STATS 510 HW3

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Problem 1

We calculate $\bar{\Phi}(z)\frac{z}{\phi(z)}$.

$$\begin{aligned}
\bar{\Phi}(z)\frac{z}{\phi(z)} &= \int_z^\infty ze^{-x^2/2+z^2/2}dx = \int_z^\infty ze^{-(x-z)(x+z)/2}dx \\
&= \int_0^\infty ze^{-u(u+2z)/2}du \quad (\text{take } u = x-z) \\
&= \int_0^\infty ze^{-t}\frac{1}{\sqrt{2t+z^2}}dt \quad (\text{take } u(u+2z) = 2t) \\
&= \int_0^\infty \frac{1}{m}e^{-t}\frac{1}{\sqrt{2t+1/m^2}}dt \quad (\text{take } z = \frac{1}{m}) \\
&= \int_0^\infty e^{-t}\frac{1}{\sqrt{2tm^2+1}}dt
\end{aligned}$$

We define

$$h(m,t) = \begin{cases} e^{-t}\frac{1}{\sqrt{2tm^2+1}}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

It's obvious that, $\forall m, t \in \mathbb{R}$

$$|h(m,t)| \leq g(t) = e^{-t}\mathbb{I}(t \geq 0)$$

and

$$\int_{-\infty}^{+\infty} g(t) < \infty$$

Therefore, by Dominated Convergence Theorem

$$\begin{aligned}
\lim_{z\to+\infty}\bar{\Phi}(z)\frac{z}{\phi(z)} &= \lim_{m\to 0}\int_0^\infty e^{-t}\frac{1}{\sqrt{2tm^2+1}}dt \\
&= \int_0^\infty \lim_{m\to 0} e^{-t}\frac{1}{\sqrt{2tm^2+1}}dt \\
&= \int_0^\infty e^{-t}dt \\
&= 1
\end{aligned}$$

Therefore,

$$\bar{\Phi}(z) \sim \frac{\phi(z)}{z}, \quad \text{as } z \to \infty$$

Problem 2

(a)

Consider the moment generating function of Z

$$\begin{aligned}
M_Z(t) = \mathbb{E}(e^{tz}) &= \int_{\mathbb{R}} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= \int_{\mathbb{R}} e^{t^2/2} \frac{1}{\sqrt{2\pi}} e^{-(z^2-2tz+t^2)/2} dz \\
&= e^{t^2/2} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-(z-t)^2/2} dz \\
&= e^{t^2/2}
\end{aligned}$$

Therefore, by the property of MGF

$$\begin{aligned}
E(Z^{2n}) &= \left[\frac{d^{2n}}{dt^{2n}} \left(e^{t^2/2} \right) \right]_{t=0} \\
&= \left[\frac{d^{2n}}{dt^{2n}} \left(\sum_{k=0}^{\infty} \frac{(t/\sqrt{2})^{2k}}{k!} \right) \right]_{t=0} \\
&= \left[\sum_{k=0}^{\infty} 2^{-k} \frac{d^{2n}}{dt^{2n}} \left(\frac{t^{2k}}{k!} \right) \right]_{t=0} \\
&= 2^{-n} \cdot 2n(2n-1) \cdots 1/k! \\
&= \frac{2^{-n}(2n)!}{n!}, n = 1, 2, \cdots
\end{aligned}$$

The commutativity of derivative and infinite sum is assured by the consistent convergence of power series.

