STATS 510 HW1

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Problem 1

We need some lemmas to complete this proof.

(1). Rational numbers are dense in the reals, i.e.

$$\forall x \in \mathbb{R}, \forall \epsilon > 0, \exists\, r \in \mathbb{Q} \quad \text{s.t.} \quad |x - r| < \epsilon$$

From this, we can get

$$\forall a < b,\ a, b \in \mathbb{R},\ \exists\, r \in \mathbb{Q} \quad \text{s.t.} \quad a < r < b$$

(2). Utilizing Lemma (1), we can construct a sequence of rational numbers converging to any given real number a. Formally

$$\begin{aligned}&\forall a \in \mathbb{R}, \exists \text{sequence } \{r_n | r_n \in \mathbb{Q},\, n \in \mathbb{N}^+\} \ \text{ s.t.}\\ &\lim_{n\to\infty} r_n = a\end{aligned}$$

In particular, there are two typical types of sequences r_n.

$$\left\{ r_n \,\middle|\, a - \frac{1}{n} < r_n < a, r_n \in \mathbb{Q} \right\} \text{or } \left\{ r_n \,\middle|\, a < r_n < a + \frac{1}{n}, r_n \in \mathbb{Q} \right\}$$

(3). Using Lemma (2), it's easy to get

$$\begin{aligned}&\forall a \in \mathbb{R},\ \text{interval } (-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, r_n],\\ &\text{where } \{r_n\} = \left\{ r_n \,\middle|\, a - \frac{1}{n} < r_n < a, r_n \in \mathbb{Q} \right\}\end{aligned}$$

Proof:

First, $\forall a_0 \in (-\infty, a)$, we have $a_0 < a$. Using Lemma (2) and the definition of limit, we take $\epsilon = a - a_0 > 0$,

$$\exists N \in \mathbb{N}, \forall n^* \geq N, |a - r_{n^*}| < a - a_0$$

then

$$a_0 - a < r_{n^*} - a < a - a_0 \longrightarrow a_0 < r_{n^*}$$

Therefore $a_0 \in \bigcup_{n=1}^{\infty}(-\infty, r_n]$.

Conversely, $\forall a_0 \in \bigcup_{n=1}^{\infty}(-\infty, r_n]$, $a_0 < r_n < a, \forall n$. So $a_0 \in (-\infty, a)$.

Hence we have $(-\infty, a) = \bigcup_{n=1}^{\infty}(-\infty, r_n]$

Similarly, we have

$$\forall a \in \mathbb{R},\ \text{interval } (-\infty, a] = \bigcap_{n=1}^{\infty}(-\infty, r_n],$$
$$\text{where } \{r_n\} = \left\{ r_n \,\middle|\, a < r_n < a + \frac{1}{n}, r_n \in \mathbb{Q} \right\}$$

Proof:

$(-\infty, a] \subseteq \bigcap_{n=1}^{\infty}(-\infty, r_n]$ is obvious.

Conversely, $\forall r \in \bigcap_{n=1}^{\infty}(-\infty, r_n]$, we employ a proof by contradiction.

If $r > a$, take $\epsilon = r - a > 0$, then

$$\exists n^* \geq N, |a - r_{n^*}| < r - a \longrightarrow r_{n^*} < r$$

However, r is an element of the infinity intersection, so

$$\forall n, r \leq r_n$$

By contradiction, $r \geq a$, i.e. $\bigcap_{n=1}^{\infty}(-\infty, r_n] \subseteq (-\infty, a]$.

(4).

$$\forall x \in \mathbb{R},\ \text{interval } (-\infty, x] = \bigcap_{n=1}^{\infty}\left(\bigcup_{m=1}^{\infty}\left(x - m, x + \frac{1}{n}\right)\right)$$

The proof is straightforward. The inner unions result in $(-\infty, x + \frac{1}{n})$. Since the right endpoint $x + \frac{1}{n} \to x$ and $x < x + \frac{1}{n}$, the outer intersections result in $(-\infty, x]$.

(5). Suppose $\{\Sigma_\alpha : \alpha \in \mathcal{A}\}$ is a collection of σ-algebras on a space X, then the intersection of a collection of σ-algebras is a σ-algebra.

Now, let's begin the proof.

