Chapter 3

Black Hole Perturbation Theory

3.1 Perturbation theory in GR

Globally speaking, when applying perturbation theory to general relativity we want to make the statement that some given spacetime $(M, g_{\mu\nu})$ is equal to some background spacetime $(\bar{M}, \bar{g}_{\mu\nu})$ plus something "small". However, the equation

$$(M, g_{\mu\nu}) = (\bar{M}, \bar{g}_{\mu\nu}) + \text{“something small”}$$

doesn't really make much mathematical sense. To make sense of such a comparison we will need a map $\phi : \bar{M} \to M$ that identifies the points of $\bar{M}$ with those of M. This makes it possible to compare any tensors (such as the metric) that "live" on M with tensors that live in $\bar{M}$ by considering the pull back of the object. E.g. we can consider the difference between the pull-back of $g_{\mu\nu}$ and compare it with $\bar{g}_{\mu\nu}$

$$\delta_\phi g_{\mu\nu} = \phi^* g_{\mu\nu} - \bar{g}_{\mu\nu},$$

and ask ourselves whether this is small. However, we should now wonder how this comparison depends on the choice of ϕ. What would change if we had chosen a different map $\varphi : \bar{M} \to M$ identify the points of $\bar{M}$ and M? To answer this question note that if ϕ and φ are both diffeomorphisms than there exists a unique automorphism $\psi : \bar{M} \to \bar{M}$ such that $\varphi = \phi \circ \psi$, which is generated by some vector field ξ^μ.

To compare $\delta_\phi g_{\mu\nu}$ and $\delta_\varphi g_{\mu\nu}$ we first note that starting from the same image point $p \in m$, the differences $\delta_\varphi g_{\mu\nu}$ and $\delta_\phi g_{\mu\nu}$ "live" at difference points

$\bar{p}, \bar{p}' \in \bar{M}$ satisfying $\bar{p}' = \psi(\bar{p})$. So to compare the two we need to pullback $\delta_\phi g_{\mu\nu}$ to $\bar{p}$ using ψ^*. In the limit that ξ^μ is small we get

$$\begin{aligned}
\delta_\varphi g_{\mu\nu} - \psi^* \delta_\phi g_{\mu\nu} &= \varphi^* g_{\mu\nu} - \bar{g}_{\mu\nu} - \psi^*(\phi^* g_{\mu\nu} - \bar{g}_{\mu\nu}) \\
&= (\phi \circ \psi)^* g_{\mu\nu} - \bar{g}_{\mu\nu} - \psi^*(\phi^* g_{\mu\nu} - \bar{g}_{\mu\nu}) \\
&= \psi^* \phi^* g_{\mu\nu} - \bar{g}_{\mu\nu} - \psi^* \phi^* g_{\mu\nu} + \psi^* \bar{g}_{\mu\nu} \\
&= \psi^* \bar{g}_{\mu\nu} - \bar{g}_{\mu\nu} \\
&= \mathcal{L}_{\xi^\mu} \bar{g}_{\mu\nu} + \mathcal{O}\left(\xi^2\right) \\
&= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \mathcal{O}\left(\xi^2\right)
\end{aligned}$$

So, fundamentally the comparison of $g_{\mu\nu}$ with $\bar{g}_{\mu\nu}$ is ambiguous up to a Lie derivative of $\bar{g}_{\mu\nu}$. In to context of perturbation theory this is known as a **gauge** ambiguity. More generally, when comparing any tensor $T_{\mu_1 \cdots \mu_n}$ with a background tensor $\bar{T}_{\mu_1 \cdots \mu_n}$ that comparison has a gauge ambiguity $\mathcal{L}_{\xi^\mu} \bar{T}_{\mu_1 \cdots \mu_n}$.

Lemma 4. *If a background tensor $\bar{T}_{\mu_1 \cdots \mu_n}$ is zero, then its perturbations are free from gauge ambiguities. A quantity free from gauge ambiguities is called **gauge invariant**.*

Note that this gauge freedom in perturbation theory is logically distinct from the general coordinate freedom of general relativity. However, in practice the two are closely related, since we tend to construct the identification map ϕ by choosing similar coordinates on $\bar{M}$ and M and identifying the points with the same coordinate values. Consequently, the ambiguity ψ in this identification map simply becomes the ambiguity of choosing the coordinates.

