STATS 510 HW4

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Problem 1

(4.1)

We convert the probability to expectation to calculate.
(a).

$$\begin{aligned}
P(X^2+Y^2<1) &= E(\mathbb{I}(X^2+Y^2<1)) \\
&= \int_{[-1,1]\times[-1,1]} \mathbb{I}(X^2+Y^2<1) f(x,y) dxdy \\
&= \frac{1}{4}\int_{x^2+y^2<1} 1\cdot dxdy \quad \text{(changing domain of double integral)} \\
&= \frac{1}{4}\text{Area}(x^2+y^2<1) \quad \text{(Defination of area)} \\
&= \frac{\pi}{4}
\end{aligned}$$

(b).
Note that the inequality $2x-y>0$ together with the square $[-1,1]\times[-1,1]$ define a trapezoid T, whose vertices are $(-0.5,-1),(1,-1),(1,1),(0.5,1)$. Similarly

$$\begin{aligned}
P(2X-Y>0) &= E(\mathbb{I}(2X-Y>0)) \\
&= \frac{1}{4}Area(\text{T}) \\
&= \frac{1}{4}\times\frac{1}{2}\times 2\times(1.5+0.5) \\
&= \frac{1}{2}
\end{aligned}$$

(c).
By triangle inequality, $|X+Y|\le|X|+|Y|$. Note that the point (X,Y) is distributed on a square $[-1,1]\times[-1,1]$, so $\max(|X|+|Y|)=1+1=2$. Therefore, the event $|X+Y|\le 2$ must happen. Note that a single point or line probability in $\mathbb{R}^2$ is zero, so

$$P(|X+Y|<2)=1$$

(4.10)

(a).
Note that

$$P(X = 1, Y = 4) = 0$$

however

$$P(X = 1) = \sum_y P(X = 1, Y = y) = 1/4$$
$$P(Y = 4) = \sum_x P(X = x, Y = 4) = 1/3$$

so

$$P(X = 1, Y = 4) \neq P(X = 1) \cdot P(Y = 4) = \frac{1}{12}$$

Therefore, X and Y are dependent.

(b).
We calculate the marginals at first and then use

$$P(U = u, V = v) = P(U = u)P(V = v)$$

to get the joint distribution table of U, V.

```
uv <- matrix(c(1/12,1/6,0,1/6,0,1/3,1/12,1/6,0), ncol=3)
colnames(uv) <- paste("u =",1:3)
rownames(uv) <- paste("v =",2:4)
#marginal
u <- colSums(uv); u
```

```
## u = 1 u = 2 u = 3
##  0.25  0.50  0.25
```

```
v <- rowSums(uv); v
```

```
##     v = 2     v = 3     v = 4
## 0.3333333 0.3333333 0.3333333
```

```
#joint distribution
uv <- tcrossprod(v, u)
colnames(uv) <- paste("u =",1:3)
rownames(uv) <- paste("v =",2:4)
uv
```

```
##            u = 1     u = 2      u = 3
## v = 2 0.08333333 0.1666667 0.08333333
## v = 3 0.08333333 0.1666667 0.08333333
## v = 4 0.08333333 0.1666667 0.08333333
```

Problem 2

(4.4)

(a).
By definition of probability

$$
\begin{aligned}
\int_{\mathbb{R}^{\nvDash}} f(x,y)dxdy = 1 \\
&= \int_0^1 \int_0^2 C(x+2y)dxdy \\
&= \int_0^1 dy \int_0^2 C(x+2y)dx \\
&= \int_0^1 C(2+4y)dy \\
&= C\left((2y+2y^2)\Big|_0^1 \right) \\
&= 4C
\end{aligned}
$$

Therefore,

$$C = \frac{1}{4}$$

(b).
Integrate over y

$$
\begin{aligned}
f(x) &= \int_0^1 f(x,y)dy \\
&= \frac{1}{4}\int_0^1 (x+2y)dy \\
&= \frac{1}{4}\left((xy+y^2)\Big|_0^1 \right) \\
&= \left(\frac{1}{4}x + \frac{1}{4} \right) \mathbb{I}(0 < x < 2)
\end{aligned}
$$

