STATS 510 HW2

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Problem 1

(a)

cdf of Y:

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)dx$$

It's easy to get

$$F_Y(y) = \begin{matrix} 0, & y \le 0 \\ \sqrt{y}, & 0 < y < 1 \\ 1, & y \ge 1 \end{matrix}$$

Therefore, pdf of Y is

$$f_Y(y) = \frac{1}{2\sqrt{y}}\mathbb{I}_{(0,1)}(y)$$

(b)

$Y = -\log X$ is invertible on $0 < x < 1$.

$$\begin{aligned} f_Y(y) &= \left|\frac{dx}{dy}\right| f_X(x) \\ &= x\frac{(n+m+1)!}{n!m!}x^n(1-x)^m \\ &= \frac{(n+m+1)!}{n!m!}e^{-(n+1)y}(1-e^{-y})^m, y > 0 \end{aligned}$$

(c)

$Y = e^X$ is invertible on $x > 0$.

$$\begin{aligned} f_Y(y) &= \left|\frac{dx}{dy}\right| f_X(x) \\ &= \frac{1}{\sigma^2}\frac{\log y}{y}e^{-((\log y)/\sigma)^2/2}, y > 0 \end{aligned}$$

Problem 2.6

(a)

Since $X \in \mathbb{R}$, $Y = |X|^3$, we have $Y \geq 0$. Note $\{Y = y\} = \{X = y^{1/3}\} \cup \{X = -y^{1/3}\}(y > 0)$. So

$$\begin{aligned} f_Y(y) &= \frac{1}{3} y^{-\frac{2}{3}} \frac{1}{2} \exp(-y^{\frac{1}{3}}) \times 2 \\ &= \frac{1}{3} y^{-\frac{2}{3}} \exp(-y^{\frac{1}{3}}), y > 0 \end{aligned}$$

As for $Y = 0$, it's equivalent to $X = 0$, which has prob. 0. So we just take $f_Y(0) = 0$. So

$$f_Y(y) = \frac{1}{3} y^{-\frac{2}{3}} \exp(-y^{\frac{1}{3}}) \mathbb{I}(y > 0)$$

Integral

$$\int_0^{+\infty} \frac{1}{3} y^{-\frac{2}{3}} \exp(-y^{\frac{1}{3}}) dy = \int_0^{+\infty} \frac{1}{3} t^{-2} e^{-t} \cdot 3t^2 dt = \int_0^{+\infty} e^{-t} dt = 1$$

(b)

Note $\{Y = y\} = \{X = \pm\sqrt{1-y}\}(0 < y < 1)$.

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sqrt{1-y}} \frac{3}{8} (\sqrt{1-y} + 1)^2 + \frac{1}{2\sqrt{1-y}} \frac{3}{8} (-\sqrt{1-y} + 1)^2 \\ &= \frac{1}{2\sqrt{1-y}} \frac{3}{8} (2(1-y) + 2) = \frac{3}{8} \frac{2-y}{\sqrt{1-y}}, 0 < y < 1 \end{aligned}$$

Similarly, take $f_Y(1) = 0$.

Integral

$$\int_0^1 \frac{3}{8} \frac{2-y}{\sqrt{1-y}} dy = -\int_0^1 \frac{3}{8} (t + \frac{1}{t})(-2t dt) = 1$$

(c)

Note $\{Y = y\} = \{X = -\sqrt{1-y}\} \cup \{X = 1-y\}(0 < y < 1)$.

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sqrt{1-y}} \frac{3}{8} (-\sqrt{1-y} + 1)^2 + \frac{3}{8} (2-y)^2 \\ &= \frac{3}{16} \left(\frac{1}{\sqrt{1-y}} + \sqrt{1-y} - 2 \right) + \frac{3}{8} (2-y)^2, 0 < y < 1 \end{aligned}$$

Similarly, take $f_Y(1) = 0$.

Integral

$$\int_0^1 f_Y(y) = \frac{3}{16} (\frac{8}{3} - 2) + \int_0^1 \frac{3}{8} (2-y)^2 dy = \frac{1}{8} + \int_{-2}^{-1} \frac{3}{8} t^2 dt = 1$$

Problem 3

(a)

1

$$\lim_{x\to-\infty} F_X(x) = \lim_{x\to-\infty} 0 = 0, \lim_{x\to+\infty} F_X(x) = 1 - \lim_{x\to+\infty} e^{-x} = 1 - 0 = 1$$

2

e^{-x} is decreasing over $x \geq 0$, so $1-e^{-x}$ is increasing over $x \geq 0$. Moreover, $\forall x \geq 0, 1-e^{-x} \geq 0$. Therefore, $F_X(x)$ is a non-decreasing function of x.

