MATH 3200: MATHEMATICAL METHODS

Pamini Thangarajah
Mount Royal University

LibreTexts™

Mount Royal University

MATH 3200: Mathematical Methods

Pamini Thangarajah

The LibreTexts libraries are Powered by NICE CXOne and are supported by the Department of Education Open Textbook Pilot Project, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

This text was compiled on 02/04/2024

TABLE OF CONTENTS

Licensing

1: Power Series

- 1.1: Power Series
  - 1.1E: Exercises
- 1.2: Properties of Power Series
  - 1.2E: Exercises
- 1.3: Taylor and Maclaurin Series
  - 1.3E: Exercises
- 1.4: Working with Taylor Series
  - 1.4E: Exercises
- 1 E: Chapter Exercises

2: Ordinary differential equations

- 2.1: Linear Second Order Homogeneous Equations
  - 2.1E: Exercises
- 2.2: Linear Second Order Constant Coefficient Homogeneous Equations
  - 2.2E: Exercises
- 2.3: Linear Second Order Nonhomogeneous Linear Equations
  - 2.3E: Exercises
- 2.4: The Method of Undetermined Coefficient
  - 2.4E: Exercises
- 2E: Exercises

3: Series Solutions of Linear Second order Equations

- 3.1: Review of Power Series
  - 3.1E: Exercises
- 3.2: Series Solutions Near an Ordinary Point I
  - 3.2E: Exercises
- 3.3: Series Solutions Near an Ordinary Point II
  - 3.3E: Exercises
- 3.4: Regular Singular Points: Euler Equations
  - 3.4E: Exercises
- 3.5: The Method of Frobenius I
  - 3.5E: Exercises
- 3.6: The Method of Frobenius II
  - 3.6E: Exercises
- 3.7: The Method of Frobenius III
  - 3.7E: Exercises

4: Linear Systems of Ordinary Differential Equations (LSODE)

- 4.1: Introduction to Systems of Differential Equations
  - 4.1E: Exercises
- 4.2: Linear Systems of Differential Equations
  - 4.2E: Exercises
- 4.3: Basic Theory of Homogeneous Linear System
  - 4.3E: Exercises
- 4.4: Constant Coefficient Homogeneous Systems I
  - 4.4E: Exercises
- 4.5: Constant Coefficient Homogeneous Systems II
  - 4.5E: Exercises
- 4.6: Constant Coefficient Homogeneous Systems III
  - 4.6E: Exercises
- 4.7: Variation of Parameters for Nonhomogeneous Linear Systems
  - 4.7E: Exercises

5: Vector-Valued Functions

- 5.1: Vector-Valued Functions and Space Curves
  - 5.1E:
- 5.2: Calculus of Vector-Valued Functions
  - 5.2E:
- 5.3: Arc Length and Curvature
  - 5.3E: Exercises
- 5.4: Motion in Space
  - 5.4E:Exercises
- 5E: Excercises

6: Differentiation of Functions of Several Variables

- 6.1: Functions of Several Variables
  - 6.1E:
- 6.2: Limits and Continuity
  - 6.2E:
- 6.3: Partial Derivatives
  - 6.3E:
- 6.4: Tangent Planes and Linear Approximations
  - 6.4E:Exercises
- 6.5: The Chain Rule for Multivariable Functions
  - 6.5E: Exercises
- 6.6: Directional Derivatives and the Gradient
  - 6.6E: Exercises
- 6.7: Maxima/Minima Problems
  - 6.7E:
- 6.8: Lagrange Multipliers
  - 6.8E:

- 6E: Chapter Review Excersies

7: Multiple Integration

- 7.1 :Double Integrals over Rectangular Regions
  - 7.1E: Exercises
- 7.2: Double Integrals over General Regions
  - 7.2E: Exercises
- 7.3: Double Integrals in Polar Coordinates
  - 7.3E:
- 7.4: Triple Integrals
  - 7.4E:
- 7.5: Triple Integrals in Cylindrical and Spherical Coordinates
  - 7.5E:
- 7.6: Calculating Centers of Mass and Moments of Inertia
  - 7.6E
- 7.7: Change of Variables in Multiple Integrals
  - 7.7E:
    - 7E: Chapter Review Excersies

8: Partial Differential Equations

- 8.1: Laplace Equations
- 8.2: The heat equation
- 8.3: Wave Equations
- 8E: Excercise

9: Vector Calculus

- 9.1: Vector Fields
  - 9.1E: Exercises
- 9.2: Line Integrals
  - 9.2E: Exercises
- 9.3: Conservative vector Fields
  - 9.3E: EXERCISES
- 9.4: Green's Theorem
  - 9.4E: EXERCISES
- 9.5: Divergence and Curl
  - 9.5E: EXERCISES
- 9.6: Surface Integrals
  - 9.6E: Exercises
- 9.7: Stoke's Theorem
  - 9.7E: EXERCISES
- 9.8: The Divergence Theorem
  - 9.8E: Exercises
- 9E: Chapter Exercises

Summary Tables

- Summary of Convergence Tests
- Summary of Theorems
- Summary table of derivatives
- Summary Table Of Integrals

Index

Glossary

Detailed Licensing

Licensing

A detailed breakdown of this resource's licensing can be found in ***Back Matter/Detailed Licensing****.*

CHAPTER OVERVIEW

1: Power Series

This page is a draft and is under active development.

When winning a lottery, sometimes an individual has an option of receiving winnings in one lump-sum payment or receiving smaller payments over fixed time intervals. For example, you might have the option of receiving 20 million dollars today or receiving 1.5 million dollars each year for the next 20 years. Which is the better deal? Certainly 1.5 million dollars over 20 years is equivalent to 30 million dollars. However, receiving the 20 million dollars today would allow you to invest the money.

Figure 1.1: If you win a lottery, do you get more money by taking a lump-sum payment or by accepting fixed payments over time? (credit: modification of work by Robert Huffstutter, Flickr)

Alternatively, what if you were guaranteed to receive 1 million dollars every year indefinitely (extending to your heirs) or receive 20 million dollars today. Which would be the better deal? To answer these questions, you need to know how to use infinite series to calculate the value of periodic payments over time in terms of today's dollars.

An infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n \tag{1.1}$$

is known as a **power series**. Since the terms contain the variable x, power series can be used to define functions. They can be used to represent given functions, but they are also important because they allow us to write functions that cannot be expressed any other way than as "infinite polynomials." In addition, power series can be easily differentiated and integrated, thus being useful in solving differential equations and integrating complicated functions. An infinite series can also be truncated, resulting in a finite polynomial that we can use to approximate functional values. Power series have applications in a variety of fields, including physics, chemistry, biology, and economics. As we will see in this chapter, representing functions using power series allows us to solve mathematical problems that cannot be solved with other techniques.

1.1 Contributors and Attributions

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

Topic hierarchy

1.1: Power Series

1.1E: Exercises

1.2: Properties of Power Series

1.2E: Exercises

1.3: Taylor and Maclaurin Series

1.3E: Exercises

1.4: Working with Taylor Series

1.4E: Exercises

1 E: Chapter Exercises

1.1: Power Series

This page is a draft and is under active development.

A power series is a type of series with terms involving a variable. More specifically, if the variable is x, then all the terms of the series involve powers of x. As a result, a power series can be thought of as an infinite polynomial. Power series are used to represent common functions and also to define new functions. In this section we define power series and show how to determine when a power series converges and when it diverges. We also show how to represent certain functions using power series.

1.1.1 Form of a Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots, \tag{1.1.1}$$

where x is a variable and the coefficients c_n are constants, is known as a **power series.** The series

$$1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \tag{1.1.2}$$

is an example of a power series. Since this series is a geometric series with ratio $r = |x|$, we know that it converges if $|x| < 1$ and diverges if $|x| \geq 1$.

Definition 1.1.1: Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots \tag{1.1.3}$$

is a power series centered at $x = 0$. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots \tag{1.1.4}$$

is a *power series* centered at $x = a$.

To make this definition precise, we stipulate that $x^0 = 1$ and $(x-a)^0 = 1$ even when $x = 0$ and $x = a$, respectively.

The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{1.1.5}$$

and

$$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \dots \tag{1.1.6}$$

are both power series centered at $x = 0$. The series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n} = 1 + \frac{x-2}{2 \cdot 3} + \frac{(x-2)^2}{3 \cdot 3^2} + \frac{(x-2)^3}{4 \cdot 3^3} + \dots \tag{1.1.7}$$

is a power series centered at $x = 2$.

1.1.2 Convergence of a Power Series

Since the terms in a power series involve a variable x, the series may converge for certain values of x and diverge for other values of x. For a power series centered at $x=a$, the value of the series at $x=a$ is given by c_0. Therefore, a power series always converges at its center. Some power series converge only at that value of x. Most power series, however, converge for more than one value of x. In that case, the power series either converges for all real numbers x or converges for all x in a finite interval. For example, the geometric series $\sum_{n=0}^{\infty} x^n$ converges for all x in the interval $(-1,1)$, but diverges for all x outside that interval. We now summarize these three possibilities for a general power series.

Note 1.1.1: Convergence of a Power Series

Consider the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$. The series satisfies exactly one of the following properties:

i. The series converges at $x=a$ and diverges for all $x\neq a$.
ii. The series converges for all real numbers x.
iii. There exists a real number $R>0$ such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$. At the values x where |x−a|=R, the series may converge or diverge.

Proof

Suppose that the power series is centered at $a=0$. (For a series centered at a value of a other than zero, the result follows by letting $y=x-a$ and considering the series

$$\sum_{n=1}^{\infty} c_n y^n.$$

We must first prove the following fact:

If there exists a real number $d\neq 0$ such that $\sum_{n=0}^{\infty} c_n d^n$ converges, then the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for all x such that $|x|<|d|$.

Since $\sum_{n=0}^{\infty} c_n d^n$ converges, the nth term $c_n d^n \to 0$ as $n\to\infty$. Therefore, there exists an integer N such that $|c_n d^n|\leq 1$ for all $n\geq N$. Writing

$$|c_n x^n| = |c_n d^n|\left|\frac{x}{d}\right|^n,$$

we conclude that, for all n≥N,

$$|c_n x^n| \leq \left|\frac{x}{d}\right|^n.$$

The series

$$\sum_{n=N}^{\infty}\left|\frac{x}{d}\right|^n$$

is a geometric series that converges if $|\frac{x}{d}| < 1$. Therefore, by the comparison test, we conclude that $\sum_{n=N}^{\infty} c_n x^n$ also converges for $|x| < |d|$. Since we can add a finite number of terms to a convergent series, we conclude that $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < |d|$.

With this result, we can now prove the theorem. Consider the series

$$\sum_{n=0}^{\infty} a_n x^n$$

and let S be the set of real numbers for which the series converges. Suppose that the set $S = 0$. Then the series falls under case i.

Suppose that the set S is the set of all real numbers. Then the series falls under case ii. Suppose that $S \neq 0$ and S is not the set of real numbers. Then there exists a real number $x* \neq 0$ such that the series does not converge. Thus, the series cannot converge for any x such that $|x| > |x*|$. Therefore, the set S must be a bounded set, which means that it must have a smallest upper bound. (This fact follows from the **Least Upper Bound Property** for the real numbers, which is beyond the scope of this text and is covered in real analysis courses.) Call that smallest upper bound R. Since $S \neq 0$, the number $R > 0$. Therefore, the series converges for all x such that $|x| < R$, and the series falls into case iii.

□

If a series $\sum_{n=0}^{\infty} c_n (x-a)^n$ falls into case iii. of Note, then the series converges for all x such that $|x-a| < R$ for some $R > 0$, and diverges for all x such that $|x-a| > R$. The series may converge or diverge at the values x where $|x-a| = R$. The set of values x for which the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges is known as the interval of convergence. Since the series diverges for all values x where $|x-a| > R$, the length of the interval is $2R$, and therefore, the radius of the interval is R. The value R is called the radius of convergence. For example, since the series $\sum_{n=0}^{\infty} x^n$ converges for all values x in the interval $(-1,1)$ and diverges for all values x such that $|x| \geq 1$, the interval of convergence of this series is $(-1,1)$. Since the length of the interval is 2, the radius of convergence is 1.

Definition: radius of convergence

Consider the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$. The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$ (Figure 1.1.1).

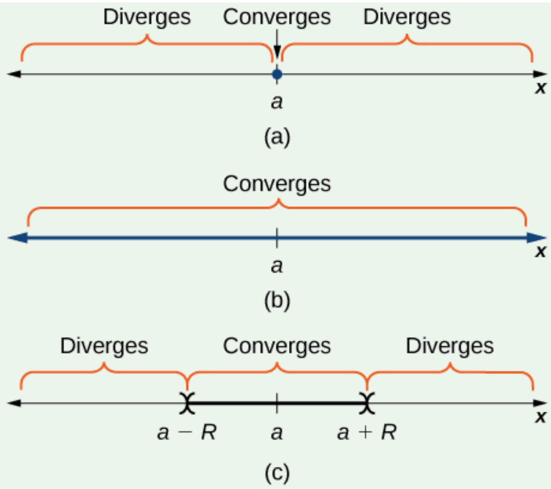


Figure 1.1.1: For a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ graph (a) shows a radius of convergence at $R=0$, graph (b) shows a radius of convergence at $R=\infty$, and graph (c) shows a radius of convergence at R. For graph (c) we note that the series may or may not converge at the endpoints $x=a+R$ and $x=a-R$.

To determine the interval of convergence for a power series, we typically apply the ratio test. In Example 1.1.1, we show the three different possibilities illustrated in Figure 1.1.1.

Example 1.1.1: Finding the Interval and Radius of Convergence

For each of the following series, find the interval and radius of convergence.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b. $\displaystyle\sum_{n=0}^{\infty} n!x^n$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

Solution

a. To check for convergence, apply the ratio test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)\cdot n!}\cdot\frac{n!}{x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| \\ &= |x|\lim_{n\to\infty}\frac{1}{n+1} \\ &= 0<1\end{aligned}$$

for all values of x. Therefore, the series converges for all real numbers x. The interval of convergence is $(-\infty,\infty)$ and the radius of convergence is $R=\infty$.

b. Apply the ratio test. For $x\neq 0$, we see that

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}|(n+1)x| \\ &= |x|\lim_{n\to\infty}(n+1) \\ &= \infty.\end{aligned}$$

Therefore, the series diverges for all $x\neq 0$. Since the series is centered at $x=0$, it must converge there, so the series converges only for $x\neq 0$. The interval of convergence is the single value $x=0$ and the radius of convergence is $R=0$.

c. In order to apply the ratio test, consider

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{\dfrac{(x-2)^{n+1}}{(n+2)3^{n+1}}}{\dfrac{(x-2)^n}{(n+1)3^n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}\cdot\frac{(n+1)3^n}{(x-2)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-2)(n+1)}{3(n+2)}\right| \\ &= \frac{|x-2|}{3}.\end{aligned}$$

The ratio $\rho < 1$ if $|x-2| < 3$. Since $|x-2| < 3$ implies that $-3 < x-2 < 3$, the series converges absolutely if $-1 < x < 5$. The ratio $\rho > 1$ if $|x-2| > 3$. Therefore, the series diverges if $x < -1$ or $x > 5$. The ratio test is inconclusive if $\rho = 1$. The ratio $\rho = 1$ if and only if $x = -1$ or $x = 5$. We need to test these values of x separately. For $x = -1$, the series is given by

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

Since this is the alternating harmonic series, it converges. Thus, the series converges at $x = -1$. For $x = 5$, the series is given by

$$\sum_{n=0}^{\infty}\frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

This is the harmonic series, which is divergent. Therefore, the power series diverges at $x = 5$. We conclude that the interval of convergence is $[-1, 5)$ and the radius of convergence is $R = 3$.

Exercise 1.1.1

Find the interval and radius of convergence for the series

$$\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}.$$

Hint

Apply the ratio test to check for absolute convergence.

Answer

The interval of convergence is $[-1, 1)$. The radius of convergence is $R = 1$.

1.1.3 Representing Functions as Power Series

Being able to represent a function by an “infinite polynomial” is a powerful tool. Polynomial functions are the easiest functions to analyze, since they only involve the basic arithmetic operations of addition, subtraction, multiplication, and division. If we can represent a complicated function by an infinite polynomial, we can use the polynomial representation to differentiate or integrate it. In addition, we can use a truncated version of the polynomial expression to approximate values of the function. So, the question is, when can we represent a function by a power series?

Consider again the geometric series

$$1+x+x^2+x^3+\ldots=\sum_{n=0}^{\infty} x^n. \tag{1.1.8}$$

Recall that the geometric series

$$a+ar+ar^2+ar^3+\ldots \tag{1.1.9}$$

converges if and only if $|r|<1$. In that case, it converges to $\dfrac{a}{1-r}$. Therefore, if $|x|<1$, the series in Example 1.1.1 converges to $\dfrac{1}{1-x}$ and we write

$$1+x+x^2+x^3+\ldots=\frac{1}{1-x} for |x|<1. \tag{1.1.10}$$

As a result, we are able to represent the function $f(x)=\dfrac{1}{1-x}$ by the power series

$$1+x+x^2+x^3+\ldots when |x|<1. \tag{1.1.11}$$

We now show graphically how this series provides a representation for the function $f(x)=\dfrac{1}{1-x}$ by comparing the graph of f with the graphs of several of the partial sums of this infinite series.

Example 1.1.2: Graphing a Function and Partial Sums of its Power Series

Sketch a graph of $f(x)=\dfrac{1}{1-x}$ and the graphs of the corresponding partial sums $S_N(x)=\sum_{n=0}^{N} x^n$ for $N=2,4,6$ on the interval $(-1,1)$. Comment on the approximation S_N as N increases.

Solution

From the graph in Figure you see that as N increases, S_N becomes a better approximation for $f(x)=\dfrac{1}{1-x}$ for x in the interval $(-1,1)$.

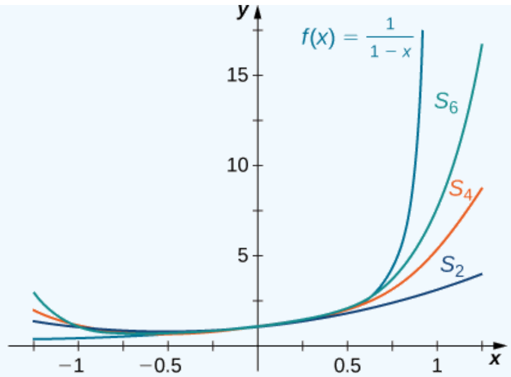


Figure 1.1.2: The graph shows a function and three approximations of it by partial sums of a power series.

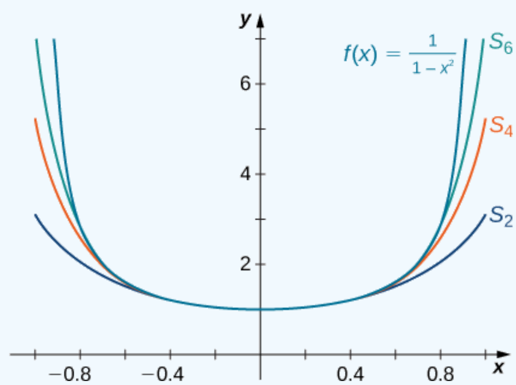
Exercise 1.1.2

Sketch a graph of $f(x) = \dfrac{1}{1-x^2}$ and the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^{2n}$ for $N = 2, 4, 6$ on the interval $(-1, 1)$.

Hint

$$S_N(x) = 1 + x^2 + \ldots + x^{2N} = \frac{1 - x^{2(N+1)}}{1 - x^2}$$

Answer



Next we consider functions involving an expression similar to the sum of a geometric series and show how to represent these functions using power series.

Example 1.1.3: Representing a Function with a Power Series

Use a power series to represent each of the following functions f. Find the interval of convergence.

a. $f(x) = \dfrac{1}{1+x^3}$

b. $f(x) = \dfrac{x^2}{4-x^2}$

Solution

a. You should recognize this function f as the sum of a geometric series, because

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}.$$

Using the fact that, for $|r|<1$, $\frac{a}{1-r}$ is the sum of the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots,$$

we see that, for $|-x^3|<1$,

$$\begin{aligned} \frac{1}{1+x^3} &= \frac{1}{1-(-x^3)} \\ &= \sum_{n=0}^{\infty}(-x^3)^n \\ &= 1 - x^3 + x^6 - x^9 + \ldots. \end{aligned}$$

Since this series converges if and only if $|-x^3|<1$, the interval of convergence is $(-1,1)$, and we have

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \ldots for |x| < 1.$$

b. This function is not in the exact form of a sum of a geometric series. However, with a little algebraic manipulation, we can relate f to a geometric series. By factoring 4 out of the two terms in the denominator, we obtain

$$\begin{aligned} \frac{x^2}{4-x^2} &= \frac{x^2}{4(\frac{1-x^2}{4})} \\ &= \frac{x^2}{4(1-(\frac{x}{2})^2)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{x^2}{4-x^2} &= \frac{x^2}{4(1-(\frac{x}{2})^2)} \\ &= \frac{\frac{x^2}{4}}{1-(\frac{x}{2})^2} \\ &= \sum_{n=0}^{\infty}\frac{x^2}{4}(\frac{x}{2})^{2n}. \end{aligned}$$

The series converges as long as $|(\frac{x}{2})^2| < 1$ (note that when $|(\frac{x}{2})^2| = 1$ the series does not converge). Solving this inequality, we conclude that the interval of convergence is $(-2, 2)$ and

$$\begin{aligned} \frac{x^2}{4-x^2} &= \sum_{n=0}^{\infty} \frac{x^{2n+2}}{4^{n+1}} \\ &= \frac{x^2}{4} + \frac{x^4}{4^2} + \frac{x^6}{4^3} + \dots \end{aligned}$$

for $|x| < 2$.

Exercise 1.1.3

Represent the function $f(x) = \frac{x^3}{2-x}$ using a power series and find the interval of convergence.

Hint

Rewrite f in the form $f(x) = \frac{g(x)}{1-h(x)}$ for some functions g and h.

Answer

$\sum_{n=0}^{\infty} \frac{x^{n+3}}{2^{n+1}}$ with interval of convergence $(-2, 2)$

In the remaining sections of this chapter, we will show ways of deriving power series representations for many other functions, and how we can make use of these representations to evaluate, differentiate, and integrate various functions.

1.1.4 Key Concepts

- For a power series centered at $x = a$, one of the following three properties hold:
  - i. The power series converges only at $x = a$. In this case, we say that the radius of convergence is $R = 0$.
  - ii. The power series converges for all real numbers x. In this case, we say that the radius of convergence is $R = \infty$.
  - iii. There is a real number R such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$. In this case, the radius of convergence is R.
- If a power series converges on a finite interval, the series may or may not converge at the endpoints.
- The ratio test may often be used to determine the radius of convergence.
- The geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$ allows us to represent certain functions using geometric series.

1.1.5 Key Equations

- **Power series centered at $x = 0$**

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots n$$

- **Power series centered at $x = a$**

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots$$

1.1.6 Glossary

interval of convergence

the set of real numbers x for which a power series converges

power series

a series of the form $\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x = a$

radius of convergence

if there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers x, the radius of convergence is $R = \infty$

1.1.7 Contributors

-

Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

1.1E: Exercises

This page is a draft and is under active development.

1.1E.1 Exercise $\mathbf{1.1E.1}$

In the following exercises, state whether each statement is true, or give an example to show that it is false.

1. If $\sum_{n=1}^{\infty} a_n x^n$ converges, then $a_n x^n \to 0$ as $n \to \infty$.

Answer

True. If a series converges then its terms tend to zero.

2. $\sum_{n=1}^{\infty} a_n x^n$ converges at $x = 0$ for any real numbers a_n.

3. Given any sequence a_n, there is always some $R > 0$, possibly very small, such that $\sum_{n=1}^{\infty} a_n x^n$ converges on $(-R, R)$.

Answer

False. It would imply that $a_n x^n \to 0$ for $|x| < R$. If $a_n = n^n$, then $a_n x^n = (nx)^n$ does not tend to zero for any $x \neq 0$.

4. If $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R > 0$ and if $|b_n| \leq |a_n|$ for all n, then the radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is greater than or equal to R

1.1E.2 Exercise $\mathbf{1.1E.2}$

1. Suppose that $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 6$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 1$

b. $x = 2$

c. $x = 3$

d. $x = 0$

e. $x = 5.99$

f. $x = 0.000001$

Answer

It must converge on $(0, 6]$ and hence at: a. $x = 1$; b. $x = 2$; c. $x = 3$; d. $x = 0$; e. $x = 5.99$; and f. $x = 0.000001$.

2. Suppose that $\sum_{n=0}^{\infty} a_n(x+1)^n$ converges at $x=-2$. At which of the following points must the series also converge? Use the fact that if $\sum a_n(x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x=2$

b. $x=-1$

c. $x=-3$

d. $x=0$

e. $x=0.99$

f. $x=0.000001$

1.1E.3 Exercise $\mathbf{1.1E.3}$

In the following exercises, suppose that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n\to\infty$. Find the radius of convergence for each series.

1. $\sum_{n=0}^{\infty} a_n 2^n x^n$

Answer

$$\left|\frac{a_{n+1}2^{n+1}x^{n+1}}{a_n 2^n x^n}\right| = 2|x|\left|\frac{a_{n+1}}{a_n}\right| \to 2|x| \text{ so } R=\frac{1}{2}$$

2. $\sum_{n=0}^{\infty} \frac{a_n x^n}{2^n}$

Answer

$$\left|\frac{a_{n+1}x^{n+1}}{2^{n+1}}\frac{2^n}{a_n x^n}\right| = \frac{|x|}{2}\left|\frac{a_{n+1}}{a_n}\right| \to \frac{|x|}{2} \text{ so } R=2$$

3. $\sum_{n=0}^{\infty} \frac{a_n \pi^n x^n}{e^n}$

Answer

$$\left|\frac{a_{n+1}\left(\frac{\pi}{e}\right)^{n+1}x^{n+1}}{a_n\left(\frac{\pi}{e}\right)^n x^n}\right| = \frac{\pi|x|}{e}\left|\frac{a_{n+1}}{a_n}\right| \to \frac{\pi|x|}{e} \text{ so } R=\frac{e}{\pi}$$

4. $\sum_{n=0}^{\infty} \frac{a_n(-1)^n x^n}{10^n}$

Answer

$$|\frac{a_{n+1}(-1)^{n+1}x^{n+1}}{10^{n+1}}\frac{10^n}{a_n(-1)^n x^n}|=\frac{|x|}{10}|\frac{a_{n+1}}{a_n}|\rightarrow\frac{|x|}{10} \text{ so } R=10$$

5. $\sum_{n=0}^{\infty} a_n(-1)^n x^{2n}$

Answer

$$\left|\frac{a_{n+1}(-1)^{n+1}x^{2n+2}}{a_n(-1)^n x^{2n}}\right|=\left|x^2\right|\left|\frac{a_{n+1}}{a_n}\right|\rightarrow\left|x^2\right| \text{ so } R=1$$

6. $\sum_{n=0}^{\infty} a_n(-4)^n x^{2n}$

Answer

$$\left|\frac{a_{n+1}(-4)^{n+1}x^{2(n+1)}}{a_n(-4)^n x^{2n}}\right|=4|x^2|\left|\frac{a_{n+1}}{a_n}\right|\rightarrow 4|x^2| \text{ so } R=\frac{1}{2}$$

1.1E.4 Exercise $\mathbf{1.1E.4}$

In the following exercises, find the radius of convergence R and interval of convergence for $\sum a_n x^n$ with the given coefficients a_n.

1. $\sum_{n=1}^{\infty}\frac{(2x)^n}{n}$

Answer

$a_n=\frac{2^n}{n}$ so $\frac{a_{n+1}x}{a_n}\rightarrow 2x$. so $R=\frac{1}{2}$. When $x=\frac{1}{2}$ the series is harmonic and diverges. When $x=-\frac{1}{2}$ the series is alternating harmonic and converges. The interval of convergence is $I=[-\frac{1}{2},\frac{1}{2})$.

2. $\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{\sqrt{n}}$

Answer

R=1

Interval of convergence (-1,1)

3. $\sum_{n=1}^{\infty}\frac{nx^n}{2^n}$

Answer

$a_n = \frac{n}{2^n}$ so $\frac{a_{n+1}x}{a_n} \to \frac{x}{2}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

4. $\sum_{n=1}^{\infty} \frac{nx^n}{e^n}$

5. $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$

Answer

$a_n = \frac{n^2}{2^n}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

6. $\sum_{k=1}^{\infty} \frac{k^e x^k}{e^k}$

7. $\sum_{k=1}^{\infty} \frac{\pi^k x^k}{k^\pi}$

Answer

$a_k = \frac{\pi^k}{k^\pi}$ so $R = \frac{1}{\pi}$. When $x = \pm\frac{1}{\pi}$ the series is an absolutely convergent p-series. The interval of convergence is $I = [-\frac{1}{\pi}, \frac{1}{\pi}]$.

8. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

9. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$

Answer

$a_n = \frac{10^n}{n!}, \frac{a_{n+1}x}{a_n} = \frac{10x}{n+1} \to 0 < 1$ so the series converges for all x by the ratio test and $I = (-\infty, \infty)$.

10. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{ln(2n)}$

1.1E.5 Exercise $\mathbf{1.1E.5}$

In the following exercises, find the radius of convergence of each series.

1. $\sum_{k=1}^{\infty} \frac{(k!)^2 x^k}{(2k)!}$

Answer

$$a_k=\frac{(k!)^2}{(2k)!} \text{ so } \frac{a_{k+1}}{a_k}=\frac{(k+1)^2}{(2k+2)(2k+1)}\to\frac{1}{4} \text{ so } R=4$$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!x^n}{n^{2n}}$
3. $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{1\cdot 3\cdot 5\cdots(2k-1)}x^k$

Answer

$$a_k=\frac{k!}{1\cdot 3\cdot 5\cdots(2k-1)} \text{ so } \frac{a_{k+1}}{a_k}=\frac{k+1}{2k+1}\to\frac{1}{2} \text{ so } R=2$$

4. $\displaystyle\sum_{k=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots 2k}{(2k)!}x^k$
5. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{\binom{2n}{n}}$ where $\binom{n}{k}=\dfrac{n!}{k!(n-k)!}$

Answer

$$a_n=\frac{1}{\binom{2n}{n}} \text{ so } \frac{a_{n+1}}{a_n}=\frac{((n+1)!)^2}{(2n+2)!}\frac{2n!}{(n!)^2}=\frac{(n+1)^2}{(2n+2)(2n+1)}\to\frac{1}{4} \text{ so } R=4$$

6. $\displaystyle\sum_{n=1}^{\infty}sin^2 nx^n$

1.1E.6 Exercise $\mathbf{1.1E.6}$

In the following exercises, use the ratio test to determine the radius of convergence of each series.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^3}{(3n)!}x^n$

Answer

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}\to\frac{1}{27} \text{ so } R=27$$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{3n}(n!)^3}{(3n)!}x^n$
3. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}x^n$

Answer

$$a_n = \frac{n!}{n^n} \text{ so } \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} = (\frac{n}{n+1})^n \to \frac{1}{e} \text{ so } R = e$$

4. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} x^n$

1.1E.7 Exercise $\mathbf{1.1E.7}$

In the following exercises, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence in $(-1,1)$, find the power series for each function with the given center a, and identify its interval of convergence.

1. $f(x) = \frac{1}{x}; a = 1$ (Hint: $\frac{1}{x} = \frac{1}{1-(1-x)}$)

Answer

$$f(x) = \sum_{n=0}^{\infty} (1-x)^n \text{ on } I = (0,2)$$

2. $f(x) = \frac{1}{1-x^2}; a = 0$
3. $f(x) = \frac{x}{1-x^2}; a = 0$

Answer

$$\sum_{n=0}^{\infty} x^{2n+1} \text{ on } I = (-1,1)$$

4. $f(x) = \frac{1}{1+x^2}; a = 0$
5. $f(x) = \frac{x^2}{1+x^2}; a = 0$

Answer

$$\sum_{n=0}^{\infty} (-1)^n x^{2n+2} \text{ on } I = (-1,1)$$

6. $f(x) = \frac{1}{2-x}; a = 1$
7. $f(x) = \frac{1}{1-2x}; a = 0.$

Answer

$$\sum_{n=0}^{\infty} 2^n x^n \text{ on } (-\frac{1}{2}, \frac{1}{2})$$

8. $f(x) = \frac{1}{1-4x^2}; a = 0$

9. $f(x) = \frac{x^2}{1-4x^2}; a = 0$

Answer

$$\sum_{n=0}^{\infty} 4^n x^{2n+2} \text{ on } (-\frac{1}{2}, \frac{1}{2})$$

10. $f(x) = \frac{x^2}{5-4x+x^2}; a = 2$

1.1E.8 Exercise $\mathbf{1.1E.8}$

Use the next exercise to find the radius of convergence of the given series in the subsequent exercises.

1. Explain why, if $|a_n|^{1/n} \to r > 0,$ then $|a_n x^n|^{1/n} \to |x| r < 1$ whenever $|x| < \frac{1}{r}$ and, therefore, the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is $R = \frac{1}{r}$.

Answer

$|a_n x^n|^{1/n} = |a_n|^{1/n}|x| \to |x|r$ as $n \to \infty$ and $|x|r < 1$ when $|x| < \frac{1}{r}$. Therefore, $\sum_{n=1}^{\infty} a_n x^n$ converges when $|x| < \frac{1}{r}$ by the *n*th root test.

2. $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

3. $\sum_{k=1}^{\infty} (\frac{k-1}{2k+3})^k x^k$

Answer

$$a_k = (\frac{k-1}{2k+3})^k \text{ so } (a_k)^{1/k} \to \frac{1}{2} < 1 \text{ so } R = 2$$

4. $\sum_{k=1}^{\infty} (\frac{2k^2-1}{k^2+3})^k x^k$

5. $\sum_{n=1}^{\infty} a_n = (n^{1/n} - 1)^n x^n$

Answer

$$a_n = (n^{1/n} - 1)^n \text{ so } (a_n)^{1/n} \to 0 \text{ so } R = \infty$$

6. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is even. Explain why $p(x) = p(-x)$.

7. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is odd. Explain why $p(x) = -p(-x)$.

Answer

We can rewrite $p(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$ and $p(x) = p(-x)$ since $x^{2n+1} = -(-x)^{2n+1}$.

8. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(Ax)$.

9. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(2x-1)$.

Answer

If $x \in [0, 1]$, then $y = 2x - 1 \in [-1, 1]$ so $p(2x-1) = p(y) = \sum_{n=0}^{\infty} a_n y^n$ converges.

1.1E.9 Exercise $\mathbf{1.1E.9}$

In the following exercises, suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ where $a_n \geq 0$ for each n. State whether each series converges on the full interval $(-1, 1)$, or if there is not enough information to draw a conclusion. Use the comparison test when appropriate.

1. $\sum_{n=0}^{\infty} a_n x^{2n}$

2. $\sum_{n=0}^{\infty} a_{2n} x^{2n}$

Answer

Converges on $(-1, 1)$ by the ratio test

3. $\sum_{n=0}^{\infty} a_{2n} x^n$ (Hint:$x = \pm\sqrt{x^2}$)

4. $\sum_{n=0}^{\infty} a_{n^2} x^{n^2}$ (Hint: Let $b_k = a_k$ if $k = n^2$ for some n, otherwise $b_k = 0$.

Answer

Consider the series $\sum b_k x^k$ where $b_k = a_k$ if $k = n^2$ and $b_k = 0$ otherwise. Then $b_k \le a_k$ and so the series converges on $(-1, 1)$ by the comparison test.

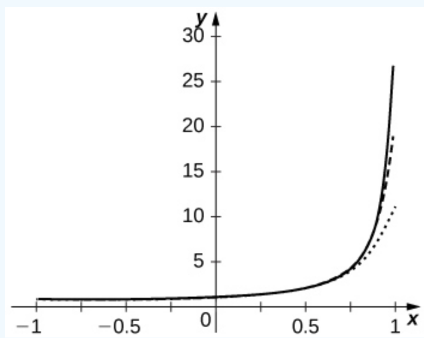
5. Suppose that $p(x)$ is a polynomial of degree N. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} p(n) x^n$.

1.1E.10 Exercise $\mathbf{1.1E.10}$

1. Plot the graphs of $\frac{1}{1-x}$ and of the partial sums $S_N = \sum_{n=0}^{N} x^n$ for $n = 10, 20, 30$ on the interval $[-0.99, 0.99]$. Comment on the approximation of $\frac{1}{1-x}$ by S_N near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation is more accurate near $x = -1$. The partial sums follow $\frac{1}{1-x}$ more closely as N increases but are never accurate near $x = 1$ since the series diverges there.

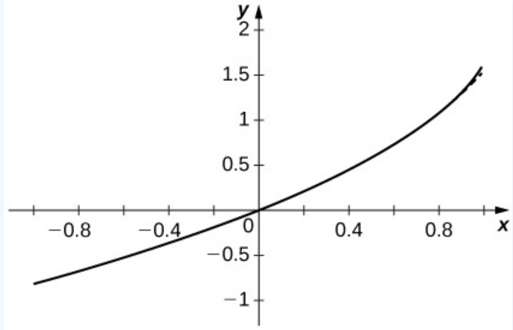


2. Plot the graphs of $-ln(1-x)$ and of the partial sums $S_N = \sum_{n=1}^{N} \frac{x^n}{n}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

3. Plot the graphs of the partial sums $S_n = \sum_{n=1}^{N} \frac{x^n}{n^2}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation appears to stabilize quickly near both $x = \pm 1$.

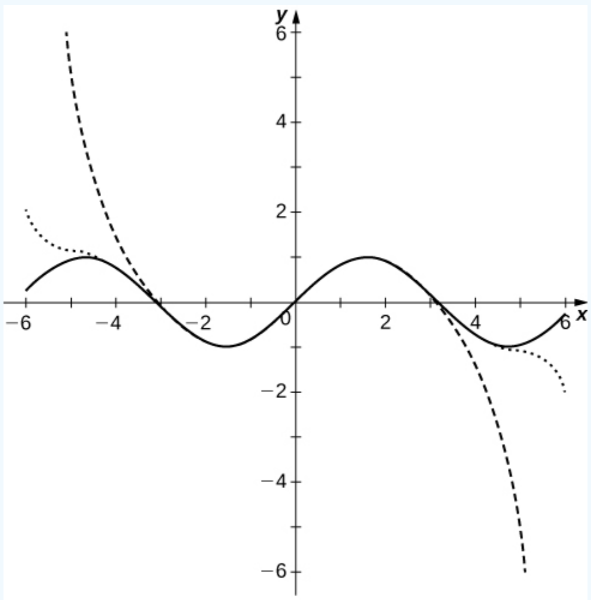


4. Plot the graphs of the partial sums $S_N = \sum_{n=1}^{N} sinnx^n$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

5. Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $sinx$ as N increases.

Answer

The polynomial curves have roots close to those of $sinx$ up to their degree and then the polynomials diverge from $sinx$.



6. Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n}}{(2n)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $cosx$ as N increases.

1.2: Properties of Power Series

This page is a draft and is under active development.

In the preceding section on power series and functions we showed how to represent certain functions using power series. In this section we discuss how power series can be combined, differentiated, or integrated to create new power series. This capability is particularly useful for a couple of reasons. First, it allows us to find power series representations for certain elementary functions, by writing those functions in terms of functions with known power series. For example, given the power series representation for $f(x) = \dfrac{1}{1-x}$, we can find a power series representation for $f'(x) = \dfrac{1}{(1-x)^2}$. Second, being able to create power series allows us to define new functions that cannot be written in terms of elementary functions. This capability is particularly useful for solving differential equations for which there is no solution in terms of elementary functions.

1.2.1 Combining Power Series

If we have two power series with the same interval of convergence, we can add or subtract the two series to create a new power series, also with the same interval of convergence. Similarly, we can multiply a power series by a power of x or evaluate a power series at x^m for a positive integer m to create a new power series. Being able to do this allows us to find power series representations for certain functions by using power series representations of other functions. For example, since we know the power series representation for $f(x) = \frac{1}{1-x}$, we can find power series representations for related functions, such as

$$y = \frac{3x}{1-x^2} \tag{1.2.1}$$

and

$$y = \frac{1}{(x-1)(x-3)}. \tag{1.2.2}$$

In Note 1.2.1, we state results regarding addition or subtraction of power series, composition of a power series, and multiplication of a power series by a power of the variable. For simplicity, we state the theorem for power series centered at $x = 0$. Similar results hold for power series centered at $x = a$.

Note: 1.2.1: Combining Power Series

Suppose that the two power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ and $\displaystyle\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I.

i. The power series $\displaystyle\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I.
ii. For any integer $m \geq 0$ and any real number b, the power series $\displaystyle\sum_{n=0}^{\infty} b x_n^m x^n$ converges to $bx^m f(x)$ on I.
iii. For any integer $m \geq 0$ and any real number b, the series $\displaystyle\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ for all x such that bx^m is in I.

Proof

We prove i. in the case of the series $\displaystyle\sum_{n=0}^{\infty} (c_n x^n + d_n x^n)$. Suppose that $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ and $\displaystyle\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on the interval I. Let x be a point in I and let $S_N(x)$ and $T_N(x)$ denote the Nth partial sums of the series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ and $\displaystyle\sum_{n=0}^{\infty} d_n x^n$, respectively. Then the sequence $S_N(x)$ converges to $f(x)$ and the sequence $T_N(x)$ converges to $g(x)$. Furthermore, the Nth partial sum of $\displaystyle\sum_{n=0}^{\infty} (c_n x^n + d_n x^n)$ is

$$\begin{aligned}\sum_{n=0}^{N}(c_nx^n+d_nx^n) &= \sum_{n=0}^{N}c_nx^n+\sum_{n=0}^{N}d_nx^n \\ &= S_N(x)+T_N(x).\end{aligned}$$

Because

$$\begin{aligned}\lim_{N\to\infty}(S_N(x)+T_N(x)) &= \lim_{N\to\infty}S_N(x)+\lim_{N\to\infty}T_N(x) \\ &= f(x)+g(x),\end{aligned}$$

we conclude that the series $\sum_{n=0}^{\infty}(c_nx^n+d_nx^n)$ converges to $f(x)+g(x)$.

□

We examine products of power series in a later theorem. First, we show several applications of Note and how to find the interval of convergence of a power series given the interval of convergence of a related power series.

Example 1.2.1: Combining Power Series

Suppose that $\sum_{n=0}^{\infty}a_nx^n$ is a power series whose interval of convergence is $(-1,1)$, and suppose that $\sum_{n=0}^{\infty}b_nx^n$ is a power series whose interval of convergence is $(-2,2)$.

a. Find the interval of convergence of the series $\sum_{n=0}^{\infty}(a_nx^n+b_nx^n)$.
b. Find the interval of convergence of the series $\sum_{n=0}^{\infty}a_n3^nx^n$.

Solution

a. Since the interval $(-1,1)$ is a common interval of convergence of the series $\sum_{n=0}^{\infty}a_nx^n$ and $\sum_{n=0}^{\infty}b_nx^n$, the interval of convergence of the series $\sum_{n=0}^{\infty}(a_nx^n+b_nx^n)$ is $(-1,1)$.
b. Since $\sum_{n=0}^{\infty}a_nx^n$ is a power series centered at zero with radius of convergence 1, it converges for all x in the interval $(-1,1)$. By Note, the series

$$\sum_{n=0}^{\infty}a_n3^nx^n=\sum_{n=0}^{\infty}a_n(3x)^n \tag{1.2.3}$$

converges if 3x is in the interval $(-1,1)$. Therefore, the series converges for all x in the interval $(-\frac{1}{3},\frac{1}{3})$.

Exercise 1.2.1

Suppose that $\sum_{n=0}^{\infty}a_nx^n$ has an interval of convergence of $(-1,1)$. Find the interval of convergence of $\sum_{n=0}^{\infty}a_n(\frac{x}{2})^n$.

Hint

Find the values of x such that $\frac{x}{2}$ is in the interval $(-1,1)$.

Answer

Interval of convergence is $(-2,2)$.

In the next example, we show how to use Note and the power series for a function f to construct power series for functions related to f. Specifically, we consider functions related to the function $f(x)=\dfrac{1}{1-x}$ and we use the fact that

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\ldots \tag{1.2.4}$$

for $|x|<1$.

Example 1.2.2: Constructing Power Series from Known Power Series

Use the power series representation for $f(x)=\dfrac{1}{1-x}$ combined with Note to construct a power series for each of the following functions. Find the interval of convergence of the power series.

a. $f(x)=\dfrac{3x}{1+x^2}$
b. $f(x)=\dfrac{1}{(x-1)(x-3)}$

Solution

a. First write $f(x)$ as

$$f(x)=3x\left(\frac{1}{1-(-x^2)}\right).$$

Using the power series representation for $f(x)=\dfrac{1}{1-x}$ and parts ii. and iii. of Note, we find that a power series representation for f is given by

$$\sum_{n=0}^{\infty}3x(-x^2)^n=\sum_{n=0}^{\infty}3(-1)^nx^{2n+1}.$$

Since the interval of convergence of the series for $\dfrac{1}{1-x}$ is $(-1,1)$, the interval of convergence for this new series is the set of real numbers x such that $|x^2|<1$. Therefore, the interval of convergence is $(-1,1)$.

b. To find the power series representation, use partial fractions to write $f(x)=\dfrac{1}{(1-x)(x-3)}$ as the sum of two fractions. We have

$$\frac{1}{(x-1)(x-3)}=\frac{-1/2}{x-1}+\frac{1/2}{x-3}=\frac{1/2}{1-x}-\frac{1/2}{3-x}=\frac{1/2}{1-x}-\frac{1/6}{1-\frac{x}{3}}.$$

First, using part ii. of Note, we obtain

$$\frac{1/2}{1-x}=\sum_{n=0}^{\infty}\frac{1}{2}x^n \text{for} |x|<1.$$

Then, using parts ii. and iii. of Note, we have

$$\frac{1/6}{1-x/3}=\sum_{n=0}^{\infty}\frac{1}{6}\left(\frac{x}{3}\right)^n \text{for} |x|<3.$$

Since we are combining these two power series, the interval of convergence of the difference must be the smaller of these two intervals. Using this fact and part i. of Note, we have

$$\frac{1}{(x-1)(x-3)}=\sum_{n=0}^{\infty}\left(\frac{1}{2}-\frac{1}{6\cdot 3^n}\right)x^n$$

where the interval of convergence is $(-1,1)$.

Exercise 1.2.2

Use the series for $f(x)=\frac{1}{1-x}$ on $|x|<1$ to construct a series for $\frac{1}{(1-x)(x-2)}$. Determine the interval of convergence.

Hint

Use partial fractions to rewrite $\frac{1}{(1-x)(x-2)}$ as the difference of two fractions.

Answer

$\sum_{n=0}^{\infty}(-1+\frac{1}{2^{n+1}})x^n$. The interval of convergence is $(-1,1)$.

In Example 1.2.2, we showed how to find power series for certain functions. In Example 1.2.3 we show how to do the opposite: given a power series, determine which function it represents.

Example 1.2.3: Finding the Function Represented by a Given Power Series

Consider the power series $\sum_{n=0}^{\infty}2^nx^n$. Find the function f represented by this series. Determine the interval of convergence of the series.

Solution

Writing the given series as

$$\sum_{n=0}^{\infty}2^nx^n=\sum_{n=0}^{\infty}(2x)^n,$$

we can recognize this series as the power series for

$$f(x)=\frac{1}{1-2x}.$$

Since this is a geometric series, the series converges if and only if $|2x|<1$. Therefore, the interval of convergence is $(-\frac{1}{2},\frac{1}{2})$.

Exercise 1.2.3

Find the function represented by the power series $\sum_{n=0}^{\infty}\frac{1}{3^n}x^n$.

Determine its interval of convergence.

Hint

Write $\frac{1}{3^n}x^n=(\frac{x}{3})^n$.

Answer

$f(x)=\frac{3}{3-x}$. The interval of convergence is $(-3,3)$.

Recall the questions posed in the chapter opener about which is the better way of receiving payouts from lottery winnings. We now revisit those questions and show how to use series to compare values of payments over time with a lump sum payment today. We will compute how much future payments are worth in terms of today's dollars, assuming we have the ability to invest winnings and earn interest. The value of future payments in terms of today's dollars is known as the *present value* of those payments.

Example 1.2.4: Present Value of Future Winnings

Suppose you win the lottery and are given the following three options:

- Receive 20 million dollars today;
- Receive 1.5 million dollars per year over the next 20 years; or
- Receive 1 million dollars per year indefinitely (being passed on to your heirs).

Which is the best deal, assuming that the annual interest rate is 5%? We answer this by working through the following sequence of questions.

a. How much is the 1.5 million dollars received annually over the course of 20 years worth in terms of today's dollars, assuming an annual interest rate of 5%?
b. Use the answer to part a. to find a general formula for the **present value** of payments of C dollars received each year over the next n years, assuming an average annual interest rate r.
c. Find a formula for the present value if annual payments of C dollars continue indefinitely, assuming an average annual interest rate r.
d. Use the answer to part c. to determine the present value of 1 million dollars paid annually indefinitely.
e. Use your answers to parts a. and d. to determine which of the three options is best.

Figure 1.2.1: (credit: modification of work by Robert Huffstutter, Flickr)

Solution

a. Consider the payment of 1.5 million dollars made at the end of the first year. If you were able to receive that payment today instead of one year from now, you could invest that money and earn 5% interest. Therefore, the present value of that money P_1 satisfies $P_1(1+0.05)=1.5$**million dollars.** We conclude that

$$P_1=\frac{1.5}{1.05}=\$1.429 \text{ **million dollars.**}$$

Similarly, consider the payment of 1.5 million dollars made at the end of the second year. If you were able to receive that payment today, you could invest that money for two years, earning 5% interest, compounded annually. Therefore, the present value of that money P_2 satisfies $P_2(1+0.05)^2=1.5$ **million dollars**. We conclude that

$$P_2=1.5(1.05)^2=\$1.361 \textbf{ million dollars.}$$

The value of the future payments today is the sum of the present values $P_1, P_2, \ldots, P_{20}$ of each of those annual payments. The present value P_k satisfies

$$P_k=\frac{1.5}{(1.05)^k}.$$

Therefore,

$$P=\frac{1.5}{1.05}+\frac{1.5}{(1.05)^2}+\ldots+\frac{1.5}{(1.05)^{20}}=\$18.693 \textbf{ million dollars.}$$

b. Using the result from part a. we see that the present value P of C dollars paid annually over the course of n years, assuming an annual interest rate r, is given by

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^n} \quad \textbf{dollars.}$$

c. Using the result from part b. we see that the present value of an annuity that continues indefinitely is given by the infinite series

$$P = \sum_{n=0}^{\infty} \frac{C}{(1+r)^{n+1}}.$$

We can view the present value as a power series in r, which converges as long as $\left|\frac{1}{1+r}\right| < 1$. Since $r > 0$, this series converges. Rewriting the series as

$$P = \frac{C}{(1+r)} \sum_{n=0}^{\infty} \left(\frac{1}{1+r}\right)^n,$$

we recognize this series as the power series for

$$f(r) = \frac{1}{1 - \left(\frac{1}{1+r}\right)} = \frac{1}{\left(\frac{r}{1+r}\right)} = \frac{1+r}{r}.$$

We conclude that the present value of this annuity is

$$P = \frac{C}{1+r} \cdot \frac{1+r}{r} = \frac{C}{r}.$$

d. From the result to part c. we conclude that the present value P of $C = 1$ **million dollars** paid out every year indefinitely, assuming an annual interest rate $r = 0.05$, is given by

$$P = \frac{1}{0.05} = 20 \quad \textbf{million dollars.}$$

e. From part a. we see that receiving \$1.5 million dollars over the course of 20 years is worth \$18.693 million dollars in today's dollars. From part d. we see that receiving \$1 million dollars per year indefinitely is worth \$20 million dollars in today's dollars. Therefore, either receiving a lump-sum payment of \$20 million dollars today or receiving \$1 million dollars indefinitely have the same present value.

1.2.2 Multiplication of Power Series

We can also create new power series by multiplying power series. Being able to multiply two power series provides another way of finding power series representations for functions. The way we multiply them is similar to how we multiply polynomials. For example, suppose we want to multiply

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots \tag{1.2.5}$$

and

$$\sum_{n=0}^{\infty} d_n x^n = d_0 + d + 1x + d_2 x^2 + \ldots. \tag{1.2.6}$$

It appears that the product should satisfy

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=-0}^{\infty} d_n x^n\right) = (c_0 + c_1 x + c_2 x^2 + \ldots) \cdot (d_0 + d_1 x + d_2 x^2 + \ldots) = c_0 d^0 + (c_1 d^0 + c_0 d^1)x \tag{1.2.7}$$
$$+ (c_2 d^0 + c_1 d^1 + c_0 d^2)x^2 + \ldots.$$

In Note, we state the main result regarding multiplying power series, showing that if $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge on a common interval I, then we can multiply the series in this way, and the resulting series also converges on the interval I.

Multiplying Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g, respectively, on a common interval I. Let

$$e_n = c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \ldots + c_{n-1} d_1 + c_n d_0 = \sum_{k=0}^{n} c_k d_{n-k}. \tag{1.2.8}$$

Then

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n \tag{1.2.9}$$

and

$$\sum_{n=0}^{\infty} e_n x^n \text{converges to} f(x) \cdot g(x) \text{on} I. \tag{1.2.10}$$

The series $\sum_{n=0}^{\infty} e_n x^n$ is known as the *Cauchy product* of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$.

We omit the proof of this theorem, as it is beyond the level of this text and is typically covered in a more advanced course. We now provide an example of this theorem by finding the power series representation for

$$f(x) = \frac{1}{(1-x)(1-x^2)} \tag{1.2.11}$$

using the power series representations for

$$y = \frac{1}{1-x} \text{and} y = \frac{1}{1-x^2} \tag{1.2.12}$$

.

Example 1.2.5: Multiplying Power Series

Multiply the power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

for $|x| < 1$ with the power series representation

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = 1 + x^2 + x^4 + x^6 + \ldots$$

for $|x| < 1$ to construct a power series for $f(x) = \dfrac{1}{(1-x)(1-x^2)}$ on the interval $(-1, 1)$.

Solution

We need to multiply

$$(1 + x + x^2 + x^3 + \ldots)(1 + x^2 + x^4 + x^6 + \ldots).$$

Writing out the first several terms, we see that the product is given by

$$(1 + x^2 + x^4 + x^6 + \ldots) + (x + x^3 + x^5 + x^7 + \ldots) + (x^2 + x^4 + x^6 + x^8 + \ldots) + (x^3 + x^5 + x^7 + x^9 + \ldots) = 1 + x \\ + (1+1)x^2 + (1+1)x^3 + (1+1+1)x^4 + (1+1+1)x^5 + \ldots = 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + \ldots.$$

Since the series for $y = \dfrac{1}{1-x}$ and $y = \dfrac{1}{1-x^2}$ both converge on the interval $(-1, 1)$, the series for the product also converges on the interval $(-1, 1)$.

Exercise 1.2.4

Multiply the series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ by itself to construct a series for $\dfrac{1}{(1-x)(1-x)}$.

Hint

Multiply the first few terms of $(1+x+x^2+x^3+\ldots)(1+x+x^2+x^3+\ldots)$

Answer

$1+2x+3x^2+4x^3+\ldots$

1.2.3 Differentiating and Integrating Power Series

Consider a power series $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$ that converges on some interval I, and let f be the function defined by this series. Here we address two questions about f.

- Is f differentiable, and if so, how do we determine the derivative f'?
- How do we evaluate the indefinite integral $\int f(x)dx$?

We know that, for a polynomial with a finite number of terms, we can evaluate the derivative by differentiating each term separately. Similarly, we can evaluate the indefinite integral by integrating each term separately. Here we show that we can do the same thing for convergent power series. That is, if

$$f(x) = c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots \tag{1.2.13}$$

converges on some interval *I*, then

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \ldots \tag{1.2.14}$$

and

$$\int f(x)dx = C + c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \ldots. \tag{1.2.15}$$

Evaluating the derivative and indefinite integral in this way is called **term-by-term differentiation of a power series** and **term-by-term integration of a power series**, respectively. The ability to differentiate and integrate power series term-by-term also allows us to use known power series representations to find power series representations for other functions. For example, given the power series for $f(x) = \dfrac{1}{1-x}$, we can differentiate term-by-term to find the power series for $f'(x) = \dfrac{1}{(1-x)^2}$. Similarly, using the power series for $g(x) = \dfrac{1}{1+x}$, we can integrate term-by-term to find the power series for $G(x) = \ln(1+x)$, an antiderivative of g. We show how to do this in Example and Example. First, we state Note, which provides the main result regarding differentiation and integration of power series.

Term-by-Term Differentiation and Integration for Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges on the interval $(a-R, a+R)$ for some $R > 0$. Let f be the function defined by the series

$$f(x) = sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \tag{1.2.16}$$

for $|x-a| < R$. Then f is differentiable on the interval $(a-R, a+R)$ and we can find f' by differentiating the series term-by-term:

$$f'(x) = \sum n = 1^{\infty} n c_n (x-a)^n - 1 = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots \tag{1.2.17}$$

for $|x-a| < R$. Also, to find $\int f(x)dx$, we can integrate the series term-by-term. The resulting series converges on $(a-R, a+R)$, and we have

$$\int f(x)dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \ldots \tag{1.2.18}$$

for $|x-a| < R$.

The proof of this result is beyond the scope of the text and is omitted. Note that although Note guarantees the same radius of convergence when a power series is differentiated or integrated term-by-term, it says nothing about what happens at the endpoints. It is possible that the differentiated and integrated power series have different behavior at the endpoints than does the original series. We see this behavior in the next examples.

Example 1.2.6: Differentiating Power Series

a. Use the power series representation

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \tag{1.2.19}$$

for $|x| < 1$ to find a power series representation for

$$g(x) = \frac{1}{(1-x)^2} \tag{1.2.20}$$

on the interval $(-1, 1)$. Determine whether the resulting series converges at the endpoints.

b. Use the result of part a. to evaluate the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{4^n}$.

Solution

a. Since $g(x) = \dfrac{1}{(1-x)^2}$ is the derivative of $f(x) = \dfrac{1}{1-x}$, we can find a power series representation for g by differentiating the power series for f term-by-term. The result is

$$g(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \frac{d}{dx}(1 + x + x^2 + x^3 + \ldots) = 0 + 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=0}^{\infty}(n+1)x^n$$

for $|x| < 1$. Note does not guarantee anything about the behavior of this series at the endpoints. Testing the endpoints by using the divergence test, we find that the series diverges at both endpoints $x = \pm 1$.Note that this is the same result found in Example.

b. From part a. we know that

$$\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}.$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{n+1}{46n}=\sum_{n=0}^{\infty}(n+1)(\frac{1}{4})^n$$

$$=\frac{1}{(1-\frac{1}{4})^2}$$

$$\frac{1}{(\frac{3}{4})^2}$$

$$\frac{16}{9}$$

Exercise 1.2.5

Differentiate the series $\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n$ term-by-term to find a power series representation for $\frac{2}{(1-x)^3}$ on the interval $(-1,1)$.

Hint

Write out the first several terms and apply the power rule.

Answer

$$\sum_{n=0}^{\infty}(n+2)(n+1)x^n$$

Example 1.2.7: Integrating Power Series

For each of the following functions *f*, find a power series representation for *f* by integrating the power series for f' and find its interval of convergence.

a. $f(x)=\ln(1+x)$
b. $f(x)=tan^{-1}x$

Solution:

a. For $f(x)=\ln(1+x)$,the derivative is $f'(x)=\frac{1}{1+x}$. We know that

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=1-x+x^2-x^3+\dots$$

for $|x|<1$. To find a power series for $f(x)=\ln(1+x)$, we integrate the series term-by-term.

$$\int f'(x)dx=\int(1-x+x^2-x^3+\dots)dx=C+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots$$

Since $f(x)=\ln(1+x)$ is an antiderivative of $\frac{1}{1+x}$, it remains to solve for the constant *C*. Since $\ln(1+0)=0$, we have $C=0$. Therefore, a power series representation for $f(x)=\ln(1+x)$ is

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

for $|x|<1$. Note does not guarantee anything about the behavior of this power series at the endpoints. However, checking the endpoints, we find that at $x=1$ the series is the alternating harmonic series, which converges. Also, at $x=-1$, the series is the harmonic series, which diverges. It is important to note that, even though this series converges at $x=1$, Note does not guarantee that the series actually converges to $\ln(2)$. In fact, the series does converge to $\ln(2)$, but showing this fact requires more advanced techniques. (Abel's theorem, covered in more advanced texts, deals with this more technical point.) The interval of convergence is $(-1,1]$.

b. The derivative of $f(x)=tan^{-1}x$ is $f'(x)=\frac{1}{1+x^2}$. We know that

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=\sum_{n=0}^{\infty}(-x^2)^n=1-x^2+x^4-x^6+\dots$$

for $|x|<1$. To find a power series for $f(x)=tan^{-1}x$, we integrate this series term-by-term.

$$\int f'(x)dx=\int(1-x^2+x^4-x^6+\dots)dx=C+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots$$

Since $tan^{-1}(0)=0$, we have $C=0$. Therefore, a power series representation for $f(x)=tan^{-1}x$ is

$$tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

for $|x|<1$. Again, Note does not guarantee anything about the convergence of this series at the endpoints. However, checking the endpoints and using the alternating series test, we find that the series converges at $x=1$ and $x=-1$. As discussed in part a., using Abel's theorem, it can be shown that the series actually converges to $tan^{-1}(1)$ and $tan^{-1}(-1)$ at $x=1$ and $x=-1$, respectively. Thus, the interval of convergence is $[-1,1]$.

Exercise $1.2.6$

Integrate the power series $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ term-by-term to evaluate $\int\ln(1+x)dx$.

Hint

Use the fact that $\frac{x^{n+1}}{(n+1)n}$ is an antiderivative of $\frac{x^n}{n}$.

Answer

$$\sum_{n=2}^{\infty}\frac{(-1)^n x^n}{n(n-1)}$$

Up to this point, we have shown several techniques for finding power series representations for functions. However, how do we know that these power series are unique? That is, given a function f and a power series for f at a, is it possible that there is a different power series for f at a that we could have found if we had used a different technique? The answer to this question is no. This fact should not seem surprising if we think of power series as polynomials with an infinite number of terms. Intuitively, if

$$c_0+c_1x+c_2x^2+\dots=d_0+d_1x+d_2x^2+\dots \tag{1.2.21}$$

for all values x in some open interval I about zero, then the coefficients c_n should equal d_n for $n\geq 0$. We now state this result formally.

Uniqueness of Power Series

Let $\sum_{n=0}^{\infty}c_n(x-a)^n$ and $\sum_{n=0}^{\infty}d_n(x-a)^n$ be two convergent power series such that

$$\sum_{n=0}^{\infty}c_n(x-a)^n=\sum_{n=0}^{\infty}d_n(x-a)^n \tag{1.2.22}$$

for all x in an open interval containing a. Then $c_n=d_n$ for all $n\geq 0$.

Proof

Let

$$\begin{aligned} f(x) &= c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\dots \\ &= d_0+d_1(x-a)+d_2(x-a)^2+d_3(x-a)^3+\dots. \end{aligned}$$

Then $f(a)=c_0=d_0$. By Note, we can differentiate both series term-by-term. Therefore,

$$\begin{aligned} f'(x) &= c_1+2c_2(x-a)+3c_3(x-a)^2+\dots \\ &= d_1+2d_2(x-a)+3d_3(x-a)^2+\dots, \end{aligned}$$

and thus, $f'(a)=c_1=d_1$. Similarly,

$$\begin{aligned} f''(x) &= 2c_2+3\cdot 2c_3(x-a)+\dots \\ &= 2d_2+3\cdot 2d_3(x-a)+\dots \end{aligned}$$

implies that $f''(a)=2c_2=2d_2$, and therefore, $c_2=d_2$. More generally, for any integer $n\geq 0$, $f^{(n)}(a)=n!c_n=n!d_n$, and consequently, $c_n=d_n$ for all $n\geq 0$.

□

In this section we have shown how to find power series representations for certain functions using various algebraic operations, differentiation, or integration. At this point, however, we are still limited as to the functions for which we can find power series representations. Next, we show how to find power series representations for many more functions by introducing Taylor series.

1.2.4 Key Concepts

- Given two power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ and $\displaystyle\sum_{n=0}^{\infty} d_n x^n$ that converge to functions f and g on a common interval I, the sum and difference of the two series converge to $f\pm g$, respectively, on I. In addition, for any real number b and integer $m\geq 0$, the series $\displaystyle\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ and the series $\displaystyle\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ whenever bx^m is in the interval I.
- Given two power series that converge on an interval $(-R,R)$, the Cauchy product of the two power series converges on the interval $(-R,R)$.
- Given a power series that converges to a function f on an interval $(-R,R)$, the series can be differentiated term-by-term and the resulting series converges to f' on $(-R,R)$. The series can also be integrated term-by-term and the resulting series converges to $\int f(x)dx$ on $(-R,R)$.

1.2.5 Glossary

term-by-term differentiation of a power series

a technique for evaluating the derivative of a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ by evaluating the derivative of each term separately to create the new power series $\displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

term-by-term integration of a power series

a technique for integrating a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ by integrating each term separately to create the new power series $C+\sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$

1.2.6 Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

1.2E: Exercises

This page is a draft and is under active development.

1.2E.1 Exercise $1.2E.1$

1. If $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and $g(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$, find the power series of $\frac{1}{2}(f(x)+g(x))$ and of $\frac{1}{2}(f(x)-g(x))$.

Answer

$$\frac{1}{2}(f(x)+g(x))=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \quad \text{and} \quad \frac{1}{2}(f(x)-g(x))=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\ .$$

2. If $C(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ and $S(x)=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$, find the power series of $C(x)+S(x)$ and of $C(x)-S(x)$.

1.2E.2 Exercise $1.2E.2$

In the following exercises, use partial fractions to find the power series of each function.

1. $\frac{4}{(x-3)(x+1)}$

Answer

$$\frac{4}{(x-3)(x+1)}=\frac{1}{x-3}-\frac{1}{x+1}=-\frac{1}{3(1-\frac{x}{3})}-\frac{1}{1-(-x)}=-\frac{1}{3}\sum_{n=0}^{\infty}(\frac{x}{3})^n-\sum_{n=0}^{\infty}(-1)^nx^n=\sum_{n=0}^{\infty}((-1)^{n+1}-\frac{1}{3n+1})x^n$$

2. $\frac{3}{(x+2)(x-1)}$

3. $\frac{5}{(x^2+4)(x^2-1)}$

Answer

$$\frac{5}{(x^2+4)(x^2-1)}=\frac{1}{x^2-1}-\frac{1}{4}\frac{1}{1+(\frac{x}{2})^2}=-\sum_{n=0}^{\infty}x^{2n}-\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(\frac{x}{2})^n=\sum_{n=0}^{\infty}((-1)+(-1)^{n+1}\frac{1}{2^{n+2}})x^{2n}$$

4. $\frac{30}{(x^2+1)(x^2-9)}$

1.2E.3 Exercise $1.2E.3$

In the following exercises, express each series as a rational function.

1. $\sum_{n=1}^{\infty}\frac{1}{x^n}$

Answer

$$\frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{x^n}=\frac{1}{x}\frac{1}{1-\frac{1}{x}}=\frac{1}{x-1}$$

2. $\sum_{n=1}^{\infty}\frac{1}{x^{2n}}$

3. $\sum_{n=1}^{\infty}\frac{1}{(x-3)^{2n-1}}$

Answer

$$\frac{1}{x-3}\frac{1}{1-\frac{1}{(x-3)^2}}=\frac{x-3}{(x-3)^2-1}$$

4. $\sum_{n=1}^{\infty}(\frac{1}{(x-3)^{2n-1}}-\frac{1}{(x-2)^{2n-1}})$

1.2E.4 Exercise $\mathbf{1.2E.4}$

The following exercises explore applications of **annuities.**

1. Calculate the present values P of an annuity in which \$10,000 is to be paid out annually for a period of 20 years, assuming interest rates of $r=0.03, r=0.05$, and $r=0.07$.

Answer

$P=P_1+\cdots+P_{20}$ where $P_k=10,000\frac{1}{(1+r)^k}$. Then $P=10,000\sum_{k=1}^{20}\frac{1}{(1+r)^k}=10,000\frac{1-(1+r)^{-20}}{r}$. When $r=0.03, P\approx 10,000\times 14.8775=148,775.$ When $r=0.05, P\approx 10,000\times 12.4622=124,622.$ When $r=0.07, P\approx 105,940.$

2. Calculate the present values P of annuities in which \$9,000 is to be paid out annually perpetually, assuming interest rates of $r=0.03, r=0.05$ and $r=0.07$.

3. Calculate the annual payouts C to be given for 20 years on annuities having present value \$100,000 assuming respective interest rates of $r=0.03, r=0.05,$ and $r=0.07$.

Answer

In general, $P=\frac{C(1-(1+r)^{-N})}{r}$ for N years of payouts, or $C=\frac{Pr}{1-(1+r)^{-N}}$. For $N=20$ and $P=100,000,$ one has $C=6721.57$ when $r=0.03$; $C=8024.26$ when $r=0.05$; and $C\approx 9439.29$ when $r=0.07$.

4. Calculate the annual payouts C to be given perpetually on annuities having present value \$100,000 assuming respective interest rates of $r=0.03, r=0.05,$ and $r=0.07$.

5. Suppose that an annuity has a present value $P=1$ **million dollars.** What interest rate r would allow for perpetual annual payouts of \$50,000?

Answer

In general, $P=\frac{C}{r}$. Thus, $r=\frac{C}{P}=5\times\frac{10^4}{10^6}=0.05.$

6. Suppose that an annuity has a present value $P=10$ **million dollars.** What interest rate r would allow for perpetual annual payouts of \$100,000?

1.2E.5 Exercise $\mathbf{1.2E.5}$

In the following exercises, express the sum of each power series in terms of geometric series, and then express the sum as a rational function.

1. $x+x^2-x^3+x^4+x^5-x^6+\cdots$ (Hint: Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

Answer

$$(x+x^2-x^3)(1+x^3+x^6+\cdots)=\frac{x+x^2-x^3}{1-x^3}$$

2. $x+x^2-x^3-x^4+x^5+x^6-x^7-x^8+\cdots$ (Hint: Group powers x^{4k}, x^{4k-1}, etc.)

3. $x-x^2-x^3+x^4-x^5-x^6+x^7-\cdots$ (Hint: Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

Answer

$$(x-x^2-x^3)(1+x^3+x^6+\cdots)=\frac{x-x^2-x^3}{1-x^3}$$

4. $\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16}+\frac{x^5}{32}-\frac{x^6}{64}+\cdots$ (Hint: Group powers $\frac{x}{2})^{3k}, (\frac{x}{2})^{3k-1}$, and $\frac{x}{2})^{3k-2}$.

1.2E.6 Exercise $\mathbf{1.2E.6}$

In the following exercises, find the power series of $f(x)g(x)$ given f and g as defined.

1. $f(x)=2\sum_{n=0}^{\infty}x^n, g(x)=\sum_{n=0}^{\infty}nx^n$

Answer

$a_n=2, b_n=n$ so $c_n=\sum_{k=0}^{n}b_ka_{n-k}=2\sum_{k=0}^{n}k=(n)(n+1)$ and $f(x)g(x)=\sum_{n=1}^{\infty}n(n+1)x^n$

2. $f(x)=\sum_{n=1}^{\infty}x^n, g(x)=\sum_{n=1}^{\infty}\frac{1}{n}x^n$. Express the coefficients of $f(x)g(x)$ in terms of $H_n=\sum_{k=1}^{n}\frac{1}{k}$.

3. $f(x)=g(x)=\sum_{n=1}^{\infty}(\frac{x}{2})^n$

Answer

$a_n=b_n=2^{-n}$ so $c_n=\sum_{k=1}^{n}b_ka_{n-k}=2^{-n}\sum_{k=1}^{n}1=\frac{n}{2^n}$ and $f(x)g(x)=\sum_{n=1}^{\infty}n(\frac{x}{2})^n$

4. $f(x)=g(x)=\sum_{n=1}^{\infty}nx^n$

1.2E.7 Exercise $\mathbf{1.2E.7}$

In the following exercises, differentiate the given series expansion of f term-by-term to obtain the corresponding series expansion for the derivative of f.

1. $f(x)=\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^nx^n$

Answer

The derivative of f is $-\frac{1}{(1+x)^2}=-\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$.

2. $f(x)=\frac{1}{1-x^2}=\sum_{n=0}^{\infty}x^{2n}$

In the following exercises, integrate the given series expansion of f term-by-term from zero to x to obtain the corresponding series expansion for the indefinite integral of f.

3. $f(x)=\frac{2x}{(1+x^2)^2}=\sum_{n=1}^{\infty}(-1)^n(2n)x^{2n-1}$

Answer

The indefinite integral of f is $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^nx^{2n}$.

4. $f(x)=\frac{2x}{1+x^2}=2\sum_{n=0}^{\infty}(-1)^nx^{2n+1}$

1.2E.8 Exercise $\mathbf{1.2E.8}$

In the following exercises, evaluate each infinite series by identifying it as the value of a derivative or integral of geometric series.

1. Evaluate $\sum_{n=1}^{\infty}\frac{n}{2^n}$ as $f'(\frac{1}{2})$ where $f(x)=\sum_{n=0}^{\infty}x^n$.

Answer

$$f(x)=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x};f'(\frac{1}{2})=\sum_{n=1}^{\infty}\frac{n}{2^{n-1}}=\frac{d}{dx}(1-x)^{-1}\mid_{x=1/2}=\frac{1}{(1-x)^2}\mid_{x=1/2}=4 \quad \text{so} \sum_{n=1}^{\infty}\frac{n}{2^n}=2.$$

2. Evaluate $\sum_{n=1}^{\infty}\frac{n}{3^n}$ as $f'(\frac{1}{3})$ where $f(x)=\sum_{n=0}^{\infty}x6n$.

3. Evaluate $\sum_{n=2}^{\infty}\frac{n(n-1)}{2^n}$ as $f''(\frac{1}{2})$ where $f(x)=\sum_{n=0}^{\infty}x^n$.

Answer

$$f(x)=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x};f''(\frac{1}{2})=\sum_{n=2}^{\infty}\frac{n(n-1)}{2^{n-2}}=\frac{d^2}{dx^2}(1-x)^{-1}\mid_{x=1/2}=\frac{2}{(1-x)^3}\mid_{x=1/2}=16 \quad \text{so} \sum_{n=2}^{\infty}n\frac{(n-1)}{2^n}=4.$$

4. Evaluate $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ as $\int_0^1 f(t)dt$ where $f(x)=\sum_{n=0}^{\infty}(-1)^n x^{2n}=\frac{1}{1+x^2}$.

1.2E.9 Exercise $\mathbf{1.2E.9}$

In the following exercises, given that $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$, use term-by-term differentiation or integration to find power series for each function centered at the given point.

1. $f(x)=lnx$ centered at $x=1$ (Hint: $x=1-(1-x)$)

Answer

$$\int\sum(1-x)^n dx=\int\sum(-1)^n(x-1)^n dx=\sum\frac{(-1)^n(x-1)^{n+1}}{n+1}$$

2. $ln(1-x)$ at $x=0$

3. $ln(1-x^2)$ at $x=0$

Answer

$$-\int_{t=0}^{x^2}\frac{1}{1-t}dt=-\sum_{n=0}^{\infty}\int_0^{x^2}t^n dx-\sum_{n=0}^{\infty}\frac{x^{2(n+1)}}{n+1}=-\sum_{n=1}^{\infty}\frac{x^{2n}}{n}$$

4. $f(x)=\frac{2x}{(1-x^2)^2}$ at $x=0$

5. $f(x)=tan^{-1}(x^2)$ at $x=0$

Answer

$$\int_0^{x^2}\frac{dt}{1+t^2}=\sum_{n=0}^{\infty}(-1)^n\int_0^{x^2}t^{2n}dt=\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{2n+1}\mid_{t=0}^{x^2}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}$$

6. $f(x)=ln(1+x^2)$ at $x=0$

7. $f(x)=\int_0^x lntdt$ where $ln(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$

Answer

Term-by-term integration gives

$$\int_0^x lntdt=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^{n+1}}{n(n+1)}=\sum_{n=1}^{\infty}(-1)^{n-1}(\frac{1}{n}-\frac{1}{n+1})(x-1)^{n+1}=(x-1)lnx+\sum_{n=2}^{\infty}(-1)^n\frac{(x-1)^n}{n}=xlnx-x.$$

1.2E.10 Exercise $\mathbf{1.2E.10}$

In the following exercises, using a substitution if indicated, express each series in terms of elementary functions and find the radius of convergence of the sum.

1. $\sum_{k=0}^{\infty}(x^k - x^{2k+1})$
2. $\sum_{k=1}^{\infty}\frac{x^{3k}}{6k}$

Answer

$\sum_{k=1}^{\infty}\frac{x^k}{k} = -ln(1-x)$ so $\sum_{k=1} 6\infty\frac{x^{3k}}{6k} = -\frac{1}{6}ln(1-x^3)$. The radius of convergence is equal to 1 by the ratio test.

3. $\sum_{k=1}^{\infty}(1+x^2)^{-k}$ using $y = \frac{1}{1+x^2}$
4. $\sum_{k=1}^{\infty}2^{-kx}$ using $y = 2^{-x}$

Answer

If $y = 2^{-x}$, then $\sum_{k=1}^{\infty}y^k = \frac{y}{1-y} = \frac{2^{-x}}{1-2^{-x}} = \frac{1}{2^x-1}$. If $a_k = 2^{-kx}$, then $\frac{a_{k+1}}{a_k} = 2^{-x} < 1$ when $x > 0$. So the series converges for all $x > 0$.

1.2E.11 Exercise **1.2*E*.11**

1. Show that, up to powers x^3 and y^3, $E(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ satisfies $E(x+y) = E(x)E(y)$.

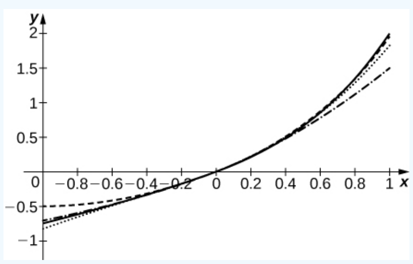
2. Differentiate the series $E(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ term-by-term to show that $E(x)$ is equal to its derivative.

3. Show that if $f(x) = \sum_{n=0}^{\infty}a_n x^n$ is a sum of even powers, that is, $a_n = 0$ if n is odd, then $F = \int_0^x f(t)dt$ is a sum of odd powers, while if *I* is a sum of odd powers, then *F* is a sum of even powers.

4. Suppose that the coefficients an of the series $\sum_{n=0}^{\infty}a_n x^n$ are defined by the recurrence relation $a_n = \frac{a_{n-1}}{n} + \frac{a_{n-2}}{n(n-1)}$. For $a_0 = 0$ and $a_1 = 1$, compute and plot the sums $S_N = \sum_{n=0}^{N}a_n x^n$ for $N = 2,3,4,5$ on $[-1,1]$.

Answer

The solid curve is S_5. The dashed curve is S_2, dotted is S_3, and dash-dotted is S_4



5. Suppose that the coefficients an of the series $\sum_{n=0}^{\infty}a_n x^n$ are defined by the recurrence relation $a_n = \frac{a_{n-1}}{\sqrt{n}} - \frac{a_{n-2}}{\sqrt{n(n-1)}}$. For $a_0 = 1$ and $a_1 = 0$, compute and plot the sums $S_N = \sum_{n=0}^{N}a_n x^n$ for $N = 2,3,4,5$ on $[-1,1]$.

6. Given the power series expansion $ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, determine how many terms N of the sum evaluated at $x=-1/2$ are needed to approximate $ln(2)$ accurate to within 1/1000. Evaluate the corresponding partial sum $\sum_{n=1}^{N}(-1)^{n-1}\frac{x^n}{n}$.

Answer

When $x=-\frac{1}{2}, -ln(2)=ln(\frac{1}{2})=-\sum_{n=1}^{\infty}\frac{1}{n2^n}$. Since $\sum_{n=11}^{\infty}\frac{1}{n2^n}<\sum_{n=11}^{\infty}\frac{1}{2^n}=\frac{1}{2^{10}}$, one has $\sum_{n=1}^{10}\frac{1}{n2^n}=0.69306\ldots$ whereas $ln(2)=0.69314\ldots$; therefore, $N=10$.

7. Given the power series expansion $tan^{-1}(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$, use the alternating series test to determine how many terms N of the sum evaluated at $x=1$ are needed to approximate $tan^{-1}(1)=\frac{\pi}{4}$ accurate to within 1/1000. Evaluate the corresponding partial sum $\sum_{k=0}^{N}(-1)^k\frac{x^{2k+1}}{2k+1}$.

8. Recall that $tan^{-1}(\frac{1}{\sqrt{3}})=\frac{\pi}{6}$. Assuming an exact value of $\frac{1}{\sqrt{3}}$), estimate $\frac{\pi}{6}$ by evaluating partial sums $S_N(\frac{1}{\sqrt{3}})$ of the power series expansion $tan^{-1}(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$ at $x=\frac{1}{\sqrt{3}}$. What is the smallest number N such that $6S_N(\frac{1}{\sqrt{3}})$ approximates π accurately to within 0.001? How many terms are needed for accuracy to within 0.00001?

Answer

$6S_N(\frac{1}{\sqrt{3}})=2\sqrt{3}\sum_{n=0}^{N}(-1)^n\frac{1}{3^n(2n+1).}$ One has $\pi-6S_4(\frac{1}{\sqrt{3}})=0.00101\ldots$ and $\pi-6S_5(\frac{1}{\sqrt{3}})=0.00028\ldots$ so $N=5$ is the smallest partial sum with accuracy to within 0.001. Also, $\pi-6S_7(\frac{1}{\sqrt{3}})=0.00002\ldots$ while $\pi-6S_8(\frac{1}{\sqrt{3}})=-0.000007\ldots$ so $N=8$ is the smallest N to give accuracy to within 0.00001.

1.3: Taylor and Maclaurin Series

This page is a draft and is under active development.

In the previous two sections we discussed how to find power series representations for certain types of functions—specifically, functions related to geometric series. Here we discuss power series representations for other types of functions. In particular, we address the following questions: Which functions can be represented by power series and how do we find such representations? If we can find a power series representation for a particular function f and the series converges on some interval, how do we prove that the series actually converges to f?

1.3.1 Overview of Taylor/Maclaurin Series

Consider a function f that has a power series representation at $x=a$. Then the series has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots. \tag{1.3.1}$$

What should the coefficients be? For now, we ignore issues of convergence, but instead focus on what the series should be, if one exists. We return to discuss convergence later in this section. If the series Equation 1.3.1 is a representation for f at $x=a$, we certainly want the series to equal $f(a)$ at $x=a$. Evaluating the series at $x=a$, we see that

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots = c_0. \tag{1.3.2}$$

Thus, the series equals $f(a)$ if the coefficient $c_0 = f(a)$. In addition, we would like the first derivative of the power series to equal $f'(a)$ at $x=a$. Differentiating Equation 1.3.2 term-by-term, we see that

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots. \tag{1.3.3}$$

Therefore, at $x=a$, the derivative is

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots = c_1. \tag{1.3.4}$$

Therefore, the derivative of the series equals $f'(a)$ if the coefficient $c_1 = f'(a)$. Continuing in this way, we look for coefficients c_n such that all the derivatives of the power series Equation 1.3.4 will agree with all the corresponding derivatives of f at $x=a$. The second and third derivatives of Equation 1.3.3 are given by

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \ldots \tag{1.3.5}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \cdots. \tag{1.3.6}$$

Therefore, at $x=a$, the second and third derivatives

$$\frac{d^2}{dx^2}(\sum_{n=0}^{\infty} c_n (x-a)^n) = 2c_2 + 3\cdot 2c_3(a-a) + 4\cdot 3c_4(a-a)^2 + \cdots = 2c_2 \tag{1.3.7}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(a-a) + 5\cdot 4\cdot 3c_5(a-a)^2 + \cdots = 3\cdot 2c_3 \tag{1.3.8}$$

equal $f''(a)$ and $f'''(a)$, respectively, if $c_2 = \dfrac{f''(a)}{2}$ and $c_3 = \dfrac{f'''(a)}{3}\cdot 2$. More generally, we see that if f has a power series representation at $x=a$, then the coefficients should be given by $c_n = \dfrac{f^{(n)}(a)}{n!}$. That is, the series should be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{1.3.9}$$

This power series for f is known as the Taylor series for f at a. If $x=0$, then this series is known as the Maclaurin series for f.