First, we prove
$$\mathcal{B} = \sigma(\mathcal{C}_{\text{intervals}}) \subseteq \sigma(\mathcal{C})$$
Note that
$$\begin{aligned}
&\forall \text{ open interval } (a,b), a, b \in \mathbb{R} \\
&(a,b) = \Big((-\infty, a]\Big)^c \bigcap (-\infty, b) \\
&\qquad = \left(\bigcap_{n=1}^{\infty} (-\infty, s_n]\right)^c \bigcap \left(\bigcup_{n=1}^{\infty} (-\infty, r_n]\right) \text{ (Lemma (3))}
\end{aligned}$$
where
$$\{s_n\} = \left\{s_n \,\middle|\, a < s_n < a + \frac{1}{n}, s_n \in \mathbb{Q}\right\} \text{and } \{r_n\} = \left\{r_n \,\middle|\, b - \frac{1}{n} < r_n < b, r_n \in \mathbb{Q}\right\}$$

By the definition of $\sigma(\mathcal{C})$ and Lemma (5), $\sigma(\mathcal{C})$ is a σ-algebra and $\mathcal{C} \subseteq \sigma(\mathcal{C})$. Therefore, $\sigma(\mathcal{C})$ is closed under countable unions, intersections and complements.

Note that both $(-\infty, s_n]$ and $(-\infty, r_n]$ are elements of $\mathcal{C}$, i.e.
$$(-\infty, s_n], (-\infty, r_n] \in \mathcal{C} \subseteq \sigma(\mathcal{C}), \forall\, n$$
so
$$(a,b) = \left(\bigcap_{n=1}^{\infty} (-\infty, s_n]\right)^c \bigcap \left(\bigcup_{n=1}^{\infty} (-\infty, r_n]\right) \in \sigma(\mathcal{C})$$
Since (a,b) is arbitrary, we have
$$\mathcal{C}_{\text{intervals}} \subseteq \sigma(\mathcal{C})$$
By the definition of $\sigma(\cdot)$, we have
$$\mathcal{B} = \sigma(\mathcal{C}_{\text{intervals}}) \subseteq \sigma(\mathcal{C})$$

Conversely, use Lemma (4),
$$\forall x \in \mathbb{Q},\ \text{interval } (-\infty, x] = \bigcap_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} \left(x - m, x + \frac{1}{n}\right)\right)$$
Note that $(x - m, x + \frac{1}{n})$ is an element of $\mathcal{C}_{\text{intervals}}$.

Similarly,
$$(-\infty, x] \in \sigma(\mathcal{C}_{\text{intervals}}) \longrightarrow \mathcal{C} \subseteq \sigma(\mathcal{C}_{\text{intervals}}) \longrightarrow \sigma(\mathcal{C}) \subseteq \sigma(\mathcal{C}_{\text{intervals}}) = \mathcal{B}$$

Combine two parts,
$$\mathcal{B} = \sigma(\mathcal{C})$$

Problem 2

(a)

Given $A_1, ..., A_n$ are mutually independent, consider $A_1^c, A_2, ..., A_n$. We use 1^* to denote the subscript of event A_1^c, i.e, $A_1^c = A_{1^*}$. We can partition the collection of any non-empty subset $S \subset 1^*, 2, 3, ..., n$ into two disjoint classes.

$$\mathcal{C}_1 = \{S | 1^* \in S\}, \mathcal{C}_2 = \{S | 1^* \notin S\},$$

For the class $\mathcal{C}_2$, $\forall S \in \mathcal{C}_2$,

$$\mathbb{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathbb{P}\left(A_i\right)$$

This is guaranteed by mutual independence.

As for the class $\mathcal{C}_1$, $\forall S \in \mathcal{C}_1$, note that

$$S = \{1^*\} \cup (S / \{1^*\})$$

Denote $\bigcap_{i \in S} A_i$ by A_S and $\bigcap_{i \in S/\{1^*\}} A_i$ by A_{S^-} , then

$$A_{S^-} = (A_{S^-} \cap A_{1^*}) \cup (A_{S^-} \cap A_1) = A_S \cup (A_{S^-} \cap A_1)$$

therefore

$$\mathbb{P}(A_S) = \mathbb{P}(A_{S^-}) - \mathbb{P}(A_{S^-} \cap A_1)$$

Note that

$$S^- \subseteq \{2, ..., n\} \subseteq \{1, ..., n\}$$

By mutually independence of $A_1, ..., A_n$,

$$\mathbb{P}(\bigcap_{i \in S} A_i) = \prod_{i \in S^-} \mathbb{P}(A_i) - \left(\prod_{i \in S^-} \mathbb{P}(A_i)\right) \cdot \mathbb{P}(A_1)$$

$$\longrightarrow \mathbb{P}(\bigcap_{i \in S} A_i) = \left(\prod_{i \in S^-} \mathbb{P}(A_i)\right)(1 - \mathbb{P}(A_1)) = \left(\prod_{i \in S^-} \mathbb{P}(A_i)\right) \mathbb{P}(A_{1^*}) = \prod_{i \in S} \mathbb{P}(A_i)$$

