```
In[1]:= Pr = (Eps (r^2 F a^2 ỳ G a E Lý^2 G DeltwQ F dL G a E Epsď^2 F mu E r^2 ỳ;

rÇĂUZẠƒ Collectỳ Pr ɪ. Delt ɸ r^2 G Rs E r F a^2 ɪ. mu ɸ Eps^2, r, Simplifyẓ

x ḌḌŬZẠƒ
-a⁝ Q + (-L⁝ H ^ ŋý^2 + ((-a Eps + L)^2 + Q) r Rs + Eps⁝ r Rs

In[2]:= Pr2 = -a^O Q + (-L^O G _ỳý^2 + ((-a Eps + L)^2 + Q) r Rs + Eps^O r^Ꝑ Rs;

RsPlus = Rs/2 + Rs/2 * Sqrt[1 - (4 a^O)/(Rs^O) ẓ

rÇĂWẠƒ Pr2
x ḌḌŴẠƒ H a^2 Q G ẑH L^2 H Q ŋ r^2 G ẑẑH a Eps G Lŋ^2 G Q ȷ r Rs G Eps^2 r^3 Rs

rÇĂ ởỞ Solve[{Pr2 == 0, r > RsPlus, Rs > 0, -Rs/2 < a < Rs/2}, r, Reals] // Simplify

ĊΣℙ ởʼ GG r τ Root[H a^2 Q G ẑ a^2 Eps^2 Rs H 2 a Eps L Rs G L^2 Rs G Q Rs) ♯1 + (-L⁝ - ^ ŋ ȷ ⌧^2 + Eps⁝ Rs ♯1&, 2] Ầ ˬMΛAZZMΛ

ýN ởỞ Solve⫽Pr2 ℂ 0, r / RsPlus, Rs = 0, +Rs/2 / a / Rs/2], r, Reals .. Simplify

ĊΣℙ ởʼ GG ý τ _ÞÞÃ H §^2 ^ G ẑ§^2 ) ðþ^2 _þ H ⁝ § ) ðþ = _þ G =^2 _þ G ^ _þ ŋ ȷ ⌧ G ẑH =^2 - Q) ♯1⁝ G ) ðþ^2 _þ ȷ ⌧^3 ⁝ Ꝑ ⌧ ⊞ ẠĆÇẶĄḌÉÇ

rÇĂ Ạƒ ] ðý¶× ĸ ( Ŋ ý ⁝ Ꝑ ýo
ĊΣℙ ởʼ 2 (-L⁝ H ^ ŋý G ẑẑH § ) ðþ G L)^2 + Q) Rs + 3 Eps⁝ r⁝ Rs

ýN ởỞ Solve⫽Pprim ℂ 0, r = RsPlus, Rs = 0, +Rs/2 / a / Rs/2], r, Reals .. Simplify

ĊΣℙ ởʼ GG ý τ

(=^2 + Q - Sqrt[L^4 + Q⁝ H §^2 ) ðþ^4 _þ^2 + 6 a Eps L Rs⁝ H ) ðþ^2 ^ _þ^2 + L⁝ ẓ ^ H ) ðþ^2 _þ^2)]) / (3 Eps⁝ Rs) ⊞ ẠĆÇẶĄḌÉÇ },

{r →
(L⁝ G ^ G Sqrt[=^4 + Q⁝ H §^2 ) ðþ^4 _þ^2 + 6 a Eps L Rs⁝ H ) ðþ^2 ^ _þ^2 + L⁝ ẓ ^ H ) ðþ^2 _þ^2)]) / (3 Eps⁝ Rs) ⊞ ẠĆÇẶĄḌÉÇ }
}

rÇĂKẠƒ PrEquat ĸ Pr2 ɪ. Q ɸ 0 ɪ. r ɸ r0
x ḌḌẊẠƒ H L^2 r0^2 G ẑH a Eps + L)^2 ý þ G ) ðþ^2 ý ^3 _þ
```

In[6]:= EquatSol(SolvekPrEquat ÿ 02, r0Q& Rule+a. , b. , ë b & Rs ë 2 M &Expand &Simplify