Definition 1.3.1: Maclaurin and Taylor series

If f has derivatives of all orders at $x=a$, then the *Taylor series* for the function f at a is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots \tag{1.3.10}$$

The Taylor series for f at 0 is known as the *Maclaurin series* for f.

Later in this section, we will show examples of finding Taylor series and discuss conditions under which the Taylor series for a function will converge to that function. Here, we state an important result. Recall that power series representations are unique. Therefore, if a function f has a power series at a, then it must be the Taylor series for f at a.

Uniqueness of Taylor Series

If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

The proof follows directly from that discussed previously.

To determine if a Taylor series converges, we need to look at its sequence of partial sums. These partial sums are finite polynomials, known as **Taylor polynomials.**

1.3.2 Taylor Polynomials

The n^{th} partial sum of the Taylor series for a function f at a is known as the n^{th} Taylor polynomial. For example, the 0^{th}, 1^{st}, 2^{nd}, and 3^{rd} partial sums of the Taylor series are given by

$$\begin{aligned}
p_0(x)&=f(a)\\
p_1(x)&=f(a)+f'(a)(x-a)\\
p_2(x)&=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2\\
p_3(x)&=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3
\end{aligned}$$

respectively. These partial sums are known as the 0^{th}, 1^{st}, 2^{nd}, and 3^{rd} Taylor polynomials of f at a, respectively. If $x=a$, then these polynomials are known as *Maclaurin polynomials* for f. We now provide a formal definition of Taylor and Maclaurin polynomials for a function f.

Definition 1.3.2: Maclaurin polynomial

If f has *n* derivatives at $x=a$, then the nth Taylor polynomial for f at a is

$$p_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{1.3.11}$$

The n^{th} Taylor polynomial for f at 0 is known as the n^{th} Maclaurin polynomial for f.

We now show how to use this definition to find several Taylor polynomials for $f(x)=\ln x$ at $x=1$.

Example 1.3.1: Finding Taylor Polynomials

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x)=\ln x$ at $x=1$. Use a graphing utility to compare the graph of f with the graphs of p_0, p_1, p_2 and p_3.

Solution

To find these Taylor polynomials, we need to evaluate f and its first three derivatives at $x=1$.

$$\begin{aligned}
f(x)&=\ln x \quad f(1)=0\\
f'(x)&=\frac{1}{x} \quad f'(1)=1\\
f''(x)&=-\frac{1}{x^2} \quad f''(1)=-1\\
f'''(x)&=\frac{2}{x^3} \quad f'''(1)=2
\end{aligned}$$

Therefore,

$$p_0(x) = f(1) = 0,$$

$$p_1(x) = f(1) + f'(1)(x-1) = x-1,$$

$$p_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2$$

$$p_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

The graphs of $y = f(x)$ and the first three Taylor polynomials are shown in Figure 1.3.1.

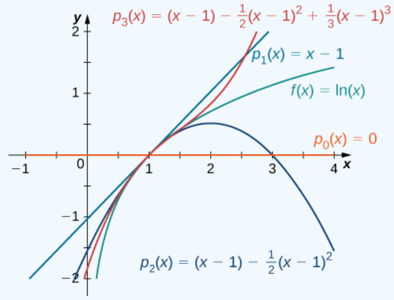


Figure 1.3.1: The function $y = \ln x$ and the Taylor polynomials p_0, p_1, p_2 and p_3 at $x = 1$ are plotted on this graph.

Exercise 1.3.1

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \frac{1}{x^2}$ at $x = 1$.

Hint

Find the first three derivatives of f and evaluate them at $x = 1$.

Answer

$$p_0(x) = 1 \tag{1.3.12}$$

$$p_1(x) = 1 - 2(x-1) \tag{1.3.13}$$

$$p_2(x) = 1 - 2(x-1) + 3(x-1)^2 \tag{1.3.14}$$

$$p_3(x) = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 \tag{1.3.15}$$

We now show how to find Maclaurin polynomials for $e^x, \sin x,$ and $\cos x$. As stated above, Maclaurin polynomials are Taylor polynomials centered at zero.

Example 1.3.2: Finding Maclaurin Polynomials

For each of the following functions, find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3. Find a formula for the *n*th Maclaurin polynomial and write it using sigma notation. Use a graphing utility to compare the graphs of p_0, p_1, p_2 and p_3 with f.

a. $f(x) = e^x$
b. $f(x) = \sin x$
c. $f(x) = \cos x$

Solution

Since $f(x) = e^x$,we know that $f(x) = f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x$ for all positive integers *n*. Therefore,

$$f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$$

for all positive integers *n*. Therefore, we have

$$p_0(x) = f(0) = 1,$$

$$p_1(x) = f(0) + f'(0)x = 1 + x,$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2}x^2 \;,$$

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 \;,$$

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!} \;.$$

The function and the first three Maclaurin polynomials are shown in Figure 2.

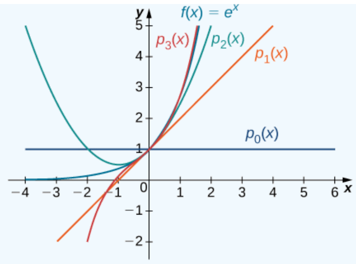


Figure 1.3.2: The graph shows the function $y = e^x$ and the Maclaurin polynomials p_0, p_1, p_2 and p_3.

b. For $f(x) = \sin x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$f(x) = \sin x \quad f(0) = 0$$
$$f'(x) = \cos x \quad f'(0) = 1$$
$$f''(x) = -\sin x \quad f''(0) = 0$$
$$f'''(x) = -\cos x \quad f'''(0) = -1$$
$$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0.$$

Since the fourth derivative is $\sin x$, the pattern repeats. That is, $f^{(2m)}(0) = 0$ and $f^{(2m+1)}(0) = (-1)^m$ for $m \geq 0$. Thus, we have

$$p_0(x) = 0,$$
$$p_1(x) = 0 + x = x,$$
$$p_2(x) = 0 + x + 0 = x,$$
$$p_3(x) = 0 + x + 0 - \frac{1}{3!}x^3 = x - \frac{x^3}{3!},$$
$$p_4(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 = x - \frac{x^3}{3!} \quad ,$$
$$p_5(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!} \quad ,$$

and for $m \geq 0$,

$$p_{2m+1}(x) = p_{2m+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} = \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad .$$

Graphs of the function and its Maclaurin polynomials are shown in Figure 3.

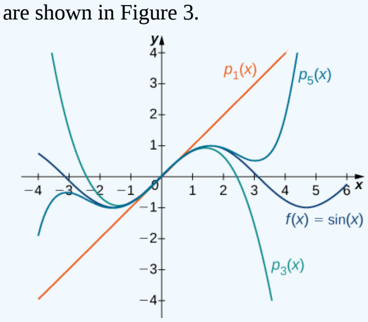


Figure 1.3.3: The graph shows the function $y = \sin x$ and the Maclaurin polynomials p_1, p_3 and p_5.

c. For $f(x) = \cos x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$f(x) = \cos x \quad f(0) = 1$$
$$f'(x) = -\sin x \quad f'(0) = 0$$
$$f''(x) = -\cos x \quad f''(0) = -1$$
$$f'''(x) = \sin x \quad f'''(0) = 0$$
$$f^{(4)}(x) = \cos x \quad f^{(4)}(0) = 1.$$

Since the fourth derivative is $\sin x$, the pattern repeats. In other words, $f^{(2m)}(0) = (-1)^m$ and $f^{(2m+1)} = 0$ for $m \geq 0$. Therefore,

$$p_0(x) = 1,$$
$$p_1(x) = 1 + 0 = 1,$$
$$p_2(x) = 1 + 0 - \frac{1}{2!}x^2 = 1 - \frac{x^2}{2!} \quad ,$$
$$p_3(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 = 1 - \frac{x^2}{2!} \quad ,$$

$$p_4(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad ,$$

$$p_5(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad ,$$

and for $n \geq 0$,

$$p_{2m}(x) = p_{2m+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} = \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!} \quad .$$

Graphs of the function and the Maclaurin polynomials appear in Figure 4.

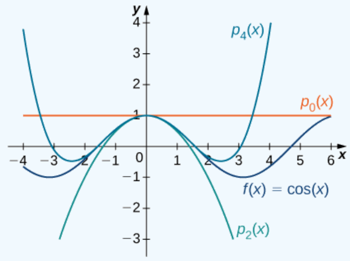


Figure 1.3.4: The function $y = \cos x$ and the Maclaurin polynomials p_0, p_2 and p_4 are plotted on this graph.

Exercise 1.3.2

Find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3 for $f(x) = \dfrac{1}{1+x}$.

Find a formula for the *n*th Maclaurin polynomial. Write your answer using sigma notation.

Hint

Evaluate the first four derivatives of f and look for a pattern.

Answer

$p_0(x) = 1; p_1(x) = 1 - x; p_2(x) = 1 - x + x^2; p_3(x) = 1 - x + x^2 - x^3; p_n(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n =_{k=0}^{n} (-1)^k x^k$

1.3.3 Taylor's Theorem with Remainder

Recall that the *n*th Taylor polynomial for a function f at a is the nth partial sum of the Taylor series for f at *a*. Therefore, to determine if the Taylor series converges, we need to determine whether the sequence of Taylor polynomials p_n converges. However, not only do we want to know if the sequence of Taylor polynomials converges, we want to know if it converges to f. To answer this question, we define the remainder $R_n(x)$ as

$$R_n(x) = f(x) - p_n(x). \tag{1.3.16}$$

For the sequence of Taylor polynomials to converge to f, we need the remainder R_n to converge to zero. To determine if R_n converges to zero, we introduce **Taylor's theorem with remainder**. Not only is this theorem useful in proving that a Taylor series converges to its related function, but it will also allow us to quantify how well the *n*th Taylor polynomial approximates the function.

Here we look for a bound on $|R_n|$. Consider the simplest case: $n = 0$. Let p_0 be the 0^{th} Taylor polynomial at a for a function f. The remainder R_0 satisfies

$$R_0(x) = f(x) - p_0(x) = f(x) - f(a).$$

If f is differentiable on an interval *I* containing a and x, then by the Mean Value Theorem there exists a real number *c* between *a* and *x* such that $f(x) - f(a) = f'(c)(x-a)$. Therefore,

$$R_0(x) = f'(c)(x-a). \tag{1.3.17}$$

Using the Mean Value Theorem in a similar argument, we can show that if f is *n* times differentiable on an interval *I* containing *a* and *x*, then the *n*th remainder R_n satisfies

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{1.3.18}$$

for some real number *c* between a and *x*. It is important to note that the value *c* in the numerator above is not the center *a*, but rather an unknown value c between *a* and *x*. This formula allows us to get a bound on the remainder R_n. If we happen to know that $\left|f^{(n+1)}(x)\right|$ is bounded by some real number *M* on this interval *I*, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{1.3.19}$$

for all *x* in the interval *I*.

We now state Taylor's theorem, which provides the formal relationship between a function f and its nth degree Taylor polynomial $p_n(x)$. This theorem allows us to bound the error when using a Taylor polynomial to approximate a function value, and will be important in proving that a Taylor series for f converges to f.

Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n+1$ times on an interval I containing the real number a. Let p_n be the nth Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x) \tag{1.3.20}$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{1.3.21}$$

.

If there exists a real number M such that $\left|f^{(n+1)}(x)\right| \leq M$ for all $x \in I$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{1.3.22}$$

for all x in I.

Proof

Fix a point $x \in I$ and introduce the function g such that

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}. \tag{1.3.23}$$

We claim that g satisfies the criteria of Rolle's theorem. Since \(g\) is a polynomial function (in t), it is a differentiable function. Also, g is zero at $t=a$ and $t=x$ because

$$g(a) = f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n - R_n(x) \tag{1.3.24}$$

$$= f(x) - p_n(x) - R_n(x) \tag{1.3.25}$$

$$= 0, \tag{1.3.26}$$

$$g(x) = f(x) - f(x) - 0 - \cdots - 0 \tag{1.3.27}$$

$$= 0. \tag{1.3.28}$$

Therefore, g satisfies Rolle's theorem, and consequently, there exists c between a and x such that $g'(c) = 0$. We now calculate g'. Using the product rule, we note that

$$\frac{d}{dt}\left[\frac{f^{(n)}(t)}{n!}(x-t)^n\right] = -\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n. \tag{1.3.29}$$

Consequently,

$$g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \cdots + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] \tag{1.3.30}$$

$$+ (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}$$

.

Notice that there is a telescoping effect. Therefore,

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}} \tag{1.3.31}$$

.

By Rolle's theorem, we conclude that there exists a number c between a and x such that $g'(c) = 0$. Since

$$g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} \tag{1.3.32}$$

we conclude that

$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n+(n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}}=0. \tag{1.3.33}$$

Adding the first term on the left-hand side to both sides of the equation and dividing both sides of the equation by $n+1$, we conclude that

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{1.3.34}$$

as desired. From this fact, it follows that if there exists *M* such that $\left|f^{(n+1)}(x)\right| \leq M$ for all *x* in *I*, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{1.3.35}$$

.

□

Not only does Taylor's theorem allow us to prove that a Taylor series converges to a function, but it also allows us to estimate the accuracy of Taylor polynomials in approximating function values. We begin by looking at linear and quadratic approximations of $f(x)=\frac{[}{3}]x$ at $x=8$ and determine how accurate these approximations are at estimating $\frac{[}{3}]11$.

Example 1.3.3: Using Linear and Quadratic Approximations to Estimate Function Values

Consider the function $f(x)=\sqrt[3]{x}$.

a. Find the first and second Taylor polynomials for f at $x=8$. Use a graphing utility to compare these polynomials with f near $x=8$.
b. Use these two polynomials to estimate $\sqrt[3]{11}$.
c. Use Taylor's theorem to bound the error.

Solution:

a. For $f(x)=\sqrt[3]{x}$, the values of the function and its first two derivatives at $x=8$ are as follows:

$$f(x)=\sqrt[3]{x} \quad f(8)=2$$
$$f'(x)=\frac{1}{3x^{2/3}} \quad f'(8)=\frac{1}{12}$$
$$f''(x)=\frac{-2}{9x^{5/3}} \quad f''(8)=-\frac{1}{144.}$$

Thus, the first and second Taylor polynomials at $x=8$ are given by

$$\begin{aligned} p_1(x) &= f(8)+f'(8)(x-8) \\ &= 2+\frac{1}{12}(x-8) \\ p_2(x) &= f(8)+f'(8)(x-8)+\frac{f''(8)}{2!}(x-8)^2 \\ &= 2+\frac{1}{12}(x-8)-\frac{1}{288}(x-8)^2 \ . \end{aligned}$$

The function and the Taylor polynomials are shown in Figure 4.

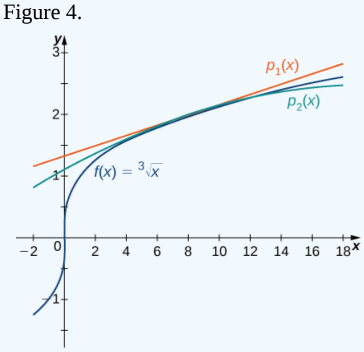


Figure 1.3.5: The graphs of $f(x)=\sqrt[3]{x}$ and the linear and quadratic approximations $p_1(x)$ and $p_2(x)$

b. Using the first Taylor polynomial at $x=8$, we can estimate

$$\frac{[}{3}]11 \approx p_1(11)=2+\frac{1}{12}(11-8)=2.25. \tag{1.3.36}$$

Using the second Taylor polynomial at $x=8$, we obtain

$$\sqrt[3]{11} \approx p_2(11) = 2 + \frac{1}{12}(11-8) - \frac{1}{288}(11-8)^2 = 2.21875. \tag{1.3.37}$$

c. By Note, there exists a c in the interval $(8,11)$ such that the remainder when approximating $\sqrt[3]{11}$ by the first Taylor polynomial satisfies

$$R_1(11) = \frac{f''(c)}{2!}(11-8)^2. \tag{1.3.38}$$

We do not know the exact value of c, so we find an upper bound on $R_1(11)$ by determining the maximum value of f'' on the interval $(8,11)$. Since $f''(x) = -\frac{2}{9x^{5/3}}$, the largest value for $|f''(x)|$ on that interval occurs at $x=8$. Using the fact that $f''(8) = -\frac{1}{144}$, we obtain

$$|R_1(11)| \le \frac{1}{144 \cdot 2!}(11-8)^2 = 0.03125.$$

Similarly, to estimate $R_2(11)$, we use the fact that

$$R_2(11) = \frac{f'''(c)}{3!}(11-8)^3.$$

Since $f'''(x) = \frac{10}{27x^{8/3}}$, the maximum value of f''' on the interval $(8,11)$ is $f'''(8) \approx 0.0014468$. Therefore, we have

$$|R_2(11)| \le \frac{0.0011468}{3!}(11-8)^3 \approx 0.0065104.$$

Exercise 1.3.3:

Find the first and second Taylor polynomials for $f(x) = \sqrt{x}$ at $x=4$. Use these polynomials to estimate $\sqrt{6}$. Use Taylor's theorem to bound the error.

Hint

Evaluate $f(4)$, $f'(4)$, and $f''(4)$.

Answer

$p_1(x) = 2 + \frac{1}{4}(x-4); p_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2; p_1(6) = 2.5; p_2(6) = 2.4375;$

$|R_1(6)| \le 0.0625; |R_2(6)| \le 0.015625$

Example 1.3.4: Approximating $\sin x$ Using Maclaurin Polynomials

From Example b., the Maclaurin polynomials for $\sin x$ are given by

$$p_{2m+1}(x) = p_{2m+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

for $m = 0, 1, 2, \ldots$.

a. Use the fifth Maclaurin polynomial for $\sin x$ to approximate $sin(\frac{\pi}{18})$ and bound the error.
b. For what values of x does the fifth Maclaurin polynomial approximate $\sin x$ to within 0.0001?

Solution

a.

The fifth Maclaurin polynomial is

$$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \tag{1.3.39}$$

.

Using this polynomial, we can estimate as follows:

$$\sin(\frac{\pi}{18}) \approx p_5(\frac{\pi}{18}) = \frac{\pi}{18} - \frac{1}{3!}(\frac{\pi}{18})^3 + \frac{1}{5!}(\frac{\pi}{18})^5 \approx 0.173648. \tag{1.3.40}$$

To estimate the error, use the fact that the sixth Maclaurin polynomial is $p_6(x) = p_5(x)$ and calculate a bound on $R_6(\frac{\pi}{18})$. By Note, the remainder is

$$R_6(\frac{\pi}{18}) = \frac{f^{(7)}(c)}{7!}(\frac{\pi}{18})^7 \tag{1.3.41}$$

for some c between 0 and $\frac{\pi}{18}$. Using the fact that $\left|f^{(7)}(x)\right| \le 1$ for all x, we find that the magnitude of the error is at most

$$\frac{1}{7!} \cdot (\frac{\pi}{18})^7 \le 9.8 \times 10^{-10}. \tag{1.3.42}$$

b.

We need to find the values of x such that

$$\frac{1}{7}!|x|^7 \leq 0.0001. \tag{1.3.43}$$

Solving this inequality for x, we have that the fifth Maclaurin polynomial gives an estimate to within 0.0001 as long as $|x| < 0.907$.

Exercise 1.3.4

Use the fourth Maclaurin polynomial for $\cos x$ to approximate $\cos(\frac{\pi}{12})$.

Hint

The fourth Maclaurin polynomial is $p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$.

Answer

0.96593

Now that we are able to bound the remainder $R_n(x)$, we can use this bound to prove that a Taylor series for f at a converges to f.

1.3.4 Representing Functions with Taylor and Maclaurin Series

We now discuss issues of convergence for Taylor series. We begin by showing how to find a Taylor series for a function, and how to find its interval of convergence.

Example 1.3.5: Finding a Taylor Series

Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 1$. Determine the interval of convergence.

Solution

For $f(x) = \frac{1}{x}$, the values of the function and its first four derivatives at $x = 1$ are

$$\begin{aligned} f(x) &= \frac{1}{x} \quad f(1) = 1 \\ f'(x) &= -\frac{1}{x^2} \quad f'(1) = -1 \\ f''(x) &= \frac{2}{x^3} \quad f''(1) = 2! \\ f'''(x) &= -\frac{3 \cdot 2}{x^4} \quad f'''(1) = -3! \\ f^{(4)}(x) &= \frac{4 \cdot 3 \cdot 2}{x^5} \quad f^{(4)}(1) = 4!\,. \end{aligned}$$

That is, we have $f^{(n)}(1) = (-1)^n n!$ for all $n \geq 0$. Therefore, the Taylor series for f at $x = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n \,.$$

To find the interval of convergence, we use the ratio test. We find that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|(-1)^{n+1}(x-1)n^{+1}|}{|(-1)^n(x-1)^n|} = |x-1|\,.$$

Thus, the series converges if $|x-1| < 1$. That is, the series converges for $0 < x < 2$. Next, we need to check the endpoints. At $x = 2$, we see that

$$\sum_{n=0}^{\infty}(-1)^n(2-1)^n = \sum_{n=0}^{\infty}(-1)^n$$

diverges by the divergence test. Similarly, at $x = 0$,

$$\sum_{n=0}^{\infty}(-1)^n(0-1)^n = \sum_{n=0}^{\infty}(-1)^{2n} = \sum_{n=0}^{\infty}1$$

diverges. Therefore, the interval of convergence is $(0, 2)$.

Exercise 1.3.5

Find the Taylor series for $f(x) = \frac{1}{2}$ at $x = 2$ and determine its interval of convergence.

Hint

$f^{(n)}(2)=\frac{(-1)^n n!}{2^{n+1}}$

Answer

$\frac{1}{2}\sum_{n=0}^{\infty}(\frac{2-x}{2})^n$. The interval of convergence is $(0,4)$.

We know that the Taylor series found in this example converges on the interval $(0,2)$, but how do we know it actually converges to f? We consider this question in more generality in a moment, but for this example, we can answer this question by writing

$$f(x)=\frac{1}{x}=\frac{1}{1-(1-x)}. \tag{1.3.44}$$

That is, f can be represented by the geometric series $\sum_{n=0}^{\infty}(1-x)^n$. Since this is a geometric series, it converges to $\frac{1}{x}$ as long as $|1-x|<1$. Therefore, the Taylor series found in Example does converge to $f(x)=\frac{1}{x}$ on $(0,2)$.

We now consider the more general question: if a Taylor series for a function f converges on some interval, how can we determine if it actually converges to f? To answer this question, recall that a series converges to a particular value if and only if its sequence of partial sums converges to that value. Given a Taylor series for f at a, the nth partial sum is given by the nth Taylor polynomial pn. Therefore, to determine if the Taylor series converges to f, we need to determine whether

$$\lim_{n\to\infty} p_n(x)=f(x).$$

Since the remainder $R_n(x)=f(x)-p_n(x)$, the Taylor series converges to f if and only if

$$\lim_{n\to\infty} R_n(x)=0.$$

We now state this theorem formally.

Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1.3.45}$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n\to\infty} R_n(x)=0 \tag{1.3.46}$$

for all x in I.

With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x)\to 0$. To prove that $R_n(x)\to 0$, we typically use the bound

$$|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1} \tag{1.3.47}$$

from Taylor's theorem with remainder.

In the next example, we find the Maclaurin series for e^x and $\sin x$ and show that these series converge to the corresponding functions for all real numbers by proving that the remainders $R_n(x)\to 0$ for all real numbers x.

Example 1.3.6: Finding Maclaurin Series

For each of the following functions, find the Maclaurin series and its interval of convergence. Use Note to prove that the Maclaurin series for f converges to f on that interval.

a. e^x
b. $\sin x$

Solution

a. Using the *n*th Maclaurin polynomial for e^x found in Example a., we find that the Maclaurin series for e^x is given by

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

To determine the interval of convergence, we use the ratio test. Since

$$\frac{|a_{n+1}|}{|a_n|}=\frac{|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{|x|^n}=\frac{|x|}{n+1},$$

we have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{|x|}{n+1}=0$$

for all x. Therefore, the series converges absolutely for all x, and thus, the interval of convergence is $(-\infty,\infty)$. To show that the series converges to e^x for all x, we use the fact that $f^{(n)}(x)=e^x$ for all $n\geq 0$ and e^x is an increasing function on $(-\infty,\infty)$. Therefore, for any real number b, the maximum value of e^x for all $|x|\leq b$ is e^b. Thus,

$$|R_n(x)|\leq\frac{e^b}{(n+1)!}|x|^{n+1}.$$

Since we just showed that

$$\sum_{n=0}^{\infty}\frac{|x|^n}{n!}$$

converges for all *x*, by the divergence test, we know that

$$\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!}=0$$

for any real number *x*. By combining this fact with the squeeze theorem, the result is $\lim_{n\to\infty}R_n(x)=0$.

b. Using the *n*th Maclaurin polynomial for $\sin x$ found in Example b., we find that the Maclaurin series for $\sin x$ is given by

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

In order to apply the ratio test, consider

$$\frac{|a_{n+1}|}{|a_n|}=\frac{|x|^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{|x|^{2n+1}}=\frac{|x|^2}{(2n+3)(2n+2)}.$$

Since

$$\lim_{n\to\infty}\frac{|x|^2}{(2n+3)(2n+2)}=0$$

for all x, we obtain the interval of convergence as $(-\infty,\infty)$. To show that the Maclaurin series converges to $\sin x$, look at $R_n(x)$. For each x there exists a real number c between 0 and x such that

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Since $\left|f^{(n+1)}(c)\right|\leq 1$ for all integers n and all real numbers *c*, we have

$$|R_n(x)|\leq\frac{|x|^{n+1}}{(n+1)!}$$

for all real numbers x. Using the same idea as in part a., the result is $\lim\limits_{n\to\infty}R_n(x)=0$ for all x, and therefore, the Maclaurin series for $\sin x$ converges to $\sin x$ for all real x.

Exercise 1.3.6

Find the Maclaurin series for $f(x)=\cos x$. Use the ratio test to show that the interval of convergence is $(-\infty,\infty)$. Show that the Maclaurin series converges to $\cos x$ for all real numbers x.

Hint

Use the Maclaurin polynomials for $\cos x$.

Answer

$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

By the ratio test, the interval of convergence is $(-\infty,\infty)$. Since $|R_n(x)|\leq\frac{|x|^{n+1}}{(n+1)!}$, the series converges to $\cos x$ for all real x.

Proving that *E* is Irrational

In this project, we use the Maclaurin polynomials for e^x to prove that e is irrational. The proof relies on supposing that e is rational and arriving at a contradiction. Therefore, in the following steps, we suppose $e=r/s$ for some integers r and s where $s\neq 0$.

1. Write the Maclaurin polynomials $p_0(x),p_1(x),p_2(x),p_3(x),p_4(x)$ for e^x. Evaluate $p_0(1),p_1(1),p_2(1),p_3(1),p_4(1)$ to estimate e.

2. Let $R_n(x)$ denote the remainder when using $p_n(x)$ to estimate e^x. Therefore, $R_n(x) = e^x - p_n(x)$, and $R_n(1) = e - p_n(1)$. Assuming that $e = \frac{r}{s}$ for integers r and s, evaluate $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$.
3. Using the results from part 2, show that for each remainder $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$, we can find an integer k such that $kR_n(1)$ is an integer for $n = 0, 1, 2, 3, 4$.
4. Write down the formula for the nth Maclaurin polynomial $p_n(x)$ for e^x and the corresponding remainder $R_n(x)$. Show that $sn!R_n(1)$ is an integer.
5. Use Taylor's theorem to write down an explicit formula for $R_n(1)$. Conclude that $R_n(1) \neq 0$, and therefore, $sn!R_n(1) \neq 0$.
6. Use Taylor's theorem to find an estimate on $R_n(1)$. Use this estimate combined with the result from part 5 to show that $|sn!R_n(1)| < \frac{se}{n+1}$. Conclude that if *n* is large enough, then $|sn!R_n(1)| < 1$. Therefore, $sn!R_n(1)$ is an integer with magnitude less than 1. Thus, $sn!R_n(1) = 0$. But from part 5, we know that $sn!R_n(1) \neq 0$. We have arrived at a contradiction, and consequently, the original supposition that e is rational must be false.

1.3.5 Key Concepts

- Taylor polynomials are used to approximate functions near a value $x = a$. Maclaurin polynomials are Taylor polynomials at $x = 0$.
- The *n*th degree Taylor polynomials for a function f are the partial sums of the Taylor series for f.
- If a function f has a power series representation at $x = a$, then it is given by its Taylor series at $x = a$.
- A Taylor series for f converges to f if and only if $\lim_{n\to\infty} R_n(x) = 0$ where $R_n(x) = f(x) - p_n(x)$.
- The Taylor series for e^x, $\sin x$, and $\cos x$ converge to the respective functions for all real x.

1.3.6 Key Equations

- **Taylor series for the function f at the point $x = a$**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

1.3.7 Glossary

Maclaurin polynomial

a Taylor polynomial centered at 0; the nth Taylor polynomial for f at 0 is the nth Maclaurin polynomial for f

Maclaurin series

a Taylor series for a function f at $x = 0$ is known as a Maclaurin series for f

Taylor polynomials

the *n*th Taylor polynomial for f at $x = a$ is $p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Taylor series

a power series at a that converges to a function f on some open interval containing a

Taylor's theorem with remainder

for a function f and the *n*th Taylor polynomial for f at $x = a$, the remainder $R_n(x) = f(x) - p_n(x)$ satisfies $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

for some *c* between *x* and a; if there exists an interval I containing a and a real number *M* such that $\left|f^{(n+1)}(x)\right| \leq M$ for all *x* in *I*, then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$

1.3.8 Contributors

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

1.3E: Exercises

This page is a draft and is under active development.

1.3E.1 Exercise $\mathbf{1.3E.1}$

In the following exercises, find the Taylor polynomials of degree two approximating the given function centered at the given point.

1. $f(x)=1+x+x^2$ at $a=1$
2. $f(x)=1+x+x^2$ at $a=-1$

Answer

$f(-1)=1; f'(-1)=-1; f''(-1)=2; f(x)=1-(x+1)+(x+1)^2$

3. $f(x)=cos(2x)$ at $a=\pi$
4. $f(x)=sin(2x)$ at $a=\frac{\pi}{2}$

Answer

$f'(x)=2cos(2x); f''(x)=-4sin(2x); p_2(x)=-2(x-\frac{\pi}{2})$

5. $f(x)=\sqrt{x}$ at $a=4$
6. $f(x)=lnx$ at $a=1$

Answer

$f'(x)=\frac{1}{x}; f''(x)=-\frac{1}{x^2}; p_2(x)=0+(x-1)-\frac{1}{2}(x-1)^2$

7. $f(x)=\frac{1}{x}$ at $a=1$
8. $f(x)=e^x$ at $a=1$

Answer

$p_2(x)=e+e(x-1)+\frac{e}{2}(x-1)^2$

1.3E.2 Exercise $\mathbf{1.3E.2}$

In the following exercises, verify that the given choice of n in the remainder estimate $|R_n|\leq\frac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $\left|f^{(n+1)}(z)\right|$ on the interval between a and the indicated point, yields $|R_n|\leq\frac{1}{1000}$. Find the value of the Taylor polynomial p_n of f at the indicated point.

1. (\displaystyle \sqrt{10};a=9,n=3\)
2. $(28)^{1/3}; a=27, n=1$

Answer

$\frac{d^2}{dx^2}x^{1/3}=-\frac{2}{9x^{5/3}}\geq-0.00092\ldots$ when $x\geq 28$ so the remainder estimate applies to the linear approximation $x^{1/3}\approx p_1(27)=3+\frac{x-27}{27}$, which gives $(28)^{1/3}\approx 3+\frac{1}{27}=3.0\bar{37}$, while $(28)^{1/3}\approx 3.03658$.

3. $sin(6); a=2\pi, n=5$
4. $e^2; a=0, n=9$

Answer

Using the estimate $\frac{2^{10}}{10!} < 0.000283$ we can use the Taylor expansion of order 9 to estimate e^x at $x = 2$. as $e^2 \approx p_9(2) = 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \cdots + \frac{2^9}{9!} = 7.3887$ … whereas $e^2 \approx 7.3891$.

5. $cos(\frac{\pi}{5}); a = 0, n = 4$

6. $ln(2); a = 1, n = 1000$

Answer

Since $\frac{d^n}{dx^n}(lnx) = (-1)^{n-1}\frac{(n-1)!}{x^n}, R_{1000} \approx \frac{1}{1001}$. One has $p_{1000}(1) = \sum_{n=1}^{1000} \frac{(-1)^{n-1}}{n} \approx 0.6936$ whereas $ln(2) \approx 0.6931 \cdots$.

1.3E.3 Exercise $\mathbf{1.3E.3}$

1. Integrate the approximation $sint \approx t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040}$ evaluated at πt to approximate $\int_0^1 \frac{sin\pi t}{\pi t} dt$.

2. Integrate the approximation $e^x \approx 1 + x + \frac{x^2}{2} + \cdots + \frac{x^6}{720}$ evaluated at $-x^2$ to approximate $\int_0^1 e^{-x^2} dx$.

Answer

$\int_0^1 (1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \frac{x^{12}}{720})dx = 1 - \frac{1^3}{3} + \frac{1^5}{10} - \frac{1^7}{42} + \frac{1^9}{9 \cdot 24} - \frac{1^{11}}{120 \cdot 11} + \frac{1^{13}}{720 \cdot 13} \approx 0.74683$ whereas $\int_0^1 e^{-x^2} dx \approx 0.74682$.

1.3E.4 Exercise $\mathbf{1.3E.4}$

In the following exercises, find the smallest value of n such that the remainder estimate $|R_n| \leq \frac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $\left|f^{(n+1)}(z)\right|$ on the interval between a and the indicated point, yields $|R_n| \leq \frac{1}{1000}$ on the indicated interval.

1. $f(x) = sinx$ on $[-\pi, \pi], a = 0$
2. $f(x) = cosx$ on $[-\frac{\pi}{2}, \frac{\pi}{2}], a = 0$

Answer

Since $f^{(n+1)}(z)$ is $sinz$ or $cosz$, we have $M = 1$. Since $|x - 0| \leq \frac{\pi}{2}$, we seek the smallest n such that $\frac{\pi^{n+1}}{2^{n+1}(n+1)!} \leq 0.001$. The smallest such value is $n = 7$. The remainder estimate is $R_7 \leq 0.00092$.

3. $f(x) = e^{-2x}$ on $[-1, 1], a = 0$
4. $f(x) = e^{-x}$ on $[-3, 3], a = 0$

Answer

Since $f^{(n+1)}(z) = \pm e^{-z}$ one has $M = e^3$. Since $|x - 0| \leq 3$, one seeks the smallest n such that $\frac{3^{n+1}e^3}{(n+1)!} \leq 0.001$. The smallest such value is $n = 14$. The remainder estimate is $R_{14} \leq 0.000220$.

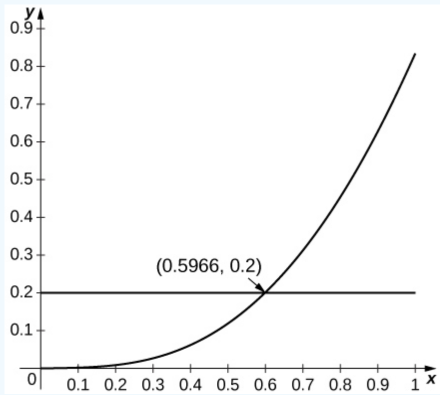
1.3E.5 Exercise $\mathbf{1.3E.5}$

In the following exercises, the maximum of the right-hand side of the remainder estimate $|R_1| \leq \frac{max|f''(z)|}{2}R^2$ on $[a-R, a+R]$ occurs at a or $a \pm R$. Estimate the maximum value of R such that $\frac{max|f''(z)|}{2}R^2 \leq 0.1$ on $[a-R, a+R]$ by plotting this maximum as a function of I.

1. e^x approximated by $1+x, a=0$
2. $sinx$ approximated by $x, a=0$

Answer

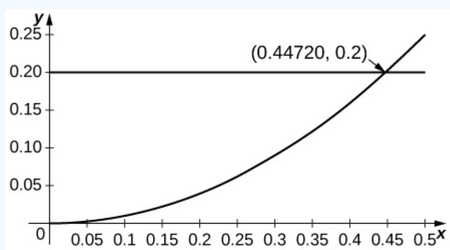
Since $sinx$ is increasing for small x and since $sin''x = -sinx$, the estimate applies whenever $R^2 sin(R) \leq 0.2$, which applies up to $R = 0.596$.



3. lnx approximated by $x-1, a=1$
4. $cosx$ approximated by $1, a=0$

Answer

Since the second derivative of $cosx$ is $-cosx$ and since $cosx$ is decreasing away from $x=0$, the estimate applies when $R^2 cosR \leq 0.2$ or $R \leq 0.447$.



1.3E.6 Exercise $\mathbf{1.3E.6}$

In the following exercises, find the Taylor series of the given function centered at the indicated point.

1. x^4 at $a=-1$
2. $1+x+x^2+x^3$ at $a=-1$

Answer

$(x+1)^3 - 2(x+1)^2 + 2(x+1)$

3. $sinx$ at $a=\pi$
4. $cosx$ at $a=2\pi$

Answer

Values of derivatives are the same as for $x=0$ so $cosx=\sum_{n=0}^{\infty}(-1)^n\frac{(x-2\pi)^{2n}}{(2n)!}$

5. $sinx$ at $x=\frac{\pi}{2}$

6. $cosx$ at $x=\frac{\pi}{2}$

Answer

$cos(\frac{\pi}{2})=0, -sin(\frac{\pi}{2})=-1$ so $cosx=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$, which is also $-cos(x-\frac{\pi}{2})$.

7. e^x at $a=-1$

8. e^x at $a=1$

Answer

The derivatives are $f^{(n)}(1)=e$ so $e^x=e\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}$.

9. $\frac{1}{(x-1)^2}$ at $a=0$ (Hint: Differentiate $\frac{1}{1-x}$.)

10. $\frac{1}{(x-1)^3}$ at $a=0$

Answer

$$\frac{1}{(x-1)^3}=-(\frac{1}{2})\frac{d^2}{dx^2}\frac{1}{1-x}=-\sum_{n=0}^{\infty}(\frac{(n+2)(n+1)x^n}{2})$$

11. $F(x)=\int_0^x cos(\sqrt{t})dt; f(t)=\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{(2n)!}$ at a=0 (Note: f is the Taylor series of $cos(\sqrt{t})$.)

1.3E.7 Exercise $\mathbf{1.3E.7}$

In the following exercises, compute the Taylor series of each function around $x=1$.

1. $f(x)=2-x$

Answer

$2-x=1-(x-1)$

2. $f(x)=x^3$

3. $f(x)=(x-2)^2$

Answer

$((x-1)-1)^2=(x-1)^2-2(x-1)+1$

4. $f(x)=lnx$

5. $f(x)=\frac{1}{x}$

Answer

$$\frac{1}{1-(1-x)}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

6. $f(x)=\frac{1}{2x-x^2}$
7. $f(x)=\frac{x}{4x-2x^2-1}$

Answer

$$x\sum_{n=0}^{\infty}2^n(1-x)^{2n}=\sum_{n=0}^{\infty}2^n(x-1)^{2n+1}+\sum_{n=0}^{\infty}2^n(x-1)^{2n}$$

8. $f(x)=e^{-x}$
9. $f(x)=e^{2x}$

Answer

$$e^{2x}=e^{2(x-1)+2}=e^2\sum_{n=0}^{\infty}\frac{2^n(x-1)^n}{n!}$$

1.3E.8 Exercise $\mathbf{1.3}\boldsymbol{E}\mathbf{.8}$

In the following exercises, identify the value of *x* such that the given series $\sum_{n=0}^{\infty}a_n$ is the value of the Maclaurin series of $f(x)$ at x. Approximate the value of $f(x)$ using $S_{10}=\sum_{n=0}^{10}a_n$.

1. $\sum_{n=0}^{\infty}\frac{1}{n!}$
2. $sum_{n=0}^{\infty}\frac{2^n}{n!}$

Answer

$$x=e^2;S_{10}=\frac{34,913}{4725}\approx 7.3889947$$

3. $\sum_{n=0}^{\infty}\frac{(-1)^n(2\pi)^{2n}}{(2n)!}$
4. $\sum_{n=0}^{\infty}\frac{(-1)^n(2\pi)^{2n+1}}{(2n+1)!}$

Answer

$$sin(2\pi)=0;S_{10}=8.27\times 10^{-5}$$

1.3E.9 Exercise $\mathbf{1.3}\boldsymbol{E}\mathbf{.9}$

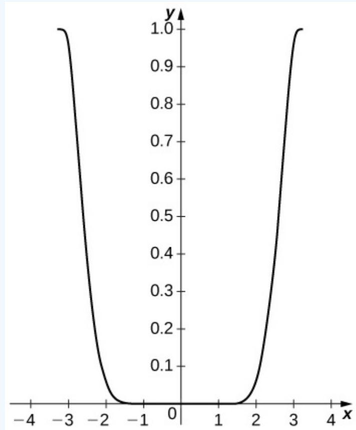
The following exercises make use of the functions $S_5(x)=x-\frac{x^3}{6}+\frac{x^5}{120}$ and $C_4(x)=1-\frac{x^2}{2}+\frac{x^4}{24}$ on $[-\pi,\pi]$.

1. Plot $sin^2x-(S_5(x))^2$ on $[-\pi,\pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $sinx$.

2. Plot $cos^2x-(C_4(x))^2$ on $[-\pi,\pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $cosx$.

Answer

The difference is small on the interior of the interval but approaches 1 near the endpoints. The remainder estimate is $|R_4| = \frac{\pi^5}{120} \approx 2.552$.

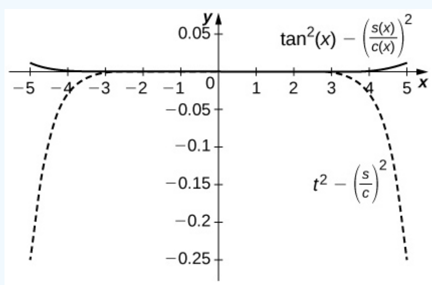


3. Plot $|2S_5(x)C_4(x) - sin(2x)|$ on $[-\pi, \pi]$.

4. Compare $\frac{S_5(x)}{C_4(x)}$ on $[-1, 1]$ to $tanx$. Compare this with the Taylor remainder estimate for the approximation of $tanx$ by $x + \frac{x^3}{3} + \frac{2x^5}{15}$.

Answer

The difference is on the order of 10^{-4} on $[-1, 1]$ while the Taylor approximation error is around 0.1 near ± 1. The top curve is a plot of $tan^2 x - (\frac{S_5(x)}{C_4(x)})^2$ and the lower dashed plot shows $t^2 - (\frac{S_5}{C_4})^2$.



5. Plot $e^x - e_4(x)$ where $e_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ on $[0, 2]$. Compare the maximum error with the Taylor remainder estimate.

1.3E.10 Exercise $\mathbf{1.3E.10}$

1. (Taylor approximations and root finding.) Recall that Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ approximates solutions of $f(x) = 0$ near the input x_0.

a. If f and g are inverse functions, explain why a solution of $g(x) = a$ is the value $f(a)$ of f.

b. Let $p_N(x)$ be the Nth degree Maclaurin polynomial of e^x. Use Newton's method to approximate solutions of $p_N(x) - 2 = 0$ for $N = 4, 5, 6$.

c. Explain why the approximate roots of $p_N(x) - 2 = 0$ are approximate values of $ln(2)$.

Answer

a. Answers will vary.

b. The following are the x_n values after 10 iterations of Newton's method to approximation a root of $p_N(x)-2=0$: for $N=4, x=0.6939...$; for $N=5, x=0.6932...$; for $N=6, x=0.69315...$; . (Note: $ln(2)=0.69314...$)

c. Answers will vary.

1.3E.11 Exercise $\mathbf{1.3E.11}$

In the following exercises, use the fact that if $q(x)=\sum_{n=1}^{\infty} a_n(x-c)^n$ converges in an interval containing c, then $\lim_{x \to c} q(x)=a_0$ to evaluate each limit using Taylor series.

1. $\lim_{x \to 0} \frac{cos x-1}{x^2}$
2. $\lim_{x \to 0} \frac{ln(1-x^2)}{x^2}$

Answer

$$\frac{ln(1-x^2)}{x^2} \to -1$$

3. $\lim_{x \to 0} \frac{e^{x^2}-x^2-1}{x^4}$
4. $\lim_{x \to 0^+} \frac{cos(\sqrt{x})-1}{2x}$

Answer

$$\frac{cos(\sqrt{x})-1}{2x} \approx \frac{(1-\frac{x}{2}+\frac{x^2}{4!}-\cdots)-1}{2x} \to -\frac{1}{4}$$

1.4: Working with Taylor Series

This page is a draft and is under active development.

In the preceding section, we defined Taylor series and showed how to find the Taylor series for several common functions by explicitly calculating the coefficients of the Taylor polynomials. In this section we show how to use those Taylor series to derive Taylor series for other functions. We then present two common applications of power series. First, we show how power series can be used to solve differential equations. Second, we show how power series can be used to evaluate integrals when the antiderivative of the integrand cannot be expressed in terms of elementary functions. In one example, we consider $\int e^{-x^2}dx$, an integral that arises frequently in probability theory.

1.4.1 The Binomial Series

Our first goal in this section is to determine the Maclaurin series for the function $f(x)=(1+x)^r$ for all real numbers r. The Maclaurin series for this function is known as the **binomial series.** We begin by considering the simplest case: r is a nonnegative integer. We recall that, for $r=0,1,2,3,4, f(x)=(1+x)^r$ can be written as

$$\begin{aligned} f(x)&=(1+x)^0=1,\\ f(x)&=(1+x)^1=1+x,\\ f(x)&=(1+x)^2=1+2x+x^2,\\ f(x)&=(1+x)^3=1+3x+3x^2+x^3\\ f(x)&=(1+x)^4=1+4x+6x^2+4x^3+x^4. \end{aligned} \tag{1.4.1}$$

The expressions on the right-hand side are known as binomial expansions and the coefficients are known as binomial coefficients. More generally, for any nonnegative integer r, the binomial coefficient of x^n in the binomial expansion of $(1+x)^r$ is given by

$$\binom{r}{n}=\frac{r!}{n!(r-n)!} \tag{1.4.2}$$

and

$$f(x)=(1+x)^r=\binom{r}{0}1+\binom{r}{1}x+\binom{r}{2}x^2+\binom{r}{3}x^3+\cdots+\binom{r}{r-1}x^{r-1}+\binom{r}{r}x^r=\sum_{n=0}^{r}\binom{r}{n}x^n. \tag{1.4.3}$$

For example, using this formula for $r=5$, we see that

$$\begin{aligned} f(x)&=(1+x)^5\\ &=\binom{5}{0}1+\binom{5}{1}x+\binom{5}{2}x^2+\binom{5}{3}x^3+\binom{5}{4}x^4+\binom{5}{5}x^5\\ &=\frac{5!}{0!5!}1+\frac{5!}{1!4!}x+\frac{5!}{2!3!}x^2+\frac{5!}{3!2!}x^3+\frac{5!}{4!1!}x^4+\frac{5!}{5!0!}x^5\\ &=1+5x+10x^2+10x^3+5x^4+x^5. \end{aligned} \tag{1.4.4}$$

We now consider the case when the exponent r

is any real number, not necessarily a nonnegative integer. If r is not a nonnegative integer, then $f(x)=(1+x)^r$ cannot be written as a finite polynomial. However, we can find a power series for f. Specifically, we look for the Maclaurin series for f. To do this, we find the derivatives of f and evaluate them at $x=0$.

$$f(x)=(1+x)^r \quad f(0)=1$$

$$f'(x)=r(1+x)^{r-1} \quad f'(0)=r$$

$$f''(x)=r(r-1)(1+x)^{r-2} \quad f''(0)=r(r-1)$$

$$f'''(x)=r(r-1)(r-2)(1+x)^{r-3} \quad f'''(0)=r(r-1)(r-2)$$

$$f(n)(x)=r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n} \quad f^{(n)}(0)=r(r-1)(r-2)\cdots(r-n+1)$$

We conclude that the coefficients in the binomial series are given by

$$\frac{f^{(n)}(0)}{n!}=\frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}. \tag{1.4.5}$$

We note that if r is a nonnegative integer, then the $(r+1)st$ derivative $f^{(r+1)}$ is the zero function, and the series terminates. In addition, if r is a nonnegative integer, then Equation for the coefficients agrees with Equation for the coefficients, and the formula for the binomial series agrees with Equation for the finite binomial expansion. More generally, to denote the binomial coefficients for any real number r, we define

$$\binom{r}{n} = \frac{(r-1)(r-2)\cdots(r-n+1)}{n!}. \tag{1.4.6}$$

With this notation, we can write the binomial series for $(1+x)^r$ as

$$\sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots. \tag{1.4.7}$$

We now need to determine the interval of convergence for the binomial series Equation. We apply the ratio test. Consequently, we consider

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|r(r-1)(r-2)\cdots(r-n)|x||^{n+1}}{(n+1)!} \cdot \frac{n}{|r(r-1)(r-2)\cdots(r-n+1)||x|^n} = \frac{|r-n||x|}{|n+1|} \tag{1.4.8}$$

.

Since

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x| < 1 \tag{1.4.9}$$

if and only if $|x|<1$, we conclude that the interval of convergence for the binomial series is $(-1,1)$. The behavior at the endpoints depends on r. It can be shown that for $r \geq 0$ the series converges at both endpoints; for $-1<r<0$, the series converges at $x=1$ and diverges at $x=-1$; and for $r<-1$, the series diverges at both endpoints. The binomial series does converge to $(1+x)^r$ in $(-1,1)$ for all real numbers r, but proving this fact by showing that the remainder $R_n(x)\to 0$ is difficult.

Definition: binomial series

For any real number r, the Maclaurin series for $f(x)=(1+x)^r$ is the binomial series. It converges to f for $|x|<1$, and we write

$$(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + r\frac{(r-1)\cdots(r-n+1)}{n!}x^n + \cdots \tag{1.4.10}$$

for $|x|<1$.

We can use this definition to find the binomial series for $f(x)=\sqrt{1+x}$ and use the series to approximate $\sqrt{1.5}$.

Example 1.4.1: Finding Binomial Series

a. Find the binomial series for $f(x)=\sqrt{1+x}$.
b. Use the third-order Maclaurin polynomial $p_3(x)$ to estimate $\sqrt{1.5}$. Use Taylor's theorem to bound the error. Use a graphing utility to compare the graphs of f and p_3.

Solution

a. Here $r=\frac{1}{2}$. Using the definition for the binomial series, we obtain

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{2!}\frac{1}{2^2}x^2 + \frac{1}{3!}\frac{1\cdot 3}{2^3}x^3 - \cdots + \frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n + \cdots \\ &= 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n.\end{aligned}$$

b. From the result in part a. the third-order Maclaurin polynomial is

$$p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 .$$

Therefore,

$$\sqrt{1.5}=\sqrt{1+0.5}\approx 1+\frac{1}{2}(0.5)-\frac{1}{8}(0.5)^2+\frac{1}{16}(0.5)^3\approx 1.2266.$$

From Taylor's theorem, the error satisfies

$$R_3(0.5)=\frac{f^{(4)}(c)}{4!}(0.5)^4$$

for some c between 0 and 0.5. Since $f^{(4)}(x)=-\frac{15}{2^4(1+x)^{7/2}}$, and the maximum value of $\left|f^{(4)}(x)\right|$ on the interval $(0,0.5)$ occurs at $x=0$, we have

$$|R_3(0.5)|\leq\frac{15}{4!2^4}(0.5)^4\approx 0.00244.$$

The function and the Maclaurin polynomial p_3 are graphed in Figure.

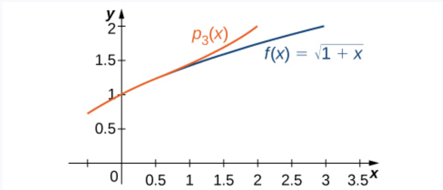


Figure 1.4.1: The third-order Maclaurin polynomial $p_3(x)$ provides a good approximation for $f(x)=\sqrt{1+x}$ for x near zero.

Exercise 1.4.1

Find the binomial series for $f(x)=\frac{1}{(1+x)^2}$.

Hint

Use the definition of binomial series for $r=-2$.S

Answer

$\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

1.4.1.1 Common Functions Expressed as Taylor Series

At this point, we have derived Maclaurin series for exponential, trigonometric, and logarithmic functions, as well as functions of the form $f(x)=(1+x)^r$. In Table, we summarize the results of these series. We remark that the convergence of the Maclaurin series for $f(x)=ln(1+x)$ at the endpoint $x=1$ and the Maclaurin series for $f(x)=tan^{-1}x$ at the endpoints $x=1$ and $x=-1$ relies on a more advanced theorem than we present here. (Refer to Abel's theorem for a discussion of this more technical point.)

Maclaurin Series for Common Functions

Function	Maclaurin Series	Interval of Convergence
$f(x)=\frac{1}{1-x}$	$\sum_{n=0}^{\infty}x^n$	$-1<x<1$
$f(x)=e^x$	$\sum_{n=0}^{\infty}\frac{x^n}{n!}$	$-\infty<x<\infty$
$f(x)=sinx$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	$-\infty<x<\infty$
\(\displaystyle f(x)=cosx\)	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$	$-\infty<x<\infty$
$f(x)=ln(1+x)$	$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n}$	$-1<x<1$
$f(x)=tan^{-1}x$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$	$-1<x<1$
$f(x)=(1+x)^r$	$\sum_{n=0}^{\infty}\binom{r}{n}x^n$	$-1<x<1$

Earlier in the chapter, we showed how you could combine power series to create new power series. Here we use these properties, combined with the Maclaurin series in Table, to create Maclaurin series for other functions.

Example 1.4.2: Deriving Maclaurin Series from Known Series

Find the Maclaurin series of each of the following functions by using one of the series listed in Table.

a. $f(x) = \cos\sqrt{x}$
b. $f(x) = sinhx$

Solution:

a. Using the Maclaurin series for $cosx$ we find that the Maclaurin series for $cos\sqrt{x}$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges to $cos\sqrt{x}$ for all x in the domain of $cos\sqrt{x}$; that is, for all $x \geq 0$.

b. To find the Maclaurin series for $sinhx$, we use the fact that

$$sinhx = \frac{e^x - e^{-x}}{2}.$$

Using the Maclaurin series for e^x, we see that the nth term in the Maclaurin series for $sinhx$ is given by

$$\frac{x^n}{n!} - \frac{(-x)^n}{n!}.$$

For n even, this term is zero. For n odd, this term is $\frac{2x^n}{n!}$. Therefore, the Maclaurin series for $sinhx$ has only odd-order terms and is given by

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

Exercise 1.4.2

Find the Maclaurin series for $sin(x^2)$.

Hint

Use the Maclaurin series for $sinx$.

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

We also showed previously in this chapter how power series can be differentiated term by term to create a new power series. In Example, we differentiate the binomial series for $\sqrt{1+x}$ term by term to find the binomial series for $\frac{1}{\sqrt{1+x}}$. Note that we could construct the binomial series for $\frac{1}{\sqrt{1+x}}$ directly from the definition, but differentiating the binomial series for $\sqrt{1+x}$ is an easier calculation.

Example 1.4.3: Differentiating a Series to Find a New Series

Use the binomial series for $\sqrt{1+x}$ to find the binomial series for $\frac{1}{\sqrt{1+x}}$.

Solution

The two functions are related by

$$\frac{d}{dx}\sqrt{1+x} = \frac{1}{2\sqrt{1+x}},$$

so the binomial series for $\frac{1}{\sqrt{1+x}}$ is given by

$$\frac{1}{\sqrt{1+x}} = 2\frac{d}{dx}\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n.$$

Exercise 1.4.3

Find the binomial series for $f(x) = \dfrac{1}{(1+x)^{3/2}}$

Hint

Differentiate the series for $\dfrac{1}{\sqrt{1+x}}$

Answer

$\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n!} \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$

In this example, we differentiated a known Taylor series to construct a Taylor series for another function. The ability to differentiate power series term by term makes them a powerful tool for solving differential equations. We now show how this is accomplished.

1.4.1.2 Solving Differential Equations with Power Series

Consider the differential equation

$$y'(x) = y. \tag{1.4.11}$$

Recall that this is a first-order separable equation and its solution is $y = Ce^x$. This equation is easily solved using techniques discussed earlier in the text. For most differential equations, however, we do not yet have analytical tools to solve them. Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$ and determine what the coefficients would need to be. In the next example, we consider an initial-value problem involving $y' = y$ to illustrate the technique.

Example 1.4.4: Power Series Solution of a Differential Equation

Use power series to solve the initial-value problem

$$y' = y, y(0) = 3. \tag{1.4.12}$$

Solution

Suppose that there exists a power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Differentiating this series term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots.$$

If *y* satisfies the differential equation, then

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = c_1 + 2c_2 x + 3c_3 x^2 + 4c_3 x^3 + \cdots.$$

Using [link] on the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$\begin{aligned} c_0 &= c_1, \\ c_1 &= 2c_2, \\ c_2 &= 3c_3, \\ c_3 &= 4c_4, \\ &\vdots. \end{aligned}$$

Using the initial condition $y(0) = 3$ combined with the power series representation

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

we find that $c_0 = 3$. We are now ready to solve for the rest of the coefficients. Using the fact that $c_0 = 3$, we have

$$\begin{aligned} c_1 &= c_0 = 3 = \frac{3}{1!}, \\ c_2 &= \frac{c_1}{2} = \frac{3}{2} = \frac{3}{2!}, \\ c_3 &= \frac{c_2}{3} = \frac{3}{3 \cdot 2} = \frac{3}{3!}, \end{aligned}$$

$$c_4 = \frac{c_3}{4} = \frac{3}{4 \cdot 3 \cdot 2} = \frac{3}{4!}.$$

Therefore,

$$y = 3[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \frac{1}{4!}x^4 + \cdots] = 3\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad .$$

You might recognize

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as the Taylor series for e^x. Therefore, the solution is $y = 3e^x$.

Exercise 1.4.4

Use power series to solve $y' = 2y, y(0) = 5$.

Hint

The equations for the first several coefficients c_n will satisfy $c_0 = 2c_1, c_1 = 2 \cdot 2c_2, c_2 = 2 \cdot 3c_3, \ldots.$ In general, for all $n \geq 0, c_n = 2(n+1)C_{n+1}$.

Answer

$y = 5e^{2x}$

We now consider an example involving a differential equation that we cannot solve using previously discussed methods. This differential equation

$$y' - xy = 0 \tag{1.4.13}$$

is known as **Airy's equation**. It has many applications in mathematical physics, such as modeling the diffraction of light. Here we show how to solve it using power series.

Example 1.4.5: Power Series Solution of Airy's Equation

Use power series to solve

$$y'' - xy = 0 \tag{1.4.14}$$

with the initial conditions $y(0) = a$ and $y'(0) = b$.

Solution

We look for a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \quad .$$

Differentiating this function term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots,$$
$$y'' = 2 \cdot 1c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + \cdots.$$

If *y* satisfies the equation $y'' = xy$, then

$$2 \cdot 1c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + \cdots = x(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots).$$

Using [link] on the uniqueness of power series representations, we know that coefficients of the same degree must be equal. Therefore,

$$2 \cdot 1c_2 = 0,$$
$$3 \cdot 2c_3 = c_0,$$
$$4 \cdot 3c_4 = c_1,$$
$$5 \cdot 4c_5 = c_2,$$
$$\vdots .$$

More generally, for $n \geq 3$, we have $n \cdot (n-1)c_n = c_{n-3}$. In fact, all coefficients can be written in terms of c_0 and c_1. To see this, first note that $c_2 = 0$. Then

$$c_3 = \frac{c_0}{3 \cdot 2},$$
$$c_4 = \frac{c_1}{4 \cdot 3}.$$

For c_5, c_6, c_7, we see that

$$c_5 = \frac{c_2}{5 \cdot 4} = 0,$$
$$c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2},$$
$$c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}.$$

Therefore, the series solution of the differential equation is given by

$$y = c_0 + c_1 x + 0 \cdot x^2 + \frac{c_0}{3 \cdot 2} x^3 + \frac{c_1}{4 \cdot 3} x^4 + 0 \cdot x^5 + \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2} x^6 + \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3} x^7 + \cdots.$$

The initial condition $y(0) = a$ implies $c_0 = a$. Differentiating this series term by term and using the fact that $y'(0) = b$, we conclude that $c_1 = b$.

Therefore, the solution of this initial-value problem is

$$y = a\left(1 + \frac{x^3}{3 \cdot 2} + \frac{x}{6 \cdot 5 \cdot 3 \cdot 2} + \cdots\right) + b\left(x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \cdots\right).$$

Exercise 1.4.5

Use power series to solve $y'' + x^2 y = 0$ with the initial condition $y(0) = a$ and $y'(0) = b$.

Hint

The coefficients satisfy $c_0 = a, c_1 = b, c_2 = 0, c_3 = 0,$ and for $n \geq 4, n(n-1)c_n = -c_{n-4}$.

Answer

$$y = a\left(1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \cdots\right) + b\left(x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \cdots\right)$$

1.4.1.3 Evaluating Nonelementary Integrals

Solving differential equations is one common application of power series. We now turn to a second application. We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions.

One integral that arises often in applications in probability theory is $\int e^{-x^2} dx$. Unfortunately, the antiderivative of the integrand e^{-x^2} is not an elementary function. By elementary function, we mean a function that can be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. We remark that the term "elementary function" is not synonymous with noncomplicated function. For example, the function $f(x) = \sqrt{x^2 - 3x} + e^{x^3} - sin(5x + 4)$ is an elementary function, although not a particularly simple-looking function. Any integral of the form $\int f(x) dx$ where the antiderivative of f cannot be written as an elementary function is considered a **nonelementary integral.**

Nonelementary integrals cannot be evaluated using the basic integration techniques discussed earlier. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term. We demonstrate this technique by considering $\int e^{-x^2} dx$.

Example 1.4.6: Using Taylor Series to Evaluate a Definite Integral

a. Express $\int e^{-x^2} dx$ as an infinite series.
b. Evaluate $\int_0^1 e^{-x^2} dx$ to within an error of 0.01.

Solution

a. The Maclaurin series for e^{-x^2} is given by

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}.$$

Therefore,

$$\int e^{-x^2}dx = \int(1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots+(-1)^n\frac{x^{2n}}{n!}+\cdots)dx = C+x-\frac{x^3}{3}+\frac{x^5}{5.2!}-\frac{x^7}{7.3!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)n!}+\cdots.$$

b. Using the result from part a. we have

$$\int_0^1 e^{-x^2}dx = 1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}+\frac{1}{216}-\cdots.$$

The sum of the first four terms is approximately 0.74. By the alternating series test, this estimate is accurate to within an error of less than $\frac{1}{216} \approx 0.0046296 < 0.01$.

Exercise 1.4.6

Express $\int cos\sqrt{x}dx$ as an infinite series. Evaluate $\int_0^1 cos\sqrt{x}dx$ to within an error of 0.01.

Hint

Use the series found in Example.

Answer

$C+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n(2n-2)!}$ The definite integral is approximately 0.514 to within an error of 0.01.

As mentioned above, the integral $\int e^{-x^2}dx$ arises often in probability theory. Specifically, it is used when studying data sets that are normally distributed, meaning the data values lie under a bell-shaped curve. For example, if a set of data values is normally distributed with mean μ and standard deviation σ, then the probability that a randomly chosen value lies between $x=a$ and $x=b$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}}\int_a^b e^{-(x-\mu)^2/(2\sigma^2)}dx. \tag{1.4.15}$$

(See Figure.)

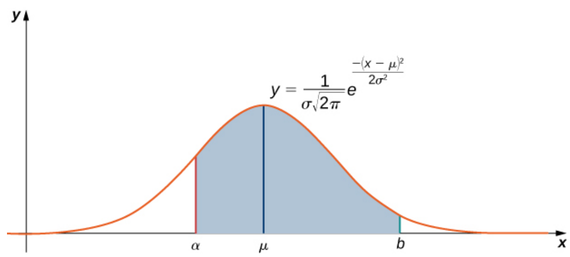


Figure 1.4.2: If data values are normally distributed with mean μ and standard deviation σ, the probability that a randomly selected data value is between a and b is the area under the curve $y=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$ between $x=a$ and $x=b$.

To simplify this integral, we typically let $z=\frac{x-\mu}{\sigma}$. This quantity z is known as the z score of a data value. With this simplification, integral Equation becomes

$$\frac{1}{\sqrt{2\pi}}\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-z^2/2}dz. \tag{1.4.16}$$

In Example, we show how we can use this integral in calculating probabilities.

Example 1.4.7: Using Maclaurin Series to Approximate a Probability

Suppose a set of standardized test scores are normally distributed with mean $\mu=100$ and standard deviation $\sigma=50$. Use Equation and the first six terms in the Maclaurin series for $e^{-x^2/2}$ to approximate the probability that a randomly selected test score is between $x=100$ and $x=200$. Use the alternating series test to determine how accurate your approximation is.

Solution

Since $\mu=100, \sigma=50,$ and we are trying to determine the area under the curve from $a=100$ to $b=200$, integral Equation becomes

$$\frac{1}{\sqrt{2\pi}}\int_0^2 e^{-z^2/2}dz.$$

The Maclaurin series for $e^{-x^2/2}$ is given by

$$e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-\frac{x^2}{2})^n}{n!} = 1 - \frac{x^2}{2^1 \cdot 1!} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n}! + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!} \quad .$$

Therefore,

$$\frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int (1 - \frac{z^2}{2^1 \cdot 1!} + \frac{z^4}{2^2 \cdot 2!} - \frac{z^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n}}{2^n \cdot n!} + \cdots) dz$$
$$= \frac{1}{\sqrt{2\pi}} (C + z - \frac{z^3}{3 \cdot 2^1 \cdot 1!} + \frac{z^5}{5 \cdot 2^2 \cdot 2!} - \frac{z^7}{7 \cdot 2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)2^n \cdot n!} + \cdots)$$
$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} (2 - \frac{8}{6} + \frac{32}{40} - \frac{128}{336} + \frac{512}{3456} - \frac{2^{11}}{11 \cdot 2^5 \cdot 5!} + \cdots)$$

Using the first five terms, we estimate that the probability is approximately **0.4922.** By the alternating series test, we see that this estimate is accurate to within

$$\frac{1}{\sqrt{2\pi}} \frac{2^{13}}{13 \cdot 2^6 \cdot 6!} \approx 0.00546. \tag{1.4.17}$$

Analysis

If you are familiar with probability theory, you may know that the probability that a data value is within two standard deviations of the mean is approximately 95 Here we calculated the probability that a data value is between the mean and two standard deviations above the mean, so the estimate should be around 47.5. The estimate, combined with the bound on the accuracy, falls within this range.

Exercise 1.4.7

Use the first five terms of the Maclaurin series for $e^{-x^2/2}$ to estimate the probability that a randomly selected test score is between 100 and 150. Use the alternating series test to determine the accuracy of this estimate.

Hint

Evaluate $\int_0^1 e^{-z^2/2} dz$ using the first five terms of the Maclaurin series for $e^{-z^2/2}$.

Answer

The estimate is approximately 0.3414. This estimate is accurate to within 0.0000094.

Another application in which a nonelementary integral arises involves the period of a pendulum. The integral is

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 sin^2 \theta}} \tag{1.4.18}$$

.

An integral of this form is known as an **elliptic integral** of the first kind. Elliptic integrals originally arose when trying to calculate the arc length of an ellipse. We now show how to use power series to approximate this integral.

Example 1.4.8: Period of a Pendulum

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle θ_{max} with the vertical, its period T is given by

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 sin^2 \theta}}$$

where g is the acceleration due to gravity and $k = sin(\frac{\theta_{max}}{2})$ (see Figure). (We note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized model is used and $sin\theta$ is approximated by θ.)

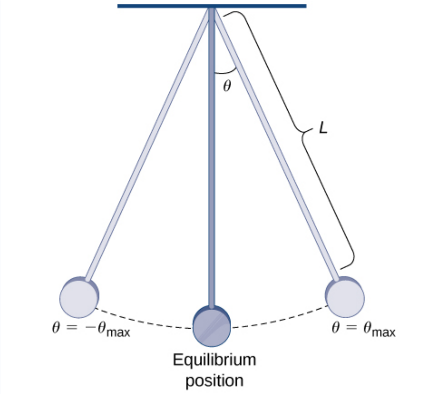


Figure 1.4.3: This pendulum has length L and makes a maximum angle θ_{max} with the vertical.

Use the binomial series

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$$

to estimate the period of this pendulum. Specifically, approximate the period of the pendulum if

a. you use only the first term in the binomial series, and
b. you use the first two terms in the binomial series.

Solution

We use the binomial series, replacing x with $-k^2 sin^2\theta$. Then we can write the period as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} (1 + \frac{1}{2}k^2 sin^2\theta + \frac{1 \cdot 3}{2!2^2} k^4 sin^4\theta + \cdots) d\theta.$$

a. Using just the first term in the integrand, the first-order estimate is

$$T \approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} d\theta = 2\pi\sqrt{\frac{L}{g}}.$$

If θ_{max} is small, then $k = sin(\frac{\theta_{max}}{2})$ is small. We claim that when k is small, this is a good estimate. To justify this claim, consider

$$\int_0^{\pi/2} (1 + \frac{1}{2}k^2 sin^2\theta + \frac{1 \cdot 3}{2!2^2} k^4 sin^4\theta + \cdots) d\theta.$$

Since $|sinx| \leq 1$, this integral is bounded by

$$\int_0^{\pi/2} (\frac{1}{2}k^2 + \frac{1.3}{2!2^2} k^4 + \cdots) d\theta < \frac{\pi}{2} (\frac{1}{2}k^2 + \frac{1 \cdot 3}{2!2^2} k^4 + \cdots).$$

Furthermore, it can be shown that each coefficient on the right-hand side is less than 1 and, therefore, that this expression is bounded by

$$\frac{\pi k^2}{2}(1 + k^2 + k^4 + \cdots) = \frac{\pi k^2}{2} \cdot \frac{1}{1-k^2},$$

which is small for k small.

b. For larger values of θ_{max}, we can approximate T by using more terms in the integrand. By using the first two terms in the integral, we arrive at the estimate

$$T \approx 4\sqrt{fracLg} \int_0^{\pi/2} (1 + \frac{1}{2}k^2 sin^2\theta) d\theta = 2\pi\sqrt{\frac{L}{g}}(1 + \frac{k^2}{4}).$$

The applications of Taylor series in this section are intended to highlight their importance. In general, Taylor series are useful because they allow us to represent known functions using polynomials, thus providing us a tool for approximating function values and estimating complicated integrals. In addition, they allow us to define new functions as power series, thus providing us with a powerful tool for solving differential equations.

1.4.2 Key Concepts

- The binomial series is the Maclaurin series for $f(x)=(1+x)^r$. It converges for $|x|<1$.
- Taylor series for functions can often be derived by algebraic operations with a known Taylor series or by differentiating or integrating a known Taylor series.
- Power series can be used to solve differential equations.
- Taylor series can be used to help approximate integrals that cannot be evaluated by other means.

1.4.3 Glossary

binomial series

the Maclaurin series for $f(x)=(1+x)^r$; it is given by

$(1+x)^r=\sum_{n=0}^{\infty}\binom{r}{n}x^n=1+rx+\frac{r(r-1)}{2!}x^2+\cdots+\frac{r(r-1)\cdots(r-n+1)}{n!}x^n+\cdots$ for $|x|<1$

nonelementary integral

an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

1.4.3.1 Contributors

-

Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

1.4E: Exercises

This page is a draft and is under active development.

1.4E.1 Exercise $1.4E.1$

In the following exercises, use appropriate substitutions to write down the Maclaurin series for the given binomial.

1. $(1-x)^{1/3}$
2. $(1+x^2)^{-1/3}$

Answer

$$(1+x^2)^{-1/3} = \sum_{n=0}^{\infty} \binom{-\frac{1}{3}}{n} x^{2n}$$

3. $(1-x)^{1.01}$
4. $(1-2x)^{2/3}$

Answer

$$(1-2x)^{2/3} = \sum_{n=0}^{\infty} (-1)^n 2^n \binom{\frac{2}{3}}{n} x^n$$

1.4E.2 Exercise $1.4E.2$

In the following exercises, use the substitution $(b+x)^r = (b+a)^r (1+\frac{x-a}{b+a})^r$ in the binomial expansion to find the Taylor series of each function with the given center.

1. $\sqrt{x+2}$ at $a=0$
2. $\sqrt{x^2+2}$ at $a=0$

Answer

$$\sqrt{2+x^2} = \sum_{n=0}^{\infty} 2^{(1/2)-n} \binom{\frac{1}{2}}{n} x^{2n}; (\mid x^2 \mid < 2)$$

3. $\sqrt{x+2}$ at $a=1$
4. $\sqrt{2x-x^2}$ at $a=1$ (Hint: $2x-x^2 = 1-(x-1)^2$)

Answer

$$\sqrt{2x-x^2} = \sqrt{1-(x-1)^2} \text{ so } \sqrt{2x-x^2} = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} (x-1)^{2n}$$

5. $(x-8)^{1/3}$ at $a=9$
6. $\sqrt{x}$ at $a=4$

Answer

$$\sqrt{x} = 2\sqrt{1+\frac{x-4}{4}} \text{ so } \sqrt{x} = \sum_{n=0}^{\infty} 2^{1-2n} \binom{\frac{1}{2}}{n} (x-4)^n$$

7. $x^{1/3}$ at $a=27$
8. $\sqrt{x}$ at $x=9$

Answer

$$\sqrt{x} = \sum_{n=0}^{\infty} 3^{1-3n} \binom{\frac{1}{2}}{n} (x-9)^n$$

1.4E.3 Exercise $1.4E.3$

In the following exercises, use the binomial theorem to estimate each number, computing enough terms to obtain an estimate accurate to an error of at most $1/1000$.

1. $(15)^{1/4}$ using $(16-x)^{1/4}$
2. $(1001)^{1/3}$ using $(1000+x)^{1/3}$

Answer

$10(1+\frac{x}{1000})^{1/3}=\sum_{n=0}^{\infty}10^{1-3n}\binom{\frac{1}{3}}{n}x^n$. Using, for example, a fourth-degree estimate at $x=1$ gives $(1001)^{1/3}\approx 10(1+\binom{\frac{1}{3}}{1}10^{-3}+\binom{\frac{1}{3}}{2}10^{-6}+\binom{\frac{1}{3}}{3}10^{-9}+\binom{\frac{1}{3}}{4}10^{-12})=10(1+\frac{1}{3.10^3}-\frac{1}{9.10^6}+\frac{5}{81.10^9}-\frac{10}{243.10^{12}})=10.00333222...$ whereas $(1001)^{1/3}=10.00332222839093....$Two terms would suffice for three-digit accuracy.

1.4E.4 Exercise $\mathbf{1.4E.4}$

In the following exercises, use the binomial approximation $\sqrt{1-x}\approx 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\frac{7x^5}{256}$ for $|x|<1$ to approximate each number. Compare this value to the value given by a scientific calculator.

1. $\frac{1}{\sqrt{2}}$ using $x=\frac{1}{2}$ in $(1-x)^{1/2}$
2. $\sqrt{5}=5\times\frac{1}{\sqrt{5}}$ using $x=\frac{4}{5}$ in $(1-x)^{1/2}$

Answer

The approximation is 2.3152; the CAS value is $2.23\ldots$.

3. $\sqrt{3}=\frac{3}{\sqrt{3}}$ using $x=\frac{2}{3}$ in $(1-x)^{1/2}$
4. $\sqrt{6}$ using $x=\frac{5}{6}$ in $(1-x)^{1/2}$

Answer

The approximation is $2.583\ldots$; the CAS value is $2.449\ldots$.

5. Integrate the binomial approximation of $\sqrt{1-x}$ to find an approximation of $\int_0^x\sqrt{1-t}\,dt$.
6. Recall that the graph of $\sqrt{1-x^2}$ is an upper semicircle of radius 1. Integrate the binomial approximation of $\sqrt{1-x^2}$ up to order 8 from $x=-1$ to $x=1$ to estimate $\frac{\pi}{2}$.

Answer

$\sqrt{1-x^2}=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}+\cdots$. Thus $\int_{-1}^{1}\sqrt{1-x^2}\,dx=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{7\cdot 16}-\frac{5x^9}{9\cdot 128}+\cdots|_{-1}^{1}\approx 2-\frac{1}{3}-\frac{1}{20}-\frac{1}{56}-\frac{10}{9\cdot 128}+error=1.590...$ whereas $\frac{\pi}{2}=1.570...$

1.4E.5 Exercise $\mathbf{1.4E.5}$

In the following exercises, use the expansion $(1+x)^{1/3}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$ to write the first five terms (not necessarily a quartic polynomial) of each expression.

1. $(1+4x)^{1/3};a=0$
2. $(1+4x)^{4/3};a=0$

Answer

$(1+x)^{4/3}=(1+x)(1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots)=1+\frac{4x}{3}+\frac{2x^2}{9}-\frac{4x^3}{81}+\frac{5x^4}{243}+\cdots$

3. $(3+2x)^{1/3};a=-1$
4. $(x^2+6x+10)^{1/3};a=-3$

Answer

$(1+(x+3)^2)^{1/3} = 1+\frac{1}{3}(x+3)^2 - \frac{1}{9}(x+3)^4 + \frac{5}{81}(x+3)^6 - \frac{10}{243}(x+3)^8 + \cdots$

5. Use $(1+x)^{1/3} = 1+\frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots$ with $x=1$ to approximate $2^{1/3}$.

6. Use the approximation $(1-x)^{2/3} = 1-\frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} + \cdots$ for $|x|<1$ to approximate $2^{1/3} = 2.2^{-2/3}$.

Answer

Twice the approximation is $1.260\ldots$ whereas $2^{1/3} = 1.2599\ldots$

7. Find the $25th$ derivative of $f(x) = (1+x^2)^{13}$ at $x=0$.

8. Find the 99 th derivative of $f(x) = (1+x^4)^{25}$.

Answer

$f^{(99)}(0) = 0$

1.4E.6 Exercise $\mathbf{1.4E.6}$

In the following exercises, find the Maclaurin series of each function.

1. $f(x) = xe^{2x}$

2. $f(x) = 2^x$

Answer

$$\sum_{n=0}^{\infty} \frac{(ln(2)x)^n}{n!}$$

3. $f(x) = \frac{sinx}{x}$

4. $f(x) = \frac{sin(\sqrt{x})}{\sqrt{x}}, (x>0),$

Answer

For $x>0, sin(\sqrt{x}) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{(2n+1)/2}}{\sqrt{x}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{(2n+1)!}$.

5. $f(x) = sin(x^2)$

6. $f(x) = e^{x^3}$

Answer

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

7. $f(x) = cos^2 x$ using the identity $cos^2 x = \frac{1}{2} + \frac{1}{2}cos(2x)$

8. $f(x) = sin^2 x$ using the identity $sin^2 x = \frac{1}{2} - \frac{1}{2}cos(2x)$

Answer

$$sin^2 x = -\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} x^{2k}}{(2k)!}$$

1.4E.7 Exercise $\mathbf{1.4E.7}$

In the following exercises, find the Maclaurin series of $F(x) = \int_0^x f(t)dt$ by integrating the Maclaurin series of f term by term. If f is not strictly defined at zero, you may substitute the value of the Maclaurin series at zero.

1. $F(x) = \int_0^x e^{-t^2} dt; f(t) = e^{-t^2} = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{n!}$

2. $F(x) = tan^{-1}x; f(t) = \frac{1}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n t^{2n}$

Answer

$$tan^{-1}x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1}$$

3. $F(x) = tanh^{-1}x; f(t) = \frac{1}{1-t^2} = \sum_{n=0}^{\infty} t^{2n}$

4. $F(x) = sin^{-1}x; f(t) = \frac{1}{\sqrt{1-t^2}} = \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{t^{2k}}{k!}$

Answer

$$sin^{-1}x = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\frac{x^{2n+1}}{(2n+1)n!}$$

5. $F(x) = \int_0^x \frac{sint}{t}dt; f(t) = \frac{sint}{t} = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n+1)!}$

6. $F(x) = \int_0^x cos(\sqrt{t})dt; f(t) = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(2n)!}$

Answer

$$F(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{(n+1)(2n)!}$$

7. $F(x) = \int_0^x \frac{1-cost}{t^2}dt; f(t) = \frac{1-cost}{t^2} = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n+2)!}$

8. $F(x) = \int_0^x \frac{ln(1+t)}{t}dt; f(t) = \sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n+1}$

Answer

$$F(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n^2}$$

1.4E.8 Exercise $\mathbf{1.4E.8}$

In the following exercises, compute at least the first three nonzero terms (not necessarily a quadratic polynomial) of the Maclaurin series of f.

1. $f(x) = sin(x+\frac{\pi}{4}) = sinxcos(\frac{\pi}{4}) + cosxsin(\frac{\pi}{4})$

2. $f(x) = tanx$

Answer

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

3. $f(x) = ln(cosx)$

4. $f(x) = e^x cosx$

Answer

$$1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

5. $f(x) = e^{sinx}$

6. $f(x) = sec^2x$

Answer

$$1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \cdots$$

7. $f(x) = tanhx$

8. $f(x) = \dfrac{tan\sqrt{x}}{\sqrt{x}}$ (see expansion for $tanx$)

Answer

Using the expansion for $tanx$ gives $1 + \dfrac{x}{3} + \dfrac{2x^2}{15}$.

1.4E.9 Exercise $\mathbf{1.4E.9}$

In the following exercises, find the radius of convergence of the Maclaurin series of each function.

1. $ln(1+x)$

2. $\dfrac{1}{1+x^2}$

Answer

$\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ so $R = 1$ by the ratio test.

3. $tan^{-1}x$

4. $ln(1+x^2)$

Answer

$ln(1+x^2) = \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n}x^{2n}$ so $R = 1$ by the ratio test.

5. Find the Maclaurin series of $sinhx = \dfrac{e^x - e^{-x}}{2}$.

6. Find the Maclaurin series of $coshx = \dfrac{e^x + e^{-x}}{2}$.

Answer

Add series of e^x and e^{-x} term by term. Odd terms cancel and $coshx = \sum_{n=0}^{\infty}\dfrac{x^{2n}}{(2n)!}$.

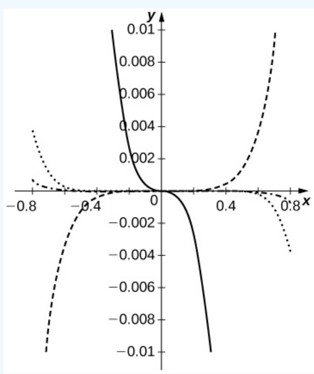
1.4E.10 Exercise $\mathbf{1.4E.10}$

1. Differentiate term by term the Maclaurin series of $sinhx$ and compare the result with the Maclaurin series of $coshx$.

2. Let $S_n(x) = \sum_{k=0}^{n}(-1)^k\dfrac{x^{2k+1}}{(2k+1)!}$ and $C_n(x) = \sum_{n=0}^{n}(-1)^k\dfrac{x^{2k}}{(2k)!}$ denote the respective Maclaurin polynomials of degree $2n+1$ of $sinx$ and degree $2n$ of $cosx$. Plot the errors $\dfrac{S_n(x)}{C_n(x)} - tanx$ for $n = 1, .., 5$ and compare them to $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} - tanx$ on $(-\dfrac{\pi}{4}, \dfrac{\pi}{4})$.

Answer

The ratio $\dfrac{S_n(x)}{C_n(x)}$ approximates $tanx$ better than does $p_7(x) = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315}$ for $N \geq 3$. The dashed curves are $\dfrac{S_n}{C_n} - tan$ for $n = 1, 2$. The dotted curve corresponds to $n = 3$, and the dash-dotted curve corresponds to $n = 4$. The solid curve is $p_7 - tanx$.



3. Use the identity $2sinxcosx = sin(2x)$ to find the power series expansion of sin^2x at $x = 0$. (Hint: Integrate the Maclaurin series of $sin(2x)$ term by term.)

4. If $y = \sum_{n=0}^{\infty} a_n x^n$, find the power series expansions of xy' and x^2y''.

Answer

By the term-by-term differentiation theorem, $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$ so $y' = \sum_{n=1}^{\infty} na_n x^{n-1} xy' = \sum_{n=1}^{\infty} na_n x^n$, whereas $y' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ so $xy'' = \sum_{n=2}^{\infty} n(n-1)a_n x^n$.