Moving forward we will drop the fancy notation involving identification maps, and work with the understanding that a suitably identification map has been chosen and used to pull back all tensors to the background manifold $\bar{M}$. Consequently, all tensors will be tensors on the manifold $\bar{M}$ and all raising and lowering operations are understood to use the background metric $\bar{g}_{\mu\nu}$. Moreover, any perturbations are understood as being ambiguous up to gauge transformations defined by gauge vectors ξ^μ

Using these conventions we are now ready to write that

$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu} + \mathcal{O}(\epsilon^2)$

where $h_{\mu\nu}$ is a symmetric rank-2 tensor living on $\bar{M}$ known as the (first-order) **metric perturbation**, and $\epsilon > 0$ is a small number that we will use to keep track of the orders in our perturbation theory. The metric perturbations are ambiguous up to transformations

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu.$$

We can try to fix this gauge freedom by imposing addition conditions on $h_{\mu\nu}$.

So, suppose now we want to know what the inverse perturbed metric $g^{\mu\nu}$ looks like. Note $g^{\mu\nu}$ is now considered a tensor of $\bar{M}$, but cannot be obtained by simply raising the indices on $g_{\mu\nu}$. We should therefore be using a different symbol. Lets write

$$g^{\mu\nu} = A^{\mu\nu} + \epsilon B^{\mu\nu} + \mathcal{O}(\epsilon^2).$$

The inverse metric will still have to satisfy

$$\delta_\mu{}^\nu = g_{\mu\alpha} g^{\alpha\nu} = (\bar{g}_{\mu\alpha} + \epsilon h_{\mu\alpha})(A^{\alpha\nu} + \epsilon B^{\alpha\nu}) = \bar{g}_{\mu\alpha} A^{\alpha\nu} + \epsilon(h_{\mu\alpha} A^{\alpha\nu} + \bar{g}_{\mu\alpha} B^{\alpha\nu}) + \mathcal{O}(\epsilon^2)$$

This equation will have to be satisfied order-by-order in ϵ. Giving us

$$\begin{aligned} \delta_\mu{}^\nu &= \bar{g}_{\mu\alpha} A^{\alpha\nu}, \quad \text{and} \\ 0 &= h_{\mu\alpha} A^{\alpha\nu} + \bar{g}_{\mu\alpha} B^{\alpha\nu}, \end{aligned}$$

which we can solve to find

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \epsilon h^{\mu\nu} + \mathcal{O}(\epsilon^2)$$

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3.2 Linearized Einstein Equation

Literature: Wald Sec. 7.5 We now want to make our way to the perturbed form of the Einstein equation. For this we first observe that both ∇_μ and $\bar{\nabla}_\mu$ are covariant derivatives, but compatible with different metrics. Consequently, we have that their difference

$$\left(\nabla_\mu - \bar{\nabla}_\mu\right) V^\nu = (\Gamma^\nu_{\mu\alpha} - \bar{\Gamma}^\nu_{\mu\alpha}) V^\alpha = \epsilon C^\nu{}_{\mu\alpha} V^\alpha$$

is represent by a rank 3-tensor $C^\nu{}_{\mu\alpha}$.

Equivalently, we could express this as

$$\nabla_\mu V^\nu = \bar{\nabla}_\mu V^\nu + \epsilon C^\nu{}_{\mu\alpha} V^\alpha.$$

This is similar as the usual expression for the covariant derivative, but with the partial derivative replaced by the background covariant derivative and the Christoffel symbols $\Gamma^\nu_{\mu\alpha}$ replaced by the tensor $C^\nu{}_{\mu\alpha}$.

