

A new gauge-invariant double copy for heavy-mass effective theory

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ABSTRACT: We propose a new form of the colour-kinematics/double-copy duality for heavy-mass effective field theories, which we apply to construct compact expressions for tree amplitudes with heavy matter particles in Yang-Mills and in gravity to leading order in the mass. In this set-up, the new BCJ numerators are fixed uniquely and directly written in terms of field strengths, making gauge invariance manifest. Furthermore, they are local and automatically satisfy the Jacobi relations and crossing symmetry. We construct these BCJ numerators explicitly up to six particles. We also discuss relations of the BCJ numerators in the heavy-mass effective theory with those in pure Yang-Mills amplitudes.

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1 Introduction

Problems in physics where the typical scale of the momenta is much smaller than the masses of the particles at play can be conveniently described using effective field theories. The prototype example is heavy-quark effective theory [1–4], where one is interested in studying the dynamics of heavy quarks exchanging momenta which are typically much smaller than their mass. The effective description is particularly convenient in that it reveals symmetries that are not present in the QCD Lagrangian, leading to a velocity superselection rule. Another example is the dynamics of black holes, which in many situations of practical relevance can be effectively considered as heavy pointlike particles exchanging momenta which are much smaller than their mass. In this context, the computation of the classical part of observable quantities, such as the deflection angle of a massless particle by a heavy black hole, is of particular interest. The classical limit is then reached by scaling the momentum of the exchanged massless gravitons as $\vec{q} = \hbar \vec{k}$ and taking the $\hbar \rightarrow 0$ limit while keeping the wavevector \vec{k} fixed, together with the masses and momenta of the black holes [5]. An approach to the effective field theory describing heavy scalars and fermions minimally coupled to gravity in the spirit of the heavy-quark effective theory was pursued in [6], with applications to Feynman-diagram computations of four-point amplitudes at one loop.

Recently, considerable effort has been devoted to applying modern amplitude methods to the computation of observable quantities in general relativity. This approach leads to significant conceptual as well as practical simplifications, since all steps of the computation are manifestly gauge invariant. Examples of this include applications of unitarity at one [7–16], two [17–19] and three loops [20] to the computation of bending angles, and classical and quantum corrections to the gravitational potential. An important tool in the amplitude arsenal is the colour-kinematics and double-copy duality [21–23], which makes explicit an intriguing interplay between colour and kinematics of amplitudes in Yang-Mills and other theories [24–38]. This also allows one to construct gravity amplitudes from Yang-Mills ones once the latter are expressed in a so-called BCJ form, in a similar spirit to the KLT relations [39]. Further applications of this duality include [40–48] and [49–59] for the construction of BCJ numerators at tree and loop level, respectively.

Intriguingly, there is strong evidence that underlying the colour-kinematics duality there must be a kinematic algebra obeyed by the BCJ numerators. This algebra was originally discovered in the self-dual sector in [60, 61], with recent important efforts to understand it beyond that sector in [62, 63]. We also mention the work of [36, 38, 64–

71] where a Lagrangian or geometric understanding of the duality was sought. With a view of applying the colour-kinematics duality to the problem of black hole scattering, one is interested in amplitudes containing two massive particles as well as gravitons, which in general can also be constructed via the double copy [72–79]. The double copy was also directly applied to the heavy-quark effective theory in [80] for particles of spin $s \leq 1$, in particular constructing three- and four-point amplitudes with two heavy spinning particles and one and two gravitons.

In this paper we systematically apply the method of [62] to obtain compact expressions for amplitudes with two massive scalars and an arbitrary number of gluons or gravitons via the double copy in a heavy-mass effective theory (HEFT) at leading order in an inverse mass expansion. These will be used in [81] to compute loop amplitudes of two heavy scalars with graviton interactions, from which one can extract classical quantities such as the bending angle and corrections to the Newtonian potential. Our HEFT amplitudes will enter the relevant unitarity cuts, crucially simplifying the loop integrations because of their special features [81].

In the approach of [62], one introduces vector and tensor currents representing the generators of the kinematic algebra, with a fusion rule among them. This fusion rule was completely determined in [62, 63] in Yang-Mills in the MHV and NMHV sectors for arbitrary multiplicity, with explicit examples up to eight particles.¹ In the present work we apply these ideas to the HEFT, computing general amplitudes up to six points and in any number of dimensions. Importantly, these are the necessary ingredients to compute the two-to-two scattering amplitude of two heavy scalars up to three loops.

This new approach is very powerful in that it leads to BCJ numerators automatically satisfying the Jacobi relations and crossing symmetry, under the assumption that the fusion rule is associative. In practice, one starts by building a function called the pre-numerator (which, despite its name, can have denominators from propagators of the heavy particles), which is written as a product of many currents multiplied using the fusion rule. The BCJ numerators are then obtained by taking appropriate anti-symmetrisations of the external particles in the pre-numerator, thereby forming nested commutator structures associated to cubic graphs. With the assumption of associativity of the fusion rule, this operation produces BCJ numerators that automatically satisfy colour-kinematics duality. We also stress that only a subset of all possible cubic graphs appears in our construction, namely those where the two massive particles connect via a single cubic vertex to the rest of the graph. Each cubic graph is in one-to-one

¹In general dimensions, by MHV amplitude in pure Yang-Mills we mean one whose numerator has the schematic form $(\epsilon \cdot \epsilon) \prod (\epsilon \cdot p)$, whereas NMHV corresponds to $(\epsilon \cdot \epsilon)^2 \prod (\epsilon \cdot p)$, and so on.

correspondence with a nested commutator.

An additional important feature of our work is that the BCJ numerators we obtain for each cubic graph are uniquely determined, manifestly gauge invariant (i.e. written in terms of field strengths) and local with respect to the massless gluons or gravitons. This is to be contrasted with the situation in Yang-Mills amplitudes, where the BCJ numerators are in general neither gauge invariant nor unique. As a byproduct of our analysis, we also show how to derive BCJ numerators in pure Yang-Mills by taking appropriate limits of our BCJ numerators.

The rest of the paper is organised as follows. In the next section and in Section 3 we briefly review basic properties of heavy-quark effective theory and of the double copy, respectively. In Section 4 we present the construction of amplitudes from the novel double copy. We briefly review the approach based on fusion rules, which we then discuss in the context of our HEFT. In particular we discuss the construction of the pre-numerator from fusion rules and from an ansatz, in terms of which the BCJ numerators are expressed. In Section 5 we systematically treat cases up to six particles. Importantly, we find unique, gauge-invariant BCJ numerators. In Section 6 we briefly discuss how to obtain pure Yang-Mills numerators from HEFT numerators in a particular limit. We present our conclusions and an outlook on future work in Section 7.

2 Elements of heavy-mass effective theory

Heavy-quark effective theory [1–4] plays an important role in hadron physics. In this set-up, the momentum of an incoming heavy quark is written as

$$p^\mu = mv^\mu , \tag{2.1}$$

where m is the heavy mass of the quark and $v^2 = 1$, which after an interaction with a soft particle becomes

$$p^\mu = mv^\mu + k^\mu . \tag{2.2}$$

In QCD, the momentum k^μ would be taken to be of order $\Lambda_{\text{QCD}} \ll m$. We are ultimately interested in applications to classical physics (discussed in the companion paper [81]), in which case it is convenient to think of the residual soft momentum as being rescaled as $k^\mu = \hbar \bar{k}^\mu$ [5], keeping \bar{k} fixed as $\hbar \rightarrow 0$. If p is the momentum of an on-shell state,

for example an outgoing heavy quark, we also get the constraint $v \cdot k = -k^2/(2m)$, which implies that $v \cdot k = 0$ in the large-mass limit. The leading terms of the effective Lagrangian are

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + i\bar{Q}_v v \cdot (\partial - igA) \frac{1 + \not{v}}{2} Q_v + \mathcal{O}(1/m) , \quad (2.3)$$

where for external fermion states one also has $\not{v}Q_v = Q_v$. If one ignores the $\mathcal{O}(1/m)$ terms, the velocity v and the polarisation of Q_v are conserved. The Feynman rules for the fermion propagator and vertex are easily found to be

$$\begin{array}{ccc} \text{Feynman diagram: } \overrightarrow{\text{line}} \text{ with } v, k & \rightarrow & \frac{i}{v \cdot k + i\varepsilon} \frac{1 + \not{v}}{2} \\ \text{Feynman diagram: } \text{wavy line} \text{ with } \mu, p_2 & \rightarrow & igT^a v_\mu \frac{1 + \not{v}}{2} \end{array} \quad (2.4)$$

which are accompanied by the standard Feynman rules for gauge fields. Importantly, the leading-order contribution in the heavy-mass limit is universal, that is the heavy quark field can be replaced by a heavy scalar or vector field without changing the amplitudes.