5. Suppose that $y = \sum_{k=0}^{\infty} a^k x^k$ satisfies $y' = -2xy$ and $y(0) = 0$. Show that $a_{2k+1} = 0$ for all k and that $a_{2k+2} = \frac{-a_{2k}}{k+1}$. Plot the partial sum S_{20} of y on the interval $[-4, 4]$.

6. Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 10$. Set up an integral that represents the probability that a test score will be between 90 and 110 and use the integral of the degree 10 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ to estimate this probability.

Answer

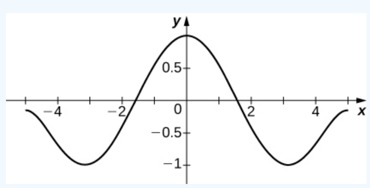
The probability is $p = \frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-x^2/2} dx$ where $a = 90$ and $b = 100$, that is, $p = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sum_{n=0}^{5} (-1)^n \frac{x^{2n}}{2^n n!} dx = \frac{2}{\sqrt{2\pi}} \sum_{n=0}^{5} (-1)^n \frac{1}{(2n+1)2^n n!} \approx 0.6827.$

7. Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 10$. Set up an integral that represents the probability that a test score will be between 70 and 130 and use the integral of the degree 50 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ to estimate this probability.

8. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 1, f'(0) = 0$, and $f''(x) = -f(x)$. Find a formula for a_n and plot the partial sum S_N for $N = 20$ on $[-5, 5]$.

Answer

As in the previous problem one obtains $a_n = 0$ if n is odd and $a_n = -(n+2)(n+1)a_{n+2}$ if n is even, so $a_0 = 1$ leads to $a_{2n} = \frac{(-1)^n}{(2n)!}$.



9. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 0, f'(0) = 1$, and $f''(x) = -f(x)$. Find a formula for an and plot the partial sum S_N for $N = 10$ on $[-5, 5]$.

10. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y' + y = 0$ where $y(0) = 1$ and $y'(0) = 0$. Find a formula that relates a_{n+2}, a_{n+1}, and an and compute $a_0, \ldots, a_5$.

Answer

$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$ and $y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$ so $y'' - y' + y = 0$ implies that $(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n = 0$ or $a_n = \frac{a_{n-1}}{n} - \frac{a_{n-2}}{n(n-1)}$ for all $n \cdot y(0) = a_0 = 1$ and $y'(0) = a_1 = 0,$ so $a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = 0$, and $a_5 = -\frac{1}{120}$.

11. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y' + y = 0$ where $y(0) = 0$ and $y'(0) = 1$. Find a formula that relates a_{n+2}, a_{n+1}, and an and compute $a_1, \ldots, a_5$.

The error in approximating the integral $\int_a^b f(t)dt$ by that of a Taylor approximation $\int_a^b Pn(t)dt$ is at most $\int_a^b R_n(t)dt$. In the following exercises, the Taylor remainder estimate $R_n \le \frac{M}{(n+1)!}|x-a|^{n+1}$ guarantees that the integral of the Taylor polynomial of the given order approximates the integral of f with an error less than $\frac{1}{10}$.

a. Evaluate the integral of the appropriate Taylor polynomial and verify that it approximates the CAS value with an error less than $\frac{1}{100}$.

b. Compare the accuracy of the polynomial integral estimate with the remainder estimate.

12. $\int_0^{\pi} \frac{sint}{t}dt; P_s = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$ (You may assume that the absolute value of the ninth derivative of $\frac{sint}{t}$ is bounded by 0.1.)

Answer

a. (Proof)

b. We have $R_s \le \frac{0.1}{(9)!}\pi^9 \approx 0.0082 < 0.01$. We have $\int_0^{\pi}(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!})dx = \pi - \frac{\pi^3}{3\cdot 3!} + \frac{\pi^5}{5\cdot 5!} - \frac{\pi^7}{7\cdot 7!} + \frac{\pi^9}{9\cdot 9!} = 1.852\ldots$, whereas $\int_0^{\pi}\frac{sint}{t}dt = 1.85194\ldots$, so the actual error is approximately 0.00006.

13. $\int_0^2 e^{-x^2}dx; p_{11} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots - \frac{x^{22}}{11!}$ (You may assume that the absolute value of the $23rd$ derivative of e^{-x^2} is less than 2×10^{14}.

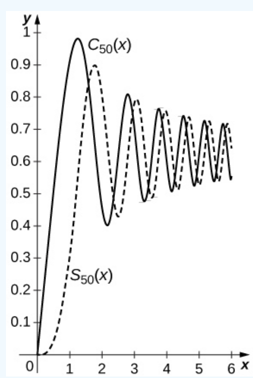
1.4E.11 Exercise $\mathbf{1.4E.11}$

The following exercises deal with **Fresnel integrals.**

1. The Fresnel integrals are defined by $C(x) = \int_0^x cos(t^2)dt$ and $S(x) = \int_0^x sin(t^2)dt$. Compute the power series of $C(x)$ and $S(x)$ and plot the sums $C_N(x)$ and $S_N(x)$ of the first $N = 50$ nonzero terms on $[0, 2\pi]$.

Answer

Since $cos(t^2) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{4n}}{(2n)!}$ and $sin(t^2) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{4n+2}}{(2n+1)!}$, one has $S(x) =_s um_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!}$ and $C(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!}$. The sums of the first 50 nonzero terms are plotted below with $C_{50}(x)$ the solid curve and $S_{50}(x)$ the dashed curve.



2. The Fresnel integrals are used in design applications for roadways and railways and other applications because of the curvature properties of the curve with coordinates $(C(t), S(t))$. Plot the curve (C_{50}, S_{50}) for $0 \le t \le 2\pi$, the coordinates of which were computed in the previous exercise.

1.4E.12 Exercise $\mathbf{1.4E.12}$

1. Estimate $\int_0^{1/4}\sqrt{x-x^2}dx$ by approximating $\sqrt{1-x}$ using the binomial approximation $1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{2128}-\frac{7x^5}{256}$.

Answer

$\int_0^{1/4}\sqrt{x}(1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\frac{7x^5}{256})dx=\frac{2}{3}2^{-3}-\frac{1}{2}\frac{2}{5}2^{-5}-\frac{1}{8}\frac{2}{7}2^{-7}-\frac{1}{16}\frac{2}{9}2^{-9}-\frac{5}{128}\frac{2}{11}2^{-11}-\frac{7}{256}\frac{2}{13}2^{-13}=0.0767732\ldots$ whereas $\int_0^{1/4}\sqrt{x-x^2}dx=0.076773.$

2. Use Newton's approximation of the binomial $\sqrt{1-x^2}$ to approximate π as follows. The circle centered at $(\frac{1}{2},0)$ with radius $\frac{1}{2}$ has upper semicircle $y=\sqrt{x}\sqrt{1-x}$. The sector of this circle bounded by the x-axis between $x=0$ and $x=\frac{1}{2}$ and by the line joining $(\frac{1}{4},\frac{\sqrt{3}}{4})$ corresponds to $\frac{1}{6}$ of the circle and has area $\frac{\pi}{24}$. This sector is the union of a right triangle with height $\frac{\sqrt{3}}{4}$ and base $\frac{1}{4}$ and the region below the graph between $x=0$ and $x=\frac{1}{4}$. To find the area of this region you can write $y=\sqrt{x}\sqrt{1-x}=\sqrt{x}\times(\text{binomial expansion of}\sqrt{1-x})$ and integrate term by term. Use this approach with the binomial approximation from the previous exercise to estimate π.

3. Use the approximation $T\approx 2\pi\sqrt{\frac{L}{g}}(1+\frac{k^2}{4})$ to approximate the period of a pendulum having length 10 meters and maximum angle $\theta_{max}=\frac{\pi}{6}$ where $k=sin(\frac{\theta_{max}}{2})$. Compare this with the small angle estimate $T\approx 2\pi\sqrt{\frac{L}{g}}$.

Answer

$T\approx 2\pi\sqrt{\frac{10}{9.8}}(1+\frac{sin^2(\theta/12)}{4})\approx 6.453$ seconds. The small angle estimate is $T\approx 2\pi\sqrt{\frac{10}{9.8}\approx 6.347}$. The relative error is around 2 percent.

4. Suppose that a pendulum is to have a period of 2 seconds and a maximum angle of $\theta_{max}=\frac{\pi}{6}$. Use $T\approx 2\pi\sqrt{\frac{L}{g}}(1+\frac{k^2}{4})$ to approximate the desired length of the pendulum. What length is predicted by the small angle estimate $T\approx 2\pi\sqrt{\frac{L}{g}}$?

5. Evaluate $\int_0^{\pi/2}sin^4\theta d\theta$ in the approximation $T=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}(1+\frac{1}{2}k^2sin^2\theta+\frac{3}{8}k^4sin^4\theta+\cdots)d\theta$ to obtain an improved estimate for T.

Answer

$\int_0^{\pi/2}sin^4\theta d\theta=\frac{3\pi}{16}$. Hence $T\approx 2\pi\sqrt{\frac{L}{g}}(1+\frac{k^2}{4}+\frac{9}{256}k^4)$.

6. An equivalent formula for the period of a pendulum with amplitude θ_max is $T(\theta_{max})=2\sqrt{2}\sqrt{\frac{L}{g}}\int_0^{\theta_{max}}\frac{d\theta}{\sqrt{cos\theta-cos(\theta_{max})}}$ where L is the pendulum length and g is the gravitational acceleration constant. When $\theta_{max}=\frac{\pi}{3}$ we get $\frac{1}{\sqrt{cost-1/2}}\approx\sqrt{2}(1+\frac{t^2}{2}+\frac{t^4}{3}+\frac{181t^6}{720})$. Integrate this approximation to estimate $T(\frac{\pi}{3})$ in terms of L and g. Assuming $g=9.806$ meters per second squared, find an approximate length L such that $T(\frac{\pi}{3})=2$ seconds.

1 E: Chapter Exercises

This page is a draft and is under active development.

1 E.1 Exercise $1E.1$

True or False? In the following exercises, justify your answer with a proof or a counterexample.

1. If the radius of convergence for a power series $\sum_{n=0}^{\infty} a_n x^n$ is 5, then the radius of convergence for the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also 5.

Answer

True

2. Power series can be used to show that the derivative of e^x is e^x. (Hint: Recall that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.)

3. For small values of $x, sinx \approx x$.

Answer

True

4. The radius of convergence for the Maclaurin series of $f(x) = 3^x$ is 3

1 E.2 Exercise $1E.2$

In the following exercises, find the radius of convergence and the interval of convergence for the given series.

1. $\sum_{n=0}^{\infty} n^2 (x-1)^n$

Answer

ROC: 1; IOC: $(0,2)$

2. $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$

3. $\sum_{n=0}^{\infty} \frac{3nx^n}{12^n}$

Answer

ROC: 12; IOC: $(-16,8)$

4. $\sum_{n=0}^{\infty} \frac{2^n}{e^n}(x-e)^n$

1 E.3 Exercise $\mathbf{1E.3}$

In the following exercises, find the power series representation for the given function. Determine the radius of convergence and the interval of convergence for that series.

1. $f(x) = \frac{x^2}{x+3}$

Answer

$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n$; ROC: 3; IOC: $(-3, 3)$

2. $f(x) = \frac{8x+2}{2x^2-3x+1}$

1 E.4 Exercise $\mathbf{1E.4}$

In the following exercises, find the power series for the given function using term-by-term differentiation or integration.

1. $f(x) = tan^{-1}(2x)$

Answer

integration: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(2x)^{2n+1}$

2. $f(x) = \frac{x}{(2+x^2)^2}$

1 E.5 Exercise $\mathbf{1E.5}$

In the following exercises, evaluate the Taylor series expansion of degree four for the given function at the specified point. What is the error in the approximation?

1. $f(x) = x^3 - 2x^2 + 4, a = -3$

Answer

$p_4(x) = (x+3)^3 - 11(x+3)^2 + 39(x+3) - 41;\quad \text{exact}$

2. $f(x) = e^{1/(4x)}, a = 4$

1 E.6 Exercise $\mathbf{1E.6}$

In the following exercises, find the Maclaurin series for the given function.

1. $f(x) = cos(3x)$

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{2n!}$$

2. $f(x) = ln(x+1)$

1 E.7 Exercise $\mathbf{1}\boldsymbol{E}.\mathbf{7}$

In the following exercises, find the Taylor series at the given value.

1. $f(x) = sinx, a = \frac{\pi}{2}$

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x - \frac{\pi}{2})^{2n}$$

2. $f(x) = \frac{3}{x}, a = 1$

1 E.8 Exercise $\mathbf{1}\boldsymbol{E}.\mathbf{8}$

In the following exercises, find the Maclaurin series for the given function.

1. $f(x) = e^{-x^2} - 1$

Answer

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

2. $f(x) = cosx - xsinx$

1 E.9 Exercise $\mathbf{1}\boldsymbol{E}.\mathbf{9}$

In the following exercises, find the Maclaurin series for $F(x) = \int_0^x f(t)dt$ by integrating the Maclaurin series of $f(x)$ term by term.

1. $f(x) = \frac{sinx}{x}$

Answer

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}$$

2. $f(x) = 1 - e^x$

3. Use power series to prove **Euler's formula:** $e^{ix} = cosx + isinx$

1 E.10 Exercise $1E.10$

The following exercises consider problems of **annuity payments.**

1. For annuities with a present value of $\$1$ million, calculate the annual payouts given over 25 years assuming interest rates of $1,$ and 10

2. A lottery winner has an annuity that has a present value of $\$10$ million. What interest rate would they need to live on perpetual annual payments of $\$250,000$?

Answer

2.5

3. Calculate the necessary present value of an annuity in order to support annual payouts of $\$15,000$ given over 25 years assuming interest rates of 1,and 10

CHAPTER OVERVIEW

2: Ordinary differential equations

This page is a draft and is under active development.

In order to apply mathematical methods to a physical or "real life" problem, we must formulate the problem in mathematical terms; that is, we must construct a mathematical model for the problem. Many physical problems concern relationships between changing quantities. Since rates of change are represented mathematically by derivatives, mathematical models often involve equations relating an unknown function and one or more of its derivatives. Such equations are differential equations. They are the subject of this chapter.

In this chapter, we study a particularly important class of second-order equations. Because of their many applications in science and engineering, second order differential equation has historically been the most thoroughly studied class of differential equations. Research on the theory of second order differential equations continues to the present day. This chapter is devoted to second order equations that can be written in the form

\[P_0(x)y"+P_1(x)y'+P_2(x)y=F(x). \[

Such equations are said to be linear. As in the case of first order linear equations, (2.1) is said to be homogeneous if $F \equiv 0$, or nonhomogeneous if $F \not\equiv 0$.

Topic hierarchy

2.1: Linear Second Order Homogeneous Equations

2.1E: Exercises

2.2: Linear Second Order Constant Coefficient Homogeneous Equations

2.2E: Exercises

2.3: Linear Second Order Nonhomogeneous Linear Equations

2.3E: Exercises

2.4: The Method of Undetermined Coefficient

2.4E: Exercises

2E: Exercises

2.1: Linear Second Order Homogeneous Equations

This page is a draft and is under active development.

A second order differential equation is said to be **linear** if it can be written as

$$y'' + p(x)y' + q(x)y = f(x). \tag{2.1.1}$$

We call the function f on the right a **forcing function**, since in physical applications it's often related to a force acting on some system modeled by the differential equation. We say that (2.1.1) is homogeneous if $f \equiv 0$ or nonhomogeneous if $f \not\equiv 0$. Since these definitions are like the corresponding definitions in 3.3: First order linear equations for the linear first order equation

$$y' + p(x)y = f(x), \tag{2.1.2}$$

it's natural to expect similarities between methods of solving (2.1.1) and (2.1.2). However, solving (2.1.1) is more difficult than solving (2.1.2). For example, while Theorem $(2.1.1)$ gives a formula for the general solution of (2.1.2) in the case where $f \equiv 0$ and Theorem 2.2.2 gives a formula for the case where $f \not\equiv 0$, there are no formulas for the general solution of (2.1.1) in either case. Therefore we must be content to solve linear second order equations of special forms.

In Section 2.1 we considered the homogeneous equation $y' + p(x)y = 0$ first, and then used a nontrivial solution of this equation to find the general solution of the nonhomogeneous equation $y' + p(x)y = f(x)$. Although the progression from the homogeneous to the nonhomogeneous case isn't that simple for the linear second order
equation, it's still necessary to solve the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1.3}$$

in order to solve the nonhomogeneous equation (2.1.1). This section is devoted to (2.1.3).

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for (???). We omit the proof.

Theorem 2.1.1

Suppose p and q are continuous on an open interval (a, b), let x_0 be any point in (a, b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \; y(x_0) = k_0, \; y'(x_0) = k_1$$

has a unique solution on (a, b).

Proof

Since $y \equiv 0$ is obviously a solution of (2.1.3) we call it the trivial solution. Any other solution is nontrivial. Under the assumptions of Theorem $(2.1.1)$, the only solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0,\ y(x_0) = 0,\ y'(x_0) = 0$$

on (a, b) is the trivial solution. (Exercise $(2.1E.24)$).

The next three examples illustrate concepts that we'll develop later in this section. You shouldn't be concerned with how to find the given solutions of the equations in these examples. This will be explained in later sections.

Example 2.1.1

The coefficients of y' and y in

$$y'' - y = 0 \tag{2.1.4}$$

are the constant functions $p \equiv 0$ and $q \equiv -1$, which are continuous on $(-\infty, \infty)$. Therefore Theorem $(2.1.1)$ implies that every initial value problem for (2.1.4) has a unique solution on $(-\infty, \infty)$.

(a) Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of (2.1.4) on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants, $y = c_1 e^x + c_2 e^{-x}$ is a solution of (2.1.4) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{2.1.5}$$

Answer

(a) If $y_1 = e^x$ then $y_1' = e^x$ and $y_1'' = e^x = y_1$, so $y_1'' - y_1 = 0$. If $y_2 = e^{-x}$, then $y_2' = -e^{-x}$ and $y_2'' = e^{-x} = y_2$, so $y_2'' - y_2 = 0$.

(b) If

$$y = c_1 e^x + c_2 e^{-x} \tag{2.1.6}$$

then

$$y' = c_1 e^x - c_2 e^{-x} \tag{2.1.7}$$

and

$$y'' = c_1 e^x + c_2 e^{-x},$$

so

$$\begin{aligned} y'' - y &= (c_1 e^x + c_2 e^{-x}) - (c_1 e^x + c_2 e^{-x}) \\ &= c_1(e^x - e^x) + c_2(e^{-x} - e^{-x}) = 0 \end{aligned}$$

for all x. Therefore $y = c_1 e^x + c_2 e^{-x}$ is a solution of (2.1.4) on $(-\infty, \infty)$.

(c) We can solve (2.1.5) by choosing c_1 and c_2 in (2.1.6) so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in (2.1.6) and (2.1.7) shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3. \end{aligned}$$

Solving these equations yields $c_1 = 2$ and $c_2 = -1$. Therefore $y = 2e^x - e^{-x}$ is the unique solution of (2.1.5) on $(-\infty, \infty)$.

Example 2.1.2

Let ω be a positive constant. The coefficients of y' and y in

$$y'' + \omega^2 y = 0 \tag{2.1.8}$$

are the constant functions $p \equiv 0$ and $q \equiv \omega^2$, which are continuous on $(-\infty, \infty)$. Therefore Theorem (2.1.1) implies that every initial value problem for (2.1.8) has a unique solution on $(-\infty, \infty)$.

(a) Verify that $y_1 = \cos\omega x$ and $y_2 = \sin\omega x$ are solutions of (2.1.8) on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 \cos\omega x + c_2 \sin\omega x$ is a solution of (2.1.8) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' + \omega^2 y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{2.1.9}$$

Answer

(a) If $y_1 = \cos\omega x$ then $y_1' = -\omega \sin\omega x$ and $y_1'' = -\omega^2 \cos\omega x = -\omega^2 y_1$, so $y_1'' + \omega^2 y_1 = 0$. If $y_2 = \sin\omega x$ then, $y_2' = \omega \cos\omega x$ and $y_2'' = -\omega^2 \sin\omega x = -\omega^2 y_2$, so $y_2'' + \omega^2 y_2 = 0$.

(b) If

$$y = c_1 \cos\omega x + c_2 \sin\omega x \tag{2.1.10}$$

then

$$y' = \omega(-c_1 \sin\omega x + c_2 \cos\omega x) \tag{2.1.11}$$

and

$$y'' = -\omega^2 (c_1 \cos\omega x + c_2 \sin\omega x),$$

so

$$\begin{aligned} y'' + \omega^2 y = & -\omega^2 (c_1 \cos\omega x + c_2 \sin\omega x) + \omega^2 (c_1 \cos\omega x + c_2 \sin\omega x) \\ = & \; c_1 \omega^2 (-\cos\omega x + \cos\omega x) + c_2 \omega^2 (-\sin\omega x + \sin\omega x) = 0 \end{aligned}$$

for all x. Therefore $y = c_1 \cos\omega x + c_2 \sin\omega x$ is a solution of (2.1.8) on $(-\infty, \infty)$.

(c) To solve (2.1.9), we must choosing c_1 and c_2 in (2.1.10) so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in (2.1.10) and (2.1.11) shows that $c_1 = 1$ and $c_2 = 3/\omega$. Therefore

$$y = \cos \omega x + \frac{3}{\omega} \sin \omega x$$

is the unique solution of (2.1.9) on $(-\infty, \infty)$.

Theorem $(2.1.1)$ implies that if k_0 and k_1 are arbitrary real numbers then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{2.1.12}$$

has a unique solution on an interval (a, b) that contains x_0, provided that P_0, P_1, and P_2 are continuous and P_0 has no zeros on (a, b). To see this, we rewrite the differential equation in (2.1.12) as

$$y'' + \frac{P_1(x)}{P_0(x)} y' + \frac{P_2(x)}{P_0(x)} y = 0$$

and apply Theorem $(2.1.1)$ with $p = P_1/P_0$ and $q = P_2/P_0$.

Example 2.1.3

The equation

$$x^2y'' + xy' - 4y = 0 \tag{2.1.13}$$

has the form of the differential equation in (2.1.12), with $P_0(x) = x^2$, $P_1(x) = x$, and $P_2(x) = -4$, which are are all continuous on $(-\infty, \infty)$. However, since $P(0) = 0$ we must consider solutions of (2.1.13) on $(-\infty, 0)$ and $(0, \infty)$. Since P_0 has no zeros on these intervals, Theorem $(2.1.1)$ implies that the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(0, \infty)$ if $x_0 > 0$, or on $(-\infty, 0)$ if $x_0 < 0$.

(a) Verify that $y_1 = x^2$ is a solution of (2.1.13) on $(-\infty, \infty)$ and $y_2 = 1/x^2$ is a solution of (2.1.13) on $(-\infty, 0)$ and $(0, \infty)$.

(b) Verify that if c_1 and c_2 are any constants then $y = c_1x^2 + c_2/x^2$ is a solution of (2.1.13) on $(-\infty, 0)$ and $(0, \infty)$.

(c) Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 0. \tag{2.1.14}$$

(d) Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 0. \tag{2.1.15}$$

Answer

(a) If $y_1 = x^2$ then $y_1' = 2x$ and $y_1'' = 2$, so

$$x^2y_1'' + xy_1' - 4y_1 = x^2(2) + x(2x) - 4x^2 = 0$$

for x in $(-\infty, \infty)$. If $y_2 = 1/x^2$, then $y_2' = -2/x^3$ and $y_2'' = 6/x^4$, so

$$x^2 y_2'' + x y_2' - 4y_2 = x^2\left(\frac{6}{x^4}\right) - x\left(\frac{2}{x^3}\right) - \frac{4}{x^2} = 0$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

(b) If

$$y = c_1 x^2 + \frac{c_2}{x^2} \tag{2.1.16}$$

then

$$y' = 2c_1 x - \frac{2c_2}{x^3} \tag{2.1.17}$$

and

$$y'' = 2c_1 + \frac{6c_2}{x^4},$$

so

$$\begin{aligned} x^2 y'' + x y' - 4y =& \ x^2\left(2c_1 + \frac{6c_2}{x^4}\right) + x\left(2c_1 x - \frac{2c_2}{x^3}\right) - 4\left(c_1 x^2 + \frac{c_2}{x^2}\right) \\ =& \ c_1(2x^2 + 2x^2 - 4x^2) + c_2\left(\frac{6}{x^2} - \frac{2}{x^2} - \frac{4}{x^2}\right) \\ =& \ c_1 \cdot 0 + c_2 \cdot 0 = 0 \end{aligned}$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

(c) To solve (2.1.14), we choose c_1 and c_2 in (2.1.16) so that $y(1) = 2$ and $y'(1) = 0$. Setting $x = 1$ in (2.1.16) and (2.1.17) shows that this is equivalent to

$$\begin{aligned} c_1 + \ \ c_2 &= 2 \\ 2c_1 - 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1 = 1$ and $c_2 = 1$. Therefore $y = x^2 + 1/x^2$ is the unique solution of (2.1.14) on $(0, \infty)$.

(d) We can solve (2.1.15) by choosing c_1 and c_2 in (2.1.16) so that $y(-1) = 2$ and $y'(-1) = 0$. Setting $x = -1$ in (2.1.16) and (2.1.17) shows that this is equivalent to

$$\begin{aligned} c_1 + \ \ c_2 &= 2 \\ -2c_1 + 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1 = 1$ and $c_2 = 1$. Therefore $y = x^2 + 1/x^2$ is the unique solution of (2.1.15) on $(-\infty, 0)$.

Although the formulas for the solutions of (2.1.14) and (2.1.15) are both $y = x^2 + 1/x^2$, you should not conclude that these two initial value problems have the same solution. Remember that a solution of an initial value problem is defined on an interval that contains the initial point; therefore, the solution of (2.1.14) is $y = x^2 + 1/x^2$ on the interval $(0, \infty)$, which contains the initial point $x_0 = 1$, while the solution of (2.1.15) is $y = x^2 + 1/x^2$ on the interval $(-\infty, 0)$, which contains the initial point $x_0 = -1$.

2.1.1 The General Solution of a Homogeneous Linear Second Order Equation

If y_1 and y_2 are defined on an interval (a, b) and c_1 and c_2 are constants, then

$$y = c_1 y_1 + c_2 y_2$$

is a linear combination of y_1 and y_2. For example, $y = 2\cos x + 7\sin x$ is a linear combination of $y_1 = \cos x$ and $y_2 = \sin x$, with $c_1 = 2$ and $c_2 = 7$.

The next theorem states a fact that we've already verified in Examples (2.1.1), (2.1.2), and (2.1.3).

Theorem 2.1.2

If y_1 and y_2 are solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1.18}$$

on (a, b), then any linear combination

$$y = c_1 y_1 + c_2 y_2 \tag{2.1.19}$$

of y_1 and y_2 is also a solution of (2.1.18) on (a, b).

Proof

If

$$y = c_1 y_1 + c_2 y_2$$

then

$$y' = c_1 y_1' + c_2 y_2' \text{ and } y'' = c_1 y_1'' + c_2 y_2''.$$

Therefore

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (c_1 y_1'' + c_2 y_2'') + p(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1 \left(y_1'' + p(x)y_1' + q(x)y_1\right) + c_2 \left(y_2'' + p(x)y_2' + q(x)y_2\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0, \end{aligned}$$

since y_1 and y_2 are solutions of (2.1.18).

We say that $\{y_1, y_2\}$ is a fundamental set of solutions of (2.1.18) on (a, b) if every solution of (2.1.18) on (a, b) can be written as a linear combination of y_1 and y_2 as in (2.1.19). In this case we say that (2.1.19) is general solution of (2.1.18) on (a, b).

2.1.2 Linear Independence

We need a way to determine whether a given set $\{y_1, y_2\}$ of solutions of (2.1.18) is a fundamental set. The next definition will enable us to state necessary and sufficient conditions for this.

We say that two functions y_1 and y_2 defined on an interval (a, b) are linearly independent on (a, b) if neither is a constant multiple of the other on (a, b). (In particular, this means that neither can be the trivial solution of (2.1.18), since, for example, if $y_1 \equiv 0$ we could write $y_1 = 0y_2$.) We'll also say that the set $\{y_1, y_2\}$ is linearly independent on (a, b).

Theorem 2.1.3

Suppose p and q are continuous on (a, b). Then a set $\{y_1, y_2\}$ of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1.20}$$

on (a, b) is a fundamental set if and only if $\{y_1, y_2\}$ is linearly independent on (a, b).

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

We'll present the proof of Theorem (2.1.3) in steps worth regarding as theorems in their own right. However, let's first interpret Theorem (2.1.3) in terms of Examples (2.1.1), (2.1.2), and (2.1.3).

Example 2.1.4:

(a) Since $e^x/e^{-x} = e^{2x}$ is nonconstant, Theorem (2.1.3) implies that $y = c_1e^x + c_2e^{-x}$ is the general solution of $y'' - y = 0$ on $(-\infty, \infty)$.

(b) Since $\cos\omega x/\sin\omega x = \cot\omega x$ is nonconstant, Theorem (2.1.3) implies that $y = c_1\cos\omega x + c_2\sin\omega x$ is the general solution of $y'' + \omega^2 y = 0$ on $(-\infty, \infty)$.

(c) Since $x^2/x^{-2} = x^4$ is nonconstant, Theorem (2.1.3) implies that $y = c_1x^2 + c_2/x^2$ is the general solution of $x^2y'' + xy' - 4y = 0$ on $(-\infty, 0)$ and $(0, \infty)$.

2.1.3 The Wronskian and Abel's Formula

To motivate a result that we need in order to prove Theorem (2.1.3), let's see what is required to prove that $\{y_1, y_2\}$ is a fundamental set of solutions of (2.1.20) on (a, b). Let x_0 be an arbitrary point in (a, b), and suppose y is an arbitrary solution of (2.1.20) on (a, b). Then y is the unique solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1; \tag{2.1.21}$$

that is, k_0 and k_1 are the numbers obtained by evaluating y and y' at x_0. Moreover, k_0 and k_1 can be any real numbers, since Theorem (2.1.1) implies that (2.1.21) has a solution no matter how k_0 and k_1

are chosen. Therefore $\{y_1, y_2\}$ is a fundamental set of solutions of (2.1.20) on (a, b) if and only if it's possible to write the solution of an arbitrary initial value problem (2.1.21) as $y = c_1 y_1 + c_2 y_2$. This is equivalent to requiring that the system

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= k_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= k_1 \end{aligned} \tag{2.1.22}$$

has a solution (c_1, c_2) for every choice of (k_0, k_1). Let's try to solve (2.1.22).

Multiplying the first equation in (2.1.22) by $y_2'(x_0)$ and the second by $y_2(x_0)$ yields

$$\begin{aligned} c_1 y_1(x_0) y_2'(x_0) + c_2 y_2(x_0) y_2'(x_0) = y_2'(x_0) k_0 \\ c_1 y_1'(x_0) y_2(x_0) + c_2 y_2'(x_0) y_2(x_0) = y_2(x_0) k_1, \end{aligned}$$

and subtracting the second equation here from the first yields

$$\left(y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)\right) c_1 = y_2'(x_0) k_0 - y_2(x_0) k_1. \tag{2.1.23}$$

Multiplying the first equation in (2.1.22) by $y_1'(x_0)$ and the second by $y_1(x_0)$ yields

$$\begin{aligned} c_1 y_1(x_0) y_1'(x_0) + c_2 y_2(x_0) y_1'(x_0) = y_1'(x_0) k_0 \\ c_1 y_1'(x_0) y_1(x_0) + c_2 y_2'(x_0) y_1(x_0) = y_1(x_0) k_1, \end{aligned}$$

and subtracting the first equation here from the second yields

$$\left(y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)\right) c_2 = y_1(x_0) k_1 - y_1'(x_0) k_0. \tag{2.1.24}$$

If

$$y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0) = 0,$$

it's impossible to satisfy (2.1.23) and (2.1.24) (and therefore (2.1.22)) unless k_0 and k_1 happen to satisfy

$$\begin{aligned} y_1(x_0) k_1 - y_1'(x_0) k_0 &= 0 \\ y_2'(x_0) k_0 - y_2(x_0) k_1 &= 0. \end{aligned}$$

On the other hand, if

$$y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0) \neq 0 \tag{2.1.25}$$

we can divide (2.1.23) and (2.1.24) through by the quantity on the left to obtain

$$\begin{aligned} c_1 &= \frac{y_2'(x_0) k_0 - y_2(x_0) k_1}{y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)} \\ c_2 &= \frac{y_1(x_0) k_1 - y_1'(x_0) k_0}{y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)}, \end{aligned} \tag{2.1.26}$$

no matter how k_0 and k_1 are chosen. This motivates us to consider conditions on y_1 and y_2 that imply (2.1.25).

Theorem 2.1.4

Suppose p and q are continuous on (a, b), let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1.27}$$

on (a, b), and define

$$W = y_1 y_2' - y_1' y_2. \tag{2.1.28}$$

Let x_0 be any point in (a, b). Then

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p(t)\,dt}, \quad a < x < b. \tag{2.1.29}$$

Therefore either W has no zeros in (a, b) or $W \equiv 0$ on (a, b).

Proof

Differentiating (2.1.28) yields

$$W' = y_1' y_2' + y_1 y_2'' - y_1' y_2' - y_1'' y_2 = y_1 y_2'' - y_1'' y_2. \tag{2.1.30}$$

Since y_1 and y_2 both satisfy (2.1.27),

$$y_1'' = -py_1' - qy_1 \text{ and } y_2'' = -py_2' - qy_2.$$

Substituting these into (2.1.30) yields

$$\begin{aligned} W' &= -y_1\left(py_2' + qy_2\right) + y_2\left(py_1' + qy_1\right) \\ &= -p(y_1 y_2' - y_2 y_1') - q(y_1 y_2 - y_2 y_1) \\ &= -p(y_1 y_2' - y_2 y_1') = -pW. \end{aligned}$$

Therefore $W' + p(x)W = 0$; that is, W is the solution of the initial value problem

$$y' + p(x)y = 0, \quad y(x_0) = W(x_0).$$

We leave it to you to verify by separation of variables that this implies (2.1.29). If $W(x_0) \neq 0$, (2.1.29) implies that W has no zeros in (a, b), since an exponential is never zero. On the other hand, if $W(x_0) = 0$, (2.1.29) implies that $W(x) = 0$ for all x in (a, b).

The function W defined in (2.1.28) is the Wronskian of $\{y_1, y_2\}$. Formula (2.1.29) is Abel's formula.

The Wronskian of $\{y_1, y_2\}$ is usually written as the determinant

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

The expressions in (2.1.26) for c_1 and c_2 can be written in terms of determinants as

$$c_1 = \frac{1}{W(x_0)} \begin{vmatrix} k_0 & y_2(x_0) \\ k_1 & y_2'(x_0) \end{vmatrix} \text{ and } c_2 = \frac{1}{W(x_0)} \begin{vmatrix} y_1(x_0) & k_0 \\ y_1'(x_0) & k_1 \end{vmatrix}.$$

If you've taken linear algebra you may recognize this as Cramer's rule.

Example 2.1.5

Verify Abel's formula for the following differential equations and the corresponding solutions, from Examples (2.1.1). (2.1.2), and (2.1.3):

(a) $y'' - y = 0; \quad y_1 = e^x,\ y_2 = e^{-x}$

(b) $y'' + \omega^2 y = 0; \qquad y_1 = \cos\omega x,\ y_2 = \sin\omega x$

(c) $x^2y'' + xy' - 4y = 0; \quad y_1 = x^2,\ y_2 = 1/x^2$

Answer

(a) Since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^x e^{-x} = -2$$

for all x.

(b) Again, since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$\begin{aligned} W(x) &= \begin{vmatrix} \cos\omega x & \sin\omega x \\ -\omega\sin\omega x & \omega\cos\omega x \end{vmatrix} \\ &= \cos\omega x(\omega\cos\omega x) - (-\omega\sin\omega x)\sin\omega x \\ &= \omega(\cos^2\omega x + \sin^2\omega x) = \omega \end{aligned}$$

for all x.

(c) Computing the Wronskian of $y_1 = x^2$ and $y_2 = 1/x^2$ directly yields

$$W = \begin{vmatrix} x^2 & 1/x^2 \\ 2x & -2/x^3 \end{vmatrix} = x^2\left(-\frac{2}{x^3}\right) - 2x\left(\frac{1}{x^2}\right) = -\frac{4}{x}. \tag{2.1.31}$$

To verify Abel's formula we rewrite the differential equation as

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$$

to see that $p(x) = 1/x$. If x_0 and x are either both in $(-\infty, 0)$ or both in $(0, \infty)$ then

$$\int_{x_0}^{x} p(t)\,dt = \int_{x_0}^{x} \frac{dt}{t} = \ln\left(\frac{x}{x_0}\right),$$

so Abel's formula becomes

$$\begin{aligned} W(x) &= W(x_0)e^{-\ln(x/x_0)} = W(x_0)\frac{x_0}{x} \\ &= -\left(\frac{4}{x_0}\right)\left(\frac{x_0}{x}\right) \text{ from (2.1.31)} \\ &= -\frac{4}{x}, \end{aligned}$$

which is consistent with (2.1.31).

The next theorem will enable us to complete the proof of Theorem (2.1.3).

Theorem 2.1.5

Suppose p and q are continuous on an open interval (a,b), let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1.32}$$

on (a,b), and let $W = y_1y_2' - y_1'y_2$. Then $y_1$$ and y_2 are linearly independent on (a,b) if and only if W has no zeros on (a,b).

Proof

We first show that if $W(x_0) = 0$ for some x_0 in (a,b), then y_1 and y_2 are linearly dependent on (a,b). Let I be a subinterval of (a,b) on which y_1 has no zeros. (If there's no such subinterval, $y_1 \equiv 0$ on (a,b), so y_1 and y_2 are linearly independent, and we're finished with this part of the proof.) Then y_2/y_1 is defined on I, and

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W}{y_1^2}. \tag{2.1.33}$$

However, if $W(x_0) = 0$, Theorem (2.1.4) implies that $W \equiv 0$ on (a,b). Therefore (2.1.33) implies that $(y_2/y_1)' \equiv 0$, so $y_2/y_1 = c$ (constant) on I. This shows that $y_2(x) = cy_1(x)$ for all x in I. However, we want to show that $y_2 = cy_1(x)$ for all x in (a,b). Let $Y = y_2 - cy_1$. Then Y is a solution of (2.1.32) on (a,b) such that $Y \equiv 0$ on I, and therefore $Y' \equiv 0$ on I. Consequently, if x_0 is chosen arbitrarily in I then Y is a solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0,$$

which implies that $Y \equiv 0$ on (a,b), by the paragraph following Theorem (2.1.1). (See also Exercise (2.1E.24). Hence, $y_2 - cy_1 \equiv 0$
on (a,b), which implies that y_1 and y_2 are not linearly independent on (a,b).

Now suppose W has no zeros on (a,b). Then y_1 can't be identically zero on (a,b) (why not?), and therefore there is a subinterval I of (a,b) on which y_1 has no zeros. Since (2.1.33) implies that y_2/y_1 is nonconstant on I, y_2 isn't a constant multiple of y_1 on (a,b). A similar argument shows that y_1 isn't a constant multiple of y_2 on (a,b), since

$$\left(\frac{y_1}{y_2}\right)' = \frac{y_1' y_2 - y_1 y_2'}{y_2^2} = -\frac{W}{y_2^2}$$

on any subinterval of (a, b) where y_2 has no zeros.

We can now complete the proof of Theorem $(2.1.3)$. From Theorem $(2.1.5)$, two solutions y_1 and y_2 of (2.1.32) are linearly independent on (a, b) if and only if W has no zeros on (a, b). From Theorem (2.1.4) and the motivating comments preceding it, $\{y_1, y_2\}$ is a fundamental set of solutions of (2.1.32) if and only if W has no zeros on (a, b). Therefore $\{y_1, y_2\}$ is a fundamental set for (2.1.32) on (a, b) if and only if $\{y_1, y_2\}$ is linearly independent on (a, b).

The next theorem summarizes the relationships among the concepts discussed in this section.

Theorem 2.1.6

Suppose p and q are continuous on an open interval (a, b) and let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1.34}$$

on (a, b). Then the following statements are equivalent; that is, they are either all true or all false.

(a) The general solution of (2.1.34) on (a, b) is $y = c_1 y_1 + c_2 y_2$.

(b) $\{y_1, y_2\}$ is a fundamental set of solutions of (2.1.34) on (a, b).

(c) $\{y_1, y_2\}$ is linearly independent on (a, b).

(d) The Wronskian of $\{y_1, y_2\}$ is nonzero at some point in (a, b).

(e) The Wronskian of $\{y_1, y_2\}$ is nonzero at all points in (a, b).

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

We can apply this theorem to an equation written as

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

on an interval (a, b) where P_0, P_1, and P_2 are continuous and P_0 has no zeros.

Theorem 2.1.7

Suppose c is in (a, b) and α and β are real numbers, not both zero. Under the assumptions of Theorem (2.1.7), suppose y_1 and y_2 are solutions of (2.1.34) such that

$$\alpha y_1(c) + \beta y_1'(c) = 0\text{\; and\; \;} \alpha y_2(c) + \beta y_2'(c) = 0. \tag{2.1.35}$$

Then $\{y_1, y_2\}$ isn't linearly independent on (a, b).

Proof

Since α and β are not both zero, (2.1.35) implies that

$$\left|\begin{array}{cc} y_1(c) & y_1'(c) \\ y_2(c) & y_2'(c) \end{array}\right| = 0, \; so\; \; \left|\begin{array}{cc} y_1(c) & y_2(c) \\ y_1'(c) & y_2'(c) \end{array}\right| = 0$$

and Theorem $(2.1.6)$ implies the stated conclusion.

2.1E: Exercises

This page is a draft and is under active development.

2.1E.1 Exercise $\boldsymbol{2.1E.1}$

(a) Verify that $y_1 = e^{2x}$ and $y_2 = e^{5x}$ are solutions of

$$y'' - 7y' + 10y = 0 \tag{2.1E.1}$$

on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 e^{2x} + c_2 e^{5x}$ is a solution of (2.1E.1) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 1.$$

(d) Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

Answer

Add texts here. Do not delete this text first.

2.1E.2 Exercise $\boldsymbol{2.1E.2}$

(a) Verify that $y_1 = e^x \cos x$ and $y_2 = e^x \sin x$ are solutions of

$$y'' - 2y' + 2y = 0 \tag{2.1E.2}$$

on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 e^x \cos x + c_2 e^x \sin x$ is a solution of (2.1E.2) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = -2.$$

(d) Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

Answer

Add texts here. Do not delete this text first.

2.1E.3 Exercise $\mathbf{2.1E.3}$

(a) Verify that $y_1 = e^x$ and $y_2 = xe^x$ are solutions of

$$y'' - 2y' + y = 0 \tag{2.1E.3}$$

on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y = e^x(c_1 + c_2x)$ is a solution of (2.1E.3) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 7, \quad y'(0) = 4.$$

(d) Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

Answer

Add texts here. Do not delete this text first.

2.1E.4 Exercise $\mathbf{2.1E.4}$

(a) Verify that $y_1 = 1/(x-1)$ and $y_2 = 1/(x+1)$ are solutions of

$$(x^2 - 1)y'' + 4xy' + 2y = 0 \tag{2.1E.4}$$

on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. What is the general solution of (2.1E.4) on each of these intervals?

(b) Solve the initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = -5, \quad y'(0) = 1.$$

What is the interval of validity of the solution?

(c) Graph the solution of the initial value problem.

(d) Verify Abel's formula for y_1 and y_2, with $x_0 = 0$.

Answer

Add texts here. Do not delete this text first.

2.1E.5 Exercise $\mathbf{2.1E.5}$

Compute the Wronskians of the given sets of functions.

(a) $\{1, e^x\}$

(b)$\{e^x, e^x \sin x\}$

(c) $\{x+1, x^2+2\}$

(d) $\{x^{1/2}, x^{-1/3}\}$

(e) $\left\{\frac{\sin x}{x}, \frac{\cos x}{x}\right\}$

(f) $\{x\ln|x|, x^2\ln|x|\}$

(g) $\{e^x\cos\sqrt{x}, e^x\sin\sqrt{x}\}$

Answer

Add texts here. Do not delete this text first.

2.1E.6 Exercise $\mathbf{2.1E.6}$

Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$y'' + 3(x^2+1)y' - 2y = 0,$$

given that $W(\pi) = 0$.

Answer

Add texts here. Do not delete this text first.

2.1E.7 Exercise $\mathbf{2.1E.7}$

Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

given that $W(0) = 1$. (This is Legendre's equation.)

Answer

Add texts here. Do not delete this text first.

2.1E.8 Exercise $\mathbf{2.1E.8}$

Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$x^2y'' + xy' + (x^2-\nu^2)y = 0,$$

given that $W(1) = 1$. (This is Bessel's equation.)

Answer

Add texts here. Do not delete this text first.

2.1E.9 Exercise $\mathbf{2.1E.9}$

(This exercise shows that if you know one nontrivial solution of $y'' + p(x)y' + q(x)y = 0$, you can use Abel's formula to find another.)

Suppose p and q are continuous and y_1 is a solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1E.5}$$

that has no zeros on (a, b). Let $P(x) = \int p(x)\,dx$ be any antiderivative of p on \)(a,b)\).

(a) Show that if K is an arbitrary nonzero constant and y_2 satisfies

$$y_1 y_2' - y_1' y_2 = Ke^{-P(x)} \tag{2.1E.6}$$

on (a, b), then y_2 also satisfies (2.1E.5) on (a, b), and $\{y_1, y_2\}$ is a fundmental set of solutions on (2.1E.5) on (a, b).

(b) Conclude from (a) that if $y_2 = uy_1$ where $u' = K\dfrac{e^{-P(x)}}{y_1^2(x)}$, then $\{y_1, y_2\}$ is a fundamental set of solutions of (2.1E.5) on (a, b).

Answer

Add texts here. Do not delete this text first.

In Exercises $(2.1E.10)$ to $(2.1E.23)$ use the method suggested by Exercise $(2.1E.9)$ to find a second solution y_2 that isn't a constant multiple of the solution y_1. Choose K conveniently to simplify y_2.

2.1E.10 Exercise $\mathbf{2.1E.10}$

$y'' - 2y' - 3y = 0; \quad y_1 = e^{3x}$

Answer

Add texts here. Do not delete this text first.

2.1E.11 Exercise $\mathbf{2.1E.11}$

$y'' - 6y' + 9y = 0; \quad y_1 = e^{3x}$

Answer

Add texts here. Do not delete this text first.

2.1E.12 Exercise $\mathbf{2.1E.12}$

$y'' - 2ay' + a^2y = 0; (a = \text{constant}); \quad y_1 = e^{ax}$

Answer

Add texts here. Do not delete this text first.

2.1E.13 Exercise $\mathbf{2.1E.13}$

$x^2y'' + xy' - y = 0; \quad y_1 = x$

Answer

Add texts here. Do not delete this text first.

2.1E.14 Exercise $\mathbf{2.1}\boldsymbol{E}.\mathbf{14}$

$x^2y'' - xy' + y = 0; \quad y_1 = x$

Answer

Add texts here. Do not delete this text first.

2.1E.15 Exercise $\mathbf{2.1}\boldsymbol{E}.\mathbf{15}$

$x^2y'' - (2a-1)xy' + a^2y = 0; (a = \text{nonzero constant}),\ x > 0; \quad y_1 = x^a$

Answer

Add texts here. Do not delete this text first.

2.1E.16 Exercise $\mathbf{2.1}\boldsymbol{E}.\mathbf{16}$

$4x^2y'' - 4xy' + (3 - 16x^2)y = 0; \quad y_1 = x^{1/2}e^{2x}$

Answer

Add texts here. Do not delete this text first.

2.1E.17 Exercise $\mathbf{2.1}\boldsymbol{E}.\mathbf{17}$

$(x-1)y'' - xy' + y = 0; \quad y_1 = e^x$

Answer

Add texts here. Do not delete this text first.

2.1E.18 Exercise $\mathbf{2.1}\boldsymbol{E}.\mathbf{18}$

$x^2y'' - 2xy' + (x^2+2)y = 0; \quad y_1 = x\cos x$

Answer

Add texts here. Do not delete this text first.

2.1E.19 Exercise $\mathbf{2.1}\boldsymbol{E}.\mathbf{19}$

$4x^2(\sin x)y'' - 4x(x\cos x + \sin x)y' + (2x\cos x + 3\sin x)y = 0; \quad y_1 = x^{1/2}$

Answer

Add texts here. Do not delete this text first.

2.1E.20 Exercise $\mathbf{2.1}\boldsymbol{E}\mathbf{.20}$

$(3x-1)y''-(3x+2)y'-(6x-8)y=0; \quad y_1=e^{2x}$

Answer

Add texts here. Do not delete this text first.

2.1E.21 Exercise $\mathbf{2.1}\boldsymbol{E}\mathbf{.21}$

$(x^2-4)y''+4xy'+2y=0; \quad y_1=\dfrac{1}{x-2}$

Answer

Add texts here. Do not delete this text first.

2.1E.22 Exercise $\mathbf{2.1}\boldsymbol{E}\mathbf{.22}$

$(2x+1)xy''-2(2x^2-1)y'-4(x+1)y=0; \quad y_1=\dfrac{1}{x}$

Answer

Add texts here. Do not delete this text first.

2.1E.23 Exercise $\mathbf{2.1}\boldsymbol{E}\mathbf{.23}$

$(x^2-2x)y''+(2-x^2)y'+(2x-2)y=0; \quad y_1=e^{x}$

Answer

Add texts here. Do not delete this text first.

2.1E.24 Exercise $\mathbf{2.1}\boldsymbol{E}\mathbf{.24}$

Suppose p and q are continuous on an open interval (a,b) and let x_0 be in (a,b). Use Theorem $(2.1.1)$ to show that the only solution of the initial value problem

$$y''+p(x)y'+q(x)y=0, \quad y(x_0)=0, \quad y'(x_0)=0$$

on (a,b) is the trivial solution $y\equiv 0$.

Answer

Add texts here. Do not delete this text first.

2.1E.25 Exercise $\mathbf{2.1}\boldsymbol{E}\mathbf{.25}$

Suppose P_0, P_1, and P_2 are continuous on (a,b) and let x_0 be in (a,b). Show that if either of the following statements is true then $P_0(x)=0$ for some x in (a,b).

(a) The initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has more than one solution on (a, b).

(b) The initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0$$

has a nontrivial solution on (a, b).

Answer

Add texts here. Do not delete this text first.

2.1E.26 Exercise $\mathbf{2.1\textit{E}.26}$

Suppose p and q are continuous on (a, b) and y_1 and y_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1E.7}$$

on (a, b). Let

$$z_1 = \alpha y_1 + \beta y_2 \quad \text{and} \quad z_2 = \gamma y_1 + \delta y_2,$$

where α, β, γ, and δ are constants. Show that if $\{z_1, z_2\}$ is a fundamental set of solutions of (2.1E.7) on (a, b) then so is $\{y_1, y_2\}$.

Answer

Add texts here. Do not delete this text first.

2.1E.27 Exercise $\mathbf{2.1\textit{E}.27}$

Suppose p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1E.8}$$

on (a, b). Let

$$z_1 = \alpha y_1 + \beta y_2 \quad \text{and} \quad z_2 = \gamma y_1 + \delta y_2,$$

where α, β, γ, and δ are constants. Show that $\{z_1, z_2\}$ is a fundamental set of solutions of (2.1E.8) on (a, b) if and only if $\alpha\gamma - \beta\delta \neq 0$.

Answer

Add texts here. Do not delete this text first.

2.1E.28 Exercise $\mathbf{2.1\textit{E}.28}$

Suppose y_1 is differentiable on an interval (a, b) and $y_2 = ky_1$, where k is a constant. Show that the Wronskian of $\{y_1, y_2\}$ is identically zero on (a, b).

Answer

Add texts here. Do not delete this text first.

2.1E.29 Exercise $\PageIndex{29}$

Let

$$y_1 = x^3 \quad \text{and} \quad y_2 = \begin{cases} x^3, & x \geq 0, \\ -x^3, & x < 0. \end{cases}$$

(a) Show that the Wronskian of $\{y_1, y_2\}$ is defined and identically zero on $(-\infty, \infty)$.

(b) Suppose $a < 0 < b$. Show that $\{y_1, y_2\}$ is linearly independent on (a, b).

(c) Use Exercise $(2.1E.25)$ part (b) to show that these results don't contradict Theorem $(2.1.5)$, because neither y_1 nor y_2 can be a solution of an equation

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) if p and q are continuous on (a, b).

Answer

Add texts here. Do not delete this text first.

2.1E.30 Exercise $\PageIndex{30}$

Suppose p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a set of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) such that either $y_1(x_0) = y_2(x_0) = 0$ or $y_1'(x_0) = y_2'(x_0) = 0$ for some x_0 in (a, b). Show that $\{y_1, y_2\}$ is linearly dependent on (a, b).

Answer

Add texts here. Do not delete this text first.

2.1E.31 Exercise $\PageIndex{31}$

Suppose p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b). Show that if $y_1(x_1) = y_1(x_2) = 0$, where $a < x_1 < x_2 < b$, then $y_2(x) = 0$ for some x in (x_1, x_2).

Hint: Show that if y_2 has no zeros in (x_1, x_2), then y_1/y_2 is either strictly increasing or strictly decreasing on (x_1, x_2), and deduce a contradiction.

Answer

Add texts here. Do not delete this text first.

2.1E.32 Exercise $\mathbf{2.1E.32}$

Suppose p and q are continuous on (a,b) and every solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1E.9}$$

on (a,b) can be written as a linear combination of the twice differentiable functions $\{y_1, y_2\}$. Use Theorem $(2.1.1)$ to show that y_1 and y_2 are themselves solutions of (2.1E.9) on (a,b).

Answer

Add texts here. Do not delete this text first.

2.1E.33 Exercise $\mathbf{2.1E.33}$

Suppose p_1, p_2, q_1, and q_2 are continuous on (a,b) and the equations

$$y'' + p_1(x)y' + q_1(x)y = 0 \quad \text{and} \quad y'' + p_2(x)y' + q_2(x)y = 0$$

have the same solutions on (a,b). Show that $p_1 = p_2$ and $q_1 = q_2$ on (a,b).

Hint: Use Abel's formula.

Answer

Add texts here. Do not delete this text first.

2.1E.34 Exercise $\mathbf{2.1E.34}$

(For this exercise you have to know about 3×3 determinants.)

Show that if y_1 and y_2 are twice continuously differentiable on (a,b) and the Wronskian W of $\{y_1, y_2\}$ has no zeros in (a,b) then the equation

$$\frac{1}{W}\begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = 0$$

can be written as

$$y'' + p(x)y' + q(x)y = 0, \tag{2.1E.10}$$

where p and q are continuous on (a,b) and $\{y_1, y_2\}$ is a fundamental set of solutions of (2.1E.10) on (a,b).

Hint: Expand the determinant by cofactors of its first column.

Answer

Add texts here. Do not delete this text first.

2.1E.35 Exercise $\mathbf{2.1}\boldsymbol{E}\mathbf{.35}$

Use the method suggested by Exercise $(2.1E.34)$ to find a linear homogeneous equation for which the given functions form a fundamental set of solutions on some interval.

(a) $e^x \cos 2x, \quad e^x \sin 2x$

(b) $x, \quad e^{2x}$

(c) $x, \quad x \ln x$

(d) $\cos(\ln x), \quad \sin(\ln x)$

(e) $\cosh x, \quad \sinh x$

(f) $x^2 - 1, \quad x^2 + 1$

Answer

Add texts here. Do not delete this text first.

2.1E.36 Exercise $\mathbf{2.1}\boldsymbol{E}\mathbf{.36}$

Suppose p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1E.11}$$

on (a, b). Show that if y is a solution of (2.1E.11) on (a, b), there's exactly one way to choose c_1 and c_2 so that $y = c_1 y_1 + c_2 y_2$ on (a, b).

Answer

Add texts here. Do not delete this text first.

2.1E.37 Exercise $\mathbf{2.1}\boldsymbol{E}\mathbf{.37}$

Suppose p and q are continuous on (a, b) and x_0 is in (a, b). Let y_1 and y_2 be the solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{2.1E.12}$$

such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0,\ y_2'(x_0) = 1.$$

(Theorem $(2.1.1)$ implies that each of these initial value problems has a unique solution on (a, b).)

(a) Show that $\{y_1, y_2\}$ is linearly independent on (a, b).

(b) Show that an arbitrary solution y of (2.1E.12) on $a, b)$ can be written as $y = y(x_0)y_1 + y'(x_0)y_2$.

(c) Express the solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2.

Answer

Add texts here. Do not delete this text first.

2.1E.38 Exercise $\mathbf{2.1E.38}$

Find solutions y_1 and y_2 of the equation $y'' = 0$ that satisfy the initial conditions

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Then use Exercise $(2.1E.37)$ (c) to write the solution of the initial value problem

$$y'' = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

as a linear combination of y_1 and y_2.

Answer

Add texts here. Do not delete this text first.

2.1E.39 Exercise $\mathbf{2.1E.39}$

Let x_0 be an arbitrary real number. Given Example $(2.1.1)$ that e^x and e^{-x} are solutions of $y'' - y = 0$, find solutions y_1 and y_2 of $y'' - y = 0$ such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \ y_2'(x_0) = 1.$$

Then use Exercise $(2.1E.37)$ (c) to write the solution of the initial value problem

$$y'' - y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2.

Answer

Add texts here. Do not delete this text first.

2.1E.40 Exercise $\mathbf{2.1E.40}$

Let x_0 be an arbitrary real number. Given Example $(2.1.2)$ that $\cos\omega x$ and $\sin\omega x$ are solutions of $y'' + \omega^2 y = 0$, find solutions of $y'' + \omega^2 y = 0\$$ such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \ y_2'(x_0) = 1.$$

Then use Exercise $(2.1E.37)$ (c) to write the solution of the initial value problem

$$y'' + \omega^2 y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2. Use the identities

$$\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \sin(A+B) &= \sin A \cos B + \cos A \sin B \end{aligned}$$

to simplify your expressions for y_1, y_2, and y.

Answer

Add texts here. Do not delete this text first.

2.1E.41 Exercise $\mathbf{2.1}\boldsymbol{E}\mathbf{.41}$

Recall from Exercise $(2.1E.4)$ that $1/(x-1)$ and $1/(x+1)$ are solutions of

$$(x^2-1)y''+4xy'+2y=0 \tag{2.1E.13}$$

on $(-1,1)$. Find solutions of (2.1E.13) such that

$$y_1(0)=1, \quad y_1'(0)=0 \quad \text{and} \quad y_2(0)=0,\ y_2'(0)=1.$$

Then use Exercise $(2.1E.37)$ (c) to write the solution of initial value problem

$$(x^2-1)y''+4xy'+2y=0, \quad y(0)=k_0, \quad y'(0)=k_1$$

as a linear combination of y_1 and y_2.

Answer

Add texts here. Do not delete this text first.

2.1E.42 Exercise $\mathbf{2.1}\boldsymbol{E}\mathbf{.42}$

(a) Verify that $y_1=x^2$ and $y_2=x^3$ satisfy

$$x^2y''-4xy'+6y=0 \tag{2.1E.14}$$

on $(-\infty,\infty)$ and that $\{y_1,y_2\}$ is a fundamental set of solutions of (2.1E.14) on $(-\infty,0)$ and $(0,\infty)$.

(b) Let a_1, a_2, b_1, and b_2 be constants. Show that

$$y=\begin{cases} a_1x^2+a_2x^3, & x\ge 0, \\ b_1x^2+b_2x^3, & x<0 \end{cases}$$

is a solution of (2.1E.14) on $(-\infty,\infty)$ if and only if $a_1=b_1$. From this, justify the statement that y is a solution of (2.1E.14) on $(-\infty,\infty)$ if and only if

$$y=\begin{cases} c_1x^2+c_2x^3, & x\ge 0, \\ c_1x^2+c_3x^3, & x<0, \end{cases}$$

where c_1, c_2, and c_3 are arbitrary constants.

(c) For what values of k_0 and k_1 does the initial value problem

$$x^2y''-4xy'+6y=0, \quad y(0)=k_0, \quad y'(0)=k_1$$

have a solution? What are the solutions?

(d) Show that if $x_0 \neq 0$ and k_0, k_1 are arbitrary constants, the initial value problem

$$x^2y'' - 4xy' + 6y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{2.1E.15}$$

has infinitely many solutions on $(-\infty, \infty)$. On what interval does (2.1E.15) have a unique solution?

Answer

Add texts here. Do not delete this text first.

2.1E.43 Exercise $\mathbf{2.1E.43}$

(a) Verify that $y_1 = x$ and $y_2 = x^2$ satisfy

$$x^2y'' - 2xy' + 2y = 0 \tag{2.1E.16}$$

on $(-\infty, \infty)$ and that $\{y_1, y_2\}$ is a fundamental set of solutions of (2.1E.16) on $(-\infty, 0)$ and $(0, \infty)$.

(b) Let a_1, a_2, b_1, and b_2 be constants. Show that

$$y = \begin{cases} a_1x + a_2x^2, & x \geq 0, \\ b_1x + b_2x^2, & x < 0 \end{cases}$$

is a solution of (2.1E.16) on $(-\infty, \infty)$ if and only if $a_1 = b_1$ and $a_2 = b_2$. From this, justify the statement that the general solution of (2.1E.16) on $(-\infty, \infty)$ is $y = c_1x + c_2x^2$, where c_1 and c_2 are arbitrary constants.

(c) For what values of k_0 and k_1 does the initial value problem

$$x^2y'' - 2xy' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

(d) Show that if $x_0 \neq 0$ and k_0, k_1 are arbitrary constants then the initial value problem

$$x^2y'' - 2xy' + 2y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(-\infty, \infty)$.

Answer

Add texts here. Do not delete this text first.

2.1E.44 Exercise $\mathbf{2.1E.44}$

(a) Verify that $y_1 = x^3$ and $y_2 = x^4$ satisfy

$$x^2y'' - 6xy' + 12y = 0 \tag{2.1E.17}$$

on $(-\infty, \infty)$, and that $\{y_1, y_2\}$ is a fundamental set of solutions of (2.1E.17) on $(-\infty, 0)$ and $(0, \infty)$.

(b) Show that y is a solution of (2.1E.17) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} a_1 x^3 + a_2 x^4, & x \geq 0, \\ b_1 x^3 + b_2 x^4, & x < 0, \end{cases}$$

where a_1, a_2, b_1, and b_2 are arbitrary constants.

(c) For what values of k_0 and k_1 does the initial value problem

$$x^2 y'' - 6xy' + 12y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

(d) Show that if $x_0 \neq 0$ and k_0, k_1 are arbitrary constants then the initial value problem

$$x^2 y'' - 6xy' + 12y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{2.1E.18}$$

has infinitely many solutions on $(-\infty, \infty)$. On what interval does (2.1E.18) have a unique solution?

Answer

Add texts here. Do not delete this text first.

2.2: Linear Second Order Constant Coefficient Homogeneous Equations

This page is a draft and is under active development.

If a, b, and c are real constants and $a \neq 0$, then

$$ay'' + by' + cy = F(x)$$

is said to be a constant coefficient equation In this section we consider the homogeneous constant coefficient equation

$$ay'' + by' + cy = 0. \tag{2.2.1}$$

As we'll see, all solutions of (2.2.1) are defined on $(-\infty, \infty)$. This being the case, we'll omit references to the interval on which solutions are defined, or on which a given set of solutions is a fundamental set, etc., since the interval will always be $(-\infty, \infty)$.

The key to solving (2.2.1) is that if $y = e^{rx}$ where r is a constant then the left side of (2.2.1) is a multiple of e^{rx}; thus, if $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$, so

$$ay'' + by' + cy = ar^2 e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx}. \tag{2.2.2}$$

The quadratic polynomial

$$p(r) = ar^2 + br + c$$

is the characteristic polynomial of (2.2.1), and $p(r) = 0$ is the characteristic equation. From (2.2.2) we can see that $y = e^{rx}$ is a solution of (2.2.1) if and only if $p(r) = 0$.

The roots of the characteristic equation are given by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.2.3}$$

We consider three cases:

Case 1. $b^2 - 4ac > 0$, so the characteristic equation has two distinct real roots.

Case 2. $b^2 - 4ac = 0$, so the characteristic equation has a repeated real root.

Case 3. $b^2 - 4ac < 0$, so the characteristic equation has complex roots.

In each case we'll start with an example.

2.2.1 Case 1: Distinct Real Roots

Example 2.2.1

(a) Find the general solution of

$$y'' + 6y' + 5y = 0. \tag{2.2.4}$$

(b) Solve the initial value problem

$$y'' + 6y' + 5y = 0, \quad y(0) = 3,\ y'(0) = -1. \tag{2.2.5}$$

Answer

(a) The characteristic polynomial of (2.2.4) is

$$p(r) = r^2 + 6r + 5 = (r+1)(r+5).$$

Since $p(-1) = p(-5) = 0$, $y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are solutions of (2.2.4). Since $y_2/y_1 = e^{-4x}$ is nonconstant, Theorem $(2.1.6)$ implies that the general solution of (2.2.4) is

$$y = c_1 e^{-x} + c_2 e^{-5x}. \tag{2.2.6}$$

(b) We must determine c_1 and c_2 in (2.2.6) so that y satisfies the initial conditions in (2.2.5). Differentiating (2.2.6) yields

$$y' = -c_1 e^{-x} - 5c_2 e^{-5x}. \tag{2.2.7}$$

Imposing the initial conditions $y(0) = 3,\ y'(0) = -1$ in (2.2.6) and (2.2.7) yields

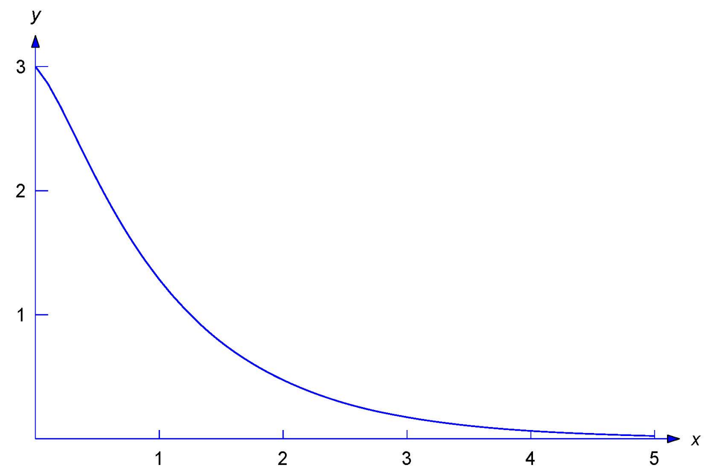
$$\begin{array}{rcr} c_1 + \ \ c_2 & = & 3 \\ -c_1 - 5c_2 & = & -1. \end{array}$$

The solution of this system is $c_1 = 7/2, c_2 = -1/2$. Therefore the solution of (2.2.5) is

$$y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}.$$

Figure $2.2.1$ is a graph of this solution.

$$y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}$$



If the characteristic equation has arbitrary distinct real roots r_1 and r_2, then $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are solutions of $ay'' + by' + cy = 0$. Since $y_2/y_1 = e^{(r_2 - r_1)x}$ is nonconstant, Theorem $(2.1.6)$ implies that $\{y_1, y_2\}$ is a fundamental set of solutions of $ay'' + by' + cy = 0$.

2.2.2 Case 2: A Repeated Real Root

Example 2.2.2

(a) Find the general solution of

$$y'' + 6y' + 9y = 0. \tag{2.2.8}$$

(b) Solve the initial value problem

$$y'' + 6y' + 9y = 0, \quad y(0) = 3, \ y'(0) = -1. \tag{2.2.9}$$

Answer

(a) The characteristic polynomial of (2.2.8) is

$$p(r) = r^2 + 6r + 9 = (r + 3)^2,$$

so the characteristic equation has the repeated real root $r_1 = -3$. Therefore $y_1 = e^{-3x}$ is a solution of (2.2.8). Since the characteristic equation has no other roots, (2.2.8) has no other solutions of the form e^{rx}. We look for solutions of the form $y = uy_1 = ue^{-3x}$, where u is a function that we'll now determine.

This should remind you of the method of variation of parameters used in Section 2.1 to solve the nonhomogeneous equation $y' + p(x)y = f(x)$, given a solution y_1 of the complementary equation $y' + p(x)y = 0$. It's also a special case of a method called reduction of order that we'll study in Section 5.6. For other ways to obtain a second solution of (2.2.8) that's not a multiple of e^{-3x}, see Exercises $(2.1E.9)$, $(2.1E.12)$, and $(2.2E.33)$.

If $y = ue^{-3x}$, then

$$y' = u'e^{-3x} - 3ue^{-3x} \quad \text{and} \quad y'' = u''e^{-3x} - 6u'e^{-3x} + 9ue^{-3x},$$

so

$$\begin{aligned} y'' + 6y' + 9y = &\ e^{-3x}\left[(u'' - 6u' + 9u) + 6(u' - 3u) + 9u\right] \\ = &\ e^{-3x}\left[u'' - (6 - 6)u' + (9 - 18 + 9)u\right] = u''e^{-3x}. \end{aligned}$$

Therefore $y = ue^{-3x}$ is a solution of (2.2.8) if and only if $u'' = 0$, which is equivalent to $u = c_1 + c_2 x$, where c_1 and c_2 are constants. Therefore any function of the form

$$y = e^{-3x}(c_1 + c_2 x) \tag{2.2.10}$$

is a solution of (2.2.8). Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-3x}$ that we already knew. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = xe^{-3x}$. Since $y_2/y_1 = x$ is nonconstant, Theorem $(2.1.6)$ implies that $\{y_1, y_2\}$ is fundamental set of solutions of (2.2.8), and (2.2.10) is the general solution.

(b) Differentiating (2.2.10) yields

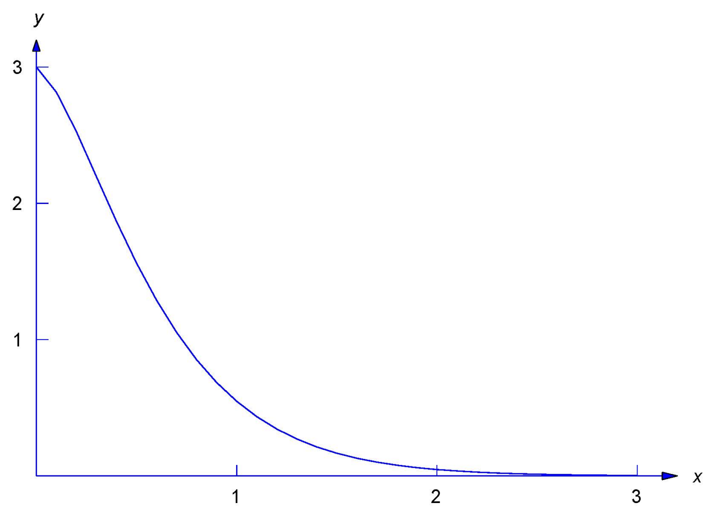
$$y' = -3e^{-3x}(c_1 + c_2x) + c_2e^{-3x}. \tag{2.2.11}$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ in (2.2.10) and (2.2.11) yields $c_1 = 3$ and $-3c_1 + c_2 = -1$, so $c_2 = 8$. Therefore the solution of (2.2.9) is

$$y = e^{-3x}(3 + 8x).$$

Figure $2.2.2$ is a graph of this solution.

$y = e^{-3x}(3 + 8x)$



If the characteristic equation of $ay'' + by' + cy = 0$ has an arbitrary repeated root r_1, the characteristic polynomial must be

$$p(r) = a(r - r_1)^2 = a(r^2 - 2r_1r + r_1^2).$$

Therefore

$$ar^2 + br + c = ar^2 - (2ar_1)r + ar_1^2,$$

which implies that $b = -2ar_1$ and $c = ar_1^2$. Therefore $ay'' + by' + cy = 0$ can be written as $a(y'' - 2r_1y' + r_1^2y) = 0$. Since $a \neq 0$ this equation has the same solutions as

$$y'' - 2r_1y' + r_1^2y = 0. \tag{2.2.12}$$

Since $p(r_1) = 0$, $y_1 = e^{r_1x}$ is a solution of $ay'' + by' + cy = 0$, and therefore of (2.2.12). Proceeding as in Example $(2.2.2)$, we look for other solutions of (2.2.12) of the form $y = ue^{r_1x}$; then

$$y' = u'e^{r_1x} + rue^{r_1x} \quad \text{and} \quad y'' = u''e^{r_1x} + 2r_1u'e^{r_1x} + r_1^2ue^{r_1x},$$

so

$$\begin{aligned} y'' - 2r_1y' + r_1^2y = & \; e^{rx}\left[(u'' + 2r_1u' + r_1^2u) - 2r_1(u' + r_1u) + r_1^2u\right] \\ = & \; e^{r_1x}\left[u'' + (2r_1 - 2r_1)u' + (r_1^2 - 2r_1^2 + r_1^2)u\right] = u''e^{r_1x}. \end{aligned}$$

Therefore $y = ue^{r_1x}$ is a solution of (2.2.12) if and only if $u'' = 0$, which is equivalent to $u = c_1 + c_2x$, where c_1 and c_2 are constants. Hence, any function of the form

$$y = e^{r_1x}(c_1 + c_2x) \tag{2.2.13}$$

is a solution of (2.2.12). Letting $c_1 = 1$ and $c_2 = 0$ here yields the solution $y_1 = e^{r_1x}$ that we already knew. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = xe^{r_1x}$. Since $y_2/y_1 = x$ is nonconstant, Theorem $(2.1.6)$ implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (2.2.12), and (2.2.13) is the general solution.

2.2.3 Case 3: Complex Conjugate Roots

Example 2.2.3

(a) Find the general solution of

$$y'' + 4y' + 13y = 0. \tag{2.2.14}$$

(b) Solve the initial value problem

$$y'' + 4y' + 13y = 0, \quad y(0) = 2, \; y'(0) = -3. \tag{2.2.15}$$

Answer

(a) The characteristic polynomial of (2.2.14) is

$$p(r) = r^2 + 4r + 13 = r^2 + 4r + 4 + 9 = (r+2)^2 + 9.$$

The roots of the characteristic equation are $r_1 = -2 + 3i$ and $r_2 = -2 - 3i$. By analogy with Case 1, it's reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions of (2.2.14). This is true (see Exercise $(2.2E.34)$); however, there are difficulties here, since you are probably not familiar with exponential functions with complex arguments, and even if you are, it's inconvenient to work with them, since they are complex--valued. We'll take a simpler approach, which we motivate as follows: the exponential notation suggests that

$$e^{(-2+3i)x} = e^{-2x}e^{3ix} \quad \text{and} \quad e^{(-2-3i)x} = e^{-2x}e^{-3ix},$$

so even though we haven't defined e^{3ix} and e^{-3ix}, it's reasonable to expect that every linear combination of $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ can be written as $y = ue^{-2x}$, where u depends upon x. To determine u, we note that if $y = ue^{-2x}$ then

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y'' = u''e^{-2x} - 4u'e^{-2x} + 4ue^{-2x},$$

so

$$\begin{aligned} y''+4y'+13y = & \; e^{-2x}\left[(u''-4u'+4u)+4(u'-2u)+13u\right] \\ = & \; e^{-2x}\left[u''-(4-4)u'+(4-8+13)u\right] = e^{-2x}(u''+9u). \end{aligned}$$

Therefore $y=ue^{-2x}$ is a solution of (2.2.14) if and only if

$$u''+9u=0.$$

From Example (2.1.2), the general solution of this equation is

$$u=c_1\cos 3x+c_2\sin 3x.$$

Therefore any function of the form

$$y=e^{-2x}(c_1\cos 3x+c_2\sin 3x) \tag{2.2.16}$$

is a solution of (2.2.14). Letting $c_1=1$ and $c_2=0$ yields the solution $y_1=e^{-2x}\cos 3x$. Letting $c_1=0$ and $c_2=1$ yields the second solution $y_2=e^{-2x}\sin 3x$. Since $y_2/y_1=\tan 3x$ is nonconstant, Theorem (2.1.6) implies that $\{y_1,y_2\}$ is a fundamental set of solutions of (2.2.14), and (2.2.16) is the general solution.

(b) Imposing the condition $y(0)=2$ in (2.2.16) shows that $c_1=2$. Differentiating (2.2.16) yields

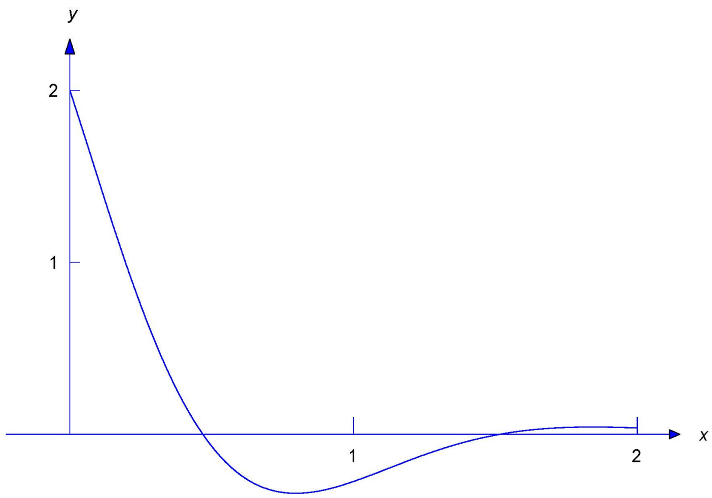
$$y'=-2e^{-2x}(c_1\cos 3x+c_2\sin 3x)+3e^{-2x}(-c_1\sin 3x+c_2\cos 3x),$$

and imposing the initial condition $y'(0)=-3$ here yields $-3=-2c_1+3c_2=-4+3c_2$, so $c_2=1/3$. Therefore the solution of (2.2.15) is

$$y=e^{-2x}(2\cos 3x+\frac{1}{3}\sin 3x).$$

Figure $2.2.3$ Is a graph of this function.

$$y=e^{-2x}(2\cos 3x+\frac{1}{3}\sin 3x)$$



Now suppose the characteristic equation of $ay'' + by' + cy = 0$ has arbitrary complex roots; thus, $b^2 - 4ac < 0$ and, from (2.2.3), the roots are

$$r_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a}, \quad r_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a},$$

which we rewrite as

$$r_1 = \lambda + i\omega, \quad r_2 = \lambda - i\omega, \tag{2.2.17}$$

with

$$\lambda = -\frac{b}{2a}, \quad \omega = \frac{\sqrt{4ac - b^2}}{2a}.$$

Don't memorize these formulas. Just remember that r_1 and r_2 are of the form (2.2.17), where λ is an arbitrary real number and ω is positive; λ and ω are the real and imaginary parts, respectively, of r_1. Similarly, λ and $-\omega$ are the real and imaginary parts of r_2. We say that r_1 and r_2 are complex conjugates, which means that they have the same real part and their imaginary parts have the same absolute values, but opposite signs.

As in Example (2.2.3), it's reasonable to to expect that the solutions of $ay'' + by' + cy = 0$ are linear combinations of $e^{(\lambda + i\omega)x}$ and $e^{(\lambda - i\omega)x}$. Again, the exponential notation suggests that

$$e^{(\lambda + i\omega)x} = e^{\lambda x} e^{i\omega x} \quad \text{and} \quad e^{(\lambda - i\omega)x} = e^{\lambda x} e^{-i\omega x},$$

so even though we haven't defined $e^{i\omega x}$ and $e^{-i\omega x}$, it's reasonable to expect that every linear combination of $e^{(\lambda + i\omega)x}$ and $e^{(\lambda - i\omega)x}$ can be written as $y = ue^{\lambda x}$, where u depends upon x. To determine u we first observe that since $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ are the roots of the characteristic equation, p must be of the form

$$\begin{aligned} p(r) &= a(r-r_1)(r-r_2) \\ &= a(r-\lambda-i\omega)(r-\lambda+i\omega) \\ &= a\left[(r-\lambda)^2+\omega^2\right] \\ &= a(r^2-2\lambda r+\lambda^2+\omega^2). \end{aligned}$$

Therefore $ay''+by'+cy=0$ can be written as

$$a\left[y''-2\lambda y'+(\lambda^2+\omega^2)y\right]=0.$$

Since $a\neq 0$ this equation has the same solutions as

$$y''-2\lambda y'+(\lambda^2+\omega^2)y=0. \tag{2.2.18}$$

To determine u we note that if $y=ue^{\lambda x}$ then

$$y'=u'e^{\lambda x}+\lambda ue^{\lambda x} \quad \text{and} \quad y''=u''e^{\lambda x}+2\lambda u'e^{\lambda x}+\lambda^2 ue^{\lambda x}.$$

Substituting these expressions into (2.2.18) and dropping the common factor $e^{\lambda x}$ yields

$$(u''+2\lambda u'+\lambda^2 u)-2\lambda(u'+\lambda u)+(\lambda^2+\omega^2)u=0,$$

which simplifies to

$$u''+\omega^2 u=0.$$

From Example (2.1.2), the general solution of this equation is

$$u=c_1\cos\omega x+c_2\sin\omega x.$$

Therefore any function of the form

$$y=e^{\lambda x}(c_1\cos\omega x+c_2\sin\omega x) \tag{2.2.19}$$

is a solution of (2.2.18). Letting $c_1=1$ and $c_2=0$ here yields the solution $y_1=e^{\lambda x}\cos\omega x$. Letting $c_1=0$ and \)c_2=1\) yields a second solution $y_2=e^{\lambda x}\sin\omega x$. Since $y_2/y_1=\tan\omega x$ is nonconstant, so Theorem (2.1.6) implies that $\{y_1,y_2\}$ is a fundamental set of solutions of (2.2.18), and (2.2.19) is the general solution.

2.2.4 Summary

The next theorem summarizes the results of this section.

Theorem 2.2.1

Let $p(r)=ar^2+br+c$ be the characteristic polynomial of

$$ay''+by'+cy=0. \tag{2.2.20}$$

Then:

(a) If $p(r)=0$ has distinct real roots r_1 and r_2, then the general solution of (2.2.20) is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

(b) If $p(r) = 0$ has a repeated root r_1, then the general solution of (2.2.20) is

$$y = e^{r_1 x}(c_1 + c_2 x).$$

(c) If $p(r) = 0$ has complex conjugate roots $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ (where $\omega > 0$), then the general solution of (2.2.20) is

$$y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x).$$

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

2.2E: Exercises

This page is a draft and is under active development.

In Exercises $(2.2E.1)$ to $(2.2E.12)$, find the general solution.

2.2E.1 Exercise $\mathbf{2.2E.1}$

$y'' + 5y' - 6y = 0$

Answer

Add texts here. Do not delete this text first.

2.2E.2 Exercise $\mathbf{2.2E.2}$

$y'' - 4y' + 5y = 0$

Answer

Add texts here. Do not delete this text first.

2.2E.3 Exercise $\mathbf{2.2E.3}$

$y'' + 8y' + 7y = 0$

Answer

Add texts here. Do not delete this text first.

2.2E.4 Exercise $\mathbf{2.2E.4}$

$y'' - 4y' + 4y = 0$

Answer

Add texts here. Do not delete this text first.

2.2E.5 Exercise $\mathbf{2.2E.5}$

$y'' + 2y' + 10y = 0$

Answer

Add texts here. Do not delete this text first.

2.2E.6 Exercise $\mathbf{2.2E.6}$

$y'' + 6y' + 10y = 0$

Answer

Add texts here. Do not delete this text first.

2.2E.7 Exercise $\mathbf{2.2E.7}$

$y'' - 8y' + 16y = 0$

Answer

Add texts here. Do not delete this text first.

2.2E.8 Exercise $\mathbf{2.2E.8}$

$y'' + y' = 0$

Answer

Add texts here. Do not delete this text first.

2.2E.9 Exercise $\mathbf{2.2E.9}$

$y'' - 2y' + 3y = 0$

Answer

Add texts here. Do not delete this text first.

2.2E.10 Exercise $\mathbf{2.2E.10}$

$y'' + 6y' + 13y = 0$

Answer

Add texts here. Do not delete this text first.

2.2E.11 Exercise $\mathbf{2.2E.11}$

$4y'' + 4y' + 10y = 0$

Answer

Add texts here. Do not delete this text first.

2.2E.12 Exercise $\mathbf{2.2E.12}$

$10y'' - 3y' - y = 0$

Answer

Add texts here. Do not delete this text first.