(c).
When $0 < y < 1$ and $0 < x < 2$

$$
\begin{aligned}
F(x,y) &= \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)dsdt \\
&= \int_0^x ds \int_0^y f(s,t)dt \\
&= \frac{1}{4}\int_0^x ds \int_0^y (s+2t)dt \\
&= \frac{1}{4}\int_0^x sy + y^2\, ds \\
&= \frac{x^2y}{8} + \frac{xy^2}{4}
\end{aligned}
$$

If $x \geq 2, 0 < y < 1$

$$\begin{aligned} F(x,y) &= \frac{1}{4}\int_0^y dt \int_0^2 (s+2t)ds \\ &= \frac{1}{4}\int_0^y 2+4t\,dt \\ &= \frac{1}{2}y + \frac{1}{2}y^2 \end{aligned}$$

If $0 < x < 2, y \geq 1$

$$\begin{aligned} F(x,y) &= \frac{1}{4}\int_0^x ds \int_0^1 (s+2t)dt \\ &= \frac{1}{4}\int_0^x s+1\,ds \\ &= \frac{1}{4}x + \frac{1}{8}x^2 \end{aligned}$$

If $x \geq 2, y \geq 1$

$$F(x,y) = 1$$

If $x \leq 0$ or $y \leq 0$

$$F(x,y) = 0$$

In total,

$$F(x,y) = \begin{cases} \frac{x^2y}{8} + \frac{xy^2}{4}, & 0 < x < 2, 0 < y < 1 \\ \frac{1}{4}x + \frac{1}{8}x^2, & 0 < x < 2, y \geq 1 \\ \frac{1}{2}y + \frac{1}{2}y^2, & x \geq 2, 0 < y < 1 \\ 1, & x \geq 2, y \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

(d).
The density of X is

$$f(x) = \left(\frac{1}{4}x + \frac{1}{4}\right)\mathbb{I}(0 < x < 2)$$

On $0 < x < 2$, the function $z = 9/(x+1)^2$ is monotone, and we have

$$x = \sqrt{\frac{9}{z}} - 1$$

So

$$\begin{aligned} f(z) &= \left|\frac{dx}{dz}\right| f(x) \\ &= \frac{3}{2}\frac{1}{z^{3/2}}\frac{1}{4}\sqrt{\frac{9}{z}} \\ &= \frac{9}{8z^2}\mathbb{I}(1 < z < 9) \end{aligned}$$

(4.5)

(a).
The domain defined by $x > \sqrt{y}, 0 \leq x \leq 1, 0 \leq y \leq 1$ is the area under the curve $y = x^2, 0 \leq x \leq 1$ in quadrant 1, denoting by D.

$$
\begin{aligned}
P(X > \sqrt{Y}) &= E(\mathbb{I}(\mathbb{X} > \sqrt{\mathbb{Y}})) \\
&= \int_D x + y \, dxdy \\
&= \int_0^1 dx \int_0^{x^2} x + y \, dy \\
&= \int_0^1 x^3 + \frac{x^4}{2} \, dx \\
&= \frac{7}{20}
\end{aligned}
$$

(b).
The domain defined by $x^2 < y < x, 0 \leq x \leq 1, 0 \leq y \leq 1$ is the area surrounded by the curve $y = x^2$ and $y = x$, denoting by D.

$$
\begin{aligned}
P(X^2 < Y < X) &= E(\mathbb{I}(\mathbb{X}^{\nvDash} < \mathbb{Y} < \mathbb{X})) \\
&= \int_D 2x \, dxdy \\
&= \int_0^1 dx \int_{x^2}^{x} 2x \, dy \\
&= \int_0^1 (2x^2 - 2x^3) dx \\
&= \frac{1}{6}
\end{aligned}
$$

Problem 3

(4.6)

We use hour as unit. Denote the difference of A and B's arrival time and 1 PM by X, Y, resp. It's known that

$$X \sim \mathcal{U}(0,1), Y \sim \mathcal{U}(0,1), X \perp\!\!\!\perp Y$$

So the joint distribution of (X, Y) is still uniform, on the square $[0,1] \times [0,1]$.
The length T of time that A waits for B is

$$T = \max\{Y - X, 0\}$$

We consider the cdf of T, $F_T(t)$.
When $t < 0$, obviously

$$F_T(t) = 0$$

When $t = 0$

$$\begin{aligned}
F_T(t) &= P(T \le t) \\
&= P(T < 0) + P(T = 0) = P(T = 0) \\
&= P(T = 0|Y < X)P(Y < X) + P(T = 0|Y \ge X)P(Y \ge X) \\
&= 1 \cdot P(Y < X) + P(Y = X|Y \ge X)P(Y \ge X)
\end{aligned}$$