Combine thes results in $\mathcal{C}_1$ and $\mathcal{C}_2$,

$$\forall S \subset 1^*, 2, 3, ..., n, \quad \mathbb{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathbb{P}\left(A_i\right)$$

Hence, $A_1^c, A_2, ..., A_n$ are also mutually independent.

Using this property, given $A_1, A_2, ..., A_n$ s.t. $0 < \mathbb{P}(A_i) < 1$, if there exists some A_k and $\mathbb{P}(A_k) > \frac{1}{2}$, we can regard this A_k as previous A_1 and then take its complement. Then, we can get $A_1, ..., A_k, ...A_n$ are mutually independent. We can repeat this procedure until we get a series of mutually independent
events, each of which satisfies $0 < \mathbb{P}(A_i) \leq \frac{1}{2}$."

(Rigorously, we must demonstrate the existence of an event A_i w.p. $\leq \frac{1}{2}$. Since $0 < \mathbb{P}(A_i) < 1$, we must have $N \geq 2$, and event A_i must encompass at least one outcome. So if we set $\mathbb{P}(A_i) = \frac{1}{N}$, we can find such an A_i.)

Now, we have mutually independent $A_1, ...A_n$ and $0 < \mathbb{P}(A_i) \leq \frac{1}{2}, \forall i$. Then

$$0 < \mathbb{P}(A_1...A_n) = \prod_{i=1}^{n} P(A_i) \leq \frac{1}{2^n}$$

Note that $0 < \mathbb{P}(A_1...A_n)$ implies $A_1...A_n \neq \emptyset$, therefore, $A_1...A_n$ must contain at least one outcome, thus

$$\frac{1}{N} \leq \mathbb{P}(A_1...A_n) \leq \frac{1}{2^n} \longrightarrow n \leq \log_2 N$$

(b)

Let's examine a random sequence of length n, where each digit is either 0 or 1, such as 000111000..., and define the sample space S as the set containing all possible sequences. It's evident that S consists of $N = 2^n$ elements.

We employ the pre-defined probability measure to select a sequence randomly from the sample space. Subsequently, we define the event A_1 as follows: "The i-th digit of the drawn sequence is 1".

$$\mathbb{P}(A_i) = \frac{|A_i|}{N} = \frac{1 \times 2^{n-1}}{2^n} = \frac{1}{2}$$

To prove that $A_1, ...A_n$ are mutually independent, for any non-empty subset $\{i_1, ...i_k\} \subseteq S$

$$\mathbb{P}(A_{i_1}...A_{i_k}) = \mathbb{P}(\text{The } i_1\text{-th}, i_2\text{-th}..., i_k\text{-th digits are all 1}) = \frac{2^{n-k}}{2^n} = \frac{1}{2^k} = \prod_{i=1}^{k} \mathbb{P}(A_{i_k})$$

Hence, the previous construction satisfies the property of mutually independence.

(c)

First, we limit the "independent events" to "non-trivial" case, i.e. assuming that distinct events $A_1, A_2, \cdots, A_n$ are mutually independent and $0 < \mathbb{P}(A_i) < 1, \forall i$. So for arbitrary i, j, we have

$$\mathbb{P}(A_i A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$$

Under uniform probability measure,

$$\frac{|A_i A_j|}{N} = \frac{|A_i|}{N} \cdot \frac{|A_j|}{N} \to \frac{N}{|A_j|} = \frac{|A_i|}{|A_i A_j|} (*)$$

We require $0 < \mathbb{P}(A_i) < 1, \forall i$, which implies $1 \leq |A_i| < N, \forall i$. Note that N is a prime number, so the only factor of N are 1 and N itself.

Now, if equation $(*)$ holds, note that $N, |A_i|, |A_j|, |A_i A_j|$ are all integers, N and $|A_j|$ must have common factors other than 1 and N. However, this is impossible when N is a prime number.