Compatibility, of ∇_μ with the metric $g_{\mu\nu}$, yields an expression for $C^{\nu}{}_{\mu\alpha}$ in the same way we obtained our expression for the Christoffel symbols of the Levi-Civita connection (see e.g. Theorem 3.1.1 in Wald)

$$\begin{aligned}\epsilon C^{\lambda}{}_{\mu\nu} &= \frac{1}{2}g^{\lambda\alpha}\left(\bar{\nabla}_\mu g_{\nu\alpha} + \bar{\nabla}_\nu g_{\alpha\mu} - \bar{\nabla}_\alpha g_{\mu\nu}\right) \\ &= \epsilon\frac{1}{2}\bar{g}^{\lambda\alpha}\left(\bar{\nabla}_\mu h_{\nu\alpha} + \bar{\nabla}_\nu h_{\alpha\mu} - \bar{\nabla}_\alpha h_{\mu\nu}\right) + \mathcal{O}(\epsilon^2).\end{aligned}$$

The Riemann tensor will now be given

$$R^{\lambda}{}_{\sigma\mu\nu} = \bar{R}^{\lambda}{}_{\sigma\mu\nu} + \epsilon\left(\bar{\nabla}_\mu C^{\lambda}{}_{\nu\sigma} - \bar{\nabla}_\nu C^{\lambda}{}_{\mu\sigma}\right) + \epsilon^2\left(C^{\lambda}{}_{\mu\alpha}C^{\alpha}{}_{\nu\sigma} - C^{\lambda}{}_{\nu\alpha}C^{\alpha}{}_{\mu\sigma}\right).$$

The Einstein equation can be written

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right),$$

where $T = g^{\alpha\beta}T_{\alpha\beta}$ the trace of the energy-momentum tensor. Lets assume we are expanding around a background metric $\bar{g}_{\mu\nu}$ that is itself a solutions to the vacuum Einstein equation (e.g. Kerr), such that $\bar{R}_{\mu\nu} = 0$. We then find

$$\begin{aligned}R_{\mu\nu} &= R^{\alpha}{}_{\mu\alpha\nu} \\ &= \epsilon\left(\bar{\nabla}_\alpha C^{\alpha}{}_{\mu\nu} - \bar{\nabla}_\nu C^{\alpha}{}_{\alpha\mu} + \mathcal{O}(\epsilon^2)\right) + \mathcal{O}(\epsilon^2) \\ &= \frac{\epsilon}{2}\bar{g}^{\alpha\beta}\big(\bar{\nabla}_\alpha\left(\bar{\nabla}_\mu h_{\nu\beta} + \bar{\nabla}_\nu h_{\beta\mu} - \bar{\nabla}_\beta h_{\mu\nu}\right) \\ &\qquad - \bar{\nabla}_\nu\left(\bar{\nabla}_\alpha h_{\mu\beta} + \bar{\nabla}_\mu h_{\beta\alpha} - \bar{\nabla}_\beta h_{\alpha\mu}\right)\big) + \mathcal{O}(\epsilon^2) \\ &= \frac{\epsilon}{2}\left(-\bar{\Box}h_{\mu\nu} - \bar{\nabla}_\nu\bar{\nabla}_\mu h + 2\bar{\nabla}^\alpha\bar{\nabla}_{(\mu}h_{\nu)\alpha}\right) + \mathcal{O}(\epsilon^2) \\ &= \frac{\epsilon}{2}\left(-\bar{\Box}h_{\mu\nu} - \bar{\nabla}_\nu\bar{\nabla}_\mu h + 2\bar{\nabla}_{(\mu}\bar{\nabla}^\alpha h_{\nu)\alpha} + 2\bar{R}^{\alpha}{}_{\mu\nu}{}^{\beta}h_{\alpha\beta}\right) + \mathcal{O}(\epsilon^2),\end{aligned}$$

where $\bar{\Box} = \bar{\nabla}^\alpha\bar{\nabla}_\alpha$, and $h = h_{\alpha\beta}\bar{g}^{\alpha\beta}$ is the trace of the metric perturbation, and in the last line we used that $\bar{\nabla}_\mu\bar{\nabla}_\nu h_{\alpha\beta} = \bar{\nabla}_\nu\bar{\nabla}_\mu h_{\alpha\beta} + \bar{R}^{\gamma}{}_{\mu\nu\beta}h_{\alpha\gamma} + \bar{R}^{\gamma}{}_{\mu\nu\alpha}h_{\beta\gamma}$. Moreover, since the background was vacuum the energy momentum tensor $T_{\mu\nu}$ must also be order ϵ. Consequently, the order ϵ of the Einstein equation becomes,

