1 Marginally Bound Geodesics

1.1

Since $\mu = -g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$, $E^2 = \mu$ implies that $E^2 = -g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} > 0$. Thus, $g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} < 0$ which means that the corresponding line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu < 0$ is timelike.

1.2

The radial equation in the general form is:

$$P(r) = (E(r^2 + a^2) - aL)^2 - \Delta(Q + (L - aE)^2 + \mu r^2) \tag{1}$$

Now replacing in 1 $\mu = E^2$, we use Mathematica to get[1]:

$$P(r) = -a^2Q + \left((-aE + L)^2 + Q\right) R_s r + (-L^2 - Q)r^2 + R_s E^2 r^3 \tag{2}$$

1.3

The radial potential, as seen from equation 2, is a third order polynomial(with real coefficients) and thus, it must have 3 roots. The number of complex roots is always even (if $z \in \mathbf{C}$ is a root then the complex conjugate $z^\star$ is also a root) so 0 or 2 in our case and therefore the $P(r)$ can have 1 or 3 real roots.

1.4

First note that from equation 2, P(r) has the following asymptotic behaviour:

$$\begin{aligned}\lim_{r\to+\infty} P(r) = R_s E^2 r^3 = +\infty \\ \lim_{r\to-\infty} P(r) = R_s E^2 r^3 = -\infty\end{aligned} \tag{3}$$

Since $E^2 > 0$ and $R_s = 2M > 0$. More over at r_+ $\Delta = 0$ thus we have:

$$P(r_+) = (\varepsilon(r^2 + a^2) - aL)^2 > 0 \tag{4}$$

Let P(r) have 1 root, $r_1 > 0$. Given the fact that it is positive both in $+\infty$ and at r_+, the case where $r_1 > r_+$ can occour only if P(r) has a minimum at r_1. However a minimum of P(r) corresponds to a double root, not a single root. Thus, $r_1 < r_+$.

[1] See the accompanied notebook

Now we consider the case where P(r) has 3 roots $0 < r_1 < r_2 < r_3 < +\infty$ and we will examine the following cases:

- all 3 roots are outside of the horizon r_+: $r_+ < r_1 < r_2 < r_3 < +\infty$
- 2 roots are outside of the horizon: $r_1 < r_+ < r_2 < r_3 < +\infty$
- 1 root is outside of the horizon: $r_1 < r_2 < r_+ < r_3 < +\infty$
- none of the roots are outside of the horizon: $r_1 < r_2 < r_3 < r_+$.

1.4.1 Case 1: $r_+ < r_1 < r_2 < r_3 < +\infty$

The only way this configuration can be feasible, given the fact that P is positive at (r_+) and at $+\infty$, is if one if the roots is a minimum of P(r). However as stated earlier a minimum corresponds to a double root and thus we would need 4 roots for that not 3. Thus **this root configuration is not possible**.

1.4.2 Case 2: $r_1 < r_+ < r_2 < r_3 < +\infty$

There is no argument against this configuration. Such geodesics, are scattering trajectories, starting from infinity and scattering off the black hole at r_3, to return back to infinity. Alternatively, solutions can be deeply bound plunges, oscillating between $r_1 < r_+$ and r_2.

Note that since marginally bound geodesics are timelike, Q can only be positive (as seen from the polar solutions), and thus P(r) is positive $\forall r \in [r_-, r_+]$. That means that the solution r_1 can only be between 0 and the inner horizon r_-.

1.4.3 Case 3: $r_1 < r_2 < r_+ < r_3 < +\infty$

Same argument as in case 1, 1.4.1. **This root configuration is not possible**

1.4.4 Case 4: $< r_1 < r_2 < r_3 < r_+$

This is also possible. Following the same argument as 1.4.2, for the positivity of Q, the three solutions can only be located between 0 and the inner horizon. Solutions corresponding to this configuration, are direct plunges coming from infinity and diving right into the black hole.