Out[6]= $$\text{GTήИ G}\frac{)^2 \% \sqrt{L^{o}\, H \acute{ı}\, v\, 8^2\, "\, GJ^4 *^2 \$ 32\, a\, Eps^K\, L\, M^{\ddot{U}}\, H \acute{ı}\, v\, "\, GJ^2\,)^2 *^2}}{o\, "\, GJ^2 *}\text{ ŧИ}$$

$$\text{G}\frac{)^2 \$ \sqrt{L^{o}\, H \acute{ı}\, v\, 8^2\, "\, GJ^4 *^2 \$ 32\, a\, Eps^K\, L\, M^{\ddot{U}}\, H \acute{ı}\, v\, "\, GJ^2\,)^2 *^2}}{o\, "\, GJ^2 *}\text{ ŧŧ}$$

In[7]:= rootsL(SolvekEquatSol˘2' ÿ EquatSol˘3'2, a, RealsQ& Rule+a. , b. , ë b

Out[7]= $$\text{GG}\boxed{\frac{H)^2 \$ 4\, Eps\, L\, M}{4\, Eps^{\ddot{U}}\, M}\ ħe\, M * 0\ \ M(\ 0\ Σ}, K\boxed{\frac{L^{\ddot{U}}\, G\, o\, "\, GJ)\, *}{o\, "\, GJ^2 *}\ \text{Ặ} *\ M\ ή\ \ *\ K\ ή}\text{ ŧŧ}$$

In[18]:= SolvekPrEquat ÿ 0 & a ë $\frac{L^{\ddot{U}}\, F\, o\, "\, GJ)\, *}{o\, "\, GJ^2 *}$ й)ʑ ɪe / J ɸ ü * ɪɪ #LCC0@DGC@=P

x D̤D̤ĀTaA̤f

KKL Ď %2 $\sqrt{Eps^{\ddot{U}}\, M\, r0}$ Σ, KL Ď 2 $\sqrt{Eps^{\ddot{U}}\, M\, r0}$ Σ, KL Ď %2 $\sqrt{Eps^{\ddot{U}}\, M\, r0}$ Σ, KL Ď 2 $\sqrt{Eps^{\ddot{U}}\, M\, r0}$ ΣΣ

rÇÃUXA̤f " HL8K0FGu̥Ü̥y ɪe 8 ɸ $\frac{)^2 \# 4\, Eps\, L\, M}{4\, Eps^{\ddot{U}}\, M}$ &Expand &FullSimplify

Out[25]=

$$\text{G}\frac{)^2}{o\, "\, GJ^2 *}\text{ ŧ}$$

In[26]:= SolvekYPrEquat ÿ 0 & L ë 2 $\sqrt{Eps^{\ddot{U}}\, M\, r0}$ ₉ , r0Q& Rule+a. , b. , ë b & Rs ë 2 M &Expand &Simplify

••• Rnkud 9 Sgdqdl ` x ad u` kt dr nesgd o` q` l dsdqr enqv ghbg rnl d nq` kkrnkt shmmr ` qd mnsu` khc-

x D̤D̤ŬYA̤f

K102, K%a $ 2 M %2 $\sqrt{M\,!\%a\,\$\,M"}$ Σ, K%a $ 2 $\left(M\,\$\,\sqrt{M\,!\%a\,\$\,M"}\right)$Σ, Ka $ 2 M %2 $\sqrt{M\,!a\,\$\,M"}$ Σ, Ka $ 2 $\left(M\,\$\,\sqrt{M\,!a\,\$\,M"}\right)$}}

••• Solve: 2 Eps $\sqrt{M\, r0}$ is not a valid variable.

Out[∘]= Solve:%4 Eps$^{\ddot{U}}$ M r0^K G " *GJ*^{2} *I ή*^{3} */ J* G *I ή* $\left(2\, Eps\, \sqrt{M\, r0}\ H\, Eps\left(-r0\, G\, 2\, \sqrt{M\, r0}\right)\right)^{⋮}$ Rs == 0, 2 Eps $\sqrt{M\, r0}$]

In[∘]:=] ý) ÷ā§Ā(®ý ((+] ý⋮ Ƥ ý, &O^ ë RET&Oý ë ýRE

Out[∘]= -*Ü*) ⋮ *I ή* + (-*8 " GJ* +))$^{\ddot{U}}$ _þ $ ESC) ðþ$^{\ddot{U}}$ ýRET _þ

In[ ]:=

Out[ ]=

In[ ]:=

Out[ ]= 12

Out[ ]= {{L → ... if condition}, {L → ... if condition}, {L → $-2\,\text{Eps}\,M + 2\sqrt{\text{Eps}^2 M (a + M)}$ ...}}

Solve[EquatDerSol[1], EquatSol[2], L, Reals]