One can now use these Feynman rules to compute directly HEFT amplitudes, at least for a small number of legs. For higher multiplicities this becomes very involved, and we will introduce more efficient techniques in the next two sections.² The three-point amplitude is given by

$$A_3^{\text{YM-M}}(123) = \text{Feynman diagram: } p_1 \rightarrow \text{wavy line} \rightarrow p_3 \text{ with } \varepsilon_2 \text{ on the wavy line} = m\varepsilon_2 \cdot v , \quad (2.5)$$

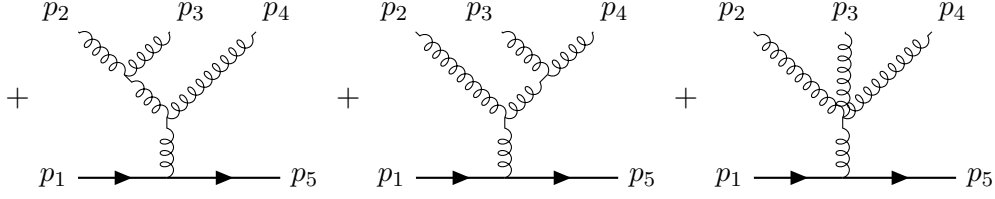
while the four-point amplitude is

$$\begin{aligned} A_4^{\text{YM-M}}(1234) &= \text{Feynman diagram: } p_1 \rightarrow \text{wavy line} \rightarrow p_4 \text{ with } p_2, p_3 \text{ on the wavy line} + \text{Feynman diagram: } p_1 \rightarrow \text{wavy line} \rightarrow p_4 \text{ with } p_2, p_3 \text{ on the wavy line} \\ &= 2m \left(-\frac{\varepsilon_2 \cdot p_3 v \cdot \varepsilon_3}{s_{23}} - \frac{\varepsilon_2 \cdot \varepsilon_3 v \cdot p_2}{s_{23}} + \frac{\varepsilon_3 \cdot p_2 v \cdot \varepsilon_2}{s_{23}} + \frac{v \cdot \varepsilon_2 v \cdot \varepsilon_3}{2v \cdot p_2} \right) . \end{aligned} \quad (2.6)$$

The Feynman diagrams contributing to the five-point amplitude are

$$\text{Feynman diagram: } p_1 \rightarrow \text{wavy line} \rightarrow p_5 \text{ with } p_2, p_3, p_4 \text{ on the wavy line} + \text{Feynman diagram: } p_1 \rightarrow \text{wavy line} \rightarrow p_5 \text{ with } p_2, p_3, p_4 \text{ on the wavy line} + \text{Feynman diagram: } p_1 \rightarrow \text{wavy line} \rightarrow p_5 \text{ with } p_2, p_3, p_4 \text{ on the wavy line}$$

²In the following we quote colour-ordered amplitudes and drop an ubiquitous factor of $i g^{n-2}$.



leading to the – somewhat lengthy – result

$$\begin{aligned}
A_5^{\text{YM-M}}(12345) = & 4m \left[\frac{v \cdot \varepsilon_2 v \cdot \varepsilon_3 v \cdot \varepsilon_4}{4v \cdot p_2 v \cdot p_{23}} + \frac{v \cdot \varepsilon_4 \left(\frac{1}{2} \varepsilon_2 \cdot \varepsilon_3 v \cdot (p_3 - p_2) + \varepsilon_3 \cdot p_2 v \cdot \varepsilon_2 - \varepsilon_2 \cdot p_3 v \cdot \varepsilon_3 \right)}{2s_{23} v \cdot p_{23}} \right. \\
& + \frac{v \cdot \varepsilon_2 \left(-\varepsilon_3 \cdot p_4 v \cdot \varepsilon_4 - \frac{1}{2} \varepsilon_3 \cdot \varepsilon_4 v \cdot p_3 + \frac{1}{2} \varepsilon_3 \cdot \varepsilon_4 v \cdot p_4 + \varepsilon_4 \cdot p_3 v \cdot \varepsilon_3 \right)}{2s_{34} v \cdot p_2} \\
& + \frac{1}{2s_{23} s_{234}} \left(v \cdot \varepsilon_4 \left(\left(\frac{s_{24} - s_{34}}{2} \right) \varepsilon_2 \cdot \varepsilon_3 - 2\varepsilon_2 \cdot p_4 \varepsilon_3 \cdot p_2 + 2\varepsilon_2 \cdot p_3 \varepsilon_3 \cdot p_4 \right) \right. \\
& + \frac{s_{23}}{2} (\varepsilon_2 \cdot \varepsilon_4 v \cdot \varepsilon_3 - \varepsilon_3 \cdot \varepsilon_4 v \cdot \varepsilon_2) - \varepsilon_4 \cdot p_{23} (\varepsilon_2 \cdot p_3 v \cdot \varepsilon_3 - \varepsilon_3 \cdot p_2 v \cdot \varepsilon_2) \\
& + 2(v \cdot p_{23} \varepsilon_3 \cdot \varepsilon_4 \varepsilon_2 \cdot p_3 - v \cdot p_{23} \varepsilon_2 \cdot \varepsilon_4 \varepsilon_3 \cdot p_2 + v \cdot p_3 \varepsilon_2 \cdot \varepsilon_3 \varepsilon_4 \cdot p_2 - v \cdot p_2 \varepsilon_2 \cdot \varepsilon_3 \varepsilon_4 \cdot p_3) \Big) \\
& + \frac{1}{s_{34} s_{234}} \left(\varepsilon_2 \cdot p_{34} (\varepsilon_3 \cdot p_4 v \cdot \varepsilon_4 - \varepsilon_4 \cdot p_3 v \cdot \varepsilon_3) + \varepsilon_2 \cdot \varepsilon_4 v \cdot p_2 \varepsilon_3 \cdot p_4 - \varepsilon_4 \cdot p_2 \varepsilon_3 \cdot p_4 v \cdot \varepsilon_2 \right. \\
& + \varepsilon_3 \cdot \varepsilon_4 v \cdot p_3 \varepsilon_2 \cdot p_{34} + \varepsilon_3 \cdot \varepsilon_4 v \cdot p_2 \varepsilon_2 \cdot p_3 - \varepsilon_2 \cdot \varepsilon_3 v \cdot p_2 \varepsilon_4 \cdot p_3 + \varepsilon_3 \cdot p_2 \varepsilon_4 \cdot p_3 v \cdot \varepsilon_2 \\
& \left. \left. - \frac{1}{4} s_{23} \varepsilon_3 \cdot \varepsilon_4 v \cdot \varepsilon_2 + \frac{1}{4} s_{24} \varepsilon_3 \cdot \varepsilon_4 v \cdot \varepsilon_2 - \frac{1}{4} s_{34} \varepsilon_2 \cdot \varepsilon_3 v \cdot \varepsilon_4 + \frac{1}{4} s_{34} \varepsilon_2 \cdot \varepsilon_4 v \cdot \varepsilon_3 \right) \right], \tag{2.7}
\end{aligned}$$

where

$$p_{i_1 i_2 \dots i_r} := p_{i_1} + p_{i_2} + \dots + p_{i_r}, \quad s_{i_1 i_2 \dots i_r} := p_{i_1 i_2 \dots i_r}^2. \tag{2.8}$$

Note that the gluon-quark amplitude in HEFT does not depend on the soft momentum of the heavy particle. One can also check that the leading order term in $1/m$ of the full five-point gluon-quark amplitude is identical to (2.7).

3 Traditional double copy construction

In the previous section we reviewed an example of a heavy-particle effective theory and presented a few tree-level amplitudes derived from the leading-order Feynman rules. We chose to work with fermions as heavy particles, however it is important to note that the results thus obtained are in fact independent of the spin of the heavy matter

fields [8]. The corresponding effective theory for gravity, studied using a form of the colour-kinematics duality and double copy of HEFT in [80], also exhibits universal behaviour at leading order in the heavy-mass limit, which is related to the universal coupling between matter and gravitons [82, 83].

Next, we discuss quark-graviton amplitudes for the full theory as computed using the traditional form of the colour-kinematics duality. The leading-order parts of these amplitudes in the heavy-mass limit can then be obtained by an expansion in the (inverse) heavy mass.³ In this form of the double copy one sums over *all* cubic graphs, and the numerators are in general *not* gauge invariant. The double copy of QCD with massive fermions was studied in [77, 80], while the double copy of massive scalar QCD was investigated in [79].

