1.1

let

$C_{\mu\alpha\beta\nu} l^{(a} n^{b)} \neq C_{\mu\alpha\beta\nu} m^{(a} \bar{m}^{b)}$. then $C_{\mu\alpha\beta\nu} l^{(a} n^{b)} - C_{\mu\alpha\beta\nu} m^{(a} \bar{m}^{b)} \neq 0$. But

$C_{\mu\alpha\beta\nu} ( l^{(a} n^{b)} - m^{(a} \bar{m}^{b)} ) = -\frac{1}{2} C_{\mu\alpha\beta\nu} (-2 l^{(a} n^{b)} + m^{(a} \bar{m}^{b)}) = -\frac{1}{2} C_{\mu\alpha\beta\nu} g^{ab} = 0$

thus $C_{\mu\alpha\beta\nu} l^{(a} n^{b)} = C_{\mu\alpha\beta\nu} m^{(a} \bar{m}^{b)}$

1.2)

$\Psi_0 = C_{\alpha\beta\gamma\delta} l^a m^b l^\gamma m^\delta \to C_{\alpha\beta\gamma\delta} l^a (m^b + \epsilon a l^b) l^\gamma (m^\delta + \epsilon a l^\delta)$

$= C_{\alpha\beta\gamma\delta} l^a m^b l^\gamma m^\delta + C_{\alpha\beta\gamma\delta} l^a (\cancel{\epsilon a l^b}) l^\gamma m^\delta$

$C_{\alpha\beta\gamma\delta} \cancel{l^a m^b} l^\gamma \epsilon a l^\delta + O(\epsilon^2)$

$C_{\alpha\beta\gamma\delta} l^a l^b = - C_{\alpha\beta\gamma\delta} l^a l^b = 0$

We have shown $\Psi_0 \to \Psi_0 + O(\epsilon^2)$

$\Psi_1 = C_{\alpha\beta\gamma\delta} l^a m^b l^\gamma n^\delta \to C_{\alpha\beta\gamma\delta} l^a (m^b + \epsilon a l^b) l^\gamma (n^\delta + \epsilon a^* m^\delta + \epsilon a \bar{m}^\delta)$

$= C_{\alpha\beta\gamma\delta} l^a m^b l^\gamma n^\delta + C_{\alpha\beta\gamma\delta} l^a m^b l^\gamma \epsilon a^* m^\delta$

$+ C_{\alpha\beta\gamma\delta} l^a m^b l^\gamma \epsilon a \bar{m}^\delta$

$= C_{\alpha\beta\gamma\delta} l^a m^b l^\gamma n^\delta + C_{\alpha\beta\gamma\delta} l^a m^b l^\gamma (a^* m^\delta + a \bar{m}^\delta)$

$$g^{\mu\nu} = -2\,l^{(\mu}n^{\nu)} + 2\,m^{(\mu}\bar{m}^{\nu)} \iff g^{\beta\delta} = -2\,l^{(\beta}n^{\delta)} + 2\,m^{(\beta}\bar{m}^{\delta)}$$

$$g^{\beta\delta}l^{\gamma} = -2\,l^{(\beta}n^{\delta)} + 2\,m^{(\beta}\bar{m}^{\delta)}l^{\gamma}$$

$$g^{\beta\delta}l^{\gamma} = -l^{\beta}n^{\delta}l^{\gamma} - l^{\delta}n^{\beta}l^{\gamma} + m^{\beta}l^{\gamma}\bar{m}^{\delta} + m^{\delta}\bar{m}^{\beta}l^{\gamma}$$

$$C_{\alpha\beta\gamma\delta}l^{\alpha}g^{\beta\delta}l^{\gamma} = -C_{\alpha\beta\gamma\delta}l^{\alpha}l^{\beta}n^{\delta}l^{\gamma} - C_{\alpha\beta\gamma\delta}l^{\alpha}l^{\delta}n^{\beta}l^{\gamma}$$
$$+ C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}l^{\gamma}\bar{m}^{\delta} + C_{\alpha\beta\gamma\delta}l^{\alpha}\bar{m}^{\beta}l^{\gamma}m^{\delta}$$

Note that $C_{\alpha\beta\gamma\delta}l^{\alpha}\bar{m}^{\beta}l^{\gamma}m^{\delta} = C_{\gamma\delta\alpha\beta}l^{\alpha}\bar{m}^{\beta}l^{\gamma}m^{\delta} \xrightarrow{\text{rename } \alpha\leftrightarrow\gamma,\ \beta\leftrightarrow\delta} C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}l^{\gamma}\bar{m}^{\delta}$

Note $C_{\alpha\beta\gamma\delta}l^{\alpha}g^{\beta\delta}l^{\gamma} = 0$ $C$ is the traceless part of $R_{\mu\nu\rho\sigma}$

More over $C_{\alpha\beta\gamma\delta} = -C_{\beta\alpha\gamma\delta}$ but $l^{\alpha}l^{\beta} = l^{\beta}l^{\alpha}$ So this contraction is 0

finally $C_{\alpha\beta\gamma\delta}l^{\alpha}n^{\beta}l^{\gamma}l^{\delta} = -C_{\alpha\beta\delta\gamma}l^{\alpha}n^{\beta}l^{\gamma}l^{\delta} \underset{\text{rename } \gamma\leftrightarrow\delta}{=\!=\!=} -C_{\alpha\beta\gamma\delta}l^{\alpha}n^{\beta}l^{\gamma}l^{\delta} = 0$

thus $C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}l^{\gamma}\bar{m}^{\delta} + C_{\alpha\beta\gamma\delta}l^{\alpha}\bar{m}^{\beta}l^{\gamma}m^{\delta} = 2C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}l^{\gamma}\bar{m}^{\delta} = 0$

