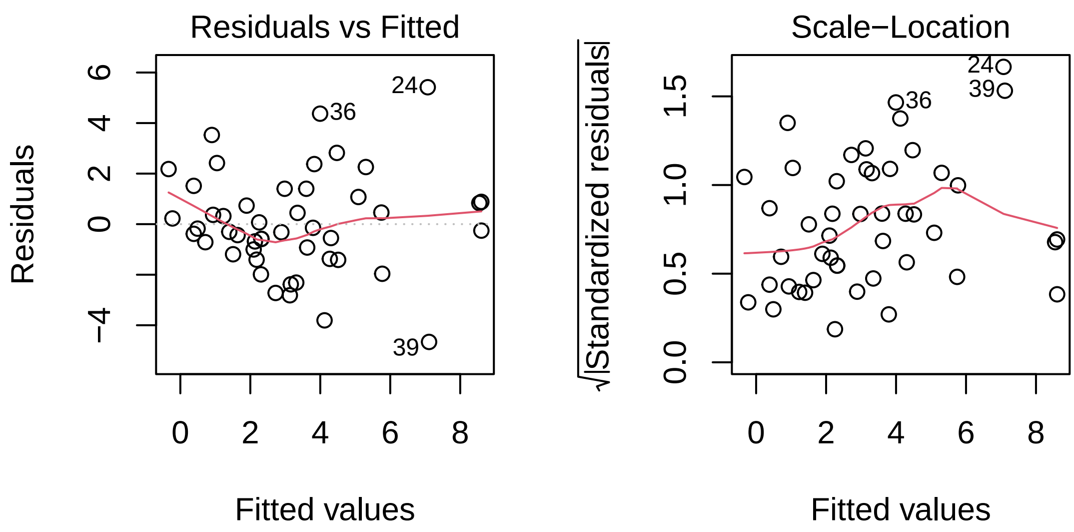



```
par(mfrow = c(1,1))
```

We take the square root of the response `gamble`.

```r
#new model
m.new <- lm(sqrt(gamble) ~ ., data=teengamb)
#summary(m.new)

par(mfrow = c(1,2))
plot(m.new, which=1)
plot(m.new, which=3)
```

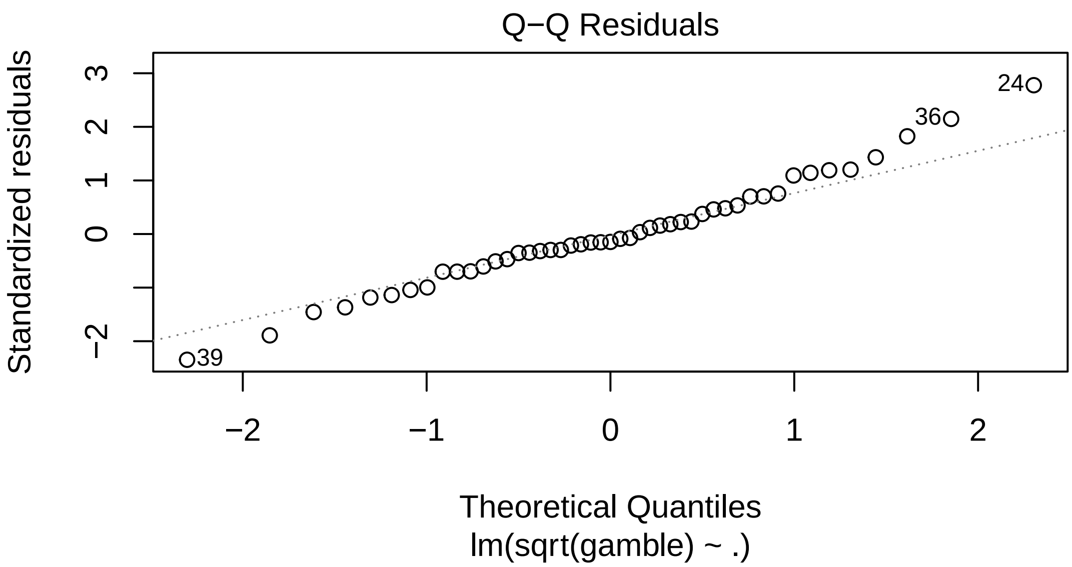



```r
par(mfrow = c(1,1))
```

In the new model, we still observe some slight heteroscedasticity in residuals, although it is not as severe as in the original model.

- Check the normality assumption.