$$\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{2}\bar{\nabla}_{(\mu}\bar{\nabla}_{\nu)}h - \bar{\nabla}_{(\mu}\bar{\nabla}^\alpha h_{\nu)\alpha} - \bar{R}^{\alpha}{}_{\mu\nu}{}^{\beta}h_{\alpha\beta} = -8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \quad (3.1)$$

We can further simplify this by using the gauge freedom in the metric perturbation. By choosing the Lorenz[1] gauge condition

$$\bar{\nabla}^\alpha h_{\mu\alpha} = \frac{1}{2}\bar{\nabla}_\mu h, \quad (3.2)$$

[1] After the 19th century Danish physicist Ludvig Lorenz, not the 19th century Dutch physicist Hendrik Lorentz.

the linearized Einstein equation becomes

$$\frac{1}{2}\Box h_{\mu\nu} - \bar{R}^{\alpha}{}_{\mu\nu}{}^{\beta} h_{\alpha\beta} = -T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} T. \tag{3.3}$$

In this gauge, the linearized Einstein equation explicitly takes the form of a wave equation. However, we should not be deceived by the apparent simplicity of this equation, it still consists of 10 coupled second order partial differential equations, and solving them is generally hard. Sometimes, the symmetries of the background help. For example, for a Schwarzschild background the equations can be solved through separation of variables, allowing the solution to be found mode-by-mode. For each mode, it then reduces to a set of 10 coupled ordinary differential equations. But even in Kerr, there is no known way the separate the variables.

It is therefore worth to consider a different approach.

3.3 The Penrose Wave Equation

Instead of a wave equation for the metric we will pursue a wave equation for the curvature instead. The Riemann curvature tensor $R_{\mu\nu\alpha\beta}$ in 4 dimensions has 20 degrees of freedom. Ten of these are encoded by the Ricci tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$, which through the Einstein equation are algebraicly determined by the energy-momentum content. The remain 10 degrees of freedom are captured by the trace-free part of the Riemann tensor, the so-called Weyl tensor

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - g_{\alpha[\mu}R_{\nu]\beta} - g_{\beta[\mu}R_{\nu]\alpha} + \frac{R}{3} g_{\alpha[\mu}g_{\nu]\beta}. \tag{3.4}$$

It is defined in such a way that any contraction of its indices produces zero, while retaining the symmetries of $R_{\mu\nu\alpha\beta}$,

$$C_{\mu\nu\alpha\beta} = C_{[\mu\nu][\alpha\beta]}, \tag{3.5}$$
$$C_{\mu\nu\alpha\beta} = C_{\alpha\beta\mu\nu}. \tag{3.6}$$
$$C_{\mu\nu\alpha\beta} = C_{\mu[\nu\alpha\beta]}. \tag{3.7}$$

Moreover, when $R_{\mu\nu} = 0$, we get that $C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta}$.

The Weyl tensor therefore must capture all the propagating curvature degrees of freedom, and it would be great if we could write a wave equation for it. To achieve this we first consider the Bianchi identity

$$\nabla_{\gamma} R_{\alpha\beta\mu\nu} + \nabla_{\alpha} R_{\beta\gamma\mu\nu} + \nabla_{\beta} R_{\gamma\alpha\mu\nu} = 0. \tag{3.8}$$

Taking the divergence of this identity yields

$$\Box R_{\alpha\beta\mu\nu} + \nabla^\gamma \nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla^\gamma \nabla_\beta R_{\gamma\alpha\mu\nu} = 0. \tag{3.9}$$