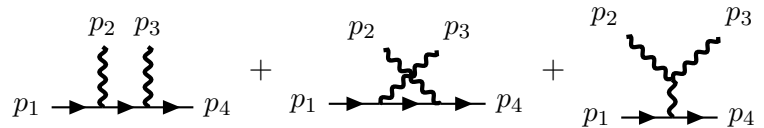
The BCJ numerators for the three-, four- and five-point amplitudes can be cast in the form [27, 62, 77]

$$\begin{aligned} N_3(123) &= \bar{Q}\not{\epsilon}_2 Q, \\ N_4(1234) &= \bar{Q}\not{\epsilon}_2(\not{p}_{12} + m)\not{\epsilon}_3 Q, \\ N_5(12345) &= \bar{Q}\not{\epsilon}_2(\not{p}_{12} + m)\not{\epsilon}_3(\not{p}_{123} + m)\not{\epsilon}_4 Q + \frac{2}{3}(p_{123}^2 - m^2) (\bar{Q}\not{\epsilon}_2\not{\epsilon}_3\not{\epsilon}_4 Q - \varepsilon_3 \cdot \varepsilon_4 \bar{Q}\not{\epsilon}_2 Q) . \end{aligned} \quad (3.1)$$

The three-point graviton amplitude is obtained by just squaring the BCJ numerator

$$A_3^{\text{GR-Q}} = [N_3(123)]^2 . \quad (3.2)$$

The four-point amplitude is obtained by summing over the following three cubic graphs with two gravitons and two fermions:



$$+ \quad + \quad + \quad . \quad (3.3)$$

The numerator of each graph is the square of the corresponding BCJ numerator. The four-point amplitude is then

$$A_4^{\text{GR-Q}} = \frac{N_4(1234)^2}{2p_1 \cdot p_2} + \frac{N_4(1324)^2}{2p_1 \cdot p_3} + \frac{N_4(1[2,3]4)^2}{s_{23}} , \quad (3.4)$$

where $N_4(1324)$ is obtained from $N_4(1234)$ by swapping the indices 2, 3 and the bracket on the external labels denotes the commutator of the indices, e.g.

$$N_4(1[2,3]4) := N_4(1234) - N_4(1324) . \quad (3.5)$$

³The quark-graviton amplitudes can also be obtained from Feynman rules [84, 85].

The four-point amplitude is manifestly invariant under the exchange of particles 2 and 3. Similarly the five-point amplitude is obtained by summing over all cubic graphs with two scalars and three external gravitons:

$$\begin{aligned}
& \begin{array}{c} p_2 \quad p_3 \quad p_4 \\ \text{wavy} \quad \text{wavy} \quad \text{wavy} \\ p_1 \rightarrow \text{---} \rightarrow p_5 \end{array} + \begin{array}{c} p_2 \quad p_3 \quad p_4 \\ \text{wavy} \quad \text{wavy} \quad \text{wavy} \\ p_1 \rightarrow \text{---} \rightarrow p_5 \end{array} + \begin{array}{c} p_2 \quad p_3 \quad p_4 \\ \text{wavy} \quad \text{wavy} \quad \text{wavy} \\ p_1 \rightarrow \text{---} \rightarrow p_5 \end{array} \\
& + \begin{array}{c} p_2 \quad p_3 \quad p_4 \\ \text{wavy} \quad \text{wavy} \quad \text{wavy} \\ p_1 \rightarrow \text{---} \rightarrow p_5 \end{array} + \begin{array}{c} p_2 \quad p_3 \quad p_4 \\ \text{wavy} \quad \text{wavy} \quad \text{wavy} \\ p_1 \rightarrow \text{---} \rightarrow p_5 \end{array} + \text{permutations of } 2, 3, 4. \quad (3.6)
\end{aligned}$$

Computing the above diagrams, one gets

$$\begin{aligned}
A_5^{\text{GR-Q}} = & \frac{N_5(12345)^2}{2p_1 \cdot p_2(2p_1 \cdot p_{23} + s_{23})} + \frac{N_5(1[2, 3]45)^2}{2s_{23}(2p_1 \cdot p_{23} + s_{23})} + \frac{N_5(12[3, 4]5)^2}{4s_{34}p_1 \cdot p_2} \\
& + \frac{N_5(1[2, [3, 4]]5)^2}{2s_{34}s_{234}} + \text{permutations of } 2, 3, 4. \quad (3.7)
\end{aligned}$$

The leading-order term in the heavy-mass expansion is obtained by simply retaining the $\mathcal{O}(m^2)$ terms in (3.7), and we have checked that it agrees with the same quantity as computed with the KLT duality.

A few comments are in order here. First, we note that in this traditional form of the double copy one has to sum over all possible cubic graphs; furthermore, gauge invariance of the amplitudes is not manifest due to the explicit appearance of polarisation vectors in the numerators N_i . Inspired by recent work on the kinematic algebra of BCJ numerators [62], in the next section we will introduce an improved version of the double copy, which will lead to much more compact expressions for the amplitudes. These will be used in [81] for the computation of classical quantities in gravity at loop level.

4 A novel double copy from gauge-invariant BCJ numerators

We now present our alternative form of the colour-kinematics duality for the gluon-matter amplitudes directly at leading order in the HEFT, avoiding the need to perform a heavy-mass expansion. This double copy is motivated by the work of one of the

present authors [62] on the algebraic structure of numerators that are consistent with the colour-kinematics duality, and allows us to generate the BCJ numerators directly. As we will see, this construction has several advantages:

1. The new BCJ numerators are automatically gauge invariant and unique.
2. Only a subset of the usual cubic diagrams contributes.
3. As a consequence, we obtain much more compact expressions for the amplitudes than those derived from the traditional BCJ construction discussed in the previous section.

We have tested this new method by explicitly constructing numerators up to six particles at tree level, although in principle the method applies to n -particle numerators.

4.1 Background for the colour-kinematics algebra from fusion rules

We now describe the new construction of BCJ numerators, which is based on the *fusion product* between two heavy-mass currents. Following the notation of [62] these currents are denoted by $J_{a_1 \otimes a_2 \otimes \dots \otimes a_r}$,⁴ and their labels a_i^μ can either be momenta or polarisation vectors. Since $\bar{Q} \not{a}_i Q \sim m v \cdot a_i$ in the HEFT, we define

$$J_a = m v \cdot a . \quad (4.1)$$

We also require that the tensor currents $J_{a_1 \otimes a_2 \dots a_i \otimes a_j \dots a_r}$ satisfy the Clifford algebra

$$J_{a_1 \otimes a_2 \dots a_i \otimes a_j \dots a_r} = -J_{a_1 \otimes a_2 \dots a_j \otimes a_i \dots a_r} + 2 a_i \cdot a_j J_{a_1 \otimes a_2 \dots \hat{a}_i \otimes \hat{a}_j \dots a_r} , \quad (4.2)$$

for each component. This property is inherited from the QCD currents.

The on-shell condition for the external quarks also gives the relations

$$\begin{aligned} J_{a_1 \otimes a_2 \otimes \dots a_n \otimes p_{12 \dots n-1}} &= m J_{a_1 \otimes a_2 \otimes \dots a_n} , \\ J_{p_1 \otimes a_1 \otimes a_2 \otimes \dots a_n} &= m J_{a_1 \otimes a_2 \otimes \dots a_n} , \end{aligned} \quad (4.3)$$

and the currents satisfy a bilinear fusion rule of the form

$$J_X \star J_Y = \sum_Z F_{XY}^Z J_Z . \quad (4.4)$$

⁴The notation for these currents is inspired by the form of tensor currents in QCD, $\bar{Q} \not{a}_1 \not{a}_2 \dots \not{a}_r Q$.

For instance [62]

$$J_{\varepsilon_2} \star J_{\varepsilon_3} := 2 \left(\frac{s_{23}}{4} \frac{v \cdot \varepsilon_3}{v \cdot p_2} J_{\varepsilon_2} - \frac{1}{2} J_{\varepsilon_2 \otimes \varepsilon_3 \otimes p_2} + \varepsilon_3 \cdot p_2 J_{\varepsilon_2} \right). \quad (4.5)$$

The fusion rule always generates a rational function which we call the *pre-numerator*:

$$\mathcal{N}_n(23 \cdots n-1, v) := \begin{array}{c} 2 \quad 3 \cdots n-1 \\ \diagdown \quad | \quad \diagup \\ \text{red box} \end{array} = J_{\varepsilon_2} \star J_{\varepsilon_3} \star \cdots \star J_{\varepsilon_{n-1}}, \quad (4.6)$$

where we always assume associativity of the \star -product. In this diagram, the red box denotes the two massive particles, while the lines correspond to the massless particles (gluons or gravitons). The general form of the fusion rule with these properties is known in pure Yang-Mills from [62] for n -point MHV amplitudes and for NMHV amplitudes up to eight points. The claim is that BCJ numerators in HEFT can be written efficiently in terms of the pre-numerator, in a way we describe below.