Therefore, if we restrict $0 < \mathbb{P}(A_i) < 1, \forall i$, we can't find solution for equation $(*)$. In other words, we can't find $A_1, \cdots, A_n$ that are mutually independent.

Next, we consider 2 trivial cases: $\mathbb{P}(A_i) = 0$ and $\mathbb{P}(A_i) = 1$, which correspond to $A_i = \emptyset$ and $A_i = S$, resp. We can choose $A_1 = \emptyset, A_2 = S$ and A_3 as arbitrary event (distinct from A_1, A_2) in $\mathcal{A} = 2^S$. It's easy to verify that A_1, A_2, A_3 are mutually Independent. However, when we introduce a new, distinct A_4, as per the previous statement, we can't find A_3, A_4 that are independent. In other words, we can't add any event to this group.

In summary, if we require non-trivial case, S can't support any independent events, and if we allow trivial case, S can support at most 3 independent events.

Problem 3

We use H_1 and T_1 to denote the first penny as head and tail, resp., and similarly, H_2 and T_2 for the second penny.

Since they are tossed independently, we have

$$p_0 = P(0 \text{ heads}) = P(T_1T_2) = (1-u)(1-w)$$

$$p_1 = P(1 \text{ heads}) = P(H_1T_2 \cup T_1H_2) = P(H_1T_2) + P(T_1H_2) = u + w - 2uw$$

$$p_1 = P(2 \text{ head}) = P(H_1H_2) = uw$$

If $p_0 = p_1 = p_2$,

$$(1-u)(1-w) = u + w - 2uw = uw$$

We can't get a real-number solution of this equation. Hence, we can't choose u, w such that $p_0 = p_1 = p_2$.

Problem 4

(a)

We number the balls and cell from 1 to n, respectively.

Sample space: $S = \{i_1, ..., i_n\}, i_j = 1, 2, ..., n$, where i_j represents the number of the cell in which the j-th ball is placed.

Since each ball can be placed in any of the n cells, $|S| = n^n$.

To ensure that exactly one cell remains empty, we begin by selecting one of the n cells to be empty, which provides us with n choices. Then, we distribute n balls into the remaining n-1 cells, ensuring that each cell contains at least one ball. Consequently, there will be exactly one cell with 2 balls.

We first choose 2 balls from the n available balls, which can be done in $\binom{n}{2}$ ways. Next, we designate one of the n-1 cells to hold these 2 balls, allowing for $n-1$ choices. Finally, we distribute the remaining balls among the n-2 cells, which can be done in $(n-2)!$ ways.

In total, there are

$$n \cdot \binom{n}{2} \cdot (n-1) \cdot (n-2)! = \binom{n}{2} n!$$

choices. Hence

$$P(\text{exactly one cell remains empty}) = \frac{\binom{n}{2} n!}{n^n}$$

(b)

We number the rings from 1 to 12 and the days of the week from 1 to 7.

Sample space: $S = \{i_1, ..., i_12\}, i_j = 1, 2, ..., 7$, where i_j represents the day number on which the j-th telephone rings.

$|S| = 7^{12}$.

To ensure that we receive at least one call each day, we need to distribute the times of rings across the days. We can achieve this by assigning a sequence to the 7 days of the week. For example, "6111111" means we receive 6 rings on Monday and only 1 call on the other days, while "5211111" implies 5 rings on Monday, 2 rings on Tuesday, and 1 call on the other days. It's important to note that, when considering the number of choices for these sequences, "6111111" and "1611111" are equivalent to some extent.

For "6111111," we start by selecting one of the seven days for 6 rings (7 choices). Then, we choose 6 rings from the 12 available ($\binom{12}{6}$ choices) and distribute the remaining ringss in 6! ways.

As another example, consider "3222111." Here, we allocate 3 rings to one day (7 choices), select 3 rings from 12 ($\binom{12}{3}$ choices), choose 3 days to have 2 rings each ($\binom{9}{3}$ choices), and then distribute the remaining rings across the selected days ($\binom{9}{2}$ choices for the first, $\binom{7}{2}$ choices for the second, and $\binom{5}{2}$ choices for the third), followed by arranging the last 3 rings (3! ways)."