3

$F_X(x)$ is continuous over $x < 0$ and $x > 0$. We only need to check $x = 0$.

$$\lim_{x\to0+} F_X(x) = \lim_{x\to0+} 1 - e^{-x} = 0 = F_X(0)$$

By 1,2,3, $F_X(x)$ is a cdf.

Inverse of cdf:
When $y = 0$, $F_X^{-1}(0) = \inf\{x : F_X(x) \geq 0\} = -\infty$.
When $0 < y < 1$, $F_X(x)$ is strictly increasing w.r.p x, so $F_X^{-1}(y) = -\log(1-y)$.
When $y = 1$, as $F_X(x) < 1, \forall x$, $F_X^{-1}(1) = +\infty$.

(b)

1

$$\lim_{x\to-\infty} F_X(x) = \lim_{x\to-\infty} \frac{e^x}{2} = 0, \lim_{x\to+\infty} F_X(x) = 1 - \lim_{x\to+\infty} \frac{e^{1-x}}{2} = 1 - 0 = 1$$

2

$\frac{e^x}{2}$ is increasing over $x < 0$ and $\frac{e^x}{2} < \frac{1}{2}, \forall x < 0$.
e^{1-x} is decreasing over $x \geq 1$, so $1 - (e^{1-x}/2)$ is increasing over $x \geq 1$. Moreover, $e^{1-x} \leq e^{1-1} = 1 \to 1 - (e^{1-x}/2) \geq \frac{1}{2}, \forall x \geq 1$.
Therefore, $F_X(x)$ is a non-decreasing function of x.

3

We check $x = 0$ and $x = 1$.

$$\lim_{x\to0+} F_X(x) = \lim_{x\to0+} \frac{1}{2} = \frac{1}{2} = F_X(0)$$

$$\lim_{x\to1+} F_X(x) = \lim_{x\to1+} 1 - \frac{e^{1-x}}{2} = \frac{1}{2} = F_X(1)$$

By 1,2,3, $F_X(x)$ is a cdf.

Inverse of cdf:
When $y = 0$, $F_X^{-1}(0) = \inf\{x : F_X(x) \geq 0\} = -\infty$.
When $0 < y < \frac{1}{2}$, $F_X(x)$ is strictly increasing w.r.p x, so $F_X^{-1}(y) = \log(2y)$.

When $y = \frac{1}{2}$, $F_X^{-1}(1/2) = \inf\{x : F_X(x) \geq 1/2\} = 0$.
When $\frac{1}{2} < y < 1$, $F_X(x)$ is strictly increasing w.r.p x, so $F_X^{-1}(y) = 1 - \log(2 - 2y)$.
When $y = 1$, as $F_X(x) < 1, \forall x$, $F_X^{-1}(1) = +\infty$.

(c)

1

$$\lim_{x\to-\infty} F_X(x) = \lim_{x\to-\infty} \frac{e^x}{4} = 0, \lim_{x\to+\infty} F_X(x) = 1 - \lim_{x\to+\infty} \frac{e^{-x}}{4} = 1 - 0 = 1$$

2

$e^x/4$ is increasing over $x < 0$ and $e^x/4 < 1/4, \forall x < 0$.
$1 - (e^{-x}/4)$ is increasing over $x \geq 0$ and $1 - (e^{-x}/4) \geq 3/4 > 1/4, \forall x \geq 0$.
Therefore, $F_X(x)$ is a non-decreasing function of x.

3

We check $x = 0$.

$$\lim_{x\to0+} F_X(x) = 1 - \lim_{x\to0+} \frac{e^{-x}}{4} = \frac{3}{4} = F_X(0)$$

By 1,2,3, $F_X(x)$ is a cdf.