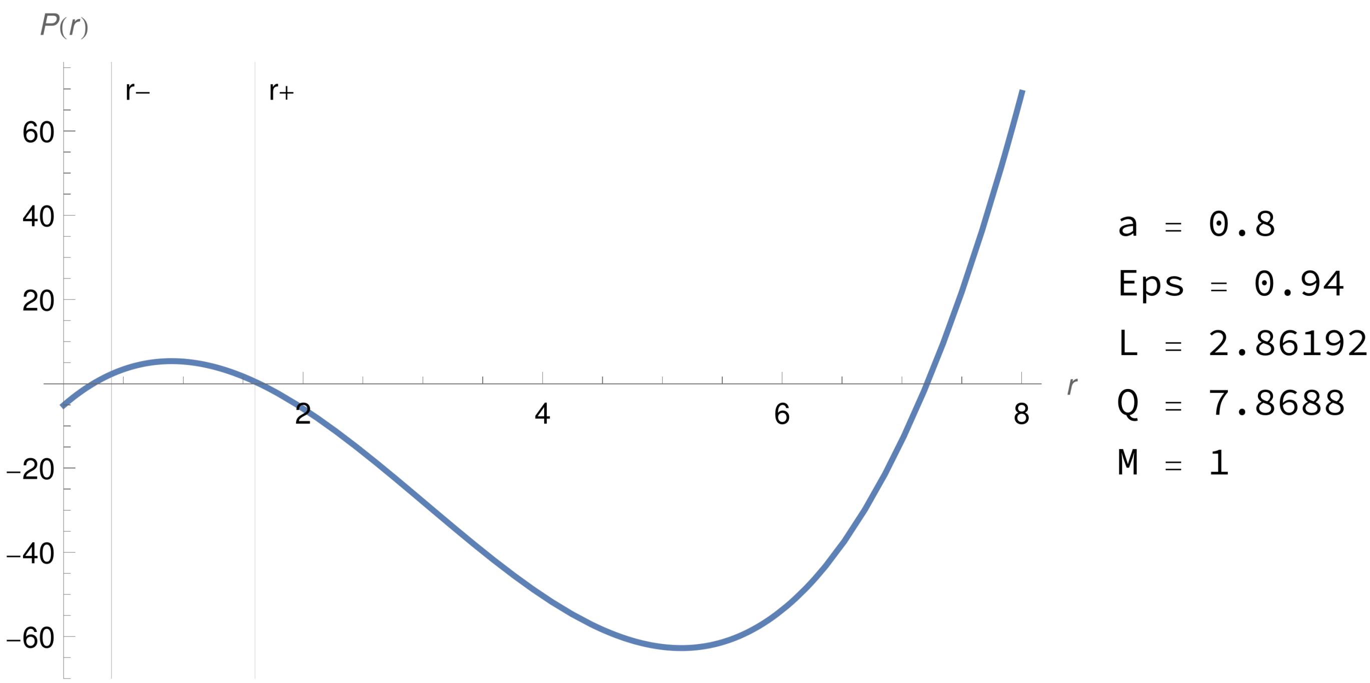

Figure 1: The radial potential Pr for a scattering trajectory

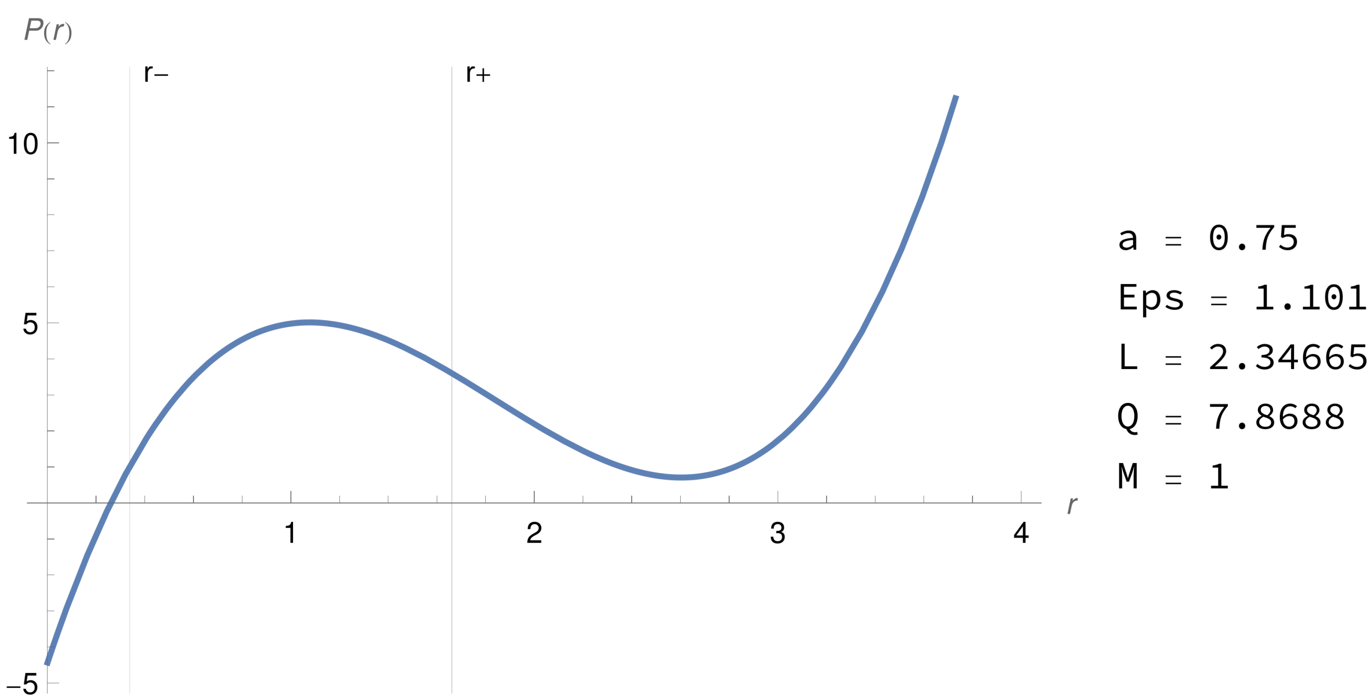

Figure 2: The radial potential Pr for a direct plunge. Ofcourse this corresponds to the case where P(r) has one root not three (the plot for three roots in this case was rather hard to produce).

1.5

In the case of a circular orbit at radius r_0, P(r) can be written as:

$$P(r) = E^2 R_s (r - r_0)^2 (r - r_1) \tag{5}$$

Of course, $r_1 < r_0$: a double root is the limit where $r_0 = r_2 = r_3 > r_+$ corresponding to a marginal case of 1.4.2. The derivatives of P are:

$$\begin{aligned} P'(r) &= 2E^2 R_s (r - r_0)(r - r_1) + E^2 R_s (r - r_0)^2 \\ P''(r) &= 2E^2 R_s (r - r_1) + 4E^2 R_s (r - r_0) \end{aligned} \tag{6}$$

At $r = r_0$ we have $P(r_0) = 2E^2 R_s (r_0 - r_1) > 0$ and since r_0 is a double root of P (by definition of a circular orbit), r_0 can only be a minimum of P which corresponds to an unstable orbit.

1.6

At the equatorial limit $Q \to 0$, equation 2 becomes:

$$P(r) = -L^2 r^2 + (-aE + L)^2 r + RsE^2 r^3 \tag{7}$$