In Exercises $(2.2E.13)$ to $(2.2E.17)$, solve the initial value problem.

2.2E.13 Exercise $\mathbf{2.2E.13}$

$y'' + 14y' + 50y = 0, \quad y(0) = 2, \quad y'(0) = -17$

Answer

Add texts here. Do not delete this text first.

2.2E.14 Exercise $\mathbf{2.2E.14}$

$6y''-y'-y=0, \quad y(0)=10, \quad y'(0)=0$

Answer

Add texts here. Do not delete this text first.

2.2E.15 Exercise $\mathbf{2.2E.15}$

$6y''+y'-y=0, \quad y(0)=-1, \quad y'(0)=3$

Answer

Add texts here. Do not delete this text first.

2.2E.16 Exercise $\mathbf{2.2E.16}$

$4y''-4y'-3y=0, \quad y(0)=\frac{13}{12}, \quad y'(0)=\frac{23}{24}$

Answer

Add texts here. Do not delete this text first.

2.2E.17 Exercise $\mathbf{2.2E.17}$

$4y''-12y'+9y=0, \quad y(0)=3, \quad y'(0)=\frac{5}{2}$

Answer

Add texts here. Do not delete this text first.

In Exercises \9(2.2E.18)\) to $(2.2E.21)$, solve the initial value problem and graph the solution.

2.2E.18 Exercise $\mathbf{2.2E.18}$

$y''+7y'+12y=0, \quad y(0)=-1, \quad y'(0)=0$

Answer

Add texts here. Do not delete this text first.

2.2E.19 Exercise $\mathbf{2.2E.19}$

$y''-6y'+9y=0, \quad y(0)=0, \quad y'(0)=2$

Answer

Add texts here. Do not delete this text first.

2.2E.20 Exercise $\mathbf{2.2}\boldsymbol{E}\mathbf{.20}$

$36y''-12y'+y=0, \quad y(0)=3, \quad y'(0)=\frac{5}{2}$

Answer

Add texts here. Do not delete this text first.

2.2E.21 Exercise $\mathbf{2.2}\boldsymbol{E}\mathbf{.21}$

$y''+4y'+10y=0, \quad y(0)=3, \quad y'(0)=-2$

Answer

Add texts here. Do not delete this text first.

2.2E.22 Exercise $\mathbf{2.2}\boldsymbol{E}\mathbf{.22}$

(a) Suppose y is a solution of the constant coefficient homogeneous equation

$$ay''+by'+cy=0. \tag{2.2E.1}$$

Let $z(x)=y(x-x_0)$, where x_0 is an arbitrary real number. Show that

$$az''+bz'+cz=0.$$

(b) Let $z_1(x)=y_1(x-x_0)$ and $z_2(x)=y_2(x-x_0)$, where $\{y_1,y_2\}$ is a fundamental set of solutions of (2.2E.1) (A). Show that $\{z_1,z_2\}$ is also a fundamental set of solutions of (2.2E.1).

(c) The statement of Theorem $(2.2.1)$ is convenient for solving an initial value problem

$$ay''+by'+cy=0, \quad y(0)=k_0, \quad y'(0)=k_1,$$

where the initial conditions are imposed at $x_0=0$. However, if the initial value problem is

$$ay''+by'+cy=0, \quad y(x_0)=k_0, \quad y'(x_0)=k_1, \tag{2.2E.2}$$

where $x_0\neq 0$, then determining the constants in

$$y=c_1e^{r_1x}+c_2e^{r_2x}, \quad y=e^{r_1x}(c_1+c_2x), \text{ or } y=e^{\lambda x}(c_1\cos\omega x+c_2\sin\omega x)$$

(whichever is applicable) is more complicated. Use part (b) to restate Theorem $(2.2.1)$ in a form more convenient for solving (2.2E.2).

Answer

Add texts here. Do not delete this text first.

In Exercises $(2.2E.23)$ tp $(2.2E.28)$, use a method suggested by Exercise $(2.2E.22)$ to solve the initial value problem.

2.2E.23 Exercise $\mathbf{2.2E.23}$

$y'' + 3y' + 2y = 0, \quad y(1) = -1, \quad y'(1) = 4$

Answer

Add texts here. Do not delete this text first.

2.2E.24 Exercise $\mathbf{2.2E.24}$

$y'' - 6y' - 7y = 0, \quad y(2) = -\frac{1}{3}, \quad y'(2) = -5$

Answer

Add texts here. Do not delete this text first.

2.2E.25 Exercise $\mathbf{2.2E.25}$

$y'' - 14y' + 49y = 0, \quad y(1) = 2, \quad y'(1) = 11$

Answer

Add texts here. Do not delete this text first.

2.2E.26 Exercise $\mathbf{2.2E.26}$

$9y'' + 6y' + y = 0, \quad y(2) = 2, \quad y'(2) = -\frac{14}{3}$

Answer

Add texts here. Do not delete this text first.

2.2E.27 Exercise $\mathbf{2.2E.27}$

$9y'' + 4y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$

Answer

Add texts here. Do not delete this text first.

2.2E.28 Exercise $\mathbf{2.2E.28}$

$y'' + 3y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -1$

Answer

Add texts here. Do not delete this text first.

2.2E.29 Exercise $\mathbf{2.2E.29}$

Prove: If the characteristic equation of

$$ay'' + by' + cy = 0 \tag{2.2E.3}$$

has a repeated negative root or two roots with negative real parts, then every solution of (2.2E.3) approaches zero as $x \to \infty$.

Answer

Add texts here. Do not delete this text first.

2.2E.30 Exercise $\mathbf{2.2}\boldsymbol{E}\mathbf{.30}$

Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has distinct real roots r_1 and r_2. Use a method suggested by Exercise $(2.2E.22)$ to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

Answer

Add texts here. Do not delete this text first.

2.2E.31 Exercise $\mathbf{2.2}\boldsymbol{E}\mathbf{.31}$

Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has a repeated real root r_1. Use a method suggested by Exercise $(2.2E.22)$ to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

Answer

Add texts here. Do not delete this text first.

2.2E.32 Exercise $\mathbf{2.2}\boldsymbol{E}\mathbf{.32}$

Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has complex conjugate roots $\lambda \pm i\omega$. Use a method suggested by Exercise $(2.2E.22)$ to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

Answer

Add texts here. Do not delete this text first.

2.2E.33 Exercise $\mathbf{2.2}\boldsymbol{E}\mathbf{.33}$

Suppose the characteristic equation of

$$ay'' + by' + cy = 0 \tag{2.2E.4}$$

has a repeated real root r_1. Temporarily, think of e^{rx} as a function of two real variables x and r.

(a) Show that

$$a\frac{\partial^2}{\partial^2 x}(e^{rx}) + b\frac{\partial}{\partial x}(e^{rx}) + ce^{rx} = a(r - r_1)^2 e^{rx}. \tag{2.2E.5}$$

(b) Differentiate (2.2E.5) with respect to r to obtain

$$a\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial^2 x}(e^{rx})\right)+b\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}(e^{rx})\right)+c(xe^{rx})=[2+(r-r_1)x]a(r-r_1)e^{rx}. \tag{2.2E.6}$$

(c) Reverse the orders of the partial differentiations in the first two terms on the left side of (2.2E.6) to obtain

$$a\frac{\partial^2}{\partial x^2}(xe^{rx})+b\frac{\partial}{\partial x}(xe^{rx})+c(xe^{rx})=[2+(r-r_1)x]a(r-r_1)e^{rx}. \tag{2.2E.7}$$

(d) Set $r=r_1$ in (2.2E.5) and (2.2E.7) to see that $y_1=e^{r_1 x}$ and $y_2=xe^{r_1 x}$ are solutions of (2.2E.4).

Answer

Add texts here. Do not delete this text first.

2.2E.34 Exercise $\mathbf{2.2E.34}$

In calculus you learned that e^u, $\cos u$, and $\sin u$ can be represented by the infinite series

$$e^u=\sum_{n=0}^{\infty}\frac{u^n}{n!}=1+\frac{u}{1!}+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots+\frac{u^n}{n!}+\cdots \tag{2.2E.8}$$

$$\cos u=\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n}}{(2n)!}=1-\frac{u^2}{2!}+\frac{u^4}{4!}+\cdots+(-1)^n\frac{u^{2n}}{(2n)!}+\cdots, \tag{2.2E.9}$$

and

$$\sin u=\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n+1}}{(2n+1)!}=u-\frac{u^3}{3!}+\frac{u^5}{5!}+\cdots+(-1)^n\frac{u^{2n+1}}{(2n+1)!}+\cdots \tag{2.2E.10}$$

for all real values of u. Even though you have previously considered (2.2E.8) only for real values of u, we can set $u=i\theta$, where θ is real, to obtain

$$e^{i\theta}=\sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}. \tag{2.2E.11}$$

Given the proper background in the theory of infinite series with complex terms, it can be shown that the series in (2.2E.11) converges for all real θ.

(a) Recalling that $i^2=-1$, write enough terms of the sequence $\{i^n\}$ to convince yourself that the sequence is repetitive:

$$1,i,-1,-i,1,i,-1,-i,1,i,-1,-i,1,i,-1,-i,\cdots.$$

Use this to group the terms in (2.2E.11) as

$$\begin{aligned}e^{i\theta}&=\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{4}+\cdots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\cdots\right)\\&=\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!}+i\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!}.\end{aligned}$$

By comparing this result with (2.2E.9) and (2.2E.10), conclude that

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{2.2E.12}$$

This is Euler's Identity.

(b) Starting from

$$e^{i\theta_1}e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1), (\cos\theta_2 + i\sin\theta_2),$$

collect the real part (the terms not multiplied by i) and the imaginary part (the terms multiplied by i) on the right, and use the trigonometric identities

$$\begin{aligned} \cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ \sin(\theta_1+\theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \end{aligned}$$

to verify that

$$e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2},$$

as you would expect from the use of the exponential notation $e^{i\theta}$.

(c) If α and β are real numbers, define

$$e^{\alpha+i\beta} = e^{\alpha}e^{i\beta} = e^{\alpha}(\cos\beta + i\sin\beta). \tag{2.2E.13}$$

Show that if $z_1 = \alpha_1 + i\beta_1$ and $z_2 = \alpha_2 + i\beta_2$ then

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

(d) Let a, b, and c be real numbers, with $a \neq 0$. Let $z = u + iv$ where u and v are real-valued functions of x. Then we say that z is a solution of

$$ay'' + by' + cy = 0 \tag{2.2E.14}$$

if u and v are both solutions of (2.2E.14). Use Theorem $(2.2.1)$ (c) to verify that if the characteristic equation of (2.2E.14) has complex conjugate roots $\lambda \pm i\omega$ then $z_1 = e^{(\lambda+i\omega)x}$ and $z_2 = e^{(\lambda-i\omega)x}$ are both solutions of (2.2E.14).

Answer

Add texts here. Do not delete this text first.

2.3: Linear Second Order Nonhomogeneous Linear Equations

This page is a draft and is under active development.

We'll now consider the nonhomogeneous linear second order equation

$$y'' + p(x)y' + q(x)y = f(x), \tag{2.3.1}$$

where the forcing function f isn't identically zero. The next theorem, an extension of Theorem $(2.1.1)$, gives sufficient conditions for existence and uniqueness of solutions of initial value problems for (2.3.1). We omit the proof, which is beyond the scope of this book.

Theorem 2.3.1

Suppose p, q, and f are continuous on an open interval (a, b), let x_0 be any point in (a, b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on (a, b).

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

To find the general solution of (2.3.1) on an interval (a, b) where p, q, and f are continuous, it's necessary to find the general solution of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{2.3.2}$$

on (a, b). We call (2.3.2) the complementary equation for (2.3.1).

The next theorem shows how to find the general solution of (2.3.1) if we know one solution y_p of (2.3.1) and a fundamental set of solutions of (2.3.2). We call y_p a particular solution of (2.3.1); it can be any solution that we can find, one way or another.

Theorem 2.3.2

Suppose p, q, and f are continuous on (a, b). Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{2.3.3}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation

$$y'' + p(x)y' + q(x)y = 0 \tag{2.3.4}$$

on (a, b). Then y is a solution of (2.3.3) on (a, b) if and only if

$$y = y_p + c_1y_1 + c_2y_2, \tag{2.3.5}$$

where c_1 and c_2 are constants.

Proof

We first show that y in (2.3.5) is a solution of (2.3.3) for any choice of the constants c_1 and c_2. Differentiating (2.3.5) twice yields

$$y' = y_p' + c_1 y_1' + c_2 y_2' \quad \text{and} \quad y'' = y_p'' + c_1 y_1'' + c_2 y_2'',$$

so

$$\begin{aligned} y'' + p(x)y' + q(x)y = & \ (y_p'' + c_1 y_1'' + c_2 y_2'') + p(x)(y_p' + c_1 y_1' + c_2 y_2') + q(x)(y_p + c_1 y_1 + c_2 y_2) \\ = & \ (y_p'' + p(x)y_p' + q(x)y_p) + c_1 (y_1'' + p(x)y_1' + q(x)y_1) + c_2 (y_2'' + p(x)y_2' + q(x)y_2) \\ = & \ f + c_1 \cdot 0 + c_2 \cdot 0 = f, \end{aligned}$$

since y_p satisfies (2.3.3) and y_1 and y_2 satisfy (2.3.4).

Now we'll show that every solution of (2.3.3) has the form (2.3.5) for some choice of the constants c_1 and c_2. Suppose y is a solution of (2.3.3). We'll show that $y - y_p$ is a solution of (2.3.4), and therefore of the form $y - y_p = c_1 y_1 + c_2 y_2$, which implies (2.3.5). To see this, we compute

$$\begin{aligned} (y - y_p)'' + p(x)(y - y_p)' + q(x)(y - y_p) &= (y'' - y_p'') + p(x)(y' - y_p') + q(x)(y - y_p) \\ &= (y'' + p(x)y' + q(x)y) - (y_p'' + p(x)y_p' + q(x)y_p) \\ &= f(x) - f(x) = 0, \end{aligned}$$

since y and y_p both satisfy (2.3.3).

We say that (2.3.5) is the general solution of (2.3.3) on (a, b).

If P_0, P_1, and F are continuous and P_0 has no zeros on (a, b), then Theorem (2.3.2) implies that the general solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{2.3.6}$$

on (a, b) is $y = y_p + c_1 y_1 + c_2 y_2$, where y_p is a particular solution of (2.3.6) on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

on (a, b). To see this, we rewrite (2.3.6) as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = \frac{F(x)}{P_0(x)}$$

and apply Theorem (2.3.2) with $p = P_1/P_0$, $q = P_2/P_0$, and $f = F/P_0$.

To avoid awkward wording in examples and exercises, we won't specify the interval (a, b) when we ask for the general solution of a specific linear second order equation, or for a fundamental set of solutions of a homogeneous linear second order equation. Let's agree that this always means that we want the general solution (or a fundamental set of solutions, as the case may be) on every open interval on which p, q, and f are continuous if the equation is of the form (2.3.3), or on which P_0, P_1, P_2, and F are continuous and P_0 has no zeros, if the equation is of the form (2.3.6). We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0, P_1, P_2, and F are all continuous on an open interval (a, b), but P_0 does have a zero in (a, b), then (2.3.6) may fail to have a general solution on (a, b) in the sense just defined. Exercises $(2.1E.42)$, $(2.1E.43)$, and $(2.1E.44)$ illustrate this point for a homogeneous equation.

In this section we limit ourselves to applications of Theorem (2.3.2) where we can guess at the form of the particular solution.

Example 2.3.1

(a) Find the general solution of

$$y'' + y = 1. \tag{2.3.7}$$

(b) Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7. \tag{2.3.8}$$

Answer

(a) We can apply Theorem (2.3.2) with $(a, b) = (-\infty, \infty)$, since the functions $p \equiv 0$, $q \equiv 1$, and $f \equiv 1$ in (2.3.7) are continuous on $(-\infty, \infty)$. By inspection we see that $y_p \equiv 1$ is a particular solution of (2.3.7). Since $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the complementary equation $y'' + y = 0$, the general solution of (2.3.7) is

$$y = 1 + c_1 \cos x + c_2 \sin x. \tag{2.3.9}$$

(b) Imposing the initial condition $y(0) = 2$ in (2.3.9) yields $2 = 1 + c_1$, so $c_1 = 1$. Differentiating (2.3.9) yields

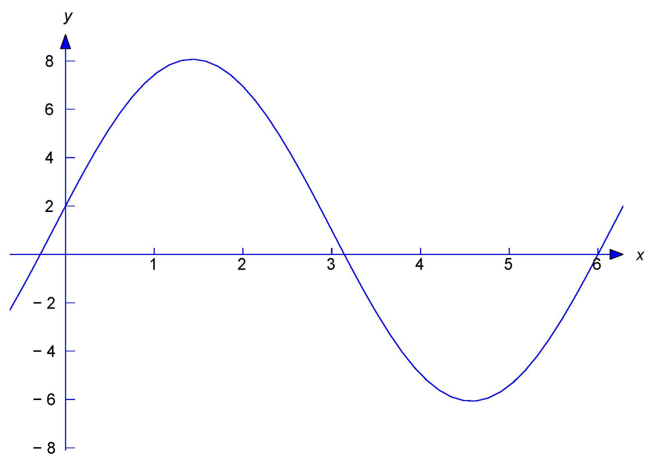
$$y' = -c_1 \sin x + c_2 \cos x.$$

Imposing the initial condition $y'(0) = 7$ here yields $c_2 = 7$, so the solution of (2.3.8) is

$$y = 1 + \cos x + 7 \sin x.$$

Figure 2.3.1 is a graph of this function.

$y = 1 + \cos x + 7 \sin x$



Example 2.3.2

(a) Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \tag{2.3.10}$$

(b) Solve the initial value problem

$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2, \quad y'(0) = 1. \tag{2.3.11}$$

Answer

(a) The characteristic polynomial of the complementary equation

$$y'' - 2y' + y = 0$$

is $r^2 - 2r + 1 = (r-1)^2$, so $y_1 = e^x$ and $y_2 = xe^x$ form a fundamental set of solutions of the complementary equation. To guess a form for a particular solution of (2.3.10), we note that substituting a second degree polynomial $y_p = A + Bx + Cx^2$ into the left side of (2.3.10) will produce another second degree polynomial with coefficients that depend upon A, B, and C. The trick is to choose A, B, and C so the polynomials on the two sides of (2.3.10) have the same coefficients; thus, if

$$y_p = A + Bx + Cx^2 \quad \text{then} \quad y_p' = B + 2Cx \text{\quad and \quad} y_p'' = 2C,$$

so

$$\begin{aligned} y_p'' - 2y_p' + y_p &= 2C - 2(B + 2Cx) + (A + Bx + Cx^2) \\ &= (2C - 2B + A) + (-4C + B)x + Cx^2 = -3 - x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} C &= 1 \\ B - 4C &= -1 \\ A - 2B + 2C &= -3, \end{aligned}$$

so $C = 1$, $B = -1 + 4C = 3$, and $A = -3 - 2C + 2B = 1$. Therefore $y_p = 1 + 3x + x^2$ is a particular solution of (2.3.10) and Theorem $(2.3.2)$ implies that

$$y = 1 + 3x + x^2 + e^x(c_1 + c_2 x) \tag{2.3.12}$$

is the general solution of (2.3.10).

(b) Imposing the initial condition $y(0) = -2$ in (2.3.12) yields $-2 = 1 + c_1$, so $c_1 = -3$. Differentiating (2.3.12) yields

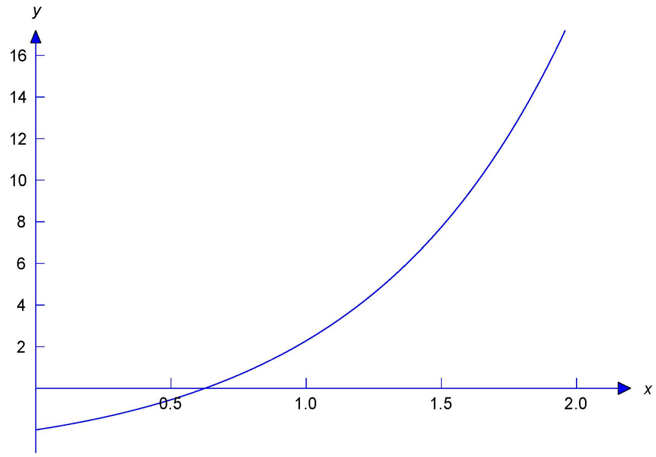
$$y' = 3 + 2x + e^x(c_1 + c_2 x) + c_2 e^x,$$

and imposing the initial condition $y'(0) = 1$ here yields $1 = 3 + c_1 + c_2$, so $c_2 = 1$. Therefore the solution of (2.3.11) is

$$y = 1 + 3x + x^2 - e^x(3 - x).$$

Figure $2.3.2$ is a graph of this solution.

$y = 1 + 3x + x^2 - e^x(3 - x)$



Example 2.3.3

Find the general solution of

$$x^2y'' + xy' - 4y = 2x^4 \tag{2.3.13}$$

on $(-\infty, 0)$ and $(0, \infty)$.

Answer

In Example (2.3.1), we verified that $y_1 = x^2$ and $y_2 = 1/x^2$ form a fundamental set of solutions of the complementary equation

$$x^2y'' + xy' - 4y = 0$$

on $(-\infty, 0)$ and $(0, \infty)$. To find a particular solution of (2.3.13), we note that if $y_p = Ax^4$, where A is a constant then both sides of (2.3.13) will be constant multiples of x^4 and we may be able to choose A so the two sides are equal. This is true in this example, since if $y_p = Ax^4$ then

$$x^2y_p'' + xy_p' - 4y_p = x^2(12Ax^2) + x(4Ax^3) - 4Ax^4 = 12Ax^4 = 2x^4$$

if $A = 1/6$; therefore, $y_p = x^4/6$ is a particular solution of (2.3.13) on $(-\infty, \infty)$. Theorem (2.3.2) implies that the general solution of (2.3.13) on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{x^4}{6} + c_1x^2 + \frac{c_2}{x^2}.$$

2.3.1 The Principle of Superposition

The next theorem enables us to break a nonhomogeneous equation into simpler parts, find a particular solution for each part, and then combine their solutions to obtain a particular solution of the original problem.

Theorem 2.3.3

Suppose y_{p_1} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on (a, b). Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

on (a, b).

Proof

If $y_p = y_{p_1} + y_{p_2}$ then

$$\begin{aligned} y_p'' + p(x)y_p' + q(x)y_p &= (y_{p_1} + y_{p_2})'' + p(x)(y_{p_1} + y_{p_2})' + q(x)(y_{p_1} + y_{p_2}) \\ &= \left(y_{p_1}'' + p(x)y_{p_1}' + q(x)y_{p_1}\right) + \left(y_{p_2}'' + p(x)y_{p_2}' + q(x)y_{p_2}\right) \\ &= f_1(x) + f_2(x). \end{aligned}$$

It's easy to generalize Theorem $(2.3.3)$ to the equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{2.3.14}$$

where

$$f = f_1 + f_2 + \cdots + f_k;$$

thus, if y_{p_i} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_i(x)$$

on (a, b) for $i = 1, 2, \ldots, k$, then $y_{p_1} + y_{p_2} + \cdots + y_{p_k}$ is a particular solution of (2.3.14) on (a, b). Moreover, by a proof similar to the proof of Theorem $(2.3.3)$ we can formulate the principle of superposition in terms of a linear equation written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$$

(Exercise $(2.3E.39)$); that is, if y_{p_1} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_2(x)$$

on (a, b), then $y_{p_1} + y_{p_2}$ is a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x) + F_2(x)$$

on (a, b).

Example 2.3.4

The function $y_{p_1} = x^4/15$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 \tag{2.3.15}$$

on $(-\infty, \infty)$ and $y_{p_2} = x^2/3$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 4x^2 \tag{2.3.16}$$

on $(-\infty, \infty)$. Use the principle of superposition to find a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 + 4x^2 \tag{2.3.17}$$

on $(-\infty, \infty)$.

Answer

The right side $F(x) = 2x^4 + 4x^2$ in (2.3.17) is the sum of the right sides

$$F_1(x) = 2x^4 \quad \text{and} \quad F_2(x) = 4x^2.$$

in (2.3.15) and (2.3.16). Therefore the principle of superposition implies that

$$y_p = y_{p_1} + y_{p_2} = \frac{x^4}{15} + \frac{x^2}{3}$$

is a particular solution of (2.3.17).

2.3E: Exercises

This page is a draft and is under active development.

In Exercises $(2.3E.1)$ to $(2.3E.6)$, find a particular solution by the method used in Example $(2.3.2)$. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

2.3E.1 Exercise $\mathbf{2.3}\boldsymbol{E}\mathbf{.1}$

$y'' + 5y' - 6y = 22 + 18x - 18x^2$

Answer

Add texts here. Do not delete this text first.

2.3E.2 Exercise $\mathbf{2.3}\boldsymbol{E}\mathbf{.2}$

$y'' - 4y' + 5y = 1 + 5x$

Answer

Add texts here. Do not delete this text first.

2.3E.3 Exercise $\mathbf{2.3}\boldsymbol{E}\mathbf{.3}$

$y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3$

Answer

Add texts here. Do not delete this text first.

2.3E.4 Exercise $\mathbf{2.3}\boldsymbol{E}\mathbf{.4}$

$y'' - 4y' + 4y = 2 + 8x - 4x^2$

Answer

Add texts here. Do not delete this text first.

2.3E.5 Exercise $\mathbf{2.3}\boldsymbol{E}\mathbf{.5}$

$y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3, \quad y(0) = 2, \quad y'(0) = 9$

Answer

Add texts here. Do not delete this text first.

2.3E.6 Exercise $\mathbf{2.3}\boldsymbol{E}\mathbf{.6}$

$y'' + 6y' + 10y = 22 + 20x, \quad y(0) = 2, \ y'(0) = -2$

Answer

Add texts here. Do not delete this text first.

2.3E.7 Exercise $\mathbf{2.3E.7}$

Show that the method used in Example $(2.3.2)$ won't yield a particular solution of

$$y''+y'=1+2x+x^2; \tag{2.3E.1}$$

that is, (2.3E.1) doesn't have a particular solution of the form $y_p=A+Bx+Cx^2$, where A, B, and C are constants.

Answer

Add texts here. Do not delete this text first.

In Exercises $(2.3E.8)$ to $(2.3E.13)$, find a particular solution by the method used in Example $(2.3.3)$.

2.3E.8 Exercise $\mathbf{2.3E.8}$

$x^2y''+7xy'+8y=\dfrac{6}{x}$

Answer

Add texts here. Do not delete this text first.

2.3E.9 Exercise $\mathbf{2.3E.9}$

$x^2y''-7xy'+7y=13x^{1/2}$

Answer

Add texts here. Do not delete this text first.

2.3E.10 Exercise $\mathbf{2.3E.10}$

$x^2y''-xy'+y=2x^3$

Answer

Add texts here. Do not delete this text first.

2.3E.11 Exercise $\mathbf{2.3E.11}$

$x^2y''+5xy'+4y=\dfrac{1}{x^3}$

Answer

Add texts here. Do not delete this text first.

2.3E.12 Exercise $\mathbf{2.3E.12}$

$x^2y'' + xy' + y = 10x^{1/3}$

Answer

Add texts here. Do not delete this text first.

2.3E.13 Exercise $\mathbf{2.3E.13}$

$x^2y'' - 3xy' + 13y = 2x^4$

Answer

Add texts here. Do not delete this text first.

2.3E.14 Exercise $\mathbf{2.3E.14}$

Show that the method suggested for finding a particular solution in Exercises $(2.3E.8)$ to $(2.3E.13)$ won't yield a particular solution of

$$x^2y'' + 3xy' - 3y = \frac{1}{x^3}; \tag{2.3E.2}$$

that is, (2.3E.2) doesn't have a particular solution of the form $y_p = A/x^3$.

Answer

Add texts here. Do not delete this text first.

2.3E.15 Exercise $\mathbf{2.3E.15}$

Prove: If a, b, c, α, and M are constants and $M \neq 0$ then

$$ax^2y'' + bxy' + cy = Mx^\alpha$$

has a particular solution $y_p = Ax^\alpha$ $(A =$ constant) if and only if $a\alpha(\alpha - 1) + b\alpha + c \neq 0$.

Answer

Add texts here. Do not delete this text first.

If a, b, c, and α are constants, then

$$a(e^{\alpha x})'' + b(e^{\alpha x})' + ce^{\alpha x} = (a\alpha^2 + b\alpha + c)e^{\alpha x}.$$

Use this in Exercises $(2.3E.16)$ to $(2.3E.21)$ to find a particular solution. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

2.3E.16 Exercise $\mathbf{2.3E.16}$

$y'' + 5y' - 6y = 6e^{3x}$

Answer

Add texts here. Do not delete this text first.

2.3E.17 Exercise $\mathbf{2.3E.17}$

$y'' - 4y' + 5y = e^{2x}$

Answer

Add texts here. Do not delete this text first.

2.3E.18 Exercise $\mathbf{2.3E.18}$

$y'' + 8y' + 7y = 10e^{-2x}, \quad y(0) = -2,\ y'(0) = 10$

Answer

Add texts here. Do not delete this text first.

2.3E.19 Exercise $\mathbf{2.3E.19}$

$y'' - 4y' + 4y = e^{x}, \quad y(0) = 2, \quad y'(0) = 0$

Answer

Add texts here. Do not delete this text first.

2.3E.20 Exercise $\mathbf{2.3E.20}$

$y'' + 2y' + 10y = e^{x/2}$

Answer

Add texts here. Do not delete this text first.

2.3E.21 Exercise $\mathbf{2.3E.21}$

$y'' + 6y' + 10y = e^{-3x}$

Answer

Add texts here. Do not delete this text first.

2.3E.22 Exercise $\mathbf{2.3E.22}$

Show that the method suggested for finding a particular solution in Exercises $(2.3E.16)$ to $(2.3E.21)$ won't yield a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}; \tag{2.3E.3}$$

that is, (2.3E.3) doesn't have a particular solution of the form $y_p = Ae^{4x}$.

Answer

Add texts here. Do not delete this text first.

2.3E.23 Exercise $\mathbf{2.3E.23}$

Prove: If α and M are constants and $M \neq 0$ then constant coefficient equation

$$ay'' + by' + cy = Me^{\alpha x}$$

has a particular solution $y_p = Ae^{\alpha x}$ ($A =$ constant) if and only if $e^{\alpha x}$ isn't a solution of the complementary equation.

Answer

Add texts here. Do not delete this text first.

If ω is a constant, differentiating a linear combination of $\cos \omega x$ and $\sin \omega x$ with respect to x yields another linear combination of $\cos \omega x$ and $\sin \omega x$. In Exercises $(2.3E.24)$ to $(2.3E.29)$ use this to find a particular solution of the equation. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

2.3E.24 Exercise $\mathbf{2.3E.24}$

$y'' - 8y' + 16y = 23\cos x - 7\sin x$

Answer

Add texts here. Do not delete this text first.

2.3E.25 Exercise $\mathbf{2.3E.25}$

$y'' + y' = -8\cos 2x + 6\sin 2x$

Answer

Add texts here. Do not delete this text first.

2.3E.26 Exercise $\mathbf{2.3E.26}$

$y'' - 2y' + 3y = -6\cos 3x + 6\sin 3x$

Answer

Add texts here. Do not delete this text first.

2.3E.27 Exercise $\mathbf{2.3E.27}$

$y'' + 6y' + 13y = 18\cos x + 6\sin x$

Answer

Add texts here. Do not delete this text first.

2.3E.28 Exercise $\textbf{2.3}\boldsymbol{E}\textbf{.28}$

$y'' + 7y' + 12y = -2\cos 2x + 36\sin 2x, \quad y(0) = -3, \quad y'(0) = 3$

Answer

Add texts here. Do not delete this text first.

2.3E.29 Exercise $\textbf{2.3}\boldsymbol{E}\textbf{.29}$

$y'' - 6y' + 9y = 18\cos 3x + 18\sin 3x, \quad y(0) = 2, \quad y'(0) = 2$

Answer

Add texts here. Do not delete this text first.

2.3E.30 Exercise $\textbf{2.3}\boldsymbol{E}\textbf{.30}$

Find the general solution of

$$y'' + \omega_0^2 y = M\cos\omega x + N\sin\omega x,$$

where M and N are constants and ω and ω_0 are distinct positive numbers.

Answer

Add texts here. Do not delete this text first.

2.3E.31 Exercise $\textbf{2.3}\boldsymbol{E}\textbf{.31}$

Show that the method suggested for finding a particular solution in Exercises $(2.3E.24)$ to $(2.3E.29)$ won't yield a particular solution of

$$y'' + y = \cos x + \sin x; \tag{2.3E.4}$$

that is, (2.3E.4) does not have a particular solution of the form $y_p = A\cos x + B\sin x$.

Answer

Add texts here. Do not delete this text first.

2.3E.32 Exercise $\textbf{2.3}\boldsymbol{E}\textbf{.32}$

Prove: If M, N are constants (not both zero) and $\omega > 0$, the constant coefficient equation

$$ay'' + by' + cy = M\cos\omega x + N\sin\omega x \tag{2.3E.5}$$

has a particular solution that's a linear combination of $\cos\omega x$ and $\sin\omega x$ if and only if the left side of (2.3E.5) is not of the form $a(y'' + \omega^2 y)$, so that $\cos\omega x$ and $\sin\omega x$ are solutions of the complementary equation.

Answer

Add texts here. Do not delete this text first.

In Exercises $(2.3E.33)$ to $2.3E.38)$, refer to the cited exercises and use the principal of superposition to find a particular solution. Then find the general solution.

2.3E.33 Exercise $\mathbf{2.3E.33}$

$y'' + 5y' - 6y = 22 + 18x - 18x^2 + 6e^{3x}$

(See Exercises $(2.3E.1)$ and $(2.3E.16)$.)

Answer

Add texts here. Do not delete this text first.

2.3E.34 Exercise $\mathbf{2.3E.34}$

$y'' - 4y' + 5y = 1 + 5x + e^{2x}$

(See Exercises $2.3E.2)$ and $(2.3E.17)$.)

Answer

Add texts here. Do not delete this text first.

2.3E.35 Exercise $\mathbf{2.3E.35}$

$y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3 + 10e^{-2x}$

(See Exercises $(2.3E.3)$ and $(2.3E.18)$.)

Answer

Add texts here. Do not delete this text first.

2.3E.36 Exercise $\mathbf{2.3E.36}$

$y'' - 4y' + 4y = 2 + 8x - 4x^2 + e^{x}$

(See Exercises $(2.3E.4)$ and $(2.3E.19)$.)

Answer

Add texts here. Do not delete this text first.

2.3E.37 Exercise $\mathbf{2.3E.37}$

$y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3 + e^{x/2}$

(See Exercises $(2.3E.5)$ and $(2.3E.20)$.)

Answer

Add texts here. Do not delete this text first.

2.3E.38 Exercise $\mathbf{2.3E.38}$

$y'' + 6y' + 10y = 22 + 20x + e^{-3x}$

(See Exercises $(2.3E.6)$ and $(2.3E.21)$.)

Answer

Add texts here. Do not delete this text first.

2.3E.39 Exercise $\mathbf{2.3E.39}$

Prove: If y_{p_1} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_2(x)$$

on (a, b), then $y_p = y_{p_1} + y_{p_2}$ is a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x) + F_2(x)$$

on (a, b).

Answer

Add texts here. Do not delete this text first.

2.3E.40 Exercise $\mathbf{2.3E.40}$

Suppose p, q, and f are continuous on (a, b). Let y_1, y_2, and y_p be twice differentiable on (a, b), such that $y = c_1y_1 + c_2y_2 + y_p$ is a solution of

$$y'' + p(x)y' + q(x)y = f$$

on (a, b) for every choice of the constants c_1, c_2. Show that y_1 and y_2 are solutions of the complementary equation on (a, b).

Answer

Add texts here. Do not delete this text first.

2.4: The Method of Undetermined Coefficient

This page is a draft and is under active development.

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x}G(x), \tag{2.4.1}$$

where α is a constant and G is a polynomial.

From Theorem $(2.3.2)$, the general solution of (2.4.1) is $y = y_p + c_1y_1 + c_2y_2$, where y_p is a particular solution of (2.4.1) and $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation

$$ay'' + by' + cy = 0.$$

In Section 2.2 we showed how to find $\{y_1, y_2\}$. In this section we'll show how to find y_p. The procedure that we'll use is called the method of undetermined coefficients.

Our first example is similar to Exercises $(2.3E.16)$ to $(2.3E.21)$.

Example 2.4.1

Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{2.4.2}$$

Then find the general solution.

Answer

Substituting $y_p = Ae^{2x}$ for y in (2.4.2) will produce a constant multiple of Ae^{2x} on the left side of (2.4.2), so it may be possible to choose A so that y_p is a solution of (2.4.2). Let's try it; if $y_p = Ae^{2x}$ then

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 2Ae^{2x} = 4e^{2x}$$

if $A = 2$. Therefore $y_p = 2e^{2x}$ is a particular solution of (2.4.2). To find the general solution, we note that the characteristic polynomial of the complementary equation

$$y'' - 7y' + 12y = 0 \tag{2.4.3}$$

is $p(r) = r^2 - 7r + 12 = (r-3)(r-4)$, so $\{e^{3x}, e^{4x}\}$ is a fundamental set of solutions of (2.4.3). Therefore the general solution of (2.4.2) is

$$y = 2e^{2x} + c_1e^{3x} + c_2e^{4x}.$$

Example 2.4.2

Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{2.4.4}$$

Then find the general solution.

Answer

Fresh from our success in finding a particular solution of 2.4.2 (where we chose $y_p = Ae^{2x}$ because the right side of 2.4.2 is a constant multiple of e^{2x}) it may seem reasonable to try $y_p = Ae^{4x}$ as a particular solution of 2.4.4. However, this won't work, since we saw in Example $(2.4.1)$ that e^{4x} is a solution of the complementary Equation 2.4.3, so substituting $y_p = Ae^{4x}$ into the left side of 2.4.4 produces zero on the left, no matter how we choose A. To discover a suitable form for y_p, we use the same approach that we used in Section 2.2 to find a second solution of

$$ay'' + by' + cy = 0$$

in the case where the characteristic equation has a repeated real root: we look for solutions of 2.4.4 in the form $y = ue^{4x}$, where u is a function to be determined. Substituting

$$y = ue^{4x}, \quad y' = u'e^{4x} + 4ue^{4x}, \quad \text{and} \quad y'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x} \tag{2.4.5}$$

into 2.4.4 and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 7(u' + 4u) + 12u = 5,$$

or

$$u'' + u' = 5.$$

By inspection we see that $u_p = 5x$ is a particular solution of this equation, so $y_p = 5xe^{4x}$ is a particular solution of 2.4.4. Therefore

$$y = 5xe^{4x} + c_1e^{3x} + c_2e^{4x}$$

is the general solution.

Example 2.4.3

Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{2.4.6}$$

Answer

Since the characteristic polynomial of the complementary equation

$$y'' - 8y' + 16y = 0 \tag{2.4.7}$$

is $p(r) = r^2 - 8r + 16 = (r-4)^2$, both $y_1 = e^{4x}$ and $y_2 = xe^{4x}$ are solutions of 2.4.7. Therefore \ref{eq:2.4.6) does not have a solution of the form \(y_p=Ae^{4x}\) or $y_p = Axe^{4x}$. As in Example $2.4.2$), we look for solutions of 2.4.6 in the form $y = ue^{4x}$, where u is a function to be determined. Substituting from 2.4.5 into 2.4.6 and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 8(u' + 4u) + 16u = 2,$$

or

$$u'' = 2.$$

Integrating twice and taking the constants of integration to be zero shows that $u_p = x^2$\$is a particular solution of this equation, so $y_p = x^2e^{4x}$ is a particular solution of 2.4.4. Therefore

$$y = e^{4x}(x^2 + c_1 + c_2x)$$

is the general solution.

The preceding examples illustrate the following facts concerning the form of a particular solution y_p of a constant coefficient equation

$$ay'' + by' + cy = ke^{\alpha x},$$

where k is a nonzero constant:

(a) If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay'' + by' + cy = 0, \tag{2.4.8}$$

then $y_p = Ae^{\alpha x}$, where A is a constant. (See Example $(2.4.1)$.)

(b) If $e^{\alpha x}$ is a solution of 2.4.8 but $xe^{\alpha x}$ is not, then $y_p = Axe^{\alpha x}$, where A is a constant. (See Example $(2.4.2)$.)

(c) If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of 2.4.8, then $y_p = Ax^2e^{\alpha x}$, where A is a constant. (See Example $(2.4.3)$.)

See Exercise $(2.4E.30)$ for the proofs of these facts.

In all three cases you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p'' + by_p' + cy_p = ke^{\alpha x},$$

and solve for the constant A, as we did in Example $(2.4.1)$. (See Exercises $(2.4E.31)$ to $(2.4E.33)$.) However, if the equation is

$$ay'' + by' + cy = ke^{\alpha x}G(x),$$

where G is a polynomial of degree greater than zero, we recommend that you use the substitution $y = ue^{\alpha x}$ as we did in Examples $(2.4.2)$ and $(2.4.3)$. The equation for u will turn out to be

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x), \tag{2.4.9}$$

where $p(r) = ar^2 + br + c$ is the characteristic polynomial of the complementary equation and $p'(r) = 2ar + b$ (Exercise $(2.4E.30)$); however, you shouldn't memorize this since it's easy to derive the equation for u in any particular case. Note, however, that if $e^{\alpha x}$ is a solution of the complementary equation then $p(\alpha)=0$, so 2.4.9 reduces to

$$au''+p'(\alpha)u'=G(x),$$

while if both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation then $p(r)=a(r-\alpha)^2$ and $p'(r)=2a(r-\alpha)$, so $p(\alpha)=p'(\alpha)=0$ and 2.4.9 reduces to

$$au''=G(x).$$

Example 2.4.4

Find a particular solution of

$$y''-3y'+2y=e^{3x}(-1+2x+x^2). \tag{2.4.10}$$

Answer

Substituting

$$y=ue^{3x},\quad y'=u'e^{3x}+3ue^{3x},\quad \text{and}\quad y''=u''e^{3x}+6u'e^{3x}+9ue^{3x}$$

into 2.4.10 and canceling e^{3x} yields

$$(u''+6u'+9u)-3(u'+3u)+2u=-1+2x+x^2,$$

or

$$u''+3u'+2u=-1+2x+x^2. \tag{2.4.11}$$

As in Example (2.3.2), in order to guess a form for a particular solution of 2.4.11, we note that substituting a second degree polynomial $u_p=A+Bx+Cx^2$ for u in the left side of 2.4.11 produces another second degree polynomial with coefficients that depend upon A, B, and C; thus,

$$\text{if}\quad u_p=A+Bx+Cx^2\quad\text{then}\quad u_p'=B+2Cx\quad\text{and}\quad u_p''=2C.$$

If u_p is to satisfy 2.4.11, we must have

$$\begin{aligned} u_p''+3u_p'+2u_p &= 2C+3(B+2Cx)+2(A+Bx+Cx^2)\\ &= (2C+3B+2A)+(6C+2B)x+2Cx^2=-1+2x+x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} 2C &= 1\\ 2B+6C &= 2\\ 2A+3B+2C &= -1. \end{aligned}$$

Solving these equations for C, B, and A (in that order) yields $C=1/2, B=-1/2, A=-1/4$. Therefore

$$u_p = -\frac{1}{4}(1 + 2x - 2x^2)$$

is a particular solution of 2.4.11, and

$$y_p = u_p e^{3x} = -\frac{e^{3x}}{4}(1 + 2x - 2x^2)$$

is a particular solution of 2.4.10.

Example 2.4.5

Find a particular solution of

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2). \tag{2.4.12}$$

Answer

Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and} \quad y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into 2.4.12 and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 4(u' + 3u) + 3u = 6 + 8x + 12x^2,$$

or

$$u'' + 2u' = 6 + 8x + 12x^2. \tag{2.4.13}$$

There's no u term in this equation, since e^{3x} is a solution of the complementary equation for 2.4.12. (See

Exercise $(2.4E.30)$.) Therefore 2.4.13 does not have a particular solution of the form $u_p = A + Bx + Cx^2$ that we used successfully in Example $(2.4.4)$, since with this choice of u_p,

$$u_p'' + 2u_p' = 2C + (B + 2Cx)$$

can't contain the last term $(12x^2)$ on the right side of 2.4.13. Instead, let's try $u_p = Ax + Bx^2 + Cx^3$ on the grounds that

$$u_p' = A + 2Bx + 3Cx^2 \quad \text{and} \quad u_p'' = 2B + 6Cx$$

together contain all the powers of x that appear on the right side of 2.4.13.

Substituting these expressions in place of u' and u'' in 2.4.13 yields

$$(2B + 6Cx) + 2(A + 2Bx + 3Cx^2) = (2B + 2A) + (6C + 4B)x + 6Cx^2 = 6 + 8x + 12x^2.$$

Comparing coefficients of like powers of x on the two sides of the last equality shows that u_p satisfies 2.4.13 if

$$\begin{array}{rcl} 6C & = & 12 \\ 4B+6C & = & 8 \\ 2A+2B & = & 6. \end{array}$$

Solving these equations successively yields $C=2$, $B=-1$, and $A=4$. Therefore

$$u_p = x(4-x+2x^2)$$

is a particular solution of 2.4.13, and

$$y_p = u_p e^{3x} = xe^{3x}(4-x+2x^2)$$

is a particular solution of 2.4.12.

Example 2.4.6

Find a particular solution of

$$4y''+4y'+y=e^{-x/2}(-8+48x+144x^2). \tag{2.4.14}$$

Answer

Substituting

$$y=ue^{-x/2},\quad y'=u'e^{-x/2}-\frac{1}{2}ue^{-x/2},\quad \text{and}\quad y''=u''e^{-x/2}-u'e^{-x/2}+\frac{1}{4}ue^{-x/2}$$

into 2.4.14 and canceling $e^{-x/2}$ yields

$$4\left(u''-u'+\frac{u}{4}\right)+4\left(u'-\frac{u}{2}\right)+u=4u''=-8+48x+144x^2,$$

or

$$u''=-2+12x+36x^2, \tag{2.4.15}$$

which does not contain u or u' because $e^{-x/2}$ and $xe^{-x/2}$ are both solutions of the complementary equation. (See Exercise $(2.4E.30)$.) To obtain a particular solution of 2.4.15 we integrate twice, taking the constants of integration to be zero; thus,

$$u_p'=-2x+6x^2+12x^3 \quad \text{and} \quad u_p=-x^2+2x^3+3x^4=x^2(-1+2x+3x^2).$$

Therefore

$$y_p=u_pe^{-x/2}=x^2e^{-x/2}(-1+2x+3x^2)$$

is a particular solution of 2.4.14.

2.4.1 Summary

The preceding examples illustrate the following facts concerning particular solutions of a constant coefficient equation of the form

$$ay''+by'+cy=e^{\alpha x}G(x),$$

where G is a polynomial (see Exercise $(2.4E.30)$):

(a) If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay''+by'+cy=0, \tag{2.4.16}$$

then $y_p=e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example $(2.4.4)$).

(b) If $e^{\alpha x}$ is a solution of 2.4.16 but $xe^{\alpha x}$ is not, then $y_p=xe^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example $(2.4.5)$.)

(c) If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of 2.4.16, then $y_p=x^2e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example $(2.4.6)$.)

In all three cases, you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p''+by_p'+cy_p=e^{\alpha x}G(x),$$

and solve for the coefficients of the polynomial Q. However, if you try this you will see that the computations are more tedious than those that you encounter by making the substitution $y=ue^{\alpha x}$ and finding a particular solution of the resulting equation for u. (See Exercises $(2.4E.34)$ to $(2.4E.36)$.) In Case (a) the equation for u will be of the form

$$au''+p'(\alpha)u'+p(\alpha)u=G(x),$$

with a particular solution of the form $u_p=Q(x)$, a polynomial of the same degree as G, whose coefficients can be found by the method used in Example $(2.4.4)$. In Case (b) the equation for u will be of the form

$$au''+p'(\alpha)u'=G(x)$$

(no u term on the left), with a particular solution of the form $u_p=xQ(x)$, where Q is a polynomial of the same degree as G whose coefficents can be found by the method used in Example $(2.4.5)$. In Case (c) the equation for u will be of the form

$$au''=G(x)$$

with a particular solution of the form $u_p=x^2Q(x)$ that can be obtained by integrating $G(x)/a$ twice and taking the constants of integration to be zero, as in Example $(2.4.6)$.

2.4.2 Using the Principle of Superposition

The next example shows how to combine the method of undetermined coefficients and Theorem $(2.3.3)$, the principle of superposition.

Example 2.4.7

Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}. \tag{2.4.17}$$

Answer

In Example $(2.4.1)$ we found that $y_{p_1} = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x},$$

and in Example $(2.4.2)$ we found that $y_{p_2} = 5xe^{4x}$ is a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

Therefore the principle of superposition implies that \(y_p=2e^{2x}+5xe^{4x}\) is a particular solution of 2.4.17.

2.4E: Exercises

This page is a draft and is under active development.

In Exercises $(2.4E.1)$ to $(2.4E.14)$, find a particular solution.

2.4E.1 Exercise $\mathbf{2.4E.1}$

$y''-3y'+2y=e^{3x}(1+x)$

Answer

Add texts here. Do not delete this text first.

2.4E.2 Exercise $\mathbf{2.4E.2}$

$y''-6y'+5y=e^{-3x}(35-8x)$

Answer

Add texts here. Do not delete this text first.

2.4E.3 Exercise $\mathbf{2.4E.3}$

$y''-2y'-3y=e^{x}(-8+3x)$

Answer

Add texts here. Do not delete this text first.

2.4E.4 Exercise $\mathbf{2.4E.4}$

$y''+2y'+y=e^{2x}(-7-15x+9x^2)$

Answer

Add texts here. Do not delete this text first.

2.4E.5 Exercise $\mathbf{2.4E.5}$

$y''+4y=e^{-x}(7-4x+5x^2)$

Answer

Add texts here. Do not delete this text first.

2.4E.6 Exercise $\mathbf{2.4E.6}$

$y''-y'-2y=e^{x}(9+2x-4x^2)$

Answer

Add texts here. Do not delete this text first.

2.4E.7 Exercise $\mathbf{2.4E.7}$

$y'' - 4y' - 5y = -6xe^{-x}$

Answer

Add texts here. Do not delete this text first.

2.4E.8 Exercise $\mathbf{2.4E.8}$

$y'' - 3y' + 2y = e^{x}(3 - 4x)$

Answer

Add texts here. Do not delete this text first.

2.4E.9 Exercise $\mathbf{2.4E.9}$

$y'' + y' - 12y = e^{3x}(-6 + 7x)$

Answer

Add texts here. Do not delete this text first.

2.4E.10 Exercise $\mathbf{2.4E.10}$

$2y'' - 3y' - 2y = e^{2x}(-6 + 10x)$

Answer

Add texts here. Do not delete this text first.

2.4E.11 Exercise $\mathbf{2.4E.11}$

$y'' + 2y' + y = e^{-x}(2 + 3x)$

Answer

Add texts here. Do not delete this text first.

2.4E.12 Exercise $\mathbf{2.4E.12}$

$y'' - 2y' + y = e^{x}(1 - 6x)$

Answer

Add texts here. Do not delete this text first.

2.4E.13 Exercise $\mathbf{2.4E.13}$

$y'' - 4y' + 4y = e^{2x}(1 - 3x + 6x^2)$

Answer

Add texts here. Do not delete this text first.

2.4E.14 Exercise $\mathbf{2.4E.14}$

$9y'' + 6y' + y = e^{-x/3}(2 - 4x + 4x^2)$

Answer

Add texts here. Do not delete this text first.

In Exercises $(2.4E.15)$ to $(2.4E.19)$, find the general solution.

2.4E.15 Exercise $\mathbf{2.4E.15}$

$y'' - 3y' + 2y = e^{3x}(1 + x)$

Answer

Add texts here. Do not delete this text first.

2.4E.16 Exercise $\mathbf{2.4E.16}$

$y'' - 6y' + 8y = e^{x}(11 - 6x)$

Answer

Add texts here. Do not delete this text first.

2.4E.17 Exercise $\mathbf{2.4E.17}$

$y'' + 6y' + 9y = e^{2x}(3 - 5x)$

Answer

Add texts here. Do not delete this text first.

2.4E.18 Exercise $\mathbf{2.4E.18}$

$y'' + 2y' - 3y = -16xe^{x}$

Answer

Add texts here. Do not delete this text first.

2.4E.19 Exercise $\mathbf{2.4E.19}$

$y'' - 2y' + y = e^{x}(2 - 12x)$

Answer

Add texts here. Do not delete this text first.

In Exercises $(2.4E.20)$ to $(2.4E.23)$, solve the initial value problem and plot the solution.

2.4E.20 Exercise $\mathbf{2.4E.20}$

$$y'' - 4y' - 5y = 9e^{2x}(1+x), \quad y(0) = 0, \quad y'(0) = -10$$

Answer

Add texts here. Do not delete this text first.

2.4E.21 Exercise $\mathbf{2.4E.21}$

$$y'' + 3y' - 4y = e^{2x}(7+6x), \quad y(0) = 2, \quad y'(0) = 8$$

Answer

Add texts here. Do not delete this text first.

2.4E.22 Exercise $\mathbf{2.4E.22}$

$$y'' + 4y' + 3y = -e^{-x}(2+8x), \quad y(0) = 1, \quad y'(0) = 2$$

Answer

Add texts here. Do not delete this text first.

2.4E.23 Exercise $\mathbf{2.4E.23}$

$$y'' - 3y' - 10y = 7e^{-2x}, \quad y(0) = 1, \quad y'(0) = -17$$

Answer

Add texts here. Do not delete this text first.

In Exercises $(2.4E.24)$ to $(2.4E.29)$, use the principle of superposition to find a particular solution.

2.4E.24 Exercise $\mathbf{2.4E.24}$

$$y'' + y' + y = xe^x + e^{-x}(1+2x)$$

Answer

Add texts here. Do not delete this text first.

2.4E.25 Exercise $\mathbf{2.4E.25}$

$$y'' - 7y' + 12y = -e^x(17-42x) - e^{3x}$$

Answer

Add texts here. Do not delete this text first.

2.4E.26 Exercise $\mathbf{2.4E.26}$

$$y'' - 8y' + 16y = 6xe^{4x} + 2 + 16x + 16x^2$$

Answer

Add texts here. Do not delete this text first.

2.4E.27 Exercise $\mathbf{2.4E.27}$

$y'' - 3y' + 2y = -e^{2x}(3 + 4x) - e^{x}$

Answer

Add texts here. Do not delete this text first.

2.4E.28 Exercise $\mathbf{2.4E.28}$

$y'' - 2y' + 2y = e^{x}(1 + x) + e^{-x}(2 - 8x + 5x^{2})$

Answer

Add texts here. Do not delete this text first.

2.4E.29 Exercise $\mathbf{2.4E.29}$

$y'' + y = e^{-x}(2 - 4x + 2x^{2}) + e^{3x}(8 - 12x - 10x^{2})$

Answer

Add texts here. Do not delete this text first.

2.4E.30 Exercise $\mathbf{2.4E.30}$

(a) Prove that y is a solution of the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x}G(x) \tag{2.4E.1}$$

if and only if $y = ue^{\alpha x}$, where u satisfies

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x) \tag{2.4E.2}$$

and $p(r) = ar^{2} + br + c$ is the characteristic polynomial of the complementary equation

$$ay'' + by' + cy = 0.$$

For the rest of this exercise, let G be a polynomial. Give the requested proofs for the case where

$$G(x) = g_0 + g_1x + g_2x^{2} + g_3x^{3}.$$

(b) Prove that if $e^{\alpha x}$ isn't a solution of the complementary equation then (2.4E.2) has a particular solution of the form $u_p = A(x)$, where A is a polynomial of the same degree as G, as in Example (2.4.4). Conclude that (2.4E.1) has a particular solution of the form $y_p = e^{\alpha x}A(x)$.

(c) Show that if $e^{\alpha x}$ is a solution of the complementary equation and $xe^{\alpha x}$ isn't, then (2.4E.2) has a particular solution of the form $u_p = xA(x)$, where A is a polynomial of the same degree as G, as in Example (2.4.5). Conclude that (2.4E.1) has a particular solution of the form $y_p = xe^{\alpha x}A(x)$.

(d) Show that if $e^{\alpha x}$ and $xe^{\alpha x}$ are both solutions of the complementary equation then (2.4E.2) has a particular solution of the form $u_p = x^2 A(x)$, where A is a polynomial of the same degree as G, and $x^2 A(x)$ can be obtained by integrating G/a twice, taking the constants of integration to be zero, as in Example $(2.4.6)$. Conclude that (2.4E.1) has a particular solution of the form $y_p = x^2 e^{\alpha x} A(x)$.

Answer

Add texts here. Do not delete this text first.

Exercises $(2.4E.31)$ to $(2.4E.36)$ treat the equations considered in Examples $(2.4.1)$ to $(2.4.6)$. Substitute the suggested form of y_p into the equation and equate the resulting coefficients of like functions on the two sides of the resulting equation to derive a set of simultaneous equations for the coefficients in y_p. Then solve for the coefficients to obtain y_p. Compare the work you've done with the work required to obtain the same results in Examples $(2.4.1)$ to $(2.4.6)$.

2.4E.31 Exercise $\mathbf{2.4E.31}$

Compare with Example $(2.4.1)$:

$$y'' - 7y' + 12y = 4e^{2x}; \quad y_p = Ae^{2x}$$

Answer

Add texts here. Do not delete this text first.

2.4E.32 Exercise $\mathbf{2.4E.32}$

Compare with Example $(2.4.2)$:

$$y'' - 7y' + 12y = 5e^{4x}; \quad y_p = Axe^{4x}$$

Answer

Add texts here. Do not delete this text first.

2.4E.33 Exercise $\mathbf{2.4E.33}$

Compare with Example $(2.4.3)$:

$$y'' - 8y' + 16y = 2e^{4x}; \quad y_p = Ax^2 e^{4x}$$

Answer

Add texts here. Do not delete this text first.

2.4E.34 Exercise $\mathbf{2.4E.34}$

Compare with Example $(2.4.4)$:

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2), \quad y_p = e^{3x}(A + Bx + Cx^2)$$

Answer

Add texts here. Do not delete this text first.

2.4E.35 Exercise $\mathbf{2.4}\boldsymbol{E}\mathbf{.35}$

Compare with Example (2.4.5):

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2), \quad y_p = e^{3x}(Ax + Bx^2 + Cx^3)$$

Answer

Add texts here. Do not delete this text first.

2.4E.36 Exercise $\mathbf{2.4}\boldsymbol{E}\mathbf{.36}$

Compare with Example (2.4.6):

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2), \quad y_p = e^{-x/2}(Ax^2 + Bx^3 + Cx^4)$$

Answer

Add texts here. Do not delete this text first.

2.4E.37 Exercise $\mathbf{2.4}\boldsymbol{E}\mathbf{.37}$

Write $y = ue^{\alpha x}$ to find the general solution.

(a) $y'' + 2y' + y = \dfrac{e^{-x}}{\sqrt{x}}$

(b) $y'' + 6y' + 9y = e^{-3x}\ln x$

(c) $y'' - 4y' + 4y = \dfrac{e^{2x}}{1+x}$

(d) $4y'' + 4y' + y = 4e^{-x/2}\left(\dfrac{1}{x} + x\right)$

Answer

Add texts here. Do not delete this text first.

2.4E.38 Exercise $\mathbf{2.4}\boldsymbol{E}\mathbf{.38}$

Suppose $\alpha \neq 0$ and k is a positive integer. In most calculus books integrals like $\int x^k e^{\alpha x}\,dx$ are evaluated by integrating by parts k times. This exercise presents another method. Let

$$y = \int e^{\alpha x} P(x)\,dx$$

with

$$P(x) = p_0 + p_1 x + \cdots + p_k x^k, \text{\quad (where } p_k \neq 0).$$

(a) Show that $y = e^{\alpha x} u$, where

$$u' + \alpha u = P(x). \tag{2.4E.3}$$

(b) Show that (2.4E.3) has a particular solution of the form

$$u_p = A_0 + A_1 x + \cdots + A_k x^k,$$

where A_k, A_{k-1}, ..., A_0 can be computed successively by equating coefficients of $x^k, x^{k-1}, \dots, 1$ on both sides of the equation

$$u_p' + \alpha u_p = P(x).$$

(c) Conclude that

$$\int e^{\alpha x} P(x)\, dx = \left(A_0 + A_1 x + \cdots + A_k x^k\right) e^{\alpha x} + c,$$

where c is a constant of integration.

Answer

Add texts here. Do not delete this text first.

2.4E.39 Exercise $\mathbf{2.4E.39}$

Use the method of Exercise $(2.4E.38)$ to evaluate the integral.

(a) $\int e^x (4 + x)\, dx$

(b) $\int e^{-x} (-1 + x^2)\, dx$

(c) $\int x^3 e^{-2x}\, dx$

(d) $\int e^x (1 + x)^2\, dx$

(e) $\int e^{3x} (-14 + 30x + 27x^2)\, dx$

(f) $\int e^{-x} (1 + 6x^2 - 14x^3 + 3x^4)\, dx$

Answer

Add texts here. Do not delete this text first.

2.4E.40 Exercise $\mathbf{2.4E.40}$

Use the method suggested in Exercise $(2.4E.38)$ to evaluate $\int x^k e^{\alpha x}\, dx$, where k is an arbitrary positive integer and $\alpha \neq 0$.

Answer

Add texts here. Do not delete this text first.

Welcome to the Mathematics Library. This Living Library is a principal hub of the LibreTexts project, which is a multi-institutional collaborative venture to develop the next generation of open-access texts to improve postsecondary education at all levels of higher learning. The LibreTexts approach is highly collaborative where an Open Access textbook environment is under constant revision by students, faculty, and outside experts to supplant conventional paper-based books.

CHAPTER OVERVIEW

3: Series Solutions of Linear Second order Equations

This page is a draft and is under active development.

In this chapter, we study a class of second order differential equations that occur in many applications, but can't be solved in closed form in terms of elementary functions. Here are some examples:

Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

which occurs in problems displaying cylindrical symmetry, such as diffraction of light through a circular aperture, propagation of electromagnetic radiation through a coaxial cable, and vibrations of a circular drum head.

Airy's equation

$$y'' - xy = 0,$$

which occurs in astronomy and quantum physics.

Legendre's equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

which occurs in problems displaying spherical symmetry, particularly in electromagnetism.

These equations and others considered in this chapter can be written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \tag{3.1}$$

where P_0, P_1, and P_2 are polynomials with no common factor. For most equations that occur in applications, these polynomials are of degree two or less. We'll impose this restriction, although the methods that we'll develop can be extended to the case where the coefficient functions are polynomials of arbitrary degree, or even power series that converge in some circle around the origin in the complex plane.

Since (3.1) does not in general have closed form solutions, we seek series representations for solutions. We'll see that if $P_0(0) \neq 0$ then solutions of (3.1) can be written as power series

$$y = \sum_{n=0}^{\infty} a_n x^n$$

that converge in an open interval centered at $x = 0$.

Topic hierarchy

3.1: Review of Power Series
3.1E: Exercises
3.2: Series Solutions Near an Ordinary Point I
3.2E: Exercises
3.3: Series Solutions Near an Ordinary Point II
3.3E: Exercises
3.4: Regular Singular Points: Euler Equations
3.4E: Exercises
3.5: The Method of Frobenius I
3.5E: Exercises
3.6: The Method of Frobenius II
3.6E: Exercises
3.7: The Method of Frobenius III
3.7E: Exercises

3.1: Review of Power Series

3.1.1 Power Series

Many applications give rise to differential equations with solutions that can't be expressed in terms of elementary functions such as polynomials, rational functions, exponential and logarithmic functions, and trigonometric functions. The solutions of some of the most important of these equations can be expressed in terms of power series. We'll study such equations in this chapter. In this section we review relevant properties of power series. We'll omit proofs, which can be found in any standard calculus text.

Theorem 3.1.1

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n, \tag{3.1.1}$$

where x_0 and $a_0, a_1, \ldots, a_n, \ldots$ are constants, is called a power series in $x-x_0$. We say that the power series (3.1.1) converges for a given x if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n (x-x_0)^n$$

exists; otherwise, we say that the power series diverges for the given x.

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

A power series in $x-x_0$ must converge if $x=x_0$, since the positive powers of $x-x_0$ are all zero in this case. This may be the only value of x for which the power series converges. However, the next theorem shows that if the power series converges for some $x \neq x_0$ then the set of all values of x for which it converges forms an interval.

Theorem 3.1.2

For any power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n,$$

exactly one of the these statements is true:

(i) The power series converges only for $x=x_0$.

(ii) The power series converges for all values of x.

(iii) There's a positive number R such that the power series converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$.

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

In case (iii), we say that R is the radius of convergence of the power series. For convenience, we include the other two cases in this definition by defining $R=0$ in case (i) and $R=\infty$ in case (ii). We define the open interval of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ to be

$$(x_0-R, x_0+R) \quad \text{if} \quad 0<R<\infty, \quad \text{or} \quad (-\infty,\infty) \quad \text{if} \quad R=\infty.$$

If R is finite, no general statement can be made concerning convergence at the endpoints $x=x_0 \pm R$ of the open interval of convergence; the series may converge at one or both points, or diverge at both.

Recall from calculus that a series of constants $\sum_{n=0}^{\infty} \alpha_n$ is said to converge absolutely if the series of absolute values $\sum_{n=0}^{\infty} |\alpha_n|$ converges. It can be shown that a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ with a positive radius of convergence R converges absolutely in its open interval of convergence; that is, the series

$$\sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

of absolute values converges if $|x-x_0|<R$. However, if $R<\infty$, the series may fail to converge absolutely at an endpoint $x_0 \pm R$, even if it converges there.

The next theorem provides a useful method for determining the radius of convergence of a power series. It's derived in calculus by applying the ratio test to the corresponding series of absolute values. For related theorems see Exercises $(3.1E.2)$ and $(3.1E.4)$.

Theorem 3.1.3

Suppose there's an integer N such that $a_n \neq 0$ if $n \geq N$ and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

where $0 \leq L \leq \infty$. Then the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $R=1/L$, which should be interpreted to mean that $R=0$ if $L=\infty$, or $R=\infty$ if $L=0$.

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

Example 3.1.1

Find the radius of convergence of the series:

(a) $\sum_{n=0}^{\infty} n!x^n$

(b) $\sum_{n=10}^{\infty} (-1)^n \frac{x^n}{n!}$

(c) $\sum_{n=0}^{\infty} 2^n n^2 (x-1)^n$

Answer

(a) Here $a_n = n!$, so

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$$

Hence, $R=0$.

(b) Here $a_n = (1)^n / n!$ for $n \geq N = 10$, so

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Hence, $R = \infty$.

(c) Here $a_n = 2^n n^2$, so

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2^{n+1}(n+1)^2}{2^n n^2} = 2 \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = 2.$$

Hence, $R = 1/2$.

3.1.2 Taylor Series

If a function f has derivatives of all orders at a point $x = x_0$, then the Taylor series of f about x_0 is defined by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

In the special case where $x_0 = 0$, this series is also called the Maclaurin series of f.

Taylor series for most of the common elementary functions converge to the functions on their open intervals of convergence. For example, you are probably familiar with the following Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty, \tag{3.1.2}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty, \tag{3.1.3}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty, \tag{3.1.4}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1. \tag{3.1.5}$$

3.1.3 Differentiation of Power Series

A power series with a positive radius of convergence defines a function

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

on its open interval of convergence. We say that the series represents f on the open interval of convergence. A function f represented by a power series may be a familiar elementary function as in (3.1.2) to (3.1.5); however, it often happens that f isn't a familiar function, so the series actually defines f.

The next theorem shows that a function represented by a power series has derivatives of all orders on the open interval of convergence of the power series, and provides power series representations of the derivatives.

Theorem 3.1.4

A power series

$$f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

with positive radius of convergence R has derivatives of all orders in its open interval of convergence, and successive derivatives can be obtained by repeatedly differentiating term by term; that is,

$$f'(x)=\sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}, \tag{3.1.6}$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}, \tag{3.1.7}$$

$$\vdots$$

$$f^{(k)}(x)=\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}. \tag{3.1.8}$$

Moreover, all of these series have the same radius of convergence R.

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

Example 3.1.2:

Let $f(x)=\sin x$. From (3.1.3),

$$f(x)=\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

From (3.1.6),

$$f'(x)=\sum_{n=0}^{\infty}(-1)^n \frac{d}{dx}\left[\frac{x^{2n+1}}{(2n+1)!}\right]=\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!},$$

which is the series (3.1.4) for $\cos x$.

3.1.4 Uniqueness of Power Series

The next theorem shows that if f is defined by a power series in $x-x_0$ with a positive radius of convergence, then the power series is the Taylor series of f about x_0.

Theorem 3.1.5

If the power series

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

has a positive radius of convergence, then

$$a_n=\frac{f^{(n)}(x_0)}{n!}; \tag{3.1.9}$$

that is, $\sum_{n=0}^{\infty}a_n(x-x_0)^n$ is the Taylor series of f about x_0.

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

This result can be obtained by setting $x=x_0$ in (3.1.8), which yields

$$f^{(k)}(x_0)=k(k-1)\cdots 1\cdot a_k=k!a_k.$$

This implies that

$$a_k=\frac{f^{(k)}(x_0)}{k!}.$$

Except for notation, this is the same as (3.1.9).

The next theorem lists two important properties of power series that follow from Theorem $(3.1.5)$.

Theorem 3.1.6

(a) If

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=\sum_{n=0}^{\infty}b_n(x-x_0)^n$$

for all x in an open interval that contains x_0, then $a_n=b_n$ for $n=0$, 1, 2, ….

(b) If

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=0$$

for all x in an open interval that contains x_0, then $a_n=0$ for $n=0$, 1, 2, ….

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

To obtain part (a) we observe that the two series represent the same function f on the open interval; hence, Theorem $(3.1.5)$ implies that

$$a_n=b_n=\frac{f^{(n)}(x_0)}{n!},\quad n=0,1,2,\dots.$$

Part (b) can be obtained from part (a) by taking $b_n=0$ for $n=0$, 1, 2, ….

3.1.5 Taylor Polynomials

If f has N derivatives at a point x_0, we say that

$$T_N(x)=\sum_{n=0}^{N}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is the Nth Taylor polynomial of f about x_0. This definition and Theorem $(3.1.5)$ imply that if

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n,$$

where the power series has a positive radius of convergence, then the Taylor polynomials of f about x_0 are given by

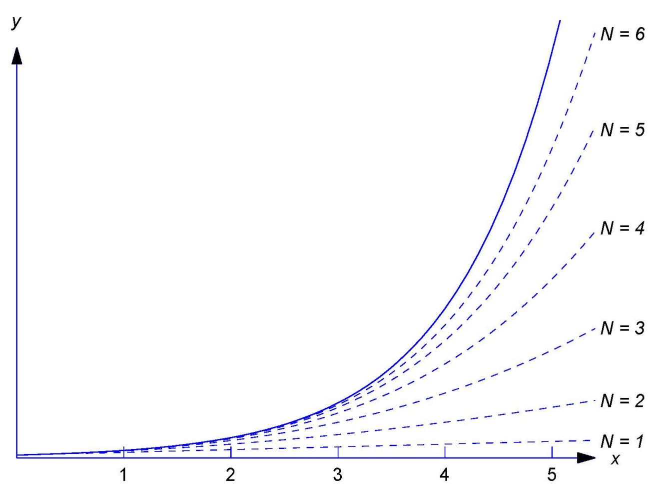
$$T_N(x)=\sum_{n=0}^{N}a_n(x-x_0)^n.$$

In numerical applications, we use the Taylor polynomials to approximate f on subintervals of the open interval of convergence of the power series. For example, (3.1.2) implies that the Taylor polynomial T_N of $f(x)=e^x$ is

$$T_N(x)=\sum_{n=0}^{N}\frac{x^n}{n!}.$$

The solid curve in Figure $3.1.1$ is the graph of $y=e^x$ on the interval $[0,5]$. The dotted curves in Figure $3.1.1$ are the graphs of the Taylor polynomials T_1, …, T_6 of $y=e^x$ about $x_0=0$. From this figure, we conclude that the accuracy of the approximation of $y=e^x$ by its Taylor polynomial T_N improves as N increases.

Approximation of $y=e^x$ by Taylor polynomials about $x=0$



3.1.6 Shifting the Summation Index

In Theorem $(3.1.1)$ of a power series in $x-x_0$, the nth term is a constant multiple of $(x-x_0)^n$. This isn't true in (3.1.6), (3.1.7), and (3.1.8), where the general terms are constant multiples of $(x-x_0)^{n-1}$,

$(x-x_0)^{n-2}$, and $(x-x_0)^{n-k}$, respectively. However, these series can all be rewritten so that their nth terms are constant multiples of $(x-x_0)^n$. For example, letting $n=k+1$ in the series in (3.1.6) yields

$$f'(x)=\sum_{k=0}^{\infty}(k+1)a_{k+1}(x-x_0)^k, \tag{3.1.10}$$

where we start the new summation index k from zero so that the first term in (3.1.10) (obtained by setting $k=0$) is the same as the first term in (3.1.6) (obtained by setting $n=1$). However, the sum of a series is independent of the symbol used to denote the summation index, just as the value of a definite integral is independent of the symbol used to denote the variable of integration. Therefore we can replace k by n in (3.1.10) to obtain

$$f'(x)=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_0)^n, \tag{3.1.11}$$

where the general term is a constant multiple of $(x-x_0)^n$.

It isn't really necessary to introduce the intermediate summation index k. We can obtain (3.1.11) directly from (3.1.6) by replacing n by $n+1$ in the general term of (3.1.6) and subtracting 1 from the lower limit of (3.1.6). More generally, we use the following procedure for shifting indices.

Shifting the Summation Index in a Power Series

For any integer k, the power series

$$\sum_{n=n_0}^{\infty} b_n(x-x_0)^{n-k}$$

can be rewritten as

$$\sum_{n=n_0-k}^{\infty} b_{n+k}(x-x_0)^n;$$

that is, replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation leaves the series unchanged.

Example 3.1.3

Rewrite the following power series from (3.1.7) and (3.1.8) so that the general term in each is a constant multiple of $(x-x_0)^n$:

(a) $\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$

(b) $\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}$.

Answer

(a) Replacing n by $n+2$ in the general term and subtracting 2 from the lower limit of summation yields

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n.$$

(b) Replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation yields

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k} = \sum_{n=0}^{\infty}(n+k)(n+k-1)\cdots(n+1)a_{n+k}(x-x_0)^n.$$

Example 3.1.4

Given that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

write the function xf'' as a power series in which the general term is a constant multiple of x^n.

Answer

From Theorem (3.1.4) with $x_0 = 0$,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$xf''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}.$$

Replacing n by $n+1$ in the general term and subtracting 1 from the lower limit of summation yields

$$xf''(x) = \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n.$$

We can also write this as

$$xf''(x) = \sum_{n=0}^{\infty}(n+1)na_{n+1}x^n,$$

since the first term in this last series is zero. (We'll see later that sometimes it's useful to include zero terms at the beginning of a series.)

3.1.7 Linear Combinations of Power Series

If a power series is multiplied by a constant, then the constant can be placed inside the summation; that is,

$$c\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} ca_n(x-x_0)^n.$$

Two power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

with positive radii of convergence can be added term by term at points common to their open intervals of convergence; thus, if the first series converges for $|x-x_0|<R_1$ and the second converges for $|x-x_0|<R_2$, then

$$f(x)+g(x)=\sum_{n=0}^{\infty}(a_n+b_n)(x-x_0)^n$$

for $|x-x_0|<R$, where R is the smaller of R_1 and R_2. More generally, linear combinations of power series can be formed term by term; for example,

$$c_1f(x)+c_2f(x)=\sum_{n=0}^{\infty}(c_1a_n+c_2b_n)(x-x_0)^n.$$

Example 3.1.5

Find the Maclaurin series for $\cosh x$ as a linear combination of the Maclaurin series for e^x and e^{-x}.

Answer

By definition,

$$\cosh x=\frac{1}{2}e^x+\frac{1}{2}e^{-x}.$$

Since

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\quad\text{and}\quad e^{-x}=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!},$$

it follows that

$$\cosh x=\sum_{n=0}^{\infty}\frac{1}{2}[1+(-1)^n]\frac{x^n}{n!}. \tag{3.1.12}$$

Since

$$\frac{1}{2}[1+(-1)^n]=\begin{cases}1 & \text{if } n=2m, \text{ an even integer},\\ 0 & \text{if } n=2m+1, \text{ an odd integer},\end{cases}$$

we can rewrite (3.1.12) more simply as

$$\cosh x=\sum_{m=0}^{\infty}\frac{x^{2m}}{(2m)!}.$$

This result is valid on $(-\infty,\infty)$, since this is the open interval of convergence of the Maclaurin series for e^x and
e^{-x}.

Example 3.1.6

Suppose

$$y=\sum_{n=0}^{\infty} a_n x^n$$

on an open interval I that contains the origin.

(a) Express

$$(2-x)y''+2y$$

as a power series in x on I.

(b) Use the result of part (a) to find necessary and sufficient conditions on the coefficients $\{a_n\}$ for y to be a solution of the homogeneous equation

$$(2-x)y''+2y=0 \tag{3.1.13}$$

on I.

Answer

(a) From (3.1.7) with $x_0=0$,

$$y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$\begin{aligned}(2-x)y''+2y &= 2y''-xy'+2y\\ &= \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2}-\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}+\sum_{n=0}^{\infty} 2a_n x^n.\end{aligned} \tag{3.1.14}$$

To combine the three series we shift indices in the first two to make their general terms constant multiples of x^n; thus,

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n \tag{3.1.15}$$

and

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}=\sum_{n=1}^{\infty}(n+1)na_{n+1}x^n=\sum_{n=0}^{\infty}(n+1)na_{n+1}x^n, \tag{3.1.16}$$

where we added a zero term in the last series so that when we substitute from (3.1.15) and (3.1.16) into (3.1.14) all three series will start with $n=0$; thus,

$$(2-x)y''+2y=\sum_{n=0}^{\infty}[2(n+2)(n+1)a_{n+2}-(n+1)na_{n+1}+2a_n]x^n. \tag{3.1.17}$$

(b) From (3.1.17) we see that y satisfies (3.1.13) on I if

$$2(n+2)(n+1)a_{n+2}-(n+1)na_{n+1}+2a_n=0, \quad n=0,1,2,\dots. \tag{3.1.18}$$

Conversely, Theorem (3.1.6) part (b) implies that if $y=\sum_{n=0}^{\infty}a_nx^n$ satisfies (3.1.13) on I, then (3.1.18) holds.

Example 3.1.7

Suppose

$$y=\sum_{n=0}^{\infty}a_n(x-1)^n$$

on an open interval I that contains $x_0=1$. Express the function

$$(1+x)y''+2(x-1)^2y'+3y \tag{3.1.19}$$

as a power series in $x-1$ on I.

Answer

Since we want a power series in $x-1$, we rewrite the coefficient of y'' in (3.1.19) as $1+x=2+(x-1)$, so (3.1.19) becomes

$$2y''+(x-1)y''+2(x-1)^2y'+3y.$$

From (3.1.6) and (3.1.7) with $x_0=1$,

$$y'=\sum_{n=1}^{\infty}na_n(x-1)^{n-1} \quad \text{and} \quad y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}.$$

Therefore

$$\begin{aligned} 2y'' &= \sum_{n=2}^{\infty}2n(n-1)a_n(x-1)^{n-2}, \\ (x-1)y'' &= \sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-1}, \\ 2(x-1)^2y' &= \sum_{n=1}^{\infty}2na_n(x-1)^{n+1}, \\ 3y &= \sum_{n=0}^{\infty}3a_n(x-1)^n. \end{aligned}$$

Before adding these four series we shift indices in the first three so that their general terms become constant multiples of $(x-1)^n$. This yields

$$
\begin{aligned}
2y'' &= \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n, &(3.1.20)\\
(x-1)y'' &= \sum_{n=0}^{\infty} (n+1)na_{n+1}(x-1)^n, &(3.1.21)\\
2(x-1)^2y' &= \sum_{n=1}^{\infty} 2(n-1)a_{n-1}(x-1)^n, &(3.1.22)\\
3y &= \sum_{n=0}^{\infty} 3a_n(x-1)^n, &(3.1.23)
\end{aligned}
$$

where we added initial zero terms to the series in (3.1.21) and (3.1.22). Adding (3.1.20) to (3.1.23) yields

$$
\begin{aligned}
(1+x)y'' + 2(x-1)^2y' + 3y &= 2y'' + (x-1)y'' + 2(x-1)^2y' + 3y\\
&= \sum_{n=0}^{\infty} b_n(x-1)^n,
\end{aligned}
$$

where

$$
\begin{aligned}
b_0 &= 4a_2 + 3a_0, &(3.1.24)\\
b_n &= 2(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + 2(n-1)a_{n-1} + 3a_n,\ n \geq 1. &(3.1.25)
\end{aligned}
$$

The formula (3.1.24) for b_0 can't be obtained by setting $n=0$ in (3.1.25), since the summation in (3.1.22) begins with $n=1$, while those in (3.1.20), (3.1.21), and (3.1.23) begin with $n=0$.

3.1E: Exercises

This page is a draft and is under active development.

3.1E.1 Exercise $3.1E.1$

For each power series use Theorem $(3.1.3)$ to find the radius of convergence R. If $R>0$, find the open interval of convergence.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n}(x-1)^n$

(b) $\displaystyle\sum_{n=0}^{\infty} 2^n n(x-2)^n$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{9^n} x^n$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{n(n+1)}{16^n}(x-2)^n$

(e) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{7^n}{n!} x^n$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}(n+1)^2}(x+7)^n$

Answer

Add texts here. Do not delete this text first.

3.1E.2 Exercise $3.1E.2$

Suppose there's an integer M such that $b_m \neq 0$ for $m \geq M$, and

$$\lim_{m \rightarrow \infty}\left|\frac{b_{m+1}}{b_m}\right|=L,$$

where $0 \leq L \leq \infty$. Show that the radius of convergence of

$$\sum_{m=0}^{\infty} b_m\left(x-x_0\right)^{2m}$$

is $R=1/\sqrt{L}$, which is interpreted to mean that $R=0$ if $L=\infty$ or $R=\infty$ if $L=0$.

Hint: Apply Theorem $(3.1.3)$ to the series $\sum_{m=0}^{\infty} b_m z^m$ and then let $z=\left(x-x_0\right)^2$.

Answer

Add texts here. Do not delete this text first.

3.1E.3 Exercise $\mathbf{3.1}\boldsymbol{E}\mathbf{.3}$

For each power series, use the result of Exercise $(3.1E.2)$ to find the radius of convergence R. If $R>0$, find the open interval of convergence.

(a) $\displaystyle\sum_{m=0}^{\infty}(-1)^m(3m+1)(x-1)^{2m+1}$

(b) $\displaystyle\sum_{m=0}^{\infty}(-1)^m\frac{m(2m+1)}{2^m}(x+2)^{2m}$

(c) $\displaystyle\sum_{m=0}^{\infty}\frac{m!}{(2m)!}(x-1)^{2m}$

(d) $\displaystyle\sum_{m=0}^{\infty}(-1)^m\frac{m!}{9^m}(x+8)^{2m}$

(e) $\displaystyle\sum_{m=0}^{\infty}(-1)^m\frac{(2m-1)}{3^m}x^{2m+1}$

(f) $\displaystyle\sum_{m=0}^{\infty}(x-1)^{2m}$

Answer

Add texts here. Do not delete this text first.

3.1E.4 Exercise $\mathbf{3.1}\boldsymbol{E}\mathbf{.4}$

Suppose there's an integer M such that $b_m\neq 0$ for \)m\ge M\), and

$$\lim_{m\to\infty}\left|\frac{b_{m+1}}{b_m}\right|=L,$$

where $0\le L\le\infty$. Let k be a positive integer. Show that the radius of convergence of

$$\sum_{m=0}^{\infty}b_m(x-x_0)^{km}$$

is $R=1/\sqrt[k]{L}$, which is interpreted to mean that $R=0$ if $L=\infty$ or $R=\infty$ if $L=0$.

Hint: Apply Theorem $(3.1.3)$ to the series $\sum_{m=0}^{\infty}b_m z^m$ and then let $z=(x-x_0)^k$.