Note that X, Y are continuous random variable, so

$$P(Y = X|Y \ge X) = 0$$

thus

$$P(T = 0) = P(Y < X)$$

The event $Y < X$ defines a triangle, so

$$F_T(t = 0) = P(Y < X) = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$

When $0 < t < 1$

$$\begin{aligned}
F_T(t) = P(T \le t) &= P(\max\{Y - X, 0\} \le t) \\
&= P(Y - X \le t, 0 \le t) \\
&= P(Y - X \le t) \\
&= 1 - P(Y - X > t) \\
&= 1 - \int_0^{1-t} dx \int_{x+t}^{1} 1\, dy \\
&= 1 - \int_0^{1-t} 1 - (x + t) dx \\
&= 1 - (1 - t - (1 - t)^2/2 - t(1 - t)) \\
&= \frac{1}{2} + t - \frac{t^2}{2}
\end{aligned}$$

When $t \ge 1$, obviously,

$$F_T(t) = 1$$

In total, the distribution of T is

$$F_T(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}, & t = 0 \\ \frac{1}{2} + t - \frac{t^2}{2}, & 0 < t < 1 \\ 1, & t \ge 1 \end{cases}$$

(4.12)

Three pieces requires two broken points. Denote them by X, Y. Without loss of generality, let the length of the stick be 1. Then

$$X \sim \mathcal{U}(0, 1), Y \sim \mathcal{U}(0, 1), X \perp\!\!\!\perp Y$$

The length of three pieces are

$$\begin{aligned}
T_1 &= \min\{X, Y\} = \frac{1}{2}(X + Y - |X - Y|) \\
T_2 &= |X - Y| \\
T_3 &= 1 - \max\{X, Y\} = 1 - \frac{1}{2}(X + Y + |X - Y|)
\end{aligned}$$

To form a triangle, it's required that

$$T_1 + T_2 \geq T_3, T_1 + T_3 \geq T_2, T_2 + T_3 \geq T_1$$

So the probability is

$$\begin{aligned}
P(\text{triangle}) &= P(T_1 + T_2 \geq T_3, T_1 + T_3 \geq T_2, T_2 + T_3 \geq T_1) \\
&= P(X + Y + |X - Y| \geq 1, |X - Y| \leq \frac{1}{2}, X + Y - |X - Y| \leq 1)
\end{aligned}$$

by removing the absolute symbol, we have

$$\begin{aligned}
&X - Y \geq 1 - X - Y \text{ or } X - Y \leq X + Y - 1, \\
&-1/2 \leq X - Y \leq 1/2 \\
&X - Y \geq X + Y - 1 \text{ or } X - Y \leq 1 - X - Y
\end{aligned}$$

simplify this, we have

$$X - Y \geq 1 - X - Y, -1/2 \leq X - Y \leq 1/2, X - Y \geq X + Y - 1 \quad (A)$$

or

$$X - Y \leq X + Y - 1, -1/2 \leq X - Y \leq 1/2, X - Y \leq X + Y - 1 \quad (B)$$

any other situations will result in zero probability. Event A defines a triangle, whose vertices are $(1/2, 0), (1/2, 1/2), (1, 1/2)$. Event B also defines a triangle, whose vertices are $(0, 1/2), (1/2, 1/2), (1/2, 1)$. Note that (X, Y) are uniform on the square, so

$$\begin{aligned}
P(\text{triangle}) &= P(A) + P(B) \\
&= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

Problem 4

(4.16)