Mathematica says that the equation $P(r) = 0$ has the following solutions:

$$\begin{aligned} r_1 &= 0 \\ r_2 &= \frac{L^2 - \sqrt{L^4 - 16a^2E^4M^2 + 32aE^3LM^2 - 16E^2L^2M^2}}{4E^2M} \\ r_3 &= \frac{L^2 + \sqrt{L^4 - 16a^2E^4M^2 + 32aE^3LM^2 - 16E^2L^2M^2}}{4E^2M} \end{aligned} \tag{8}$$

As stated earlier, the radius r_0 of the circular orbit is a double root of P(r). Given the three roots of P(r), and the fact that it wouldn't be of much physical significance to have any of the roots r_1 or r_2 equal to 0, we require. $r_0 = r_2 = r_3$ which is equivalent to the equation:

$$L^4 - 16a^2E^4M^2 + 32aE^3LM^2 - 16E^2L^2M^2 = 0 \tag{9}$$

Solving this equation, using Mathematica, for a we get:

$$a = \frac{\pm L^2 + 4ELM}{4E^2M} \tag{10}$$

Plugin this back to equation 7 and solving for L we get:

$$L = \pm 2\sqrt{E^2Mr_0} \Rightarrow L^2 = 4E^2Mr_0 = 4\mu Mr_0 \tag{11}$$

Finally, plugging once again the result of equation 11 back to 7 and solving for r, with the requirement that $r > r_+$, we get the following solutions:

$$\begin{aligned} r_1 &= -a + 2M + 2\sqrt{(M(-a + M))} \\ r_2 &= a + 2M + 2\sqrt{(M(a + M))} \end{aligned} \tag{12}$$