To begin with, it is useful to introduce the notion of ordered and un-ordered nested commutators. In the case of ordered nested commutators, which is relevant for colour-ordered amplitudes, the order of a set of indices is fixed while commutators are applied in all possible ways. For example, for $n = 5$, the ordered set $\{2, 3, 4\}$ gives rise to two ordered, nested commutators: $[[2, 3], 4]$ and $[2, [3, 4]]$. In the case of graviton amplitudes, we need to include also un-ordered nested commutators (omitting numerators that differ by minus signs): $[[2, 3], 4]$, $[[2, 4], 3]$ and $[[3, 4], 2]$. Then the gluon-matter and graviton-matter amplitudes are given by the following expressions:

$$\begin{aligned} A_n^{\text{YM-M}}(12 \cdots n) &= \sum_{\Gamma \in \text{ordered commutators } \{2, 3, \dots, n-1\}} \frac{\mathcal{N}_n(\Gamma, v)}{d_\Gamma}, \\ A_n^{\text{GR-M}}(12 \cdots n) &= \sum_{\Gamma \in \text{non-ordered commutators } \{2, 3, \dots, n-1\}} \frac{[\mathcal{N}_n(\Gamma, v)]^2}{d_\Gamma}, \end{aligned} \quad (4.7)$$

where particles 1 and n are heavy and all others are massless. Each nested commutator (and hence BCJ numerator) is in one-to-one correspondence with a specific cubic graph, from which one can also read off the relevant massless scalar propagators, denoted as d_Γ in (4.7). For instance, the nested commutator $[[2, 3], 4]$ and the associated BCJ numerator corresponds to the cubic graph

$$\mathcal{N}_5([[2, 3], 4], v) \longleftrightarrow \begin{array}{c} 2 \quad 3 \quad 4 \\ \diagdown \quad \bullet \quad \diagup \\ \diagdown \quad \bullet \quad \diagup \\ \text{red box} \end{array}, \quad (4.8)$$

where in this case $d_{[[2,3],4]} = s_{23}s_{234}$. We also note an important feature of this BCJ representation, namely that we only sum over cubic graphs corresponding to nested commutators. In particular, only massless propagators appear within the cubic graph – the two massive particles always connect to the graph via a single cubic vertex.

Each BCJ numerator $\mathcal{N}_n(\Gamma, v)$ can be conveniently obtained by acting with specific operators on the pre-numerator. For instance, a left-nested commutator is written as

$$\mathcal{N}_n([\dots [[2, 3], 4], \dots, n-1], v) := \mathbb{L}(2, 3, 4, \dots, n-1) \circ \mathcal{N}_n(234 \dots n-1, v), \quad (4.9)$$

where $\mathbb{L}(2, 3, 4, \dots, n-1)$ is the \mathbb{L} -operator introduced in [63] as a group algebra element [86, 87] and in [88] as a free Lie algebra element,

$$\mathbb{L}(i_1, i_2, \dots, i_r) := \left[\mathbb{I} - \mathbb{P}_{(i_2 i_1)} \right] \left[\mathbb{I} - \mathbb{P}_{(i_3 i_2 i_1)} \right] \cdots \left[\mathbb{I} - \mathbb{P}_{(i_r \dots i_2 i_1)} \right]. \quad (4.10)$$

Here $\mathbb{P}_{(j_1 j_2 j_3 \dots j_m)}$ denotes the cyclic permutation $j_1 \rightarrow j_2 \rightarrow j_3 \rightarrow \dots \rightarrow j_m \rightarrow j_1$. For instance

$$\begin{aligned} \mathbb{I} \circ \mathcal{N}_6(2345, v) &= \mathcal{N}_6(2345, v), \\ \mathbb{P}_{(432)} \circ \mathcal{N}_6(2345, v) &= \mathcal{N}_6(4235, v). \end{aligned} \quad (4.11)$$

Importantly, as we will discuss in more detail later, for each choice of Γ the numerator $\mathcal{N}_n(\Gamma, v)$ is consistent with colour-kinematics duality.

We have successfully checked this conjecture up to six points. An important observation is that the numerators we obtain in this approach are all gauge invariant, as will become clear in concrete examples presented later. The general method to determine the fusion rules will be described elsewhere, while in this paper we will focus on describing a method to construct the pre-numerator from an ansatz, and from this, the BCJ numerators $\mathcal{N}_n(\Gamma, v)$.

4.2 General method to construct the pre-numerator

Since the fusion rule introduced in (4.4) is not yet known in general, our strategy consists in writing down a general ansatz for the pre-numerator, in terms of which the BCJ numerators $\mathcal{N}_n(\Gamma, v)$ are written. We then require that these BCJ numerators $\mathcal{N}_n(\Gamma, v)$ generate the correct amplitudes when inserted in (4.7). In all cases we have considered, this procedure leads to a unique, gauge-invariant answer for the BCJ numerators (while in general the pre-numerator has undetermined coefficients).

The building blocks of the ansatz are products of the following Lorentz-invariant quantities:⁵

$$\frac{1}{v \cdot p_{i_1 \dots i_r}}, \quad \varepsilon_i \cdot \varepsilon_j, \quad \varepsilon_i \cdot p_j, \quad v \cdot \varepsilon_i, \quad v \cdot p_i, \quad s_{ij}, \quad J_{\varepsilon_i}, \quad J_{p_i}. \quad (4.12)$$

In fact, in constructing the ansatz we can restrict the vector currents J to only those involving J_{p_2} and J_{ε_2} , since the others can be generated with the $\mathbb{L}(2, 3, 4, \dots, n-1)$ operator.

There are several constraints to impose on our ansatz. First, for an n -point Yang-Mills amplitude, there are at most $n-3$ propagators of the heavy particles and no double poles. Second, it follows from the Feynman rules (2.4) that the overall scaling degree in v should be one. Third, we have observed that it is sufficient to restrict the power of s_{ij} in a particular term in the ansatz to be the same as the total number of massive propagators in that term. Hence a general term in the ansatz for the pre-numerator $\mathcal{N}^{(\text{ans})}(234 \dots n-1, v)$ has the form

$$\left(\prod_{h=1}^d \frac{1}{v \cdot p_{i_1^{(h)} \dots i_{r_h}^{(h)}}} \right) \left(\prod_{h=1}^d s_{i^{(h)} j^{(h)}} \right) \left(\prod_{h=1}^d v \cdot a_{i^{(h)}} \right) \left(\prod_{h=1}^{d'} \varepsilon_{i^{(h)}} \cdot a_{j^{(h)}} \right) J_{a_2}, \quad (4.13)$$

where $d \in [1, n-3]$, and a denotes either ε or p . Finally, the ansatz is further constrained by power counting and multi-linearity with respect to external polarisation vectors (which fixes d').

An important comment is in order here. The terms without any denominator are fixed by the MHV sector in the pure Yang-Mills theory⁶ with ϵ_1 replaced by mv . In fact, we have determined the MHV fusion rules, and hence the Yang-Mills pre-numerator for this sector, which has the following form:

$$\begin{aligned} \mathcal{N}_n^{(0)}(234 \dots n-1, v) &= 2^{n-3} \left(\left(\prod_{j=3}^{n-1} \varepsilon_j \cdot p_{2 \dots j} \right) J_{\varepsilon_2} \right. \\ &\quad \left. - \frac{1}{2} \sum_{h>\ell=2}^{n-1} (-1)^\ell \left(\prod_{j=\ell+1, j \neq h}^{n-1} \varepsilon_j \cdot p_{2 \dots j} \right) \varepsilon_2 \cdot H_3 \cdot H_4 \cdots H_{\ell-1} \cdot p_{1 \dots \ell} J_{\varepsilon_\ell \otimes \varepsilon_h \otimes p_2} \right), \end{aligned} \quad (4.14)$$

⁵Although J_a is proportional to $v \cdot a$, we prefer to treat these two quantities as separate as their origin in the QCD amplitude is different, specifically the J_a terms arise from fermion bilinears while the $v \cdot a$ terms from expanding propagators. Note that in constructing the ansatz, we always fix J_a to be either J_{p_2} or J_{ε_2} from the requirement of full crossing symmetry with respect to the gravitons.

⁶This is further discussed in Section 6.

where $H_i^{\mu\nu} = p_{2\dots i}^\mu \varepsilon_i^\nu - \eta^{\mu\nu} \varepsilon_i \cdot p_{2\dots i}$. Furthermore, when $\ell = 3$ in the sum in the second line of (4.14), the product of H factors should be replaced by $\eta_{\mu\nu}$, while when $\ell = 2$, $\varepsilon_2 \cdot H_3 \cdot H_4 \cdots H_{\ell-1} \cdot p_{1\dots\ell}$ is replaced by 1.

We will find it convenient to write the ansatz for the n -particle pre-numerator as the sum of the purely polynomial MHV part and a yet to be determined remainder containing massive propagators:

$$\mathcal{N}_n(234 \cdots n-1, v) := \mathcal{N}_n^{(0)}(234 \cdots n-1, v) + \mathcal{N}_n^{(\text{ans})}(234 \cdots n-1, v). \quad (4.15)$$

In the next section we will apply this general procedure to construct BCJ numerators in explicit examples.

5 Applications of the new double copy

We now give examples of the construction of gauge-invariant BCJ numerators up to six-point amplitudes. We remind the reader that in what follows $p_1 = mv \simeq -p_n$ always denote the momenta of the hard particles.