Inverse of cdf:
When $y = 0$, $F_X^{-1}(0) = \inf\{x : F_X(x) \geq 0\} = -\infty$.
When $0 < y < \frac{1}{4}$, $F_X(x)$ is strictly increasing w.r.p x, so $F_X^{-1}(y) = \log(4y)$.
When $\frac{1}{4} \leq y < \frac{3}{4}$, $F_X^{-1}(1/2) = \inf\{x : F_X(x) \geq 1/2\} = 0$.
When $\frac{3}{4} \leq y < 1$, $F_X(x)$ is strictly increasing w.r.p x, so $F_X^{-1}(y) = -\log(4 - 4y)$.
When $y = 1$, as $F_X(x) < 1, \forall x$, $F_X^{-1}(1) = +\infty$.

Problem 4

(a)

$$\begin{aligned}
EX^2 &= \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{x^2}{2} e^{-x^2/2} d\left(\frac{x}{\sqrt{2}}\right) \\
&= \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}} t(te^{-t^2}) dt \\
&= \frac{2}{\sqrt{\pi}} \left[t\frac{-e^{-t^2}}{2} \bigg|_{-\infty}^{+\infty} - \int_{\mathbb{R}} \frac{-e^{-t^2}}{2} dt \right] \\
&= \frac{2}{\sqrt{\pi}} \left(0 + \frac{1}{2} \int_{\mathbb{R}} e^{-t^2} dx \right) \\
&= \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \\
&= 1
\end{aligned}$$

By Example 2.1.7

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}} + \frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}}\right] = \frac{1}{\sqrt{2\pi y}}e^{\frac{-y}{2}}$$

so

$$\begin{aligned} EY &= \int_0^{+\infty} y\frac{1}{\sqrt{2\pi y}}e^{\frac{-y}{2}}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} y^{\frac{1}{2}}e^{\frac{-y}{2}}dy \\ &= \frac{2\sqrt{2}}{\sqrt{2\pi}}\int_0^{+\infty} t^{\frac{1}{2}}e^{-t}dt \\ &= \frac{2}{\sqrt{\pi}}\Gamma(\frac{3}{2}) \\ &= \frac{2}{\sqrt{\pi}}\frac{1}{2}\Gamma(\frac{1}{2}) = 1 \end{aligned}$$

(b)

$\{Y = y\} = \{X = \pm y\}$, so

$$f_Y(y) = \frac{2}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}, y \geq 0$$

Mean

$$EY = \int_0^{+\infty} y\frac{2}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy = \sqrt{\frac{2}{\pi}}\int_0^{+\infty} e^{-t}dt = \sqrt{\frac{2}{\pi}}$$

Variance

$$\begin{aligned} EY^2 &= \int_0^{+\infty} y^2\frac{2}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\ &= \sqrt{\frac{2}{\pi}}\int_0^{+\infty} 2t\, e^{-t}(2t)^{-1/2}dt \\ &= 2\sqrt{\frac{1}{\pi}}\int_0^{+\infty} t^{1/2}\, e^{-t}dt \\ &= 2\sqrt{\frac{1}{\pi}}\Gamma(\frac{3}{2}) = 1 \end{aligned}$$

Therefore

$$Var(Y) = EY^2 - (EY)^2 = 1 - \frac{2}{\pi}$$

Problem 5

(a)

2.14 (a)

We use Fubini theorem.

$$
\begin{aligned}
\int_0^{+\infty}[1-F_X(x)]dx &= \int_0^{+\infty}\left[\int_{-\infty}^{+\infty} f(t)dt - \int_{-\infty}^{x} f(t)dt\right]dx \\
&= \int_0^{+\infty}\left[\int_{x}^{+\infty} f(t)dt\right]dx \\
&= \int_0^{+\infty}\left[\int_{0}^{t} f(t)dx\right]dt \\
&= \int_0^{+\infty} f(t)\left[\int_{0}^{t} 1dx\right]dt \\
&= \int_0^{+\infty} tf(t)dt \\
&= EX
\end{aligned}
$$

2.14 (b)

$$
\begin{aligned}
\sum_{k=0}^{\infty}(1-F_X(k)) &= \sum_{k=0}^{\infty} P(X>k) \\
&= \sum_{k=0}^{\infty}\sum_{t=k+1}^{\infty} P(X=t) \\
&= \sum_{t=1}^{\infty}\sum_{k=1}^{t} P(X=t) \;\; \text{(change order of double summation)} \\
&= \sum_{t=1} tP(X=t) \\
&= EX
\end{aligned}
$$

Compare:
Obviously, Part (b) is a discrete version of the conclusion in Part (a).