(b)

$$f(x) = 2 \times \left| \frac{dz}{dx} \right| \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = \frac{1}{\sqrt{2\pi x}} e^{-x/2} \mathbb{I}(x > 0)$$

(c)

We have

$$\mathbb{P}[X \le x_p] = \mathbb{P}[|Z| \le x_p] = \mathbb{P}[-x_p \le Z \le x_p] = p \in (0, 1)$$

The second equal sign is a result of knowing that $p > 0$, which assures us that $x_p \ge 0$; otherwise, we would have $p = 0$. Note that $\mathcal{N}(0, 1)$ is symmetric about $x = 0$, so

$$\begin{aligned}
&\mathbb{P}[-x_p \le Z \le x_p] = \Phi(x_p) - \Phi(-x_p) = 2\Phi(x_p) - 1 = p \\
\to\ &\mathbb{P}[Z \le x_p] = \Phi(x_p) = \frac{p+1}{2}
\end{aligned}$$

Problem 3

Let $\bar{F}(x) = 1 - F(x) = \mathbb{P}(X > x)$, then

$$\begin{aligned}\mathbb{P}(X - t > s | X > t) &= \frac{\mathbb{P}(X - t > s, X > s)}{\mathbb{P}(X > t)} \\ &= \frac{\bar{F}(t+s)}{\bar{F}(t)} \\ &= \mathbb{P}(X > s) = \bar{F}(s)\end{aligned}$$

i.e.

$$\bar{F}(t+s) = \bar{F}(t)\bar{F}(s), \forall\, t, s > 0$$

So for all integer $m, n \geq 1$ and all positive real number t, we have

$$\begin{aligned}\bar{F}(mt) &= \bar{F}(n \times \frac{m}{n}t) = \left(\bar{F}(\frac{m}{n}t)\right)^n \\ \bar{F}(mt) &= \bar{F}(t + t + \cdots + t) = \left(\bar{F}(t)\right)^m\end{aligned}$$

thus

$$\bar{F}\left(\frac{m}{n}t\right) = \left(\bar{F}(t)\right)^{\frac{m}{n}} \qquad (*)$$

Note that here $\frac{m}{n}$ is a positive rational number, it's known that for any real number, there exists some rational sequence that converges to it ref:(click). Therefore, we can construct $\{m/n\}$ as such a convergent rational sequence, i.e. we have

$$\lim_{n\to+\infty} x_n = x, \quad \bar{F}\left(x_n t\right) = \left(\bar{F}(t)\right)^{x_n}$$

in which x_n is a positive rational number and $x > 0$. By continuity of the CDF $F(x)$ and, subsequently, $\bar{F}(x)$, we take the limit $n \to +infty$ on both sides of the equation above,

$$\begin{aligned}\lim_{n\to+\infty} \bar{F}\left(x_n t\right) &= \bar{F}\left(\lim_{n\to+\infty} x_n t\right) = \bar{F}\left(xt\right) \\ &= \lim_{n\to+\infty}\left(\bar{F}(t)\right)^{x_n} = \bar{F}\left(t\right)^x\end{aligned}$$

thus, for all $x > 0, t > 0$

$$\bar{F}\left(xt\right) = \bar{F}\left(t\right)^x$$

Take $t = 1$, and let $\lambda = -\ln \bar{F}(1)$

$$\bar{F}\left(x\right) = \bar{F}\left(1\right)^x = e^{\ln \bar{F}(1)\cdot x} = e^{-\lambda x}, x > 0$$

Hence

$$F(x) = 1 - e^{-\lambda x}, x > 0, \lambda = -\ln(1 - F(1))$$

For $x \leq 0$, it's given that X is a positive r.v., so $F(x) = 0,$ when $x \leq 0$.
It's evident that X follows an exponential distribution.

Problem 4

(a)

Suppose there are n indistinguishable balls to be placed into k distinguishable cells (e.g. numbered 1 to k) without any restriction (i.e. any number of balls can be placed into a cell). It is known that the total number of ways of distribution is

$$\binom{n+k-1}{k-1}$$

We can also distribute the balls in two steps. First, we place $i, 0 \le i \le n$ balls into the first cell. Then, for the remaining $n-i$ balls, we place them into the remaining $k-1$ cells as before. Therefore, the total number of ways of distribution is,

$$\sum_{i=0}^{n} \binom{n-i+k-2}{k-2}$$

These two values should be equal,

$$\binom{n+k-1}{k-1} = \sum_{i=0}^{n} \binom{n-i+k-2}{k-2}$$

Now, we substitute $n \to n-r$, $k \to r+2$, then

$$\binom{n+1}{r+1} = \sum_{i=0}^{n-r} \binom{n-i}{r} = \sum_{i=r}^{n} \binom{i}{r}$$