{{L → a Eps}}

{{a → $-\sqrt{3}\ \sqrt{M}\ \sqrt{r0}$}, {a → $\sqrt{3}\ \sqrt{M}\ \sqrt{r0}$}}

In[ ]:= %KO; N3I F& , ë ; " %JM& 3KLN; . " <. ᴘë ; ^Ü %9

Out[ ]= x4ή5й 4ή5й x $\frac{8}{Ü *}$ èè

In[ ]:= 3I FP?k%KO; N$?L3I F ' ÿ %KO; N$?L3I F ' , 2?; FM

Rxw @ KK, Ď 2I I N% ; Ü %JM -Ü ' üo 8 " GJ * ÿ % %JM -Ü í $ ÿ º , kh condition èй

x) ∴ /FFK (í ü 8 " GJ * $; %JM -Ü í (í ü " GJ * ÿ Ü ' í έй üн he frqglwlrq Σ

K, Ď 2I I N% ; Ü %JM -Ü ' üo 8 " GJ * ÿ % %JM -Ü í $ ÿ º , kh condition èй

x) ∴ /FFK (í ü 8 " GJ * $; %JM -Ü í (í ü " GJ * ÿ Ü ' í έй oн he frqglwlrq ΣΣ

Rxw @ KK $\frac{, Ü (\sqrt{)} \; \% \; ; Ü \%JM -Ü ' κ ü 8 " GJ) * \% \%JM , Ü -Ü}{\%JM -}$ Σ Ġ

K $\frac{, Ü (\sqrt{)} \; \% \; ; Ü \%JM -Ü ' üo 8 " GJ) * \% \%JM , Ü -Ü}{\%JM -}$ ΣΣ

Rxw @ KK $\frac{, Ü ' \sqrt{)} \; \% \; ; Ü \%JM -Ü ' κ ü 8 " GJ) * \% \%JM , Ü -Ü}{\%JM -}$ Σ Ġ

K $\frac{, Ü ' \sqrt{)} \; \% \; ; Ü \%JM -Ü ' üo 8 " GJ) * \% \%JM , Ü -Ü}{\%JM -}$ ΣΣ

Lq @ /<; L: <#C8KK<E " Hí й * , ήй " GJ , ήм й) й /<8CJ
/<; L: <#C8KK<E 3 " Hü5 й) й /<8CJ

Out[ ]= - * %JM* , Ġ ; %JM

In[ ]:= 3I FP?k0L%KO; Nÿ & й ˘ ' ; Q & 2Më -

Out[ ]= ф8 ∴ ($\sqrt{Ü}$ $\sqrt{*}$ $\sqrt{I ή}$ } й { 8 → $\sqrt{Ü}$ $\sqrt{*}$ $\sqrt{I ή}$ } }

In[ ]:= 8 í ήü .e 8 í ήμ1Ü .e / LC<@8 ^ й 9 ^[| 9 &O) ðþ^Ü | DL

ĊΣR ð kk2 M mu r0}}

In[ ]:= [illegible]

Out[ ]= [illegible] condition [illegible]

[illegible] condition [illegible]

[illegible] condition [illegible]

[illegible] if [illegible]

In[ ]:= [illegible]

Out[ ]= [illegible] if [illegible]

[illegible] $\left(2 M \ldots \sqrt{\ldots}\ \sqrt{\ldots} \ldots - \sqrt{2}\ \sqrt{M \ldots r 0}\right)$ if [illegible]

[illegible] if [illegible]

$\left\{4 \, Eps \ldots \left(\ldots{}^2 + \sqrt{2}\ \sqrt{M \ldots r 0} - 2 M \sqrt{M \ldots \sqrt{\ldots}\ \sqrt{\ldots{}^3 \ldots}}\right)\right.$ if [illegible]

In[ ]:= Solve[D[PrEquat, r] == 0, r] // Simplify

$$\text{Out[ ]=}\ \left\{\left\{r \to \frac{L^2 - \sqrt{L^4 - 3\,Eps^2 \ldots a\,Eps \ldots L^2\,Rs^2}}{3\,Eps^2\,Rs}\right\}, \left\{r \to \frac{L^2 + \sqrt{L^4 - 3\,Eps^2 \ldots a\,Eps \ldots L^2\,Rs^2}}{3\,Eps^2\,Rs}\right\}\right\}$$

$$\text{In[ ]:=}\ \text{Solve}\left[\frac{L^2 + \sqrt{L^4 + 4\,Eps^2 \ldots a\,Eps \ldots L^2\,Rs^2}}{2\,Eps^2\,Rs} == r,\ L\right] \text{// FullSimplify}$$

