Black holes and their perturbations

Exercise sheet week 1

This exercise sheet will **not** be graded.

1 Kruskal-Szekeres coordinates

The Schwarschild metric in Kruskal-Szekeres coordinates is given by

$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} (-d\underline{t}^2 + d\underline{r}^2) + r(\underline{t},\underline{r})^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

with $r(\underline{t},\underline{r})$ defined implicitly by

$(\underline{t}-\underline{r})(\underline{t}+\underline{r}) =$

$$\underline{t}^2 - \underline{r}^2 = \left(1 - \frac{r}{2M}\right) e^{\frac{r}{2M}}.$$

$(1 - \frac{r}{2M})$

In region I of the Schwarzschild spacetime, the Kruskal coordinates $\underline{t}$ and $\underline{r}$ are related to the Schwarzschild coordinates t and r trough

$1 - r/2M$ cosh $r = 2M$

$$\underline{t} = \left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) \qquad (1)$$

$\leftarrow t > 0$

$$\underline{r} = \left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right) \qquad (2)$$

$1 - r/2M$

a. Show by explicit substitution that (1) and (2) relate the Kruskal metric in region I ($0 < \underline{t} + \underline{r} < \infty$ and $0 > \underline{t} - \underline{r} > -\infty$) is isometric to the exterior ($r > 2M$) patch of the Schwarzschild metric. Pay explicit attention to the ranges of the coordinates.
b. The coordinate transformation (1) and (2) does not make sense when $r < 2M$. Why? Find the coordinate transformation that relates the Schwarzschild interior patch ($r < 2M$) to region II ($0 < \underline{t} + \underline{r} < \infty$ and $0 < \underline{t} - \underline{r} < +\infty$).
c. Find the coordinate transformations that relate regions III ($0 > \underline{t} + \underline{r} > -\infty$ and $0 > \underline{t} - \underline{r} > -\infty$) and IV ($0 > \underline{t} + \underline{r} > \infty$ and $0 < \underline{t} - \underline{r} < -\infty$) of the Kruskal metric to parts of the Schwarzschild metric. Are these regions isometric to the interior or exterior patch of the Schwarzschild metric?

2 Conformal transformations

a. The angle α between two spacelike vectors x^μ and y^ν is defined by

$$\cos\alpha = \frac{x^\mu y^\nu g_{\mu\nu}}{\sqrt{x^\mu x_\mu y^\nu y_\nu}}. \qquad (3)$$

Show that α remains invariant when change the metric by a conformal transformation.

$$\cos\alpha = \frac{x^\mu y^\nu \omega^2 g_{\mu\nu}}{\sqrt{x^\mu x_\mu y^\nu y_\nu}} = \frac{x^\mu y^\nu g_{\mu\nu}\omega^2}{\sqrt{x^\mu x^\alpha \omega^2 g_{\mu\alpha}\, y^\nu y^\beta \omega^2 g_{\nu\beta}}} = \frac{x^\mu y^\nu \cancel{\omega^2} g_{\mu\nu}}{\cancel{\omega^2}\sqrt{x^\mu x_\mu y^\nu y_\nu}}$$

$\to \underline{t} = \left(\frac{r}{2m} - 1\right)^{1/2} e^{r/4m} \sinh\left(\frac{t}{4m}\right) \quad \begin{array}{l} 0 \le \underline{t} - \underline{r} \le \infty \\ 0 \le \underline{t} + \underline{r} \le \infty \end{array}$

$\underline{r} = \left(\frac{r}{2m} - 1\right)^{1/2} e^{r/4m} \cosh\left(\frac{t}{4m}\right) \quad \sim r \ t$

$\underbrace{r < 2m}$ $\to \left(1 - \frac{r}{2m}\right)^{1/2} e^{r/4m} \cosh$

$\underline{\mathrm{II}}$

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}\right)$$

$$\to \Gamma'^{\mu}_{\alpha\beta} = \frac{1}{2} w^{2} g^{\mu\sigma}\Big(g_{\beta\sigma}(\partial_\alpha w^2) + w^2 \partial_\alpha g_{\beta\sigma} + g_{\alpha\sigma}(\partial_\beta w^2) + w^2 \partial_\beta g_{\alpha\sigma} - g_{\alpha\beta}(\partial_\sigma w^2) + w^2 \partial_\sigma g_{\alpha\beta} \Big)$$

$$= \frac{1}{2} w^2 g^{\mu\sigma}\left(g_{\beta\sigma}\partial_\alpha w^2 + g_{\alpha\sigma}\partial_\beta w^2 - g_{\alpha\beta}\partial_\sigma w^2\right)$$

$$+ \frac{1}{2} w^4 \Gamma^{\mu}_{\alpha\beta}$$

b. In the lectures we showed that

$$\bar{\eta}_{\mu\nu} = -d\bar{t}^2 + d\bar{r}^2 \sin^2 \bar{r}(d\theta^2 + \sin^2\theta d\phi^2) \tag{4}$$

is related to the Minkowski metric by a conformal transformation. Calculate the Ricci tensor $R_{\mu\nu}$ for $\bar{\eta}_{\mu\nu}$. Does $\bar{\eta}_{\mu\nu}$ satisfy the vacuum Einstein equation?

3 Negative Kerr

In the lecture we discussed that in the maximally extended Kerr solution we can access the $r < 0$ region by passing through the ring singularity at $r = 0$.
a. Show that the $r < 0$ patch of the Kerr metric is isometric to the an $r > 0$ patch of the Kerr metric with negative mass.
b. Consider the closed loop described by taking t, r, and θ constant. Calculate the norm of tangent vector of this loop.
c. Show that for small negative values of r this norm can become negative. What does this imply?

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \xrightarrow{t,\,r,\,\theta \text{ const}} ds^2 = \left(r^2 + a^2 + \frac{r_s r a^2}{\Sigma}\sin^2\theta\right) d\varphi^2$$

tangent vector to geodesic : $T \to g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$. In our case choose $\lambda = \varphi$

$$\therefore \quad T = g_{\mu\nu}\frac{dx^\mu}{d\varphi}\frac{dx^\nu}{d\varphi}\, d\varphi^2 = \frac{ds^2}{d\varphi^2} = \left(r^2 + a^2 + \frac{r_s r a^2}{\Sigma}\sin^2\theta\right)$$