Now we notice that r_1 corresponds to the solution for porsitive angular momentum where:

$$r_2 = -a + 2M + 2M\sqrt{1 - a/M} \tag{13}$$

and r_2 corresponds to the solution for the negative angulare momentum as for $a \to -a$, $r_2 \to r_1$

2 Kerry Metric

2.1

We use Taylor's theorem to expand each component of the metric around $a = 0$[2]:

$$
\begin{aligned}
g_{tt} &= -\left(1 - \frac{R_s r}{\Sigma}\right) \approx (-1 + \frac{R_s}{r}) + O(a^2) \\
g_{rr} &= \frac{\Sigma}{\Delta} \approx \frac{r}{r - R_s} + O(a^2) \\
g_{\theta\theta} &= \Sigma \approx r^2 + O(a^2) \\
g_{\phi\phi} &= \frac{((r^2+a^2)^2 - \Delta a^2 \sin^2(\theta))}{\Sigma} \sin^2(\theta) \approx r^2 \sin^2(\theta) + O(a^2) \\
g_{t\phi} &= -\frac{2aR_s r}{\Sigma} \sin^2(\theta) \approx -\frac{2R_s \sin^2(\theta) a}{r} + O(a^2)
\end{aligned}
\tag{14}
$$

Where

$$
\begin{aligned}
\Sigma &= r^2 + a^2 \cos^2(u) \\
\Delta &= r^2 - R_s r + a^2
\end{aligned}
\tag{15}
$$

Comparing these results to the Schwarzschild metric, $ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, we see that the only non zero element of the perturbed metric $h_{\mu\nu}$ is $h_{t\phi} = \frac{-2R_s(\sin^2\theta)}{r}$.

The Lorentz condition is:

$$
\nabla^\alpha h_{\mu\alpha} = \frac{1}{2}\nabla_\mu h \tag{16}
$$

Since that the only non zero element of h is $h_{t\phi}$, the only non zero element in $\nabla^\alpha h_{\mu\alpha}$ is: $\nabla^\alpha h_{t\alpha} = \nabla^t h_{tt} + \nabla^r h_{tr} + \nabla^\phi h_{t\phi} + \nabla^\theta h_{t\theta} = \nabla^\phi h_{t\phi} = 0$. $h_{t\phi}$ doesn't depend on ϕ, and thus, we have: $(\nabla^\alpha) h_{\mu\alpha} = 0$. On the other hand, the trace $h = h_{\alpha\beta}(g^{\text{schw}})^{\alpha\beta} = 0$, given the fact that h is off-diagonal and g^{schw} is diagonal. Therefore, $\nabla^\alpha h_{\mu\alpha} = \frac{1}{2}\nabla_\mu h = 0$.

2.2

The only components of the Schwarchild metric that depend on mass are g_{tt} and g_{rr}. Thus expanding these components around M_0 yields:

$$
\begin{aligned}
g_{tt} &= -(1 - \frac{2M}{r}) \approx (-1 + \frac{2M_0}{r}) + \frac{2}{r}\delta M + O(\delta M^2) \\
g_{rr} &= \frac{1}{1 - \frac{2M}{r}} \approx \frac{1}{1 - \frac{2M_0}{r}} + \frac{2r}{(2M_0 - r)^2}\delta M + O(\delta M^2)
\end{aligned}
\tag{17}
$$

[2] See Mathematica notebook

The perturbed metric is :

$$h_{\mu\nu} = \begin{pmatrix} \frac{2}{r} & 0 & 0 & 0 \\ 0 & \frac{2r}{(2M_0-r)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

Note that

$$h = h_{\mu\nu} g^{\mu\nu} = \frac{2}{r}\frac{r}{2M_0 - r} + \frac{2r}{(2M_0-r)^2}\frac{r-2M_0}{r} = 0$$

However

$$\nabla^\alpha h_{\mu\alpha} = \begin{pmatrix} 0 \\ -\frac{2}{2M_0 r - r^2} \\ 0 \\ 0 \end{pmatrix}$$

Therefore, $h_{\mu\nu}$ doesn't satisfy the Lorentz condition in this case.