(b)

Mean duration is expectation of T.

$$
\begin{aligned}
ET &= \int_0^{+\infty}[1-F_T(t)]dt \\
&= \int_0^{+\infty} P(T>t)dt \\
&= \int_0^{+\infty} ae^{-\lambda t} + (1-a)e^{-\mu t}dt \\
&= \left.\frac{-ae^{-\lambda t}}{\lambda} - \frac{(1-a)e^{-\mu t}}{\mu}\right|_0^{\infty} \\
&= \frac{a}{\lambda} + \frac{1-a}{\mu}
\end{aligned}
$$

Problem 6

(a)

We use law of total expectation.

$$\begin{aligned} LHS &= E((|X-Y|)|X \geq Y)P(X \geq Y) + E((|X-Y|)|X < Y)P(X < Y) \\ &= E(X-Y|X \geq Y)P(X \geq Y) + E(Y-X)|X < Y)P(X < Y) \end{aligned}$$

$$\begin{aligned} RHS &= E(X) + E(Y) - 2\left(E(X \wedge Y|X \geq Y)P(X \geq Y) + E(X \wedge Y|X < Y)P(X < Y)\right) \\ &= E(X) + E(Y) - 2\left(E(Y|X \geq Y)P(X \geq Y) + E(X|X < Y)P(X < Y)\right) \\ &= E(X|X \geq Y)P(X \geq Y) - E(X|X < Y)P(X < Y) \\ &\quad + E(Y|X < Y)P(X < Y) - E(Y|X \geq Y)P(X \geq Y) \\ &= E(X-Y|X \geq Y)P(X \geq Y) + E(Y-X)|X < Y)P(X < Y) \\ &= LHS \end{aligned}$$

(b)

$$\begin{aligned} E(|X-a|) &= \int_{-\infty}^{+\infty} |x-a| f(x)dx \\ &= \int_{-\infty}^{+a} (a-x)f(x)dx + \int_{a}^{+\infty} (x-a)f(x)dx \\ &= aF(a) - \int_{-\infty}^{+a} xf(x)dx + \int_{a}^{+\infty} xf(x)dx - a(1-F(a)) \\ &= 2aF(a) - a - \int_{-\infty}^{+a} xf(x)dx + \int_{a}^{+\infty} xf(x)dx \end{aligned}$$

We take derivative of $E(|X-a|)$ w.r.t a and set it to zero.

$$\frac{dE(|X-a|)}{da} = 2F(a) + 2af(a) - 1 - af(a) - af(a) = 0$$

We get $a = m$ where $F(m) = \frac{1}{2}$, i.e. $P(X \leq m) = \frac{1}{2}$. Furthermore, the second derivative is

$$\frac{d^2E(|X-a|)}{da^2} = 2f(a) > 0$$

therefore $a = m$ can minimize $E|X-a|$.

(c)

$$\begin{aligned} E(\alpha(X-c)_- + \beta(X-c)_+) &= E(\alpha(c - X \wedge c) + \beta(X - X \wedge c)) \\ &= E(\alpha c + \beta X - (\alpha+\beta)(X \wedge c)) \\ &= \alpha c + \beta \int_{-\infty}^{+\infty} xf(x)dx - (\alpha+\beta)\left(\int_{-\infty}^{c} xf(x)dx + c\int_{c}^{+\infty} f(x)dx\right) \end{aligned}$$

We take derivative w.r.t c and set it to zero.

$$\begin{aligned}\frac{dE(\alpha(X-c)_- + \beta(X-c)_+)}{dc} &= \alpha - (\alpha+\beta)(cf(c) + c(-f(c)) + (1-F(c))) \\ &= \alpha - (\alpha+\beta)(1-F(c))) \\ &= 0\end{aligned}$$

We get $c = c_p$ where $F(c_p) = p = \frac{\beta}{\alpha+\beta}$, i.e. $P(X \leq c_p) = \frac{\beta}{\alpha+\beta}$. Furthermore, the second derivative is

$$\frac{d^2E(\alpha(X-c)_- + \beta(X-c)_+)}{dc^2} = (\alpha+\beta)f(c) > 0$$

therefore $c = c_p$ can minimize $E(\alpha(X-c)_- + \beta(X-c)_+)$.