The box operator already gives this the appearance of a wave equation. We can commute the two covariant derivatives using the Ricci identity

$$\begin{aligned} &\nabla_\mu \nabla_\nu R_{\alpha\beta\gamma\delta} - \nabla_\nu \nabla_\mu R_{\alpha\beta\gamma\delta} = \\ &\qquad R_{\mu\nu\alpha}{}^\lambda R_{\lambda\beta\gamma\delta} + R_{\mu\nu\beta}{}^\lambda R_{\alpha\lambda\gamma\delta} + R_{\mu\nu\gamma}{}^\lambda R_{\alpha\beta\lambda\delta} + R_{\mu\nu\delta}{}^\lambda R_{\alpha\beta\gamma\lambda} \end{aligned} \tag{3.10}$$

This yields

$$\begin{aligned} &\Box R_{\alpha\beta\mu\nu} + 2R^\gamma{}_{\alpha\mu}{}^\lambda R_{\gamma\beta\nu\lambda} - 2R^\gamma{}_{\alpha\nu}{}^\lambda R_{\gamma\beta\mu\lambda} + 2R^\gamma{}_{\alpha\beta}{}^\lambda R_{\gamma\lambda\mu\nu} \\ &\qquad - R_\alpha{}^\lambda R_{\lambda\beta\mu\nu} + R_\beta{}^\lambda R_{\lambda\alpha\mu\nu} - \nabla_\alpha \nabla^\gamma R_{\gamma\beta\mu\nu} + \nabla_\beta \nabla^\gamma R_{\gamma\alpha\mu\nu} = 0. \end{aligned} \tag{3.11}$$

Next by contracting the Bianchi identity itself we find the following identity for the divergence of the Riemann tensor

$$\nabla^\gamma R_{\gamma\nu\alpha\beta} = \nabla_\alpha R_{\beta\nu} - \nabla_\beta R_{\alpha\nu} = 2\nabla_{[\alpha} R_{\beta]\nu}. \tag{3.12}$$

Using this last identity, we can eliminate the divergences of the Riemann curvature in favour of the Ricci tensor to obtain the **Penrose wave equation**

$$\begin{aligned} &\Box R_{\alpha\beta\mu\nu} + 2R^\gamma{}_{\alpha\mu}{}^\lambda R_{\gamma\beta\nu\lambda} - 2R^\gamma{}_{\alpha\nu}{}^\lambda R_{\gamma\beta\mu\lambda} + 2R^\gamma{}_{\alpha\beta}{}^\lambda R_{\gamma\lambda\mu\nu} \\ &\qquad - R_\alpha{}^\lambda R_{\lambda\beta\mu\nu} + R_\beta{}^\lambda R_{\lambda\alpha\mu\nu} - 2\nabla_\alpha \nabla_{[\mu} R_{\nu]\beta} + 2\nabla_\beta \nabla_{[\mu} R_{\nu]\alpha} = 0. \end{aligned} \tag{3.13}$$

This equation is build from identities that hold for any pseudo-Riemannian manifold, and therefore this equation is satisfied for the curvature tensor obtained from any metric. If in addition we impose that the metric satisfies the vacuum Einstein equation $R_{\mu\nu} = 0$, all the terms in the second line vanish. So, for vacuum solutions this turns into a wave equation for the Weyl tensor.

$$\Box C_{\alpha\beta\mu\nu} + 2C^\gamma{}_{\alpha\mu}{}^\lambda C_{\gamma\beta\nu\lambda} - 2C^\gamma{}_{\alpha\nu}{}^\lambda C_{\gamma\beta\mu\lambda} + 2C^\gamma{}_{\alpha\beta}{}^\lambda C_{\gamma\lambda\mu\nu} = 0. \tag{3.14}$$

More generally, we could replace all occurrences of the Ricci tensor with the trace reversed energy-momentum tensor using the Einstein equation to find the wave equation for the curvature coupled to matter.

Let us now consider perturbations around a vacuum solution of the Einstein equation with an energy-momentum source that is order ϵ. Writting $C_{\alpha\beta\mu\nu} = \bar{C}_{\alpha\beta\mu\nu} + \epsilon\, \delta C_{\alpha\beta\mu\nu} + \mathcal{O}(\epsilon^2)$, the background Weyl curvature will satisfy Eq. (3.14).