Answer

Add texts here. Do not delete this text first.

3.1E.5 Exercise $\mathbf{3.1}\boldsymbol{E}\mathbf{.5}$

For each power series use the result of Exercise $(3.1E.4)$ to find the radius of convergence R. If $R > 0$, find the open interval of convergence.

(a) $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{(27)^m}(x-3)^{3m+2}$

(b) $\displaystyle\sum_{m=0}^{\infty} \frac{x^{7m+6}}{m}$

(c) $\displaystyle\sum_{m=0}^{\infty} \frac{9^m(m+1)}{(m+2)}(x-3)^{4m+2}$

(d) $\displaystyle\sum_{m=0}^{\infty} (-1)^m \frac{2^m}{m!} x^{4m+3}$

(e) $\displaystyle\sum_{m=0}^{\infty} \frac{m!}{(26)^m}(x+1)^{4m+3}$

(f) $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m m(m+1)}(x-1)^{3m+1}$

Answer

Add texts here. Do not delete this text first.

3.1E.6 Exercise $\mathbf{3.1E.6}$

Graph $y = \sin x$ and the Taylor polynomial

$$T_{2M+1}(x) = \sum_{n=0}^{M} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

on the interval $(-2\pi, 2\pi)$ for $M = 1, 2, 3, \ldots$, until you find a value of M for which there's no perceptible difference between the two graphs.

Answer

Add texts here. Do not delete this text first.

3.1E.7 Exercise $\mathbf{3.1E.7}$

Graph $y = \cos x$ and the Taylor polynomial

$$T_{2M}(x) = \sum_{n=0}^{M} \frac{(-1)^n x^{2n}}{(2n)!}$$

on the interval $(-2\pi, 2\pi)$ for $M = 1, 2, 3, \ldots$, until you find a value of M for which there's no perceptible difference between the two graphs.

Answer

Add texts here. Do not delete this text first.

3.1E.8 Exercise $\mathbf{3.1E.8}$

Graph $y=1/(1-x)$ and the Taylor polynomial

$$T_N(x)=\sum_{n=0}^{N} x^n$$

on the interval $[0,.95]$ for $N=1,\ 2,\ 3,\ \dots$, until you find a value of N for which there's no perceptible difference between the two graphs. Choose the scale on the y-axis so that $0\le y\le 20$.

Answer

Add texts here. Do not delete this text first.

3.1E.9 Exercise $\mathbf{3.1E.9}$

Graph $y=\cosh x$ and the Taylor polynomial

$$T_{2M}(x)=\sum_{n=0}^{M} \frac{x^{2n}}{(2n)!}$$

on the interval $(-5,5)$ for $M=1,\ 2,\ 3,\ \dots$, until you find a value of M for which there's no perceptible difference between the two graphs. Choose the scale on the y-axis so that $0\le y\le 75$.

Answer

Add texts here. Do not delete this text first.

3.1E.10 Exercise $\mathbf{3.1E.10}$

Graph $y=\sinh x$ and the Taylor polynomial

$$T_{2M+1}(x)=\sum_{n=0}^{M} \frac{x^{2n+1}}{(2n+1)!}$$

on the interval $(-5,5)$ for $M=0,\ 1,\ 2,\ \dots$, until you find a value of M for which there's no perceptible difference between the two graphs. Choose the scale on the y-axis so that $-75\le y\le 75$.

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.1E.11)$ to $(3.1E.15)$, find a power series solution $y(x)=\sum_{n=0}^{\infty} a_n x^n$.

3.1E.11 Exercise $\mathbf{3.1E.11}$

$(2+x)y''+xy'+3y$

Answer

Add texts here. Do not delete this text first.

3.1E.12 Exercise $\mathbf{3.1E.12}$

$(1+3x^2)y''+3x^2y'-2y$

Answer

Add texts here. Do not delete this text first.

3.1E.13 Exercise $\mathbf{3.1E.13}$

$(1+2x^2)y''+(2-3x)y'+4y$

Answer

Add texts here. Do not delete this text first.

3.1E.14 Exercise $\mathbf{3.1E.14}$

$(1+x^2)y''+(2-x)y'+3y$

Answer

Add texts here. Do not delete this text first.

3.1E.15 Exercise $\mathbf{3.1E.15}$

$(1+3x^2)y''-2xy'+4y$

Answer

Add texts here. Do not delete this text first.

3.1E.16 Exercise $\mathbf{3.1E.16}$

Suppose $y(x)=\sum_{n=0}^{\infty}a_n(x+1)^n$ on an open interval that contains $x_0=-1$. Find a power series in $x+1$ for

$$xy''+(4+2x)y'+(2+x)y.$$

Answer

Add texts here. Do not delete this text first.

3.1E.17 Exercise $\PageIndex{17}$

Suppose $y(x)=\sum_{n=0}^\infty a_n(x-2)^n$ on an open interval that contains $x_0=2$. Find a power series in $x-2$ for

$$x^2y''+2xy'-3xy.$$

Answer

Add texts here. Do not delete this text first.

3.1E.18 Exercise $\PageIndex{18}$

Do the following experiment for various choices of real numbers a_0 and a_1.

(a) Use differential equations software to solve the initial value problem

$$(2-x)y''+2y=0,\quad y(0)=a_0,\quad y'(0)=a_1,$$

numerically on $(-1.95,1.95)$. Choose the most accurate method your software package provides. (See Section 3.1 for a brief discussion of one such method.)

(b) For $N=2$, 3, 4, …, compute a_2, …, a_N from Equation $(3.1.18)$ and graph

$$T_N(x)=\sum_{n=0}^N a_nx^n$$

and the solution obtained in part (a) on the same axes. Continue increasing N until it's obvious that there's no point in continuing. (This sounds vague, but you'll know when to stop.)

Answer

Add texts here. Do not delete this text first.

3.1E.19 Exercise $\PageIndex{19}$

Follow the directions of Exercise $(3.1.18)$ for the initial value problem

$$(1+x)y''+2(x-1)^2y'+3y=0,\quad y(1)=a_0,\quad y'(1)=a_1,$$

on the interval $(0,2)$. Use Equations $(3.1.24)$ and $(3.1.25)$ to compute $\{a_n\}$.

Answer

Add texts here. Do not delete this text first.

3.1E.20 Exercise $\PageIndex{20}$

Suppose the series $\sum_{n=0}^\infty a_nx^n$ converges on an open interval $(-R,R)$, let r be an arbitrary real number, and define

$$y(x)=x^r\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+r}$$

on $(0,R)$. Use Theorem $(3.1.4)$ and the rule for differentiating the product of two functions to show that

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}, \\ y''(x) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}, \\ &\vdots \\ y^{(k)}(x) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)\cdots(n+r-k)a_nx^{n+r-k} \end{aligned}$$

on $(0,R)$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.1E.21)$ to $(3.1E.26)$, let y be as defined in Exercise $(3.1E.20)$, and write the given expression in the form $x^r\sum_{n=0}^{\infty}b_nx^n$.

3.1E.21 Exercise $\mathbf{3.1}\boldsymbol{E}\mathbf{.21}$

$x^2(1-x)y''+x(4+x)y'+(2-x)y$

Answer

Add texts here. Do not delete this text first.

3.1E.22 Exercise $\mathbf{3.1}\boldsymbol{E}\mathbf{.22}$

$x^2(1+x)y''+x(1+2x)y'-(4+6x)y$

Answer

Add texts here. Do not delete this text first.

3.1E.23 Exercise $\mathbf{3.1}\boldsymbol{E}\mathbf{.23}$

$x^2(1+x)y''-x(1-6x-x^2)y'+(1+6x+x^2)y$

Answer

Add texts here. Do not delete this text first.

3.1E.24 Exercise $\mathbf{3.1E.24}$

$x^2(1+3x)y''+x(2+12x+x^2)y'+2x(3+x)y$

Answer

Add texts here. Do not delete this text first.

3.1E.25 Exercise $\mathbf{3.1E.25}$

$x^2(1+2x^2)y''+x(4+2x^2)y'+2(1-x^2)y$

Answer

Add texts here. Do not delete this text first.

3.1E.26 Exercise $\mathbf{3.1E.26}$

$x^2(2+x^2)y''+2x(5+x^2)y'+2(3-x^2)y$

Answer

Add texts here. Do not delete this text first.

3.2: Series Solutions Near an Ordinary Point I

This page is a draft and is under active development.

3.2.1 Series Solutions Near an Ordinary Point

Many physical applications give rise to second order homogeneous linear differential equations of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \tag{3.2.1}$$

where P_0, P_1, and P_2 are polynomials. Usually the solutions of these equations can't be expressed in terms of familiar elementary functions. Therefore we'll consider the problem of representing solutions of (3.2.1) with series.

We assume throughout that P_0, P_1 and P_2 have no common factors. Then we say that x_0 is an ordinary point of (3.2.1) if $P_0(x_0) \neq 0$, or a singular point if $P_0(x_0) = 0$. For Legendre's equation,

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \tag{3.2.2}$$

$x_0 = 1$ and $x_0 = -1$ are singular points and all other points are ordinary points. For Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

$x_0 = 0$ is a singular point and all other points are ordinary points. If P_0 is a nonzero constant as in Airy's equation,

$$y'' - xy = 0, \tag{3.2.3}$$

then every point is an ordinary point.

Since polynomials are continuous everywhere, P_1/P_0 and P_2/P_0 are continuous at any point x_0 that isn't a zero of P_0. Therefore, if x_0 is an ordinary point of (3.2.1) and a_0 and a_1 are arbitrary real numbers, then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = a_0, \quad y'(x_0) = a_1 \tag{3.2.4}$$

has a unique solution on the largest open interval that contains x_0 and does not contain any zeros of P_0. To see this, we rewrite the differential equation in (3.2.4) as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem (2.1.1) with $p = P_1/P_0$ and $q = P_2/P_0$. In this section and the next we consider the problem of representing solutions of (3.2.1) by power series that converge for values of x near an ordinary point x_0.

We state the next theorem without proof.

Theorem 3.2.1

Suppose P_0, P_1, and P_2 are polynomials with no common factor and P_0 isn't identically zero. Let x_0 be a point such that $P_0(x_0) \neq 0$, and let ρ be the distance from x_0 to the nearest zero of P_0 in the complex plane. (If P_0 is constant, then $\rho = \infty$.) Then every solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{3.2.5}$$

can be represented by a power series

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{3.2.6}$$

that converges at least on the open interval $(x_0 - \rho, x_0 + \rho)$. (If P_0 is nonconstant, so that ρ is necessarily finite, then the open interval of convergence of (3.2.6) may be larger than $(x_0 - \rho, x_0 + \rho)$. If P_0 is constant then $\rho = \infty$ and $(x_0 - \rho, x_0 + \rho) = (-\infty, \infty)$.)

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

We call (3.2.6) a power series solution in $x - x_0$ of (3.2.5). We'll now develop a method for finding power series solutions of (3.2.5). For this purpose we write (3.2.5) as $Ly = 0$, where

$$Ly = P_0 y'' + P_1 y' + P_2 y. \tag{3.2.7}$$

Theorem (3.2.1) implies that every solution of $Ly = 0$ on $(x_0 - \rho, x_0 + \rho)$ can be written as

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Setting $x = x_0$ in this series and in the series

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

shows that $y(x_0) = a_0$ and $y'(x_0) = a_1$. Since every initial value problem (3.2.4) has a unique solution, this means that a_0 and a_1 can be chosen arbitrarily, and $a_2, a_3, \ldots$ are uniquely determined by them.

To find $a_2, a_3, \ldots$, we write P_0, P_1, and P_2 in powers of $x - x_0$, substitute

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$$

into (3.2.7), and collect the coefficients of like powers of $x - x_0$. This yields

$$Ly = \sum_{n=0}^{\infty} b_n(x-x_0)^n, \tag{3.2.8}$$

where $\{b_0, b_1, \ldots, b_n, \ldots\}$ are expressed in terms of $\{a_0, a_1, \ldots, a_n, \ldots\}$ and the coefficients of P_0, P_1, and P_2, written in powers of $x - x_0$. Since (3.2.8) and part (a) of Theorem (3.1.6) imply that $Ly = 0$ if and only if $b_n = 0$ for $n \geq 0$, all power series solutions in $x - x_0$ of $Ly = 0$ can be obtained by choosing a_0 and a_1 arbitrarily and computing $a_2, a_3, \ldots$, successively so that $b_n = 0$ for $n \geq 0$. For simplicity, we call the power series obtained this way the power series in $x - x_0$ for the general solution of $Ly = 0$, without explicitly identifying the open interval of convergence of the series.

Example 3.2.1

Let x_0 be an arbitrary real number. Find the power series in $x - x_0$ for the general solution of

$$y'' + y = 0. \tag{3.2.9}$$

Answer

Here

$$Ly = y'' + y.$$

If

$$y = \sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2},$$

so

$$Ly = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + \sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

To collect coefficients of like powers of $x - x_0$, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n + \sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n,$$

with

$$b_n = (n+2)(n+1)a_{n+2} + a_n.$$

Therefore $Ly = 0$ if and only if

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0, \tag{3.2.10}$$

where a_0 and a_1 are arbitrary. Since the indices on the left and right sides of (3.2.10) differ by two, we write (3.2.10) separately for n even $(n = 2m)$ and n odd $(n = 2m+1)$. This yields

$$a_{2m+2} = \frac{-a_{2m}}{(2m+2)(2m+1)}, \quad m \geq 0, \tag{3.2.11}$$

$$a_{2m+3} = \frac{-a_{2m+1}}{(2m+3)(2m+2)}, \quad m \geq 0. \tag{3.2.12}$$

Computing the coefficients of the even powers of $x - x_0$ from (3.2.11) yields

$$\begin{aligned} a_2 &= -\frac{a_0}{2 \cdot 1} \\ a_4 &= -\frac{a_2}{4 \cdot 3} = -\frac{1}{4 \cdot 3}\left(-\frac{a_0}{2 \cdot 1}\right) = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}, \\ a_6 &= -\frac{a_4}{6 \cdot 5} = -\frac{1}{6 \cdot 5}\left(\frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}\right) = -\frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}, \end{aligned}$$

and, in general,

$$a_{2m} = (-1)^m \frac{a_0}{(2m)!}, \quad m \geq 0. \tag{3.2.13}$$

Computing the coefficients of the odd powers of $x - x_0$ from (3.2.12) yields

$$\begin{aligned} a_3 &= -\frac{a_1}{3 \cdot 2} \\ a_5 &= -\frac{a_3}{5 \cdot 4} = -\frac{1}{5 \cdot 4}\left(-\frac{a_1}{3 \cdot 2}\right) = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, \\ a_7 &= -\frac{a_5}{7 \cdot 6} = -\frac{1}{7 \cdot 6}\left(\frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}\right) = -\frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}, \end{aligned}$$

and, in general,

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!} \quad m \geq 0. \tag{3.2.14}$$

Thus, the general solution of (3.2.9) can be written as

$$y = \sum_{m=0}^{\infty} a_{2m}(x - x_0)^{2m} + \sum_{m=0}^{\infty} a_{2m+1}(x - x_0)^{2m+1},$$

or, from (3.2.13) and (3.2.14), as

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m+1}}{(2m+1)!}. \tag{3.2.15}$$

If we recall from calculus that

$$\sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m}}{(2m)!} = \cos(x-x_0) \quad \text{and} \quad \sum_{m=0}^{\infty} (-1)^m \frac{(x-x_0)^{2m+1}}{(2m+1)!} = \sin(x-x_0),$$

then (3.2.15) becomes

$$y = a_0 \cos(x-x_0) + a_1 \sin(x-x_0),$$

which should look familiar.

Equations like (3.2.10), (3.2.11), and (3.2.12), which define a given coefficient in the sequence $\{a_n\}$ in terms of one or more coefficients with lesser indices are called recurrence relations. When we use a recurrence relation to compute terms of a sequence we're computing recursively.

In the remainder of this section we consider the problem of finding power series solutions in $x-x_0$ for equations of the form

$$\left(1+\alpha(x-x_0)^2\right) y'' + \beta(x-x_0)y' + \gamma y = 0. \tag{3.2.16}$$

Many important equations that arise in applications are of this form with $x_0 = 0$, including Legendre's equation (3.2.2), Airy's equation (3.2.3), Chebyshev's equation,

$$(1-x^2)y'' - xy' + \alpha^2 y = 0,$$

and Hermite's equation,

$$y'' - 2xy' + 2\alpha y = 0.$$

Since

$$P_0(x) = 1 + \alpha(x-x_0)^2$$

in (3.2.16), the point x_0 is an ordinary point of (3.2.16), and Theorem $(3.2.1)$ implies that the solutions of (3.2.16) can be written as power series in $x-x_0$ that converge on the interval $(x_0 - 1/\sqrt{|\alpha|}, x_0 + 1/\sqrt{|\alpha|})$ if $\alpha \neq 0$, or on $(-\infty, \infty)$ if $\alpha = 0$. We'll see that the coefficients in these power series can be obtained by methods similar to the one used in Example $(3.2.1)$.

To simplify finding the coefficients, we introduce some notation for products:

$$\prod_{j=r}^{s} b_j = b_r b_{r+1} \cdots b_s \quad \text{if} \quad s \geq r.$$

Thus,

$$\prod_{j=2}^{7} b_j = b_2 b_3 b_4 b_5 b_6 b_7,$$

$$\prod_{j=0}^{4} (2j+1) = (1)(3)(5)(7)(9) = 945,$$

and

$$\prod_{j=2}^{2} j^2 = 2^2 = 4.$$

We define

$$\prod_{j=r}^{s} b_j = 1 \quad \text{if} \quad s < r,$$

no matter what the form of b_j.

Example 3.2.2

Find the power series in x for the general solution of

$$(1+2x^2)y'' + 6xy' + 2y = 0. \tag{3.2.17}$$

Answer

Here

$$Ly = (1+2x^2)y'' + 6xy' + 2y.$$

If

$$y = \sum_{n=0}^{\infty} a_n x^n$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

so

$$\begin{aligned} Ly = & \ (1+2x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 6x\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n \\ = & \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty}\left[2n(n-1)+6n+2\right]a_n x^n \\ = & \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=0}^{\infty}(n+1)^2 a_n x^n. \end{aligned}$$

To collect coefficients of x^n, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2\sum_{n=0}^{\infty}(n+1)^2 a_n x^n = \sum_{n=0}^{\infty} b_n x^n,$$

with

$$b_n = (n+2)(n+1)a_{n+2} + 2(n+1)^2 a_n, \quad n \geq 0.$$

To obtain solutions of (3.2.17), we set $b_n = 0$ for $n \geq 0$. This is equivalent to the recurrence relation

$$a_{n+2} = -2\frac{n+1}{n+2}a_n, \quad n \geq 0. \tag{3.2.18}$$

Since the indices on the left and right differ by two, we write (3.2.18) separately for $n = 2m$ and $n = 2m+1$, as in Example (3.2.1). This yields

$$\begin{aligned} a_{2m+2} &= -2\frac{2m+1}{2m+2}a_{2m} = -\frac{2m+1}{m+1}a_{2m}, \quad m \geq 0, && (3.2.19) \\ a_{2m+3} &= -2\frac{2m+2}{2m+3}a_{2m+1} = -4\frac{m+1}{2m+3}a_{2m+1}, \quad m \geq 0. && (3.2.20) \end{aligned}$$

Computing the coefficients of even powers of x from (3.2.19) yields

$$\begin{aligned} a_2 &= -\frac{1}{1}a_0, \\ a_4 &= -\frac{3}{2}a_2 = \left(-\frac{3}{2}\right)\left(-\frac{1}{1}\right)a_0 = \frac{1\cdot 3}{1\cdot 2}a_0, \\ a_6 &= -\frac{5}{3}a_4 = -\frac{5}{3}\left(\frac{1\cdot 3}{1\cdot 2}\right)a_0 = -\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}a_0, \\ a_8 &= -\frac{7}{4}a_6 = -\frac{7}{4}\left(-\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\right)a_0 = \frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 3\cdot 4}a_0. \end{aligned}$$

In general,

$$a_{2m} = (-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!}a_0, \quad m \geq 0. \tag{3.2.21}$$

(Note that (3.2.21) is correct for $m = 0$ because we defined $\prod_{j=1}^{0} b_j = 1$ for any b_j.)

Computing the coefficients of odd powers of x from (3.2.20) yields

$$\begin{aligned} a_3 &= -4\,\frac{1}{3}a_1, \\ a_5 &= -4\,\frac{2}{5}a_3 = -4\,\frac{2}{5}\left(-4\frac{1}{3}\right)a_1 = 4^2\,\frac{1\cdot 2}{3\cdot 5}a_1, \\ a_7 &= -4\,\frac{3}{7}a_5 = -4\,\frac{3}{7}\left(4^2\,\frac{1\cdot 2}{3\cdot 5}\right)a_1 = -4^3\,\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}a_1, \\ a_9 &= -4\,\frac{4}{9}a_7 = -4\,\frac{4}{9}\left(4^3\,\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)a_1 = 4^4\,\frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9}a_1. \end{aligned}$$

In general,

$$a_{2m+1} = \frac{(-1)^m 4^m m!}{\prod_{j=1}^{m}(2j+1)}a_1, \quad m \ge 0. \tag{3.2.22}$$

From (3.2.21) and (3.2.22),

$$y = a_0\sum_{m=0}^{\infty}(-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!}x^{2m} + a_1\sum_{m=0}^{\infty}(-1)^m \frac{4^m m!}{\prod_{j=1}^{m}(2j+1)}x^{2m+1}.$$

is the power series in x for the general solution of (3.2.17). Since $P_0(x) = 1+2x^2$ has no real zeros, Theorem (2.1.1) implies that every solution of (3.2.17) is defined on $(-\infty,\infty)$. However, since $P_0(\pm i/\sqrt{2}) = 0$, Theorem (3.2.1) implies only that the power series converges in $(-1/\sqrt{2}, 1/\sqrt{2})$ for any choice of a_0 and a_1.

The results in Examples (3.2.1) and (3.2.2) are consequences of the following general theorem.

Theorem 3.2.2

The coefficients $\{a_n\}$ in any solution $y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ of

$$\left(1+\alpha(x-x_0)^2\right)y'' + \beta(x-x_0)y' + \gamma y = 0 \tag{3.2.23}$$

satisfy the recurrence relation

$$a_{n+2} = -\frac{p(n)}{(n+2)(n+1)}a_n, \quad n \ge 0, \tag{3.2.24}$$

where

$$p(n) = \alpha n(n-1) + \beta n + \gamma. \tag{3.2.25}$$

Moreover, the coefficients of the even and odd powers of $x - x_0$ can be computed separately as

$$a_{2m+2} = -\frac{p(2m)}{(2m+2)(2m+1)} a_{2m}, \quad m \geq 0 \tag{3.2.26}$$

$$a_{2m+3} = -\frac{p(2m+1)}{(2m+3)(2m+2)} a_{2m+1}, \quad m \geq 0, \tag{3.2.27}$$

where a_0 and a_1 are arbitrary.

Proof

Here

$$Ly = (1+\alpha(x-x_0)^2)y'' + \beta(x-x_0)y' + \gamma y.$$

If

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n,$$

then

$$y' = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}.$$

Hence,

$$\begin{aligned} Ly &= \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + \sum_{n=0}^{\infty} \left[\alpha n(n-1) + \beta n + \gamma\right] a_n (x-x_0)^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + \sum_{n=0}^{\infty} p(n) a_n (x-x_0)^n, \end{aligned}$$

from (3.2.25). To collect coefficients of powers of $x-x_0$, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + p(n)a_n\right](x-x_0)^n.$$

Thus, $Ly = 0$ if and only if

$$(n+2)(n+1)a_{n+2} + p(n)a_n = 0, \quad n \geq 0,$$

which is equivalent to (3.2.24). Writing (3.2.24) separately for the cases where $n = 2m$ and $n = 2m+1$ yields (3.2.26) and (3.2.27).

Example $3.2.3$

Find the power series in $x-1$ for the general solution of

$$(2+4x-2x^2)y'' - 12(x-1)y' - 12y = 0. \tag{3.2.28}$$

Answer

We must first write the coefficient $P_0(x) = 2 + 4x - x^2$ in powers of $x-1$. To do this, we write $x = (x-1)+1$ in $P_0(x)$ and then expand the terms, collecting powers of $x-1$; thus,

$$\begin{aligned} 2+4x-2x^2 &= 2+4[(x-1)+1]-2[(x-1)+1]^2 \\ &= 4-2(x-1)^2. \end{aligned}$$

Therefore we can rewrite (3.2.28) as

$$\left(4-2(x-1)^2\right)y'' - 12(x-1)y' - 12y = 0,$$

or, equivalently,

$$\left(1-\frac{1}{2}(x-1)^2\right)y'' - 3(x-1)y' - 3y = 0.$$

This is of the form (3.2.23) with $\alpha = -1/2$, $\beta = -3$, and $\gamma = -3$. Therefore, from (3.2.25)

$$p(n) = -\frac{n(n-1)}{2} - 3n - 3 = -\frac{(n+2)(n+3)}{2}.$$

Hence, Theorem $(3.2.2)$ implies that

$$\begin{aligned} a_{2m+2} &= -\frac{p(2m)}{(2m+2)(2m+1)}a_{2m} \\ &= \frac{(2m+2)(2m+3)}{2(2m+2)(2m+1)}a_{2m} = \frac{2m+3}{2(2m+1)}a_{2m}, \quad m \geq 0 \\ a_{2m+3} &= -\frac{p(2m+1)}{(2m+3)(2m+2)}a_{2m+1} \\ &= \frac{(2m+3)(2m+4)}{2(2m+3)(2m+2)}a_{2m+1} = \frac{m+2}{2(m+1)}a_{2m+1}, \quad m \geq 0. \end{aligned}$$

We leave it to you to show that

$$a_{2m} = \frac{2m+1}{2^m}a_0 \quad \text{and} \quad a_{2m+1} = \frac{m+1}{2^m}a_1, \quad m \geq 0,$$

which implies that the power series in $x-1$ for the general solution of (3.2.28) is

$$y = a_0\sum_{m=0}^{\infty}\frac{2m+1}{2^m}(x-1)^{2m} + a_1\sum_{m=0}^{\infty}\frac{m+1}{2^m}(x-1)^{2m+1}.$$

In the examples considered so far we were able to obtain closed formulas for coefficients in the power series solutions. In some cases this is impossible, and we must settle for computing a finite number of terms in the series. The next example illustrates this with an initial value problem.

Example 3.2.4

Compute $a_0, a_1, \ldots, a_7$ in the series solution $y=\sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1+2x^2)y''+10xy'+8y=0, \quad y(0)=2, \quad y'(0)=-3. \tag{3.2.29}$$

Answer

Since $\alpha=2$, $\beta=10$, and $\gamma=8$ in (3.2.29),

$$p(n)=2n(n-1)+10n+8=2(n+2)^2.$$

Therefore

$$a_{n+2}=-2\frac{(n+2)^2}{(n+2)(n+1)}a_n=-2\frac{n+2}{n+1}a_n, \quad n\geq 0.$$

Writing this equation separately for $n=2m$ and $n=2m+1$ yields

$$a_{2m+2}=-2\frac{(2m+2)}{2m+1}a_{2m}=-4\frac{m+1}{2m+1}a_{2m}, \quad m\geq 0 \tag{3.2.30}$$

$$a_{2m+3}=-2\frac{2m+3}{2m+2}a_{2m+1}=-\frac{2m+3}{m+1}a_{2m+1}, \quad m\geq 0. \tag{3.2.31}$$

Starting with $a_0=y(0)=2$, we compute a_2, a_4, and a_6 from (3.2.30):

$$\begin{aligned} a_2 &= -4\,\frac{1}{1}2=-8, \\ a_4 &= -4\,\frac{2}{3}(-8)=\frac{64}{3}, \\ a_6 &= -4\,\frac{3}{5}\left(\frac{64}{3}\right)=-\frac{256}{5}. \end{aligned}$$

Starting with $a_1=y'(0)=-3$, we compute a_3, a_5 and a_7 from (3.2.31):

$$\begin{aligned} a_3 &= -\frac{3}{1}(-3)=9, \\ a_5 &= -\frac{5}{2}9=-\frac{45}{2}, \\ a_7 &= -\frac{7}{3}\left(-\frac{45}{2}\right)=\frac{105}{2}. \end{aligned}$$

Therefore the solution of (3.2.29) is

$$y=2-3x-8x^2+9x^3+\frac{64}{3}x^4-\frac{45}{2}x^5-\frac{256}{5}x^6+\frac{105}{2}x^7+\cdots.$$

Computing coefficients recursively as in Example $(3.2.4)$ is tedious. We recommend that you do this kind of computation by writing a short program to implement the appropriate recurrence relation on a calculator or computer. You may wish to do this in verifying examples and doing exercises (identified by the symbol \Cex) in this chapter that call for numerical computation of the coefficients in series solutions. We obtained the answers to these exercises by using software that can produce answers in the form of rational numbers. However, it's perfectly acceptable - and more practical - to get your answers in decimal form. You can always check them by converting our fractions to decimals.

If you're interested in actually using series to compute numerical approximations to solutions of a differential equation, then whether or not there's a simple closed form for the coefficients is essentially irrelevant. For computational purposes it's usually more efficient to start with the given coefficients $a_0 = y(x_0)$ and $a_1 = y'(x_0)$, compute $a_2, \dots, a_N$ recursively, and then compute approximate values of the solution from the Taylor polynomial

$$T_N(x) = \sum_{n=0}^{N} a_n (x - x_0)^n.$$

The trick is to decide how to choose N so the approximation $y(x) \approx T_N(x)$ is sufficiently accurate on the subinterval of the interval of convergence that you're interested in. In the computational exercises in this and the next two sections, you will often be asked to obtain the solution of a given problem by numerical integration with software of your choice (see Section 3.1) for a brief discussion of one such method), and to compare the solution obtained in this way with the approximations obtained with T_N for various values of N. This is a typical textbook kind of exercise, designed to give you insight into how the accuracy of the approximation $y(x) \approx T_N(x)$ behaves as a function of N and the interval that you're working on. In real life, you would choose one or the other of the two methods (numerical integration or series solution). If you choose the method of series solution, then a practical procedure for determining a suitable value of N is to continue increasing N until the maximum of $|T_N - T_{N-1}|$ on the interval of interest is within the margin of error that you're willing to accept.

In doing computational problems that call for numerical solution of differential equations you should choose the most accurate numerical integration procedure your software supports, and experiment with the step size until you're confident that the numerical results are sufficiently accurate for the problem at hand.

3.2E: Exercises

This page is a draft and is under active development.

In Exercises $(3.2E.1)$ to $(3.2E.8)$, find the power series in x for the general solution.

3.2E.1 Exercise $\mathbf{3.2E.1}$

$$(1+x^2)y''+6xy'+6y=0$$

Answer

Add texts here. Do not delete this text first.

3.2E.2 Exercise $\mathbf{3.2E.2}$

$$(1+x^2)y''+2xy'-2y=0$$

Answer

Add texts here. Do not delete this text first.

3.2E.3 Exercise $\mathbf{3.2E.3}$

$$(1+x^2)y''-8xy'+20y=0$$

Answer

Add texts here. Do not delete this text first.

3.2E.4 Exercise $\mathbf{3.2E.4}$

$$(1-x^2)y''-8xy'-12y=0$$

Answer

Add texts here. Do not delete this text first.

3.2E.5 Exercise $\mathbf{3.2E.5}$

$$(1+2x^2)y''+7xy'+2y=0$$

Answer

Add texts here. Do not delete this text first.

3.2E.6 Exercise $\mathbf{3.2E.6}$

$$(1+x^2)y''+2xy'+\frac{1}{4}y=0$$

Answer

Add texts here. Do not delete this text first.

3.2E.7 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.7}$

$(1-x^2)y''-5xy'-4y=0$

Answer

Add texts here. Do not delete this text first.

3.2E.8 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.8}$

$(1+x^2)y''-10xy'+28y=0$

Answer

Add texts here. Do not delete this text first.

3.2E.9 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.9}$

(a) Find the power series in x for the general solution of $y''+xy'+2y=0$.

(b) For several choices of a_0 and a_1, use differential equations software to solve the initial value problem

$$y''+xy'+2y=0, \quad y(0)=a_0, \quad y'(0)=a_1, \tag{3.2E.1}$$

numerically on $(-5,5)$.

(c) For fixed r in $\{1,2,3,4,5\}$ graph

$$T_N(x)=\sum_{n=0}^N a_n x^n$$

and the solution obtained in part (a) on $(-r,r)$. Continue increasing N until there's no perceptible difference between the two graphs.

Answer

Add texts here. Do not delete this text first.

3.2E.10 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.10}$

Follow the directions of Exercise $(3.2E.9)$ for the differential equation

$$y''+2xy'+3y=0.$$

Answer

Add texts here. Do not delete this text first.

In Exercises (3.2E.11) to (3.2E.13), find a_0, ..., a_N for N at least 7 in the power series solution $y=\sum_{n=0}^{\infty} a_n x^n$ of the initial value problem.

3.2E.11 Exercise **3.2*E*.11**

$(1+x^2)y''+xy'+y=0, \quad y(0)=2, \quad y'(0)=-1$

Answer

Add texts here. Do not delete this text first.

3.2E.12 Exercise **3.2*E*.12**

$(1+2x^2)y''-9xy'-6y=0, \quad y(0)=1, \quad y'(0)=-1$

Answer

Add texts here. Do not delete this text first.

3.2E.13 Exercise **3.2*E*.13**

$(1+8x^2)y''+2y=0, \quad y(0)=2, \quad y'(0)=-1$

Answer

Add texts here. Do not delete this text first.

3.2E.14 Exercise **3.2*E*.14**

Do the next experiment for various choices of real numbers a_0, a_1, and r, with $0<r<1/\sqrt{2}$.

(a) Use differential equations software to solve the initial value problem

$$(1-2x^2)y''-xy'+3y=0, \quad y(0)=a_0, \quad y'(0)=a_1, \tag{3.2E.2}$$

numerically on $(-r,r)$.

(b) For $N=2,3,4,\ldots$, compute a_2, ..., a_N in the power series solution $y=\sum_{n=0}^{\infty} a_n x^n$ of (3.2E.2), and graph

$$T_N(x)=\sum_{n=0}^{N} a_n x^n$$

and the solution obtained in part (a) on $(-r,r)$. Continue increasing N until there's no perceptible difference between the two graphs.

Answer

Add texts here. Do not delete this text first.

3.2E.15 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.15}$

Do part (a) and part (b) for several values of r in $(0,1)$:

(a) Use differential equations software to solve the initial value problem

$$(1+x^2)y''+10xy'+14y=0, \quad y(0)=5, \quad y'(0)=1, \tag{3.2E.3}$$

numerically on $(-r,r)$.

(b) For $N=2,\ 3,\ 4,\ \ldots$, compute $a_2,\ \ldots,\ a_N$ in the power series solution $y=\sum_{n=0}^{\infty} a_n x^n$ of (3.2E.3), and graph

$$T_N(x)=\sum_{n=0}^{N} a_n x^n$$

and the solution obtained in part (a) on $(-r,r)$. Continue increasing N until there's no perceptible difference between the two graphs. What happens to the required N as $r \to 1$?

(c) Try part (a) and part (b) with $r=1.2$. Explain your results.

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.2E.16)$ to $(3.2E.20)$, find the power series in $-x_0$ for the general solution.

3.2E.16 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.16}$

$y''-y=0; \quad x_0=3$

Answer

Add texts here. Do not delete this text first.

3.2E.17 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.17}$

$y''-(x-3)y'-y=0; \quad x_0=3$

Answer

Add texts here. Do not delete this text first.

3.2E.18 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.18}$

$(1-4x+2x^2)y''+10(x-1)y'+6y=0; \quad x_0=1$

Answer

Add texts here. Do not delete this text first.

3.2E.19 Exercise $\mathbf{3.2E.19}$

$(11-8x+2x^2)y''-16(x-2)y'+36y=0; \quad x_0=2$

Answer

Add texts here. Do not delete this text first.

3.2E.20 Exercise $\mathbf{3.2E.20}$

$(5+6x+3x^2)y''+9(x+1)y'+3y=0; \quad x_0=-1$

Answer

Add texts here. Do not delete this text first.

In Exercises (3.2E.21) to (3.2E.26), find $a_0, \ldots, a_N$ for N at least 7 in the power series $y=\sum_{n=0}^{\infty} a_n(x-x_0)^n$ for the solution of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

3.2E.21 Exercise $\mathbf{3.2E.21}$

$(x^2-4)y''-xy'-3y=0, \quad y(0)=-1, \quad y'(0)=2$

Answer

Add texts here. Do not delete this text first.

3.2E.22 Exercise $\mathbf{3.2E.22}$

$y''+(x-3)y'+3y=0, \quad y(3)=-2, \quad y'(3)=3$

Answer

Add texts here. Do not delete this text first.

3.2E.23 Exercise $\mathbf{3.2E.23}$

$(5-6x+3x^2)y''+(x-1)y'+12y=0, \quad y(1)=-1, \quad y'(1)=1$

Answer

Add texts here. Do not delete this text first.

3.2E.24 Exercise $\mathbf{3.2E.24}$

$(4x^2-24x+37)y''+y=0, \quad y(3)=4, \quad y'(3)=-6$

Answer

Add texts here. Do not delete this text first.

3.2E.25 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.25}$

$$(x^2-8x+14)y''-8(x-4)y'+20y=0,\quad y(4)=3,\quad y'(4)=-4$$

Answer

Add texts here. Do not delete this text first.

3.2E.26 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.26}$

$$(2x^2+4x+5)y''-20(x+1)y'+60y=0,\quad y(-1)=3,\quad y'(-1)=-3$$

Answer

Add texts here. Do not delete this text first.

3.2E.27 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.27}$

(a) Find a power series in x for the general solution of

$$(1+x^2)y''+4xy'+2y=0. \tag{3.2E.4}$$

(b) Use (a) and the formula

$$\frac{1}{1-r}=1+r+r^2+\cdots+r^n+\cdots \quad (-1<r<1)$$

for the sum of a geometric series to find a closed form expression for the general solution of (3.2E.4) on $(-1,1)$.

(c) Show that the expression obtained in part (b) is actually the general solution of (3.2E.4) on $(-\infty,\infty)$.

Answer

Add texts here. Do not delete this text first.

3.2E.28 Exercise $\mathbf{3.2}\boldsymbol{E}\mathbf{.28}$

Use Theorem $(3.2.2)$ to show that the power series in x for the general solution of

$$(1+\alpha x^2)y''+\beta xy'+\gamma y=0$$

is

$$y=a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}p(2j)\right]\frac{x^{2m}}{(2m)!}+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}p(2j+1)\right]\frac{x^{2m+1}}{(2m+1)!}.$$

Answer

Add texts here. Do not delete this text first.

3.2E.29 Exercise $\mathbf{3.2}\boldsymbol{E}.\mathbf{29}$

Use Exercise $(3.2E.28)$ to show that all solutions of

$$(1+\alpha x^2)y''+\beta xy'+\gamma y=0$$

are polynomials if and only if

$$\alpha n(n-1)+\beta n+\gamma=\alpha(n-2r)(n-2s-1),$$

where r and s are nonnegative integers.

Answer

Add texts here. Do not delete this text first.

3.2E.30 Exercise $\mathbf{3.2}\boldsymbol{E}.\mathbf{30}$

(a) Use Exercise $(3.2E.28)$ to show that the power series in x for the general solution of

$$(1-x^2)y''-2bxy'+\alpha(\alpha+2b-1)y=0$$

is $y=a_0y_1+a_1y_2$, where

$$y_1=\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j-\alpha)(2j+\alpha+2b-1)\right]\frac{x^{2m}}{(2m)!}$$

and

$$y_2=\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j+1-\alpha)(2j+\alpha+2b)\right]\frac{x^{2m+1}}{(2m+1)!}.$$

(b) Suppose $2b$ isn't a negative odd integer and k is a nonnegative integer. Show that y_1 is a polynomial of degree $2k$ such that $y_1(-x)=y_1(x)$ if $\alpha=2k$, while y_2 is a polynomial of degree $2k+1$ such that $y_2(-x)=-y_2(-x)$ if $\alpha=2k+1$. Conclude that if n is a nonnegative integer, then there's a polynomial P_n of degree n such that $P_n(-x)=(-1)^nP_n(x)$ and

$$(1-x^2)P_n''-2bxP_n'+n(n+2b-1)P_n=0. \tag{3.2E.5}$$

(c) Show that (3.2E.5) implies that

$$[(1-x^2)^bP_n']'=-n(n+2b-1)(1-x^2)^{b-1}P_n,$$

and use this to show that if m and n are nonnegative integers, then

$$\begin{aligned}&[(1-x^2)^bP_n']'P_m-[(1-x^2)^bP_m']'P_n=\\&[m(m+2b-1)-n(n+2b-1)]\,(1-x^2)^{b-1}P_mP_n.\end{aligned} \tag{3.2E.6}$$

(d) Now suppose $b>0$. Use (3.2E.6) and integration by parts to show that if $m\neq n$, then

$$\int_{-1}^{1}(1-x^2)^{b-1}P_m(x)P_n(x)\,dx=0.$$

(We say that P_m and P_n are orthogonal on $(-1,1)$ with respect to the weighting function $(1-x^2)^{b-1}$.)
\end{alist}

Answer

Add texts here. Do not delete this text first.

3.2E.31 Exercise $\mathbf{3.2E.31}$

(a) Use Exercise $(3.2E.28)$ to show that the power series in x for the general solution of Hermite's equation

$$y''-2xy'+2\alpha y=0$$

is $y=a_0y_1+a_1y_1$, where

$$y_1=\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j-\alpha)\right]\frac{2^m x^{2m}}{(2m)!}$$

and

$$y_2=\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j+1-\alpha)\right]\frac{2^m x^{2m+1}}{(2m+1)!}.$$

(b) Suppose k is a nonnegative integer. Show that y_1 is a polynomial of degree $2k$ such that $y_1(-x)=y_1(x)$ if $\alpha=2k$, while y_2 is a polynomial of degree $2k+1$ such that $y_2(-x)=-y_2(-x)$ if $\alpha=2k+1$. Conclude that if n is a nonnegative integer then there's a polynomial P_n of degree n such that $P_n(-x)=(-1)^nP_n(x)$ and

$$P_n''-2xP_n'+2nP_n=0. \tag{3.2E.7}$$

(c) Show that (3.2E.7) implies that

$$[e^{-x^2}P_n']'=-2ne^{-x^2}P_n,$$

and use this to show that if m and n are nonnegative integers, then

$$[e^{-x^2}P_n']'P_m-[e^{-x^2}P_m']'P_n=2(m-n)e^{-x^2}P_mP_n. \tag{3.2E.8}$$

(d) Use (3.2E.8) and integration by parts to show that if $m\neq n$, then

$$\int_{-\infty}^{\infty}e^{-x^2}P_m(x)P_n(x)\,dx=0.$$

(We say that P_m and P_n are orthogonal on $(-\infty,\infty)$ with respect to the weighting function e^{-x^2}.)

Answer

Add texts here. Do not delete this text first.

3.2E.32 Exercise $\mathbf{3.2}\boldsymbol{E}.\mathbf{32}$

Consider the equation

$$\left(1+\alpha x^{3}\right) y''+\beta x^{2} y'+\gamma x y=0, \tag{3.2E.9}$$

and let $p(n)=\alpha n(n-1)+\beta n+\gamma$. (The special case $y''-xy=0$ of (3.2E.9) is Airy's equation.)

(a) Modify the argument used to prove Theorem $(3.2.2)$ to show that

$$y=\sum_{n=0}^{\infty} a_{n} x^{n}$$

is a solution of (3.2E.9) if and only if $a_2=0$ and

$$a_{n+3}=-\frac{p(n)}{(n+3)(n+2)} a_{n}, \quad n \geq 0.$$

(b) Show from (a) that $a_n=0$ unless $n=3m$ or $n=3m+1$ for some nonnegative integer m, and that

$$\begin{aligned} a_{3m+3} &= -\frac{p(3m)}{(3m+3)(3m+2)} a_{3m}, \quad m \geq 0, \\ \text{and} & \\ a_{3m+4} &= -\frac{p(3m+1)}{(3m+4)(3m+3)} a_{3m+1}, \quad m \geq 0, \end{aligned}$$

where a_0 and a_1 may be specified arbitrarily.

(c) Conclude from (b) that the power series in x for the general solution of (3.2E.9) is

$$\begin{aligned} y= & a_{0} \sum_{m=0}^{\infty}(-1)^{m}\left[\prod_{j=0}^{m-1} \frac{p(3j)}{3j+2}\right] \frac{x^{3m}}{3^{m} m!} \\ & +a_{1} \sum_{m=0}^{\infty}(-1)^{m}\left[\prod_{j=0}^{m-1} \frac{p(3j+1)}{3j+4}\right] \frac{x^{3m+1}}{3^{m} m!}. \end{aligned}$$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.2E.33)$ to $(3.2E.37)$, use the method of Exercise $(3.2E.32)$ to find the power series in x for the general solution.

3.2E.33 Exercise $\mathbf{3.2}\boldsymbol{E}.\mathbf{33}$

$y''-xy=0$

Answer

Add texts here. Do not delete this text first.

3.2E.34 Exercise $\mathbf{3.2E.34}$

$(1-2x^3)y''-10x^2y'-8xy=0$

Answer

Add texts here. Do not delete this text first.

3.2E.35 Exercise $\mathbf{3.2E.35}$

$(1+x^3)y''+7x^2y'+9xy=0$

Answer

Add texts here. Do not delete this text first.

3.2E.36 Exercise $\mathbf{3.2E.36}$

$(1-2x^3)y''+6x^2y'+24xy=0$

Answer

Add texts here. Do not delete this text first.

3.2E.37 Exercise $\mathbf{3.2E.37}$

$(1-x^3)y''+15x^2y'-63xy=0$

Answer

Add texts here. Do not delete this text first.

3.2E.38 Exercise $\mathbf{3.2E.38}$

Consider the equation

$$\left(1+\alpha x^{k+2}\right)y''+\beta x^{k+1}y'+\gamma x^k y=0, \tag{3.2E.10}$$

where k is a positive integer, and let $p(n)=\alpha n(n-1)+\beta n+\gamma$.

(a) Modify the argument used to prove Theorem $(3.2.2)$ to show that

$$y=\sum_{n=0}^{\infty}a_n x^n$$

is a solution of (3.2E.10) if and only if $a_n=0$ for $2\le n\le k+1$ and

$$a_{n+k+2}=-\frac{p(n)}{(n+k+2)(n+k+1)}a_n,\quad n\ge 0.$$

(b) Show from (a) that $a_n = 0$ unless $n = (k+2)m$ or $n = (k+2)m+1$ for some nonnegative integer m, and that

$$a_{(k+2)(m+1)} = -\frac{p((k+2)m)}{(k+2)(m+1)[(k+2)(m+1)-1]} a_{(k+2)m}, \quad m \geq 0,$$

and

$$a_{(k+2)(m+1)+1} = -\frac{p((k+2)m+1)}{[(k+2)(m+1)+1](k+2)(m+1)} a_{(k+2)m+1}, \quad m \geq 0,$$

where a_0 and a_1 may be specified arbitrarily.

(c) Conclude from (b) that the power series in x for the general solution of (3.2E.10) is

$$\begin{aligned} y = a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{p((k+2)j)}{(k+2)(j+1)-1} \right] \frac{x^{(k+2)m}}{(k+2)^m m!} \\ + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{p((k+2)j+1)}{(k+2)(j+1)+1} \right] \frac{x^{(k+2)m+1}}{(k+2)^m m!}. \end{aligned}$$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.2E.39)$ to $(3.2E.44)$, use the method of Exercise $(3.2E.38)$ to find the power series in x for the general solution.

3.2E.39 Exercise $\mathbf{3.2E.39}$

$(1+2x^5)y'' + 14x^4y' + 10x^3y = 0$

Answer

Add texts here. Do not delete this text first.

3.2E.40 Exercise $\mathbf{3.2E.40}$

$y'' + x^2y = 0$

Answer

Add texts here. Do not delete this text first.

3.2E.41 Exercise $\mathbf{3.2E.41}$

$y'' + x^6y' + 7x^5y = 0$

Answer

Add texts here. Do not delete this text first.

3.2E.42 Exercise $3.2E.42$

$(1+x^8)y'' - 16x^7y' + 72x^6y = 0$

Answer

Add texts here. Do not delete this text first.

3.2E.43 Exercise $3.2E.43$

$(1-x^6)y'' - 12x^5y' - 30x^4y = 0$

Answer

Add texts here. Do not delete this text first.

3.2E.44 Exercise $3.2E.44$

$y'' + x^5y' + 6x^4y = 0$

Answer

Add texts here. Do not delete this text first.

3.3: Series Solutions Near an Ordinary Point II

This page is a draft and is under active development.

In this section we continue to find series solutions

$$y=\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

of initial value problems

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0,\quad y(x_0)=a_0,\quad y'(x_0)=a_1, \tag{3.3.1}$$

where P_0, P_1, and P_2 are polynomials and $P_0(x_0)\neq 0$, so x_0 is an ordinary point of (3.3.1). However, here we consider cases where the differential equation in (3.3.1) is not of the form

$$\left(1+\alpha(x-x_0)^2\right)y''+\beta(x-x_0)y'+\gamma y=0,$$

so Theorem $(3.2.2)$ does not apply, and the computation of the coefficients $\{a_n\}$ is more complicated. For the equations considered here it's difficult or impossible to obtain an explicit formula for a_n in terms of n. Nevertheless, we can calculate as many coefficients as we wish. The next three examples illustrate this.

3.3.1 Example **3.3.1**

Find the coefficients $a_0, \ldots, a_7$ in the series solution $y=\sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1+x+2x^2)y''+(1+7x)y'+2y=0,\quad y(0)=-1,\quad y'(0)=-2. \tag{3.3.2}$$

Answer

Here

$$Ly=(1+x+2x^2)y''+(1+7x)y'+2y.$$

The zeros $(-1\pm i\sqrt{7})/4$ of $P_0(x)=1+x+2x^2$ have absolute value $1/\sqrt{2}$, so Theorem $(3.2.2)$ implies that the series solution converges to the solution of (3.3.2) on $(-1/\sqrt{2}, 1/\sqrt{2})$. Since

$$y=\sum_{n=0}^{\infty} a_n x^n,\quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1}\quad \text{and}\quad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},$$

$$\begin{aligned} Ly= & \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}+2\sum_{n=2}^{\infty} n(n-1)a_n x^n \\ & +\sum_{n=1}^{\infty} n a_n x^{n-1}+7\sum_{n=1}^{\infty} n a_n x^n+2\sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Shifting indices so the general term in each series is a constant multiple of x^n yields

$$\begin{aligned} Ly = & \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)na_{n+1}x^n + 2\sum_{n=0}^{\infty}n(n-1)a_nx^n \\ & + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 7\sum_{n=0}^{\infty}na_nx^n + 2\sum_{n=0}^{\infty}a_nx^n = \sum_{n=0}^{\infty}b_nx^n, \end{aligned}$$

where

$$b_n = (n+2)(n+1)a_{n+2} + (n+1)^2a_{n+1} + (n+2)(2n+1)a_n.$$

Therefore $y = \sum_{n=0}^{\infty} a_nx^n$ is a solution of $Ly = 0$ if and only if

$$a_{n+2} = -\frac{n+1}{n+2}\,a_{n+1} - \frac{2n+1}{n+1}\,a_n,\ n \geq 0. \tag{3.3.3}$$

From the initial conditions in (3.3.2), $a_0 = y(0) = -1$ and $a_1 = y'(0) = -2$. Setting $n = 0$ in (3.3.3) yields

$$a_2 = -\frac{1}{2}a_1 - a_0 = -\frac{1}{2}(-2) - (-1) = 2.$$

Setting $n = 1$ in (3.3.3) yields

$$a_3 = -\frac{2}{3}a_2 - \frac{3}{2}a_1 = -\frac{2}{3}(2) - \frac{3}{2}(-2) = \frac{5}{3}.$$

We leave it to you to compute a_4, a_5, a_6, a_7 from (3.3.3) and show that

$$y = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \frac{61}{8}x^6 - \frac{443}{56}x^7 + \cdots.$$

We also leave it to you in Exercise $(3.3E.13)$ to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^{N} a_nx^n$ converge to the solution of (3.3.2) on $(-1/\sqrt{2}, 1/\sqrt{2})$.

3.3.2 Example **3.3.2**

Find the coefficients $a_0, \dots, a_5$ in the series solution

$$y = \sum_{n=0}^{\infty} a_n(x+1)^n$$

of the initial value problem

$$(3+x)y'' + (1+2x)y' - (2-x)y = 0, \quad y(-1) = 2, \quad y'(-1) = -3. \tag{3.3.4}$$

Answer

Since the desired series is in powers of $x+1$ we rewrite the differential equation in (3.3.4) as $Ly = 0$, with

$$Ly = (2 + (x+1))\,y'' - (1 - 2(x+1))\,y' - (3 - (x+1))\,y.$$

Since

$$y=\sum_{n=0}^{\infty} a_n(x+1)^n, \quad y'=\sum_{n=1}^{\infty} n a_n(x+1)^{n-1} \quad \text{and} \quad y''=\sum_{n=2}^{\infty} n(n-1)a_n(x+1)^{n-2},$$

$$\begin{aligned}
Ly= & 2\sum_{n=2}^{\infty} n(n-1)a_n(x+1)^{n-2}+\sum_{n=2}^{\infty} n(n-1)a_n(x+1)^{n-1} \\
& -\sum_{n=1}^{\infty} n a_n(x+1)^{n-1}+2\sum_{n=1}^{\infty} n a_n(x+1)^n \\
& -3\sum_{n=0}^{\infty} a_n(x+1)^n+\sum_{n=0}^{\infty} a_n(x+1)^{n+1}.
\end{aligned}$$

Shifting indices so that the general term in each series is a constant multiple of $(x+1)^n$ yields

$$\begin{aligned}
Ly= & 2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x+1)^n+\sum_{n=0}^{\infty}(n+1)n a_{n+1}(x+1)^n \\
& -\sum_{n=0}^{\infty}(n+1)a_{n+1}(x+1)^n+\sum_{n=0}^{\infty}(2n-3)a_n(x+1)^n+\sum_{n=1}^{\infty} a_{n-1}(x+1)^n \\
= & \sum_{n=0}^{\infty} b_n(x+1)^n,
\end{aligned}$$

where

$$b_0=4a_2-a_1-3a_0$$

and

$$b_n=2(n+2)(n+1)a_{n+2}+(n^2-1)a_{n+1}+(2n-3)a_n+a_{n-1}, \quad n\geq 1.$$

Therefore $y=\sum_{n=0}^{\infty} a_n(x+1)^n$ is a solution of $Ly=0$ if and only if

$$a_2=\frac{1}{4}(a_1+3a_0) \tag{3.3.5}$$

and

$$a_{n+2}=-\frac{1}{2(n+2)(n+1)}\left[(n^2-1)a_{n+1}+(2n-3)a_n+a_{n-1}\right], \quad n\geq 1. \tag{3.3.6}$$

From the initial conditions in (3.3.4), $a_0=y(-1)=2$ and $a_1=y'(-1)=-3$. We leave it to you to compute $a_2,\ldots,a_5$ with (3.3.5) and (3.3.6) and show that the solution of (3.3.4) is

$$y=-2-3(x+1)+\frac{3}{4}(x+1)^2-\frac{5}{12}(x+1)^3+\frac{7}{48}(x+1)^4-\frac{1}{60}(x+1)^5+\cdots.$$

We also leave it to you in Exercise $(3.3E.14)$ to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^{N} a_n x^n$ converge to the solution of (3.3.4) on the interval of convergence of the power series solution.

3.3.3 Example $\mathbf{3.3.3}$

Find the coefficients $a_0, \dots, a_5$ in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$y'' + 3xy' + (4 + 2x^2)y = 0, \quad y(0) = 2, \quad y'(0) = -3. \tag{3.3.7}$$

Answer

Here

$$Ly = y'' + 3xy' + (4 + 2x^2)y.$$

Since

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

$$Ly = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3 \sum_{n=1}^{\infty} n a_n x^n + 4 \sum_{n=0}^{\infty} a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^{n+2}.$$

Shifting indices so that the general term in each series is a constant multiple of x^n yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (3n+4) a_n x^n + 2 \sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=0}^{\infty} b_n x^n$$

where

$$b_0 = 2a_2 + 4a_0, \quad b_1 = 6a_3 + 7a_1,$$

and

$$b_n = (n+2)(n+1) a_{n+2} + (3n+4) a_n + 2a_{n-2}, \quad n \geq 2.$$

Therefore $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ if and only if

$$a_2 = -2a_0, \quad a_3 = -\frac{7}{6} a_1, \tag{3.3.8}$$

and

$$a_{n+2} = -\frac{1}{(n+2)(n+1)} \left[(3n+4) a_n + 2a_{n-2} \right], \quad n \geq 2. \tag{3.3.9}$$

From the initial conditions in (3.3.7), $a_0 = y(0) = 2$ and $a_1 = y'(0) = -3$. We leave it to you to compute $a_2, \dots, a_5$ with (3.3.8) and (3.3.9) and show that the solution of (3.3.7) is

$$y=2-3x-4x^2+\frac{7}{2}x^3+3x^4-\frac{79}{40}x^5+\cdots.$$

We also leave it to you in Exercise $(3.3E.15)$ to verify numerically that the Taylor polynomials $T_N(x)=\sum_{n=0}^N a_nx^n$ converge to the solution of (3.3.9) on the interval of convergence of the power series solution.

3.3E: Exercises

This page is a draft and is under active development.

In Exercises $(3.3E.1)$ to $(3.3E.12)$, find the coefficients $a_0, \dots, a_N$ for N at least 7 in the series solution $y=\sum_{n=0}^{\infty} a_n x^n$ of the initial value problem.

3.3E.1 Exercise $\mathbf{3.3E.1}$

$(1+3x)y''+xy'+2y=0, \quad y(0)=2, \quad y'(0)=-3$

Answer

Add texts here. Do not delete this text first.

3.3E.2 Exercise $\mathbf{3.3E.2}$

$(1+x+2x^2)y''+(2+8x)y'+4y=0, \quad y(0)=-1, \quad y'(0)=2$

Answer

Add texts here. Do not delete this text first.

3.3E.3 Exercise $\mathbf{3.3E.3}$

$(1-2x^2)y''+(2-6x)y'-2y=0, \quad y(0)=1, \quad y'(0)=0$

Answer

Add texts here. Do not delete this text first.

3.3E.4 Exercise $\mathbf{3.3E.4}$

$(1+x+3x^2)y''+(2+15x)y'+12y=0, \quad y(0)=0, \quad y'(0)=1$

Answer

Add texts here. Do not delete this text first.

3.3E.5 Exercise $\mathbf{3.3E.5}$

$(2+x)y''+(1+x)y'+3y=0, \quad y(0)=4, \quad y'(0)=3$

Answer

Add texts here. Do not delete this text first.

3.3E.6 Exercise $\mathbf{3.3E.6}$

$(3+3x+x^2)y''+(6+4x)y'+2y=0, \quad y(0)=7, \quad y'(0)=3$

Answer

Add texts here. Do not delete this text first.

3.3E.7 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.7}$

$$(4+x)y''+(2+x)y'+2y=0, \quad y(0)=2, \quad y'(0)=5$$

Answer

Add texts here. Do not delete this text first.

3.3E.8 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.8}$

$$(2-3x+2x^2)y''-(4-6x)y'+2y=0, \quad y(1)=1, \quad y'(1)=-1$$

Answer

Add texts here. Do not delete this text first.

3.3E.9 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.9}$

$$(3x+2x^2)y''+10(1+x)y'+8y=0, \quad y(-1)=1, \quad y'(-1)=-1$$

Answer

Add texts here. Do not delete this text first.

3.3E.10 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.10}$

$$(1-x+x^2)y''-(1-4x)y'+2y=0, \quad y(1)=2, \quad y'(1)=-1$$

Answer

Add texts here. Do not delete this text first.

3.3E.11 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.11}$

$$(2+x)y''+(2+x)y'+y=0, \quad y(-1)=-2, \quad y'(-1)=3$$

Answer

Add texts here. Do not delete this text first.

3.3E.12 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.12}$

$$x^2y''-(6-7x)y'+8y=0, \quad y(1)=1, \quad y'(1)=-2$$

Answer

Add texts here. Do not delete this text first.

3.3E.13 Exercise $\mathbf{3.3E.13}$

Do the following experiment for various choices of real numbers a_0, a_1, and r, with $0 < r < 1/\sqrt{2}$.

(a) Use differential equations software to solve the initial value problem

$$(1+x+2x^2)y'' + (1+7x)y' + 2y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \tag{3.3E.1}$$

numerically on $(-r, r)$. (See Example (3.3.1).)

(b) For $N = 2, 3, 4, \ldots$, compute $a_2, \ldots, a_N$ in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of (3.3E.1), and graph

$$T_N(x) = \sum_{n=0}^{N} a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs.

Answer

Add texts here. Do not delete this text first.

3.3E.14 Exercise $\mathbf{3.3E.14}$

Do the following experiment for various choices of real numbers a_0, a_1, and r, with $0 < r < 2$.

(a) Use differential equations software to solve the initial value problem

$$(3+x)y'' + (1+2x)y' - (2-x)y = 0, \quad y(-1) = a_0, \quad y'(-1) = a_1, \tag{3.3E.2}$$

numerically on $(-1-r, -1+r)$. (See Example (3.3.2). Why this interval?)

(b) For $N = 2, 3, 4, \ldots$, compute $a_2, \ldots, a_N$ in the power series solution

$$y = \sum_{n=0}^{\infty} a_n (x+1)^n$$

of (3.3E.2), and graph

$$T_N(x) = \sum_{n=0}^{N} a_n (x+1)^n$$

and the solution obtained in (a) on $(-1-r, -1+r)$. Continue increasing N until there's no perceptible difference between the two graphs.

Answer

Add texts here. Do not delete this text first.

3.3E.15 Exercise $\mathbf{3.3E.15}$

Do the following experiment for several choices of a_0, a_1, and r, with $r > 0$.

(a) Use differential equations software to solve the initial value problem

$$y'' + 3xy' + (4 + 2x^2)y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \tag{3.3E.3}$$

numerically on $(-r, r)$. (See Example (3.3.3).)

(b) Find the coefficients a_0, a_1, ..., a_N in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of (3.3E.3), and graph

$$T_N(x) = \sum_{n=0}^{N} a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs.

Answer

Add texts here. Do not delete this text first.

3.3E.16 Exercise $\mathbf{3.3E.16}$

Do the following experiment for several choices of a_0 and a_1.

(a) Use differential equations software to solve the initial value problem

$$(1 - x)y'' - (2 - x)y' + y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \tag{3.3E.4}$$

numerically on $(-r, r)$.

(b) Find the coefficients a_0, a_1, ..., a_N in the power series solution $y = \sum_{n=0}^{N} a_n x^n$ of (3.3E.4), and graph

$$T_N(x) = \sum_{n=0}^{N} a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs. What happens as you let $r \to 1$?

Answer

Add texts here. Do not delete this text first.

3.3E.17 Exercise $\mathbf{3.3E.17}$

Follow the directions of Exercise $(3.3E.16)$ for the initial value problem

$$(1 + x)y'' + 3y' + 32y = 0, \quad y(0) = a_0, \quad y'(0) = a_1.$$

Answer

Add texts here. Do not delete this text first.

3.3E.18 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.18}$

Follow the directions of Exercise $(3.3E.16)$ for the initial value problem

$$(1+x^2)y''+y'+2y=0, \quad y(0)=a_0, \quad y'(0)=a_1.$$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.3E.19)$ to $(3.3E.28)$, find the coefficients $a_0, \ldots, a_N$ for N at least 7 in the series solution

$$y=\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

3.3E.19 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.19}$

$(2+4x)y''-4y'-(6+4x)y=0, \quad y(0)=2, \quad y'(0)=-7$

Answer

Add texts here. Do not delete this text first.

3.3E.20 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.20}$

$(1+2x)y''-(1-2x)y'-(3-2x)y=0, \quad y(1)=1, \quad y'(1)=-2$

Answer

Add texts here. Do not delete this text first.

3.3E.21 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.21}$

$(5+2x)y''-y'+(5+x)y=0, \quad y(-2)=2, \quad y'(-2)=-1$

Answer

Add texts here. Do not delete this text first.

3.3E.22 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.22}$

$(4+x)y''-(4+2x)y'+(6+x)y=0, \quad y(-3)=2, \quad y'(-3)=-2$

Answer

Add texts here. Do not delete this text first.

3.3E.23 Exercise $\mathbf{3.3}\boldsymbol{E}.\mathbf{23}$

$(2+3x)y''-xy'+2xy=0,\quad y(0)=-1,\quad y'(0)=2$

Answer

Add texts here. Do not delete this text first.

3.3E.24 Exercise $\mathbf{3.3}\boldsymbol{E}.\mathbf{24}$

$(3+2x)y''+3y'-xy=0,\quad y(-1)=2,\quad y'(-1)=-3$

Answer

Add texts here. Do not delete this text first.

3.3E.25 Exercise $\mathbf{3.3}\boldsymbol{E}.\mathbf{25}$

$(3+2x)y''-3y'-(2+x)y=0,\quad y(-2)=-2,\quad y'(-2)=3$

Answer

Add texts here. Do not delete this text first.

3.3E.26 Exercise $\mathbf{3.3}\boldsymbol{E}.\mathbf{26}$

$(10-2x)y''+(1+x)y=0,\quad y(2)=2,\quad y'(2)=-4$

Answer

Add texts here. Do not delete this text first.

3.3E.27 Exercise $\mathbf{3.3}\boldsymbol{E}.\mathbf{27}$

$(7+x)y''+(8+2x)y'+(5+x)y=0,\quad y(-4)=1,\quad y'(-4)=2$

Answer

Add texts here. Do not delete this text first.

3.3E.28 Exercise $\mathbf{3.3}\boldsymbol{E}.\mathbf{28}$

$(6+4x)y''+(1+2x)y=0,\quad y(-1)=-1,\quad y'(-1)=2$

Answer

Add texts here. Do not delete this text first.

3.3E.29 Exercise $\mathbf{3.3}\boldsymbol{E}.\mathbf{29}$

Show that the coefficients in the power series in x for the general solution of

$$(1+\alpha x+\beta x^2)y''+(\gamma+\delta x)y'+\epsilon y=0$$

satisfy the recurrence relation

$$a_{n+2}=-\frac{\gamma+\alpha n}{n+2}\,a_{n+1}-\frac{\beta n(n-1)+\delta n+\epsilon}{(n+2)(n+1)}\,a_n.$$

Answer

Add texts here. Do not delete this text first.

3.3E.30 Exercise $\mathbf{3.3E.30}$

(a) Let α and β be constants, with $\beta \neq 0$. Show that $y=\sum_{n=0}^{\infty} a_n x^n$ is a solution of

$$(1+\alpha x+\beta x^2)y''+(2\alpha+4\beta x)y'+2\beta y=0 \tag{3.3E.5}$$

if and only if

$$a_{n+2}+\alpha a_{n+1}+\beta a_n=0,\quad n\geq 0. \tag{3.3E.6}$$

An equation of this form is called a second order homogeneous linear difference equation. The polynomial $p(r)=r^2+\alpha r+\beta$ is called the characteristic polynomial of (3.3E.6). If r_1 and r_2 are the zeros of p, then $1/r_1$ and $1/r_2$ are the zeros of

$$P_0(x)=1+\alpha x+\beta x^2.$$

(b) Suppose $p(r)=(r-r_1)(r-r_2)$ where r_1 and r_2 are real and distinct, and let ρ be the smaller of the two numbers $\{1/|r_1|, 1/|r_2|\}$. Show that if c_1 and c_2 are constants then the sequence

$$a_n=c_1r_1^n+c_2r_2^n,\quad n\geq 0$$

satisfies (3.3E.6). Conclude from this that any function of the form

$$y=\sum_{n=0}^{\infty}(c_1r_1^n+c_2r_2^n)x^n$$

is a solution of (3.3E.5) on $(-\rho,\rho)$.

(c) Use (b) and the formula for the sum of a geometric series to show that the functions

$$y_1=\frac{1}{1-r_1x}\quad\text{and}\quad y_2=\frac{1}{1-r_2x}$$

form a fundamental set of solutions of (3.3E.5) on $(-\rho,\rho)$.

(d) Show that $\{y_1,y_2\}$ is a fundamental set of solutions of (3.3E.5) on any interval that doesn't contain either $1/r_1$ or $1/r_2$.

(e) Suppose $p(r)=(r-r_1)^2$, and let $\rho=1/|r_1|$. Show that if c_1 and c_2 are constants then the sequence

$$a_n=(c_1+c_2n)r_1^n,\quad n\geq 0$$

satisfies (3.3E.6). Conclude from this that any function of the form

$$y=\sum_{n=0}^{\infty}(c_1+c_2n)r_1^nx^n$$

is a solution of (3.3E.5) on $(-\rho,\rho)$.

(f) Use (e) and the formula for the sum of a geometric series to show that the functions

$$y_1=\frac{1}{1-r_1x}\quad \text{and} \quad y_2=\frac{x}{(1-r_1x)^2}$$

form a fundamental set of solutions of (3.3E.5) on $(-\rho,\rho)$.

(g) Show that $\{y_1,y_2\}$ is a fundamental set of solutions of (3.3E.5) on any interval that does not contain $1/r_1$.

Answer

Add texts here. Do not delete this text first.

3.3E.31 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.31}$

Use the results of Exercise $(3.3E.30)$ to find the general solution of the given equation on any interval on which polynomial multiplying y'' has no zeros.

(a) $(1+3x+2x^2)y''+(6+8x)y'+4y=0$

(b) $(1-5x+6x^2)y''-(10-24x)y'+12y=0$

(c) $(1-4x+4x^2)y''-(8-16x)y'+8y=0$

(d) $(4+4x+x^2)y''+(8+4x)y'+2y=0$

(e) $(4+8x+3x^2)y''+(16+12x)y'+6y=0$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.3E.32)$ to $(3.3E.38)$, find the coefficients a_0, …, a_N for N at least 7 in the series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of the initial value problem.

3.3E.32 Exercise $\mathbf{3.3}\boldsymbol{E}\mathbf{.32}$

$y''+2xy'+(3+2x^2)y=0,\quad y(0)=1,\quad y'(0)=-2$

Answer

Add texts here. Do not delete this text first.

3.3E.33 Exercise $\mathbf{3.3E.33}$

$y'' - 3xy' + (5 + 2x^2)y = 0, \quad y(0) = 1, \quad y'(0) = -2$

Answer

Add texts here. Do not delete this text first.

3.3E.34 Exercise $\mathbf{3.3E.34}$

$y'' + 5xy' - (3 - x^2)y = 0, \quad y(0) = 6, \quad y'(0) = -2$

Answer

Add texts here. Do not delete this text first.

3.3E.35 Exercise $\mathbf{3.3E.35}$

$y'' - 2xy' - (2 + 3x^2)y = 0, \quad y(0) = 2, \quad y'(0) = -5$

Answer

Add texts here. Do not delete this text first.

3.3E.36 Exercise $\mathbf{3.3E.36}$

$y'' - 3xy' + (2 + 4x^2)y = 0, \quad y(0) = 3, \quad y'(0) = 6$

Answer

Add texts here. Do not delete this text first.

3.3E.37 Exercise $\mathbf{3.3E.37}$

$2y'' + 5xy' + (4 + 2x^2)y = 0, \quad y(0) = 3, \quad y'(0) = -2$

Answer

Add texts here. Do not delete this text first.

3.3E.38 Exercise $\mathbf{3.3E.38}$

$3y'' + 2xy' + (4 - x^2)y = 0, \quad y(0) = -2, \quad y'(0) = 3$

Answer

Add texts here. Do not delete this text first.

3.3E.39 Exercise $\mathbf{3.3E.39}$

Find power series in x for the solutions y_1 and y_2 of

$$y'' + 4xy' + (2 + 4x^2)y = 0$$

such that $y_1(0)=1$, $y_1'(0)=0$, $y_2(0)=0$, $y_2'(0)=1$, and identify y_1 and y_2 in terms of familiar elementary functions.

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.3E.40)$ tp $(3.3E.49)$, find the coefficients a_0, ..., a_N for N at least 7 in the series solution

$$y=\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

3.3E.40 Exercise $\mathbf{3.3E.40}$

$(1+x)y''+x^2y'+(1+2x)y=0, \quad y(0)-2, \quad y'(0)=3$

Answer

Add texts here. Do not delete this text first.

3.3E.41 Exercise $\mathbf{3.3E.41}$

$y''+(1+2x+x^2)y'+2y=0, \quad y(0)=2, \quad y'(0)=3$

Answer

Add texts here. Do not delete this text first.

3.3E.42 Exercise $\mathbf{3.3E.42}$

$(1+x^2)y''+(2+x^2)y'+xy=0, \quad y(0)=-3, \quad y'(0)=5$

Answer

Add texts here. Do not delete this text first.

3.3E.43 Exercise $\mathbf{3.3E.43}$

$(1+x)y''+(1-3x+2x^2)y'-(x-4)y=0, \quad y(1)=-2, \quad y'(1)=3$

Answer

Add texts here. Do not delete this text first.

3.3E.44 Exercise $\mathbf{3.3E.44}$

$y''+(13+12x+3x^2)y'+(5+2x), \quad y(-2)=2, \quad y'(-2)=-3$

Answer

Add texts here. Do not delete this text first.

3.3E.45 Exercise $3.3E.45$

$$(1+2x+3x^2)y''+(2-x^2)y'+(1+x)y=0, \quad y(0)=1, \quad y'(0)=-2$$

Answer

Add texts here. Do not delete this text first.

3.3E.46 Exercise $3.3E.46$

$$(3+4x+x^2)y''-(5+4x-x^2)y'-(2+x)y=0, \quad y(-2)=2, \quad y'(-2)=-1$$

Answer

Add texts here. Do not delete this text first.

3.3E.47 Exercise $3.3E.47$

$$(1+2x+x^2)y''+(1-x)y=0, \quad y(0)=2, \quad y'(0)=-1$$

Answer

Add texts here. Do not delete this text first.

3.3E.48 Exercise $3.3E.48$

$$(x-2x^2)y''+(1+3x-x^2)y'+(2+x)y=0, \quad y(1)=1, \quad y'(1)=0$$

Answer

Add texts here. Do not delete this text first.

3.3E.49 Exercise $3.3E.49$

$$(16-11x+2x^2)y''+(10-6x+x^2)y'-(2-x)y, \quad y(3)=1, \quad y'(3)=-2$$

Answer

Add texts here. Do not delete this text first.

3.4: Regular Singular Points: Euler Equations

This page is a draft and is under active development.

3.4.1 Regular Singular Points

In the next three sections we'll continue to study equations of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{3.4.1}$$

where P_0, P_1, and P_2 are polynomials, but the emphasis will be different from that of Sections 3.2 and 3.3, where we obtained solutions of (3.4.1) near an ordinary point x_0 in the form of power series in $x - x_0$. If x_0 is a singular point of (3.4.1) (that is, if $P(x_0) = 0$), the solutions can't in general be represented by power series in $x - x_0$. Nevertheless, it's often necessary in physical applications to study the behavior of solutions of (3.4.1) near a singular point. Although this can be difficult in the absence of some sort of assumption on the nature of the singular point, equations that satisfy the requirements of the next definition can be solved by series methods discussed in the next three sections. Fortunately, many equations arising in applications satisfy these requirements.

Definition 3.4.1:

Let P_0, P_1, and P_2 be polynomials with no common factor and suppose $P_0(x_0) = 0$. Then x_0 is a regular singular point of the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{3.4.2}$$

if (3.4.2) can be written as

$$(x - x_0)^2 A(x)y'' + (x - x_0)B(x)y' + C(x)y = 0 \tag{3.4.3}$$

where A, B, and C are polynomials and $A(x_0) \neq 0$; otherwise, x_0 is an irregular singular point of (3.4.2).

Example 3.4.1:

Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \tag{3.4.4}$$

has the singular point $x_0 = 0$. Since this equation is of the form (3.4.3) with $x_0 = 0$, $A(x) = 1$, $B(x) = 1$, and $C(x) = x^2 - \nu^2$, it follows that $x_0 = 0$ is a regular singular point of (3.4.4).

Example 3.4.2:

Legendre's equation,

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \tag{3.4.5}$$

has the singular points $x_0 = \pm 1$. Multiplying through by $1 - x$ yields

$$(x-1)^2(x+1)y'' + 2x(x-1)y' - \alpha(\alpha+1)(x-1)y = 0,$$

which is of the form (3.4.3) with $x_0 = 1$, $A(x) = x+1$, $B(x) = 2x$, and $C(x) = -\alpha(\alpha+1)(x-1)$. Therefore $x_0 = 1$ is a regular singular point of (3.4.5). We leave it to you to show that $x_0 = -1$ is also a regular singular point of (3.4.5).

Example 3.4.3:

The equation

$$x^3y'' + xy' + y = 0$$

has an irregular singular point at $x_0 = 0$. (Verify.)

For convenience we restrict our attention to the case where $x_0 = 0$ is a regular singular point of (3.4.2). This isn't really a restriction, since if $x_0 \neq 0$ is a regular singular point of (3.4.2) then introducing the new independent variable $t = x - x_0$ and the new unknown $Y(t) = y(t + x_0)$ leads to a differential equation with polynomial coefficients that has a regular singular point at $t_0 = 0$. This is illustrated in Exercise $(3.4E.22)$ for Legendre's equation, and in Exercise $(3.4E.23)$ for the general case.

3.4.2 Euler Equations

The simplest kind of equation with a regular singular point at $x_0 = 0$ is the Euler equation, defined as follows.

Definition 3.4.2:

An Euler equation is an equation that can be written in the form

$$ax^2y'' + bxy' + cy = 0, \tag{3.4.6}$$

where a, b, and c are real constants and $a \neq 0$.

Theorem $(2.1.1)$ implies that (3.4.6) has solutions defined on $(0, \infty)$ and $(-\infty, 0)$, since (3.4.6) can be rewritten as

$$ay'' + \frac{b}{x}y' + \frac{c}{x^2}y = 0.$$

For convenience we'll restrict our attention to the interval $(0, \infty)$. (Exercise $(3.4E.19)$ deals with solutions of (3.4.6) on $(-\infty, 0)$.) The key to finding solutions on $(0, \infty)$ is that if $x > 0$ then x^r is defined as a real-valued function on $(0, \infty)$ for all values of r, and substituting $y = x^r$ into (3.4.6) produces

$$\begin{aligned} ax^2(x^r)'' + bx(x^r)' + cx^r &= ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r \\ &= [ar(r-1) + br + c]x^r. \end{aligned} \tag{3.4.7}$$

The polynomial

$$p(r) = ar(r-1) + br + c$$

is called the indicial polynomia of (3.4.6), and $p(r) = 0$ is its indicial equation. From (3.4.7) we can see that $y = x^r$ is a solution of (3.4.6) on $(0, \infty)$ if and only if $p(r) = 0$. Therefore, if the indicial equation has distinct real roots r_1 and r_2 then $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$ form a fundamental set of solutions of (3.4.6) on $(0, \infty)$, since $y_2/y_1 = x^{r_2 - r_1}$ is nonconstant. In this case

$$y = c_1 x^{r_1} + c_2 x^{r_2}$$

is the general solution of (3.4.6) on $(0, \infty)$.

Example 3.4.4

Find the general solution of

$$x^2 y'' - x y' - 8y = 0 \tag{3.4.8}$$

on $(0, \infty)$.

Answer

The indicial polynomial of (3.4.8) is

$$p(r) = r(r-1) - r - 8 = (r-4)(r+2).$$

Therefore $y_1 = x^4$ and $y_2 = x^{-2}$ are solutions of (3.4.8) on $(0, \infty)$, and its general solution on $(0, \infty)$ is

$$y = c_1 x^4 + \frac{c_2}{x^2}.$$

Example 3.4.5

Find the general solution of

$$6x^2 y'' + 5x y' - y = 0 \tag{3.4.9}$$

on $(0, \infty)$.

Answer

The indicial polynomial of (3.4.9) is

$$p(r) = 6r(r-1) + 5r - 1 = (2r-1)(3r+1).$$

Therefore the general solution of (3.4.9) on $(0, \infty)$ is

$$y = c_1 x^{1/2} + c_2 x^{-1/3}.$$

If the indicial equation has a repeated root r_1, then $y_1 = x^{r_1}$ is a solution of

$$a x^2 y'' + b x y' + c y = 0, \tag{3.4.10}$$

on $(0,\infty)$, but (3.4.10) has no other solution of the form $y=x^r$. If the indicial equation has complex conjugate zeros then (3.4.10) has no real-valued solutions of the form $y=x^r$. Fortunately we can use the results of Section 3.2 for constant coefficient equations to solve (3.4.10) in any case.

Theorem 3.4.3

Suppose the roots of the indicial equation

$$ar(r-1)+br+c=0 \tag{3.4.11}$$

are r_1 and r_2. Then the general solution of the Euler equation

$$ax^2y''+bxy'+cy=0 \tag{3.4.12}$$

on $(0,\infty)$ is

$$\begin{aligned} y= & \; c_1x^{r_1}+c_2x^{r_2} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers };\\ y= & \; x^{r_1}(c_1+c_2\ln x) \text{ if } r_1=r_2 ;\\ y= & \; x^{\lambda}\left[c_1\cos(\omega\ln x)+c_2\sin(\omega\ln x)\right] \text{ if } r_1,r_2=\lambda\pm i\omega \text{ with } \omega>0. \end{aligned}$$

Proof

We first show that $y=y(x)$ satisfies (3.4.12) on $(0,\infty)$ if and only if $Y(t)=y(e^t)$ satisfies the constant coefficient equation

$$a\frac{d^2Y}{dt^2}+(b-a)\frac{dY}{dt}+cY=0 \tag{3.4.13}$$

on $(-\infty,\infty)$. To do this, it's convenient to write $x=e^t$, or, equivalently, $t=\ln x$; thus, $Y(t)=y(x)$, where $x=e^t$. From the chain rule,

$$\frac{dY}{dt}=\frac{dy}{dx}\frac{dx}{dt}$$

and, since

$$\frac{dx}{dt}=e^t=x,$$

it follows that

$$\frac{dY}{dt}=x\frac{dy}{dx}. \tag{3.4.14}$$

Differentiating this with respect to t and using the chain rule again yields

$$\begin{aligned}\frac{d^2Y}{dt^2} &= \frac{d}{dt}\left(\frac{dY}{dt}\right) = \frac{d}{dt}\left(x\frac{dy}{dx}\right) \\ &= \frac{dx}{dt}\frac{dy}{dx} + x\frac{d^2y}{dx^2}\frac{dx}{dt} \\ &= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} \quad \left(\text{since } \frac{dx}{dt} = x\right).\end{aligned}$$

From this and (3.4.14),

$$x^2\frac{d^2y}{dx^2} = \frac{d^2Y}{dt^2} - \frac{dY}{dt}.$$

Substituting this and (3.4.14) into (3.4.12) yields (3.4.13). Since (3.4.11) is the characteristic equation of (3.4.13), Theorem (3.2.1) implies that the general solution of (3.4.13) on $(-\infty, \infty)$ is

$$\begin{aligned}&Y(t) = c_1e^{r_1t} + c_2e^{r_2t} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers;} \\ &Y(t) = e^{r_1t}(c_1 + c_2t) \text{ if } r_1 = r_2; \\ &Y(t) = \ e^{\lambda t}(c_1\cos\omega t + c_2\sin\omega t) \text{ if } r_1, r_2 = \lambda \pm i\omega \text{ with } \omega \neq 0.\end{aligned}$$

Since $Y(t) = y(e^t)$, substituting $t = \ln x$ in the last three equations shows that the general solution of (3.4.12) on $(0, \infty)$ has the form stated in the theorem.

Example 3.4.6

Find the general solution of

$$x^2y'' - 5xy' + 9y = 0 \tag{3.4.15}$$

on $(0, \infty)$.

Answer

The indicial polynomial of (3.4.15) is

$$p(r) = r(r-1) - 5r + 9 = (r-3)^2.$$

Therefore the general solution of (3.4.15) on $(0, \infty)$ is

$$y = x^3(c_1 + c_2\ln x).$$

Example 3.4.7

Find the general solution of

$$x^2y'' + 3xy' + 2y = 0 \tag{3.4.16}$$

on $(0, \infty)$.