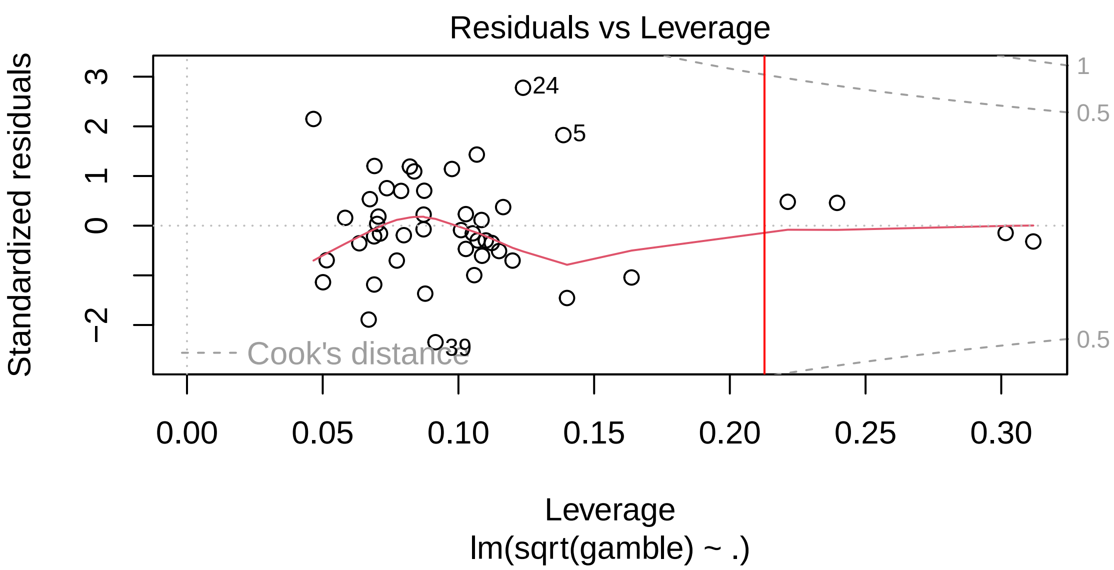
```r
plot(m.new, which=2)
```



From cases No.39,36 and 24, we observe that the residuals are slightly heavy-tailed compared to a normal distribution. So, there is a slight violation of the normality assumption.

- Check for large leverage points.

```
plot(m.new, which=5)
abline(v=2*length(m.new$coefficients)/nrow(teengamb), col='red')
```



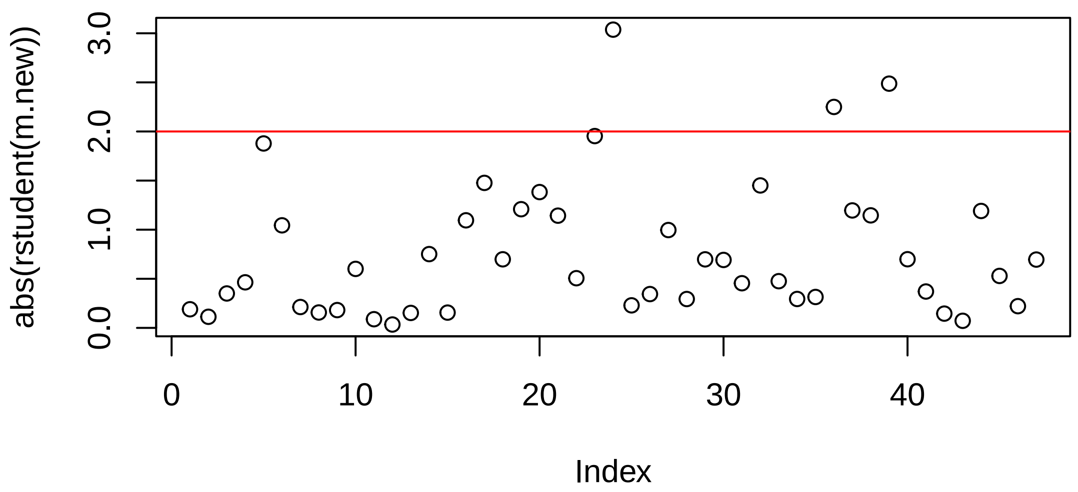
```
hatvalues(m.ori)>2*length(m.new$coefficients)/nrow(teengamb)
```

```
    1     2     3     4     5     6     7     8     9    10    11    12    13
FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE
   14    15    16    17    18    19    20    21    22    23    24    25    26
FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE
   27    28    29    30    31    32    33    34    35    36    37    38    39
FALSE FALSE FALSE FALSE  TRUE FALSE  TRUE FALSE  TRUE FALSE FALSE FALSE FALSE
   40    41    42    43    44    45    46    47
FALSE FALSE  TRUE FALSE FALSE FALSE FALSE FALSE
```

When we add a vertical line $x = \frac{2(p+1)}{n}$ on the `Residuals vs Leverage` plot, we can identify 4 points with high leverage. Upon direct calculation, case No.31, 33, 35 and 42 are large leverage points.

- Check for outliers.

```
plot(abs(rstudent(m.new)))
abline(h=2, col='red')
```



We can plot all absolute values of studentized deleted residuals. One empirical rule of outliers is

$$|t_i| > 2$$

We find it's likely that there are three outliers.