Take $r = 2$,

$$\binom{n+1}{r+1} = \frac{(n+1)n(n-1)}{6} = \sum_{i=r}^{n} \binom{i}{r} = \sum_{i=2}^{n} \frac{i(i-1)}{2}$$

so

$$\begin{aligned}
\sum_{i=1}^{n} i^2 &= 1 + 2\sum_{i=2}^{n} \frac{i(i-1)}{2} + \sum_{i=2}^{n} i \\
&= 1 + \frac{(n+1)2n(n-1)}{6} + \frac{(2+n)(n-1)}{2} \\
&= \frac{n(2n^2+3n+1)}{6} = \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

(b)

a.

$$\begin{aligned}
EX &= \int_0^1 x \cdot a x^{a-1} dx = \int_0^1 a x^a dx = \left.\frac{a x^{a+1}}{a+1}\right|_0^1 = \frac{a}{a+1} \\
EX^2 &= \int_0^1 x^2 \cdot a x^{a-1} dx = \int_0^1 a x^{a+1} dx = \left.\frac{a x^{a+2}}{a+2}\right|_0^1 = \frac{a}{a+2} \\
Var(X) &= EX^2 - (EX)^2 = \frac{a}{(a+1)^2(a+2)}
\end{aligned}$$

b.

$$
\begin{aligned}
EX &= \sum_{x=1}^{n} x\frac{1}{n} = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2} \\
EX^2 &= \sum_{x=1}^{n} x^2\frac{1}{n} = \frac{1}{n}\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6} \\
Var(X) &= EX^2 - (EX)^2 = \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4} = \frac{n^2+1}{12}
\end{aligned}
$$

c.

$$
\begin{aligned}
EX &= \int_0^2 x\cdot\frac{3}{2}(x-1)^2dx = \frac{3}{2}\int_0^2\left(x^3-2x^2+x\right)dx = 1 \\
EX^2 &= \int_0^2 x^2\cdot\frac{3}{2}(x-1)^2dx = \frac{3}{2}\int_0^2\left(x^4-2x^3+x^2\right)dx = \frac{8}{5} \\
Var(X) &= EX^2 - (EX)^2 = \frac{8}{5} - 1 = \frac{3}{5}
\end{aligned}
$$

Problem 5

(a)

$$
E\left(e^{tX}\right) = \int_0^c e^{tx}\frac{1}{c}dx = \frac{1}{ct}e^{tx}\Big|_0^c = \frac{1}{ct}\left(e^{tc}-1\right)
$$

(b)

$$
E(e^{tX}) = \int_0^c e^{tx}\frac{2x}{c^2}dx = \frac{2}{c^2}\int_0^c xe^{tx}dx = \frac{2xe^{tx}}{c^2}\Bigg|_0^c - \frac{2}{c^2}\int_0^c e^{tx}dx = \frac{2(1-e^{ct}+cte^{ct})}{c^2t^2}
$$

(c)

$$
\begin{aligned}
E(e^{tX}) &= \frac{1}{2\beta}\left(\int_{-\infty}^{\alpha} e^{tx}e^{(x-\alpha)/\beta}dx + \int_{\alpha}^{+\infty} e^{tx}e^{-(x-\alpha)/\beta}dx\right) \\
&= \frac{1}{2\beta}\left(\frac{e^{\frac{x-\alpha+tx\beta}{\beta}}\beta}{1+t\beta}\Bigg|_{-\infty}^{\alpha} + \frac{e^{\frac{-x+\alpha+tx\beta}{\beta}}\beta}{-1+t\beta}\Bigg|_{\alpha}^{+\infty}\right) \\
&= \frac{1}{2\beta}\left(\frac{e^{t\alpha}\beta}{1+t\beta} + \frac{e^{t\alpha}\beta}{1-t\beta}\right) \\
&= \frac{e^{t\alpha}}{2\beta}\frac{2\beta}{1-t^2\beta^2} \\
&= \frac{e^{t\alpha}}{1-t^2\beta^2}, -\frac{1}{\beta} < t < \frac{1}{\beta}
\end{aligned}
$$