5.1 Three-point numerator

At three points, the pre-numerator is simply

$$\mathcal{N}_3(2, v) := \begin{array}{c} 2 \\ | \\ \blacksquare \end{array} = J_{\varepsilon_2} = mv \cdot \varepsilon_2, \quad (5.1)$$

and the amplitudes with a gluon or a graviton are

$$A_3^{\text{YM-M}}(123) = \mathcal{N}_3(2, v), \quad A_3^{\text{GR-M}}(123) = [\mathcal{N}_3(2, v)]^2. \quad (5.2)$$

5.2 Four-point numerator

At four points, we know from (4.5) that the pre-numerator is

$$\mathcal{N}_4(23, v) := \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ \blacksquare \end{array} = 2 \left(\frac{s_{23} v \cdot \varepsilon_3}{4v \cdot p_2} J_{\varepsilon_2} - \frac{1}{2} J_{\varepsilon_2 \otimes \varepsilon_3 \otimes p_2} + \varepsilon_3 \cdot p_2 J_{\varepsilon_2} \right). \quad (5.3)$$

Then the amplitudes with two gluons or two gravitons are

$$A_4^{\text{YM-M}} = \frac{\mathcal{N}_4([2, 3], v)}{s_{23}}, \quad A_4^{\text{GR-M}} = \frac{[\mathcal{N}_4([2, 3], v)]^2}{s_{23}}, \quad (5.4)$$

where

$$\mathcal{N}_4([2, 3], v) := \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ \bullet \\ \textcolor{red}{\square} \end{array} := \mathbb{L}(2, 3) \circ \mathcal{N}_4(23, v) = 2m \left(\frac{v \cdot F_2 \cdot F_3 \cdot v}{v \cdot p_3} \right). \quad (5.5)$$

Here $F_i^{\mu\nu} = p_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu p_i^\nu$, and we have used the fact that $J_a = mv \cdot a$ for a vector current. Note that terms containing tensor currents always cancel out among themselves after using the Clifford algebra. For instance, the current $J_{\varepsilon_2 \otimes \varepsilon_3 \otimes p_2}$ appears in the form of a commutator, which we can recast as

$$\begin{aligned} J_{\varepsilon_2 \otimes \varepsilon_3 \otimes p_2} - J_{\varepsilon_3 \otimes \varepsilon_2 \otimes p_3} &= 2\varepsilon_2 \cdot \varepsilon_3 J_{p_2} - J_{\varepsilon_3 \otimes \varepsilon_2 \otimes p_{23}} = 2\varepsilon_2 \cdot \varepsilon_3 J_{p_2} - J_{\varepsilon_3 \otimes \varepsilon_2 \otimes p_{123}} + J_{\varepsilon_3 \otimes \varepsilon_2 \otimes p_1} \\ &= 2\varepsilon_2 \cdot \varepsilon_3 J_{p_2} + 2\varepsilon_2 \cdot p_1 J_{\varepsilon_3} - 2p_1 \cdot \varepsilon_3 J_{\varepsilon_2} - J_{\varepsilon_3 \otimes \varepsilon_2 \otimes p_{123}} + J_{p_1 \otimes \varepsilon_3 \otimes \varepsilon_2} \\ &= 2\varepsilon_2 \cdot \varepsilon_3 J_{p_2} + 2\varepsilon_2 \cdot p_1 J_{\varepsilon_3} - 2p_1 \cdot \varepsilon_3 J_{\varepsilon_2} + m J_{\varepsilon_3 \otimes \varepsilon_2} - m J_{\varepsilon_3 \otimes \varepsilon_2} \\ &= 2\varepsilon_2 \cdot \varepsilon_3 J_{p_2} + 2\varepsilon_2 \cdot p_1 J_{\varepsilon_3} - 2p_1 \cdot \varepsilon_3 J_{\varepsilon_2}, \end{aligned} \quad (5.6)$$

where we have repeatedly used (4.2) and the on-shell conditions (4.3). Note the absence of tensor currents in the last line of (5.6). We also observe that (5.5) is expressed in terms of gauge-invariant quantities only.

5.3 Five-point numerator

From five points on, we need to use the method described in Section 4.2 to determine the BCJ numerators. According to (4.14), the part of the pre-numerator without linear propagators, that is $\mathcal{N}_5^{(0)}(234, v)$, is given by

$$\begin{aligned} \mathcal{N}_5^{(0)}(234, v) &= 4 \left(p_2 \cdot \varepsilon_3 p_{23} \cdot \varepsilon_4 J_{\varepsilon_2} \right. \\ &\quad \left. - \frac{1}{2} p_{23} \cdot \varepsilon_4 J_{\varepsilon_2 \otimes \varepsilon_3 \otimes p_2} - \frac{1}{2} p_{23} \cdot \varepsilon_3 J_{\varepsilon_2 \otimes \varepsilon_4 \otimes p_2} + \frac{1}{2} p_{23} \cdot \varepsilon_2 J_{\varepsilon_3 \otimes \varepsilon_4 \otimes p_2} \right). \end{aligned} \quad (5.7)$$

We now write the ansatz for the remaining terms in the pre-numerator, as per (4.15). We divide these terms into three sectors of the form

$$v \cdot \varepsilon \, v \cdot \varepsilon, \quad \varepsilon \cdot \varepsilon \, v \cdot \varepsilon, \quad v \cdot \varepsilon \, v \cdot \varepsilon \, p \cdot \varepsilon. \quad (5.8)$$

The most general ansatz made of terms of this form contains 42 parameters, denoted by $x_{i,j}$ below:

$$\begin{aligned}
\mathcal{N}_5^{(\text{ans}_1)}(234, v) &= \left(\frac{x_{1,1} s_{23}^2 v \cdot \epsilon_3 v \cdot \epsilon_4}{v \cdot p_2 v \cdot p_{23}} + \frac{x_{1,2} s_{24} s_{23} v \cdot \epsilon_3 v \cdot \epsilon_4}{v \cdot p_2 v \cdot p_{23}} + \frac{x_{1,3} s_{24}^2 v \cdot \epsilon_3 v \cdot \epsilon_4}{v \cdot p_2 v \cdot p_{23}} \right. \\
&\quad \left. + \frac{x_{1,4} s_{34} s_{23} v \cdot \epsilon_3 v \cdot \epsilon_4}{v \cdot p_2 v \cdot p_{23}} + \frac{x_{1,5} s_{24} s_{34} v \cdot \epsilon_3 v \cdot \epsilon_4}{v \cdot p_2 v \cdot p_{23}} + \frac{x_{1,6} s_{34}^2 v \cdot \epsilon_3 v \cdot \epsilon_4}{v \cdot p_2 v \cdot p_{23}} \right) J_{\epsilon_2}, \\
\mathcal{N}_5^{(\text{ans}_2)}(234, v) &= \left(\frac{x_{2,1} s_{23} \epsilon_3 \cdot \epsilon_4 v \cdot p_3}{v \cdot p_2} + 11 \text{ terms} \right) J_{\epsilon_2} \\
\mathcal{N}_5^{(\text{ans}_3)}(234, v) &= \left(\frac{x_{3,1} s_{23} \epsilon_4 \cdot p_3 v \cdot \epsilon_3}{v \cdot p_2} + 23 \text{ terms} \right) J_{\epsilon_2}. \tag{5.9}
\end{aligned}$$

We solve for the parameters according to (4.7), and arrive at a pre-numerator with 11 undetermined parameters:

$$\begin{aligned}
\mathcal{N}_5(234, v) &:= \begin{array}{c} 2 \quad 3 \quad 4 \\ \diagdown \quad | \quad \diagup \\ \blacksquare \end{array} \tag{5.10} \\
&= 4 \left(\frac{s_{23} \epsilon_4 \cdot p_3 v \cdot \epsilon_3}{2v \cdot p_2} + \frac{s_{34} \epsilon_3 \cdot p_2 v \cdot \epsilon_4}{2v \cdot p_{23}} + \frac{s_{23} \epsilon_3 \cdot \epsilon_4 v \cdot p_4}{2v \cdot p_2} + \frac{s_{23} s_{34} v \cdot \epsilon_3 v \cdot \epsilon_4}{8v \cdot p_2 v \cdot p_{23}} \right) J_{\epsilon_2} \\
&\quad + \mathcal{N}_5^{(0)}(234, v) + x_{1,1} \left(\frac{s_{23}^2 v \cdot \epsilon_3 v \cdot \epsilon_4}{v \cdot p_2 v \cdot p_{23}} - \frac{s_{34}^2 v \cdot \epsilon_3 v \cdot \epsilon_4}{v \cdot p_2 v \cdot p_{23}} \right) J_{\epsilon_2} + \dots,
\end{aligned}$$

where we have indicated for brevity only one of the terms with a free parameter. Remarkably, we find that the terms with undetermined parameters vanish under the action of $\mathbb{L}(2, 3, 4)$:

$$\mathbb{L}(2, 3, 4) \circ \left[x_{1,1} \left(\frac{s_{23}^2 v \cdot \epsilon_3 v \cdot \epsilon_4}{v \cdot p_2 v \cdot p_{23}} - \frac{s_{34}^2 v \cdot \epsilon_3 v \cdot \epsilon_4}{v \cdot p_2 v \cdot p_{23}} \right) J_{\epsilon_2} + \dots \right] = 0. \tag{5.11}$$

As a consequence, and in contradistinction with the expectation from pure Yang-Mills amplitudes, we arrive at a unique solution for the BCJ numerator.

