Black holes and their perturbations

Exercise sheet week 1

This exercise sheet will **not** be graded.

1 Kruskal-Szekeres coordinates

The Schwarschild metric in Kruskal-Szekeres coordinates is given by

$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} (-d\underline{t}^2 + d\underline{r}^2) + r(\underline{t}, \underline{r})^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

with $r(\underline{t}, \underline{r})$ defined implicitly by

$$\underline{t}^2 - \underline{r}^2 = \left(1 - \frac{r}{2M}\right) e^{\frac{r}{2M}}.$$

In region I of the Schwarzschild spacetime, the Kruskal coordinates $\underline{t}$ and $\underline{r}$ are related to the Schwarzschild coordinates t and r trough

$$\underline{t} = \left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) \tag{1}$$

$$\underline{r} = \left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right) \tag{2}$$

a. Show by explicit substitution that (1) and (2) relate the Kruskal metric in region I ($0 < \underline{t} + \underline{r} < \infty$ and $0 > \underline{t} - \underline{r} > -\infty$) is isometric to the exterior ($r > 2M$) patch of the Schwarzschild metric. Pay explicit attention to the ranges of the coordinates.
b. The coordinate transformation (1) and (2) does not make sense when $r < 2M$. Why? Find the coordinate transformation that relates the Schwarzschild interior patch ($r < 2M$) to region II ($0 < \underline{t} + \underline{r} < \infty$ and $0 < \underline{t} - \underline{r} < -\infty$).
c. Find the coordinate transformations that relate regions III ($0 > \underline{t} + \underline{r} > \infty$ and $0 > \underline{t} - \underline{r} > -\infty$) and IV ($0 > \underline{t} + \underline{r} > \infty$ and $0 < \underline{t} - \underline{r} < -\infty$) of the Kruskal metric to parts of the Schwarzschild metric. Are these regions isometric to the interior or exterior patch of the Schwarzschild metric?

2 Conformal transformations

a.The angle α between two spacelike vectors x^μ and x^ν is defined by

$$\cos\alpha = \frac{x^\mu y^\nu g_{\mu\nu}}{\sqrt{x^\mu x_\mu y^\nu y_\nu}}. \tag{3}$$

Show that α remains invariant when change the metric by a conformal transformation.

b. In the lectures we showed that

$$\bar{\eta}_{\mu\nu} = -d\bar{t}^2 + d\bar{r}^2 \sin^2 \bar{r}(d\theta^2 + \sin^2 \theta d\phi^2) \tag{4}$$

is related to the Minkowski metric by a conformal transformation. Calculate the Ricci tensor $R_{\mu\nu}$ for $\bar{\eta}_{\mu\nu}$. Does $\bar{\eta}_{\mu\nu}$ satisfy the vacuum Einstein equation?

3 Negative Kerr

In the lecture we discussed that in the maximally extended Kerr solution we can access the $r < 0$ region by passing through the ring singularity at $r = 0$.
a. Show that the $r < 0$ patch of the Kerr metric is isometric to the an $r > 0$ patch of the Kerr metric with negative mass.
b. Consider the closed loop described by taking t, r, and θ constant. Calculate the norm of tangent vector of this loop.
c. Show that for small negative values of r this norm can become negative. What does this imply?