1 Killing-Yano tensors

An anti-symmetric rank-2 tensor $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is called a **Killing-Yano tensor** if it satisfies

$$\nabla_\alpha \omega_{\beta\gamma} + \nabla_\beta \omega_{\alpha\gamma} = 0.$$

a. Show that for any Killing-Yano tensor $\omega_{\alpha\beta}$, the symmetric tensor $\omega_{\alpha\gamma}\omega_\beta{}^\gamma$ is a Killing tensor. (In some sense Killing-Yano tensors can be thought of as the "square root" of a Killing tensor.))

b. Show that for any Killing-Yano tensor $\omega_{\mu\nu}$ and geodesic $x^\mu(s)$, the (co-)vector $V_\mu = \omega_{\mu\alpha}\frac{dx^\alpha}{ds}$ is parallel transported only the geodesic, i.e.

$$\frac{dx^\alpha}{ds}\nabla_\alpha V^\mu = 0.$$

Note: Kerr also has a Killing-Yano tensor.

a) $w_\beta{}^\gamma = g^{\gamma\sigma} w_{\beta\sigma} \to \nabla_a w_\beta{}^\gamma + \nabla_\beta w_a{}^\gamma = g^{\gamma\sigma}\nabla_a w_{\beta\sigma} + g^{\gamma\sigma}\nabla_\beta w_{a\sigma}$

$= g^{\gamma\sigma}(\nabla_a w_{\beta\sigma} + \nabla_\beta w_{a\sigma}) = 0$

thus $\nabla_a(w_{\beta\sigma} w_\gamma{}^\sigma) + \nabla_\beta(w_{a\sigma} w_\gamma{}^\sigma) + \nabla_a(w_{\gamma\sigma} w_\beta{}^\sigma) + \nabla_\beta(w_{\gamma\sigma} w_a{}^\sigma)$

$+ \nabla_\gamma(w_{a\sigma} w_\beta{}^\sigma) + \nabla_\gamma(w_{\beta\sigma} w_a{}^\sigma)$

$= w_\gamma{}^\sigma(\nabla_a w_{\beta\sigma} + \nabla_\beta w_{a\sigma}) + (w_{\beta\sigma}\nabla_a w_\gamma{}^\sigma + w_{a\sigma}\nabla_\beta w_\gamma{}^\sigma)$

$+ w_\beta{}^\sigma(\nabla_a w_{\gamma\sigma}) + w_a{}^\sigma(\nabla_\beta w_{\gamma\sigma}) + w_{\gamma\sigma}(\nabla_a w_\beta{}^\sigma + \nabla_\beta w_a{}^\sigma)$

$+ w_\beta{}^\sigma \nabla_\gamma w_{a\sigma} + w_{a\sigma}\nabla_\gamma w_\beta{}^\sigma + w_a{}^\sigma \nabla_\gamma w_{\beta\sigma} + w_{\beta\sigma}\nabla_\gamma w_a{}^\sigma = 0$

b) Nts $\frac{dx^a}{ds}\nabla_a V^\mu = 0$ $\quad V^\mu = w_{\mu a}\frac{dx^a}{ds}$

$\nabla_\sigma V^\mu = \nabla_\sigma\left(w_{\mu a}\frac{dx^a}{d\sigma}\right) = (\nabla_\sigma w_{\mu a})\frac{dx^a}{d\sigma} + w_{\mu a}\left(\nabla_\sigma \frac{dx^a}{d\sigma}\right)$

$\frac{dx^\sigma}{d\sigma}\nabla_\sigma V^\mu = \frac{dx^\sigma}{d\sigma}(\nabla_\sigma w_{\mu a})\frac{dx^a}{d\tau} + w_{\mu a}\frac{dx^\sigma}{d\sigma}\left(\nabla_\sigma \frac{dx^a}{d\sigma}\right)$ $\to$ $\frac{dx^\sigma}{d\sigma}\left(\nabla_\sigma \frac{dx^a}{d\tau}\right) = \frac{dx^\sigma}{d\sigma}\partial_\sigma \frac{dx^a}{d\tau} + \Gamma^a_{\sigma\lambda}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$