••• Solve: There may be values of the parameters for which some or all solutions are not valid.

$$\text{Out[ ]=}\ \left\{\left\{L \to -\frac{a\,Eps\,Rs + \sqrt{Eps^2\,r\,Rs\left(a^2 + \ldots{}^2 - r\,Rs\right)}}{r - Rs}\right\}, \left\{L \to \frac{-a\,Eps\,Rs + \sqrt{Eps^2\,r\left(a^2 + r\,(r - Rs)\right)Rs}}{r - Rs}\right\}\right\}$$

$$\text{In[ ]:=}\ L = \frac{a\,Eps\,Rs - \sqrt{Eps^2\,r\,Rs\left(a^2 - r^2 + r\,Rs\right)}}{r - Rs} \text{ /. } Rs \to 2\,M \text{ // Expand // Simplify}$$

$$\text{Out[ ]=}\ -\frac{2\,a\,Eps\,M - \sqrt{2}\ \sqrt{Eps^2\,M\,r\left(a^2 + r\,(-2\,M + r)\right)}}{2\,M - r}$$

In[•]:=
```
- 8 " GJ / J + √(" GJO I / J (8O # ý^Ü - I / J)) / (I - / J) - (-8 " GJ / J + √(" GJO I (8O # ý ý $ ` Þ/E ` Þ)) / (ý $ ` Þ) &* ĆðÝ¬ &
+āĐĐ{ ¶×ðĐ¶° ć
```

Out[•]=
```
. Ü √(" GJO I / J 8O $ ý^Ü . I / J.) / (I . / J)
```

```
(-Ü √(" GJO I / J (8O # ý^Ü - I / J)))^O & ` Þ ë O@& § ë < / @ &* ĆðÝ¬ &{ ¶×ðĐ¶° ć
```

Out[•]=
```
Ü * ðÞ^Ü ý ∝^Ü - *O I . Ü * - I .. / *
```

```
OFCM[ ()O $ √(>o - K " GJO B$ § * ðÞ # >E^O / JO)) / (K " GJO / J) == ()O # √(>o - K " GJO B$ § * ðÞ # >E^O / JO)) / (K " GJO / J) й ) ] Ie / J → Ü * II
" OG8E; II #LCCO@DGC@=P
```

```
KK> Ď %*ðÞ(√P @$ √@ √(O √P § $ P@)) й ≡) —"GJ (% √P @$ √@ √(O √P § $ P@)) й
≡) —√K " GJ * . √(" GJO * . Ü √K 8 - K * .) й ≡) —√K " GJ * - √(" GJO * . Ü √K 8 - K * .)
```

In[•]:=
```
^ý & _ ë M& ) ®ĐĀ ë ý^Ü - / J * I + 8O & ×ā ë * ðÞ^Ü Ie ) → √(Ü DL * * * I) II " OG8E; II #LCCO@DGC@=P
I (" GJ (8O " GJ - " GJ I O % O √O § √(@×ā ý)) ` Þ $ O@×ā ý !%ý $ ` Þ")
```

In[•]:=
```
{ ÞĐĂ®>o - o " GJO B$ § * ðÞ # >E^O / JO ÿ M[ENTER]Q &{ ¶×ðĐ¶° ć
```

Out[•]=
```
≡≡) —"GJ / J . √(" GJO / J . Ü 8 - / J.) й ≡) —"GJ / J - √(" GJO / J . Ü 8 - / J.) й
≡) —"GJ / J . √(" GJO / J Ü 8 - / J.) й ≡) —"GJ / J - √(EpsO Rs 2 a - Rs.)
```

Pr2

Kerry Metric

Define auxiliary terms

In[•]:=
```
Clear[Delt, Sigm, a, r]
```

```
O@>D K I^2 # a^Ü Cos[u]^2 p
! < CK K I^2 $ Rs " r # a^Ü;
```

Define the metric components

In[ ]:= g+t t, ($ (í G /J E I / 0@>D) p

>N I OK 0@>D / ! < CK p

>NLLOK 0@>Dp œL JK8E; J =FI K?<K8ED

>NMMOK (þI ʾ # a^ij G ! < CK E 8 ʾ " Si n+u, ʾ / 0@>D) 0@ENLő K "v stands for phi"!