Answer

The indicial polynomial of (3.4.16) is

$$p(r)=r(r-1)+3r+2=(r+1)^2+1.$$

The roots of the indicial equation are $r=-1\pm i$ and the general solution of (3.4.16) on $(0,\infty)$ is

$$y=\frac{1}{x}\left[c_1\cos(\ln x)+c_2\sin(\ln x)\right].$$

3.4E: Exercises

This page is a draft and is under active development.

In Exercises $(3.4E.1)$ to $(3.4E.18)$, find the general solution of the given Euler equation on $(0, \infty)$.

3.4E.1 Exercise $\mathbf{3.4E.1}$

$x^2y'' + 7xy' + 8y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.2 Exercise $\mathbf{3.4E.2}$

$x^2y'' - 7xy' + 7y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.3 Exercise $\mathbf{3.4E.3}$

$x^2y'' - xy' + y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.4 Exercise $\mathbf{3.4E.4}$

$x^2y'' + 5xy' + 4y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.5 Exercise $\mathbf{3.4E.5}$

$x^2y'' + xy' + y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.6 Exercise $\mathbf{3.4E.6}$

$x^2y'' - 3xy' + 13y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.7 Exercise $\mathbf{3.4}\boldsymbol{E}\mathbf{.7}$

$x^2y'' + 3xy' - 3y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.8 Exercise $\mathbf{3.4}\boldsymbol{E}\mathbf{.8}$

$12x^2y'' - 5xy'' + 6y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.9 Exercise $\mathbf{3.4}\boldsymbol{E}\mathbf{.9}$

$4x^2y'' + 8xy' + y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.10 Exercise $\mathbf{3.4}\boldsymbol{E}\mathbf{.10}$

$3x^2y'' - xy' + y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.11 Exercise $\mathbf{3.4}\boldsymbol{E}\mathbf{.11}$

$2x^2y'' - 3xy' + 2y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.12 Exercise $\mathbf{3.4}\boldsymbol{E}\mathbf{.12}$

$x^2y'' + 3xy' + 5y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.13 Exercise $\mathbf{3.4}\boldsymbol{E}\mathbf{.13}$

$9x^2y'' + 15xy' + y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.14 Exercise $\mathbf{3.4E.14}$

$x^2y'' - xy' + 10y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.15 Exercise $\mathbf{3.4E.15}$

$x^2y'' - 6y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.16 Exercise $\mathbf{3.4E.16}$

$2x^2y'' + 3xy' - y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.17 Exercise $\mathbf{3.4E.17}$

$x^2y'' - 3xy' + 4y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.18 Exercise $\mathbf{3.4E.18}$

$2x^2y'' + 10xy' + 9y = 0$

Answer

Add texts here. Do not delete this text first.

3.4E.19 Exercise $\mathbf{3.4E.19}$

(a) Adapt the proof of Theorem $(3.4.3)$ to show that $y = y(x)$ satisfies the Euler equation

$$ax^2y'' + bxy' + cy = 0 \tag{3.4E.1}$$

on $(-\infty, 0)$ if and only if $Y(t) = y(-e^t)$

$$a\frac{d^2Y}{dt^2} + (b-a)\frac{dY}{dt} + cY = 0.$$

on $(-\infty, \infty)$.

(b) Use part (a) to show that the general solution of (3.4E.1) on $(-\infty, 0)$ is

$$y = c_1|x|^{r_1} + c_2|x|^{r_2} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers;}$$
$$y = |x|^{r_1}(c_1 + c_2 \ln|x|) \text{ if } r_1 = r_2;$$
$$y = |x|^{\lambda}\left[c_1 \cos(\omega \ln|x|) + c_2 \sin(\omega \ln|x|)\right] \text{ if } r_1, r_2 = \lambda \pm i\omega \text{ with } \omega > 0.$$

Answer

Add texts here. Do not delete this text first.

3.4E.20 Exercise $\mathbf{3.4E.20}$

Use reduction of order to show that if

$$ar(r-1) + br + c = 0$$

has a repeated root r_1 then $y = x^{r_1}(c_1 + c_2 \ln x)$ is the general solution of

$$ax^2y'' + bxy' + cy = 0$$

on $(0, \infty)$.

Answer

Add texts here. Do not delete this text first.

3.4E.21 Exercise $\mathbf{3.4E.21}$

A nontrivial solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

is said to be oscillatory on an interval (a, b) if it has infinitely many zeros on (a, b). Otherwise y is said to be nonoscillatory on (a, b). Show that the equation

$$x^2y'' + ky = 0 \quad (k = \text{ constant})$$

has oscillatory solutions on $(0, \infty)$ if and only if $k > 1/4$.

Answer

Add texts here. Do not delete this text first.

3.4E.22 Exercise $\mathbf{3.4E.22}$

In Example $(3.4.2)$ we saw that $x_0 = 1$ and $x_0 = -1$ are regular singular points of Legendre's equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0. \tag{3.4E.2}$$

(a) Introduce the new variables $t = x - 1$ and $Y(t) = y(t+1)$, and show that y is a solution of (3.4E.2) if and only if Y is a solution of

$$t(2+t)\frac{d^2Y}{dt^2} + 2(1+t)\frac{dY}{dt} - \alpha(\alpha+1)Y = 0,$$

which has a regular singular point at $t_0 = 0$.

(b) Introduce the new variables $t = x + 1$ and $Y(t) = y(t-1)$, and show that y is a solution of (3.4E.2) if and only if \(Y\(is a solution of

$$t(2-t)\frac{d^2Y}{dt^2} + 2(1-t)\frac{dY}{dt} + \alpha(\alpha+1)Y = 0,$$

which has a regular singular point at $t_0 = 0$.

Answer

Add texts here. Do not delete this text first.

3.4E.23 Exercise $\mathbf{3.4E.23}$

Let P_0, P_1, and P_2 be polynomials with no common factor, and suppose $x_0 \neq 0$ is a singular point of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{3.4E.3}$$

Let $t = x - x_0$ and $Y(t) = y(t + x_0)$.

(a) Show that y is a solution of (3.4E.3) if and only if Y is a solution of

$$R_0(t)\frac{d^2Y}{dt^2} + R_1(t)\frac{dY}{dt} + R_2(t)Y = 0. \tag{3.4E.4}$$

where

$$R_i(t) = P_i(t + x_0), \quad i = 0, 1, 2.$$

(b) Show that R_0, R_1, and R_2 are polynomials in t with no common factors, and $R_0(0) = 0$; thus, $t_0 = 0$ is a singular point of (3.4E.4).

Answer

Add texts here. Do not delete this text first.

3.5: The Method of Frobenius I

This page is a draft and is under active development.

3.5.1 The Method of Frobenius

In this section we begin to study series solutions of a homogeneous linear second order differential equation with a regular singular point at $x_0 = 0$, so it can be written as

$$x^2 A(x)y'' + xB(x)y' + C(x)y = 0, \tag{3.5.1}$$

where A, B, C are polynomials and $A(0) \neq 0$.

We'll see that (3.5.1) always has at least one solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

where $a_0 \neq 0$ and r is a suitably chosen number. The method we will use to find solutions of this form and other forms that we'll encounter in the next two sections is called the method of Frobenius, and we'll call them Frobenius solutions.

It can be shown that the power series $\sum_{n=0}^{\infty} a_n x^n$ in a Frobenius solution of (3.5.1) converges on some open interval $(-\rho, \rho)$, where $0 < \rho \leq \infty$. However, since x^r may be complex for negative x or undefined if $x = 0$, we'll consider solutions defined for positive values of x. Easy modifications of our results yield solutions defined for negative values of x. (Exercise $(3.5E.54)$).

We'll restrict our attention to the case where A, B, and C are polynomials of degree not greater than two, so (3.5.1) becomes

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0, \tag{3.5.2}$$

where α_i, β_i, and γ_i are real constants and $\alpha_0 \neq 0$. Most equations that arise in applications can be written this way. Some examples are

$$\begin{gathered}
\alpha x^2 y'' + \beta x y' + \gamma y = 0 \quad \text{(Euler's equation)}, \\
x^2 y'' + x y' + (x^2 - \nu^2)y = 0 \quad \text{(Bessel's equation)}, \\
\text{and} \\
xy'' + (1 - x)y' + \lambda y = 0, \quad \text{(Laguerre's equation)}\,,
\end{gathered}$$

where we would multiply the last equation through by x to put it in the form (3.5.2). However, the method of Frobenius can be extended to the case where A, B, and C are functions that can be represented by power series in x on some interval that contains zero, and $A_0(0) \neq 0$ (Exercises $(3.5E.57)$ and $(3.5E.38)$).

The next two theorems will enable us to develop systematic methods for finding Frobenius solutions of (3.5.2).

Theorem 3.5.1

Let

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y,$$

and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1) + \beta_0 r + \gamma_0, \\ p_1(r) &= \alpha_1 r(r-1) + \beta_1 r + \gamma_1, \\ p_2(r) &= \alpha_2 r(r-1) + \beta_2 r + \gamma_2. \end{aligned}$$

Suppose the series

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{3.5.3}$$

converges on $(0, \rho)$. Then

$$Ly = \sum_{n=0}^{\infty} b_n x^{n+r} \tag{3.5.4}$$

on $(0, \rho)$, where

$$\begin{aligned} b_0 &= p_0(r)a_0, \\ b_1 &= p_0(r+1)a_1 + p_1(r)a_0, \\ b_n &= p_0(n+r)a_n + p_1(n+r-1)a_{n-1} + p_2(n+r-2)a_{n-2}, \quad n \geq 2. \end{aligned} \tag{3.5.5}$$

Proof

We begin by showing that if y is given by (3.5.3) and α, β, and γ are constants, then

$$\alpha x^2 y'' + \beta x y' + \gamma y = \sum_{n=0}^{\infty} p(n+r)a_n x^{n+r}, \tag{3.5.6}$$

where

$$p(r) = \alpha r(r-1) + \beta r + \gamma.$$

Differentiating (3) twice yields

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \tag{3.5.7}$$

and

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}. \tag{3.5.8}$$

Multiplying (3.5.7) by x and (3.5.8) by x^2 yields

$$xy' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

and

$$x^2y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}.$$

Therefore

$$\begin{aligned}\alpha x^2y'' + \beta xy' + \gamma y &= \sum_{n=0}^{\infty}\left[\alpha(n+r)(n+r-1) + \beta(n+r) + \gamma\right] a_n x^{n+r} \\ &= \sum_{n=0}^{\infty} p(n+r)a_n x^{n+r},\end{aligned}$$

which proves (3.5.6).

Multiplying (3.5.6) by x yields

$$x(\alpha x^2y'' + \beta xy' + \gamma y) = \sum_{n=0}^{\infty} p(n+r)a_n x^{n+r+1} = \sum_{n=1}^{\infty} p(n+r-1)a_{n-1}x^{n+r}. \tag{3.5.9}$$

Multiplying (3.5.6) by x^2 yields

$$x^2(\alpha x^2y'' + \beta xy' + \gamma y) = \sum_{n=0}^{\infty} p(n+r)a_n x^{n+r+2} = \sum_{n=2}^{\infty} p(n+r-2)a_{n-2}x^{n+r}. \tag{3.5.10}$$

To use these results, we rewrite

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y$$

as

$$\begin{aligned}Ly \quad = \quad & \left(\alpha_0 x^2y'' + \beta_0 xy' + \gamma_0 y\right) + x\left(\alpha_1 x^2y'' + \beta_1 xy' + \gamma_1 y\right) \\ & + x^2\left(\alpha_2 x^2y'' + \beta_2 xy' + \gamma_2 y\right).\end{aligned} \tag{3.5.11}$$

From (3.5.6) with $p = p_0$,

$$\alpha_0 x^2y'' + \beta_0 xy' + \gamma_0 y = \sum_{n=0}^{\infty} p_0(n+r)a_n x^{n+r}.$$

From (3.5.9) with $p = p_1$,

$$x\left(\alpha_1 x^2y'' + \beta_1 xy' + \gamma_1 y\right) = \sum_{n=1}^{\infty} p_1(n+r-1)a_{n-1}x^{n+r}.$$

From (3.5.10) with $p = p_2$,

$$x^2\left(\alpha_2x^2y''+\beta_2xy'+\gamma_2y\right)=\sum_{n=2}^{\infty}p_2(n+r-2)a_{n-2}x^{n+r}.$$

Therefore we can rewrite (3.5.11) as

$$\begin{aligned} Ly = \sum_{n=0}^{\infty}p_0(n+r)a_nx^{n+r}+\sum_{n=1}^{\infty}p_1(n+r-1)a_{n-1}x^{n+r} \\ +\sum_{n=2}^{\infty}p_2(n+r-2)a_{n-2}x^{n+r}, \end{aligned}$$

or

$$\begin{aligned} Ly = &\ p_0(r)a_0x^r+\left[p_0(r+1)a_1+p_1(r)a_2\right]x^{r+1} \\ &+\sum_{n=2}^{\infty}\left[p_0(n+r)a_n+p_1(n+r-1)a_{n-1}+p_2(n+r-2)a_{n-2}\right]x^{n+r}, \end{aligned}$$

which implies (3.5.4) with $\{b_n\}$ defined as in (3.5.5).

Theorem 3.5.2

Let

$$Ly=x^2(\alpha_0+\alpha_1x+\alpha_2x^2)y''+x(\beta_0+\beta_1x+\beta_2x^2)y'+(\gamma_0+\gamma_1x+\gamma_2x^2)y,$$

where $\alpha_0\neq0,$ and define

$$\begin{aligned} p_0(r) &= \alpha_0r(r-1)+\beta_0r+\gamma_0, \\ p_1(r) &= \alpha_1r(r-1)+\beta_1r+\gamma_1, \\ p_2(r) &= \alpha_2r(r-1)+\beta_2r+\gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n. Define

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1}(r)+p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)}, \quad n\geq2. \end{aligned} \tag{3.5.12}$$

Then the Frobenius series

$$y(x,r)=x^r\sum_{n=0}^{\infty}a_n(r)x^n \tag{3.5.13}$$

converges and satisfies

$$Ly(x,r)=p_0(r)x^r \tag{3.5.14}$$

on the interval $(0,\rho)$, where ρ is the distance from the origin to the nearest zero of $A(x)=\alpha_0+\alpha_1x+\alpha_2x^2$ in the complex plane. (If A is constant, then $\rho=\infty$.)

Proof

If $\{a_n(r)\}$ is determined by the recurrence relation (3.5.12) then substituting $a_n=a_n(r)$ into (3.5.5) yields $b_0=p_0(r)$ and $b_n=0$ for $n\geq 1$, so (3.5.4) reduces to (3.5.14). We omit the proof that the series (3.5.13) converges on $(0,\rho)$.

If $\alpha_i=\beta_i=\gamma_i=0$ for $i=1,2$, then $Ly=0$ reduces to the Euler equation

$$\alpha_0x^2y''+\beta_0xy'+\gamma_0y=0.$$

Theorem (3.4.3) shows that the solutions of this equation are determined by the zeros of the indicial polynomial

$$p_0(r)=\alpha_0r(r-1)+\beta_0r+\gamma_0.$$

Since (3.5.14) implies that this is also true for the solutions of $Ly=0$, we'll also say that p_0 is the indicial polynomial of (3.5.2), and that $p_0(r)=0$ is the indicial equation of $Ly=0$. We'll consider only cases where the indicial equation has real roots r_1 and r_2, with $r_1\geq r_2$.

Theorem 3.5.3

Let L and $\{a_n(r)\}$ be as in Theorem (3.5.2), and suppose the indicial equation $p_0(r)=0$ of $Ly=0$ has real roots r_1 and r_2, where $r_1\geq r_2$. Then

$$y_1(x)=y(x,r_1)=x^{r_1}\sum_{n=0}^{\infty}a_n(r_1)x^n$$

is a Frobenius solution of $Ly=0$. Moreover, if r_1-r_2 isn't an integer then

$$y_2(x)=y(x,r_2)=x^{r_2}\sum_{n=0}^{\infty}a_n(r_2)x^n$$

is also a Frobenius solution of $Ly=0$, and $\{y_1,y_2\}$ is a fundamental set of solutions.

Proof

Since r_1 and r_2 are roots of $p_0(r)=0$, the indicial polynomial can be factored as

$$p_0(r)=\alpha_0(r-r_1)(r-r_2). \tag{3.5.15}$$

Therefore

$$p_0(n+r_1)=n\alpha_0(n+r_1-r_2),$$

which is nonzero if $n>0$, since $r_1-r_2\geq 0$. Therefore the assumptions of Theorem (3.5.2) hold with $r=r_1$, and
(3.5.14) implies that $Ly_1=p_0(r_1)x^{r_1}=0$.

Now suppose $r_1 - r_2$ isn't an integer. From (3.5.15),

$$p_0(n+r_2) = n\alpha_0(n - r_1 + r_2) \neq 0 \quad \text{if} \quad n = 1, 2, \cdots.$$

Hence, the assumptions of Theorem (3.5.2) hold with $r = r_2$, and (3.5.14) implies that $Ly_2 = p_0(r_2)x^{r_2} = 0$. We leave the proof that $\{y_1, y_2\}$ is a fundamental set of solutions as an exercise (Exercise (3.5E.52)).

It isn't always possible to obtain explicit formulas for the coefficients in Frobenius solutions. However, we can always set up the recurrence relations and use them to compute as many coefficients as we want. The next example illustrates this.

Example 3.5.1

Find a fundamental set of Frobenius solutions of

$$2x^2(1+x+x^2)y'' + x(9+11x+11x^2)y' + (6+10x+7x^2)y = 0. \tag{3.5.16}$$

Compute just the first six coefficients $a_0, \ldots, a_5$ in each solution.

Answer

For the given equation, the polynomials defined in Theorem (3.5.2) are

$$\begin{array}{rclcl} p_0(r) & = & 2r(r-1)+9r+6 & = & (2r+3)(r+2), \\ p_1(r) & = & 2r(r-1)+11r+10 & = & (2r+5)(r+2), \\ p_2(r) & = & 2r(r-1)+11r+7 & = & (2r+7)(r+1). \end{array}$$

The zeros of the indicial polynomial p_0 are $r_1 = -3/2$ and $r_2 = -2$, so $r_1 - r_2 = 1/2$. Therefore Theorem (3.5.3) implies that

$$y_1 = x^{-3/2}\sum_{n=0}^{\infty} a_n(-3/2)x^n \quad \text{and} \quad y_2 = x^{-2}\sum_{n=0}^{\infty} a_n(-2)x^n \tag{3.5.17}$$

form a fundamental set of Frobenius solutions of (3.5.16). To find the coefficients in these series, we use the recurrence relation of Theorem (3.5.2); thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)} = -\frac{(2r+5)(r+2)}{(2r+5)(r+3)} = -\frac{r+2}{r+3}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1} + p_2(n+r-2)a_{n-2}}{p_0(n+r)} \\ &= -\frac{(n+r+1)(2n+2r+3)a_{n-1}(r) + (n+r-1)(2n+2r+3)a_{n-2}(r)}{(n+r+2)(2n+2r+3)} \\ &= -\frac{(n+r+1)a_{n-1}(r) + (n+r-1)a_{n-2}(r)}{n+r+2}, \quad n \geq 2. \end{aligned}$$

Setting $r=-3/2$ in these equations yields

$$\begin{aligned} a_0(-3/2) &= 1, \\ a_1(-3/2) &= -1/3, \\ a_n(-3/2) &= -\frac{(2n-1)a_{n-1}(-3/2)+(2n-5)a_{n-2}(-3/2)}{2n+1}, \quad n\geq 2, \end{aligned} \tag{3.5.18}$$

and setting $r=-2$ yields

$$\begin{aligned} a_0(-2) &= 1, \\ a_1(-2) &= 0, \\ a_n(-2) &= -\frac{(n-1)a_{n-1}(-2)+(n-3)a_{n-2}(-2)}{n}, \quad n\geq 2. \end{aligned} \tag{3.5.19}$$

Calculating with (3.5.18) and (3.5.19) and substituting the results into (3.5.17) yields the fundamental set of Frobenius solutions

$$\begin{aligned} y_1 &= x^{-3/2}\left(1-\frac{1}{3}x+\frac{2}{5}x^2-\frac{5}{21}x^3+\frac{7}{135}x^4+\frac{76}{1155}x^5+\cdots\right), \\ y_2 &= x^{-2}\left(1+\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{8}x^4+\frac{1}{30}x^5+\cdots\right). \end{aligned}$$

3.5.2 Special Cases With Two Term Recurrence Relations

For $n\geq 2$, the recurrence relation (3.5.12) of Theorem $(3.5.2)$ involves the three coefficients $a_n(r)$, $a_{n-1}(r)$, and $a_{n-2}(r)$. We'll now consider some special cases where (3.5.12) reduces to a two term recurrence relation; that is, a relation involving only $a_n(r)$ and $a_{n-1}(r) or only \(a_n(r)$ and $a_{n-2}(r)$. This simplification often makes it possible to obtain explicit formulas for the coefficents of Frobenius solutions.

We first consider equations of the form

$$x^2(\alpha_0+\alpha_1x)y''+x(\beta_0+\beta_1x)y'+(\gamma_0+\gamma_1x)y=0$$

with $\alpha_0\neq 0$. For this equation, $\alpha_2=\beta_2=\gamma_2=0$, so $p_2\equiv 0$ and the recurrence relations in Theorem $(3.5.2)$ simplify to

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r), \quad n\geq 1. \end{aligned} \tag{3.5.20}$$

Example $3.5.2$

Find a fundamental set of Frobenius solutions of

$$x^2(3+x)y''+5x(1+x)y'-(1-4x)y=0. \tag{3.5.21}$$

Give explicit formulas for the coefficients in the solutions.

Answer

For this equation, the polynomials defined in Theorem $(3.5.2)$ are

$$\begin{array}{rcccc} p_0(r) & = & 3r(r-1)+5r-1 & = & (3r-1)(r+1), \\ p_1(r) & = & r(r-1)+5r+4 & = & (r+2)^2, \\ p_2(r) & = & 0. & & \end{array}$$

The zeros of the indicial polynomial p_0 are $r_1 = 1/3$ and $r_2 = -1$, so $r_1 - r_2 = 4/3$. Therefore Theorem $(3.5.3)$ implies that

$$y_1 = x^{1/3}\sum_{n=0}^{\infty} a_n(1/3)x^n \quad \text{and} \quad y_2 = x^{-1}\sum_{n=0}^{\infty} a_n(-1)x^n$$

form a fundamental set of Frobenius solutions of (3.5.21). To find the coefficients in these series, we use the recurrence relations (3.5.20); thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r) \\ &= -\frac{(n+r+1)^2}{(3n+3r-1)(n+r+1)}a_{n-1}(r) \\ &= -\frac{n+r+1}{3n+3r-1}a_{n-1}(r), \quad n \geq 1. \end{aligned} \tag{3.5.22}$$

Setting $r = 1/3$ in (3.5.22) yields

$$\begin{aligned} a_0(1/3) &= 1, \\ a_n(1/3) &= -\frac{3n+4}{9n}a_{n-1}(1/3), \quad n \geq 1. \end{aligned}$$

By using the product notation introduced in Section 3.2 and proceeding as we did in the examples in that section yields

$$a_n(1/3) = \frac{(-1)^n \prod_{j=1}^{n}(3j+4)}{9^n n!}, \quad n \geq 0.$$

Therefore

$$y_1 = x^{1/3}\sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(3j+4)}{9^n n!}x^n$$

is a Frobenius solution of (3.5.21).

Setting $r = -1$ in (3.5.22) yields

$$
\begin{aligned}
a_0(-1) &= 1, \\
a_n(-1) &= -\frac{n}{3n-4}a_{n-1}(-1), \quad n \geq 1,
\end{aligned}
$$

so

$$a_n(-1) = \frac{(-1)^n n!}{\prod_{j=1}^{n}(3j-4)}.$$

Therefore

$$y_2 = x^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n n!}{\prod_{j=1}^{n}(3j-4)}x^n$$

is a Frobenius solution of (3.5.21), and $\{y_1, y_2\}$ is a fundamental set of solutions.

We now consider equations of the form

$$x^2(\alpha_0+\alpha_2x^2)y''+x(\beta_0+\beta_2x^2)y'+(\gamma_0+\gamma_2x^2)y=0 \tag{3.5.23}$$

with $\alpha_0 \neq 0$. For this equation, $\alpha_1=\beta_1=\gamma_1=0$, so $p_1\equiv 0$ and the recurrence relations in Theorem (3.5.2) simplify to

$$
\begin{aligned}
a_0(r) &= 1, \\
a_1(r) &= 0, \\
a_n(r) &= -\frac{p_2(n+r-2)}{p_0(n+r)}a_{n-2}(r), \quad n \geq 2.
\end{aligned}
$$

Since $a_1(r)=0$, the last equation implies that $a_n(r)=0$ if n is odd, so the Frobenius solutions are of the form

$$y(x,r) = x^r\sum_{m=0}^{\infty}a_{2m}(r)x^{2m},$$

where

$$
\begin{aligned}
a_0(r) &= 1, \\
a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r), \quad m \geq 1.
\end{aligned} \tag{3.5.24}
$$

Example 3.5.3

Find a fundamental set of Frobenius solutions of

$$x^2(2-x^2)y''-x(3+4x^2)y'+(2-2x^2)y=0. \tag{3.5.25}$$

Give explicit formulas for the coefficients in the solutions.

Answer

For this equation, the polynomials defined in Theorem (3.5.2) are

$$\begin{array}{rcccl} p_0(r) & = & 2r(r-1)-3r+2 & = & (r-2)(2r-1), \\ p_1(r) & = & 0 & & \\ p_2(r) & = & -[r(r-1)+4r+2] & = & -(r+1)(r+2). \end{array}$$

The zeros of the indicial polynomial p_0 are $r_1=2$ and $r_2=1/2$, so $r_1-r_2=3/2$. Therefore Theorem (3.5.3) implies that

$$y_1 = x^2\sum_{m=0}^{\infty} a_{2m}(1/3)x^{2m} \quad \text{and} \quad y_2 = x^{1/2}\sum_{m=0}^{\infty} a_{2m}(1/2)x^{2m}$$

form a fundamental set of Frobenius solutions of (3.5.25). To find the coefficients in these series, we use the recurrence relation (3.5.24); thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r) \\ &= \frac{(2m+r)(2m+r-1)}{(2m+r-2)(4m+2r-1)}a_{2m-2}(r), \quad m \geq 1. \end{aligned} \tag{3.5.26}$$

Setting $r=2$ in (3.5.26) yields

$$\begin{aligned} a_0(2) &= 1, \\ a_{2m}(2) &= \frac{(m+1)(2m+1)}{m(4m+3)}a_{2m-2}(2), \quad m \geq 1, \end{aligned}$$

so

$$a_{2m}(2) = (m+1)\prod_{j=1}^{m}\frac{2j+1}{4j+3}.$$

Therefore

$$y_1 = x^2\sum_{m=0}^{\infty}(m+1)\left(\prod_{j=1}^{m}\frac{2j+1}{4j+3}\right)x^{2m}$$

is a Frobenius solution of (3.5.25).

Setting $r=1/2$ in (3.5.26) yields

$$\begin{aligned} a_0(1/2) &= 1, \\ a_{2m}(1/2) &= \frac{(4m-1)(4m+1)}{8m(4m-3)}a_{2m-2}(1/2), \quad m \geq 1, \end{aligned}$$

so

$$a_{2m}(1/2)=\frac{1}{8^m m!}\prod_{j=1}^{m}\frac{(4j-1)(4j+1)}{4j-3}.$$

Therefore

$$y_2=x^{1/2}\sum_{m=0}^{\infty}\frac{1}{8^m m!}\left(\prod_{j=1}^{m}\frac{(4j-1)(4j+1)}{4j-3}\right)x^{2m}$$

is a Frobenius solution of (3.5.25) and $\{y_1,y_2\}$ is a fundamental set of solutions.

Thus far, we considered only the case where the indicial equation has real roots that don't differ by an integer, which allows us to apply Theorem $(3.5.3)$. However, for equations of the form (3.5.23), the sequence $\{a_{2m}(r)\}$ in (3.5.24) is defined for $r=r_2$ if r_1-r_2 isn't an even integer. It can be shown (Exercise $(3.5E.56)$) that in this case

$$y_1=x^{r_1}\sum_{m=0}^{\infty}a_{2m}(r_1)x^{2m} \quad \text{and} \quad y_2=x^{r_2}\sum_{m=0}^{\infty}a_{2m}(r_2)x^{2m}$$

form a fundamental set Frobenius solutions of (3.5.23)}.

As we said at the end of Section 3.2, if you're interested in actually using series to compute numerical approximations to solutions of a differential equation, then whether or not there's a simple closed form for the coefficients is essentially irrelevant; recursive computation is usually more efficient. Since it's also laborious, we encourage you to write short programs to implement recurrence relations on a calculator or computer, even in exercises where this is not specifically required.

In practical use of the method of Frobenius when $x_0=0$ is a regular singular point, we're interested in how well the functions

$$y_N(x,r_i)=x^{r_i}\sum_{n=0}^{N}a_n(r_i)x^n, \quad i=1,2,$$

approximate solutions to a given equation when r_i is a zero of the indicial polynomial. In dealing with the corresponding problem for the case where $x_0=0$ is an ordinary point, we used numerical integration to solve the differential equation subject to initial conditions $y(0)=a_0, \quad y'(0)=a_1$, and compared the result with values of the Taylor polynomial

$$T_N(x)=\sum_{n=0}^{N}a_nx^n.$$

We can't do that here, since in general we can't prescribe arbitrary initial values for solutions of a differential equation at a singular point. Therefore, motivated by Theorem $(3.5.2)$ (specifically, (3.5.14)), we suggest the following procedure.

Verification Procedure

Let L and $Y_n(x;r_i)$ be defined by

$$Ly = x^2(\alpha_0+\alpha_1x+\alpha_2x^2)y'' + x(\beta_0+\beta_1x+\beta_2x^2)y' + (\gamma_0+\gamma_1x+\gamma_2x^2)y$$

and

$$y_N(x;r_i) = x^{r_i}\sum_{n=0}^{N} a_n(r_i)x^n,$$

where the coefficients $\{a_n(r_i)\}_{n=0}^{N}$ are computed as in (3.5.12), Theorem $(3.5.2)$. Compute the error

$$E_N(x;r_i) = x^{-r_i}Ly_N(x;r_i)/\alpha_0 \tag{3.5.27}$$

for various values of N and various values of x in the interval $(0,\rho)$, with ρ as defined in Theorem $(3.5.2)$.

The multiplier x^{-r_i}/α_0 on the right of (3.5.27) eliminates the effects of small or large values of x^{r_i} near $x=0$, and of multiplication by an arbitrary constant. In some exercises you will be asked to estimate the maximum value of $E_N(x;r_i)$ on an interval $(0,\delta]$ by computing $E_N(x_m;r_i)$ at the M points $x_m = m\delta/M,\ m=1\,,2,\,...,\,M$, and finding the maximum of the absolute values:

$$\sigma_N(\delta) = \max\{|E_N(x_m;r_i)|,\ m=1,2,\dots,M\}. \tag{3.5.28}$$

(For simplicity, this notation ignores the dependence of the right side of the equation on i and M.)

To implement this procedure, you'll have to write a computer program to calculate $\{a_n(r_i)\}$ from the applicable recurrence relation, and to evaluate $E_N(x;r_i)$.

The next exercise set contains five exercises specifically identified by \Lex that ask you to implement the verification procedure. These particular exercises were chosen arbitrarily you can just as well formulate such laboratory problems for any of the equations in any of the Exercises $(3.5E.1)$ to $(3.5E.10)$, $(3.5E.14)$ to $(3.5E.25)$, and $(3.5E.28)$ to $(3.5E.51)$.

3.5E: Exercises

This page is a draft and is under active development.

This set contains exercises specifically identified by \Lex that ask you to implement the verification procedure. These particular exercises were chosen arbitrarily you can just as well formulate such laboratory problems for any of the equations in Exercises $(3.5E.1)$ to $(3.5E.10)$, $(3.5E.14)$ to $(3.5E.25)$, and $(3.5E.28)$ to $(3.5E.51)$.

In Exercises $(3.5E.1)$ to $(3.5E.10)$, find a fundamental set of Frobenius solutions. Compute a_0, a_1 $\ldots$, a_N for N at least 7 in each solution.

3.5E.1 Exercise 3.5E.1

$2x^2(1+x+x^2)y''+x(3+3x+5x^2)y'-y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.2 Exercise 3.5E.2

$3x^2y''+2x(1+x-2x^2)y'+(2x-8x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.3 Exercise 3.5E.3

$x^2(3+3x+x^2)y''+x(5+8x+7x^2)y'-(1-2x-9x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.4 Exercise 3.5E.4

$4x^2y''+x(7+2x+4x^2)y'-(1-4x-7x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.5 Exercise 3.5E.5

$12x^2(1+x)y''+x(11+35x+3x^2)y'-(1-10x-5x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.6 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{6}$

$x^2(5+x+10x^2)y''+x(4+3x+48x^2)y'+(x+36x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.7 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{7}$

$8x^2y''-2x(3-4x-x^2)y'+(3+6x+x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.8 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{8}$

$18x^2(1+x)y''+3x(5+11x+x^2)y'-(1-2x-5x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.9 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{9}$

$x(3+x+x^2)y''+(4+x-x^2)y'+xy=0$

Answer

Add texts here. Do not delete this text first.

3.5E.10 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{10}$

$10x^2(1+x+2x^2)y''+x(13+13x+66x^2)y'-(1+4x+10x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.11 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{11}$

\Lex

The Frobenius solutions of

$$2x^2(1+x+x^2)y''+x(9+11x+11x^2)y'+(6+10x+7x^2)y=0$$

obtained in Example $(3.5.1)$ are defined on $(0,\rho)$, where ρ is defined in Theorem $(3.5.2)$. Find ρ. Then do the following experiments for each Frobenius solution, with $M=20$ and $\delta=.5\rho$, $.7\rho$, and $.9\rho$ in the verification procedure described at the end of this section.

(a) Compute $\sigma_N(\delta)$ (see Equation $(3.5.28)$) for $N=5$, 10, 15, …, 50.

(b) Find N such that $\sigma_N(\delta)<10^{-5}$.

(c) Find N such that $\sigma_N(\delta) < 10^{-10}$.

Answer

Add texts here. Do not delete this text first.

3.5E.12 Exercise $\mathbf{3.5E.12}$

\Lex

By Theorem $(3.5.2)$, the Frobenius solutions of the equation in Exercise $(3.5E.4)$ are defined on $(0, \infty)$. Do experiments (a), (b), and (c) of Exercise $(3.5E.11)$ for each Frobenius solution, with $M = 20$ and $\delta = 1$, 2, and 3 in the verification procedure described at the end of this section.

Answer

Add texts here. Do not delete this text first.

3.5E.13 Exercise $\mathbf{3.5E.13}$

\Lex

The Frobenius solutions of the equation in Exercise $(3.5E.6)$ are defined on $(0, \rho)$, where ρ is defined in Theorem $(3.5.2)$. Find ρ and do experiments (a), (b), and (c), of Exercise $(3.5E.11)$ for each Frobenius solution, with $M = 20$ and $\delta = .3\rho$, $.4\rho$, and $.5\rho$, in the verification procedure described at the end of this section.

Answer

Add texts here. Do not delete this text first.

In Exercises \9(3.5E.14)\) to $(3.5E.25)$, find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients in each solution.

3.5E.14 Exercise $\mathbf{3.5E.14}$

$2x^2y'' + x(3+2x)y' - (1-x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.15 Exercise $\mathbf{3.5E.15}$

$x^2(3+x)y'' + x(5+4x)y' - (1-2x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.16 Exercise $\mathbf{3.5E.16}$

$2x^2y'' + x(5+x)y' - (2-3x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.17 Exercise $\mathbf{3.5E.17}$

$3x^2y'' + x(1+x)y' - y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.18 Exercise $\mathbf{3.5E.18}$

$x^2y'' - xy' + (1-2x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.19 Exercise $\mathbf{3.5E.19}$

$9x^2y'' + 9xy' - (1+3x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.20 Exercise $\mathbf{3.5E.20}$

$3x^2y'' + x(1+x)y' - (1+3x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.21 Exercise $\mathbf{3.5E.21}$

$2x^2(3+x)y'' + x(1+5x)y' + (1+x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.22 Exercise $\mathbf{3.5E.22}$

$x^2(4+x)y'' - x(1-3x)y' + y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.23 Exercise $\mathbf{3.5E.23}$

$2x^2y'' + 5xy' + (1+x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.24 Exercise $\mathbf{3.5E.24}$

$x^2(3+4x)y'' + x(5+18x)y' - (1-12x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.25 Exercise $\mathbf{3.5E.25}$

$6x^2y'' + x(10-x)y' - (2+x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.26 Exercise $\mathbf{3.5E.26}$

\Lex

By Theorem $(3.5.2)$ the Frobenius solutions of the equation in Exercise $(3.5E.17)$ are defined on $(0, \infty)$. Do experiments (a), (b), and (c) of Exercise $(3.5E.11)$ for each Frobenius solution, with $M = 20$ and $\delta = 3$, 6, 9, and 12 in the verification procedure described at the end of this section.

Answer

Add texts here. Do not delete this text first.

3.5E.27 Exercise $\mathbf{3.5E.27}$

\Lex

The Frobenius solutions of the equation in Exercise $(3.5E.22)$ are defined on $(0, \rho)$, where ρ is defined in Theorem $(3.5.2)$. Find ρ and do experiments (a), (b), and (c) of Exercise $(3.5E.11)$ for each Frobenius solution, with $M = 20$ and $\delta = .25\rho$, $.5\rho$, and $.75\rho$ in the verification procedure described at the end of this section.

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.5E.28)$ to $(3.5E.32)$, find a fundamental set of Frobenius solutions. Compute coefficients $a_0, \ldots, a_N$ for N at least 7 in each solution.

3.5E.28 Exercise $\mathbf{3.5E.28}$

$x^2(8+x)y'' + x(2+3x)y' + (1+x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.29 Exercise $\mathbf{3.5E.29}$

$x^2(3+4x)y'' + x(11+4x)y' - (3+4x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.30 Exercise $\mathbf{3.5E.30}$

$2x^2(2+3x)y'' + x(4+11x)y' - (1-x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.31 Exercise $\mathbf{3.5E.31}$

$x^2(2+x)y'' + 5x(1-x)y' - (2-8x)y$

Answer

Add texts here. Do not delete this text first.

3.5E.32 Exercise $\mathbf{3.5E.32}$

$x^2(6+x)y'' + x(11+4x)y' + (1+2x)y = 0$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.5E.33)$ to $(3.5E.46)$, find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients in each solution.

3.5E.33 Exercise $\mathbf{3.5E.33}$

$8x^2y'' + x(2+x^2)y' + y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.34 Exercise $\mathbf{3.5E.34}$

$8x^2(1-x^2)y''+2x(1-13x^2)y'+(1-9x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.35 Exercise $\mathbf{3.5E.35}$

$x^2(1+x^2)y''-2x(2-x^2)y'+4y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.36 Exercise $\mathbf{3.5E.36}$

$x(3+x^2)y''+(2-x^2)y'-8xy=0$

Answer

Add texts here. Do not delete this text first.

3.5E.37 Exercise $\mathbf{3.5E.37}$

$4x^2(1-x^2)y''+x(7-19x^2)y'-(1+14x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.38 Exercise $\mathbf{3.5E.38}$

$3x^2(2-x^2)y''+x(1-11x^2)y'+(1-5x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.39 Exercise $\mathbf{3.5E.39}$

$2x^2(2+x^2)y''-x(12-7x^2)y'+(7+3x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.40 Exercise $\mathbf{3.5E.40}$

$2x^2(2+x^2)y''+x(4+7x^2)y'-(1-3x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.41 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{41}$

$2x^2(1+2x^2)y''+5x(1+6x^2)y'-(2-40x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.42 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{42}$

$3x^2(1+x^2)y''+5x(1+x^2)y'-(1+5x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.43 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{43}$

$x(1+x^2)y''+(4+7x^2)y'+8xy=0$

Answer

Add texts here. Do not delete this text first.

3.5E.44 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{44}$

$x^2(2+x^2)y''+x(3+x^2)y'-y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.45 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{45}$

$2x^2(1+x^2)y''+x(3+8x^2)y'-(3-4x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.46 Exercise $\mathbf{3.5}\boldsymbol{E}.\mathbf{46}$

$9x^2y''+3x(3+x^2)y'-(1-5x^2)y=0$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.5E.47)$ to $(3.5E.51)$, find a fundamental set of Frobenius solutions. Compute the coefficients $a_0, \ldots, a_{2M}$ for M at least 7 in each solution.

3.5E.47 Exercise $\mathbf{3.5}\boldsymbol{E}\mathbf{.47}$

$6x^2y'' + x(1+6x^2)y' + (1+9x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.48 Exercise $\mathbf{3.5}\boldsymbol{E}\mathbf{.48}$

$x^2(8+x^2)y'' + 7x(2+x^2)y' - (2-9x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.49 Exercise $\mathbf{3.5}\boldsymbol{E}\mathbf{.49}$

$9x^2(1+x^2)y'' + 3x(3+13x^2)y' - (1-25x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.50 Exercise $\mathbf{3.5}\boldsymbol{E}\mathbf{.50}$

$4x^2(1+x^2)y'' + 4x(1+6x^2)y' - (1-25x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.51 Exercise $\mathbf{3.5}\boldsymbol{E}\mathbf{.51}$

$8x^2(1+2x^2)y'' + 2x(5+34x^2)y' - (1-30x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.5E.52 Exercise $\mathbf{3.5}\boldsymbol{E}\mathbf{.52}$

Suppose $r_1 > r_2$, $a_0 = b_0 = 1$, and the Frobenius series

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2 = x^{r_2}\sum_{n=0}^{\infty} b_n x^n$$

both converge on an interval $(0,\rho)$.

(a) Show that y_1 and y_2 are linearly independent on $(0,\rho)$.

Hint: Show that if c_1 and c_2 are constants such that $c_1y_1 + c_2y_2 \equiv 0$ on $(0,\rho)$, then

$$c_1 x^{r_1-r_2}\sum_{n=0}^{\infty} a_n x^n + c_2\sum_{n=0}^{\infty} b_n x^n = 0,\quad 0<x<\rho.$$

Then let $x \to 0+$ to conclude that $c_2 = 0$.

(b) Use the result of part (a) to complete the proof of Theorem $(3.5.3)$.

Answer

Add texts here. Do not delete this text first.

3.5E.53 Exercise $\mathbf{3.5E.53}$

The equation

$$x^2y'' + xy' + (x^2-\nu^2)y = 0 \tag{3.5E.1}$$

is Bessel's equation of order ν. (Here ν is a parameter, and this use of "order" should not be confused with its usual use as in "the order of the equation.") The solutions of $(3.5\text{E}.1)$ are Bessel functions of order ν.

(a) Assuming that ν isn't an integer, find a fundamental set of Frobenius solutions of $(3.5\text{E}.1)$.

(b) If $\nu = 1/2$, the solutions of $(3.5\text{E}.1)$ reduce to familiar elementary functions. Identify these functions.

Answer

Add texts here. Do not delete this text first.

3.5E.54 Exercise $\mathbf{3.5E.54}$

(a) Verify that

$$\frac{d}{dx}\left(|x|^r x^n\right) = (n+r)|x|^r x^{n-1} \quad \text{and} \quad \frac{d^2}{dx^2}\left(|x|^r x^n\right) = (n+r)(n+r-1)|x|^r x^{n-2}$$

if $x \neq 0$.

(b) Let

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0.$$

Show that if $x^r \sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ on $(0,\rho)$ then $|x|^r \sum_{n=0}^{\infty} a_n x^n$ is a solution on $(-\rho, 0)$ and $(0,\rho)$.

Answer

Add texts here. Do not delete this text first.

3.5E.55 Exercise $\mathbf{3.5E.55}$

(a) Deduce from Equation $(3.5.20)$ that

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{p_1(j+r-1)}{p_0(j+r)}.$$

(b) Conclude that if $p_0(r) = \alpha_0(r-r_1)(r-r_2)$ where $r_1 - r_2$ is not an integer, then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n \quad \text{and} \quad y_2 = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2)x^n$$

form a fundamental set of Frobenius solutions of

$$x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y = 0.$$

(c) Show that if p_0 satisfies the hypotheses of part (b) then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^{n}(j + r_1 - r_2)} \left(\frac{\gamma_1}{\alpha_0}\right)^n x^n$$

and

$$y_2 = x^{r_2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^{n}(j + r_2 - r_1)} \left(\frac{\gamma_1}{\alpha_0}\right)^n x^n$$

form a fundamental set of Frobenius solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_1 x)y = 0.$$

Answer

Add texts here. Do not delete this text first.

3.5E.56 Exercise $\mathbf{3.5E.56}$

Let

$$Ly = x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y = 0$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_2(r) = \alpha_2 r(r-1) + \beta_2 r + \gamma_2.$$

(a) Use Theorem $(3.5.2)$ to show that if

$$\begin{aligned} a_0(r) &= 1, \\ p_0(2m+r)a_{2m}(r) + p_2(2m+r-2)a_{2m-2}(r) &= 0, \quad m \geq 1, \end{aligned} \tag{3.5E.2}$$

then the Frobenius series $y(x,r) = x^r \sum_{m=0}^{\infty} a_{2m} x^{2m}$ satisfies $Ly(x,r) = p_0(r)x^r$.

(b) Deduce from (3.5E.2) that if $p_0(2m+r)$ is nonzero for every positive integer m then

$$a_{2m}(r) = (-1)^m \prod_{j=1}^{m} \frac{p_2(2j+r-2)}{p_0(2j+r)}.$$

(c) Conclude that if $p_0(r) = \alpha_0(r-r_1)(r-r_2)$ where $r_1 - r_2$ is not an even integer, then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{2m}(r_1)x^{2m} \quad \text{and} \quad y_2 = x^{r_2} \sum_{m=0}^{\infty} a_{2m}(r_2)x^{2m}$$

form a fundamental set of Frobenius solutions of $Ly = 0$.

(d) Show that if p_0 satisfies the hypotheses of part (c) then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m! \prod_{j=1}^{m}(2j+r_1-r_2)} \left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m}$$

and

$$y_2 = x^{r_2} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m! \prod_{j=1}^{m}(2j+r_2-r_1)} \left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m}$$

form a fundamental set of Frobenius solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_2 x^2) y = 0.$$

Answer

Add texts here. Do not delete this text first.

3.5E.57 Exercise $\mathbf{3.5E.57}$

Let

$$Ly = x^2 q_0(x) y'' + x q_1(x) y' + q_2(x) y,$$

where

$$q_0(x) = \sum_{j=0}^{\infty} \alpha_j x^j, \quad q_1(x) = \sum_{j=0}^{\infty} \beta_j x^j, \quad q_2(x) = \sum_{j=0}^{\infty} \gamma_j x^j,$$

and define

$$p_j(r) = \alpha_j r(r-1) + \beta_j r + \gamma_j, \quad j = 0, 1, \ldots.$$

Let $y = x^r \sum_{n=0}^{\infty} a_n x^n$. Show that

$$Ly = x^r \sum_{n=0}^{\infty} b_n x^n,$$

where

$$b_n = \sum_{j=0}^{n} p_j(n+r-j)a_{n-j}.$$

Answer

Add texts here. Do not delete this text first.

3.5E.58 Exercise $3.5E.58$

(a) Let L be as in Exercise $(3.5E.57)$. Show that if

$$y(x,r) = x^r \sum_{n=0}^{\infty} a_n(r)x^n$$

where

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{1}{p_0(n+r)} \sum_{j=1}^{n} p_j(n+r-j)a_{n-j}(r), \quad n \geq 1, \end{aligned}$$

then

$$Ly(x,r) = p_0(r)x^r.$$

(b) Conclude that if

$$p_0(r) = \alpha_0(r-r_1)(r-r_2)$$

where $r_1 - r_2$ isn't an integer then $y_1 = y(x,r_1)$ and $y_2 = y(x,r_2)$ are solutions of $Ly = 0$.

Answer

Add texts here. Do not delete this text first.

3.5E.59 Exercise $3.5E.59$

Let

$$Ly = x^2(\alpha_0 + \alpha_q x^q)y'' + x(\beta_0 + \beta_q x^q)y' + (\gamma_0 + \gamma_q x^q)y$$

where q is a positive integer, and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_q(r) = \alpha_q r(r-1) + \beta_q r + \gamma_q.$$

(a) Show that if

$$y(x,r) = x^r \sum_{m=0}^{\infty} a_{qm}(r)x^{qm}$$

where

$$\begin{aligned} a_0(r) &= 1, \\ a_{qm}(r) &= -\frac{p_q\left(q(m-1)+r\right)}{p_0(qm+r)} a_{q(m-1)}(r), \quad m \geq 1, \end{aligned} \tag{3.5E.3}$$

then

$$Ly(x,r) = p_0(r)x^r.$$

(b) Deduce from (3.5E.3) that

$$a_{qm}(r) = (-1)^m \prod_{j=1}^{m} \frac{p_q\left(q(j-1)+r\right)}{p_0(qj+r)}.$$

(c) Conclude that if $p_0(r) = \alpha_0(r-r_1)(r-r_2)$ where $r_1 - r_2$ is not an integer multiple of q, then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{qm}(r_1)x^{qm} \quad \text{and} \quad y_2 = x^{r_2} \sum_{m=0}^{\infty} a_{qm}(r_2)x^{qm}$$

form a fundamental set of Frobenius solutions of $Ly = 0$.

(d) Show that if p_0 satisfies the hypotheses of part (c) then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{q^m m! \prod_{j=1}^{m}(qj+r_1-r_2)} \left(\frac{\gamma_q}{\alpha_0}\right)^m x^{qm}$$

and

$$y_2 = x^{r_2} \sum_{m=0}^{\infty} \frac{(-1)^m}{q^m m! \prod_{j=1}^{m}(qj+r_2-r_1)} \left(\frac{\gamma_q}{\alpha_0}\right)^m x^{qm}$$

form a fundamental set of Frobenius solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_q x^q)y = 0.$$

Answer

Add texts here. Do not delete this text first.

3.5E.60 Exercise $\mathbf{3.5E.60}$

(a) Suppose α_0, α_1, and α_2 are real numbers with $\alpha_0 \neq 0$, and $\{a_n\}_{n=0}^{\infty}$ is defined by

$$\alpha_0 a_1 + \alpha_1 a_0 = 0$$

and

$$\alpha_0 a_n + \alpha_1 a_{n-1} + \alpha_2 a_{n-2} = 0, \quad n \geq 2.$$

Show that

$$(\alpha_0+\alpha_1x+\alpha_2x^2)\sum_{n=0}^{\infty}a_nx^n=\alpha_0a_0,$$

and infer that

$$\sum_{n=0}^{\infty}a_nx^n=\frac{\alpha_0a_0}{\alpha_0+\alpha_1x+\alpha_2x^2}.$$

(b) With α_0, α_1, and α_2 as in part (a), consider the equation

$$x^2(\alpha_0+\alpha_1x+\alpha_2x^2)y''+x(\beta_0+\beta_1x+\beta_2x^2)y'+(\gamma_0+\gamma_1x+\gamma_2x^2)y=0, \tag{3.5E.4}$$

and define

$$p_j(r)=\alpha_jr(r-1)+\beta_jr+\gamma_j, \quad j=0,1,2.$$

Suppose

$$\frac{p_1(r-1)}{p_0(r)}=\frac{\alpha_1}{\alpha_0}, \qquad \frac{p_2(r-2)}{p_0(r)}=\frac{\alpha_2}{\alpha_0},$$

and

$$p_0(r)=\alpha_0(r-r_1)(r-r_2),$$

where $r_1>r_2$. Show that

$$y_1=\frac{x^{r_1}}{\alpha_0+\alpha_1x+\alpha_2x^2} \quad \text{and} \quad y_2=\frac{x^{r_2}}{\alpha_0+\alpha_1x+\alpha_2x^2}$$

form a fundamental set of Frobenius solutions of (3.5E.4) on any interval $(0,\rho)$ on which $\alpha_0+\alpha_1x+\alpha_2x^2$ has no zeros.

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.5E.61)$ to $(3.5E.68)$, use the method suggested by Exercise $(3.5E.60)$ to find the general solution on some interval $(0,\rho)$.

3.5E.61 Exercise $\mathbf{3.5E.61}$

$2x^2(1+x)y''-x(1-3x)y'+y=0$

Answer

Add texts here. Do not delete this text first.

3.5E.62 Exercise $\mathbf{3.5E.62}$

$$6x^2(1+2x^2)y'' + x(1+50x^2)y' + (1+30x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.5E.63 Exercise $\mathbf{3.5E.63}$

$$28x^2(1-3x)y'' - 7x(5+9x)y' + 7(2+9x)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.5E.64 Exercise $\mathbf{3.5E.64}$

$$9x^2(5+x)y'' + 9x(5+3x)y' - (5-8x)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.5E.65 Exercise $\mathbf{3.5E.65}$

$$8x^2(2-x^2)y'' + 2x(10-21x^2)y' - (2+35x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.5E.66 Exercise $\mathbf{3.5E.66}$

$$4x^2(1+3x+x^2)y'' - 4x(1-3x-3x^2)y' + 3(1-x+x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.5E.67 Exercise $\mathbf{3.5E.67}$

$$3x^2(1+x)^2y'' - x(1-10x-11x^2)y' + (1+5x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.5E.68 Exercise $\mathbf{3.5E.68}$

$$4x^2(3+2x+x^2)y'' - x(3-14x-15x^2)y' + (3+7x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.6: The Method of Frobenius II

This page is a draft and is under active development.

In this section we discuss a method for finding two linearly independent Frobenius solutions of a homogeneous linear second order equation near a regular singular point in the case where the indicial equation has a repeated real root. As in the preceding section, we consider equations that can be written as

$$x^2(\alpha_0+\alpha_1x+\alpha_2x^2)y''+x(\beta_0+\beta_1x+\beta_2x^2)y'+(\gamma_0+\gamma_1x+\gamma_2x^2)y=0 \tag{3.6.1}$$

where $\alpha_0\neq 0$. We assume that the indicial equation $p_0(r)=0$ has a repeated real root r_1. In this case Theorem $(3.5.3)$ implies that (3.6.1) has one solution of the form

$$y_1=x^{r_1}\sum_{n=0}^{\infty}a_nx^n,$$

but does not provide a second solution y_2 such that $\{y_1,y_2\}$ is a fundamental set of solutions. The following extension of Theorem $(3.5.2)$ provides a way to find a second solution.

3.6.1 Theorem **3.6.1**

Let

$$Ly=x^2(\alpha_0+\alpha_1x+\alpha_2x^2)y''+x(\beta_0+\beta_1x+\beta_2x^2)y'+(\gamma_0+\gamma_1x+\gamma_2x^2)y, \tag{3.6.2}$$

where $\alpha_0\neq 0$ and define

$$\begin{aligned}p_0(r)=&\ \alpha_0r(r-1)+\beta_0r+\gamma_0,\\ p_1(r)=&\ \alpha_1r(r-1)+\beta_1r+\gamma_1,\\ p_2(r)=&\ \alpha_2r(r-1)+\beta_2r+\gamma_2.\end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n, and define

$$\begin{aligned}a_0(r)&=1,\\ a_1(r)&=-\frac{p_1(r)}{p_0(r+1)},\\ a_n(r)&=-\frac{p_1(n+r-1)a_{n-1}(r)+p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)},\quad n\geq 2.\end{aligned}$$

Then the Frobenius series

$$y(x,r)=x^r\sum_{n=0}^{\infty}a_n(r)x^n \tag{3.6.3}$$

satisfies

$$Ly(x,r)=p_0(r)x^r. \tag{3.6.4}$$

Moreover,

$$\frac{\partial y}{\partial r}(x,r)=y(x,r)\ln x+x^r\sum_{n=1}^{\infty}a_n'(r)x^n, \tag{3.6.5}$$

and

$$L\left(\frac{\partial y}{\partial r}(x,r)\right)=p_0'(r)x^r+x^r p_0(r)\ln x. \tag{3.6.6}$$

Proof

Theorem $(3.5.2)$ implies (3.6.4). Differentiating formally with respect to r in (3.6.3) yields

$$\begin{aligned}\frac{\partial y}{\partial r}(x,r) &= \frac{\partial}{\partial r}(x^r)\sum_{n=0}^{\infty}a_n(r)x^n + x^r\sum_{n=1}^{\infty}a_n'(r)x^n \\ &= x^r\ln x\sum_{n=0}^{\infty}a_n(r)x^n + x^r\sum_{n=1}^{\infty}a_n'(r)x^n \\ &= y(x,r)\ln x + x^r\sum_{n=1}^{\infty}a_n'(r)x^n,\end{aligned}$$

which proves (3.6.5).

To prove that $\partial y(x,r)/\partial r$ satisfies (3.6.6), we view y in (3.6.2) as a function $y=y(x,r)$ of two variables, where the prime indicates partial differentiation with respect to x; thus,

$$y'=y'(x,r)=\frac{\partial y}{\partial x}(x,r) \quad \text{and} \quad y''=y''(x,r)=\frac{\partial^2 y}{\partial x^2}(x,r).$$

With this notation we can use (3.6.2) to rewrite (3.6.4) as

$$x^2q_0(x)\frac{\partial^2 y}{\partial x^2}(x,r)+xq_1(x)\frac{\partial y}{\partial x}(x,r)+q_2(x)y(x,r)=p_0(r)x^r, \tag{3.6.7}$$

where

$$\begin{aligned}q_0(x) &= \alpha_0+\alpha_1x+\alpha_2x^2,\\ q_1(x) &= \beta_0+\beta_1x+\beta_2x^2,\\ q_2(x) &= \gamma_0+\gamma_1x+\gamma_2x^2.\end{aligned}$$

Differentiating both sides of (3.6.7) with respect to r yields

$$x^2q_0(x)\frac{\partial^3 y}{\partial r\partial x^2}(x,r)+xq_1(x)\frac{\partial^2 y}{\partial r\partial x}(x,r)+q_2(x)\frac{\partial y}{\partial r}(x,r)=p_0'(r)x^r+p_0(r)x^r\ln x.$$

By changing the order of differentiation in the first two terms on the left we can rewrite this as

$$x^2q_0(x)\frac{\partial^3 y}{\partial x^2\partial r}(x,r)+xq_1(x)\frac{\partial^2 y}{\partial x\partial r}(x,r)+q_2(x)\frac{\partial y}{\partial r}(x,r)=p_0'(r)x^r+p_0(r)x^r\ln x,$$

or

$$x^2q_0(x)\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial r}(x,r)\right)+xq_1(x)\frac{\partial}{\partial r}\left(\frac{\partial y}{\partial x}(x,r)\right)+q_2(x)\frac{\partial y}{\partial r}(x,r)=p_0'(r)x^r+p_0(r)x^r\ln x,$$

which is equivalent to (3.6.6).

3.6.2 Theorem 3.6.2

Let L be as in Theorem $(3.6.1)$ and suppose the indicial equation $p_0(r) = 0$ has a repeated real root r_1. Then

$$y_1(x) = y(x, r_1) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$$

and

$$y_2(x) = \frac{\partial y}{\partial r}(x, r_1) = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n \tag{3.6.8}$$

form a fundamental set of solutions of $Ly = 0$.

Proof

Since r_1 is a repeated root of $p_0(r) = 0$, the indicial polynomial can be factored as

$$p_0(r) = \alpha_0 (r - r_1)^2,$$

so

$$p_0(n + r_1) = \alpha_0 n^2,$$

which is nonzero if $n > 0$. Therefore the assumptions of Theorem $(3.6.1)$ hold with $r = r_1$, and (3.6.4) implies that $Ly_1 = p_0(r_1) x^{r_1} = 0$. Since

$$p_0'(r) = 2\alpha(r - r_1)$$

it follows that $p_0'(r_1) = 0$, so (3.6.6) implies that

$$Ly_2 = p_0'(r_1) x^{r_1} + x^{r_1} p_0(r_1) \ln x = 0.$$

This proves that y_1 and y_2 are both solutions of $Ly = 0$. We leave the proof that $\{y_1, y_2\}$ is a fundamental set as an exercise (Exercise $(3.6E.53)$).

3.6.3 Example 3.6.1

Find a fundamental set of solutions of

$$x^2(1 - 2x + x^2) y'' - x(3 + x) y' + (4 + x) y = 0. \tag{3.6.9}$$

Compute just the terms involving x^{n+r_1}, where $0 \le n \le 4$ and r_1 is the oot of the indicial equation.

Answer

For the given equation, the polynomials defined in Theorem $(3.6.1)$ are

$$\begin{array}{rclcl} p_0(r) & = & r(r-1) - 3r + 4 & = & (r-2)^2, \\ p_1(r) & = & -2r(r-1) - r + 1 & = & -(r-1)(2r+1), \\ p_2(r) & = & r(r-1). & & \end{array}$$

Since $r_1 = 2$ is a repeated root of the indicial polynomial p_0, Theorem $(3.6.2)$ implies that

$$y_1 = x^2 \sum_{n=0}^{\infty} a_n(2) x^n \quad \text{and} \quad y_2 = y_1 \ln x + x^2 \sum_{n=1}^{\infty} a_n'(2) x^n \tag{3.6.10}$$

form a fundamental set of Frobenius solutions of (3.6.9). To find the coefficients in these series, we use the recurrence formulas from Theorem (3.6.1):

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)} = -\frac{(r-1)(2r+1)}{(r-1)^2} = \frac{2r+1}{r-1}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1}(r)+p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)} \\ &= \frac{(n+r-2)\left[(2n+2r-1)a_{n-1}(r)-(n+r-3)a_{n-2}(r)\right]}{(n+r-2)^2} \\ &= \frac{(2n+2r-1)}{(n+r-2)}a_{n-1}(r) - \frac{(n+r-3)}{(n+r-2)}a_{n-2}(r), n \geq 2. \end{aligned} \tag{3.6.11}$$

Differentiating yields

$$\begin{aligned} a_1'(r) &= -\frac{3}{(r-1)^2}, \\ a_n'(r) &= \frac{2n+2r-1}{n+r-2}a_{n-1}'(r) - \frac{n+r-3}{n+r-2}a_{n-2}'(r) \\ &\quad -\frac{3}{(n+r-2)^2}a_{n-1}(r) - \frac{1}{(n+r-2)^2}a_{n-2}(r), \quad n \geq 2. \end{aligned} \tag{3.6.12}$$

Setting $r=2$ in (3.6.11) and (3.6.12) yields

$$\begin{aligned} a_0(2) &= 1, \\ a_1(2) &= 5, \\ a_n(2) &= \frac{(2n+3)}{n}a_{n-1}(2) - \frac{(n-1)}{n}a_{n-2}(2), \quad n \geq 2 \end{aligned} \tag{3.6.13}$$

and

$$\begin{aligned} a_1'(2) &= -3, \\ a_n'(2) &= \frac{2n+3}{n}a_{n-1}'(2) - \frac{n-1}{n}a_{n-2}'(2) - \frac{3}{n^2}a_{n-1}(2) - \frac{1}{n^2}a_{n-2}(2), \quad n \geq 2. \end{aligned} \tag{3.6.14}$$

Computing recursively with (3.6.13) and (3.6.14) yields

$$a_0(2)=1,\ a_1(2)=5,\ a_2(2)=17,\ a_3(2)=\frac{143}{3},\ a_4(2)=\frac{355}{3},$$

and

$$a_1'(2)=-3,\ a_2'(2)=-\frac{29}{2},\ a_3'(2)=-\frac{859}{18},\ a_4'(2)=-\frac{4693}{36}.$$

Substituting these coefficients into (3.6.10) yields

$$y_1 = x^2\left(1+5x+17x^2+\frac{143}{3}x^3+\frac{355}{3}x^4+\cdots\right)$$

and

$$y_2 = y_1 \ln x - x^3\left(3 + \frac{29}{2}x + \frac{859}{18}x^2 + \frac{4693}{36}x^3 + \cdots\right).$$

Since the recurrence formula (3.6.11) involves three terms, it's not possible to obtain a simple explicit formula for the coefficients in the Frobenius solutions of (3.6.9). However, as we saw in the preceding sections, the recurrence formula for $\{a_n(r)\}$ involves only two terms if either $\alpha_1 = \beta_1 = \gamma_1 = 0$ or $\alpha_2 = \beta_2 = \gamma_2 = 0$ in (3.6.1). In this case, it's often possible to find explicit formulas for the coefficients. The next two examples illustrate this.

3.6.4 Example **3.6.2**

Find a fundamental set of Frobenius solutions of

$$2x^2(2+x)y'' + 5x^2y' + (1+x)y = 0. \tag{3.6.15}$$

Give explicit formulas for the coefficients in the solutions.

Answer

For the given equation, the polynomials defined in Theorem $(3.6.1)$ are

$$\begin{array}{rcccl} p_0(r) & = & 4r(r-1)+1 & = & (2r-1)^2, \\ p_1(r) & = & 2r(r-1)+5r+1 & = & (r+1)(2r+1), \\ p_2(r) & = & 0. & & \end{array}$$

Since $r_1 = 1/2$ is a repeated zero of the indicial polynomial p_0, Theorem $(3.6.2)$ implies that

$$y_1 = x^{1/2}\sum_{n=0}^{\infty} a_n(1/2)x^n \tag{3.6.16}$$

and

$$y_2 = y_1 \ln x + x^{1/2}\sum_{n=1}^{\infty} a_n'(1/2)x^n \tag{3.6.17}$$

form a fundamental set of Frobenius solutions of (3.6.15). Since $p_2 \equiv 0$, the recurrence formulas in Theorem $(3.6.1)$ reduce to

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r), \\ &= -\frac{(n+r)(2n+2r-1)}{(2n+2r-1)^2}a_{n-1}(r), \\ &= -\frac{n+r}{2n+2r-1}a_{n-1}(r), \quad n \geq 0. \end{aligned}$$

We leave it to you to show that

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{j+r}{2j+2r-1}, \quad n \geq 0. \tag{3.6.18}$$

Setting $r = 1/2$ yields

$$\begin{aligned} a_n(1/2) &= (-1)^n \prod_{j=1}^{n} \frac{j+1/2}{2j} = (-1)^n \prod_{j=1}^{n} \frac{2j+1}{4j}, \\ &= \frac{(-1)^n \prod_{j=1}^{n}(2j+1)}{4^n n!}, \quad n \ge 0. \end{aligned} \tag{3.6.19}$$

Substituting this into (3.6.16) yields

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j+1)}{4^n n!} x^n.$$

To obtain y_2 in (3.6.17), we must compute $a_n'(1/2)$ for $n = 1, 2, \ldots$. We'll do this by logarithmic differentiation. From (3.6.18),

$$|a_n(r)| = \prod_{j=1}^{n} \frac{|j+r|}{|2j+2r-1|}, \quad n \ge 1.$$

Therefore

$$\ln |a_n(r)| = \sum_{j=1}^{n} \left(\ln |j+r| - \ln |2j+2r-1| \right).$$

Differentiating with respect to r yields

$$\frac{a_n'(r)}{a_n(r)} = \sum_{j=1}^{n} \left(\frac{1}{j+r} - \frac{2}{2j+2r-1} \right).$$

Therefore

$$a_n'(r) = a_n(r) \sum_{j=1}^{n} \left(\frac{1}{j+r} - \frac{2}{2j+2r-1} \right).$$

Setting $r = 1/2$ here and recalling (3.6.19) yields

$$a_n'(1/2) = \frac{(-1)^n \prod_{j=1}^{n}(2j+1)}{4^n n!} \left(\sum_{j=1}^{n} \frac{1}{j+1/2} - \sum_{j=1}^{n} \frac{1}{j} \right). \tag{3.6.20}$$

Since

$$\frac{1}{j+1/2} - \frac{1}{j} = \frac{j-j-1/2}{j(j+1/2)} = -\frac{1}{j(2j+1)},$$

(3.6.20) can be rewritten as

$$a_n'(1/2) = -\frac{(-1)^n \prod_{j=1}^{n}(2j+1)}{4^n n!} \sum_{j=1}^{n} \frac{1}{j(2j+1)}.$$

Therefore, from (3.6.17),

$$y_2 = y_1 \ln x - x^{1/2} \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j+1)}{4^n n!} \left(\sum_{j=1}^{n} \frac{1}{j(2j+1)} \right) x^n.$$

3.6.5 Example $\mathbf{3.6.3}$

Find a fundamental set of Frobenius solutions of

$$x^2(2-x^2)y''-2x(1+2x^2)y'+(2-2x^2)y=0. \tag{3.6.21}$$

Give explicit formulas for the coefficients in the solutions.

Answer

For (3.6.21), the polynomials defined in Theorem $(3.6.1)$ are

$$\begin{array}{ccccc} p_0(r) & = & 2r(r-1)-2r+2 & = & 2(r-1)^2, \\ p_1(r) & = & 0, & & \\ p_2(r) & = & -r(r-1)-4r-2 & = & -(r+1)(r+2). \end{array}$$

As in Section 3.5, since $p_1 \equiv 0$, the recurrence formulas of Theorem $(3.6.1)$ imply that $a_n(r)=0$ if n is odd, and

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r) \\ &= \frac{(2m+r-1)(2m+r)}{2(2m+r-1)^2}a_{2m-2}(r) \\ &= \frac{2m+r}{2(2m+r-1)}a_{2m-2}(r), \quad m \geq 1. \end{aligned}$$

Since $r_1=1$ is a repeated root of the indicial polynomial p_0, Theorem $(3.6.2)$ implies that

$$y_1 = x\sum_{m=0}^{\infty} a_{2m}(1)x^{2m} \tag{3.6.22}$$

and

$$y_2 = y_1 \ln x + x\sum_{m=1}^{\infty} a'_{2m}(1)x^{2m} \tag{3.6.23}$$

form a fundamental set of Frobenius solutions of (3.6.21). We leave it to you to show that

$$a_{2m}(r) = \frac{1}{2^m}\prod_{j=1}^{m}\frac{2j+r}{2j+r-1}. \tag{3.6.24}$$

Setting $r=1$ yields

$$a_{2m}(1) = \frac{1}{2^m}\prod_{j=1}^{m}\frac{2j+1}{2j} = \frac{\prod_{j=1}^{m}(2j+1)}{4^m m!}, \tag{3.6.25}$$

and substituting this into (3.6.22) yields

$$y_1 = x\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{m}(2j+1)}{4^m m!}x^{2m}.$$

To obtain y_2 in (3.6.23), we must compute $a'_{2m}(1)$ for $m = 1, 2, \ldots$. Again we use logarithmic differentiation. From (3.6.24),

$$|a_{2m}(r)| = \frac{1}{2^m} \prod_{j=1}^{m} \frac{|2j+r|}{|2j+r-1|}.$$

Taking logarithms yields

$$\ln|a_{2m}(r)| = -m\ln 2 + \sum_{j=1}^{m} \left(\ln|2j+r| - \ln|2j+r-1|\right).$$

Differentiating with respect to r yields

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = \sum_{j=1}^{m} \left(\frac{1}{2j+r} - \frac{1}{2j+r-1}\right).$$

Therefore

$$a'_{2m}(r) = a_{2m}(r) \sum_{j=1}^{m} \left(\frac{1}{2j+r} - \frac{1}{2j+r-1}\right).$$

Setting $r = 1$ and recalling (3.6.25) yields

$$a'_{2m}(1) = \frac{\prod_{j=1}^{m}(2j+1)}{4^m m!} \sum_{j=1}^{m} \left(\frac{1}{2j+1} - \frac{1}{2j}\right). \tag{3.6.26}$$

Since

$$\frac{1}{2j+1} - \frac{1}{2j} = -\frac{1}{2j(2j+1)},$$

(3.6.26) can be rewritten as

$$a'_{2m}(1) = -\frac{\prod_{j=1}^{m}(2j+1)}{2 \cdot 4^m m!} \sum_{j=1}^{m} \left(\frac{1}{(2j+1)}\right).$$

Substituting this into (3.6.23) yields

$$y_2 = y_1 \ln x - \frac{x}{2} \sum_{m=1}^{\infty} \frac{\prod_{j=1}^{m}(2j+1)}{4^m m!} \left(\sum_{j=1}^{m} \frac{1}{j(2j+1)}\right) x^{2m}.$$

If the solution $y_1 = y(x, r_1)$ of $Ly = 0$ reduces to a finite sum, then there's a difficulty in using logarithmic differentiation to obtain the coefficients $\{a'_n(r_1)\}$ in the second solution. The next example illustrates this difficulty and shows how to overcome it.

3.6.6 Example **3.6.4**

Find a fundamental set of Frobenius solutions of

$$x^2 y'' - x(5-x)y' + (9-4x)y = 0. \tag{3.6.27}$$

Give explicit formulas for the coefficients in the solutions.

Answer

For (3.6.27) the polynomials defined in Theorem $(3.6.1)$ are

$$\begin{aligned} p_0(r) &= r(r-1)-5r+9 &= (r-3)^2, \\ p_1(r) &= r-4, \\ p_2(r) &= 0. \end{aligned}$$

Since $r_1 = 3$ is a repeated zero of the indicial polynomial p_0, Theorem $(3.6.2)$ implies that

$$y_1 = x^3 \sum_{n=0}^{\infty} a_n(3)x^n \tag{3.6.28}$$

and

$$y_2 = y_1 \ln x + x^3 \sum_{n=1}^{\infty} a_n'(3)x^n \tag{3.6.29}$$

are linearly independent Frobenius solutions of (3.6.27). To find the coefficients in (3.6.28) we use the recurrence formulas

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ &= -\frac{n+r-5}{(n+r-3)^2} a_{n-1}(r), \quad n \geq 1. \end{aligned}$$

We leave it to you to show that

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{j+r-5}{(j+r-3)^2}. \tag{3.6.30}$$

Setting $r = 3$ here yields

$$a_n(3) = (-1)^n \prod_{j=1}^{n} \frac{j-2}{j^2},$$

so $a_1(3) = 1$ and $a_n(3) = 0$ if $n \geq 2$. Substituting these coefficients into (3.6.28) yields

$$y_1 = x^3(1+x).$$

To obtain y_2 in (3.6.29) we must compute $a_n'(3)$ for $n = 1, 2, \ldots$. Let's first try logarithmic differentiation. From (3.6.30),

$$|a_n(r)| = \prod_{j=1}^{n} \frac{|j+r-5|}{|j+r-3|^2}, \quad n \geq 1,$$

so

$$\ln|a_n(r)| = \sum_{j=1}^{n} (\ln|j+r-5| - 2\ln|j+r-3|).$$

Differentiating with respect to r yields

$$\frac{a_n'(r)}{a_n(r)} = \sum_{j=1}^{n} \left(\frac{1}{j+r-5} - \frac{2}{j+r-3}\right).$$

Therefore

$$a_n'(r) = a_n(r) \sum_{j=1}^{n} \left(\frac{1}{j+r-5} - \frac{2}{j+r-3}\right). \tag{3.6.31}$$

However, we can't simply set $r=3$ here if $n \geq 2$, since the bracketed expression in the sum corresponding to $j=2$ contains the term $1/(r-3)$. In fact, since $a_n(3)=0$ for $n \geq 2$, the formula (3.6.31) for $a_n'(r)$ is actually an indeterminate form at $r=3$.

We overcome this difficulty as follows. From (3.6.30) with $n=1$,

$$a_1(r) = -\frac{r-4}{(r-2)^2}.$$

Therefore

$$a_1'(r) = \frac{r-6}{(r-2)^3},$$

so

$$a_1'(3) = -3. \tag{3.6.32}$$

From (3.6.30) with $n \geq 2$,

$$a_n(r) = (-1)^n (r-4)(r-3)\frac{\prod_{j=3}^{n}(j+r-5)}{\prod_{j=1}^{n}(j+r-3)^2} = (r-3)c_n(r),$$

where

$$c_n(r) = (-1)^n (r-4)\frac{\prod_{j=3}^{n}(j+r-5)}{\prod_{j=1}^{n}(j+r-3)^2}, \quad n \geq 2.$$

Therefore

$$a_n'(r) = c_n(r) + (r-3)c_n'(r), \quad n \geq 2,$$

which implies that $a_n'(3) = c_n(3)$ if $n \geq 3$. We leave it to you to verify that

$$a_n'(3) = c_n(3) = \frac{(-1)^{n+1}}{n(n-1)n!}, \quad n \geq 2.$$

Substituting this and (3.6.32) into (3.6.29) yields

$$y_2 = x^3(1+x)\ln x - 3x^4 - x^3 \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)n!} x^n.$$

3.6E: Exercises

This page is a draft and is under active development.

In Exercises $(3.6E.1)$ to $(3.6E.11)$, find a fundamental set of Frobenius solutions. Compute the terms involving x^{n+r_1}, where $0 \le n \le N$ (N at least 7) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $N > 7$.

3.6E.1 Exercise $\mathbf{3.6E.1}$

$x^2y'' - x(1-x)y' + (1-x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.2 Exercise $\mathbf{3.6E.2}$

$x^2(1+x+2x^2)y' + x(3+6x+7x^2)y' + (1+6x-3x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.3 Exercise $\mathbf{3.6E.3}$

$x^2(1+2x+x^2)y'' + x(1+3x+4x^2)y' - x(1-2x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.4 Exercise $\mathbf{3.6E.4}$

$4x^2(1+x+x^2)y'' + 12x^2(1+x)y' + (1+3x+3x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.5 Exercise $\mathbf{3.6E.5}$

$x^2(1+x+x^2)y'' - x(1-4x-2x^2)y' + y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.6 Exercise $\mathbf{3.6E.6}$

$9x^2y'' + 3x(5+3x-2x^2)y' + (1+12x-14x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.7 Exercise $\mathbf{3.6E.7}$

$x^2y'' + x(1+x+x^2)y' + x(2-x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.8 Exercise $\mathbf{3.6E.8}$

$x^2(1+2x)y'' + x(5+14x+3x^2)y' + (4+18x+12x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.9 Exercise $\mathbf{3.6E.9}$

$4x^2y'' + 2x(4+x+x^2)y' + (1+5x+3x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.10 Exercise $\mathbf{3.6E.10}$

$16x^2y'' + 4x(6+x+2x^2)y' + (1+5x+18x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.11 Exercise $\mathbf{3.6E.11}$

$9x^2(1+x)y'' + 3x(5+11x-x^2)y' + (1+16x-7x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.6E.12)$ to $(3.6E.22)$, find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

3.6E.12 Exercise $\mathbf{3.6E.12}$

$4x^2y'' + (1+4x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.13 Exercise $\mathbf{3.6E.13}$

$36x^2(1-2x)y'' + 24x(1-9x)y' + (1-70x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.14 Exercise $\mathbf{3.6E.14}$

$x^2(1+x)y'' - x(3-x)y' + 4y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.15 Exercise $\mathbf{3.6E.15}$

$x^2(1-2x)y'' - x(5-4x)y' + (9-4x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.16 Exercise $\mathbf{3.6E.16}$

$25x^2y'' + x(15+x)y' + (1+x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.17 Exercise $\mathbf{3.6E.17}$

$2x^2(2+x)y'' + x^2y' + (1-x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.18 Exercise $\mathbf{3.6E.18}$

$x^2(9+4x)y'' + 3xy' + (1+x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.19 Exercise $\mathbf{3.6E.19}$

$x^2y'' - x(3-2x)y' + (4+3x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.20 Exercise $\mathbf{3.6E.20}$

$x^2(1-4x)y''+3x(1-6x)y'+(1-12x)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.21 Exercise $\mathbf{3.6E.21}$

$x^2(1+2x)y''+x(3+5x)y'+(1-2x)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.22 Exercise $\mathbf{3.6E.22}$

$2x^2(1+x)y''-x(6-x)y'+(8-x)y=0$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.6E.23)$ to $(3.6E.27)$, find a fundamental set of Frobenius solutions. Compute the terms involving x^{n+r_1}, where $0 \le n \le N$ (N at least 7) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $N > 7$.

3.6E.23 Exercise $\mathbf{3.6E.23}$

$x^2(1+2x)y''+x(5+9x)y'+(4+3x)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.24 Exercise $\mathbf{3.6E.24}$

$x^2(1-2x)y''-x(5+4x)y'+(9+4x)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.25 Exercise $\mathbf{3.6E.25}$

$x^2(1+4x)y''-x(1-4x)y'+(1+x)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.26 Exercise $\mathbf{3.6}\boldsymbol{E}\mathbf{.26}$

$x^2(1+x)y''+x(1+2x)y'+xy=0$

Answer

Add texts here. Do not delete this text first.

3.6E.27 Exercise $\mathbf{3.6}\boldsymbol{E}\mathbf{.27}$

$x^2(1-x)y''+x(7+x)y'+(9-x)y=0$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.6E.28)$ to $(3.6E.38)$, find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

3.6E.28 Exercise $\mathbf{3.6}\boldsymbol{E}\mathbf{.28}$

$x^2y''-x(1-x^2)y'+(1+x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.29 Exercise $\mathbf{3.6}\boldsymbol{E}\mathbf{.29}$

$x^2(1+x^2)y''-3x(1-x^2)y'+4y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.30 Exercise $\mathbf{3.6}\boldsymbol{E}\mathbf{.30}$

$4x^2y''+2x^3y'+(1+3x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.31 Exercise $\mathbf{3.6}\boldsymbol{E}\mathbf{.31}$

$x^2(1+x^2)y''-x(1-2x^2)y'+y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.32 Exercise $\textbf{3.6}\boldsymbol{E}\textbf{.32}$

$2x^2(2+x^2)y''+7x^3y'+(1+3x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.33 Exercise $\textbf{3.6}\boldsymbol{E}\textbf{.33}$

$x^2(1+x^2)y''-x(1-4x^2)y'+(1+2x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.34 Exercise $\textbf{3.6}\boldsymbol{E}\textbf{.34}$

$4x^2(4+x^2)y''+3x(8+3x^2)y'+(1-9x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.35 Exercise $\textbf{3.6}\boldsymbol{E}\textbf{.35}$

$3x^2(3+x^2)y''+x(3+11x^2)y'+(1+5x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.36 Exercise $\textbf{3.6}\boldsymbol{E}\textbf{.36}$

$4x^2(1+4x^2)y''+32x^3y'+y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.37 Exercise $\textbf{3.6}\boldsymbol{E}\textbf{.37}$

$9x^2y''-3x(7-2x^2)y'+(25+2x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.38 Exercise $\textbf{3.6}\boldsymbol{E}\textbf{.38}$

$x^2(1+2x^2)y''+x(3+7x^2)y'+(1-3x^2)y=0$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.6E.39)$ to $(3.6E.43)$, find a fundamental set of Frobenius solutions. Compute the terms involving x^{2m+r_1}, where $0 \leq m \leq M$ (M at least 3) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $M > 3$.

3.6E.39 Exercise **3.6E.39**

$x^2(1+x^2)y'' + x(3+8x^2)y' + (1+12x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.40 Exercise **3.6E.40**

$x^2y'' - x(1-x^2)y' + (1+x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.41 Exercise **3.6E.41**

$x^2(1-2x^2)y'' + x(5-9x^2)y' + (4-3x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.42 Exercise **3.6E.42**

$x^2(2+x^2)y'' + x(14-x^2)y' + 2(9+x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.43 Exercise **3.6E.43**

$x^2(1+x^2)y'' + x(3+7x^2)y' + (1+8x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.6E.44)$ to $(3.6E.52)$, find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

3.6E.44 Exercise $\mathbf{3.6E.44}$

$x^2(1-2x)y''+3xy'+(1+4x)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.45 Exercise $\mathbf{3.6E.45}$

$x(1+x)y''+(1-x)y'+y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.46 Exercise $\mathbf{3.6E.46}$

$x^2(1-x)y''+x(3-2x)y'+(1+2x)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.47 Exercise $\mathbf{3.6E.47}$

$4x^2(1+x)y''-4x^2y'+(1-5x)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.48 Exercise $\mathbf{3.6E.48}$

$x^2(1-x)y''-x(3-5x)y'+(4-5x)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.49 Exercise $\mathbf{3.6E.49}$

$x^2(1+x^2)y''-x(1+9x^2)y'+(1+25x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.50 Exercise $\mathbf{3.6E.50}$

$9x^2y''+3x(1-x^2)y'+(1+7x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.51 Exercise $\mathbf{3.6E.51}$

$x(1+x^2)y''+(1-x^2)y'-8xy=0$

Answer

Add texts here. Do not delete this text first.

3.6E.52 Exercise $\mathbf{3.6E.52}$

$4x^2y''+2x(4-x^2)y'+(1+7x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.53 Exercise $\mathbf{3.6E.53}$

Under the assumptions of Theorem $(3.6.2)$, suppose the power series

$$\sum_{n=0}^{\infty} a_n(r_1)x^n \quad \text{and} \quad \sum_{n=1}^{\infty} a_n'(r_1)x^n$$

converge on $(-\rho,\rho)$.