(a).
The support of the joint distribution (U, V) is $\{u = 1, 2, \cdots, v = 0, \pm 1, \pm 2\}$. We calculate the pmf $P(U = u, V = v)$.
It's known that

$$X \sim \text{Geo}(p), Y \sim \text{Geo}(p), X \perp\!\!\!\perp Y$$

When $v > 0$, we have $X - Y = v > 0$, then $U = Y$, so

$$\begin{aligned} P(u, v) &= P(Y = u, X - Y = v) = P(Y = u, X = u + v) \\ &= (1-p)^{u+v-1}p(1-p)^{u-1}p \\ &= (1-p)^{2u+v-2}p^2 \end{aligned}$$

When $v < 0$, we have $X - Y = v < 0$, then $U = X$, so

$$\begin{aligned} P(u, v) &= P(X = u, X - Y = v) = P(X = u, Y = u - v) \\ &= (1-p)^{u-1}p(1-p)^{u-v-1}p \\ &= (1-p)^{2u-v-2}p^2 \end{aligned}$$

When $v = 0$, we have $X = Y$

$$\begin{aligned} P(u, v) &= P(X = Y = u) \\ &= (1-p)^{2u-2}p^2 \end{aligned}$$

In all three cases, we can write the pmf as

$$P(u, v) = (1-p)^{2u+|v|-2}p^2 = \left(p^2(1-p)^{2u}\right)(1-p)^{|v|-2}, \forall u, v$$

By Lemma 4.2.7, U and V are independent.
(b).
Note that both X, Y are positive integers, so the value of Z must be a positive rational number and less than 1. Denote it by $\frac{m}{n}, m < n$. Let $r = m, s = n - m$. We consider the X, Y pairs $(r, s), (2r, 2s), \cdots, (kr, ks), \cdots$. By the property of rational number, we have $Z = m/n$ iff. $(X, Y) = (kr, ks), k = 1, 2, \cdots$. Therefore

$$\begin{aligned} P(Z = \frac{m}{n}) &= \sum_{k=1}^{+\infty} P(X = kr, Y = ks) \\ &= \sum_{k=1}^{+\infty} P(X = kr)P(Y = ks) \quad (\text{by } X \perp\!\!\!\perp Y) \\ &= \sum_{k=1}^{+\infty}(1-p)^{kr-1}p(1-p)^{ks-1}p \\ &= \sum_{k=1}^{+\infty}(1-p)^{kr+ks-2}p^2 \\ &= \sum_{k=1}^{+\infty}(1-p)^{kn-2}p^2 \quad (\text{geometric sequence}) \\ &= \frac{p^2(1-p)^{n-2}}{1-(1-p)^n}, n = 2, 3, \cdots \end{aligned}$$

(c).
The support of $(X, X + Y)$ is $\{(s, t) | 1 \leq s < t, \ s, t \text{ are integers } \}$, so

$$\begin{aligned} P(X = s, X + Y = t) &= P(X = s, Y = t - s) \\ &= P(X = s)P(Y = t - s) \\ &= (1-p)^{s-1}p(1-p)^{t-s-1}p \\ &= (1-p)^{t-2}p^2 \end{aligned}$$

(4.17)

By definition of Y, we have

$$\begin{aligned}P(Y=i+1) &= P(i \leq X \leq i+1)\\ &= \int_i^{i+1} e^{-x}dx\\ &= -e^{-x}|_i^{i+1} = (1-e^{-1})e^{-i}\end{aligned}$$

Note that

$$(1-e^{-1})e^{-i} = (1-(1-e^{-1}))^i(1-e^{-1})$$

Therefore

$$Y \sim \text{Geo}(1-e^{-1})$$

(b).
$Y \geq 5$ means $Y=5, Y=6, Y=7, \cdots$, which is equivalent to

$$4 \leq X < 5, 5 \leq X < 6, \cdots \Rightarrow X \geq 4$$

Therefore, we consider the cdf of $X-4|Y \geq 5$

$$\begin{aligned}P(X-4 \leq x|Y \geq 5) &= P(X-4 \leq x|X \geq 4)\\ &= 1-P(X > x+4|X \geq 4)\end{aligned}$$

Note that X is exponential distribution, by its memorylessness

$$P(X-4 \leq x|Y \geq 5) = 1-P(X > x) = P(X \leq x) = 1-e^{-x}$$

So, the conditional distribution of $X-4$ given $Y \geq 5$ is exponential(1).