$\therefore\ \Psi_1 \to \Psi_1 + \varepsilon a^* \Psi_0$

$\Psi_2 = C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}\bar{m}^{\gamma}n^{\delta} \longrightarrow C_{\alpha\beta\gamma\delta}l^{\alpha}(m^{\beta} + \varepsilon a l^{\beta})(\bar{m}^{\gamma} + \varepsilon a^* l^{\gamma})(n^{\delta} + a^*\varepsilon m^{\delta} + a\varepsilon\bar{m}^{\delta})$

$= C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}\bar{m}^{\gamma}n^{\delta} + C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}\bar{m}^{\gamma}a^*\varepsilon n^{\delta} + C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}\cancel{\bar{m}^{\gamma}}a\varepsilon\bar{m}^{\delta}$ 0 Symmetric $\bar{m}^{\gamma}\bar{m}^{\delta}$

$+ C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}\varepsilon a^* l^{\gamma}n^{\delta}$

$C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}\bar{m}^{\gamma}m^{\delta}$ $\quad g^{\beta\gamma} = -2\,l^{(\beta}n^{\gamma)} + 2\,m^{(\beta}\bar{m}^{\gamma)}$

$$C_{\alpha\beta\gamma\delta}l^{\alpha}g^{\beta\gamma}m^{\delta} = -C_{\alpha\beta\gamma\delta}l^{\alpha}l^{\beta}n^{\gamma}m^{\delta} - C_{\alpha\beta\gamma\delta}l^{\alpha}n^{\beta}l^{\gamma}m^{\delta}$$
$$+ C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}\bar{m}^{\gamma}m^{\delta} + C_{\alpha\beta\gamma\delta}l^{\alpha}\bar{m}^{\beta}\cancel{m^{\gamma}}m^{\delta}$$
0

$\therefore\ C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta \bar{m}^\gamma m^\delta = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta l^\gamma m^\delta \underset{\beta \leftrightarrow \delta}{=} C_{\alpha\delta\gamma\beta}\, l^\alpha m^\beta l^\gamma n^\delta$

$$\overset{\alpha\leftrightarrow\gamma}{=} C_{\gamma\delta\alpha\beta}\, l^\alpha m^\beta l^\gamma n^\delta = C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma n^\delta$$

$\therefore\ \Psi_2 \to \Psi_2 + 2a^*\varepsilon\,\Psi_1$

$$\Psi_3 = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta \to C_{\alpha\beta\gamma\delta}\, l^\alpha (n^\beta + a^*\varepsilon m^\beta + a\varepsilon \bar{m}^\beta)(\bar{m}^\gamma + a^*\varepsilon l^\gamma)(n^\delta + a^*\varepsilon m^\delta + a\varepsilon\bar{m}^\delta)$$

$$C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma a^*\varepsilon m^\delta + \overset{0}{\cancel{C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma a\varepsilon\bar{m}^\delta}}$$

$$+ C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta a^*\varepsilon l^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha a^*\varepsilon m^\beta \bar{m}^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha a\varepsilon \bar{m}^\beta \bar{m}^\gamma n^\delta$$

Note $C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta l^\gamma n^\delta = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta l^\gamma n^\delta + \underbrace{C_{\alpha\beta\gamma\delta}\, l^\alpha l^\beta n^\gamma n^\delta}_{\text{this is 0 so I can add it with no problem}} = 2 C_{\alpha\beta\gamma\delta}\, l^\alpha l^{(\beta} n^{\gamma)} n^\delta$

$= 2 C_{\alpha\beta\gamma\delta}\, l^\alpha m^{(\beta} \bar{m}^{\gamma)} n^\delta$

focus on these terms

$$C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta + 2C_{\alpha\beta\gamma\delta}\, l^\alpha \bar{m}^\beta m^\gamma n^\delta + \underset{0}{\cancel{C_{\alpha\beta\gamma\delta}\, l^\alpha \bar{m}^\beta \bar{m}^\gamma n^\delta}}$$

$$g^{\alpha\delta} = -l^\alpha n^\delta - n^\alpha l^\delta + m^\alpha \bar{m}^\delta + \bar{m}^\alpha m^\delta$$

$$C_{\alpha\beta\gamma\delta}\, g^{\alpha\delta} \bar{m}^\beta \bar{m}^\gamma = -C_{\alpha\beta\gamma\delta}\, l^\alpha \bar{m}^\beta \bar{m}^\gamma n^\delta - C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta \bar{m}^\gamma l^\delta + \overset{0}{\cancel{C_{\alpha\beta\gamma\delta}\, m^\alpha \bar{m}^\beta \bar{m}^\gamma \bar{m}^\delta}} + \underset{0}{\cancel{C_{\alpha\beta\gamma\delta}\, \bar{m}^\alpha \bar{m}^\beta \bar{m}^\gamma m^\delta}}$$

$$C_{\alpha\beta\gamma\delta}\, l^\alpha \bar{m}^\beta \bar{m}^\gamma n^\delta = -C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta \bar{m}^\gamma l^\delta$$