We can also perform a test.

```r
library(car)
```

```
Loading required package: carData

Attaching package: 'car'

The following objects are masked from 'package:faraway':

    logit, vif
```

```r
outlierTest(m.new)
```

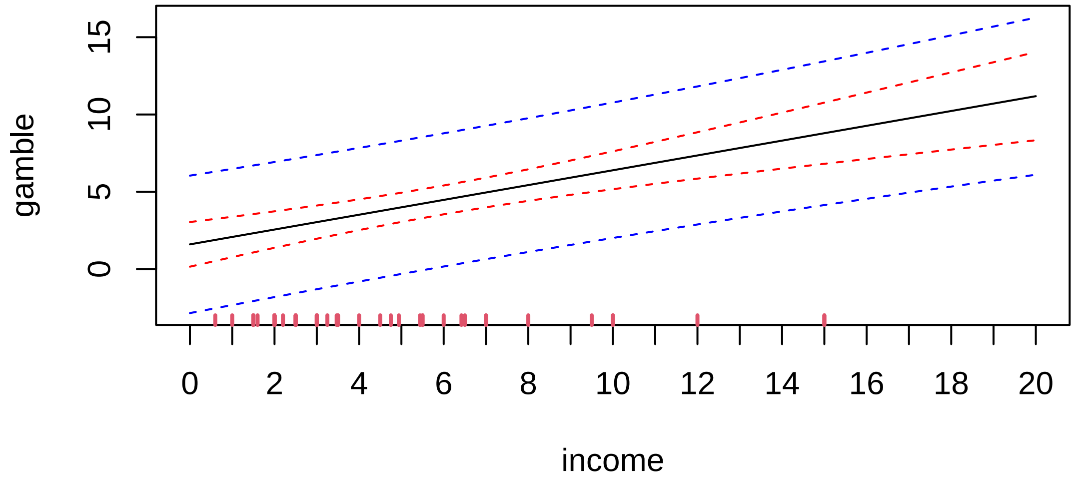
```
No Studentized residuals with Bonferroni p < 0.05
Largest |rstudent|:
   rstudent unadjusted p-value Bonferroni p
24 3.037005          0.0041428      0.19471
```

The test result tells that there are no outliers.

(b)

We generate the pointwise confidence/prediction band (ref:click).

```r
newx <- data.frame(
            sex = 0,
            income = seq(0,20),
            status = 43,
            verbal = 7
)
conf <- predict(m.new, newdata=newx, interval="confidence")
pred <- predict(m.new, newdata=newx, interval="prediction")
matplot(newx$income, cbind(conf, pred[,2:3]),
        lty=c(1,2,2,2,2),
        col=c(1, 'red', 'red', 'blue', 'blue'), type="l",
        xlab="income", ylab="gamble", xaxt="n")
axis(1, at = seq(0, 20))
rug(teengamb$income,col=2, lwd=2)
```



Problem 2

(a)

Note that if $Z \sim \mathcal{N}(0,1)$, then $X = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$ and the cdf of normal distribution is strictly monotonic. By definition

$$\begin{aligned}\mathbb{P}(X \leq F^{-1}(q)) &= q \\ &= \mathbb{P}(\mu + \sigma Z \leq F^{-1}(q)) \\ &= \mathbb{P}(Z \leq \frac{F^{-1}(q) - \mu}{\sigma})\end{aligned}$$

i.e.

$$\mathbb{P}(Z \leq \frac{F^{-1}(q) - \mu}{\sigma}) = q$$

Use the definition of quantile function again

$$\frac{F^{-1}(q) - \mu}{\sigma} = \Phi^{-1}(q) \rightarrow F^{-1}(q) = \mu + \sigma \Phi^{-1}(q)$$

(b)

In fact, we need the converse version of result in (a). That is, for a r.v. X with cdf F and quantile function F^{-1}, if, for all p,

$$F^{-1}(q) = \mu + \sigma \Phi^{-1}(q)$$

then $X \sim N(\mu, \sigma^2)$

Proof

$$\mathbb{P}(X \leq F^{-1}(q)) = \mathbb{P}(X \leq \mu + \sigma\Phi^{-1}(q)) = \mathbb{P}(\frac{X-\mu}{\sigma} \leq \Phi^{-1}(q)) = q, \forall 0 < q < 1$$

Therefore, $\Phi^{-1}(q)$ must be the quantile function of $\frac{X-\mu}{\sigma}$, i.e.

$$\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1) \rightarrow X \sim N(\mu, \sigma^2)$$

In QQ plots for normality test, we plot

$$\left\{y = r_{[i]}, x = \Phi^{-1}\left(\frac{i}{n+1}\right)\right\}$$

where $n+1$ is the correction for continuous distribution.

After sorting r_i to $r_{[i]}$, this sequence consists of quantile points of the underlying distribution.
By identifying the corresponding quantiles in Φ^{-1}, if we observe $y - x$ forms a line or closely resembles a line, then we can conclude that the distribution of residual satisfies

$$F^{-1}(q) = \mu + \sigma\Phi^{-1}(q)$$

therefore, we conclude that the residuals follow a normal distribution.