$= 0$

The only thing that remains is $\frac{dx^\sigma}{d\sigma}(\nabla_\sigma \omega_{\mu a})\frac{dx^a}{d\sigma}$

$$\nabla_\sigma \omega_{\mu a} = -\nabla_\mu \omega_{\sigma a} = \nabla_\mu \omega_{a\sigma} = -\nabla_a \omega_{\mu\sigma}$$

$$(\nabla_\sigma \omega_{\mu a})\frac{dx^a}{d\sigma}\frac{dx^\sigma}{d\sigma} = -\nabla_a \omega_{\mu\sigma}\frac{dx^a}{d\sigma}\frac{dx^\sigma}{d\sigma} \quad \overrightarrow{\underset{\sigma\to a}{a\to\sigma}} = -\nabla_\sigma \omega_{\mu a}\frac{dx^a}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

2 Vortical solutions revisited

In class we introduced "vortical" geodesic solutions as solutions for which $z = \cos\theta$ oscillates between $0 \le z_1 \le z_2 \le 1$. We argued that these solutions exist only when

$$\mathcal{E}^2 - \mu > 0 \qquad \underline{\mathcal{E}^2 > \mu} \tag{1}$$

$$\mathcal{L}^2 \le a^2(\mathcal{E}^2 - \mu) \qquad \mathcal{E}^2 = \mu \tag{2}$$

$$-(|\mathcal{L}| - |a|\sqrt{\mathcal{E}^2 - \mu}|)^2 \le \mathcal{Q} \le 0 \qquad \mathcal{E} = \pm\sqrt{\mu} \tag{3}$$

Let's examine these solutions a little further.

$\mathcal{E} > \sqrt{\mu}$ $\qquad 9 > 4$

$3 > 2$

$-3 < -2$

a. Define

$$\hat{Q} = \frac{Q}{a^2(E^2 + \mu)}$$

$$\hat{L}^2 = \frac{L^2}{a^2(E^2 + \mu)}$$

$$u = \cos^2\theta$$

$$\hat{P}_\theta = -\frac{P_\theta}{a^2(E^2 + \mu)}$$

Show that we this notation we get

$$\hat{P}_\theta(u) = u^2 + \left(\hat{Q} + \hat{L}^2 - 1\right) u - \hat{Q}.$$

b. Find the zeroes of $\hat{P}_\theta(u)$.

c. Under what conditions are both roots u_1 and u_2 real lie in the range $0 < u_1 < u_2 < 1$? Can you recover the mentioned conditions for vortical solutions. We now turn the radial part of vortical solution.

d. Show that for vortical solutions

$$P_r \ge 2M(Q + (L - aE)^2 + \mu r^2).$$

e. Show that for vortical null geodesics ($\mu = 0$) that $P_r > 0$. What does this mean for their radial solutions? Sketch a vortical null trajectory in the maximally extended Penrose diagram for Kerr.

f. Show that for vortical timelike geodesics ($\mu > 0$) we always have that $P_r > 0$ outside r_-.

$$u^2 + (\hat{Q} + \hat{L}^2 - 1)u - \hat{Q} = 0$$

$$\Delta = \sqrt{(\hat{Q} + \hat{L}^2 - 1)^2 + 4\hat{Q}} \qquad u_{1,2} = \frac{-(\hat{Q} + \hat{L}^2 - 1) \pm \sqrt{(\hat{Q} + \hat{L}^2 - 1)^2 + 4\hat{Q}}}{2}$$

$$\sqrt{(\hat{Q} + \hat{L}^2 - 1)^2 + 4\hat{Q}} = Q^2 + 2(L^2 + 1) + (L^2 - 1)^2$$

$$(Q + 2A + 1)(Q + B - 1) \longrightarrow \quad A + B = \underbrace{(L - 1)}_{A} + \underbrace{(L + 1)}_{B}$$

$$AB = (L + 1)^2 (L - 1)^2$$

Real solutions when $(Q+L^2-1)^2+4Q>0$

$(Q+L^2-1)^2>-4Q \longrightarrow Q<0$

$$0<\frac{-(\hat{Q}+\hat{L}^2-1)\pm\sqrt{(\hat{Q}+\hat{L}^2-1)^2+4\hat{Q}}}{2}<1$$

$$-(Q+L^2-1)-\sqrt{(Q+L^2-1)^2+4Q}>0$$

$$\sqrt{(Q+L^2-1)^2+4Q}>-(Q+L^2-1)$$