But $C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta \bar{m}^\gamma l^\delta = C_{\gamma\delta\alpha\beta}\, n^\alpha \bar{m}^\beta \bar{m}^\gamma l^\delta$

$= C_{\alpha\beta\gamma\delta}\, l^\alpha \bar{m}^\beta \bar{m}^\gamma n^\delta$

$$\therefore\ C_{\alpha\beta\gamma\delta}\, l^\alpha \bar{m}^\beta \bar{m}^\gamma n^\delta = -C_{\alpha\beta\gamma\delta}\, l^\alpha \bar{m}^\beta \bar{m}^\gamma n^\delta = 0$$

Now

$$C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta + 2C_{\alpha\beta\gamma\delta}\, l^\alpha \bar{m}^\beta m^\gamma n^\delta = 0$$

since $C_{\alpha[\beta\gamma\delta]} = 0$

and $C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar m^\gamma m^\delta + 2C_{\alpha\beta\gamma\delta}\, l^\alpha \bar m^\beta m^\gamma n^\delta =$

$C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar m^\gamma m^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha \bar m^\beta m^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha m^\gamma n^\delta \bar m^\beta =$

$= C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar m^\gamma m^\delta + C_{\alpha\gamma\delta\beta}\, \bar m^\gamma m^\delta n^\beta + C_{\alpha\gamma\delta\beta}\, l^\alpha \bar m^\gamma m^\delta n^\beta = 0$

So what is left is

$2C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta \bar m^\gamma n^\delta \cdot a^*\epsilon + C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar m^\gamma n^\delta a^*\epsilon + \Psi_3$

$\therefore \Psi_3 \to \Psi_3 + 3a^*\epsilon\, C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta \bar m^\gamma n^\delta = \Psi_3 + 3a^*\epsilon\,\Psi_2$

$\Psi_4 = C_{\alpha\beta\gamma\delta}\, n^\alpha \bar m^\beta n^\gamma \bar m^\delta$

$\Psi_4 \to C_{\alpha\beta\gamma\delta}\,(n^\alpha + a^*\epsilon m^\alpha + a\epsilon \bar m^\alpha)(\bar m^\beta + \epsilon a^* l^\beta)(n^\gamma + a^*\epsilon m^\gamma + a\epsilon \bar m^\gamma)(\bar m^\delta + a^*\epsilon l^\delta)$

$= C_{\alpha\beta\gamma\delta}\, n^\alpha \bar m^\beta n^\gamma \bar m^\delta + \overset{(1)}{C_{\alpha\beta\gamma\delta}\, n^\alpha \bar m^\beta n^\gamma a^*\epsilon l^\delta} + C_{\alpha\beta\gamma\delta}\, n^\alpha \bar m^\beta a^*\epsilon m^\gamma \bar m^\delta$ (3)

$+ C_{\alpha\beta\gamma\delta}\, n^\alpha \bar m^\beta \overset{0}{\cancel{a\epsilon \bar m^\gamma}} \bar m^\delta + \underset{(2)}{C_{\alpha\beta\gamma\delta}\, n^\alpha \epsilon a^* l^\beta n^\gamma \bar m^\delta} + C_{\alpha\beta\gamma\delta}\, a^*\epsilon m^\alpha \bar m^\beta n^\gamma \bar m^\delta$ (4)

$+ C_{\alpha\beta\gamma\delta}\, a\epsilon \underbrace{\bar m^\alpha \bar m^\beta}_{0} n^\gamma \bar m^\delta$

(1) $C_{\alpha\beta\gamma\delta}\, n^\alpha \bar m^\beta n^\gamma l^\delta = C_{\gamma\delta\alpha\beta}\, n^\gamma l^\delta n^\alpha \bar m^\beta = C_{\delta\gamma\beta\alpha}\, l^\delta n^\gamma \bar m^\beta n^\alpha = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar m^\gamma n^\delta$

(2) $C_{\alpha\beta\gamma\delta}\, n^\alpha l^\beta n^\gamma \bar m^\delta = C_{\beta\alpha\delta\gamma}\, l^\beta n^\alpha \bar m^\delta n^\gamma = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar m^\gamma n^\delta$

0 symmetry

0 symmetry

(3) $C_{\alpha\beta\gamma\delta}\, n^\alpha \bar m^\beta m^\gamma \bar m^\delta = -C_{\alpha\beta\gamma\delta}\, n^\alpha n^\beta \bar m^\gamma \bar m^\delta + C_{\alpha\beta\gamma\delta}\, n^\alpha l^\beta n^\gamma \bar m^\delta + C_{\alpha\beta\gamma\delta}\, n^\alpha n^\beta l^\gamma \bar m^\delta$

$= C_{\alpha\beta\gamma\delta}\, n^\alpha l^\beta n^\gamma \bar m^\delta = C_{\beta\alpha\delta\gamma}\, l^\beta n^\alpha \bar m^\delta n^\gamma = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar m^\gamma n^\delta$

(4) $C_{\alpha\beta\gamma\delta}\, m^\alpha \bar m^\beta n^\gamma \bar m^\delta = C_{\gamma\delta\alpha\beta}\, n^\gamma \bar m^\delta m^\alpha \bar m^\beta = C_{\alpha\beta\gamma\delta}\, n^\alpha \bar m^\beta m^\gamma \bar m^\delta = (3)$