(d)

$$\begin{aligned} E(e^{tX}) &= \sum_{x=0}^{+\infty} e^{tx} \binom{r+x-1}{x} p^r (1-p)^x \\ &= \left(\frac{p}{1-e^t+pe^t}\right)^r \sum_{x=0}^{+\infty} \binom{r+x-1}{x} (1-e^t+pe^t)^r (1-(1-e^t+pe^t))^x \\ &= \left(\frac{p}{1-e^t+pe^t}\right)^r, t < -\ln(1-p) \end{aligned}$$

The last equal sign results from treating $1-e^t+pe^t$ as the parameter of the negative binomial distribution and applying the normalization of PMF.

(e) 2.31

By the definition of MGF, for any distribution of X, we must have

$$M_X(0) = E(e^{0X}) = E(1) = 1$$

However, in $M_X(t) = t/(t-1)$, $M_X(0) = 0$. Therefore, $M_X(t)$ can't be an MGF and such a distribution X doesn't exist.

Problem 6

Leibniz integral rule tells that

$$\begin{aligned} &\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) \\ =&f(x,b(x)) \cdot \frac{d}{dx}b(x) - f(x,a(x)) \cdot \frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt \end{aligned}$$

We use it for this problem.

(a)

$$\frac{d}{dx}\int_0^x e^{-\lambda t}dt = e^{-\lambda x}$$

(b)

$$\frac{d}{d\lambda}\int_0^{+\infty} e^{-\lambda t}dt = \int_0^{+\infty} \frac{\partial}{\partial \lambda} e^{-\lambda t}dt = \int_0^{+\infty} -te^{-\lambda t}dt = -\frac{1}{\lambda^2}$$

(c)

$$\frac{d}{dt}\int_t^1 \frac{1}{x^2}dx = -\frac{1}{t^2}$$

(d)

$$
\begin{aligned}
\frac{d}{dt}\int_{1}^{+\infty}\frac{1}{(x-t)^2}dx &= +\int_{1}^{+\infty}\frac{\partial}{\partial t}\frac{1}{(x-t)^2}dx \\
&= \int_{1}^{+\infty}\frac{2}{(x-t)^3}dx \\
&= -\frac{1}{(x-t)^2}\Bigg|_{1}^{+\infty} \\
&= \frac{1}{(1-t)^2}
\end{aligned}
$$

Justify:

The Leibniz integral rule requires that both $f(x,t)$ and its partial derivative are continuous. All integrands in this context are elementary functions, ensuring continuity. Additionally, this rule necessitates that the upper and lower limits are continuous, which is satisfied in this case. Therefore, we can indeed use the Leibniz integral rule.

Alternatively, we can verify by direct calculation.

(a)

$$\frac{d}{dx}\left(\int_{0}^{x} e^{-\lambda t}dt\right) = \frac{d}{dx}\left(-\frac{1}{\lambda}e^{-\lambda x}+\frac{1}{\lambda}\right) = e^{-\lambda x}$$

(b)

$$\frac{d}{d\lambda}\int_{0}^{\infty} e^{-\lambda t}dt = \frac{d}{d\lambda}\frac{1}{\lambda} = -\frac{1}{\lambda^2}$$

(c)

$$\frac{d}{dt}\left(\int_{t}^{1}\frac{1}{x^2}dx\right) = \frac{d}{dt}\left(-1+\frac{1}{t}\right) = -\frac{1}{t^2}$$

(d)

$$\frac{d}{dt}\int_{1}^{\infty}(x-t)^{-2}dx = \frac{d}{dt}\left(-\left.(x-t)^{-1}\right|_{1}^{\infty}\right) = \frac{d}{dt}\frac{1}{1-t} = \frac{1}{(1-t)^2}$$