We use a table to list all sequences and choices

Equivalent Sequence	Choices
6111111	$7\binom{12}{6}6!$
5211111	$7\binom{12}{5}6\binom{7}{2}5!$
4221111	$7\binom{12}{4}\binom{6}{2}\binom{8}{2}\binom{6}{2}4!$
4311111	$7\binom{12}{4}6\binom{8}{3}5!$
3321111	$\binom{7}{2}\binom{12}{3}\binom{9}{3}5\binom{6}{2}4!$
3222111	$7\binom{12}{3}\binom{6}{3}\binom{9}{2}\binom{7}{2}\binom{5}{2}3!$
2222211	$\binom{7}{5}\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}2!$

```r
(7 * choose(12, 6) * factorial(6) +
7 * choose(12, 5) * 6 * choose(7, 2) * factorial(5) +
7 * choose(12, 4) * choose(6, 2) * choose(8, 2) * choose(6, 2) * factorial(4) +
7 * choose(12, 4) * 6 * choose(8, 3) * factorial(5) +
choose(7, 2) * choose(12, 3) * choose(9, 3) * 5 * choose(6, 2) * factorial(4) +
7 * choose(12, 3) * choose(6, 3) * choose(9, 2) * choose(7, 2) *
choose(5, 2) * factorial(3) +
choose(7, 5) * choose(12, 2) * choose(10, 2) * choose(8, 2) *
choose(6, 2) * choose(4, 2) * factorial(2))/7^12
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## [1] 0.2284524
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The probability is 0.2285.

Problem 5

We use $k_i, i = 1, 2..., n$ to denote the number of occurrences of x_i in the sample we obtain. Note that in this problem, sampling with replacement is equivalent to placing n identical balls into n distinguishable cells without restrictions. We can also use k_i to represent the number of balls in cell i.

All possible ways of placing are n^n. When given $k_1, ...k_n$ is given, to get this sample, we first choose k_1 balls from the n balls to place into cell 1, then choose k_2 balls from the remaining $n - k_1$ balls to place into cell 2, and so on. The total number of ways to do this is

$$\frac{n!}{k_1!k_2!\cdots k_n!}$$

Given that $x_1, x_2, \ldots, x_n$ are all distinct values, in order to obtain the average $(x_1 + x_2 + \cdots + x_n)/n$, the only valid configuration is to draw exactly one occurrence for each x_i, which means $k_1 = k_2 = \cdots = k_n = 1$. Therefore, the probability of obtaining this specific average is:

$$\frac{n!}{n^n}$$

To demonstrate that this probability is the highest, consider that any other choices of k_i must lead to at least one k_i being greater than 1, with some k_i being zero. Since $0! = 1! = 1$, the configurations with $k_i = 0$ can be considered as equivalent to $k_i = 1$ in the previous configuration.

Now, when there exists at least one $k_i > 1$, i.e., $k_i \geq 2$, any other choice will result in a larger denominator than the configuration with $\frac{n!}{1!1!\cdots 1!}$. Therefore, the probability associated with the configuration where all k_i are equal to 1 is the largest.

Hence, the average $(x_1 + x_2 + \cdots + x_n)/n$ is the most likely and have the probability $\frac{n!}{n^n}$

(b)

Stirling's Formula is

$$n! \approx \sqrt{2\pi} n^{n+(1/2)} e^{-n}, n \to \infty$$

so

$$\frac{n!}{n^n} \approx \sqrt{2\pi} n^{1/2} e^{-n} = \frac{\sqrt{2n\pi}}{e^n}$$

(c)

Without loss of generality, let's assume that x_1 is the missing value. Using the ball-cell model, this implies that we do not place any balls in cell 1 and randomly distribute n balls among the remaining $n-1$ cells. Each ball has $n-1$ cells to choose from. Therefore, the probability is:

$$\frac{(n-1)^n}{n^n} = \left(1 - \frac{1}{n}\right)^n$$

When $n \to \infty$, by $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$,

$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1 + \left(-\frac{1}{n}\right)\right)^{-n}} = e^{-1}$$

Problem 6

Let X denote the number of correct questions in the 20 questions. It's obvious that

$$X \sim B(20, \frac{1}{4})$$

Therefore

$$P(X \geq 10) = \sum_{i=10}^{20} P(X = i) = \sum_{i=10}^{20} \binom{20}{i} \left(\frac{1}{4}\right)^i \left(\frac{3}{4}\right)^{20-i}$$

The answer is

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sum(choose(20, seq(10,20))*(1/4)^(seq(10,20))*(3/4)^(seq(10,0)))
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## [1] 0.01386442
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