$\therefore (3) + (4) = 2C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar m^\gamma n^\delta$

$\therefore (1)+(2)+(3)+(4) = 4\overbrace{C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar m^\gamma n^\delta}^{\Psi_3}$ thus, $\Psi_4 \to \Psi_4 + a^*\epsilon\, 4\Psi_3$

c)

$\Psi_1 = C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma n^\delta$

$\Psi_1 \to C_{\alpha\beta\gamma\delta}\,(l^\alpha + \epsilon b^* m^\alpha + \epsilon b \bar{m}^\alpha)(m^\beta + \epsilon b n^\beta)(l^\gamma + \epsilon b^* m^\gamma + \epsilon b \bar{m}^\gamma)(n^\delta + \epsilon b n^\delta)$

0 Symmetry

(1)

$= C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma m^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma \epsilon b n^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha m^b \cancel{\epsilon b^* m^\gamma} m^\delta$

(2) (3) 0 Symmetry

$+ C_{\alpha\beta\gamma\delta}\, l^\alpha m^b \epsilon b \bar{m}^\gamma m^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha \epsilon b n^\beta l^\gamma m^\delta + C_{\alpha\beta\gamma\delta}\, \epsilon b^* \cancel{m^\alpha} m^\beta l^\gamma m^\delta$

$+ C_{\alpha\beta\gamma\delta}\, \epsilon b \bar{m}^\alpha m^\beta l^\gamma m^\delta$ (4)

(1) $C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma n^\delta = \Psi_1$

0 Symmetry 0 Symmetry

(2) $C_{\alpha\beta\gamma\delta}\, l^\alpha m^b \bar{m}^\gamma m^\delta = -C_{\alpha\beta\gamma\delta}\, l^\alpha \cancel{n^\gamma} m^b m^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha \cancel{l^b} n^\gamma m^\delta$

$+ C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta l^\gamma m^\delta = C_{\gamma\delta\alpha\beta}\, l^\gamma m^\delta l^\alpha n^\beta = C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma n^\delta$
$= \Psi_1$

(3) $C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta l^\gamma m^\delta = C_{\gamma\delta\alpha\beta}\, l^\gamma m^\delta l^\alpha n^\beta = C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma n^\delta = \Psi_1$

(4) $C_{\alpha\beta\gamma\delta}\, \bar{m}^\alpha m^\beta l^\gamma m^\delta = C_{\gamma\delta\alpha\beta}\, l^\gamma m^\delta \bar{m}^\alpha m^\beta = C_{\alpha\beta\gamma\delta}\, l^\alpha m^b \bar{m}^\gamma m^\delta = (2)$

$\therefore$ (1)+(2)+(3)+(4) $= 4\Psi_1$

thus $\Psi_0 \to \Psi_0 + \epsilon b \Psi_1$

$\Psi_1 = C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma n^\delta \to C_{\alpha\beta\gamma\delta}\,(l^\alpha + \epsilon b^* m^\alpha + \epsilon b \bar{m}^\alpha)(m^\beta + \epsilon b n^\beta)(l^\gamma + \epsilon b^* m^\gamma + \epsilon b \bar{m}^\gamma) \times n^\delta$

(1) (2)

$= C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta \epsilon b^* m^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta \epsilon b \bar{m}^\gamma n^\delta +$

(3) (n)

$C_{\alpha\beta\gamma\delta}\, l^\alpha \epsilon b n^\beta l^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, \epsilon b^* \cancel{m^\alpha} m^\beta l^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, \epsilon b \bar{m}^\alpha m^\beta l^\gamma n^\delta$

0 Symmetry

$(1)$: $C_{\alpha\beta\gamma\delta} l^\alpha m^\beta m^\gamma n^\delta = C_{\gamma\delta\alpha\beta} m^\gamma n^\delta l^\alpha m^\beta = C_{\alpha\beta\gamma\delta} m^\alpha n^\beta l^\gamma m^\delta$

0 symmetry  0 Symmetry

$= - C_{\alpha\beta\gamma\delta} m^\alpha l^\beta n^\gamma m^\delta + C_{\alpha\beta\gamma\delta} m^\alpha \cancel{m^\beta} \bar{m}^\gamma m^\delta + C_{\alpha\beta\gamma\delta} m^\alpha \bar{m}^\beta \cancel{m^\gamma} m^\delta$

$\therefore C_{\alpha\beta\gamma\delta} m^\alpha n^\beta l^\gamma m^\delta = - C_{\alpha\beta\gamma\delta} m^\alpha l^\beta n^\gamma m^\delta$

but $C_{\alpha\beta\gamma\delta} m^\alpha n^\beta l^\gamma m^\delta = C_{\beta\alpha\delta\gamma} n^\beta m^\alpha m^\delta l^\gamma$ rename $a \to \delta$ $= C_{\beta\delta\alpha\gamma} n^\beta n^\delta m^\alpha l^\gamma$

$= C_{\alpha\gamma\beta\delta} m^\alpha l^\gamma n^\beta m^\delta$

$= C_{\alpha\beta\gamma\delta} m^\alpha l^\beta n^\gamma m^\delta$

$\therefore C_{\alpha\beta\gamma\delta} m^\alpha l^\beta n^\gamma m^\delta = - C_{\alpha\beta\gamma\delta} m^\alpha l^\beta n^\gamma m^\delta = 0$ thus $(1) = 0$