Summarising, the BCJ numerator for the left-nested commutator is

$$\mathcal{N}_5([2, 3], 4, v) := \begin{array}{c} 2 \quad 3 \quad 4 \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \blacksquare \end{array} := \mathbb{L}(2, 3, 4) \circ \mathcal{N}_5(234, v). \tag{5.12}$$

Importantly, we find that this BCJ numerator can be rewritten in a manifestly gauge-invariant form as

$$\mathcal{N}_5([2, 3], 4, v) = \mathbb{L}(2, 3, 4) \circ \left[4m \frac{v \cdot F_2 \cdot F_3 \cdot V_3 \cdot F_4 \cdot v}{v \cdot p_3 v \cdot p_4} \right], \tag{5.13}$$

where $V_i^{\mu\nu} = v^\mu p_i^\nu$. Then the colour-ordered amplitude in the novel colour-kinematics duality form is

$$A_5^{\text{YM-M}}(12345) = \frac{\mathcal{N}_5([2, 3], 4, v)}{s_{234}s_{23}} + \frac{\mathcal{N}_5([2, [3, 4]], v)}{s_{234}s_{34}}. \quad (5.14)$$

Note that these two terms are related by exchanging $2 \leftrightarrow 4$. By the double copy, the gravity amplitude is obtained immediately as

$$A_5^{\text{GR-M}}(12345) = \frac{[\mathcal{N}_5([2, 3], 4, v)]^2}{s_{234}s_{23}} + \frac{[\mathcal{N}_5([2, 4], 3, v)]^2}{s_{234}s_{24}} + \frac{[\mathcal{N}_5([3, 4], 2, v)]^2}{s_{234}s_{34}}. \quad (5.15)$$

We have checked that (5.15) agrees with the $\mathcal{O}(m^2)$ term of (3.7). It is useful to pause and contrast these two expressions to appreciate the simplicity of our approach. First, (5.15) is much more compact than (3.7); furthermore, expanding (3.7) to $\mathcal{O}(m^2)$ gives rise to a very large number of terms lacking any particular structure. The compactness, and manifest gauge invariance and locality of our HEFT amplitudes are of considerable advantage when these expressions are fed into unitarity cuts.

We also comment that the numerators we have constructed automatically satisfy the Jacobi relations. For instance, by definition we have

$$\begin{aligned} \mathcal{N}_5([2, 3], 4, v) &= \mathcal{N}_5(234, v) - \mathcal{N}_5(324, v) - \mathcal{N}_5(423, v) + \mathcal{N}_5(432, v) \\ \mathcal{N}_5([2, 4], 3, v) &= \mathcal{N}_5(243, v) - \mathcal{N}_5(423, v) - \mathcal{N}_5(324, v) + \mathcal{N}_5(342, v) \\ \mathcal{N}_5([2, [3, 4]], v) &= \mathcal{N}_5(234, v) - \mathcal{N}_5(243, v) - \mathcal{N}_5(342, v) + \mathcal{N}_5(432, v), \end{aligned} \quad (5.16)$$

which makes the Jacobi relation manifest,

$$\mathcal{N}_5([2, [3, 4]], v) = \mathcal{N}_5([2, 3], 4, v) - \mathcal{N}_5([2, 4], 3, v). \quad (5.17)$$

Finally, note that one can construct $\mathcal{N}_5([2, 4], 3, v)$ from $\mathcal{N}_5([2, 3], 4, v)$ by simply swapping labels 3 and 4.

5.4 Six-point numerator

Similarly to what was done at five points, using again (4.14) we find that, at six points, the polynomial part of the pre-numerator, $\mathcal{N}_6^{(0)}(2345, v)$, has the form

$$\mathcal{N}_6^{(0)}(2345, v) = 8 \left(J_{\varepsilon_2} p_{2 \cdot \varepsilon_3} p_{23 \cdot \varepsilon_4} p_{234 \cdot \varepsilon_5} \right) \quad (5.18)$$

$$\begin{aligned}
& -\frac{1}{2}p_{23}\cdot\epsilon_4p_{234}\cdot\epsilon_5J_{\epsilon_2\otimes\epsilon_3\otimes p_2} - \frac{1}{2}p_2\cdot\epsilon_3p_{234}\cdot\epsilon_5J_{\epsilon_2\otimes\epsilon_4\otimes p_2} - \frac{1}{2}p_2\cdot\epsilon_3p_{23}\cdot\epsilon_4J_{\epsilon_2\otimes\epsilon_5\otimes p_2} \\
& + \frac{1}{2}p_3\cdot\epsilon_2p_{234}\cdot\epsilon_5J_{\epsilon_3\otimes\epsilon_4\otimes p_2} + \frac{1}{2}p_3\cdot\epsilon_2p_{23}\cdot\epsilon_4J_{\epsilon_3\otimes\epsilon_5\otimes p_2} + \frac{1}{2}(\epsilon_2\cdot p_4\epsilon_3\cdot p_2 - \epsilon_2\cdot p_3\epsilon_3\cdot p_4)J_{\epsilon_4\otimes\epsilon_5\otimes p_2} \Big).
\end{aligned}$$

The most general ansatz for the remainder of the pre-numerator can be decomposed into six sectors. The number of free parameters and physical constraints arising from (4.7) are shown in the table below:

sector	$v\cdot\epsilon\ v\cdot\epsilon\ v\cdot\epsilon\ v\cdot\epsilon$	$v\cdot\epsilon\ v\cdot\epsilon\ v\cdot\epsilon\ p\cdot\epsilon$	$v\cdot\epsilon\ v\cdot\epsilon\ \epsilon\cdot\epsilon$	$v\cdot\epsilon\ v\cdot\epsilon\ p\cdot\epsilon\ p\cdot\epsilon$	$v\cdot\epsilon\ \epsilon\cdot\epsilon\ p\cdot\epsilon$	$\epsilon\cdot\epsilon\ \epsilon\cdot\epsilon$
# of parameters	280	1134	945	1128	1134	378
# of constraints	176	601	438	570	810	108

(5.19)

Thus we arrive at a pre-numerator with 2296 free parameters:

$$\begin{aligned}
\mathcal{N}_6(2345, v) := & \begin{array}{c} 2 \quad 3 \quad 4 \quad 5 \\ \diagdown \quad \diagup \quad \diagup \quad \diagup \\ \text{red square} \end{array} = 8 \left(\frac{s_{34}v\cdot\epsilon_3v\cdot\epsilon_4v\cdot\epsilon_5}{8v\cdot p_2v\cdot p_{23}} \left(\frac{s_{23}s_{45}}{2v\cdot p_{234}} + \frac{s_{24}s_{45}}{3v\cdot p_{234}} - \frac{s_{23}s_{35}}{3v\cdot p_4} \right) J_{\epsilon_2} \right. \\
& + \left(\frac{s_{23}s_{34}\epsilon_5\cdot p_4v\cdot\epsilon_3v\cdot\epsilon_4}{4v\cdot p_2v\cdot p_{23}} + \frac{s_{45}s_{34}\epsilon_3\cdot p_2v\cdot\epsilon_4v\cdot\epsilon_5}{4v\cdot p_{23}v\cdot p_{234}} - \frac{s_{23}s_{34}\epsilon_5\cdot p_3v\cdot\epsilon_3v\cdot\epsilon_4}{8v\cdot p_2v\cdot p_4} \right. \\
& \quad \left. - \frac{s_{35}s_{34}\epsilon_3\cdot p_2v\cdot\epsilon_4v\cdot\epsilon_5}{4v\cdot p_4v\cdot p_{234}} - \frac{s_{23}s_{45}\epsilon_3\cdot p_4v\cdot\epsilon_4v\cdot\epsilon_5}{4v\cdot p_2v\cdot p_{234}} - \frac{s_{24}s_{45}\epsilon_3\cdot p_4v\cdot\epsilon_4v\cdot\epsilon_5}{8v\cdot p_2v\cdot p_{234}} \right) J_{\epsilon_2} \\
& + \left(\frac{s_{23}s_{34}\epsilon_4\cdot\epsilon_5v\cdot p_5v\cdot\epsilon_3}{4v\cdot p_2v\cdot p_{23}} - \frac{s_{23}s_{34}\epsilon_3\cdot\epsilon_5v\cdot p_5v\cdot\epsilon_4}{8v\cdot p_2v\cdot p_4} + \frac{s_{23}s_{35}\epsilon_3\cdot\epsilon_4v\cdot p_4v\cdot\epsilon_5}{8v\cdot p_{23}v\cdot p_{234}} \right. \\
& \quad \left. + \frac{s_{23}s_{45}\epsilon_3\cdot\epsilon_4v\cdot\epsilon_5}{8v\cdot p_{234}} \left(\frac{v\cdot p_4}{v\cdot p_2} + \frac{v\cdot p_5}{v\cdot p_{23}} \right) + \frac{s_{24}s_{45}\epsilon_3\cdot\epsilon_4v\cdot p_4v\cdot\epsilon_5}{8v\cdot p_{234}} \left(\frac{1}{v\cdot p_2} - \frac{1}{v\cdot p_{23}} \right) \right) J_{\epsilon_2} \\
& + \left(\frac{(s_{23} - s_{45})\epsilon_3\cdot p_4\epsilon_5\cdot p_2v\cdot\epsilon_4}{2v\cdot p_{34}} + s_{23}\epsilon_4\cdot p_3v\cdot\epsilon_3 \left(\frac{\epsilon_5\cdot p_{34}}{2v\cdot p_2} - \frac{\epsilon_5\cdot p_2}{2v\cdot p_{34}} \right) + \frac{s_{25}\epsilon_3\cdot p_2\epsilon_5\cdot p_4v\cdot\epsilon_4}{2v\cdot p_{23}} \right. \\
& \quad \left. + \frac{(s_{35} + s_{45})\epsilon_3\cdot p_2\epsilon_4\cdot p_{23}v\cdot\epsilon_5}{2v\cdot p_{234}} - \frac{s_{34}\epsilon_3\cdot p_2\epsilon_5\cdot p_{23}v\cdot\epsilon_4}{2v\cdot p_4} - \frac{(s_{25} + s_{34})\epsilon_3\cdot p_2\epsilon_4\cdot p_5v\cdot\epsilon_5}{4v\cdot p_{23}} \right) J_{\epsilon_2} \\
& + \left(\frac{s_{23}\epsilon_3\cdot\epsilon_5v\cdot p_5\epsilon_4\cdot p_2}{2v\cdot p_{24}} + \frac{s_{23}\epsilon_3\cdot\epsilon_5v\cdot p_5\epsilon_4\cdot p_3}{2v\cdot p_2} + \frac{s_{23}\epsilon_3\cdot\epsilon_4v\cdot p_4\epsilon_5\cdot p_3}{2v\cdot p_2} + \frac{s_{23}\epsilon_3\cdot\epsilon_4v\cdot p_4\epsilon_5\cdot p_4}{2v\cdot p_2} \right. \\
& \quad \left. - \frac{s_{23}\epsilon_4\cdot\epsilon_5v\cdot p_5\epsilon_3\cdot p_4}{2v\cdot p_2} - \frac{s_{23}\epsilon_3\cdot\epsilon_4v\cdot p_4\epsilon_5\cdot p_2}{2v\cdot p_{34}} + \frac{(s_{24} + s_{34})\epsilon_4\cdot\epsilon_5v\cdot p_5\epsilon_3\cdot p_2}{2v\cdot p_{23}} \right) J_{\epsilon_2} \\
& + \left(\frac{s_{25}\epsilon_2\cdot\epsilon_3\epsilon_4\cdot\epsilon_5v\cdot p_4}{4v\cdot p_{23}} + \frac{s_{35}\epsilon_2\cdot\epsilon_3\epsilon_4\cdot\epsilon_5v\cdot p_5}{4v\cdot p_{234}} + \frac{s_{45}\epsilon_2\cdot\epsilon_4\epsilon_3\cdot\epsilon_5v\cdot p_5}{4v\cdot p_{234}} \right. \\
& \quad \left. - s_{45}\epsilon_2\cdot\epsilon_5\epsilon_3\cdot\epsilon_4 \left(\frac{v\cdot p_4}{2v\cdot p_{34}} + \frac{v\cdot p_5}{4v\cdot p_{234}} \right) \right) J_{p_2} \Big) + \mathcal{N}_6^{(0)}(2345, v) + \dots, \tag{5.20}
\end{aligned}$$

where we omit all the terms with free parameters. Then the BCJ numerator from the left-nested commutator is obtained by acting with the operator $\mathbb{L}(2, 3, 4, 5)$:

$$\mathcal{N}_6([[[2, 3], 4], 5], v) := \begin{array}{c} 2 \quad 3 \quad 4 \quad 5 \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \\ \color{red}{\blacksquare} \end{array} := \mathbb{L}(2, 3, 4, 5) \circ \mathcal{N}_6(2345, v). \quad (5.21)$$

The BCJ numerator thus obtained does not contain any free parameter and is unique:

$$\begin{aligned} \mathcal{N}_6([[[2, 3], 4], 5], v) = 8m \mathbb{L}(2, 3, 4, 5) \circ \Big[& \quad (5.22) \\ & \frac{v \cdot F_2 \cdot F_3 \cdot V_3 \cdot F_4 \cdot F_5 \cdot v}{2v \cdot p_3 v \cdot p_{45}} + \frac{v \cdot F_2 \cdot F_3 \cdot V_3 \cdot F_4 \cdot V_4 \cdot F_5 \cdot v}{2v \cdot p_3 v \cdot p_5 v \cdot p_{45}} + \frac{v \cdot F_4 \cdot F_5 \cdot V_3 \cdot F_2 \cdot V_4 \cdot F_3 \cdot v}{2v \cdot p_3 v \cdot p_4 v \cdot p_{345}} \\ & - \frac{v \cdot F_3 \cdot F_5 \cdot V_3 \cdot F_2 \cdot V_3 \cdot F_4 \cdot v}{2v \cdot p_3 v \cdot p_4 v \cdot p_{345}} - \frac{v \cdot F_3 \cdot F_4 \cdot V_3 \cdot F_2 \cdot V_3 \cdot F_5 \cdot v}{2v \cdot p_3 v \cdot p_5 v \cdot p_{345}} - \frac{v \cdot p_{45} v \cdot F_3 \cdot F_4 \cdot V_3 \cdot F_2 \cdot V_4 \cdot F_5 \cdot v}{2v \cdot p_3 v \cdot p_4 v \cdot p_5 v \cdot p_{345}} \Big]. \end{aligned}$$

The colour-ordered amplitude in the novel colour-kinematics duality form is then

$$\begin{aligned} A_6^{\text{YM-M}}(123456) = & \frac{\mathcal{N}_6([[[2, 3], 4], 5], v)}{s_{2345} s_{234} s_{23}} + \frac{\mathcal{N}_6([2, [3, [4, 5]]], v)}{s_{2345} s_{345} s_{45}} + \frac{\mathcal{N}_6([2, 3], [4, 5], v)}{s_{2345} s_{23} s_{45}} \\ & + \frac{\mathcal{N}_6([2, [3, 4]], 5], v)}{s_{2345} s_{234} s_{34}} + \frac{\mathcal{N}_6([2, [[3, 4], 5]], v)}{s_{2345} s_{34} s_{345}}. \end{aligned} \quad (5.23)$$

The gravity amplitude is constructed similarly from the double copy:

$$\begin{aligned} A_6^{\text{GR-M}}(123456) = & \frac{[\mathcal{N}_6([2, 3], [4, 5], v)]^2}{s_{23} s_{45} s_{2345}} + \frac{[\mathcal{N}_6([2, [3, [4, 5]]], v)]^2}{s_{345} s_{45} s_{2345}} + \frac{[\mathcal{N}_6([2, 4], [3, 5], v)]^2}{s_{24} s_{35} s_{2345}} \\ & + \frac{[\mathcal{N}_6([2, [4, [3, 5]]], v)]^2}{s_{345} s_{35} s_{2345}} + \frac{[\mathcal{N}_6([2, 5], [3, 4], v)]^2}{s_{25} s_{34} s_{2345}} + \frac{[\mathcal{N}_6([2, [5, [3, 4]]], v)]^2}{s_{345} s_{34} s_{2345}} \\ & + \frac{[\mathcal{N}_6([3, [2, [4, 5]]], v)]^2}{s_{245} s_{45} s_{2345}} + \frac{[\mathcal{N}_6([3, [4, [2, 5]]], v)]^2}{s_{245} s_{25} s_{2345}} + \frac{[\mathcal{N}_6([3, [5, [2, 4]]], v)]^2}{s_{24} s_{245} s_{2345}} \\ & + \frac{[\mathcal{N}_6([4, [2, [3, 5]]], v)]^2}{s_{235} s_{35} s_{2345}} + \frac{[\mathcal{N}_6([4, [3, [2, 5]]], v)]^2}{s_{235} s_{25} s_{2345}} + \frac{[\mathcal{N}_6([4, [5, [2, 3]]], v)]^2}{s_{23} s_{235} s_{2345}} \\ & + \frac{[\mathcal{N}_6([5, [2, [3, 4]]], v)]^2}{s_{234} s_{34} s_{2345}} + \frac{[\mathcal{N}_6([5, [3, [2, 4]]], v)]^2}{s_{24} s_{234} s_{2345}} + \frac{[\mathcal{N}_6([5, [4, [2, 3]]], v)]^2}{s_{23} s_{234} s_{2345}} \\ = & \frac{[\mathcal{N}_6([2, 3], [4, 5], v)]^2}{8s_{23} s_{45} s_{2345}} + \frac{[\mathcal{N}_6([2, [3, [4, 5]]], v)]^2}{2s_{345} s_{45} s_{2345}} + \text{permutations of } 2, 3, 4, 5. \end{aligned} \quad (5.24)$$

Note that gauge invariance, crossing symmetry and locality for the massless particles in the BCJ numerator (5.24) are manifest.