g+t v, ($ ʾ " a " Rs " r / Si gm Si n+u, ʾ p

Expand each component around a=0

In[ ]:= Ser i es+g+t t, И 0a И , И ι 2Q

x ÐДA Ã (Hí G /J / I) $ O,a- ʾ

In[ ]:= Ser i es+g+r r, И xa И , И ι 8Q

x ÐДA Ã I / I H /J G, @8P

rĆA Ãf 0<I @<J>NLLOй γ8й ήй í Ιζε

Out[ ]= r G, @8P

rĆA Ãf 0<I @<J>NMMOй γ8й ήй í Ιζε

Out[ ]= r Si n,u- ʾ $ O,a- ʾ

In[ ]:= Ser i es+g+t v, И 0a И , И ι 2,

x ÐДA Ã % ʾ dRs Si n,u- ʾ ua / r $ O,a- ʾ

FD G8I< K?@KF K?< J: ?N8IQ ?@ D <K@KF =@; K?8Kh$_K$) HY/JJ@^{2} ¥ / r 9+FK< K?8K8CK?< I<JKF=K?< : FD GF[E<EKJF=? 8I< Ų91?LJ0@ Ú$_8$ h$_©$ the only non zero term is ψ ?$_t$ L Ú h$_K$ G Ú h$_K$ G Ú h$_K$ $ ψ ?$_t$ L Ú h$_K$) 0 , since the >$_t$ tern doesn't depend on ᵒ . Therefore we have ψ ?$_{ĖĚ}$ Ų9, E K?< FK?<I ?8E; K?< K8: < h) h$_{/ }$ B>^{schw} Ŧ) 0, since h is off diagonal and >^{schw} @ ; @>FE8@1?<I<=FI<0Ú$_©$h) 0. 6 ψ ?$_{ĖĚ}$ $\frac{1}{2}$ ψ$_Ė$? L Ų9

Schwarzschild metric around * L * 0

In[ ]:= Rs (ʾ MK

In[∘]:= ±+ĀĀ,

Ċ ΣΡ ŏ %⍃$ ⋮ > ý / (ý^ΰ + 8⋮ FJ[L]^ΰ)

ýΝ ŏ >J@KK[< +(⍃$ _þ/ý)Ŝ

±þ+ýý, (⍃ / (⍃$ _þ/ý) p

>J@EF[< l⋮ p ')F JK8E; J F= FD<>8)(

In[∘]:= `®ý¶®þ±þ+ýý,Ƥ x>Ƥ >␤Ƥ ⍃Q

Ċ ΣΡ ŏ ί / (ί − ΰ*ή / l) + ΰ l (* − *ή) / (ΰ *ή − l)^ΰ +, [* − *ή]^ΰ

ýΝ ŏ 0<l @<J=>J@KK[й *й *ήй ί],

Out[∘]= (%⍃$ ⋮ >␤ / ý) + ΰ (* − *ή) / l +, [* − *ή]^ΰ

' §ĒŒ-ŒĀŖĀ-®©þĂ§ýŢ§ÝĀ-®ýŢĂ§ĀŢĂ®þ°|?$_{\mu\nu}$

#@JK: 8CLC8K< K?< ?l@KFц<CJPD 9FCJ=FI K?< 0: ?N8IQJ: ?@ D<KI@

```
>J: ? < \\>J@KK[й ήй ή й ή]й \ήй >J@l[й ήй ή]й \ήй ήй l², 0в, ꞁ0, 0, 0, Sinuσ²вв;
œu stands for thetaED
Gsch κ Tableỳgschġii₁ġjj₁, ꞁii, 1, 4в, ꞁjj, 1, 4вz;
Gsch ɪɪMatrixForm
```

xD̤D̤Ă A̤RRv ĂD̤C̤ĂDoĆC̤B̤ f

$$\begin{pmatrix} H1\,G\,\frac{2M}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & \mathrm{Sin}\,u^2 \end{pmatrix}$$

```
Ginv κ InverseɴGschɴ;
XX κ ꞁt, r, u, vв;
```

```
Christ κ 1/2 Tableỳ
    Sumỳ Ginvɴm, lɴᴇ ᴡ DỳGschụbụ,ụlụ, XXụaụzF DỳGschụaụ,ụlụ, XXụbụz G DỳGschụaụ,ụbụ, XXụlụzỳ, ꞁl, 1, 4вz,
    ꞁm, 1, 4в, ꞁa, 1, 4}, {b, 1, 4}
  ];
```

Calculate the covariant derivative

Calculate the covariant derivative of the trace