$(2)$: $C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta = \Psi_2$

0 Symetry

$(3)$: $C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma n^\delta = - C_{\alpha\beta\gamma\delta} l^\alpha \cancel{l^\beta} n^\gamma n^\delta + \underbrace{C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta}_{\Psi_2}$

$+ C_{\alpha\beta\gamma\delta} l^\alpha \bar{m}^\beta m^\gamma n^\delta$

this is (3)

$(4)$: $C_{\alpha\beta\gamma\delta} \bar{m}^\alpha m^\beta l^\gamma n^\delta = - C_{\alpha\beta\gamma\delta} m^\alpha \bar{m}^\beta l^\gamma n^\delta + C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma n^\delta$

$+ C_{\alpha\beta\gamma\delta} l^\alpha \cancel{l^\beta} n^\gamma n^\delta$

0 Symmetry

$= - C_{\alpha\beta\gamma\delta} m^\alpha \bar{m}^\beta l^\gamma n^\delta + \underbrace{C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta}_{\Psi_2}$

$+ C_{\alpha\beta\gamma\delta} l^\alpha \bar{m}^\beta m^\gamma n^\delta$

$(3)+(4)$

$\therefore C_{\alpha\beta\gamma\delta} \bar{m}^\alpha m^\beta l^\gamma n^\delta + 2 C_{\alpha\beta\gamma\delta} l^\alpha \bar{m}^\beta m^\gamma n^\delta = 0$ Same as question b

thus $\Psi_1 \to \Psi_1 + 3\varepsilon b \Psi_2$

$\Psi_2 = C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta = C_{\alpha\beta\gamma\delta} (l^\alpha + b\varepsilon^* m^\alpha + b\varepsilon \bar{m}^\alpha)(m^\beta + \varepsilon b n^\beta)(\bar{m}^\gamma + \varepsilon b^* n^\gamma) n^\delta$

0 Symmetry  (1)

$= C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta + C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \varepsilon \cancel{b^*} n^\gamma n^\delta + C_{\alpha\beta\gamma\delta} l^\alpha \varepsilon b n^\beta \bar{m}^\gamma n^\delta$

0 Symmetry

$+ C_{\alpha\beta\gamma\delta} b \varepsilon^* \cancel{m^\alpha} m^\beta \bar{m}^\gamma n^\delta + C_{\alpha\beta\gamma\delta} b\varepsilon \bar{m}^\alpha m^\beta \bar{m}^\gamma n^\delta$ (2)

(1) : $C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta = \Psi_3$ 0 Symmetry 0 Symmetry

(2) : $C_{\alpha\beta\gamma\delta}\, \bar{m}^\alpha m^\beta \bar{m}^\gamma n^\delta = -C_{\alpha\beta\gamma\delta}\, \bar{m}^\alpha \bar{m}^\beta m^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, \bar{m}^\alpha l^\beta n^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, \bar{m}^\alpha n^\beta l^\gamma n^\delta$

$= C_{\gamma\delta\alpha\beta}\, l^\gamma n^\delta \bar{m}^\alpha n^\beta = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta = \Psi_3$

(1) + (2) $= 2\Psi_3$

$\therefore \Psi_2 \to \Psi_2 + 2b\varepsilon\Psi_3$

$\Psi_3 = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta \to C_{\alpha\beta\gamma\delta}\, (l^\alpha + \varepsilon b^* m^\alpha + \varepsilon b \bar{m}^\alpha)\, n^\beta\, (\bar{m}^\gamma + \varepsilon b^* n^\gamma)\, n^\delta$

0 Symmetry

$= C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \varepsilon b^* n^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, \varepsilon b^* m^\alpha n^\beta \bar{m}^\gamma n^\delta$ (1)

$+ C_{\alpha\beta\gamma\delta}\, \varepsilon b \bar{m}^\alpha n^\beta \bar{m}^\gamma n^\delta$ (2)

(1) $C_{\alpha\beta\gamma\delta}\, m^\alpha n^\beta \bar{m}^\gamma n^\delta = -C_{\beta\alpha\gamma\delta}\, n^\beta m^\alpha \bar{m}^\gamma n^\delta \overset{\text{rename}}{=} -C_{\alpha\beta\gamma\delta}\, n^\alpha m^\beta \bar{m}^\gamma n^\delta$

$= C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta m^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, n^\alpha l^\beta n^\gamma n^\delta + C_{\alpha\beta\gamma\delta}\, n^\alpha n^\beta l^\gamma n^\delta$

0 Symmetry 0 Symmetry

$\therefore C_{\alpha\beta\gamma\delta}\, n^\alpha m^\beta \bar{m}^\gamma n^\delta = -C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta m^\gamma n^\delta$

But $C_{\alpha\beta\gamma\delta}\, n^\alpha m^\beta \bar{m}^\gamma n^\delta = C_{\gamma\delta\alpha\beta}\, \bar{m}^\gamma n^\delta n^\alpha m^\beta = C_{\delta\gamma\beta\alpha}\, n^\delta \bar{m}^\gamma m^\beta n^\alpha$

rename $= C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta m^\gamma n^\delta$ $\therefore C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta m^\gamma n^\delta = -C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta m^\gamma n^\delta = 0$