6 Local BCJ numerators for pure Yang-Mills theory

The numerators we have constructed in the HEFT are closely related to the local BCJ numerators for pure Yang-Mills theory by the following equation

$$N_{n-1}^{\text{YM}}(234 \cdots (n-1)1) = \mathcal{N}_n([\cdots [[2, 3], 4], \cdots, n-1], v)|_{v \rightarrow \varepsilon_1, p_{234 \cdots n-1}^2 = 0}. \quad (6.1)$$

We have checked this relation at five and six points, and we will now illustrate it in detail in the five-point case.

At five points, the main observation is to impose the on-shell condition for the leg with momentum p_{234} . Doing so, we get

$$N_4^{\text{YM}}(2341) = \mathcal{N}_5([[2, 3], 4], v)|_{mv \rightarrow \varepsilon_1, p_{234}^2 = 0}. \quad (6.2)$$

We expand $\mathcal{N}_5([[2, 3], 4], v)$ in (5.13) and get

$$\begin{aligned} N_4^{\text{YM}}(2341) = 4 & \left(\varepsilon_1 \cdot \varepsilon_4 \varepsilon_2 \cdot p_3 \varepsilon_3 \cdot p_4 - \varepsilon_1 \cdot \varepsilon_4 \varepsilon_2 \cdot p_4 \varepsilon_3 \cdot p_2 + \varepsilon_2 \cdot \varepsilon_4 \varepsilon_1 \cdot p_4 \varepsilon_3 \cdot p_2 + \varepsilon_1 \cdot \varepsilon_2 \varepsilon_4 \cdot p_{23} \varepsilon_3 \cdot p_2 \right. \\ & - \varepsilon_3 \cdot \varepsilon_4 \varepsilon_1 \cdot p_4 \varepsilon_2 \cdot p_3 - \varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot p_3 \varepsilon_4 \cdot p_{23} + \varepsilon_2 \cdot \varepsilon_3 \varepsilon_1 \cdot p_3 \varepsilon_4 \cdot p_{23} + \varepsilon_2 \cdot \varepsilon_3 \varepsilon_1 \cdot p_4 \varepsilon_4 \cdot p_3 \\ & \left. + p_2 \cdot p_{34} \varepsilon_1 \cdot \varepsilon_4 \varepsilon_2 \cdot \varepsilon_3 - \frac{1}{4} s_{23} \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 + \frac{1}{2} s_{23} \mathcal{I}(3, 4) \right), \end{aligned} \quad (6.3)$$

where

$$\begin{aligned} \mathcal{I}(3, 4) = & -\frac{1}{2} \frac{\varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4}{\varepsilon_1 \cdot p_{34}} (\varepsilon_1 \cdot p_4 - \varepsilon_1 \cdot p_3) + \frac{\varepsilon_1 \cdot \varepsilon_2 \varepsilon_1 \cdot \varepsilon_3 \varepsilon_1 \cdot \varepsilon_4}{\varepsilon_1 \cdot p_{34}} \left(\frac{s_{24}}{\varepsilon_1 \cdot p_3} - \frac{s_{23}}{\varepsilon_1 \cdot p_4} \right) \\ & + \varepsilon_1 \cdot \varepsilon_3 \varepsilon_1 \cdot \varepsilon_4 \left(\frac{\varepsilon_2 \cdot p_4}{\varepsilon_1 \cdot p_3} - \frac{\varepsilon_2 \cdot p_3}{\varepsilon_1 \cdot p_4} \right) + \frac{\varepsilon_1 \cdot \varepsilon_2}{\varepsilon_1 \cdot p_{34}} (\varepsilon_1 \cdot \varepsilon_4 \varepsilon_3 \cdot p_4 - \varepsilon_1 \cdot \varepsilon_3 \varepsilon_4 \cdot p_3) \\ & + \frac{\varepsilon_1 \cdot \varepsilon_4 \varepsilon_2 \cdot \varepsilon_3 \varepsilon_1 \cdot p_3}{\varepsilon_1 \cdot p_4} + \frac{\varepsilon_1 \cdot \varepsilon_2 \varepsilon_1 \cdot \varepsilon_4 \varepsilon_3 \cdot p_2}{\varepsilon_1 \cdot p_4} - \frac{\varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot \varepsilon_4 \varepsilon_1 \cdot p_4}{\varepsilon_1 \cdot p_3} - \frac{\varepsilon_1 \cdot \varepsilon_2 \varepsilon_1 \cdot \varepsilon_3 \varepsilon_4 \cdot p_2}{\varepsilon_1 \cdot p_3}. \end{aligned} \quad (6.4)$$

As $\frac{1}{2} \mathbb{L}(3, 4) \circ \mathcal{I}(3, 4) = \mathcal{I}(3, 4)$, then $s_{23} \mathcal{I}(3, 4)$ is a pure gauge term⁷ which does not contribute to the Yang-Mills amplitude. By removing pure gauge terms we arrive at an expression which is manifestly local,

$$\begin{aligned} N_4^{\text{YM}}(2341) = 4 & \left(\varepsilon_1 \cdot \varepsilon_4 \varepsilon_2 \cdot p_3 \varepsilon_3 \cdot p_4 - \varepsilon_1 \cdot \varepsilon_4 \varepsilon_2 \cdot p_4 \varepsilon_3 \cdot p_2 + \varepsilon_2 \cdot \varepsilon_4 \varepsilon_1 \cdot p_4 \varepsilon_3 \cdot p_2 + \varepsilon_1 \cdot \varepsilon_2 \varepsilon_4 \cdot p_{23} \varepsilon_3 \cdot p_2 \right. \\ & - \varepsilon_3 \cdot \varepsilon_4 \varepsilon_1 \cdot p_4 \varepsilon_2 \cdot p_3 - \varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot p_3 \varepsilon_4 \cdot p_{23} + \varepsilon_2 \cdot \varepsilon_3 \varepsilon_1 \cdot p_3 \varepsilon_4 \cdot p_{23} + \varepsilon_2 \cdot \varepsilon_3 \varepsilon_1 \cdot p_4 \varepsilon_4 \cdot p_3 \\ & \left. + p_2 \cdot p_{34} \varepsilon_1 \cdot \varepsilon_4 \varepsilon_2 \cdot \varepsilon_3 - \frac{1}{4} s_{23} \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 \right). \end{aligned} \quad (6.5)$$

⁷We comment that pure gauge terms are classified by the \mathbb{L} -invariants \mathcal{I} as discussed in [63].

Finally, we have checked that the numerator (6.5) generates the correct amplitude. This numerator by itself is no longer gauge invariant.

We have also confirmed that the six-point numerator in HEFT is directly related to the five-point BCJ numerator in pure Yang-Mills theory,

$$N_5^{\text{YM}}(23451) = \mathcal{N}_6([[[2, 3], 4], 5], v)|_{v \rightarrow \varepsilon_1, p_{2345}^2=0}. \quad (6.6)$$

This result is in agreement with [62].

7 Conclusions

We conclude by summarising a number of interesting properties of the new BCJ numerators we have constructed in the HEFT:

1. **Gauge invariance.** We have expressed all the BCJ numerators $\mathcal{N}_n(\Gamma, v)$ in terms of field strengths, hence they are gauge invariant even if the on-shell and transversality conditions $p_i \cdot \varepsilon_i = 0$ are not imposed.
2. **Locality with respect to the massless particles.** Our new BCJ numerators do not contain spurious poles. This is particularly convenient when constructing loop integrands via generalised unitarity, as we will demonstrate in several examples at loop level in [81].
3. **Crossing symmetry and Jacobi relation are manifest.** As all the BCJ numerators are generated from the pre-numerator, the crossing symmetry with respect to all the massless particles is manifest. Moreover, based on the assumption of associativity of the fusion product, the Jacobi relations are automatically satisfied by the BCJ numerators.
4. **Numerators for pure Yang-Mills theory.** Our BCJ numerators are directly related to local BCJ numerators for pure Yang-Mills theory.

Note that usually BCJ numerators are manifestly local but not gauge invariant, while using KLT relations one arrives at expressions which are gauge invariant but non-local; here we have both locality and gauge invariance. We have checked the above properties up to six particles, and we conjecture them to be valid for arbitrary number of particles.

There are a few directions for future work. First, it would be desirable to find a proof of the gauge invariance of the BCJ numerators for any multiplicity. Is this a property of the HEFT or only of its leading term considered in this paper? A pressing question is to find a closed form of the pre-numerator, which would require a full understanding of the fusion rule for any number of particles. Finally, it would also be important to understand the subleading terms in the inverse mass expansion as well as higher-spin effects, to ascertain how much of the structures we have uncovered survive the expansion.

Acknowledgements

We would like to thank Henrik Johansson, Rodolfo Russo, Fei Teng and Tianheng Wang for interesting discussions. This work was supported by the Science and Technology Facilities Council (STFC) Consolidated Grants ST/P000754/1 “*String theory, gauge theory & duality*” and ST/T000686/1 “*Amplitudes, strings & duality*” and by the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 764850 “*SAGEX*”. CW is supported by a Royal Society University Research Fellowship No. UF160350.

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