(a) Show that

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n \quad \text{and} \quad y_2 = y_1\ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n$$

are linearly independent on $(0,\rho)$.

Hint: Show that if c_1 and c_2 are constants such that $c_1y_1+c_2y_2\equiv 0$ on $(0,\rho)$, then

$$(c_1+c_2\ln x)\sum_{n=0}^{\infty} a_n(r_1)x^n + c_2\sum_{n=1}^{\infty} a_n'(r_1)x^n = 0, \quad 0<x<\rho.$$

Then let $x\to 0+$ to conclude that $c_2=0$.

(b) Use the result of part (a) to complete the proof of Theorem $(3.6.2)$.

Answer

Add texts here. Do not delete this text first.

3.6E.54 Exercise $\mathbf{3.6E.54}$

Let

$$Ly = x^2(\alpha_0+\alpha_1x)y''+x(\beta_0+\beta_1x)y'+(\gamma_0+\gamma_1x)y$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_1(r) = \alpha_1 r(r-1) + \beta_1 r + \gamma_1.$$

Theorem $(3.6.1)$ and Exercise $(3.5E.55)$ part (a) imply that if

$$y(x,r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n$$

where

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{p_1(j+r-1)}{p_0(j+r)},$$

then

$$Ly(x,r) = p_0(r) x^r.$$

Now suppose $p_0(r) = \alpha_0 (r-r_1)^2$ and $p_1(k+r_1) \neq 0$ if k is a nonnegative integer.

(a) Show that $Ly = 0$ has the solution

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n,$$

where

$$a_n(r_1) = \frac{(-1)^n}{\alpha_0^n (n!)^2} \prod_{j=1}^{n} p_1(j+r_1-1).$$

(b) Show that $Ly = 0$ has the second solution

$$y_2 = y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n(r_1) J_n x^n,$$

where

$$J_n = \sum_{j=1}^{n} \frac{p_1'(j+r_1-1)}{p_1(j+r_1-1)} - 2 \sum_{j=1}^{n} \frac{1}{j}.$$

(c) Conclude from part (a) and part (b) that if $\gamma_1 \neq 0$ then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\gamma_1}{\alpha_0}\right)^n x^n$$

and

$$y_2 = y_1 \ln x - 2x^{r_1} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\gamma_1}{\alpha_0}\right)^n \left(\sum_{j=1}^{n} \frac{1}{j}\right) x^n$$

are solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_1 x) y = 0.$$

(The conclusion is also valid if $\gamma_1 = 0$. Why?)

Answer

Add texts here. Do not delete this text first.

3.6E.55 Exercise $\mathbf{3.6E.55}$

Let

$$Ly = x^2(\alpha_0 + \alpha_q x^q) y'' + x(\beta_0 + \beta_q x^q) y' + (\gamma_0 + \gamma_q x^q) y$$

where q is a positive integer, and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_q(r) = \alpha_q r(r-1) + \beta_q r + \gamma_q.$$

Suppose

$$p_0(r) = \alpha_0 (r - r_1)^2 \quad \text{and} \quad p_q(r) \not\equiv 0.$$

(a) Recall from Exercise $(3.5E.59)$ that $Ly = 0$ has the solution

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{qm}(r_1) x^{qm},$$

where

$$a_{qm}(r_1) = \frac{(-1)^m}{(q^2 \alpha_0)^m (m!)^2} \prod_{j=1}^{m} p_q\left(q(j-1) + r_1\right).$$

(b) Show that $Ly = 0$ has the second solution

$$y_2 = y_1 \ln x + x^{r_1} \sum_{m=1}^{\infty} a'_{qm}(r_1) J_m x^{qm},$$

where

$$J_m = \sum_{j=1}^{m} \frac{p'_q\left(q(j-1) + r_1\right)}{p_q\left(q(j-1) + r_1\right)} - \frac{2}{q} \sum_{j=1}^{m} \frac{1}{j}.$$

(c) Conclude from part (a) and part (b) that if $\gamma_q \neq 0$ then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\gamma_q}{q^2\alpha_0}\right)^m x^{qm}$$

and

$$y_2 = y_1 \ln x - \frac{2}{q} x^{r_1} \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\gamma_q}{q^2\alpha_0}\right)^m \left(\sum_{j=1}^{m} \frac{1}{j}\right) x^{qm}$$

are solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_q x^q) y = 0.$$

Answer

Add texts here. Do not delete this text first.

3.6E.56 Exercise $\mathbf{3.6E.56}$

The equation

$$xy'' + y' + xy = 0$$

is Bessel's equation of order zero. (See Exercise $(3.5E.53)$.) Find two linearly independent Frobenius solutions of this equation.

Answer

Add texts here. Do not delete this text first.

3.6E.57 Exercise $\mathbf{3.6E.57}$

Suppose the assumptions of Exercise $(3.5E.53)$ hold, except that

$$p_0(r) = \alpha_0 (r - r_1)^2.$$

Show that

$$y_1 = \frac{x^{r_1}}{\alpha_0 + \alpha_1 x + \alpha_2 x^2} \quad \text{and} \quad y_2 = \frac{x^{r_1} \ln x}{\alpha_0 + \alpha_1 x + \alpha_2 x^2}$$

are linearly independent Frobenius solutions of

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2) y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2) y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2) y = 0$$

on any interval $(0, \rho)$ on which $\alpha_0 + \alpha_1 x + \alpha_2 x^2$ has no zeros.

Answer

Add texts here. Do not delete this text first.

In Exercises $(3.6E.58)$ to $(3.6E.65)$, use the method suggested by Exercise $(3.6E.57)$ to find the general solution on some interval $(0, \rho)$.

3.6E.58 Exercise $\mathbf{3.6E.58}$

$4x^2(1+x)y'' + 8x^2y' + (1+x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.59 Exercise $\mathbf{3.6E.59}$

$9x^2(3+x)y'' + 3x(3+7x)y' + (3+4x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.60 Exercise $\mathbf{3.6E.60}$

$x^2(2-x^2)y'' - x(2+3x^2)y' + (2-x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.61 Exercise $\mathbf{3.6E.61}$

$16x^2(1+x^2)y'' + 8x(1+9x^2)y' + (1+49x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.62 Exercise $\mathbf{3.6E.62}$

$x^2(4+3x)y'' - x(4-3x)y' + 4y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.63 Exercise $\mathbf{3.6E.63}$

$4x^2(1+3x+x^2)y'' + 8x^2(3+2x)y' + (1+3x+9x^2)y = 0$

Answer

Add texts here. Do not delete this text first.

3.6E.64 Exercise $\mathbf{3.6E.64}$

$x^2(1-x)^2y''-x(1+2x-3x^2)y'+(1+x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.65 Exercise $\mathbf{3.6E.65}$

$9x^2(1+x+x^2)y''+3x(1+7x+13x^2)y'+(1+4x+25x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.6E.66 Exercise $\mathbf{3.6E.66}$

(a) Let L and $y(x,r)$ be as in Exercises $(3.5E.57)$ and $(3.5E.58)$. Extend Theorem $(3.6.1)$ by showing that

$$L\left(\frac{\partial y}{\partial r}(x,r)\right)=p_0'(r)x^r+x^rp_0(r)\ln x.$$

(b) Show that if

$$p_0(r)=\alpha_0(r-r_1)^2$$

then

$$y_1=y(x,r_1)\quad \text{and} \quad y_2=\frac{\partial y}{\partial r}(x,r_1)$$

are solutions of $Ly=0$.

Answer

Add texts here. Do not delete this text first.

3.7: The Method of Frobenius III

This page is a draft and is under active development.

In Sections 3.5 and 3.6 we discussed methods for finding Frobenius solutions of a homogeneous linear second order equation near a regular singular point in the case where the indicial equation has a repeated root or distinct real roots that don't differ by an integer. In this section we consider the case where the indicial equation has distinct real roots that differ by an integer. We'll limit our discussion to equations that can be written as

$$x^2(\alpha_0+\alpha_1x)y''+x(\beta_0+\beta_1x)y'+(\gamma_0+\gamma_1x)y=0 \tag{3.7.1}$$

or

$$x^2(\alpha_0+\alpha_2x^2)y''+x(\beta_0+\beta_2x^2)y'+(\gamma_0+\gamma_2x^2)y=0,$$

where the roots of the indicial equation differ by a positive integer.

We begin with a theorem that provides a fundamental set of solutions of equations of the form (3.7.1).

3.7.1 Theorem $3.7.1$

Let

$$Ly=x^2(\alpha_0+\alpha_1x)y''+x(\beta_0+\beta_1x)y'+(\gamma_0+\gamma_1x)y,$$

where $\alpha_0\neq 0,$ and define

$$\begin{aligned} p_0(r)=&\ \alpha_0r(r-1)+\beta_0r+\gamma_0,\\ p_1(r)=&\ \alpha_1r(r-1)+\beta_1r+\gamma_1.\end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers $n,$ and define

$$\begin{aligned} a_0(r) &= 1,\\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r),\quad n\geq 1.\end{aligned} \tag{3.7.2}$$

Let r_1 and r_2 be the roots of the indicial equation $p_0(r)=0,$ and suppose $r_1=r_2+k,$ where k is a positive integer. Then

$$y_1=x^{r_1}\sum_{n=0}^\infty a_n(r_1)x^n$$

is a Frobenius solution of $Ly=0$. Moreover, if we define

$$\begin{aligned} a_0(r_2) &= 1,\\ a_n(r_2) &= -\frac{p_1(n+r_2-1)}{p_0(n+r_2)}a_{n-1}(r_2),\quad 1\leq n\leq k-1,\end{aligned} \tag{3.7.3}$$

and

$$C = -\frac{p_1(r_1-1)}{k\alpha_0} a_{k-1}(r_2), \tag{3.7.4}$$

then

$$y_2 = x^{r_2} \sum_{n=0}^{k-1} a_n(r_2)x^n + C\left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1)x^n\right) \tag{3.7.5}$$

is also a solution of $Ly = 0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Proof

Theorem (3.5.3) implies that $Ly_1 = 0$. We'll now show that $Ly_2 = 0$. Since L is a linear operator, this is equivalent to showing that

$$L\left(x^{r_2} \sum_{n=0}^{k-1} a_n(r_2)x^n\right) + CL\left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1)x^n\right) = 0. \tag{3.7.6}$$

To verify this, we'll show that

$$L\left(x^{r_2} \sum_{n=0}^{k-1} a_n(r_2)x^n\right) = p_1(r_1-1)a_{k-1}(r_2)x^{r_1} \tag{3.7.7}$$

and

$$L\left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1)x^n\right) = k\alpha_0 x^{r_1}. \tag{3.7.8}$$

This will imply that $Ly_2 = 0$, since substituting (3.7.7) and (3.7.8) into (3.7.6) and using (3.7.4) yields

$$\begin{aligned} Ly_2 &= [p_1(r_1-1)a_{k-1}(r_2) + Ck\alpha_0]\, x^{r_1} \\ &= [p_1(r_1-1)a_{k-1}(r_2) - p_1(r_1-1)a_{k-1}(r_2)]\, x^{r_1} = 0. \end{aligned}$$

We'll prove (3.7.8) first. From Theorem (3.6.1),

$$L\left(y(x,r)\ln x + x^r \sum_{n=1}^{\infty} a_n'(r)x^n\right) = p_0'(r)x^r + x^r p_0(r)\ln x.$$

Setting $r = r_1$ and recalling that $p_0(r_1) = 0$ and $y_1 = y(x, r_1)$ yields

$$L\left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1)x^n\right) = p_0'(r_1)x^{r_1}. \tag{3.7.9}$$

Since r_1 and r_2 are the roots of the indicial equation, the indicial polynomial can be written as

$$p_0(r)=\alpha_0(r-r_1)(r-r_2)=\alpha_0\left[r^2-(r_1+r_2)r+r_1r_2\right].$$

Differentiating this yields

$$p_0'(r)=\alpha_0(2r-r_1-r_2).$$

Therefore $p_0'(r_1)=\alpha_0(r_1-r_2)=k\alpha_0$, so (3.7.9) implies (3.7.8).

Before proving (3.7.7), we first note $a_n(r_2)$ is well defined by (3.7.3) for $1\le n\le k-1$, since $p_0(n+r_2)\ne 0$ for these values of n. However, we can't define $a_n(r_2)$ for $n\ge k$ with (3.7.3), since $p_0(k+r_2)=p_0(r_1)=0$. For convenience, we define $a_n(r_2)=0$ for $n\ge k$. Then, from Theorem (3.5.1),

$$L\left(x^{r_2}\sum_{n=0}^{k-1}a_n(r_2)x^n\right)=L\left(x^{r_2}\sum_{n=0}^{\infty}a_n(r_2)x^n\right)=x^{r_2}\sum_{n=0}^{\infty}b_nx^n, \tag{3.7.10}$$

where $b_0=p_0(r_2)=0$ and

$$b_n=p_0(n+r_2)a_n(r_2)+p_1(n+r_2-1)a_{n-1}(r_2),\quad n\ge 1.$$

If $1\le n\le k-1$, then (3.7.3) implies that $b_n=0$. If $n\ge k+1$, then $b_n=0$ because $a_{n-1}(r_2)=a_n(r_2)=0$. Therefore (3.7.10) reduces to

$$L\left(x^{r_2}\sum_{n=0}^{k-1}a_n(r_2)x^n\right)=\left[p_0(k+r_2)a_k(r_2)+p_1(k+r_2-1)a_{k-1}(r_2)\right]x^{k+r_2}.$$

Since $a_k(r_2)=0$ and $k+r_2=r_1$, this implies (3.7.7).

We leave the proof that $\{y_1,y_2\}$ is a fundamental set as an exercise (Exercise (3.7.41)).

3.7.2 Example **3.7.1**

Find a fundamental set of Frobenius solutions of

$$2x^2(2+x)y''-x(4-7x)y'-(5-3x)y=0.$$

Give explicit formulas for the coefficients in the solutions.

Answer

For the given equation, the polynomials defined in Theorem (3.7.1) are

$$\begin{array}{rcccl} p_0(r) & = & 4r(r-1)-4r-5 & = & (2r+1)(2r-5),\\ p_1(r) & = & 2r(r-1)+7r+3 & = & (r+1)(2r+3).\end{array}$$

The roots of the indicial equation are $r_1=5/2$ and $r_2=-1/2$, so $k=r_1-r_2=3$. Therefore Theorem (3.7.1) implies that

$$y_1=x^{5/2}\sum_{n=0}^{\infty}a_n(5/2)x^n \tag{3.7.11}$$

and

$$y_2 = x^{-1/2} \sum_{n=0}^{2} a_n(-1/2) + C \left(y_1 \ln x + x^{5/2} \sum_{n=1}^{\infty} a_n'(5/2) x^n \right) \tag{3.7.12}$$

(with C as in (3.7.4)) form a fundamental set of solutions of $Ly = 0$. The recurrence formula (3.7.2) is

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ &= -\frac{(n+r)(2n+2r+1)}{(2n+2r+1)(2n+2r-5)} a_{n-1}(r), \\ &= -\frac{n+r}{2n+2r-5} a_{n-1}(r),\ n \geq 1, \end{aligned} \tag{3.7.13}$$

which implies that

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{j+r}{2j+2r-5},\ n \geq 0. \tag{3.7.14}$$

Therefore

$$a_n(5/2) = \frac{(-1)^n \prod_{j=1}^{n}(2j+5)}{4^n n!}. \tag{3.7.15}$$

Substituting this into (3.7.11) yields

$$y_1 = x^{5/2} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j+5)}{4^n n!} x^n.$$

To compute the coefficients $a_0(-1/2), a_1(-1/2)$ and $a_2(-1/2)$ in y_2, we set $r = -1/2$ in (3.7.13) and apply the resulting recurrence formula for $n = 1, 2$; thus,

$$\begin{aligned} a_0(-1/2) &= 1, \\ a_n(-1/2) &= -\frac{2n-1}{4(n-3)} a_{n-1}(-1/2),\ n = 1, 2. \end{aligned}$$

The last formula yields

$$a_1(-1/2) = 1/8 \quad \text{and} \quad a_2(-1/2) = 3/32.$$

Substituting $r_1 = 5/2, r_2 = -1/2, k = 3$, and $\alpha_0 = 4$ into (3.7.4) yields $C = -15/128$. Therefore, from (3.7.12),

$$y_2 = x^{-1/2}\left(1 + \frac{1}{8}x + \frac{3}{32}x^2\right) - \frac{15}{128}\left(y_1 \ln x + x^{5/2} \sum_{n=1}^{\infty} a_n'(5/2) x^n\right). \tag{3.7.16}$$

We use logarithmic differentiation to obtain obtain $a_n'(r)$. From (3.7.14),

$$|a_n(r)| = \prod_{j=1}^{n} \frac{|j+r|}{|2j+2r-5|},\ n \geq 1.$$

Therefore

$$\ln|a_n(r)| = \sum_{j=1}^{n} \left(\ln|j+r| - \ln|2j+2r-5|\right).$$

Differentiating with respect to r yields

$$\frac{a_n'(r)}{a_n(r)} = \sum_{j=1}^{n} \left(\frac{1}{j+r} - \frac{2}{2j+2r-5}\right).$$

Therefore

$$a_n'(r) = a_n(r) \sum_{j=1}^{n} \left(\frac{1}{j+r} - \frac{2}{2j+2r-5}\right).$$

Setting $r = 5/2$ here and recalling (3.7.15) yields

$$a_n'(5/2) = \frac{(-1)^n \prod_{j=1}^{n}(2j+5)}{4^n n!} \sum_{j=1}^{n} \left(\frac{1}{j+5/2} - \frac{1}{j}\right). \tag{3.7.17}$$

Since

$$\frac{1}{j+5/2} - \frac{1}{j} = -\frac{5}{j(2j+5)},$$

we can rewrite (3.7.17) as

$$a_n'(5/2) = -5 \frac{(-1)^n \prod_{j=1}^{n}(2j+5)}{4^n n!} \left(\sum_{j=1}^{n} \frac{1}{j(2j+5)}\right).$$

Substituting this into (3.7.16) yields

$$\begin{aligned} y_2 = {} & x^{-1/2}\left(1 + \frac{1}{8}x + \frac{3}{32}x^2\right) - \frac{15}{128} y_1 \ln x \\ & + \frac{75}{128} x^{5/2} \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j+5)}{4^n n!} \left(\sum_{j=1}^{n} \frac{1}{j(2j+5)}\right) x^n. \end{aligned}$$

If $C = 0$ in (3.7.4), there's no need to compute

$$y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n$$

in the formula (3.7.5) for y_2. Therefore it's best to compute C before computing $\{a_n'(r_1)\}_{n=1}^{\infty}$. This is illustrated in the next example. (See also Exercises $(3.7E.44)$ and $(3.7E.45)$.)

3.7.3 Example $\mathbf{3.7.2}$

Find a fundamental set of Frobenius solutions of

$$x^2(1-2x)y''+x(8-9x)y'+(6-3x)y=0.$$

Give explicit formulas for the coefficients in the solutions.

Answer

For the given equation, the polynomials defined in Theorem $(3.7.1)$ are

$$\begin{array}{ccccc} p_0(r) & = & r(r-1)+8r+6 & = & (r+1)(r+6) \\ p_1(r) & = & -2r(r-1)-9r-3 & = & -(r+3)(2r+1). \end{array}$$

The roots of the indicial equation are $r_1=-1$ and $r_2=-6$, so $k=r_1-r_2=5$. Therefore Theorem $(3.7.1)$ implies that

$$y_1=x^{-1}\sum_{n=0}^{\infty}a_n(-1)x^n \tag{3.7.18}$$

and

$$y_2=x^{-6}\sum_{n=0}^{4}a_n(-6)+C\left(y_1\ln x+x^{-1}\sum_{n=1}^{\infty}a_n'(-1)x^n\right) \tag{3.7.19}$$

(with C as in (3.7.4)) form a fundamental set of solutions of $Ly=0$. The recurrence formula (3.7.2) is

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r) \\ &= \frac{(n+r+2)(2n+2r-1)}{(n+r+1)(n+r+6)}a_{n-1}(r),\ n\geq 1, \end{aligned} \tag{3.7.20}$$

which implies that

$$\begin{aligned} a_n(r) &= \prod_{j=1}^{n}\frac{(j+r+2)(2j+2r-1)}{(j+r+1)(j+r+6)} \\ &= \left(\prod_{j=1}^{n}\frac{j+r+2}{j+r+1}\right)\left(\prod_{j=1}^{n}\frac{2j+2r-1}{j+r+6}\right). \end{aligned} \tag{3.7.21}$$

Since

$$\prod_{j=1}^{n} \frac{j+r+2}{j+r+1} = \frac{(r+3)(r+4)\cdots(n+r+2)}{(r+2)(r+3)\cdots(n+r+1)} = \frac{n+r+2}{r+2}$$

because of cancellations, (3.7.21) simplifies to

$$a_n(r) = \frac{n+r+2}{r+2} \prod_{j=1}^{n} \frac{2j+2r-1}{j+r+6}.$$

Therefore

$$a_n(-1) = (n+1) \prod_{j=1}^{n} \frac{2j-3}{j+5}.$$

Substituting this into (3.7.18) yields

$$y_1 = x^{-1} \sum_{n=0}^{\infty} (n+1) \left(\prod_{j=1}^{n} \frac{2j-3}{j+5} \right) x^n.$$

To compute the coefficients $a_0(-6), \ldots, a_4(-6)$ in y_2, we set $r=-6$ in (3.7.20) and apply the resulting recurrence formula for $n=1, 2, 3, 4$; thus,

$$\begin{aligned} a_0(-6) &= 1, \\ a_n(-6) &= \frac{(n-4)(2n-13)}{n(n-5)} a_{n-1}(-6),\ n = 1,2,3,4. \end{aligned}$$

The last formula yields

$$a_1(-6) = -\frac{33}{4},\ a_2(-6) = \frac{99}{4},\ a_3(-6) = -\frac{231}{8},\ a_4(-6) = 0.$$

Since $a_4(-6) = 0$, (3.7.4) implies that the constant C in (3.7.19) is zero. Therefore (3.7.19) reduces to

$$y_2 = x^{-6} \left(1 - \frac{33}{4} x + \frac{99}{4} x^2 - \frac{231}{8} x^3 \right).$$

We now consider equations of the form

$$x^2(\alpha_0 + \alpha_2 x^2) y'' + x(\beta_0 + \beta_2 x^2) y' + (\gamma_0 + \gamma_2 x^2) y = 0,$$

where the roots of the indicial equation are real and differ by an even integer. The case where the roots are real and differ by an odd integer can be handled by the method discussed in Exercise $(3.7E.46)$.

The proof of the next theorem is similar to the proof of Theorem $(3.7.1)$ (Exercise $(3.7E.43)$).

3.7.4 Theorem $\mathbf{3.7.2}$

Let

$$Ly = x^2(\alpha_0+\alpha_2x^2)y'' + x(\beta_0+\beta_2x^2)y' + (\gamma_0+\gamma_2x^2)y,$$

where $\alpha_0 \neq 0$, and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1)+\beta_0 r+\gamma_0, \\ p_2(r) &= \alpha_2 r(r-1)+\beta_2 r+\gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(2m+r)$ is nonzero for all positive integers m, and define

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r), \quad m \geq 1. \end{aligned} \tag{3.7.22}$$

Let r_1 and r_2 be the roots of the indicial equation $p_0(r)=0$, and suppose $r_1 = r_2+2k$, where k is a positive integer. Then

$$y_1 = x^{r_1}\sum_{m=0}^{\infty} a_{2m}(r_1)x^{2m}$$

is a Frobenius solution of $Ly=0$. Moreover, if we define

$$\begin{aligned} a_0(r_2) &= 1, \\ a_{2m}(r_2) &= -\frac{p_2(2m+r_2-2)}{p_0(2m+r_2)} a_{2m-2}(r_2), \quad 1 \leq m \leq k-1 \end{aligned}$$

and

$$C = -\frac{p_2(r_1-2)}{2k\alpha_0} a_{2k-2}(r_2), \tag{3.7.23}$$

then

$$y_2 = x^{r_2}\sum_{m=0}^{k-1} a_{2m}(r_2)x^{2m} + C\left(y_1 \ln x + x^{r_1}\sum_{m=1}^{\infty} a'_{2m}(r_1)x^{2m}\right) \tag{3.7.24}$$

is also a solution of $Ly=0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

3.7.5 Example **3.7.3**

Find a fundamental set of Frobenius solutions of

$$x^2(1+x^2)y'' + x(3+10x^2)y' - (15-14x^2)y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Answer

For the given equation, the polynomials defined in Theorem $(3.7.2)$ are

$$\begin{array}{rcccl} p_0(r) & = & r(r-1)+3r-15 & = & (r-3)(r+5) \\ p_2(r) & = & r(r-1)+10r+14 & = & (r+2)(r+7). \end{array}$$

The roots of the indicial equation are $r_1=3$ and $r_2=-5$, so $k=(r_1-r_2)/2=4$. Therefore Theorem $(3.7.2)$ implies that

$$y_1=x^3\sum_{m=0}^{\infty}a_{2m}(3)x^{2m} \tag{3.7.25}$$

and

$$y_2=x^{-5}\sum_{m=0}^{3}a_{2m}(-5)x^{2m}+C\left(y_1\ln x+x^3\sum_{m=1}^{\infty}a'_{2m}(3)x^{2m}\right)$$

(with C as in (3.7.23)) form a fundamental set of solutions of $Ly=0$. The recurrence formula (3.7.22) is

$$\begin{array}{rcl} a_0(r) & = & 1, \\ a_{2m}(r) & = & -\dfrac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r) \\ & = & -\dfrac{(2m+r)(2m+r+5)}{(2m+r-3)(2m+r+5)}a_{2m-2}(r) \\ & = & -\dfrac{2m+r}{2m+r-3}a_{2m-2}(r),\ m\geq 1, \end{array} \tag{3.7.26}$$

which implies that

$$a_{2m}(r)=(-1)^m\prod_{j=1}^{m}\frac{2j+r}{2j+r-3},\ m\geq 0. \tag{3.7.27}$$

Therefore

$$a_{2m}(3)=\frac{(-1)^m\prod_{j=1}^{m}(2j+3)}{2^m m!}. \tag{3.7.28}$$

Substituting this into (3.7.25) yields

$$y_1=x^3\sum_{m=0}^{\infty}\frac{(-1)^m\prod_{j=1}^{m}(2j+3)}{2^m m!}x^{2m}.$$

To compute the coefficients $a_2(-5)$, $a_4(-5)$, and $a_6(-5)$ in y_2, we set $r=-5$ in (3.7.26) and apply the resulting recurrence formula for $m=1,2,3$; thus,

$$a_{2m}(-5) = -\frac{2m-5}{2(m-4)} a_{2m-2}(-5),\ m = 1, 2, 3.$$

This yields

$$a_2(-5) = -\frac{1}{2},\ a_4(-5) = \frac{1}{8},\ a_6(-5) = \frac{1}{16}.$$

Substituting $r_1 = 3,\ r_2 = -5,\ k = 4,$ and $\alpha_0 = 1$ into (3.7.23) yields $C = -3/16$. Therefore, from (3.7.24),

$$y_2 = x^{-5}\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6\right) - \frac{3}{16}\left(y_1 \ln x + x^3 \sum_{m=1}^{\infty} a'_{2m}(3) x^{2m}\right). \tag{3.7.29}$$

To obtain $a'_{2m}(r)$ we use logarithmic differentiation. From (3.7.27),

$$|a_{2m}(r)| = \prod_{j=1}^{m} \frac{|2j+r|}{|2j+r-3|},\ m \geq 1.$$

Therefore

$$\ln |a_{2m}(r)| = \sum_{j=1}^{n} (\ln |2j+r| - \ln |2j+r-3|).$$

Differentiating with respect to r yields

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = \sum_{j=1}^{m} \left(\frac{1}{2j+r} - \frac{1}{2j+r-3}\right).$$

Therefore

$$a'_{2m}(r) = a_{2m}(r) \sum_{j=1}^{n} \left(\frac{1}{2j+r} - \frac{1}{2j+r-3}\right).$$

Setting $r = 3$ here and recalling (3.7.28) yields

$$a'_{2m}(3) = \frac{(-1)^m \prod_{j=1}^{m}(2j+3)}{2^m m!} \sum_{j=1}^{m} \left(\frac{1}{2j+3} - \frac{1}{2j}\right). \tag{3.7.30}$$

Since

$$\frac{1}{2j+3} - \frac{1}{2j} = -\frac{3}{2j(2j+3)},$$

we can rewrite (3.7.30) as

$$a'_{2m}(3) = -\frac{3}{2} \frac{(-1)^n \prod_{j=1}^{m}(2j+3)}{2^m m!} \left(\sum_{j=1}^{n} \frac{1}{j(2j+3)}\right).$$

Substituting this into (3.7.29) yields

$$\begin{aligned} y_2 = & x^{-5}\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{16}x^6\right)-\frac{3}{16}y_1\ln x \\ & +\frac{9}{32}x^3\sum_{m=1}^{\infty}\frac{(-1)^m\prod_{j=1}^{m}(2j+3)}{2^m m!}\left(\sum_{j=1}^{m}\frac{1}{j(2j+3)}\right)x^{2m}. \end{aligned}$$

3.7.6 Example **3.7.4**

Find a fundamental set of Frobenius solutions of

$$x^2(1-2x^2)y''+x(7-13x^2)y'-14x^2y=0.$$

Give explicit formulas for the coefficients in the solutions.

Answer

For the given equation, the polynomials defined in Theorem $(3.7.2)$ are

$$\begin{aligned} p_0(r) &= r(r-1)+7r &&= r(r+6), \\ p_2(r) &= -2r(r-1)-13r-14 &&= -(r+2)(2r+7). \end{aligned}$$

The roots of the indicial equation are $r_1=0$ and $r_2=-6$, so $k=(r_1-r_2)/2=3$. Therefore Theorem $(3.7.2)$ implies that

$$y_1=\sum_{m=0}^{\infty}a_{2m}(0)x^{2m}, \tag{3.7.31}$$

and

$$y_2=x^{-6}\sum_{m=0}^{2}a_{2m}(-6)x^{2m}+C\left(y_1\ln x+\sum_{m=1}^{\infty}a'_{2m}(0)x^{2m}\right) \tag{3.7.32}$$

(with C as in (3.7.23)) form a fundamental set of solutions of $Ly=0$. The recurrence formulas (3.7.22) are

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r) \\ &= \frac{(2m+r)(4m+2r+3)}{(2m+r)(2m+r+6)}a_{2m-2}(r) \\ &= \frac{4m+2r+3}{2m+r+6}a_{2m-2}(r),\ m\geq 1, \end{aligned} \tag{3.7.33}$$

which implies that

$$a_{2m}(r) = \prod_{j=1}^{m} \frac{4j+2r+3}{2j+r+6}.$$

Setting $r = 0$ yields

$$a_{2m}(0) = 6\frac{\prod_{j=1}^{m}(4j+3)}{2^m(m+3)!}.$$

Substituting this into (3.7.31) yields

$$y_1 = 6\sum_{m=0}^{\infty} \frac{\prod_{j=1}^{m}(4j+3)}{2^m(m+3)!}x^{2m}.$$

To compute the coefficients $a_0(-6)$, $a_2(-6)$, and $a_4(-6)$ in y_2, we set $r = -6$ in (3.7.33) and apply the resulting recurrence formula for $m = 1, 2$; thus,

$$\begin{aligned} a_0(-6) &= 1, \\ a_{2m}(-6) &= \frac{4m-9}{2m}a_{2m-2}(-6),\ m = 1, 2. \end{aligned}$$

The last formula yields

$$a_2(-6) = -\frac{5}{2} \quad \text{and} \quad a_4(-6) = \frac{5}{8}.$$

Since $p_2(-2) = 0$, the constant C in (3.7.23) is zero. Therefore (3.7.32) reduces to

$$y_2 = x^{-6}\left(1 - \frac{5}{2}x^2 + \frac{5}{8}x^4\right).$$

3.7E: Exercises

This page is a draft and is under active development.

In Exercises $(3.7E.1)$ to $(3.7E.40)$, find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

3.7E.1 Exercise $\mathbf{3.7E.1}$

$x^2y'' - 3xy' + (3 + 4x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.2 Exercise $\mathbf{3.7E.2}$

$xy'' + y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.3 Exercise $\mathbf{3.7E.3}$

$4x^2(1 + x)y'' + 4x(1 + 2x)y' - (1 + 3x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.4 Exercise $\mathbf{3.7E.4}$

$xy'' + xy' + y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.5 Exercise $\mathbf{3.7E.5}$

$2x^2(2 + 3x)y'' + x(4 + 21x)y' - (1 - 9x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.6 Exercise $\mathbf{3.7E.6}$

$x^2y'' + x(2 + x)y' - (2 - 3x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.7 Exercise $\mathbf{3.7E.7}$

$4x^2y'' + 4xy' - (9-x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.8 Exercise $\mathbf{3.7E.8}$

$x^2y'' + 10xy' + (14+x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.9 Exercise $\mathbf{3.7E.9}$

$4x^2(1+x)y'' + 4x(3+8x)y' - (5-49x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.10 Exercise $\mathbf{3.7E.10}$

$x^2(1+x)y'' - x(3+10x)y' + 30xy = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.11 Exercise $\mathbf{3.7E.11}$

$x^2y'' + x(1+x)y' - 3(3+x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.12 Exercise $\mathbf{3.7E.12}$

$x^2y'' + x(1-2x)y' - (4+x)y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.13 Exercise $\mathbf{3.7E.13}$

$x(1+x)y'' - 4y' - 2y = 0$

Answer

Add texts here. Do not delete this text first.

3.7E.14 Exercise $\mathbf{3.7E.14}$

$$x^2(1+2x)y'' + x(9+13x)y' + (7+5x)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.15 Exercise $\mathbf{3.7E.15}$

$$4x^2y'' - 2x(4-x)y' - (7+5x)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.16 Exercise $\mathbf{3.7E.16}$

$$3x^2(3+x)y'' - x(15+x)y' - 20y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.17 Exercise $\mathbf{3.7E.17}$

$$x^2(1+x)y'' + x(1-10x)y' - (9-10x)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.18 Exercise $\mathbf{3.7E.18}$

$$x^2(1+x)y'' + 3x^2y' - (6-x)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.19 Exercise $\mathbf{3.7E.19}$

$$x^2(1+2x)y'' - 2x(3+14x)y' + (6+100x)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.20 Exercise $\mathbf{3.7E.20}$

$$x^2(1+x)y'' - x(6+11x)y' + (6+32x)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.21 Exercise $\mathbf{3.7E.21}$

$4x^2(1+x)y''+4x(1+4x)y'-(49+27x)y=0$

Answer

Add texts here. Do not delete this text first.

3.7E.22 Exercise $\mathbf{3.7E.22}$

$x^2(1+2x)y''-x(9+8x)y'-12xy=0$

Answer

Add texts here. Do not delete this text first.

3.7E.23 Exercise $\mathbf{3.7E.23}$

$x^2(1+x^2)y''-x(7-2x^2)y'+12y=0$

Answer

Add texts here. Do not delete this text first.

3.7E.24 Exercise $\mathbf{3.7E.24}$

$x^2y''-x(7-x^2)y'+12y=0$

Answer

Add texts here. Do not delete this text first.

3.7E.25 Exercise $\mathbf{3.7E.25}$

$xy''-5y'+xy=0$

Answer

Add texts here. Do not delete this text first.

3.7E.26 Exercise $\mathbf{3.7E.26}$

$x^2y''+x(1+2x^2)y'-(1-10x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.7E.27 Exercise $\mathbf{3.7E.27}$

$x^2y''-xy'-(3-x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.7E.28 Exercise $\mathbf{3.7}\boldsymbol{E}\mathbf{.28}$

$$4x^2y'' + 2x(8+x^2)y' + (5+3x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.29 Exercise $\mathbf{3.7}\boldsymbol{E}\mathbf{.29}$

$$x^2y'' + x(1+x^2)y' - (1-3x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.30 Exercise $\mathbf{3.7}\boldsymbol{E}\mathbf{.30}$

$$x^2y'' + x(1-2x^2)y' - 4(1+2x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.31 Exercise $\mathbf{3.7}\boldsymbol{E}\mathbf{.31}$

$$4x^2y'' + 8xy' - (35-x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.32 Exercise $\mathbf{3.7}\boldsymbol{E}\mathbf{.32}$

$$9x^2y'' - 3x(11+2x^2)y' + (13+10x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.33 Exercise $\mathbf{3.7}\boldsymbol{E}\mathbf{.33}$

$$x^2y'' + x(1-2x^2)y' - 4(1-x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.34 Exercise $\mathbf{3.7}\boldsymbol{E}\mathbf{.34}$

$$x^2y'' + x(1-3x^2)y' - 4(1-3x^2)y = 0$$

Answer

Add texts here. Do not delete this text first.

3.7E.35 Exercise $\mathbf{3.7E.35}$

$x^2(1+x^2)y''+x(5+11x^2)y'+24x^2y=0$

Answer

Add texts here. Do not delete this text first.

3.7E.36 Exercise $\mathbf{3.7E.36}$

$4x^2(1+x^2)y''+8xy'-(35-x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.7E.37 Exercise $\mathbf{3.7E.37}$

$x^2(1+x^2)y''-x(5-x^2)y'-(7+25x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.7E.38 Exercise $\mathbf{3.7E.38}$

$x^2(1+x^2)y''+x(5+2x^2)y'-21y=0$

Answer

Add texts here. Do not delete this text first.

3.7E.39 Exercise $\mathbf{3.7E.39}$

$x^2(1+2x^2)y''-x(3+x^2)y'-2x^2y=0$

Answer

Add texts here. Do not delete this text first.

3.7E.40 Exercise $\mathbf{3.7E.40}$

$4x^2(1+x^2)y''+4x(2+x^2)y'-(15+x^2)y=0$

Answer

Add texts here. Do not delete this text first.

3.7E.41 Exercise $\mathbf{3.7E.41}$

(a) Under the assumptions of Theorem $(3.7.1)$, show that

$$y_1=x^{r_1}\sum_{n=0}^{\infty}a_n(r_1)x^n$$

and

$$y_2 = x^{r_2} \sum_{n=0}^{k-1} a_n(r_2)x^n + C\left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1)x^n\right)$$

are linearly independent.

Hint: Show that if c_1 and c_2 are constants such that $c_1y_1 + c_2y_2 \equiv 0$ on an interval $(0, \rho)$, then

$$x^{-r_2}(c_1y_1(x) + c_2y_2(x)) = 0, \quad 0 < x < \rho.$$

Then let $x \to 0+$ to conclude that c_2=0.

(b) Use the result of part (a) to complete the proof of Theorem $(3.7.1)$.

Answer

Add texts here. Do not delete this text first.

3.7E.42 Exercise $\mathbf{3.7E.42}$

Find a fundamental set of Frobenius solutions of Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

in the case where ν is a positive integer.

Answer

Add texts here. Do not delete this text first.

3.7E.43 Exercise $\mathbf{3.7E.43}$

Prove Theorem $(3.7.2)$.

Answer

Add texts here. Do not delete this text first.

3.7E.44 Exercise $\mathbf{3.7E.44}$

Under the assumptions of Theorem $(3.7.1)$, show that $C = 0$ if and only if $p_1(r_2 + \ell) = 0$ for some integer ℓ in $\{0, 1, \ldots, k-1\}$.

Answer

Add texts here. Do not delete this text first.

3.7E.45 Exercise $\mathbf{3.7E.45}$

Under the assumptions of Theorem $(3.7.2)$, show that $C = 0$ if and only if $p_2(r_2 + 2\ell) = 0$ for some integer ℓ in $\{0, 1, \ldots, k-1\}$.

Answer

Add texts here. Do not delete this text first.

3.7E.46 Exercise $\mathbf{3.7E.46}$

Let

$$Ly = \alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_1 x) y$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0.$$

Show that if

$$p_0(r) = \alpha_0 (r - r_1)(r - r_2)$$

where $r_1 - r_2 = k$, a positive integer, then $Ly = 0$ has the solutions

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^{n} (j+k)} \left(\frac{\gamma_1}{\alpha_0}\right)^n x^n$$

and

$$\begin{aligned} y_2 = {} & x^{r_2} \sum_{n=0}^{k-1} \frac{(-1)^n}{n! \prod_{j=1}^{n} (j-k)} \left(\frac{\gamma_1}{\alpha_0}\right)^n x^n \\ & - \frac{1}{k!(k-1)!} \left(\frac{\gamma_1}{\alpha_0}\right)^k \left(y_1 \ln x - x^{r_1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \prod_{j=1}^{n} (j+k)} \left(\frac{\gamma_1}{\alpha_0}\right)^n \left(\sum_{j=1}^{n} \frac{2j+k}{j(j+k)} \right) x^n \right). \end{aligned}$$

Answer

Add texts here. Do not delete this text first.

3.7E.47 Exercise $\mathbf{3.7E.47}$

Let

$$Ly = \alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_2 x^2) y$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0.$$

Show that if

$$p_0(r) = \alpha_0 (r - r_1)(r - r_2)$$

where $r_1 - r_2 = 2k$, an even positive integer, then $Ly = 0$ has the solutions

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m! \prod_{j=1}^{m} (j+k)} \left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m}$$

and

$$y_2 = x^{r_2}\sum_{m=0}^{k-1}\frac{(-1)^m}{4^m m!\prod_{j=1}^m(j-k)}\left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m}$$
$$-\frac{2}{4^k k!(k-1)!}\left(\frac{\gamma_2}{\alpha_0}\right)^k\left(y_1\ln x-\frac{x^{r_1}}{2}\sum_{m=1}^{\infty}\frac{(-1)^m}{4^m m!\prod_{j=1}^m(j+k)}\left(\frac{\gamma_2}{\alpha_0}\right)^m\left(\sum_{j=1}^m\frac{2j+k}{j(j+k)}\right)x^{2m}\right).$$

Answer

Add texts here. Do not delete this text first.

3.7E.48 Exercise $\mathbf{3.7E.48}$

Let L be as in Exercises $(3.5E.57)$ and $(3.5E.58)$, and suppose the indicial polynomial of $Ly=0$ is

$$p_0(r)=\alpha_0(r-r_1)(r-r_2),$$

with $k=r_1-r_2$, where k is a positive integer. Define $a_0(r)=1$ for all r. If r is a real number such that $p_0(n+r)$
is nonzero for all positive integers n, define

$$a_n(r)=-\frac{1}{p_0(n+r)}\sum_{j=1}^n p_j(n+r-j)a_{n-j}(r),\ n\geq 1,$$

and let

$$y_1=x^{r_1}\sum_{n=0}^{\infty}a_n(r_1)x^n.$$

Define

$$a_n(r_2)=-\frac{1}{p_0(n+r_2)}\sum_{j=1}^n p_j(n+r_2-j)a_{n-j}(r_2) \text{ if } n\geq 1 \text{ and } n\neq k,$$

and let $a_k(r_2)$ be arbitrary.

(a) Conclude from Exercise $(3.6E.66)$ that

$$L\left(y_1\ln x+x^{r_1}\sum_{n=1}^{\infty}a_n'(r_1)x^n\right)=k\alpha_0x^{r_1}.$$

(b) Conclude from Exercise $(3.5E.57)$ that

$$L\left(x^{r_2}\sum_{n=0}^{\infty}a_n(r_2)x^n\right)=Ax^{r_1},$$

where

$$A=\sum_{j=1}^k p_j(r_1-j)a_{k-j}(r_2).$$

(c) Show that y_1 and

$$y_2 = x^{r_2}\sum_{n=0}^{\infty} a_n(r_2)x^n - \frac{A}{k\alpha_0}\left(y_1 \ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n\right)$$

form a fundamental set of Frobenius solutions of $Ly = 0$.

(d) Show that choosing the arbitrary quantity $a_k(r_2)$ to be nonzero merely adds a multiple of y_1 to y_2. Conclude that we may as well take $a_k(r_2) = 0$.

Answer

Add texts here. Do not delete this text first.

LibreTexts

CHAPTER OVERVIEW

4: Linear Systems of Ordinary Differential Equations (LSODE)

This page is a draft and is under active development.

Definition of Linear Systems of Ordinary Differential Equations (LSODE)

$$\begin{aligned} y_1'(t) = & \ \alpha_{11}(t)y_1(t) \ +\alpha_{12}(t)y_2 + \ldots + \alpha_{1n}(t)y_n + f_1(t) \\ y_2'(t) = & \ \alpha_{21}(t)y_1(t) \ +\alpha_{22}(t)y_2 + \ldots + \alpha_{2n}(t)y_n + f_2(t) \\ y_n'(t) = & \ \alpha_{n1}(t)y_1(t) \ +\alpha_{n2}(t)y_2 + \ldots + \alpha_{nn}(t)y_n + f_n(t), \end{aligned}$$

where $y_i'(t) := \dfrac{dy_i(t)}{dt}$ denotes the first derivative of functions $y_i(t),\ i = 1, 2, \ldots,\ n$, with respect to t.

Matrix Form of LSODE

$\lfloor_{y_n'(t)}|^{y_2'(t)}\lceil 1 \ldots \rceil = \lfloor \alpha_{n1}(t)|^{\alpha_{21}.(t)}\lceil \alpha_{11}.(t)\alpha_{n2}(t)\alpha_{22}.(t)\alpha_{12}.(t).\cdot.\cdot.\ \alpha_{nn}(t)\alpha_{2n}.(t)\alpha_{1n}.(t)\rceil \lfloor y_n(t)|^{y_2(t)}\lceil y_1.(t)\rceil + \lfloor_{f_n(t)}|^{f_2(t)}\lceil f_1.(t)\rceil$

,

or in matrix form

$$y^{\prec}(t) = A(t)\vec{y}(t) + f^{\rightarrow}(t),$$

where

$\vec{y}(t)$, (3), $A(t)$in (3)

$= |\lceil y_2'(ty_1'(t)\rceil_1, A(t)$

$= |\lceil \alpha_{21}.(t\alpha_{11}.(t)\alpha_{22}.(t\alpha_{12}.(t)\alpha_{2n}.(t\alpha_{1n}.(t)\rceil_1, f^{\rightarrow}(t) = |\lceil f_2(tf_1(t)\rceil_1 \lfloor y_n'(t)\rfloor \lfloor \alpha_{n1}(t)\alpha_{n2}(t)\alpha_{nn}(t)\rfloor \lfloor f_n(t)\rfloor$

is called coefficient matrix of (2) and $f^{\rightarrow}(t)$

Topic hierarchy

4.1: Introduction to Systems of Differential Equations
- 4.1E: Exercises

4.2: Linear Systems of Differential Equations
- 4.2E: Exercises

4.3: Basic Theory of Homogeneous Linear System
- 4.3E: Exercises

4.4: Constant Coefficient Homogeneous Systems I
- 4.4E: Exercises

4.5: Constant Coefficient Homogeneous Systems II
- 4.5E: Exercises

4.6: Constant Coefficient Homogeneous Systems III
- 4.6E: Exercises

4.7: Variation of Parameters for Nonhomogeneous Linear Systems
- 4.7E: Exercises

4.1: Introduction to Systems of Differential Equations

This page is a draft and is under active development.

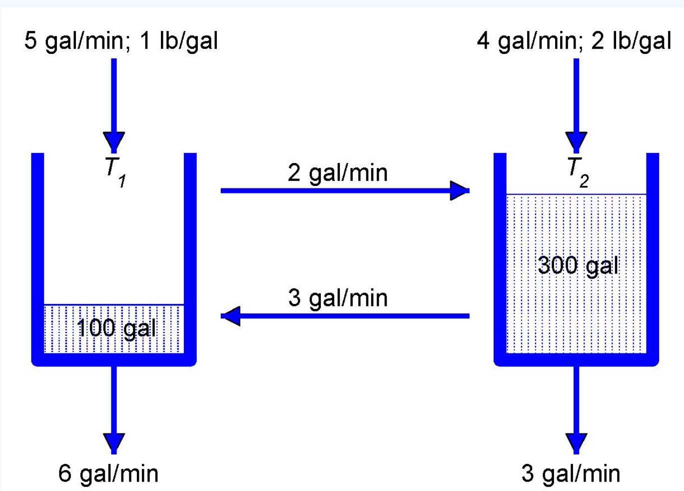
4.1.1 Introduction to Systems of Differential Equations

Many physical situations are modeled by systems of n differential equations in n unknown functions, where $n \geq 2$. The next three examples illustrate physical problems that lead to systems of differential equations. In these examples and throughout this chapter we'll denote the independent variable by t.

Example 4.1.1

Tanks T_1 and T_2 contain 100 gallons and 300 gallons of salt solutions, respectively. Salt solutions are simultaneously added to both tanks from external sources, pumped from each tank to the other, and drained from both tanks (Figure 4.1.1). A solution with 1 pound of salt per gallon is pumped into T_1 from an external source at 5 gal/min, and a solution with 2 pounds of salt per gallon is pumped into T_2 from an external source at 4 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 3 gal/min. T_1 is drained at 6 gal/min and T_2 is drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2. Assume that both mixtures are well stirred.

4.1.1.0.1 Figure **4.1.1**:



Answer

As in Section 4.2, let rate in and rate out denote the rates (lb/min) at which salt enters and leaves a tank; thus,

$$\begin{aligned} Q_1' &= (\text{rate in})_1 - (\text{rate out})_1, \\ Q_2' &= (\text{rate in})_2 - (\text{rate out})_2. \end{aligned}$$

Note that the volumes of the solutions in T_1 and T_2 remain constant at 100 gallons and 300 gallons, respectively.

T_1 receives salt from the external source at the rate of

$$(1\text{ lb/gal}) \times (5\text{ gal/min}) = 5\text{ lb/min},$$

and from T_2 at the rate of

$$\begin{aligned}(\text{lb/gal in } T_2) \times (3\text{ gal/min}) &= \frac{1}{300}Q_2 \times 3 \\ &= \frac{1}{100}Q_2\text{ lb/min.}\end{aligned}$$

Therefore

$$(\text{rate in})_1 = 5 + \frac{1}{100}Q_2. \tag{4.1.1}$$

Solution leaves T_1 at the rate of 8 gal/min, since 6 gal/min are drained and 2 gal/min are pumped to T_2; hence,

$$(\text{rate out})_1 = (\text{ lb/gal in T}_1) \times (8\text{ gal/min}) = \frac{1}{100}Q_1 \times 8 = \frac{2}{25}Q_1. \tag{4.1.2}$$

Equations (4.1.1) and (4.1.2) imply that

$$Q_1' = 5 + \frac{1}{100}Q_2 - \frac{2}{25}Q_1. \tag{4.1.3}$$

T_2 receives salt from the external source at the rate of

$$(2\text{ lb/gal}) \times (4\text{ gal/min}) = 8\text{ lb/min},$$

and from T_1 at the rate of

$$\begin{aligned}(\text{lb/gal in } T_1) \times (2\text{ gal/min}) &= \frac{1}{100}Q_1 \times 2 \\ &= \frac{1}{50}Q_1\text{ lb/min.}\end{aligned}$$

Therefore

$$(\text{rate in})_2 = 8 + \frac{1}{50}Q_1. \tag{4.1.4}$$

Solution leaves T_2 at the rate of 6 gal/min, since 3 gal/min are drained and 3 gal/min are pumped to T_1; hence,

$$(\text{rate out})_2 = (\text{ lb/gal in T}_2) \times (6\text{ gal/min}) = \frac{1}{300}Q_2 \times 6 = \frac{1}{50}Q_2. \tag{4.1.5}$$

Equations (4.1.4) and (4.1.5) imply that

$$Q_2' = 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2. \tag{4.1.6}$$

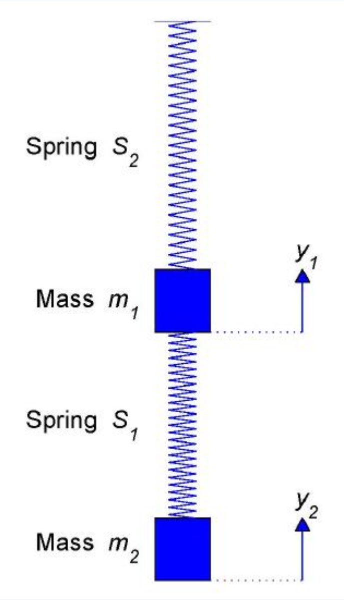
We say that (4.1.3) and (4.1.6) form a system of two first order equations in two unknowns , and write them together as

$$\begin{aligned} Q_1' &= 5 - \frac{2}{25}Q_1 + \frac{1}{100}Q_2 \\ Q_2' &= 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2. \end{aligned}$$

Example 4.1.2

A mass m_1 is suspended from a rigid support on a spring S_1 and a second mass m_2 is suspended from the first on a spring S_2 (Figure 4.1.2). The springs obey Hooke's law, with spring constants k_1 and k_2. Internal friction causes the springs to exert damping forces proportional to the rates of change of their lengths, with damping constants c_1 and c_2. Let $y_1 = y_1(t)$ and $y_2 = y_2(t)$ be the displacements of the two masses from their equilibrium positions at time t, measured positive upward. Derive a system of differential equations for y_1 and y_2, assuming that the masses of the springs are negligible and that vertical external forces F_1 and F_2 also act on the objects.

4.1.1.0.1 Figure $\mathbf{4.1.2}$:



Answer

In equilibrium, S_1 supports both m_1 and m_2 and S_2 supports only m_2. Therefore, if $\Delta\ell_1$ and $\Delta\ell_2$ are the elongations of the springs in equilibrium then

$$(m_1 + m_2)g = k_1\Delta\ell_1 \quad \text{and} \quad m_2 g = k_2\Delta\ell_2. \tag{4.1.7}$$

Let H_1 be the Hooke's law force acting on m_1, and let D_1 be the damping force on m_1. Similarly, let H_2 and D_2 be the Hooke's law and damping forces acting on m_2. According to Newton's second law of motion,

$$\begin{aligned} m_1 y_1'' &= -m_1 g + H_1 + D_1 + F_1, \\ m_2 y_2'' &= -m_2 g + H_2 + D_2 + F_2. \end{aligned} \tag{4.1.8}$$

When the displacements are y_1 and y_2, the change in length of S_1 is $-y_1 + \Delta\ell_1$ and the change in length of S_2 is $-y_2 + y_1 + \Delta\ell_2$. Both springs exert Hooke's law forces on m_1, while only S_2 exerts a

Hooke's law force on m_2. These forces are in directions that tend to restore the springs to their natural lengths. Therefore

$$H_1 = k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \quad \text{and} \quad H_2 = k_2(-y_2 + y_1 + \Delta\ell_2). \tag{4.1.9}$$

When the velocities are y_1' and y_2', S_1 and S_2 are changing length at the rates $-y_1'$ and $-y_2' + y_1'$, respectively. Both springs exert damping forces on m_1, while only S_2 exerts a damping force on m_2. Since the force due to damping exerted by a spring is proportional to the rate of change of length of the spring and in a direction that opposes the change, it follows that

$$D_1 = -c_1 y_1' + c_2(y_2' - y_1') \quad \text{and} \quad D_2 = -c_2(y_2' - y_1'). \tag{4.1.10}$$

From (4.1.8), (4.1.9), and (4.1.10),

$$\begin{aligned} m_1 y_1'' &= -m_1 g + k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \\ &\quad -c_1 y_1' + c_2(y_2' - y_1') + F_1 \\ &= -(m_1 g - k_1\Delta\ell_1 + k_2\Delta\ell_2) - k_1 y_1 + k_2(y_2 - y_1) \\ &\quad -c_1 y_1' + c_2(y_2' - y_1') + F_1 \end{aligned} \tag{4.1.11}$$

and

$$\begin{aligned} m_2 y_2'' &= -m_2 g + k_2(-y_2 + y_1 + \Delta\ell_2) - c_2(y_2' - y_1') + F_2 \\ &= -(m_2 g - k_2\Delta\ell_2) - k_2(y_2 - y_1) - c_2(y_2' - y_1') + F_2. \end{aligned} \tag{4.1.12}$$

From (4.1.7),

$$m_1 g - k_1\Delta\ell_1 + k_2\Delta\ell_2 = -m_2 g + k_2\Delta\ell_2 = 0.$$

Therefore we can rewrite (4.1.11) and (4.1.12) as

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2)y_1' + c_2 y_2' - (k_1 + k_2)y_1 + k_2 y_2 + F_1 \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + F_2. \end{aligned}$$

Example 4.1.3:

Let $\mathbf{X} = \mathbf{X}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ be the position vector at time t of an object with mass m, relative to a rectangular coordinate system with origin at Earth's center (Figure 4.1.3). According to Newton's law of gravitation, Earth's gravitational force $\mathbf{F} = \mathbf{F}(x, y, z)$ on the object is inversely proportional to the square of the distance of the object from Earth's center, and directed toward the center; thus,

$$\mathbf{F} = \frac{K}{\|\mathbf{X}\|^2}\left(-\frac{\mathbf{X}}{\|\mathbf{X}\|}\right) = -K\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} \tag{4.1.13}$$

where K is a constant. To determine K, we observe that the magnitude of $\mathbf{F}$ is

$$\|\mathbf{F}\| = K\frac{\|\mathbf{X}\|}{\|\mathbf{X}\|^3} = \frac{K}{\|\mathbf{X}\|^2} = \frac{K}{(x^2 + y^2 + z^2)}.$$

Let R be Earth's radius. Since $\|\mathbf{F}\| = mg$ when the object is at Earth's surface,

$$mg = \frac{K}{R^2}, \quad \text{so} \quad K = mgR^2.$$

Therefore we can rewrite (4.1.13) as

$$\mathbf{F} = -mgR^2 \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\left(x^2+y^2+z^2\right)^{3/2}}.$$

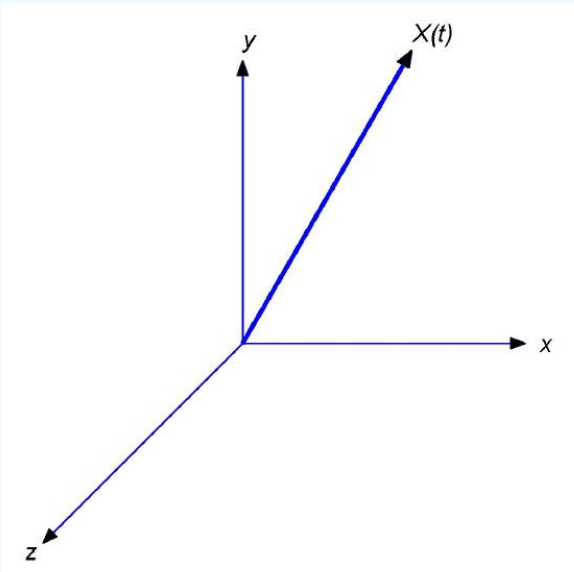
Now suppose $\mathbf{F}$ is the only force acting on the object. According to Newton's second law of motion, $\mathbf{F} = m\mathbf{X}''$; that is,

$$m(x''\,\mathbf{i} + y''\,\mathbf{j} + z''\,\mathbf{k}) = -mgR^2 \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\left(x^2+y^2+z^2\right)^{3/2}}.$$

Cancelling the common factor m and equating components on the two sides of this equation yields the system

$$\begin{aligned} x'' &= -\frac{gR^2x}{(x^2+y^2+z^2)^{3/2}} \\ y'' &= -\frac{gR^2y}{(x^2+y^2+z^2)^{3/2}} \\ z'' &= -\frac{gR^2z}{(x^2+y^2+z^2)^{3/2}}. \end{aligned} \tag{4.1.14}$$

4.1.1.0.1 Figure $\mathbf{4.1.3}$:



4.1.2 Rewriting Higher Order Systems as First Order Systems

A system of the form

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2, \ldots, y_n) \\ y_2' &= g_2(t, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= g_n(t, y_1, y_2, \ldots, y_n) \end{aligned} \tag{4.1.15}$$

is called a first order system, since the only derivatives occurring in it are first derivatives. The derivative of each of the unknowns may depend upon the independent variable and all the unknowns, but not on the derivatives of other unknowns. When we wish to emphasize the number of unknown functions in (4.1.15) we will say that (4.1.15) is an $n \times n$ system.

Systems involving higher order derivatives can often be reformulated as first order systems by introducing additional unknowns. The next two examples illustrate this.

Example 4.1.4

Rewrite the system

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2) y_1' + c_2 y_2' - (k_1 + k_2) y_1 + k_2 y_2 + F_1 \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + F_2. \end{aligned} \tag{4.1.16}$$

derived in Example $(4.1.2)$ as a system of first order equations.

Answer

If we define $v_1 = y_1'$ and $v_2 = y_2'$, then $v_1' = y_1''$ and $v_2' = y_2''$, so (4.1.16) becomes

$$\begin{aligned} m_1 v_1' &= -(c_1 + c_2) v_1 + c_2 v_2 - (k_1 + k_2) y_1 + k_2 y_2 + F_1 \\ m_2 v_2' &= c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + F_2. \end{aligned} \tag{4.1.17}$$

Therefore $\{y_1, y_2, v_1, v_2\}$ satisfies the 4×4 first order system

$$\begin{aligned} y_1' &= v_1 \\ y_2' &= v_2 \\ v_1' &= \frac{1}{m_1}\left[-(c_1 + c_2) v_1 + c_2 v_2 - (k_1 + k_2) y_1 + k_2 y_2 + F_1\right] \\ v_2' &= \frac{1}{m_2}\left[c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + F_2\right]. \end{aligned} \tag{4.1.18}$$

The difference in form between(4.1.15) and (4.1.18), due to the way in which the unknowns are denoted in the two systems, isn't important; (4.1.18) is a first order system, in that each equation in (4.1.18) expresses the first derivative of one of the unknown functions in a way that does not involve derivatives of any of the other unknowns.

Example 4.1.5

Rewrite the system

$$\begin{aligned} x'' &= f(t, x, x', y, y', y'') \\ y''' &= g(t, x, x', y, y' y'') \end{aligned} \tag{4.1.19}$$

as a first order system.

Answer

We regard x, x', y, y', and y'' as unknown functions, and rename them

$$x = x_1,\ x' = x_2, \quad y = y_1, \quad y' = y_2, \quad y'' + y_3.$$

These unknowns satisfy the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= f(t, x_1, x_2, y_1, y_2, y_3) \\ y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= g(t, x_1, x_2, y_1, y_2, y_3). \end{aligned} \tag{4.1.20}$$

4.1.3 Rewriting Scalar Differential Equations as Systems

In this chapter we'll refer to differential equations involving only one unknown function as scalar differential equations. Scalar differential equations can be rewritten as systems of first order equations by the method illustrated in the next two examples.

Example 4.1.6

Rewrite the equation

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0 \tag{4.1.21}$$

as a 4×4 first order system.

Answer

We regard y, y', y'', and y''' as unknowns and rename them

\begin{eqnarray*}
y=y_1,\quad y'=y_2,\quad y''=y_3, \quad \mbox{and}\quad y'''=y_4.
\end{eqnarray*}

Then $y^{(4)} = y_4'$, so (4.1.21) can be written as

\begin{eqnarray*}
y_4'+4y_4+6y_3+4y_2+y_1=0.
\end{eqnarray*}

Therefore $\{y_1, y_2, y_3, y_4\}$ satisfies the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= -4y_4 - 6y_3 - 4y_2 - y_1. \end{aligned}$$

Example 4.1.7

Rewrite

$$x''' = f(t, x, x', x'')$$

as a system of first order equations.

Answer

We regard x, x', and x'' as unknowns and rename them

$$x = y_1, \quad x' = y_2, \quad \text{and} \quad x'' = y_3.$$

Then

$$y_1' = x' = y_2, \quad y_2' = x'' = y_3, \quad \text{and} \quad y_3' = x'''.$$

Therefore $\{y_1, y_2, y_3\}$ satisfies the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= f(t, y_1, y_2, y_3). \end{aligned} \tag{4.1.22}$$

Since systems of differential equations involving higher derivatives can be rewritten as first order systems by the method used in
Examples $(4.1.5)$ to $(4.1.7)$, we'll consider only first order systems.

4.1.4 Numerical Solution of Systems

The numerical methods that we studied in Chapter 3 can be extended to systems, and most differential equation software packages include programs to solve systems of equations. We won't go into detail on numerical methods for systems; however, for illustrative purposes we'll describe the Runge-Kutta method for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20} \end{aligned}$$

at equally spaced points t_0, t_1, …, $t_n = b$ in an interval $[t_0, b]$. Thus,

$$t_i = t_0 + ih, \quad i = 0, 1, \dots, n,$$

where

$$h = \frac{b - t_0}{n}.$$

We'll denote the approximate values of y_1 and y_2 at these points by $y_{10}, y_{11}, \dots, y_{1n}$ and $y_{20}, y_{21}, \dots, y_{2n}$. The Runge-Kutta method computes these approximate values as follows: given y_{1i} and y_{2i}, compute

$$
\begin{aligned}
I_{1i} &= g_1(t_i, y_{1i}, y_{2i}), \\
J_{1i} &= g_2(t_i, y_{1i}, y_{2i}), \\
I_{2i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{1i}, y_{2i} + \frac{h}{2} J_{1i}\right), \\
J_{2i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{1i}, y_{2i} + \frac{h}{2} J_{1i}\right), \\
I_{3i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{2i}, y_{2i} + \frac{h}{2} J_{2i}\right), \\
J_{3i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{2i}, y_{2i} + \frac{h}{2} J_{2i}\right), \\
I_{4i} &= g_1(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}), \\
J_{4i} &= g_2(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}),
\end{aligned}
$$

and

$$
\begin{aligned}
y_{1,i+1} &= y_{1i} + \frac{h}{6}(I_{1i} + 2I_{2i} + 2I_{3i} + I_{4i}), \\
y_{2,i+1} &= y_{2i} + \frac{h}{6}(J_{1i} + 2J_{2i} + 2J_{3i} + J_{4i})
\end{aligned}
$$

for $i = 0, \ldots, n-1$. Under appropriate conditions on g_1 and g_2, it can be shown that the global truncation error for the Runge-Kutta method is $O(h^4)$, as in the scalar case considered in Section 3.3.

4.1E: Exercises

This page is a draft and is under active development.

4.1E.1 Exercise $4.1E.1$

Tanks T_1 and T_2 contain 50 gallons and 100 gallons of salt solutions, respectively. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 1 gal/min, and a solution with 3 pounds of salt per gallon is pumped into T_2 from an external source at 2 gal/min. The solution from T_1 is pumped into T_2 at 3 gal/min, and the solution from T_2 is pumped into T_1 at 4 gal/min. T_1 is drained at 2 gal/min and T_2 is drained at 1 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t>0$. Derive a system of differential equations for Q_1 and Q_2. Assume that both mixtures are well stirred.

Answer

Add texts here. Do not delete this text first.

4.1E.2 Exercise $4.1E.2$

Two 500 gallon tanks T_1 and T_2 initially contain 100 gallons each of salt solution. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 6 gal/min, and a solution with 1 pound of salt per gallon is pumped into T_2 from an external source at 5 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 1 gal/min. Both tanks are drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t>0$. Derive a system of differential equations for Q_1 and Q_2 that's valid until a tank is about to overflow. Assume that both mixtures are well stirred.

Answer

Add texts here. Do not delete this text first.

4.1E.3 Exercise $4.1E.3$

A mass m_1 is suspended from a rigid support on a spring S_1 with spring constant k_1 and damping constant c_1. A second mass m_2 is suspended from the first on a spring S_2 with spring constant k_2 and damping constant c_2, and a third mass m_3 is suspended from the second on a spring S_3 with spring constant k_3 and damping constant c_3. Let $y_1=y_1(t)$, $y_2=y_2(t)$, and $y_3=y_3(t)$ be the displacements of the three masses from their equilibrium positions at time t, measured positive upward. Derive a system of differential equations for y_1, y_2 and y_3, assuming that the masses of the springs are negligible and that vertical external forces F_1, F_2, and F_3 also act on the masses.

Answer

Add texts here. Do not delete this text first.

4.1E.4 Exercise $\mathbf{4.1} \boldsymbol{E}.\mathbf{4}$

Let $\mathbf{X} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector of an object with mass m, expressed in terms of a rectangular coordinate system with origin at Earth's center (Figure $(4.1.3)$). Derive a system of differential equations for x, y, and z, assuming that the object moves under Earth's gravitational force (given by Newton's law of gravitation, as in Example $(4.1.3)$) and a resistive force proportional to the speed of the object. Let α be the constant of proportionality.

Answer

Add texts here. Do not delete this text first.

4.1E.5 Exercise $\mathbf{4.1} \boldsymbol{E}.\mathbf{5}$

Rewrite the given system as a first order system.

(a)

$$\begin{aligned} x''' &= f(t, x, y, y') \\ y'' &= g(t, y, y') \end{aligned} \tag{4.1E.1}$$

(b)

$$\begin{aligned} u' &= f(t, u, v, v', w') \\ v'' &= g(t, u, v, v', w) \\ w'' &= h(t, u, v, v', w, w') \end{aligned} \tag{4.1E.2}$$

(c) $y''' = f(t, y, y', y'')$

(d) $y^{(4)} = f(t, y)$

(e)

$$\begin{aligned} x'' &= f(t, x, y) \\ y'' &= g(t, x, y) \end{aligned} \tag{4.1E.3}$$

Answer

Add texts here. Do not delete this text first.

4.1E.6 Exercise $\mathbf{4.1} \boldsymbol{E}.\mathbf{6}$

Rewrite the system Equation $(4.1.14)$ of differential equations derived in Example $(4.1.3)$ as a first order system.

Answer

Add texts here. Do not delete this text first.

4.1E.7 Exercise $\mathbf{4.1} \boldsymbol{E}.\mathbf{7}$

Formulate a version of Euler's method (Section 3.1) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20}, \end{aligned} \tag{4.1E.4}$$

on an interval $[t_0, b]$.

Answer

Add texts here. Do not delete this text first.

4.1E.8 Exercise $4.1E.8$

Formulate a version of the improved Euler method (Section 3.2) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20}, \end{aligned} \tag{4.1E.5}$$

on an interval $[t_0, b]$.

Answer

Add texts here. Do not delete this text first.

4.2: Linear Systems of Differential Equations

This page is a draft and is under active development.

A first order system of differential equations that can be written in the form

$$\begin{array}{ccc} y_1' & = & a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t) \\ y_2' & = & a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t) \\ & \vdots & \\ y_n' & = & a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_n(t) \end{array} \tag{4.2.1}$$

is called a linear system.

The linear system (4.2.1) can be written in matrix form as

$$y'_n = A_{nn}y_n + f_n,$$

or more briefly as

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \tag{4.2.2}$$

where

$$\mathbf{y} = y_n, \quad A(t) = A_{nn}, \quad \text{and} \quad \mathbf{f}(t) = f_n.$$

We call A the coefficient matrix of (4.2.2) and $\mathbf{f}$ the forcing function. We'll say that A and $\mathbf{f}$ are continuous if their entries are continuous. If $\mathbf{f} = \mathbf{0}$, then (4.2.2) is homogeneous; otherwise, (4.2.2) is nonhomogeneous.

An initial value problem for (4.2.2) consists of finding a solution of (4.2.2) that equals a given constant vector

$$\mathbf{k} = k_n.$$

at some initial point t_0. We write this initial value problem as

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) - \mathbf{k}.$$

The next theorem gives sufficient conditions for the existence of solutions of initial value problems for (4.2.2). We omit the proof.

4.2.1 Theorem **4.2.1**

Suppose the coefficient matrix A and the forcing function $\mathbf{f}$ are continuous on (a,b), let t_0 be in (a,b), and let $\mathbf{k}$ be an arbitrary constant n-vector. Then the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}$$

has a unique solution on (a,b).

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

4.2.2 Example 4.2.1

(a) Write the system

$$\begin{aligned} y_1' &= \ \ y_1 + 2y_2 + 2e^{4t} \\ y_2' &= 2y_1 + \ \ y_2 + \ \ e^{4t} \end{aligned} \tag{4.2.3}$$

in matrix form and conclude from Theorem $(4.2.1)$ that every initial value problem for (4.2.3) has a unique solution on $(-\infty, \infty)$.

(b) Verify that

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8 \\ 7e^{4t} \end{bmatrix} + c_1 \begin{bmatrix} 11 \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1e^{-t} \end{bmatrix} \tag{4.2.4}$$

is a solution of (4.2.3) for all values of the constants c_1 and c_2.

(c) Find the solution of the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 21 \\ e^{4t} \end{bmatrix}, \quad \mathbf{y}(0) = \frac{1}{5}\begin{bmatrix} 3 \\ 22 \end{bmatrix}. \tag{4.2.5}$$

Answer

(a) The system (4.2.3) can be written in matrix form as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 21 \\ e^{4t} \end{bmatrix}.$$

An initial value problem for (4.2.3) can be written as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 21 \\ e^{4t} \end{bmatrix}, \quad y(t_0) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

Since the coefficient matrix and the forcing function are both continuous on $(-\infty, \infty)$, Theorem $(4.2.1)$ implies that this problem has a unique solution on $(-\infty, \infty)$.

(b) If $\mathbf{y}$ is given by (4.2.4), then

$$\begin{aligned} A\mathbf{y} + \mathbf{f} &= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 7e^{4t} \end{bmatrix} + c_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 11 \\ e^{3t} \end{bmatrix} \\ &\quad + c_2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1e^{-t} \end{bmatrix} + \begin{bmatrix} 2 \\ 1e^{4t} \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 22 \\ 23e^{4t} \end{bmatrix} + c_1 \begin{bmatrix} 3 \\ 3e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1e^{-t} \end{bmatrix} + \begin{bmatrix} 2 \\ 1e^{4t} \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 32 \\ 28e^{4t} \end{bmatrix} + 3c_1 \begin{bmatrix} 11 \\ e^{3t} \end{bmatrix} - c_2 \begin{bmatrix} 1 \\ -1e^{-t} \end{bmatrix} = \mathbf{y}'. \end{aligned}$$

(c) We must choose c_1 and c_2 in (4.2.4) so that

$$\frac{1}{5}\begin{bmatrix}8\\7\end{bmatrix}+c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}=\frac{1}{5}\begin{bmatrix}3\\22\end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}-1\\3\end{bmatrix}.$$

Solving this system yields $c_1=1$, $c_2=-2$, so

$$\mathbf{y}=\frac{1}{5}\begin{bmatrix}8\\7e^{4t}\end{bmatrix}+\begin{bmatrix}11\\e^{3t}-2\end{bmatrix}\begin{bmatrix}1\\-1e^{-t}\end{bmatrix}$$

is the solution of (4.2.5).

The theory of $n\times n$ linear systems of differential equations is analogous to the theory of the scalar nth order equation

$$P_0(t)y^{(n)}+P_1(t)y^{(n-1)}+\cdots+P_n(t)y=F(t), \tag{4.2.6}$$

as developed in Sections 3.1. For example, by rewriting (4.2.6) as an equivalent linear system it can be shown that Theorem $(4.2.1)$ implies Theorem $(3.1.1)$ (Exercise $(4.2E.12)$).

4.2E: Exercises

This page is a draft and is under active development.

4.2E.1 Exercise $\textbf{4.2}\boldsymbol{E}\textbf{.1}$

Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 and c_2.

(a) $\begin{array}{ccc} y_1' & = & 2y_1+4y_2 \\ y_2' & = & 4y_1+2y_2; \end{array}$ $\quad \mathbf{y}=c_1 11 e^{6t}+c_2 1{-}1 e^{-2t}$

(b) $\begin{array}{ccc} y_1' & = & -2y_1-2y_2 \\ y_2' & = & -5y_1+\ \ y_2; \end{array}$ $\quad \mathbf{y}=c_1 11 e^{-4t}+c_2 {-}25 e^{3t}$

(c) $\begin{array}{ccc} y_1' & = & -4y_1-10y_2 \\ y_2' & = & 3y_1+\ \ 7y_2; \end{array}$ $\quad \mathbf{y}=c_1 {-}53 e^{2t}+c_2 2{-}1 e^{t}$

(d) $\begin{array}{ccc} y_1' & = & 2y_1+\ \ y_2 \\ y_2' & = & y_1+2y_2; \end{array}$ $\quad \mathbf{y}=c_1 11 e^{3t}+c_2 1{-}1 e^{t}$

Answer

Add texts here. Do not delete this text first.

4.2E.2 Exercise $\textbf{4.2}\boldsymbol{E}\textbf{.2}$

Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1, c_2, and c_3.

(a) $\begin{array}{ccc} y_1' & = & -y_1+2y_2+3y_3 \\ y_2' & = & y_2+6y_3 \\ y_3' & = & -2y_3; \end{array}$

$\mathbf{y}=c_1 110 e^{t}+c_2 100 e^{-t}+c_3 1{-}21 e^{-2t}$

(b) $\begin{array}{ccc} y_1' & = & 2y_2+2y_3 \\ y_2' & = & 2y_1\qquad +2y_3 \\ y_3' & = & 2y_1+2y_2; \end{array}$

$\mathbf{y}=c_1 {-}101 e^{-2t}+c_2 0{-}11 e^{-2t}+c_3 111 e^{4t}$

(c) $\begin{array}{ccc} y_1' & = & -y_1+2y_2+2y_3 \\ y_2' & = & 2y_1-\ \ y_2+2y_3 \\ y_3' & = & 2y_1+2y_2-\ \ y_3; \end{array}$

$\mathbf{y}=c_1 {-}101 e^{-3t}+c_2 0{-}11 e^{-3t}+c_3 111 e^{3t}$

$$\text{(d)}\quad \begin{aligned} y_1' &= \quad 3y_1 - \ \ y_2 - \ \ y_3 \\ y_2' &= -2y_1 + 3y_2 + 2y_3 \\ y_3' &= \quad 4y_1 - \ \ y_2 - 2y_3; \end{aligned}$$

$\mathbf{y} = c_1 101e^{2t} + c_2 1{-}11e^{3t} + c_3 1{-}37e^{-t}$

Answer

Add texts here. Do not delete this text first.

4.2E.3 Exercise $\mathbf{4.2E.3}$

Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

(a)

$$\begin{aligned} y_1' &= \quad y_1 + \ \ y_2 \\ y_2' &= -2y_1 + 4y_2, \end{aligned}$$

$$\begin{aligned} y_1(0) &= 1 \\ y_2(0) &= 0; \end{aligned}$$

$\mathbf{y} = 211e^{2t} - 12e^{3t}$

(b)

$$\begin{aligned} y_1' &= 5y_1 + 3y_2 \\ y_2' &= -y_1 + y_2, \end{aligned}$$

$$\begin{aligned} y_1(0) &= 12 \\ y_2(0) &= -6; \end{aligned}$$

$\mathbf{y} = 31{-}1e^{2t} + 33{-}1e^{4t}$

Answer

Add texts here. Do not delete this text first.

4.2E.4 Exercise $\mathbf{4.2E.4}$

Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

(a)

$$\begin{aligned} y_1' &= 6y_1 + 4y_2 + 4y_3 \\ y_2' &= -7y_1 - 2y_2 - y_3, \\ y_3' &= 7y_1 + 4y_2 + 3y_3, \end{aligned}$$

$$\begin{aligned} y_1(0) &= 3 \\ y_2(0) &= -6 \\ y_3(0) &= 4 \end{aligned}$$

$\mathbf{y} = 1{-}11e^{6t} + 21{-}21e^{2t} + 0{-}11e^{-t}$

(b)

$$\begin{aligned} y_1' &= 8y_1 + 7y_2 + 7y_3 \\ y_2' &= -5y_1 - 6y_2 - 9y_3, \\ y_3' &= 5y_1 + 7y_2 + 10y_3, \end{aligned}$$

$$\begin{aligned} y_1(0) &= 2 \\ y_2(0) &= -4 \\ y_3(0) &= 3 \end{aligned}$$

$\mathbf{y} = 1{-}11e^{8t} + 0{-}11e^{3t} + 1{-}21e^{t}$

Answer

Add texts here. Do not delete this text first.

4.2E.5 Exercise $4.2E.5$

Rewrite the system in matrix form and verify that the given vector function satisfies the ystem for any choice of the constants c_1 and c_2.

(a)

$$\begin{aligned} y_1' &= -3y_1 + 2y_2 + 3 - 2t \\ y_2' &= -5y_1 + 3y_2 + 6 - 3t \end{aligned}$$

$\mathbf{y} = c_1 \begin{bmatrix} \cos t \\ 3\cos t - \sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ 3\sin t + \cos t \end{bmatrix} + 1t$

(b)

$$\begin{aligned} y_1' &= 3y_1 + y_2 - 5e^t \\ y_2' &= -y_1 + y_2 + e^t \end{aligned}$$

$\mathbf{y} = c_1{-}11e^{2t} + c_2 \begin{bmatrix} +t \\ -t \end{bmatrix} e^{2t} + 13e^t$

(c)

$$\begin{aligned} y_1' &= -y_1 - 4y_2 + 4e^t + 8te^t \\ y_2' &= -y_1 - y_2 + e^{3t} + (4t+2)e^t \end{aligned}$$

$\mathbf{y} = c_1 21e^{-3t} + c_2{-}21e^{t} + \begin{bmatrix} e^{3t} \\ 2te^t \end{bmatrix}$

(d)

$$\begin{aligned} y_1' &= -6y_1 - 3y_2 + 14e^{2t} + 12e^t \\ y_2' &= \quad y_1 - 2y_2 + 7e^{2t} - 12e^t \end{aligned}$$

$$\mathbf{y} = c_1 {-3 \atop 1} e^{-5t} + c_2 {-1 \atop 1} e^{-3t} + \begin{bmatrix} e^{2t} + 3e^t \\ 2e^{2t} - 3e^t \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.2E.6 Exercise $\mathbf{4.2E.6}$

Convert the linear scalar equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y(t) = F(t) \tag{4.2E.1}$$

into an equivalent $n \times n$ system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t),$$

and show that A and $\mathbf{f}$ are continuous on an interval (a,b) if and only if (4.2E.1) is normal on (a,b).

Answer

Add texts here. Do not delete this text first.

4.2E.7 Exercise $\mathbf{4.2E.7}$

A matrix function

$$Q(t) = q_{rs}$$

is said to be differentiable if its entries $\{q_{ij}\}$ are differentiable. Then the derivative Q' is defined by

$$Q'(t) = q'_{rs}.$$

(a) Prove: If P and Q are differentiable matrices such that $P+Q$ is defined and if c_1 and c_2 are constants, then

$$(c_1P + c_2Q)' = c_1P' + c_2Q'.$$

(b) Prove: If P and Q are differentiable matrices such that PQ is defined, then

$$(PQ)' = P'Q + PQ'.$$

Answer

Add texts here. Do not delete this text first.

4.2E.8 Exercise $\mathbf{4.2E.8}$

Verify that $Y' = AY$.

(a) $Y = \begin{bmatrix} e^{6t} & e^{-2t} \\ e^{6t} & -e^{-2t} \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

(b) $Y = \begin{bmatrix} e^{-4t} & -2e^{3t} \\ e^{-4t} & 5e^{3t} \end{bmatrix}, \quad A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}$

(c) $Y = \begin{bmatrix} -5e^{2t} & 2e^{t} \\ 3e^{2t} & -e^{t} \end{bmatrix}, \quad A = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix}$

(d) $Y = \begin{bmatrix} e^{3t} & e^{t} \\ e^{3t} & -e^{t} \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(e) $Y = \begin{bmatrix} e^{t} & e^{-t} & e^{-2t} \\ e^{t} & 0 & -2e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}$

(f) $Y = \begin{bmatrix} -e^{-2t} & -e^{-2t} & e^{4t} \\ 0 & e^{-2t} & e^{4t} \\ e^{-2t} & 0 & e^{4t} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

(g) $Y = \begin{bmatrix} e^{3t} & e^{-3t} & 0 \\ e^{3t} & 0 & -e{-3t} \\ e^{3t} & e^{-3t} & e^{-3t} \end{bmatrix}, \quad A = \begin{bmatrix} -9 & 6 & 6 \\ -6 & 3 & 6 \\ -6 & 6 & 3 \end{bmatrix}$

(h) $Y = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{-3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix}$

Answer

Add texts here. Do not delete this text first.

4.2E.9 Exercise $\mathbf{4.2E.9}$

Suppose $\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix}$ are solutions of the homogeneous system

$$\mathbf{y}' = A(t)\mathbf{y}, \tag{4.2E.2}$$

and define $Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$.

(a) Show that $Y' = AY$.

(b) Show that if $\mathbf{c}$ is a constant vector then $\mathbf{y} = Y\mathbf{c}$ is a solution of (4.2E.2).