$\therefore (1) = 0$

(2) : $C_{\alpha\beta\gamma\delta}\, \varepsilon b \bar{m}^\alpha n^\beta \bar{m}^\gamma n^\delta = C_{\gamma\delta\alpha\beta}\, \bar{m}^\gamma n^\delta \bar{m}^\alpha n^\beta = C_{\delta\gamma\beta\alpha}\, n^\delta \bar{m}^\gamma n^\beta \bar{m}^\alpha$

rename $= C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta = \Psi_4$

$\therefore \Psi_3 \to \Psi_3 + \varepsilon b \Psi_4$

$$\Psi_4 = C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta \rightarrow C_{\alpha\beta\gamma\delta} n^\alpha (\bar{m}^\beta + i b^* n^\beta) n^\gamma (\bar{m}^\delta + i b^* n^\delta)$$

0 Symmetry

$$= C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta + C_{\alpha\beta\gamma\delta} n^\alpha i b^* \cancel{n^\beta n^\gamma} \bar{m}^\delta + C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta \cancel{n^\gamma} i n b^* n^\delta$$

0 Symmetry

$$= C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta = \Psi_4 + O(\varepsilon^2)$$

d)

$$\Psi_0 = C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta \rightarrow C_{\alpha\beta\gamma\delta}(l^\alpha - \varepsilon c l^\alpha)(m^\beta + i\varepsilon d m^\beta)(l^\gamma - \varepsilon c l^\gamma)(m^\delta + i\varepsilon d m^\delta)$$

$$= C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta + C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma i\varepsilon d m^\delta - C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \varepsilon c l^\gamma m^\delta$$

$$+ C_{\alpha\beta\gamma\delta} l^\alpha i\varepsilon d m^\beta l^\gamma m^\delta - C_{\alpha\beta\gamma\delta} \varepsilon c l^\alpha m^\beta l^\gamma m^\delta$$

$$= \Psi_0 + i\varepsilon d\Psi_0 - \varepsilon c\Psi_0 + i\varepsilon d\Psi_0 - \varepsilon c\Psi_0 = \Psi_0 + 2(i\varepsilon d - \varepsilon c)\Psi_0$$

$$\Psi_1 = C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma n^\delta = C_{\alpha\beta\gamma\delta}(l^\alpha - \varepsilon c l^\alpha)(m^\beta + i\varepsilon d m^\beta)(l^\gamma - \varepsilon c l^\gamma)(n^\delta + \varepsilon c n^\delta)$$

$$= C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma n^\delta + C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma \varepsilon c n^\delta + C_{\alpha\beta\gamma\delta} l^\alpha m^\beta (-\varepsilon c l^\gamma) n^\delta$$

$$+ C_{\alpha\beta\gamma\delta} l^\alpha i d\varepsilon m^\beta l^\gamma n^\delta + C_{\alpha\beta\gamma\delta}(-\varepsilon c l^\alpha) m^\beta l^\gamma n^\delta$$

$$= \Psi_1 + \varepsilon c\cancel{\Psi_1} - \cancel{\varepsilon} c\Psi_1 + i d\varepsilon\Psi_1 - \varepsilon c\Psi_1 = \Psi_1 - \varepsilon(c - id)\Psi_1$$

$$\Psi_2 = C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta = C_{\alpha\beta\gamma\delta}(l^\alpha - \varepsilon c l^\alpha)(m^\beta + i\varepsilon d m^\beta)(\bar{m}^\gamma - i\varepsilon d \bar{m}^\gamma)(n^\delta + \varepsilon c n^\delta)$$

$$= C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta + C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \cancel{\bar{m}^\gamma} \underline{\varepsilon c} n^\delta + C_{\alpha\beta\gamma\delta} l^\alpha \cancel{m^\beta}(-i\varepsilon d)\bar{m}^\gamma n^\delta$$

$$+ \cancel{C_{\alpha\beta\gamma\delta} l^\alpha i\varepsilon d m^\beta \bar{m}^\gamma n^\delta} - \varepsilon c \cancel{C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta} = \Psi_2$$

$$\Psi_3 = C_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta = C_{\alpha\beta\gamma\delta}(l^\alpha - \varepsilon c l^\alpha)(n^\beta + \varepsilon c n^\beta)(\bar{m}^\gamma - i\varepsilon d \bar{m}^\gamma)(n^\delta + \varepsilon c n^\delta)$$

$$= C_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta + C_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma \varepsilon c n^\delta + C_{\alpha\beta\gamma\delta} l^\alpha n^\beta(-i\varepsilon d)\bar{m}^\gamma n^\delta + C_{\alpha\beta\gamma\delta} l^\alpha \varepsilon c n^\beta \bar{m}^\gamma n^\delta$$

$$+ C_{\alpha\beta\gamma\delta}(-\varepsilon c l^\alpha) n^\beta \bar{m}^\gamma n^\delta = \Psi_3 + \varepsilon c\Psi_3 - i\varepsilon d\Psi_3 + \varepsilon c\cancel{\Psi_3} - \cancel{\varepsilon} c\Psi_3 = \Psi_3 + \varepsilon(c - id)\Psi_3$$