Problem 5 (4.19)

(a)

From

$$X_1, X_2 \overset{\text{i.i.d}}{\sim} N(0,1)$$

by the property of normal distribution, we have

$$X_1 - X_2 \sim N(0,2)$$

thus

$$\frac{1}{\sqrt{2}}(X_1-X_2) \sim N(0,1)$$

By the relationship between normal distribution and chi-squared distribution, we have

$$\frac{(X_1-X_2)^2}{2} \sim \chi^2(1)$$

(b) It's known that

$$X_1 \sim \Gamma(\alpha_1, 1), X_2 \sim \Gamma(\alpha_2, 1), X_1 \perp\!\!\!\perp X_2$$

so the joint distribution is

$$f(x_1, x_2) = \frac{x_1^{\alpha_1 - 1} e^{-x_1}}{\Gamma(\alpha_1)} \frac{x_2^{\alpha_2 - 1} e^{-x_2}}{\Gamma(\alpha_2)}$$

Let

$$Y_1 = \frac{X_1}{X_1 + X_2}, Y_2 = X_1 + X_2$$

then

$$X_1 = Y_1 Y_2, X_2 = Y_2(1 - Y_1)$$

So, we derive a one-to-one transformation between (X_1, X_2) and (Y_1, Y_2). By (4.3.2),

$$f(y_1, y_2) = f(x_1, x_2)|J|$$

in which

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{vmatrix} = |y_2| = y_2$$

Therefore

$$\begin{aligned} f(y_1, y_2) &= \frac{x_1^{\alpha_1 - 1} e^{-x_1}}{\Gamma(\alpha_1)} \frac{x_2^{\alpha_2 - 1} e^{-x_2}}{\Gamma(\alpha_2)} y_2 \\ &= \frac{(y_1 y_2)^{\alpha_1 - 1} e^{-y_1 y_2}}{\Gamma(\alpha_1)} \frac{(y_2(1 - y_1))^{\alpha_2 - 1} e^{-y_2(1 - y_1)}}{\Gamma(\alpha_2)} y_2 \\ &= \frac{y_1^{\alpha_1 - 1}(1 - y_1)^{\alpha_2 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)/\Gamma(\alpha_1 + \alpha_2)} \frac{y_2^{\alpha_1 + \alpha_2 - 1} e^{-y_2}}{\Gamma(\alpha_1 + \alpha_2)} \\ &= \frac{y_1^{\alpha_1 - 1}(1 - y_1)^{\alpha_2 - 1}}{\mathrm{B}(\alpha_1, \alpha_2)} \frac{y_2^{\alpha_1 + \alpha_2 - 1} e^{-y_2}}{\Gamma(\alpha_1 + \alpha_2)} \end{aligned}$$

Note that the joint distribution can be factorized to two parts and we can read out the marginal distribution of Y_1 and Y_2. That is

$$Y_1 \sim \text{Beta}(\alpha_1, \alpha_2), \quad Y_2 \sim \Gamma(\alpha_1 + \alpha_2)$$

Moreover, note that

$$\frac{X_2}{X_1 + X_2} = 1 - Y_1 := U$$

so

$$\begin{aligned} f_U(u) = f(y_1) \left| \frac{dy_1}{du} \right| \\ &= \frac{y_1^{\alpha_1 - 1}(1 - y_1)^{\alpha_2 - 1}}{\mathrm{B}(\alpha_1, \alpha_2)} \cdot 1 \\ &= \frac{u^{\alpha_2 - 1}(1 - u)^{\alpha_1 - 1}}{\mathrm{B}(\alpha_1, \alpha_2)} \\ &\sim \text{Beta}(\alpha_2, \alpha_1) \end{aligned}$$

Overall, the marginal distribution of $X_1/(X_1 + X_2)$ is $\text{Beta}(\alpha_1, \alpha_2)$ and the marginal distribution of $X_2/(X_1 + X_2)$ is $\text{Beta}(\alpha_2, \alpha_1)$.

Problem 6 (4.26)

(a)

It's known that

$$X \sim \mathcal{E}(\lambda), Y \sim \mathcal{E}(\mu), X \perp\!\!\!\perp Y$$

so the joint distribution of (X, Y) is

$$f(x,y) = f(x)f(y) = \lambda e^{-\lambda x} \mu e^{-\mu y}, x, y \geq 0$$

We consider the cdf $P(Z \leq z, W \leq w)$ of this distribution. Obviously, if $w < 0$ or $z < 0$, $P(Z \leq z, W \leq w) = 0$.