2.3

We make a gauge transformation to the perturbed metric:

$$h \rightarrow \tilde{h}_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{19}$$

where $h_{\mu\nu}$ is given by 18. We want the gauge transformed metric to obey the Lorentz condition thus we have:

$$\overline{\nabla}^\alpha \tilde{h}_{\mu\alpha} = \frac{1}{2}\overline{\nabla}_\mu h \tag{20}$$

with $\overline{\nabla}_\mu$ being the covariant derivative in the background Schwarzchild metric $g_\mu \nu(M_0)$ while ∇_μ is the covariant derivative in the full metric with $\nabla_\mu V^\nu = \overline{\nabla}_m u V^\nu + \epsilon C^\nu_{\mu\alpha} V^\alpha$[3].

Expanding equation 20 we get:

$$g^{\alpha\beta}\left(\overline{\nabla}_\alpha h_{\mu\beta} - \frac{1}{2}\overline{\nabla}_\mu h_{\alpha\beta}\right) + g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta \xi_\mu + \delta M \left(\frac{1}{2} g^{\alpha\beta}\overline{\nabla}_\mu C^\rho_\alpha \beta \xi_\rho - g^{\alpha\beta}\overline{\nabla}_\alpha C^\rho_{\mu\beta}\xi_\rho\right) = 0 \tag{21}$$

of course this equation needs to be satisfied order by order in δM and to 0th order, noting that $\frac{1}{2} g^{\alpha\beta}\overline{\nabla}_\mu h_{\alpha\beta} = 0$, we get (once again we drop the bars in the derivatives for notional simplicity) :

$$\nabla_\alpha \nabla^\alpha \xi_\mu = -\nabla^\alpha h_{\mu\alpha} \tag{22}$$

The right hand side of equation 22, is known. Expanding the left hand side we get:

$$g^{\alpha\beta}\left(\partial_\alpha\partial_\beta \xi_\mu - (\partial_\alpha \Gamma^\lambda_{\beta\mu})\xi_\lambda - \Gamma^\rho_{\beta\mu}\partial_\alpha \xi_\rho - \Gamma^\rho_{\alpha\beta}\partial_\rho \xi_\mu - \Gamma^\rho_{\alpha\mu}\partial_\beta \xi_\rho + \Gamma^\rho_{\alpha\beta}\Gamma^\lambda_{\rho\mu}\xi_\lambda + \Gamma^\rho_{\alpha\mu}\Gamma^\lambda_{\rho\beta}\xi_\lambda\right) = -\nabla^\alpha h_{\mu\alpha} \tag{23}$$

[3] In the previous sections of these exercise, we only had to use $\overline{\nabla}$ so for simplicity in notation the bar was neglected

This is a differential equation for the components of ξ_μ. By inserting the ansatz $\xi_r = f(r)$, $\xi_t = \xi_\theta = \xi_\phi = 0$, equation 23 reduces to

$$\frac{(4M_0 - 2r)}{r^3} f(r) + \frac{2}{r} f'(r) + \frac{(r - 2M_0)}{r} f''(r) = \frac{2}{2M_0 r - r^2} \tag{24}$$

With solution

$$f(r) = \frac{C_1}{2M_0 r - r^2} - \frac{r^2 C_2}{3(-2M_0 + r)} + \frac{4M_0^2 r + M_0 r^2 + r^3 \log(r) + 8M_0^3 \log(-2M_0 + r) - r^3 \log(-2M_0 + r)}{3M_0 r(-2M_0 + r)} \tag{25}$$

To fix the coefficients C_1 and C_2 we will need to impose boundary conditions. There is an infinite number of boundary conditions that could be imposed which means that the Lorentz condition, in equation 16, does not uniquely specify ξ.

One such boundar condition, could be the requirement of $\tilde{h} = g^{\alpha\beta}\tilde{h}_{\alpha\beta} = 0$.