$$\psi_4 = C_{\alpha\beta\gamma\delta}\, l^\alpha \bar{m}^\beta n^\gamma m^\delta = C_{\alpha\beta\gamma\delta}(n^\alpha + \varepsilon c n^\alpha)(\bar{m}^\beta - i\varepsilon d \bar{m}^\beta)(n^\gamma + \varepsilon c n^\gamma)(\bar{m}^\delta - i\varepsilon d \bar{m}^\delta)$$

$$= C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta - C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma i\varepsilon d \bar{m}^\delta + C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta \varepsilon c n^\gamma \bar{m}^\delta$$

$$+ C_{\alpha\beta\gamma\delta} n^\alpha(-i\varepsilon d \bar{m}^\beta) n^\gamma \bar{m}^\delta + \varepsilon c\, C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$$

$$= \psi_4 - i\varepsilon d \psi_4 + \varepsilon c \psi_4 - i\varepsilon d \psi_4 + \varepsilon c \psi_4 = \psi_4 + 2\varepsilon(c - id)\psi_4$$

e)

First of all, as was discussed in class Kerr spacetime has two double principal null vectors and thus, one can always choose both l and n such that ψ0=ψ1=ψ3=ψ4=0. Looking at the prevous questions, one can see that rotations of ψ0 combine only the background values (0th order in ε) of ψ1 and ψ0 into the preturbed value (1st order in ε) of ψ0. Thus one concludes that, since both the background values of ψ0 and ψ1 vanish, ψ0 remains invariant under infinitesimal rotaions.

One sees that rotations of ψ1(type II) and ψ3 (type I) combine background values of ψ2, which doesn't vanish, into their perturbed values. Thus, these two cannot be invariant under infinitesimal rotations. Since the background value of ψ2 is non vanishing, we do not consider it as invariant under infinitesimal rotations, and finally as for ψ0, infinitesimal rotations of ψ4 mix only the background values of ψ4 and ψ3 into the preturbed value of ψ4, and since both vanish, ψ4 remains invariant under infinitesimal rotations as well.

f)

Since the ψ fields are transform as scalars and the background values of ψ0, ψ1, ψ3, ψ4 vanish, normal gauge transformations do not affect these scalars. Thus, they are gauge invariant.

g) $\psi_2 = \psi_2^{kerr} + \delta\psi_2$ In class we saw how do gauge transformations work with metric. for $g_{\mu\nu} = g'_{\mu\nu} + \varepsilon h_{\mu\nu}$ then a gauge transformation

$\underset{\text{perturbation}}{h_{\mu\nu}}$

acted on $h_{\mu\nu}$ as $h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$

Similarly here, the gauge transformatiom will act on $\delta\psi$ like $\delta\psi \to \delta\psi + \mathcal{L}_\xi \psi^{kerr}$ with $\mathcal{L}_\xi \psi^{kerr} = \xi^r \partial_r \psi^{kerr} + \xi^\theta \partial_\theta \psi^{kerr}$. To have $\psi_2 = \psi_2^{kerr}$, $\delta\psi_2$ has to vanish after the gauge transformation

$\therefore \quad \xi^r \partial_r \psi^{\text{kerr}} + \xi^\theta \partial_\theta \psi^{\text{kerr}} = -\delta\psi$ assuming $\delta\psi = \psi'(r,\theta)$

$$\Leftrightarrow \quad \xi^r \frac{3M}{(r - ia\cos\theta)^4} + \xi^\theta \frac{3iaM\sin\theta}{(r - ia\cos\theta)^4} = -\psi'(r,\theta)$$

$$\Leftrightarrow \quad \xi^r \frac{3M(r + ia\cos\theta)^4}{(r^2 + a^2\cos^2\theta)^4} + \xi^\theta \frac{3iaM\sin\theta\,(r + ia\cos\theta)^4}{(r^2 + a^2\cos^2\theta)^4} = -\psi'(r,\theta)$$

Assuming $\xi^r, \xi^\theta, r, \theta, M, a$ to be real, we can find two equations for the 2 unknowns $\xi^r$ and $\xi^\theta$ By equating $\mathrm{Re}[LHS] = \mathrm{Re}(\psi'(r,\theta))$ and $\mathrm{Im}[LHS] = \mathrm{Im}[\psi'(r,\theta)]$

See Mathematica for the solution

Problem 2)

a) $h_+ = \frac{4}{d_L}\left(\frac{GM_z}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}(t)}{c}\right)^{2/3}\left(\frac{1+\cos^2 \iota}{2}\right)\cos(\phi(t))$

$h_- = \frac{4}{d_L}\left(\frac{GM_z}{c^2}\right)\left(\frac{\pi f_{gw}(t)}{c}\right)^{2/3}\cos\iota \sin\phi(t)$. See the mathematica notebook for the a solution

b) In the previous question we showed that $h(t) = \frac{4}{d_{eff}}\left(\frac{GM_z}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}(t)}{c}\right)^{2/3}\cos[\phi(t)+a]$

for simplicity I will absorb $a$ in the constant $\phi_c$ of $\phi(t)$

then we can define $A(t) = \frac{4}{d_{eff}}\left(\frac{GM_z}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}(t)}{c}\right)^{2/3} \rightarrow h(t) = A(t)\cos(\phi(t))$

in lecture notes we calculated the fourier tranform of a function which has this form using the stationary phase approx: $\tilde{h} = \int dt\, A(t)\cos\phi(t)\, e^{2i\pi f t}$

the stationary phase approx, assumes $\frac{d\ln A}{dt} \ll \dot\phi$, $\ddot\phi \ll (\dot\phi)^2$. then we rewrite $\cos\phi = \frac{1}{2}(e^{i\phi}+e^{-i\phi})$ and we keep only the first term since the second term cannot create a stationary poin. For the first term we have $\dot\phi(t_*) = \pi f$. finally we evaluate everything in retarded time since we are far enough to need here redshift

stationary point

$\leftarrow$ taylor expand around $t_*$ and use $\int dx\, e^{-ix^2} = \sqrt{\pi}\, e^{i\pi/4}$