If $0 \leq w < 1, z \geq 0$, we have

$$\begin{aligned} P(Z \leq z, W \leq w) &= P(Z \leq z, W < 0) \\ &\quad + P(Z \leq z, W = 0) \\ &\quad + P(Z \leq z, 0 < W \leq w) \\ &= P(Z \leq z, W = 0) \end{aligned}$$

then

$$\begin{aligned} P(Z \leq z, W = 0) &= P(\min\{X, Y\} \leq z, Z = Y) \\ &= P(Y \leq z, Y \leq X) \\ &= \int_0^z \int_y^{+\infty} f(x,y) dx dy \\ &= \int_0^z \mu e^{-\mu y} e^{-\lambda y} dy \\ &= \frac{\mu}{\mu + \lambda} \left(1 - e^{-(\lambda+\mu)z}\right) \end{aligned}$$

so

$$P(Z \leq z, W \leq w) = \frac{\mu}{\mu + \lambda} \left(1 - e^{-(\lambda+\mu)z}\right), 0 \leq w < 1, z \geq 0$$

If $w \geq 1, z \geq 0$, we have

$$\begin{aligned} P(Z \leq z, W \leq w) &= P(Z \leq z, W < 0) + P(Z \leq z, W = 0) \\ &\quad + P(Z \leq z, 0 < W < 1) + P(Z \leq z, W = 1) \\ &\quad + P(Z \leq z, 1 < W \leq w) \\ &= P(Z \leq z, W = 0) + P(Z \leq z, W = 1) \end{aligned}$$

we calculate $P(Z \leq z, W = 1)$.

$$\begin{aligned} P(Z \leq z, W = 1) &= P(\min\{X, Y\} \leq z, Z = X) \\ &= P(X \leq z, X \leq Y) \\ &= \int_0^z \int_x^{+\infty} f(x,y) dy dx \\ &= \int_0^z \lambda e^{-\lambda x} e^{-\mu x} dx \\ &= \frac{\lambda}{\mu + \lambda} \left(1 - e^{-(\lambda+\mu)z}\right) \end{aligned}$$

So

$$P(Z \le z, W \le w) = 1 - e^{-(\lambda+\mu)z}, 1 \le w, z \ge 0$$

Overall, the cdf of (Z, W) is

$$F_{Z,W}(z,w) = \begin{cases} \frac{\mu}{\mu+\lambda}\left(1 - e^{-(\lambda+\mu)z}\right), & 0 \le w < 1, z \ge 0 \\ 1 - e^{-(\lambda+\mu)z}, & 1 \le w, z \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

(b)

From (a)

$$P(Z \le z) = P(Z \le z, W = 0) + P(Z \le z, W = 1) = 1 - e^{-(\lambda+\mu)z}$$

$$\begin{aligned} P(Z \le z | W = 0) &= \frac{P(Z \le z, W = 0)}{P(W = 0)} \\ &= \frac{P(Z \le z, W = 0)}{P(Z \le +\infty, W = 0)} \\ &= \frac{\frac{\mu}{\mu+\lambda}\left(1 - e^{-(\lambda+\mu)z}\right)}{\frac{\mu}{\mu+\lambda}} \\ &= 1 - e^{-(\lambda+\mu)z} = P(Z \le z) \end{aligned}$$

$$\begin{aligned} P(Z \le z | W = 1) &= \frac{P(Z \le z, W = 1)}{P(W = 1)} \\ &= \frac{P(Z \le z, W = 1)}{P(Z \le +\infty, W = 1)} \\ &= \frac{\frac{\lambda}{\mu+\lambda}\left(1 - e^{-(\lambda+\mu)z}\right)}{\frac{\lambda}{\mu+\lambda}} \\ &= 1 - e^{-(\lambda+\mu)z} = P(Z \le z) \end{aligned}$$

Therefore,

$$P(Z \le z, W = i) = P(Z \le z)P(W = i), i = 0, 1$$

so Z and W are independent.