To obtain the linear in ϵ part of the equation (3.13) we make the following observations

- The linear in ϵ part of the second line in Eq. (3.13) is constructed entirely from $\bar{C}_{\alpha\beta\mu\nu}$, $T_{\mu\nu}$, the background metric $\bar{g}_{\mu\nu}$, and the background covariant derivative $\bar{\nabla}_\mu$.

- The linear in ϵ part of the Riemann curvature tensor $\delta R_{\alpha\beta\mu\nu}$, consists of $\delta C_{\alpha\beta\mu\nu}$ plus terms constructed from $T_{\mu\nu}$ and the background metric $\bar{g}_{\mu\nu}$.

- We can write the action of the box operator on a generic 4-tensor $T_{\alpha\beta\mu\nu}$ as $\Box T_{\alpha\beta\mu\nu} = \bar{\Box} T_{\alpha\beta\mu\nu} + \epsilon B[h]_{\alpha\beta\mu\nu}{}^{\alpha'\beta'\mu'\nu'} T_{\alpha'\beta'\mu'\nu'}$ where $B[h]_{\alpha\beta\mu\nu}{}^{\alpha'\beta'\mu'\nu'}$ is formed as a linear operator on $h_{\mu\nu}$ constructed entirely from $\bar{g}_{\mu\nu}$ and $\bar{\nabla}_\mu$.

- Combining the last two observations we note that we can write the linear in ϵ part of $\Box R_{\alpha\beta\mu\nu}$ as $\bar{\Box}\delta C_{\alpha\beta\mu\nu} + B[h]_{\alpha\beta\mu\nu}{}^{\alpha'\beta'\mu'\nu'} \bar{C}_{\alpha'\beta'\mu'\nu'}$ plus terms depending only on

So, if in the linear in ϵ part of the equation (3.13), we take all the terms that depend only on $T_{\mu\nu}$, the background metric $\bar{g}_{\mu\nu}$, and the background covariant derivative $\bar{\nabla}_\mu$, move them to the right hand side and collectively call them $S[T]_{\alpha\beta\mu\nu}$, we get

$$
\begin{aligned}
\bar{\Box}\delta C_{\alpha\beta\mu\nu} + B[h]_{\alpha\beta\mu\nu}{}^{\alpha'\beta'\mu'\nu'} \bar{C}_{\alpha'\beta'\mu'\nu'} + 2\bar{C}^{\gamma}{}_{\beta\nu}{}^{\lambda}\delta C_{\gamma\alpha\mu\lambda} + 2\bar{C}^{\gamma}{}_{\alpha\mu}{}^{\lambda}\delta C_{\gamma\beta\nu\lambda} \\
-2\bar{C}^{\gamma}{}_{\beta\mu}{}^{\lambda}\delta C_{\gamma\alpha\nu\lambda} - 2\bar{C}^{\gamma}{}_{\alpha\nu}{}^{\lambda}\delta C_{\gamma\beta\mu\lambda} + 2\bar{C}^{\gamma}{}_{\mu\nu}{}^{\lambda}\delta C_{\gamma\lambda\alpha\beta} + 2\bar{C}^{\gamma}{}_{\alpha\beta}{}^{\lambda}\delta C_{\gamma\lambda\mu\nu} \\
= S[T]_{\alpha\beta\mu\nu}.
\end{aligned}
\tag{3.15}
$$

This does not seem like much progress. While it looks like a wave equation for $\delta C_{\alpha\beta\mu\nu}$ it mixes with the equally unknown metric perturbation $h_{\mu\nu}$ in a non-trivial way. To make this useful in someway, we are going to need a miracle.

3.4 The Weyl scalars

The Weyl curvature tensor with its many components seems a highly inefficient way of capturing 10 degrees of freedom. The Newman-Penrose formalism provides a more compact way of describing these degrees of freedom.

Bibliography

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[3] Michael P. Ryan. Teukolsky equation and Penrose wave equation. **Phys. Rev. D**, 10:1736–1740, 1974.