$\tilde{h} \approx \frac{1}{2} e^{\frac{\pi i d f}{c}} \int dt_{ret}\, A(t_{ret}) e^{\pi i f t_{ret} - i\phi(t_{ret})}$ $\quad t_{ret} \rightarrow t \quad dt_{ret} = dt$

to get $\tilde{h} = \frac{1}{2} A(t_*)\left(\frac{\pi}{\ddot\phi(t_*)}\right)^{1/2} e^{i(\pi f t_* - \phi(t_*) - \pi/4 + \pi f d/c)}$

now we have to relate $t_*$ and $f$: $\phi(t) = -2\left(\frac{5GM_c}{c^3}\right)^{-5/8}(t_c - t)^{5/8} + \phi_0$

$\dot\phi(t_*) = n f \to -\cancel{2}\left(\frac{5GM_c}{c^3}\right)^{-5/8} \cdot \frac{5}{8}(t_c - t)^{-3/8} = \cancel{2}nf$

$\to t_* - t_c = -\frac{5}{256}\frac{1}{(\pi f)^{8/3}}\left(\frac{GM_c}{c^3}\right)^{-5/3}$

$$\phi(t_*) = \phi_c - \frac{c^5}{15 f^{5/3} G^{5/3} M_c^{5/3} n^{5/3}}$$

the full phase is $\psi = nf(t_* + d_{eff}/c) - \phi(t_*) - \pi/4$ using $t_* = \underbrace{t_* - t_c}_{\text{I know this}} + t_c$

$$\psi = nf\left((t_* - t_c) + \frac{d_{eff}}{c}\right) - \phi(t_*) - \pi/4 + nf t_c$$

mathematica

$$\psi = -\phi_c + \frac{13\, c^5}{128 f^{5/3} G^{5/3} M_c^{5/3} n^{5/3}} - \frac{\pi}{4} - nf\frac{d_{eff}}{c} + nf t_c$$

$$\psi = \varphi_0 + nf\left(t_c - \frac{d_{eff}}{c}\right) + \frac{13\, c^5}{128 f^{5/3}(GM_c n)^{5/3}}$$

we see reference time $t_0 : t_c$
effective distance $d_{eff}$
reference phase $\varphi_0 = -\phi_c - \pi/4$
$M_c$ should be $M_z$?
or $M_c = \frac{M_z}{1+z}$ but then we have another parameter $z$

$$\tilde{A}(f) = \frac{1}{2}A(t_*)\left(\frac{n}{\ddot\phi(t_*)}\right)^{1/2}$$

for the rest of this problem in $\phi : M_c \to M_z$

$$= \frac{2}{d_{eff}}\left(\frac{GM_z}{c^2}\right)^{5/3}\left(\frac{n f_{gw}(t_*)}{c}\right)^{2/3}\left(\frac{n}{\ddot\phi(t_*)}\right)^{1/2}$$

During the inspiral phase

$$= \frac{2}{d_{eff}}\sqrt{\frac{5}{6}}\left(\frac{GM_z}{c^2}\right)^{5/3}\left(\frac{n f_{gw}(t_*)}{c}\right)^{2/3}\frac{c^{5/2}}{4 f^{11/6}(GM_z)^{5/6} n^{4/3}}$$

used to be $M_c$

$$f(t) = \frac{1}{(1+z)\pi}\left(\frac{5}{256}\frac{1}{t_c - t}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

$$= \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{t_c - t}\right)^{3/8}\left(\frac{GM_z}{c^3}\right)^{-5/8}$$

and without $\tilde{A} = \left(\frac{5}{2}\right)^{1/4}\frac{(GM_z)^{5/6}}{d_{eff}\, 2\sqrt{3}\, c^{3/2}\, n^{2/3}} f^{-7/6}$

this fourier expansion is valid untill the system reaches its ISCO thus as long as the emission is dominated by quadruple radiation $f_{max} = 2 f_{ISCO} = \frac{1}{3\pi\sqrt{6}}\frac{c^3}{GM_{tot}}$

1) $$h = \underbrace{\left(\frac{5}{2}\right)^{1/2} \frac{G^{5/6}}{2\sqrt{3}\, c^{3/2} \pi^{2/3}}}_{G_1} M_z^{5/6} f^{-7/6} \exp\left[i\left(\varphi_0 + \pi f\left(t_c - \frac{d_{eff}}{c}\right) + \frac{13\, c^5 f^{-5/3}}{\underbrace{128 (G\pi)^{5/3}}_{G_2} M_z^{5/3}}\right)\right]$$

$$h = G_1 M_z^{5/6} f^{-7/6} \exp\left[i\left(\varphi_0 + \pi f\left(t_0 - \frac{d_0}{c}\right) + G_2 f^{-5/3} M_z^{-5/3}\right)\right]$$

See mathematica