(c) State generalizations of part (a) and part (b) for $n \times n$ systems.

Answer

Add texts here. Do not delete this text first.

4.2E.10 Exercise $\mathbf{4.2E.10}$

Suppose Y is a differentiable square matrix.

(a) Find a formula for the derivative of Y^2.

(b) Find a formula for the derivative of Y^n, where n is any positive integer.

(c) State how the results obtained in part (a\) and part (b\) are analogous to results from calculus concerning scalar functions.

Answer

Add texts here. Do not delete this text first.

4.2E.11 Exercise $\mathbf{4.2}\boldsymbol{E}\mathbf{.11}$

It can be shown that if Y is a differentiable and invertible square matrix function, then Y^{-1} is differentiable.

(a) Show that $(Y^{-1})' = -Y^{-1}Y'Y^{-1}$.
Hint: Differentiate the identity $Y^{-1}Y = I$.

(b) Find the derivative of $Y^{-n} = \left(Y^{-1}\right)^n$, where n is a positive integer.

(c) State how the results obtained in part (a) and part (b) are analogous to results from calculus concerning scalar functions.

Answer

Add texts here. Do not delete this text first.

4.2E.12 Exercise $\mathbf{4.2}\boldsymbol{E}\mathbf{.12}$

Show that Theorem $(4.2.1)$ implies Theorem $(3.1.1)$.

Hint: Write the scalar equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x)$$

as an $n \times n$ system of linear equations.

Answer

Add texts here. Do not delete this text first.

4.2E.13 Exercise $\mathbf{4.2}\boldsymbol{E}\mathbf{.13}$

Suppose $\mathbf{y}$ is a solution of the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ on (a,b), and that the $n \times n$ matrix P is invertible and differentiable on (a,b). Find a matrix B such that the function $\mathbf{x} = P\mathbf{y}$ is a solution of $\mathbf{x}' = B\mathbf{x}$ on (a,b).

Answer

Add texts here. Do not delete this text first.

4.3: Basic Theory of Homogeneous Linear System

This page is a draft and is under active development.

In this section we consider homogeneous linear systems $\mathbf{y}' = A(t)\mathbf{y}$, where $A = A(t)$ is a continuous $n \times n$ matrix function on an interval (a, b). The theory of linear homogeneous systems has much in common with the theory of linear homogeneous scalar equations, which we considered in Sections 2.1 and 3.1.

Whenever we refer to solutions of $\mathbf{y}' = A(t)\mathbf{y}$ we'll mean solutions on (a, b). Since $\mathbf{y} \equiv \mathbf{0}$ is obviously a solution of $\mathbf{y}' = A(t)\mathbf{y}$, we call it the trivial solution. Any other solution is nontrivial.

If $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ are vector functions defined on an interval (a, b) and c_1, c_2, …, c_n are constants, then

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n \tag{4.3.1}$$

is a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$. It's easy show that if $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), then so is any linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ (Exercise $(4.3E.1)$). We say that $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) on if every solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) can be written as a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$, as in (4.3.1). In this case we say that (4.3.1) is the general solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

It can be shown that if A is continuous on (a, b) then $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on (a, b) (Exercises $(4.3E.15)$ and $(4.3E.16)$). The next definition will help to characterize fundamental sets of solutions of $\mathbf{y}' = A(t)\mathbf{y}$.

We say that a set $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ of n-vector functions is linearly independent on (a, b) if the only constants c_1, c_2, …, c_n such that

$$c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t) = 0, \quad a < t < b, \tag{4.3.2}$$

are $c_1 = c_2 = \cdots = c_n = 0$. If (4.3.2) holds for some set of constants c_1, c_2, …, c_n that are not all zero, then $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is linearly dependent on (a, b).

The next theorem is analogous to Theorems $(2.1.3)$ and $(3.1.2)$.

4.3.1 Theorem $\mathbf{4.3.1}$

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b). Then a set $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ of n solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is a fundamental set if and only if it's linearly independent on (a, b).

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

4.3.2 Example $\PageIndex{1}$

Show that the vector functions

$$\mathbf{y}_1=\begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix},\quad \text{and}\quad \mathbf{y}_3=\begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix}$$

are linearly independent on every interval (a,b).

Answer

Suppose

$$c_1\begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}+c_2\begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix}+c_3\begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\quad a<t<b.$$

We must show that $c_1=c_2=c_3=0$. Rewriting this equation in matrix form yields

$$\begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\quad a<t<b.$$

Expanding the determinant of this system in cofactors of the entries of the first row yields

$$\begin{aligned} \left|\begin{array}{ccc} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{array}\right| &= e^t\left|\begin{array}{cc} e^{3t} & e^{3t} \\ 1 & 0 \end{array}\right|-0\left|\begin{array}{cc} 0 & e^{3t} \\ e^{-t} & 0 \end{array}\right|+e^{2t}\left|\begin{array}{cc} 0 & e^{3t} \\ e^{-t} & 1 \end{array}\right| \\ &= e^t(-e^{3t})+e^{2t}(-e^{2t})=-2e^{4t}. \end{aligned}$$

Since this determinant is never zero, $c_1=c_2=c_3=0$.

We can use the method in Example (4.3.1) to test n solutions $\{\mathbf{y}_1,\mathbf{y}_2,\dots,\mathbf{y}_n\}$ of any $n\times n$ system $\mathbf{y}'=A(t)\mathbf{y}$ for linear independence on an interval (a,b) on which A is continuous. To explain this (and for other purposes later), it's useful to write a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ in a different way. We first write the vector functions in terms of their components as

$$\mathbf{y}_1=\begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix},\ \dots,\quad \mathbf{y}_n=\begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}.$$

If

$$\mathbf{y}=c_1\mathbf{y}_1+c_2\mathbf{y}_2+\cdots+c_n\mathbf{y}_n$$

then

$$\mathbf{y}= c_1\begin{bmatrix} y_{11}\\ y_{21}\\ \vdots\\ y_{n1}\end{bmatrix}+c_2\begin{bmatrix} y_{12}\\ y_{22}\\ \vdots\\ y_{n2}\end{bmatrix}+\cdots+c_n\begin{bmatrix} y_{1n}\\ y_{2n}\\ \vdots y_{nn}\end{bmatrix} =\begin{bmatrix} y_{11}&y_{12}&\cdots&y_{1n}\\ y_{21}&y_{22}&\cdots&y_{2n}\\ \vdots&\vdots&\vdots&\vdots\\ y_{1n}&y_{2n}&\cdots&y_{nn}\end{bmatrix}.$$

This shows that

$$c_1\mathbf{y}_1+c_2\mathbf{y}_2+\cdots+c_n\mathbf{y}_n=Y\mathbf{c}, \tag{4.3.3}$$

where

$$\mathbf{c}=c_n$$

and

$$Y=[\mathbf{y}_1\ \mathbf{y}_2\ \cdots\ \mathbf{y}_n]=\begin{bmatrix} y_{11}&y_{12}&\cdots&y_{1n}\\ y_{21}&y_{22}&\cdots&y_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ y_{n1}&y_{n2}&\cdots&y_{nn}\end{bmatrix}; \tag{4.3.4}$$

that is, the columns of Y are the vector functions $\mathbf{y}_1,\mathbf{y}_2,\dots,\mathbf{y}_n$.

For reference below, note that

$$\begin{aligned} Y' &= [\mathbf{y}_1'\ \mathbf{y}_2'\ \cdots\ \mathbf{y}_n'] \\ &= [A\mathbf{y}_1\ A\mathbf{y}_2\ \cdots\ A\mathbf{y}_n] \\ &= A[\mathbf{y}_1\ \mathbf{y}_2\ \cdots\ \mathbf{y}_n]=AY; \end{aligned}$$

that is, Y satisfies the matrix differential equation

$$Y'=AY.$$

The determinant of Y,

$$W=\begin{vmatrix} y_{11}&y_{12}&\cdots&y_{1n}\\ y_{21}&y_{22}&\cdots&y_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ y_{n1}&y_{n2}&\cdots&y_{nn}\end{vmatrix} \tag{4.3.5}$$

is called the Wronskian of $\{\mathbf{y}_1,\mathbf{y}_2,\dots,\mathbf{y}_n\}$. It can be shown (Exercises $(4.3E.2)$ and $(4.3E.3)$) that this definition is analogous to definitions of the Wronskian of scalar functions given in Sections 2.1 and 3.1. The next theorem is analogous to Theorems $(2.1.4)$ and $(3.1.3)$. The proof is sketched in Exercise $(4.3E.4)$ for $n=2$ and in Exercise $(4.3E.5)$ for general n.

4.3.3 Theorem - ABEL'S FORMULA $\mathbf{4.3.2}$

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a,b), let $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a,b), and let t_0 be in (a,b). Then the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is given by

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t}\left[a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)\right] ds\right), \quad a < t < b. \tag{4.3.6}$$

Therefore, either W has no zeros in (a,b) or $W \equiv 0$ on (a,b).

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

The sum of the diagonal entries of a square matrix A is called the trace of A, denoted by $\mathrm{tr}(A)$. Thus, for an $n \times n$ matrix A,

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn},$$

and (4.3.6) can be written as

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t}\mathrm{tr}(A)(s))\, ds\right), \quad a < t < b.$$

The next theorem is analogous to Theorems $(2.1.6)$ and $(3.1.4)$.

4.3.4 Theorem $\mathbf{4.3.3}$

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a,b) and let $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a,b). Then the following statements are equivalent; that is, they are either all true or all false:

(a) The general solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a,b) is $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$, where c_1, c_2, …, c_n are arbitrary constants.

(b) $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a,b).

(c) $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is linearly independent on (a,b).

(d) The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is nonzero at some point in (a,b).

(e) The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is nonzero at all points in (a,b).

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

We say that Y in (4.3.4) is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ if any (and therefore all) of the statements $(\mathbf{a}) - (\mathbf{e})$ of Theorem $(4.3.2)$ are true for the columns of Y. In this case, (4.3.3) implies that the general solution of $\mathbf{y}' = A(t)\mathbf{y}$ can be written as $\mathbf{y} = Y\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant n vector.

4.3.5 Example $\mathbf{4.3.2}$

The vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

are solutions of the constant coefficient system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y} \tag{4.3.7}$$

on $(-\infty, \infty)$. (Verify.)

(a) Compute the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$ directly from the definition (4.3.5)

(b) Verify Abel's formula (4.3.6) for the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.

(c) Find the general solution of (4.3.7).

(d) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}. \tag{4.3.8}$$

Answer

(a) From (4.3.5)

$$W(t) = \begin{vmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{vmatrix} = e^{2t}e^{-t} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = e^t. \tag{4.3.9}$$

(b) Here

$$A = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix}$$

so $\operatorname{tr}(A) = -4 + 5 = 1$. If t_0 is an arbitrary real number then (4.3.6) implies that

$$W(t) = W(t_0)\exp\left(\int_{t_0}^t 1\,ds\right) = \begin{vmatrix} -e^{2t_0} & e^{-t_0} \\ 2e^{2t_0} & e^{-t_0} \end{vmatrix} e^{(t-t_0)} = e^{t_0}e^{t-t_0} = e^t,$$

which is consistent with (4.3.9).

(c) Since $W(t) \neq 0$, Theorem $(4.3.3)$ implies that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions of (4.3.7) and

$$Y = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix for (4.3.7). Therefore the general solution of (4.3.7) is

$$\mathbf{y}=c_1\mathbf{y}_1+c_2\mathbf{y}_2=c_1\begin{bmatrix}-e^{2t}\\2e^{2t}\end{bmatrix}+c_2\begin{bmatrix}-e^{-t}\\e^{-t}\end{bmatrix}=\begin{bmatrix}-e^{2t}&-e^{-t}\\2e^{2t}&e^{-t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}. \tag{4.3.10}$$

(d) Setting $t=0$ in (4.3.10) and imposing the initial condition in (4.3.8) yields

$$C_1\begin{bmatrix}-1\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}4\\-5\end{bmatrix}.$$

Thus,

$$\begin{aligned}-c_1-c_2&=4\\2c_1+c_2&=-5.\end{aligned}$$

The solution of this system is $c_1=-1$, $c_2=-3$. Substituting these values into (4.3.10) yields

$$\mathbf{y}=-\begin{bmatrix}-e^{2t}\\2e^{2t}\end{bmatrix}-3\begin{bmatrix}-e^{-t}\\e^{-t}\end{bmatrix}=\begin{bmatrix}e^{2t}+3e^{-t}\\-2e^{2t}-3e^{-t}\end{bmatrix}$$

as the solution of (4.3.8).

4.3E: Exercises

This page is a draft and is under active development.

4.3E.1 Exercise $\boldsymbol{4.3E.1}$

Prove: If $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a,b), then any linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$ is also a solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a,b).

Answer

Add texts here. Do not delete this text first.

4.3E.2 Exercise $\boldsymbol{4.3E.2}$

In Section 2.1 the Wronskian of two solutions y_1 and y_2 of the scalar second order equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{4.3E.1}$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

(a) Rewrite (4.3E.1) as a system of first order equations and show that W is the Wronskian (as defined in this section) of two solutions of this system.

(b) Apply Equation $(4.3.6)$ to the system derived in part (a), and show that

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x} \frac{P_1(s)}{P_0(s)}\,ds\right\},$$

which is the form of Abel's formula given in Theorem $(3.1.3)$

Answer

Add texts here. Do not delete this text first.

4.3E.3 Exercise $\boldsymbol{4.3E.3}$

In Section 3.1, the Wronskian of n solutions y_1, y_2, ..., y_n of the nth order equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0 \tag{4.3E.2}$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

(a) Rewrite (4.3E.2) as a system of first order equations and show that W is the Wronskian (as defined in this section) of n solutions of this system.

(b) Apply Equation $(4.3.6)$ to the system derived in part (a), and show that

$$W(x)=W(x_0)\exp\left\{-\int_{x_0}^{x}\frac{P_1(s)}{P_0(s)}\,ds\right\},$$

which is the form of Abel's formula given in Theorem $(3.1.3)$.

Answer

Add texts here. Do not delete this text first.

4.3E.4 Exercise $\mathbf{4.3E.4}$

Suppose

$$\mathbf{y}_1=\begin{bmatrix}y_{11}\\ y_{21}\end{bmatrix}\quad\text{and}\quad\mathbf{y}_2=\begin{bmatrix}y_{12}\\ y_{22}\end{bmatrix}$$

are solutions of the 2×2 system $\mathbf{y}'=A\mathbf{y}$ on (a,b), and let

$$Y=\begin{bmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix}\quad\text{and}\quad W=\begin{vmatrix}11 & y_{12}\\ y_{21} & y_{22}\end{vmatrix};$$

thus, W is the Wronskian of $\{\mathbf{y}_1,\mathbf{y}_2\}$.

(a) Deduce from the definition of determinant that

$$W'=\begin{vmatrix}y'_{11} & y'_{12}\\ y_{21} & y_{22}\end{vmatrix}+\begin{vmatrix}y_{11} & y_{12}\\ y'_{21} & y'_{22}\end{vmatrix}.$$

(b) Use the equation $Y'=A(t)Y$ and the definition of matrix multiplication to show that

$$[y'_{11}\quad y'_{12}]=a_{11}[y_{11}\quad y_{12}]+a_{12}[y_{21}\quad y_{22}]$$

and

$$[y'_{21}\quad y'_{22}]=a_{21}[y_{11}\quad y_{12}]+a_{22}[y_{21}\quad y_{22}].$$

(c) Use properties of determinants to deduce from part (a) and part (b) that

$$\begin{vmatrix}y'_{11} & y'_{12}\\ y_{21} & y_{22}\end{vmatrix}=a_{11}W\quad\text{and}\quad\begin{vmatrix}y_{11} & y_{12}\\ y'_{21} & y'_{22}\end{vmatrix}=a_{22}W.$$

(d) Conclude from part (c) that

$$W'=(a_{11}+a_{22})W,$$

and use this to show that if $a<t_0<b$ then

$$W(t)=W(t_0)\exp\left(\int_{t_0}^{t}[a_{11}(s)+a_{22}(s)]\,ds\right)\quad a<t<b.$$

Answer

Add texts here. Do not delete this text first.

4.3E.5 Exercise $\mathbf{4.3E.5}$

Suppose the $n\times n$ matrix $A=A(t)$ is continuous on (a,b). Let

$$Y=\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n}\\ y_{21} & y_{22} & \cdots & y_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ y_{n1} & y_{n2} & \cdots & y_{nn}\end{bmatrix},$$

where the columns of Y are solutions of $\mathbf{y}'=A(t)\mathbf{y}$. Let

$$r_i=[y_{i1}\,y_{i2}\,\ldots\,y_{in}]$$

be the ith row of Y, and let W be the determinant of Y.

(a) Deduce from the definition of determinant that

$$W'=W_1+W_2+\cdots+W_n,$$

where, for $1\le m\le n$, the ith row of W_m is r_i if $i\ne m$, and r'_m if $i=m$.

(b) Use the equation $Y'=AY$ and the definition of matrix multiplication to show that

$$r'_m=a_{m1}r_1+a_{m2}r_2+\cdots+a_{mn}r_n.$$

(c) Use properties of determinants to deduce from part (b) that

$$\det(W_m)=a_{mm}W.$$

(d) Conclude from part (a) and part (c) that

$$W'=(a_{11}+a_{22}+\cdots+a_{nn}W,$$

and use this to show that if $a<t_0<b$ then

$$W(t)=W(t_0)\exp\left(\int_{t_0}^{t}[a_{11}(s)+a_{22}(s)+\cdots+a_{nn}(s)]\,ds\right),\quad a<t<b.$$

Answer

Add texts here. Do not delete this text first.

4.3E.6 Exercise $\mathbf{4.3E.6}$

Suppose the $n \times n$ matrix A is continuous on (a, b) and t_0 is a point in (a, b). Let Y be a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

(a) Show that $Y(t_0)$ is invertible.

(b) Show that if $\mathbf{k}$ is an arbitrary n-vector then the solution of the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y}, \quad \mathbf{y}(t_0) = \mathbf{k}$$

is

$$\mathbf{y} = Y(t)Y^{-1}(t_0)\mathbf{k}.$$

Answer

Add texts here. Do not delete this text first.

4.3E.7 Exercise $\mathbf{4.3E.7}$

Let

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \quad \mathbf{y}_1 = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}.$$

(a) Verify that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$.

(b) Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{k}. \tag{4.3E.3}$$

(c) Use the result of Exercise $(4.3E.6)$ part (b) to find a formula for the solution of (4.3E.3) for an arbitrary initial vector $\mathbf{k}$.

Answer

Add texts here. Do not delete this text first.

4.3E.8 Exercise $\mathbf{4.3E.8}$

Repeat Exercise $(4.3E7)$ with

$$A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -2e^{3t} \\ 5e^{3t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 10 \\ -4 \end{bmatrix}.$$

Answer

Add texts here. Do not delete this text first.

4.3E.9 Exercise $\mathbf{4.3E.9}$

Repeat Exercise $(4.3E.7)$ with

$$A=\begin{bmatrix}-4 & -10\\ 3 & 7\end{bmatrix},\quad \mathbf{y}_1=\begin{bmatrix}-5e^{2t}\\ 3e^{2t}\end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix}2e^{t}\\ -e^{t}\end{bmatrix},\quad \mathbf{k}=\begin{bmatrix}-19\\ 11\end{bmatrix}.$$

Answer

Add texts here. Do not delete this text first.

4.3E.10 Exercise $\mathbf{4.3E.10}$

Repeat Exercise $(4.3E.7)$ with

$$A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix},\quad \mathbf{y}_1=\begin{bmatrix}e^{3t}\\ e^{3t}\end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix}e^{t}\\ -e^{t}\end{bmatrix},\quad \mathbf{k}=\begin{bmatrix}2\\ 8\end{bmatrix}.$$

Answer

Add texts here. Do not delete this text first.

4.3E.11 Exercise $\mathbf{4.3E.11}$

Let

$$A=\begin{bmatrix}3 & -1 & -1\\ -2 & 3 & 2\\ 4 & -1 & -2\end{bmatrix},$$
$$\mathbf{y}_1=\begin{bmatrix}e^{2t}\\ 0\\ e^{2t}\end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix}e^{3t}\\ -e^{3t}\\ e^{3t}\end{bmatrix},\quad \mathbf{y}_3=\begin{bmatrix}e^{-t}\\ -3e^{-t}\\ 7e^{-t}\end{bmatrix},\quad \mathbf{k}=\begin{bmatrix}2\\ -7\\ 20\end{bmatrix}.$$

(a) Verify that $\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\}$ is a fundamental set of solutions for $\mathbf{y}'=A\mathbf{y}$.

(b) Solve the initial value problem

$$\mathbf{y}'=A\mathbf{y},\quad \mathbf{y}(0)=\mathbf{k}. \tag{4.3E.4}$$

(c) Use the result of Exercise $(4.3E.6)$ part (b) to find a formula for the solution of (4.3E.4) for an arbitrary initial vector $\mathbf{k}$.

Answer

Add texts here. Do not delete this text first.

4.3E.12 Exercise $\mathbf{4.3E.12}$

Repeat Exercise $(4.3E.11)$ with

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} -e^{-2t} \\ 0 \\ e^{-2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-2t} \\ e^{-2t} \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ -9 \\ 12 \end{bmatrix}.$$

Answer

Add texts here. Do not delete this text first.

4.3E.13 Exercise $\mathbf{4.3E.13}$

Repeat Exercise $(4.3E.11)$ with

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

Answer

Add texts here. Do not delete this text first.

4.3E.14 Exercise $\mathbf{4.3E.14}$

Suppose Y and Z are fundamental matrices for the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$. Then some of the four matrices YZ^{-1}, $Y^{-1}Z$, $Z^{-1}Y$, ZY^{-1} are necessarily constant. Identify them and prove that they are constant.

Answer

Add texts here. Do not delete this text first.

4.3E.15 Exercise $\mathbf{4.3E.15}$

Suppose the columns of an $n \times n$ matrix Y are solutions of the $n \times n$ system $\mathbf{y}' = A\mathbf{y}$ and C is an $n \times n$ constant matrix.

(a) Show that the matrix $Z = YC$ satisfies the differential equation $Z' = AZ$.

(b) Show that Z is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ if and only if C is invertible and Y is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$.

Answer

Add texts here. Do not delete this text first.

4.3E.16 Exercise $\mathbf{4.3}\boldsymbol{E}\mathbf{.16}$

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) and t_0 is in (a, b). For $i = 1, 2, \ldots, n$, let $\mathbf{y}_i$ be the solution of the initial value problem $\mathbf{y}_i' = A(t)\mathbf{y}_i, \ \mathbf{y}_i(t_0) = \mathbf{e}_i$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

that is, the jth component of $\mathbf{e}_i$ is 1 if $j = i$, or 0 if $j \neq i$.

(a) Show that $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

(b) Conclude from part (a) and Exercise $(4.3E.15)$ that $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on (a, b).

Answer

Add texts here. Do not delete this text first.

4.3E.17 Exercise $\mathbf{4.3}\boldsymbol{E}\mathbf{.17}$

Show that Y is a fundamental matrix for the system $\mathbf{y}' = A(t)\mathbf{y}$ if and only if Y^{-1} is a fundamental matrix for $\mathbf{y}' = -A^T(t)\mathbf{y}$, where A^T denotes the transpose of A.
Hint: See Exercise $(4.2E.11)$.

Answer

Add texts here. Do not delete this text first.

4.3E.18 Exercise $\mathbf{4.3}\boldsymbol{E}\mathbf{.18}$

Let Z be the fundamental matrix for the constant coefficient system $\mathbf{y}' = A\mathbf{y}$ such that $Z(0) = I$.

(a) Show that $Z(t)Z(s) = Z(t + s)$ for all s and t.
Hint: For fixed s let $\Gamma_1(t) = Z(t)Z(s)$ and $\Gamma_2(t) = Z(t + s)$. Show that Γ_1 and Γ_2 are both solutions of the matrix initial value problem $\Gamma' = A\Gamma, \quad \Gamma(0) = Z(s)$. Then conclude from Theorem $(4.2.1)$ that $\Gamma_1 = \Gamma_2$.

(b) Show that $(Z(t))^{-1} = Z(-t)$.

(c) The matrix Z defined above is sometimes denoted by e^{tA}. Discuss the motivation for this notation.

Answer

Add texts here. Do not delete this text first.

4.4: Constant Coefficient Homogeneous Systems I

This page is a draft and is under active development.

4.4.1 Constant Coefficient Homogenous Systems I

We'll now begin our study of the homogeneous system

$$\mathbf{y}'=A\mathbf{y}, \tag{4.4.1}$$

where A is an $n\times n$ constant matrix. Since A is continuous on $(-\infty,\infty)$, Theorem $(4.2.1)$ implies that all solutions of (4.4.1) are defined on $(-\infty,\infty)$. Therefore, when we speak of solutions of $\mathbf{y}'=A\mathbf{y}$, we'll mean solutions on $(-\infty,\infty)$.

In this section we assume that all the eigenvalues of A are real and that A has a set of n linearly independent eigenvectors. In the next two sections we consider the cases where some of the eigenvalues of A are complex, or where A does not have n linearly independent eigenvectors.

In Example $(4.3.2)$ we showed that the vector functions

$$\mathbf{y}_1=\begin{bmatrix}-e^{2t}\\ 2e^{2t}\end{bmatrix}\quad\text{and}\quad \mathbf{y}_2=\begin{bmatrix}-e^{-t}\\ e^{-t}\end{bmatrix}$$

form a fundamental set of solutions of the system

$$\mathbf{y}'=\begin{bmatrix}-4&-3\\ 6&5\end{bmatrix}\mathbf{y}, \tag{4.4.2}$$

but we did not show how we obtained $\mathbf{y}_1$ and $\mathbf{y}_2$ in the first place. To see how these solutions can be obtained we write (4.4.2) as

$$\begin{aligned} y_1' &= -4y_1-3y_2\\ y_2' &= 6y_1+5y_2 \end{aligned} \tag{4.4.3}$$

and look for solutions of the form

$$y_1=x_1e^{\lambda t}\quad\text{and}\quad y_2=x_2e^{\lambda t}, \tag{4.4.4}$$

where x_1, x_2, and λ are constants to be determined. Differentiating (4.4.4) yields

$$y_1'=\lambda x_1e^{\lambda t}\quad\text{and}\quad y_2'=\lambda x_2e^{\lambda t}.$$

Substituting this and (4.4.4) into (4.4.3) and canceling the common factor $e^{\lambda t}$ yields

$$\begin{aligned} -4x_1-3x_2 &= \lambda x_1\\ 6x_1+5x_2 &= \lambda x_2. \end{aligned}$$

For a given λ, this is a homogeneous algebraic system, since it can be rewritten as

$$\begin{aligned} (-4-\lambda)x_1 - 3x_2 &= 0 \\ 6x_1 + (5-\lambda)x_2 &= 0. \end{aligned} \tag{4.4.5}$$

The trivial solution $x_1 = x_2 = 0$ of this system isn't useful, since it corresponds to the trivial solution $y_1 \equiv y_2 \equiv 0$ of (4.4.3), which can't be part of a fundamental set of solutions of (4.4.2). Therefore we consider only those values of λ for which (4.4.5) has nontrivial solutions. These are the values of λ for which the determinant of (4.4.5) is zero; that is,

$$\begin{aligned} \begin{vmatrix} -4-\lambda & -3 \\ 6 & 5-\lambda \end{vmatrix} &= (-4-\lambda)(5-\lambda)+18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda-2)(\lambda+1) = 0, \end{aligned}$$

which has the solutions $\lambda_1 = 2$ and $\lambda_2 = -1$.

Taking $\lambda = 2$ in (4.4.5) yields

$$\begin{aligned} -6x_1 - 3x_2 &= 0 \\ 6x_1 + 3x_2 &= 0, \end{aligned}$$

which implies that $x_1 = -x_2/2$, where x_2 can be chosen arbitrarily. Choosing $x_2 = 2$ yields the solution $y_1 = -e^{2t}$,
$y_2 = 2e^{2t}$ of (4.4.3). We can write this solution in vector form as

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ 2e^{2t} \end{bmatrix}. \tag{4.4.6}$$

Taking $\lambda = -1$ in (4.4.5) yields the system

$$\begin{aligned} -3x_1 - 3x_2 &= 0 \\ 6x_1 + 6x_2 &= 0, \end{aligned}$$

so $x_1 = -x_2$. Taking |(x_2=1\) here yields the solution $y_1 = -e^{-t}$, $y_2 = e^{-t}$ of (4.4.3). We can write this solution in vector form as

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1e^{-t} \end{bmatrix}. \tag{4.4.7}$$

In (4.4.6) and (4.4.7) the constant coefficients in the arguments of the exponential functions are the eigenvalues of the coefficient matrix in (4.4.2), and the vector coefficients of the exponential functions are associated eigenvectors. This illustrates the next theorem.

4.4.1.1 Theorem **4.4.1**

Suppose the $n \times n$ constant matrix A has n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (*which need not be distinct)* with associated linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Then the functions

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t},\ \mathbf{y}_2 = \mathbf{x}_2 e^{\lambda_2 t},\ \ldots,\ \mathbf{y}_n = \mathbf{x}_n e^{\lambda_n t}$$

form a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$; that is, the general solution of this system is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{x}_n e^{\lambda_n t}.$$

Proof

Differentiating $\mathbf{y}_i = \mathbf{x}_i e^{\lambda_i t}$ and recalling that $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ yields

$$\mathbf{y}_i' = \lambda_i\mathbf{x}_i e^{\lambda_i t} = A\mathbf{x}_i e^{\lambda_i t} = A\mathbf{y}_i.$$

This shows that $\mathbf{y}_i$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is

$$\left|\begin{array}{cccc} x_{11}e^{\lambda_1 t} & x_{12}e^{\lambda_2 t} & \cdots & x_{1n}e^{\lambda_n t} \\ x_{21}e^{\lambda_1 t} & x_{22}e^{\lambda_2 t} & \cdots & x_{2n}e^{\lambda_n t} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}e^{\lambda_1 t} & x_{n2}e^{\lambda_2 t} & \cdots & x_{nn}e^{\lambda x_n} \end{array}\right| = e^{\lambda_1 t}e^{\lambda_2 t}\cdots e^{\lambda_n t}\left|\begin{array}{cccc} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right|.$$

Since the columns of the determinant on the right are $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_n$, which are assumed to be linearly independent, the determinant is nonzero. Therefore Theorem (4.3.3) implies that $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$.

Example 4.4.1

(a) Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}\mathbf{y}. \tag{4.4.8}$$

(b) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}. \tag{4.4.9}$$

Answer

(a) The characteristic polynomial of the coefficient matrix A in (4.4.8) is

$$\begin{aligned} \left|\begin{array}{cc} 2-\lambda & 4 \\ 4 & 2-\lambda \end{array}\right| &= (\lambda-2)^2 - 16 \\ &= (\lambda-2-4)(\lambda-2+4) \\ &= (\lambda-6)(\lambda+2). \end{aligned}$$

Hence, $\lambda_1 = 6$ and $\lambda_2 = -2$ are eigenvalues of A. To obtain the eigenvectors, we must solve the system

$$\begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.4.10}$$

with $\lambda = 6$ and $\lambda = -2$. Setting $\lambda = 6$ in (4.4.10) yields

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

is a solution of (4.4.8). Setting $\lambda = -2$ in (4.4.10) yields

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = -x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

is a solution of (4.4.8). From Theorem (4.4.1), the general solution of (4.4.8) is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}. \tag{4.4.11}$$

(b) To satisfy the initial condition in (4.4.9), we must choose c_1 and c_2 in (4.4.11) so that

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

This is equivalent to the system

$$\begin{aligned} c_1 - c_2 &= 5 \\ c_1 + c_2 &= -1, \end{aligned}$$

so $c_1 = 2, c_2 = -3$. Therefore the solution of (4.4.9) is

$$\mathbf{y} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} - 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

or, in terms of components,

$$y_1 = 2e^{6t} + 3e^{-2t}, \quad y_2 = 2e^{6t} - 3e^{-2t}.$$

Example 4.4.2

(a) Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}. \tag{4.4.12}$$

(b) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}. \tag{4.4.13}$$

Answer

(a) The characteristic polynomial of the coefficient matrix A in (4.4.12) is

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{vmatrix} = -(\lambda-2)(\lambda-3)(\lambda+1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -1$. To find the eigenvectors, we must solve the system

$$\begin{bmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.4.14}$$

with $\lambda = 2, 3, -1$. With $\lambda = 2$, the augmented matrix of (4.4.14) is

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & 0 \\ -2 & 1 & 2 & \vdots & 0 \\ 4 & -1 & -4 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

as a solution of (4.4.12). With $\lambda = 3$, the augmented matrix of (4.4.14) is

$$\begin{bmatrix} 0 & -1 & -1 & \vdots & 0-2 & 0 & 2 & \vdots & 0 \\ 4 & -1 & -5 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t}$$

as a solution of (4.4.12). With $\lambda = -1$, the augmented matrix of (4.4.14) is

$$\begin{bmatrix} 4 & -1 & -1 & \vdots & 0 \\ -2 & 4 & 2 & \vdots & 0 \\ 4 & -1 & -1 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -\frac{1}{7} & \vdots & 0 \\ 0 & 1 & \frac{3}{7} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_3/7$ and $x_2 = -3x_3/7$. Taking $x_3 = 7$ yields

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$$

as a solution of (4.4.12). By Theorem (4.4.1), the general solution of (4.4.12) is

$$\mathbf{y}=c_1\begin{bmatrix}1\\0\\1\end{bmatrix}e^{2t}+c_2\begin{bmatrix}1\\-1\\1\end{bmatrix}e^{3t}+c_3\begin{bmatrix}1\\-3\\7\end{bmatrix}e^{-t},$$

which can also be written as

$$\mathbf{y}=\begin{bmatrix}e^{2t}&e^{3t}&e^{-t}\\0&-e^{3t}&-3e^{-t}\\e^{2t}&e^{3t}&7e^{-t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}. \tag{4.4.15}$$

(b) To satisfy the initial condition in (4.4.13) we must choose c_1, c_2, c_3 in (4.4.15) so that

$$\begin{bmatrix}1&1&1\\0&-1&-3\\1&1&7\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}2\\-1\\8\end{bmatrix}.$$

Solving this system yields $c_1=3$, $c_2=-2$, $c_3=1$. Hence, the solution of (4.4.13) is

$$\begin{aligned}\mathbf{y}&=\begin{bmatrix}e^{2t}&e^{3t}&e^{-t}\\0&-e^{3t}&-3e^{-t}\\e^{2t}&e^{3t}&7e^{-t}\end{bmatrix}\begin{bmatrix}3\\-2\\1\end{bmatrix}\\&=3\begin{bmatrix}1\\0\\1\end{bmatrix}e^{2t}-2\begin{bmatrix}1\\-1\\1\end{bmatrix}e^{3t}+\begin{bmatrix}1\\-3\\7\end{bmatrix}e^{-t}.\end{aligned}$$

Example 4.4.3

Find the general solution of

$$\mathbf{y}'=\begin{bmatrix}-3&2&2\\2&-3&2\\2&2&-3\end{bmatrix}\mathbf{y}. \tag{4.4.16}$$

Answer

The characteristic polynomial of the coefficient matrix A in (4.4.16) is

$$\begin{bmatrix}-3-\lambda&2&2\\2&-3-\lambda&2\\2&2&-3-\lambda\end{bmatrix}=-(\lambda-1)(\lambda+5)^2.$$

Hence, $\lambda_1=1$ is an eigenvalue of multiplicity 1, while $\lambda_2=-5$ is an eigenvalue of multiplicity 2. Eigenvectors associated with $\lambda_1=1$ are solutions of the system with augmented matrix

$$\begin{bmatrix} -4 & 2 & 2 & \vdots & 0 \\ 2 & -4 & 2 \vdots & 0 \\ 2 & 2 & -4 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_2 = x_3$, and we choose $x_3 = 1$ to obtain the solution

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t \tag{4.4.17}$$

of (4.4.16). Eigenvectors associated with $\lambda_2 = -5$ are solutions of the system with augmented matrix

$$\begin{bmatrix} 2 & 2 & 2 & \vdots & 0 \\ 2 & 2 & 2 & \vdots & 0 \\ 2 & 2 & 2 & \vdots & 0 \end{bmatrix}.$$

Hence, the components of these eigenvectors need only satisfy the single condition

$$x_1 + x_2 + x_3 = 0.$$

Since there's only one equation here, we can choose x_2 and x_3 arbitrarily. We obtain one eigenvector by choosing $x_2 = 0$ and $x_3 = 1$, and another by choosing $x_2 = 1$ and $x_3 = 0$. In both cases $x_1 = -1$. Therefore

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \, quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent eigenvectors associated with $\lambda_2 = -5$, and the corresponding solutions of (4.4.16) are

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Because of this and (4.4.17), Theorem $(4.4.1)$ implies that the general solution of (4.4.16) is

$$\mathbf{y}=c_1\begin{bmatrix}1\\1\\1\end{bmatrix}e^t+c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{-5t}+c_3\begin{bmatrix}-1\\1\\0\end{bmatrix}e^{-5t}.$$

4.4.2 Geometric Properties of Solutions when $n=2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\begin{bmatrix}y_1'\\y_2'\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}. \tag{4.4.18}$$

It is convenient to think of a ``y_1-y_2 plane," where a point is identified by rectangular coordinates (y_1,y_2). If $\mathbf{y}=\begin{bmatrix}y_1\\y_2\end{bmatrix}$ is a non-constant solution of (4.4.18), then the point $(y_1(t),y_2(t))$ moves along a curve C in the y_1-y_2 plane as t varies from $-\infty$ to ∞. We call C the trajectory of $\mathbf{y}$. (We also say that C is a trajectory of the system (4.4.18).) It's important to note that C is the trajectory of infinitely many solutions of (4.4.18), since if τ is any real number, then $\mathbf{y}(t-\tau)$ is a solution of (4.4.18) (Exercise $(4.4E.28)$ part (b)), and $(y_1(t-\tau),y_2(t-\tau))$ also moves along C as t varies from $-\infty$ to ∞. Moreover, Exercise $(4.4E.28)$ part (c) implies that distinct trajectories of (4.4.18) can't intersect, and that two solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of (4.4.18) have the same trajectory if and only if $\mathbf{y}_2(t)=\mathbf{y}_1(t-\tau)$ for some τ.

From Exercise $(4.4E.28)$ part (a), a trajectory of a nontrivial solution of (4.4.18) can't contain $(0,0)$, which we define to be the trajectory of the trivial solution $\mathbf{y}\equiv 0$. More generally, if $\mathbf{y}=\begin{bmatrix}k_1\\k_2\end{bmatrix}\neq\mathbf{0}$ is a constant solution of (4.4.18) (which could occur if zero is an eigenvalue of the matrix of (4.4.18)), we define the trajectory of $\mathbf{y}$ to be the single point (k_1,k_2).

To be specific, this is the question: What do the trajectories look like, and how are they traversed? In this section we'll answer this question, assuming that the matrix

$$A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$$

of (4.4.18) has real eigenvalues λ_1 and λ_2 with associated linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then the general solution of (4.4.18) is

$$\mathbf{y}=c_1\mathbf{x}_1e^{\lambda_1 t}+c_2\mathbf{x}_2e^{\lambda_2 t}. \tag{4.4.19}$$

We'll consider other situations in the next two sections.

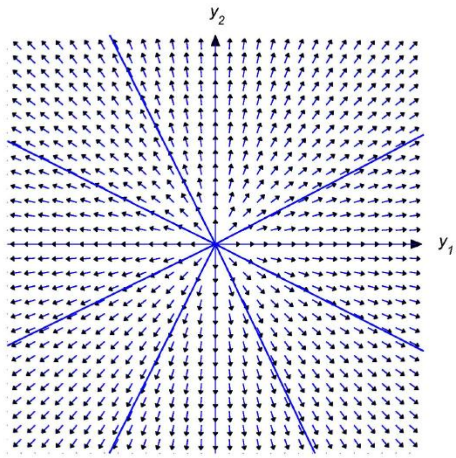
We leave it to you (Exercise $(4.4E.35)$) to classify the trajectories of (4.4.18) if zero is an eigenvalue of A. We'll confine our attention here to the case where both eigenvalues are nonzero. In this case the simplest situation is where $\lambda_1=\lambda_2\neq 0$, so (4.4.19) becomes

$$\mathbf{y}=(c_1\mathbf{x}_1+c_2\mathbf{x}_2)e^{\lambda_1 t}.$$

Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, an arbitrary vector $\mathbf{x}$ can be written as $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. Therefore the general solution of (4.4.18) can be written as $\mathbf{y} = \mathbf{x}e^{\lambda_1 t}$ where $\mathbf{x}$ is an arbitrary 2-vector, and the trajectories of nontrivial solutions of (4.4.18) are half-lines through (but not including) the origin. The direction of motion is away from the origin if $\lambda_1 > 0$ (Figure 4.4.1), toward it if $\lambda_1 < 0$ (Figure 4.4.2). (In these and the next figures an arrow through a point indicates the direction of motion along the trajectory through the point.)

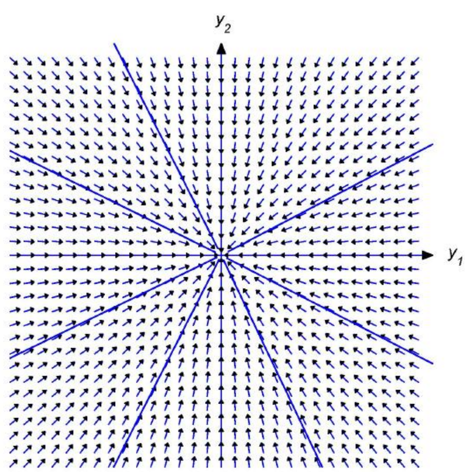
4.4.2.1 Figure **4.4.1**

Trajectories of a 2×2 system with a repeated positive



4.4.2.2 Figure **4.4.2**

Trajectories of a 2×2 system with a repeated negative



Now suppose $\lambda_2 > \lambda_1$, and let L_1 and L_2 denote lines through the origin parallel to $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. By a half-line of L_1 (or L_2), we mean either of the rays obtained by removing the origin from L_1 (or L_2).

Letting $c_2 = 0$ in (4.4.19) yields $\mathbf{y} = c_1\mathbf{x}_1e^{\lambda_1 t}$. If $c_1 \neq 0$, the trajectory defined by this solution is a half-line of L_1. The direction of motion is away from the origin if $\lambda_1 > 0$, toward the origin if $\lambda_1 < 0$. Similarly, the trajectory of $\mathbf{y} = c_2\mathbf{x}_2e^{\lambda_2 t}$ with $c_2 \neq 0$ is a half-line of L_2.

Henceforth, we assume that c_1 and c_2 in (4.4.19) are both nonzero. In this case, the trajectory of (4.4.19) can't intersect L_1 or L_2, since every point on these lines is on the trajectory of a solution for which either $c_1 = 0$ or $c_2 = 0$. (Remember: distinct trajectories can't intersect!). Therefore the trajectory of (4.4.19) must lie entirely in one of the four open sectors bounded by L_1 and L_2, but do not any point on L_1 or L_2. Since the initial point $(y_1(0), y_2(0))$ defined by

$$\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

is on the trajectory, we can determine which sector contains the trajectory from the signs of c_1 and c_2, as shown in Figure 4.4.3.

The direction of $\mathbf{y}(t)$ in (4.4.19) is the same as that of

$$e^{-\lambda_2 t}\mathbf{y}(t) = c_1\mathbf{x}_1 e^{-(\lambda_2-\lambda_1)t} + c_2\mathbf{x}_2 \tag{4.4.20}$$

and of

$$e^{-\lambda_1 t}\mathbf{y}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 e^{(\lambda_2-\lambda_1)t}. \tag{4.4.21}$$

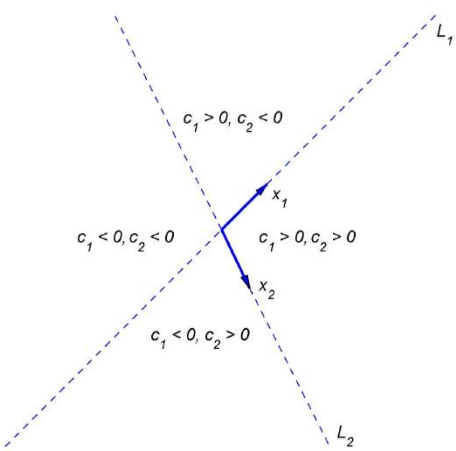
Since the right side of (4.4.20) approaches $c_2\mathbf{x}_2$ as $t \to \infty$, the trajectory is asymptotically parallel to L_2 as $t \to \infty$. Since the right side of (4.4.21) approaches $c_1\mathbf{x}_1$ as $t \to -\infty$, the trajectory is asymptotically parallel to L_1 as $t \to -\infty$.

The shape and direction of traversal of the trajectory of (4.4.19) depend upon whether λ_1 and λ_2 are both positive, both negative, or of opposite signs. We'll now analyze these three cases.

Henceforth $\|\mathbf{u}\|$ denote the length of the vector $\mathbf{u}$.

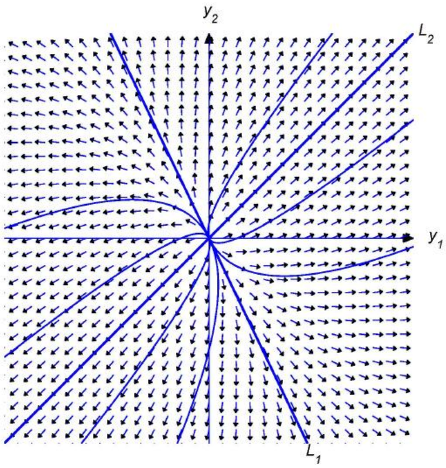
4.4.2.3 Figure $\mathbf{4.4.3}$

Four open sectors bounded by L_1 and L_2



4.4.2.4 Figure $\mathbf{4.4.4}$

Two positive eigenvalues; motion away from origin



4.4.3 Case 1: $\lambda_2 > \lambda_1 > 0$

Figure 4.4.4 shows some typical trajectories. In this case, $\lim_{t\to-\infty} \|\mathbf{y}(t)\| = 0$, so the trajectory is not only asymptotically parallel to L_1 as $t \to -\infty$, but is actually asymptotically tangent to L_1 at the origin. On the other hand, $\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t\to\infty} \left\|\mathbf{y}(t) - c_2\mathbf{x}_2 e^{\lambda_2 t}\right\| = \lim_{t\to\infty} \|c_1\mathbf{x_1} e^{\lambda_1 t}\| = \infty,$$

so, although the trajectory is asymptotically parallel to L_2 as $t \to \infty$, it's not asymptotically tangent to L_2. The direction of motion along each trajectory is away from the origin.

4.4.4 Case 2: $0 > \lambda_2 > \lambda_1$

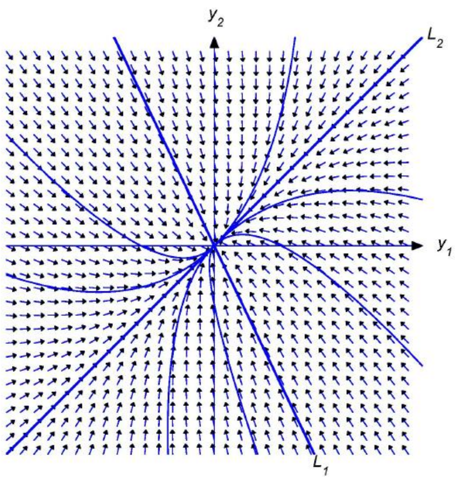
Figure 4.4.5 shows some typical trajectories. In this case, $\lim_{t\to\infty} \|\mathbf{y}(t)\| = 0$, so the trajectory is asymptotically tangent to L_2 at the origin as $t \to \infty$. On the other hand, $\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t\to-\infty} \left\|\mathbf{y}(t) - c_1\mathbf{x}_1 e^{\lambda_1 t}\right\| = \lim_{t\to-\infty} \|c_2\mathbf{x}_2 e^{\lambda_2 t}\| = \infty,$$

so, although the trajectory is asymptotically parallel to L_1 as $t \to -\infty$, it's not asymptotically tangent to it. The direction of motion along each trajectory is toward the origin.

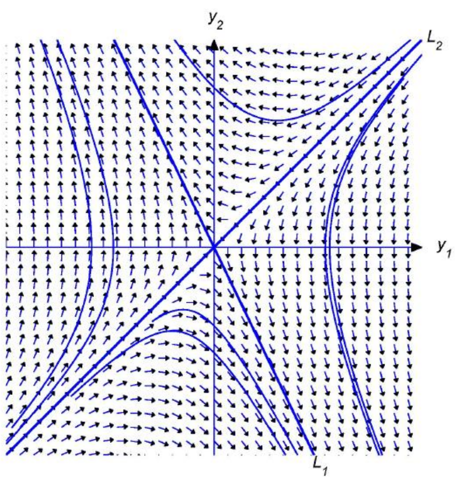
4.4.4.1 Figure $4.4.5$

Two negative eigenvalues; motion toward the origin



4.4.4.2 Figure $\mathbf{4.4.6}$

Eigenvalues of different signs



4.4.5 Case 3: $\boldsymbol{\lambda_2 > 0 > \lambda_1}$

Figure 4.4.6 shows some typical trajectories. In this case,

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim |t \to \infty \left\|\mathbf{y}(t) - c_2\mathbf{x}_2 e^{\lambda_2 t}\right\| = \lim_{t\to\infty} \|c_1\mathbf{x}_1 e^{\lambda_1 t}\| = 0,$$

so the trajectory is asymptotically tangent to L_2 as $t \to \infty$. Similarly,

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \left\|\mathbf{y}(t) - c_1\mathbf{x}_1 e^{\lambda_1 t}\right\| = \lim_{t\to\infty} \|c_2\mathbf{x}_2 e^{\lambda_2 t}\| = 0,$$

so the trajectory is asymptotically tangent to L_1 as $t \to -\infty$. The direction of motion is toward the origin on L_1 and away from the origin on L_2. The direction of motion along any other trajectory is away from L_1, toward L_2.

4.4E: Exercises

This page is a draft and is under active development.

In Exercises $(4.4E.1)$ to $(4.4E.15)$, find the general solution.

4.4E.1 Exercise $\mathbf{4.4E.1}$

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.2 Exercise $\mathbf{4.4E.2}$

$$\mathbf{y}' = \frac{1}{4} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.3 Exercise $\mathbf{4.4E.3}$

$$\mathbf{y}' = \frac{1}{5} \begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.4 Exercise $\mathbf{4.4E.4}$

$$\mathbf{y}' = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.5 Exercise $\mathbf{4.4E.5}$

$$\mathbf{y}' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.6 Exercise $\mathbf{4.4E.6}$

$$\mathbf{y}'=\begin{bmatrix}4 & -3\\ 2 & -1\end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.7 Exercise $\mathbf{4.4E.7}$

$$\mathbf{y}'=\begin{bmatrix}-6 & -3\\ 1 & -2\end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.8 Exercise $\mathbf{4.4E.8}$

$$\mathbf{y}'=\begin{bmatrix}1 & -1 & -2\\ 1 & -2 & -3\\ -4 & 1 & -1\end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.9 Exercise $\mathbf{4.4E.9}$

$$\mathbf{y}'=\begin{bmatrix}-6 & -4 & -8\\ -4 & 0 & -4\\ -8 & -4 & -6\end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.10 Exercise $\mathbf{4.4E.10}$

$$\mathbf{y}'=\begin{bmatrix}3 & 5 & 8\\ 1 & -1 & -2\\ -1 & -1 & -1\end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.11 Exercise $\mathbf{4.4E.11}$

$$\mathbf{y}' = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.12 Exercise $\mathbf{4.4}\boldsymbol{E}\mathbf{.12}$

$$\mathbf{y}' = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.13 Exercise $\mathbf{4.4}\boldsymbol{E}\mathbf{.13}$

$$\mathbf{y}' = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.14 Exercise $\mathbf{4.4}\boldsymbol{E}\mathbf{.14}$

$$\mathbf{y}' = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.15 Exercise $\mathbf{4.4}\boldsymbol{E}\mathbf{.15}$

$$\mathbf{y}' = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

In Exercises $(4.4E.16)$ to $(4.4E.27)$, solve the initial value problem.

4.4E.16 Exercise $\mathbf{4.4}\boldsymbol{E}\mathbf{.16}$

$$\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.17 Exercise $\mathbf{4.4}\boldsymbol{E}\mathbf{.17}$

$$\mathbf{y}' = \frac{1}{6}\begin{bmatrix} 7 & 2 \\ -2 & 2 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.18 Exercise $\mathbf{4.4}\boldsymbol{E}\mathbf{.18}$

$$\mathbf{y}' = \begin{bmatrix} 21 & -12 \\ 24 & -15 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.19 Exercise $\mathbf{4.4}\boldsymbol{E}\mathbf{.19}$

$$\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.20 Exercise $\mathbf{4.4}\boldsymbol{E}\mathbf{.20}$

$$\mathbf{y}' = \frac{1}{6}\begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.21 Exercise $\mathbf{4.4}\boldsymbol{E}\mathbf{.21}$

$$\mathbf{y}' = \frac{1}{3}\begin{bmatrix} 2 & -2 & 3 \\ -4 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.22 Exercise $\mathbf{4.4E.22}$

$$\mathbf{y}' = \begin{bmatrix} 6 & -3 & -8 \\ 2 & 1 & -2 \\ 3 & -3 & -5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.23 Exercise $\mathbf{4.4E.23}$

$$\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 2 & 4 & -7 \\ 1 & 5 & -5 \\ -4 & 4 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.24 Exercise $\mathbf{4.4E.24}$

$$\mathbf{y}' = \begin{bmatrix} 3 & 0 & 1 \\ 11 & -2 & 7 \\ 1 & 0 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.25 Exercise $\mathbf{4.4E.25}$

$$\mathbf{y}' = \begin{bmatrix} -2 & -5 & -1 \\ -4 & -1 & 1 \\ 4 & 5 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ -10 \\ -4 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.26 Exercise $\mathbf{4.4E.26}$

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & 0 \\ 4 & -2 & 0 \\ 4 & -4 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 7 \\ 10 \\ 2 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.27 Exercise $\mathbf{4.4E.27}$

$$\mathbf{y}' = \begin{bmatrix} -2 & 2 & 6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ -10 \\ 7 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.4E.28 Exercise $\mathbf{4.4E.28}$

Let A be an $n \times n$ constant matrix. Then Theorem $(4.2.1)$ implies that the solutions of

$$\mathbf{y}' = A\mathbf{y} \tag{4.4E.1}$$

are all defined on $(-\infty, \infty)$.

(a) Use Theorem $(4.2.1)$ to show that the only solution of (4.4E.1) that can ever equal the zero vector is $\mathbf{y} \equiv \mathbf{0}$.

(b) Suppose $\mathbf{y}_1$ is a solution of (4.4E.1) and $\mathbf{y}_2$ is defined by $\mathbf{y}_2(t) = \mathbf{y}_1(t-\tau)$, where τ is an arbitrary real number. Show that $\mathbf{y}_2$ is also a solution of (4.4E.1).

(c) Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of (4.4E.1) and there are real numbers t_1 and t_2 such that $\mathbf{y}_1(t_1) = \mathbf{y}_2(t_2)$. Show that $\mathbf{y}_2(t) = \mathbf{y}_1(t-\tau)$ for all t, where $\tau = t_2 - t_1$.
Hint: Show that $\mathbf{y}_1(t-\tau)$ and $\mathbf{y}_2(t)$ are solutions of the same initial value problem for (4.4E.1), and apply the uniqueness assertion of Theorem $(4.2.1)$.

Answer

Add texts here. Do not delete this text first.

In Exercises $(4.4E.29)$ to $(4.4E.34)$, describe and graph trajectories of the given system.

4.4E.29 Exercise $\mathbf{4.4E.29}$

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.30 Exercise $\mathbf{4.4E.30}$

$$\mathbf{y}' = \begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.31 Exercise $\mathbf{4.4E.31}$

$$\mathbf{y}' = \begin{bmatrix} 9 & -3 \\ -1 & 11 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.32 Exercise $\mathbf{4.4E.32}$

$$\mathbf{y}' = \begin{bmatrix} -1 & -10 \\ -5 & 4 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.33 Exercise $\mathbf{4.4E.33}$

$$\mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 1 & 10 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.34 Exercise $\mathbf{4.4E.34}$

$$\mathbf{y}' = \begin{bmatrix} -7 & 1 \\ 3 & -5 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.35 Exercise $\mathbf{4.4E.35}$

Suppose the eigenvalues of the 2×2 matrix A are $\lambda = 0$ and $\mu \neq 0$, with corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Let L_1 be the line through the origin parallel to $\mathbf{x}_1$.

(a) Show that every point on L_1 is the trajectory of a constant solution of $\mathbf{y}' = A\mathbf{y}$.

(b) Show that the trajectories of nonconstant solutions of $\mathbf{y}' = A\mathbf{y}$ are half-lines parallel to $\mathbf{x}_2$ and on either side of L_1, and that the direction of motion along these trajectories is away from L_1 if $\mu > 0$, or toward L_1 if $\mu < 0$.

Answer

Add texts here. Do not delete this text first.

The matrices of the systems in Exercises $(4.4E.36)$ to $(4.4E.41)$ are singular. Describe and graph the trajectories of nonconstant solutions of the given systems.

4.4E.36 Exercise $\mathbf{4.4E.36}$

$$\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.37 Exercise $\mathbf{4.4E.37}$

$$\mathbf{y}' = \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.38 Exercise $\mathbf{4.4E.38}$

$$\mathbf{y}' = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.39 Exercise $\mathbf{4.4E.39}$

$$\mathbf{y}' = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.40 Exercise $\mathbf{4.4E.40}$

$$\mathbf{y}' = \begin{bmatrix} -4 & -4 \\ 1 & 1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.41 Exercise $\mathbf{4.4E.41}$

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.4E.42 Exercise $\mathbf{4.4E.42}$

Let $P = P(t)$ and $Q = Q(t)$ be the populations of two species at time t, and assume that each population would grow exponentially if the other didn't exist; that is, in the absence of competition,

$$P' \; aP \quad \text{and} \quad Q' = bQ, \tag{4.4E.2}$$

where a and b are positive constants. One way to model the effect of competition is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so (4.4E.2) is replaced by

$$\begin{aligned} P' &= \quad aP - \alpha Q \\ Q' &= -\beta P + bQ, \end{aligned}$$

where α and β are positive constants. (Since negative population doesn't make sense, this system holds only while P and Q are both positive.) Now suppose $P(0) = P_0 > 0$ and $Q(0) = Q_0 > 0$.

(a) For several choices of a, b, α, and β, verify experimentally (by graphing trajectories of (4.4E.2) in the P-Q plane) that there's a constant $\rho > 0$ (depending upon a, b, α, and β) with the following properties:
(i) If $Q_0 > \rho P_0$, then P decreases monotonically to zero in finite time, during which Q remains positive.
(ii) If $Q_0 < \rho P_0$, then Q decreases monotonically to zero in finite time, during which P remains positive.

(b) Conclude from part (a) that exactly one of the species becomes extinct in finite time if $Q_0 \neq \rho P_0$. Determine experimentally what happens if $Q_0 = \rho P_0$.

(c) Confirm your experimental results and determine γ by expressing the eigenvalues and associated eigenvectors of

$$A = \begin{bmatrix} a & -\alpha \\ -\beta & b \end{bmatrix}$$

in terms of a, b, α, and β, and applying the geometric arguments developed at the end of this section.

Answer

Add texts here. Do not delete this text first.

4.5: Constant Coefficient Homogeneous Systems II

This page is a draft and is under active development.

4.5.1 Constant Coefficient Homogeneous Systems II

We saw in Section 4.4 that if an $n \times n$ constant matrix A has n real eigenvalues λ_1, λ_2, …, λ_n (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$, then the general solution of $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} = c_2\mathbf{x}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{x}_n e^{\lambda_n t}.$$

In this section we consider the case where A has n real eigenvalues, but does not have n linearly independent eigenvectors. It is shown in linear algebra that this occurs if and only if A has at least one eigenvalue of multiplicity $r > 1$ such that the associated eigenspace has dimension less than r. In this case A is said to be defective. Since it's beyond the scope of this book to give a complete analysis of systems with defective coefficient matrices, we will restrict our attention to some commonly occurring special cases.

Example 4.5.1

Show that the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y} \tag{4.5.1}$$

does not have a fundamental set of solutions of the form $\{\mathbf{x}_1 e^{\lambda_1 t}, \mathbf{x}_2 e^{\lambda_2 t}\}$, where λ_1 and λ_2 are eigenvalues of the coefficient matrix A of (4.5.1) and $\mathbf{x}_1$, and $\mathbf{x}_2$ are associated linearly independent eigenvectors.

Answer

The characteristic polynomial of A is

$$\begin{aligned} \begin{bmatrix} 11-\lambda & -25 \\ 4 & -9-\lambda \end{bmatrix} &= (\lambda - 11)(\lambda + 9) + 100 \\ &= \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2. \end{aligned}$$

Hence, $\lambda = 1$ is the only eigenvalue of A. The augmented matrix of the system $(A - I)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{ccccc} 10 & -25 & \vdots & 0 \\ 4 & -10 & \vdots \$0 \end{array}\right],$$

which is row equivalent to

$$\begin{bmatrix} 1 & -\frac{5}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = 5x_2/2$ where x_2 is arbitrary. Therefore all eigenvectors of A are scalar multiples of $\mathbf{x}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, so A does not have a set of two linearly independent eigenvectors.

From Example (4.5.1), we know that all scalar multiples of $\mathbf{y}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t$ are solutions of (4.5.1); however, to find the general solution we must find a second solution $\mathbf{y}_2$ such that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is linearly independent. Based on your recollection of the procedure for solving a constant coefficient scalar equation

$$ay'' + by' + cy = 0$$

in the case where the characteristic polynomial has a repeated root, you might expect to obtain a second solution of (4.5.1) by multiplying the first solution by t. However, this yields $\mathbf{y}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t$, which doesn't work, since

$$\mathbf{y}_2' = \begin{bmatrix} 5 \\ 2 \end{bmatrix} (te^t + e^t), \quad \text{while} \quad \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t.$$

The next theorem shows what to do in this situation.

Theorem 4.5.1

Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 2 and the associated eigenspace has dimension 1; that is, all λ_1-eigenvectors of A are scalar multiples of an eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \tag{4.5.2}$$

Moreover, if $\mathbf{u}$ is any such vector then

$$\mathbf{y}_1 = \mathbf{x}e^{\lambda_1 t} \quad \text{and} \quad \mathbf{y}_2 = \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t} \tag{4.5.3}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

A complete proof of this theorem is beyond the scope of this book. The difficulty is in proving that there's a vector $\mathbf{u}$ satisfying (4.5.2), since $\det(A - \lambda_1 I) = 0$. We'll take this without proof and verify the other assertions of the theorem.

We already know that $\mathbf{y}_1$ in (4.5.3) is a solution of $\mathbf{y}' = A\mathbf{y}$. To see that $\mathbf{y}_2$ is also a solution, we compute

$$\begin{aligned} \mathbf{y}_2' - A\mathbf{y}_2 &= \lambda_1 \mathbf{u} e^{\lambda_1 t} + \mathbf{x} e^{\lambda_1 t} + \lambda_1 \mathbf{x} t e^{\lambda_1 t} - A\mathbf{u} e^{\lambda_1 t} - A\mathbf{x} t e^{\lambda_1 t} \\ &= (\lambda_1 \mathbf{u} + \mathbf{x} - A\mathbf{u}) e^{\lambda_1 t} + (\lambda_1 \mathbf{x} - A\mathbf{x}) t e^{\lambda_1 t}. \end{aligned}$$

Since $A\mathbf{x} = \lambda_1 \mathbf{x}$, this can be written as

$$\mathbf{y}_2' - A\mathbf{y}_2 = -\left((A - \lambda_1 I)\mathbf{u} - \mathbf{x}\right) e^{\lambda_1 t},$$

and now (4.5.2) implies that $\mathbf{y}_2' = A\mathbf{y}_2$.

To see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent, suppose c_1 and c_2 are constants such that

$$c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 = c_1 \mathbf{x} e^{\lambda_1 t} + c_2 (\mathbf{u} e^{\lambda_1 t} + \mathbf{x} t e^{\lambda_1 t}) = \mathbf{0}. \tag{4.5.4}$$

We must show that $c_1 = c_2 = 0$. Multiplying (4.5.4) by $e^{-\lambda_1 t}$ shows that

$$c_1 \mathbf{x} + c_2 (\mathbf{u} + \mathbf{x} t) = \mathbf{0}. \tag{4.5.5}$$

By differentiating this with respect to t, we see that $c_2 \mathbf{x} = \mathbf{0}$, which implies $c_2 = 0$, because $\mathbf{x} \neq \mathbf{0}$. Substituting $c_2 = 0$ into (4.5.5) yields $c_1 \mathbf{x} = \mathbf{0}$, which implies that $c_1 = 0$, again because $\mathbf{x} \neq \mathbf{0}$

Example 4.5.2

Use Theorem (4.5.1) to find the general solution of the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y} \tag{4.5.6}$$

considered in Example (4.5.1).

Answer

In Example (4.5.1) we saw that $\lambda_1 = 1$ is an eigenvalue of multiplicity 2 of the coefficient matrix A in (4.5.6), and that all of the eigenvectors of A are multiples of

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t$$

is a solution of (4.5.6). From Theorem (4.5.1), a second solution is given by $\mathbf{y}_2 = \mathbf{u} e^t + \mathbf{x} t e^t$, where $(A - I)\mathbf{u} = \mathbf{x}$. The augmented matrix of this system is

$$\left[\begin{array}{rrcr} 10 & -25 & \vdots & 5 \\ 4 & -10 & \vdots & 2 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc}1 & -\frac{5}{2} & \vdots & \frac{1}{2} \\ 0 & 0 & \vdots & 0\end{array}\right].$$

Therefore the components of $\mathbf{u}$ must satisfy

$$u_1-\frac{5}{2}u_2=\frac{1}{2},$$

where u_2 is arbitrary. We choose $u_2=0$, so that $u_1=1/2$ and

$$\mathbf{u}=\left[\begin{array}{c}\frac{1}{2} \\ 0\end{array}\right].$$

Thus,

$$\mathbf{y}_2=\left[\begin{array}{l}1 \\ 0\end{array}\right]\frac{e^t}{2}+\left[\begin{array}{l}5 \\ 2\end{array}\right]e^t.$$

Since $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent by Theorem $(4.5.1)$, they form a fundamental set of solutions of (4.5.6). Therefore the general solution of (4.5.6) is

$$\mathbf{y}=c_1\left[\begin{array}{l}5 \\ 2\end{array}\right]e^t+c_2\left(\left[\begin{array}{l}1 \\ 0\end{array}\right]\frac{e^t}{2}+\left[\begin{array}{l}5 \\ 2\end{array}\right]te^t\right)$$

Note that choosing the arbitrary constant u_2 to be nonzero is equivalent to adding a scalar multiple of $\mathbf{y}_1$ to the second solution $\mathbf{y}_2$ (Exercise $(4.5E.33)$).

Example $4.5.3$

Find the general solution of

$$\mathbf{y}'=\left[\begin{array}{ccc}3 & 4 & -10 \\ 2 & 1 & -2 \\ 2 & 2 & -5\end{array}\right]\mathbf{y}. \tag{4.5.7}$$

Answer

The characteristic polynomial of the coefficient matrix A in (4.5.7) is

$$\left[\begin{array}{ccc}3-\lambda & 4 & -10 \\ 2 & 1-\lambda & -2 \\ 2 & 2 & -5-\lambda\end{array}\right]=-(\lambda-1)(\lambda+1)^2.$$

Hence, the eigenvalues are $\lambda_1=1$ with multiplicity 1 and $\lambda_2=-1$ with multiplicity 2.

Eigenvectors associated with $\lambda_1=1$ must satisfy $(A-I)\mathbf{x}=\mathbf{0}$. The augmented matrix of this system is

$$\begin{bmatrix} 2 & 4 & -10 & \vdots & 0 \\ 2 & 0 & -2 & \vdots & 0 \\ 2 & 2 & -6 & \vdots & 0 \end{bmatrix}$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_3$ and $x_2 = 2x_3$, where x_3 is arbitrary. Choosing $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^t$$

is a solution of (4.5.7).

Eigenvectors associated with $\lambda_2 = -1$ satisfy $(A + I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\begin{bmatrix} 4 & 4 & -10 & \vdots & 0 \\ 2 & 2 & -2 & \vdots & 0 \\ 2 & 2 & -4 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_3 = 0$ and $x_1 = -x_2$, where x_2 is arbitrary. Choosing $x_2 = 1$ yields the eigenvector

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

so

$$\mathbf{y}_2=\begin{bmatrix}-1\\1\\0\end{bmatrix}e^{-t}$$

is a solution of (4.5.7).

Since all the eigenvectors of A associated with $\lambda_2=-1$ are multiples of $\mathbf{x}_2$, we must now use Theorem $(4.5.1)$ to find a third solution of (4.5.7) in the form

$$\mathbf{y}_3=\mathbf{u}e^{-t}+\begin{bmatrix}-1\\1\\0\end{bmatrix}te^{-t}, \tag{4.5.8}$$

where $\mathbf{u}$ is a solution of $(A+I)\mathbf{u}=\mathbf{x}_2$. The augmented matrix of this system is

$$\begin{bmatrix}4 & 4 & -10 & \vdots & -1\\ 2 & 2 & -2 & \vdots & 1\\ 2 & 2 & -4 & \vdots & 0\end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix}1 & 1 & 0 & \vdots & 0\\ 0 & 0 & 1 & \vdots & \frac{1}{2}\\ 0 & 0 & 0 & \vdots & 0\end{bmatrix}.$$

Hence, $u_3=1/2$ and $u_1=1-u_2$, where u_2 is arbitrary. Choosing $u_2=0$ yields

$$\mathbf{u}=\begin{bmatrix}1\\0\\\frac{1}{2}\end{bmatrix},$$

and substituting this into (4.5.8) yields the solution

$$\mathbf{y}_3=\begin{bmatrix}2\\0\\1\end{bmatrix}\frac{e^{-t}}{2}+\begin{bmatrix}-1\\1\\0\end{bmatrix}te^{-t}$$

of (4.5.7).

Since the Wronskian of $\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\}$ at $t=0$ is

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2},$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (4.5.7). Therefore the general solution of (4.5.7) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t} \right).$$

Theorem 4.5.2

Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 3 and the associated eigenspace is one-dimensional; that is, all eigenvectors associated with λ_1 are scalar multiples of the eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}, \tag{4.5.9}$$

and, if $\mathbf{u}$ is any such vector, there are infinitely many vectors $\mathbf{v}$ such that

$$(A - \lambda_1 I)\mathbf{v} = \mathbf{u}. \tag{4.5.10}$$

If $\mathbf{u}$ satisfies {\rm(4.5.9)} and $\mathbf{v}$ satisfies (4.5.10), then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{v}e^{\lambda_1 t} + \mathbf{u}te^{\lambda_1 t} + \mathbf{x}\frac{t^2 e^{\lambda_1 t}}{2} \end{aligned}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

Again, it's beyond the scope of this book to prove that there are vectors $\mathbf{u}$ and $\mathbf{v}$ that satisfy (4.5.9) and (4.5.10). Theorem (4.5.1) implies that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{y}' = A\mathbf{y}$. We leave the rest of the proof to you (Exercise $(4.5E.34)$).

Example 4.5.4

Use Theorem (4.5.2) to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} \mathbf{y}. \tag{4.5.11}$$

Answer

The characteristic polynomial of the coefficient matrix A in (4.5.11) is

$$\left|\begin{array}{lll}1-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 0 & 2 & -\lambda\end{array}\right|=-(\lambda-2)^3.$$

Hence, $\lambda_1=2$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A-2I)\mathbf{x}=\mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr}-1 & 1 & 1 & \vdots & 0 \\ 1 & 1 & -1 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 0\end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr}1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0\end{array}\right].$$

Hence, $x_1=x_3$ and $x_2=0$, so the eigenvectors are all scalar multiples of

$$\mathbf{x}_1=\left[\begin{array}{l}1 \\ 0 \\ 1\end{array}\right].$$

Therefore

$$\mathbf{y}_1=\left[\begin{array}{l}1 \\ 0 \\ 1\end{array}\right] e^{2t}$$

is a solution of (4.5.11).

We now find a second solution of (4.5.11) in the form

$$\mathbf{y}_2=\mathbf{u} e^{2t}+\left[\begin{array}{l}1 \\ 0 \\ 1\end{array}\right] t e^{2t},$$

where $\mathbf{u}$ satisfies $(A-2I)\mathbf{u}=\mathbf{x}_1$. The augmented matrix of this system is

$$\begin{bmatrix} -1 & 1 & 1 & \vdots & 1 \\ 1 & 1 & -1 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 1 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & -\frac{1}{2} \\ 0 & 1 & 0 & \vdots & \frac{1}{2} \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Letting $u_3 = 0$ yields $u_1 = -1/2$ and $u_2 = 1/2$; hence,

$$\mathbf{u} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\frac{e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{2t}$$

is a solution of (4.5.11).

We now find a third solution of (4.5.11) in the form

$$\mathbf{y}_3 = \mathbf{v}e^{2t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\frac{te^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\frac{t^2e^{2t}}{2}$$

where $\mathbf{v}$ satisfies $(A-2I)\mathbf{v} = \mathbf{u}$. The augmented matrix of this system is

$$\begin{bmatrix} -1 & 1 & 1 & \vdots & -\frac{1}{2} \\ 1 & 1 & -1 & \vdots & \frac{1}{2} \\ 0 & 2 & 0 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & \frac{1}{2} \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Letting $v_3 = 0$ yields $v_1 = 1/2$ and $v_2 = 0$; hence,

$$\mathbf{v} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\frac{te^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\frac{t^2e^{2t}}{2}$$

is a solution of (4.5.11). Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem (4.5.2), they form a fundamental set of solutions of (4.5.11). Therefore the general solution of (4.5.11) is

$$\begin{aligned} \mathbf{y} = &\ c_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{2t} + c_2\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\frac{e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}te^{2t}\right) \\ &+ c_3\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\frac{te^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\frac{t^2e^{2t}}{2}\right). \end{aligned}$$

Theorem 4.5.3

Suppose the $n\times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 3 and the associated eigenspace is two-dimensional; that is, all eigenvectors of A associated with λ_1 are linear combinations of two linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then there are constants α and β (not both zero) such that if

$$\mathbf{x}_3 = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \tag{4.5.12}$$

then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}_3. \tag{4.5.13}$$

If $\mathbf{u}$ satisfies (4.5.13), then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}_1 e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{x}_2 e^{\lambda_1 t}, \text{and} \\ \mathbf{y}_3 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}_3 te^{\lambda_1 t}, \end{aligned} \tag{4.5.14}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Proof

We omit the proof of this theorem.

Example 4.5.5

Use Theorem $(4.5.3)$ to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} \mathbf{y}. \tag{4.5.15}$$

Answer

The characteristic polynomial of the coefficient matrix A in (4.5.15) is

$$\begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & 1 \\ -1 & 0 & 2-\lambda \end{vmatrix} = -(\lambda-1)^3.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A - I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{rrrcr} -1 & 0 & 1 & \vdots & 0 \\ -1 & 0 & 1 & \vdots & 0 \\ -1 & 0 & 1 & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and x_2 is arbitrary, so the eigenvectors are of the form

$$\mathbf{x}_1 = \begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{4.5.16}$$

form a basis for the eigenspace, and

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t$$

are linearly independent solutions of (4.5.15).

To find a third linearly independent solution of (4.5.15), we must find constants α and β (not both zero) such that the system

$$(A-I)\mathbf{u}=\alpha\mathbf{x}_1+\beta\mathbf{x}_2 \tag{4.5.17}$$

has a solution $\mathbf{u}$. The augmented matrix of this system is

$$\begin{bmatrix} -1 & 0 & 1 & \vdots & \alpha \\ -1 & 0 & 1 & \vdots & \beta \\ -1 & 0 & 1 & \vdots & \alpha \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & -\alpha \\ 0 & 0 & 0 & \vdots & \beta-\alpha \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}. \tag{4.5.18}$$

Therefore (4.5.17) has a solution if and only if $\beta=\alpha$, where α is arbitrary. If $\alpha=\beta=1$ then (4.5.12) and (4.5.16) yield

$$\mathbf{x}_3=\mathbf{x}_1+\mathbf{x}_2=\begin{bmatrix}1\\0\\1\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix},$$

and the augmented matrix (4.5.18) becomes

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

This implies that $u_1=-1+u_3$, while u_2 and u_3 are arbitrary. Choosing $u_2=u_3=0$ yields

$$\mathbf{u}=\begin{bmatrix}-1\\0\\0\end{bmatrix}.$$

Therefore (4.5.14) implies that

$$\mathbf{y}_3=\mathbf{u}e^t+\mathbf{x}_3te^t=\begin{bmatrix}-1\\0\\0\end{bmatrix}e^t+\begin{bmatrix}1\\1\\1\end{bmatrix}te^t$$

is a solution of (4.5.15). Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem $(4.5.3)$, they form a fundamental set of solutions for (4.5.15). Therefore the general solution of (4.5.15) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t \right)$$

4.5.2 Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{4.5.19}$$

under the assumptions of this section; that is, when the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a repeated eigenvalue λ_1 and the associated eigenspace is one-dimensional. In this case we know from Theorem $(4.5.1)$ that the general solution of (4.5.19) is

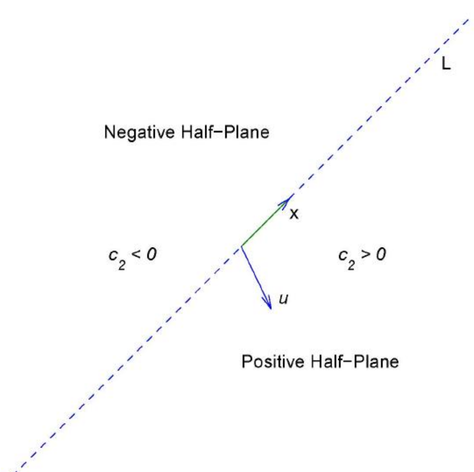
$$\mathbf{y} = c_1 \mathbf{x} e^{\lambda_1 t} + c_2 (\mathbf{u} e^{\lambda_1 t} + \mathbf{x} t e^{\lambda_1 t}), \tag{4.5.20}$$

where $\mathbf{x}$ is an eigenvector of A and $\mathbf{u}$ is any one of the infinitely many solutions of

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \tag{4.5.21}$$

We assume that $\lambda_1 \neq 0$.

Positive and negative half-planes



Let L denote the line through the origin parallel to $\mathbf{x}$. By a half-line of L we mean either of the rays obtained by removing the origin from L. Equation (4.5.20) is a parametric equation of the half-line of L

in the direction of $\mathbf{x}$ if $c_1 > 0$, or of the half-line of |(L\) in the direction of $-\mathbf{x}$ if $c_1 < 0$. The origin is the trajectory of the trivial solution $\mathbf{y} \equiv \mathbf{0}$.

Henceforth, we assume that $c_2 \neq 0$. In this case, the trajectory of (4.5.20) can't intersect L, since every point of L is on a trajectory obtained by setting $c_2 = 0$. Therefore the trajectory of (4.5.20) must lie entirely in one of the open half-planes bounded by L, but does not contain any point on L. Since the initial point $(y_1(0), y_2(0))$ defined by $\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{u}$ is on the trajectory, we can determine which half-plane contains the trajectory from the sign of c_2, as shown in Figure 4.5.1. For convenience we'll call the half-plane where $c_2 > 0$ the positive half-plane. Similarly, the-half plane where $c_2 < 0$ is the negative half-plane. You should convince yourself (Exercise $(4.5E.35)$) that even though there are infinitely many vectors $\mathbf{u}$ that satisfy (4.5.21), they all define the same positive and negative half-planes. In the figures simply regard $\mathbf{u}$ as an arrow pointing to the positive half-plane, since wen't attempted to give $\mathbf{u}$ its proper length or direction in comparison with $\mathbf{x}$. For our purposes here, only the relative orientation of $\mathbf{x}$ and $\mathbf{u}$ is important; that is, whether the positive half-plane is to the right of an observer facing the direction of $\mathbf{x}$ (as in Figures 4.5.2 and 4.5.5), or to the left of the observer (as in Figures 4.5.3 and 4.5.4).

Multiplying (4.5.20) by $e^{-\lambda_1 t}$ yields

$$e^{-\lambda_1 t}\mathbf{y}(t) = c_1\mathbf{x} + c_2\mathbf{u} + c_2 t\mathbf{x}.$$

Since the last term on the right is dominant when $|t|$ is large, this provides the following information on the direction of $\mathbf{y}(t)$:

(a) Along trajectories in the positive half-plane ($c_2 > 0$), the direction of $\mathbf{y}(t)$ approaches the direction of $\mathbf{x}$ as $t \to \infty$ and the direction of $-\mathbf{x}$ as $t \to -\infty$.

(b) Along trajectories in the negative half-plane ($c_2 < 0$), the direction of $\mathbf{y}(t)$ approaches the direction of $-\mathbf{x}$ as $t \to \infty$ and the direction of $\mathbf{x}$ as $t \to -\infty$.

Since

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \mathbf{y}(t) = \mathbf{0} \quad \text{if} \quad \lambda_1 > 0,$$

or

$$\lim_{t-\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0} \quad \text{if} \quad \lambda_1 < 0,$$

there are four possible patterns for the trajectories of (4.5.19), depending upon the signs of c_2 and λ_1. Figures 4.5.2 to 4.5.5 illustrate these patterns, and reveal the following principle:

If λ_1 and c_2 have the same sign then the direction of the trajectory approaches the direction of $-\mathbf{x}$ as $\|\mathbf{y}\| \to 0$ and the direction of $\mathbf{x}$ as $\|\mathbf{y}\| \to \infty$ If λ_1 and c_2 have opposite signs then the direction of the trajectory approaches the direction of $\mathbf{x}$ as $\|\mathbf{y}\| \to 0$ and the direction of $-\mathbf{x}$ as $\|\mathbf{y}\| \to \infty$.

Positive eigenvalue; motion away from the origin

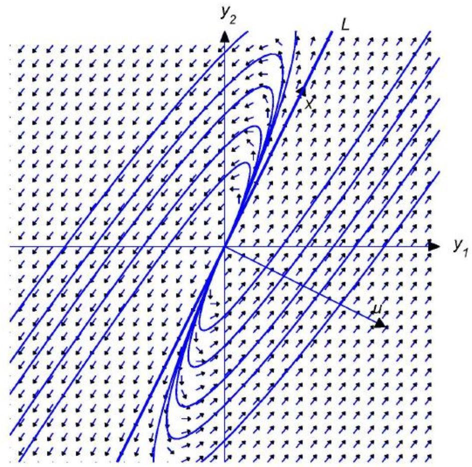


Figure 4.5.3

Positive eigenvalue; motion away from the origin

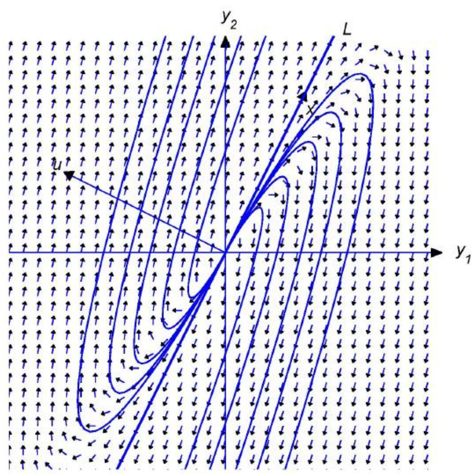
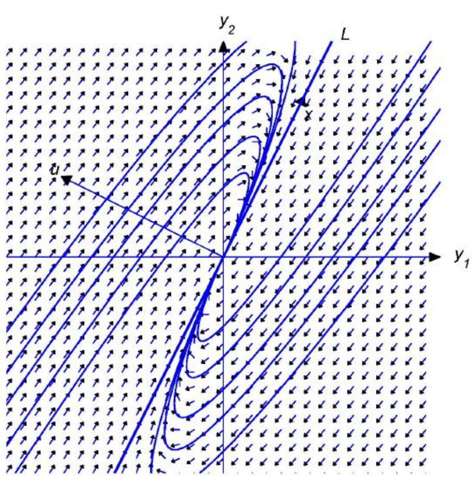
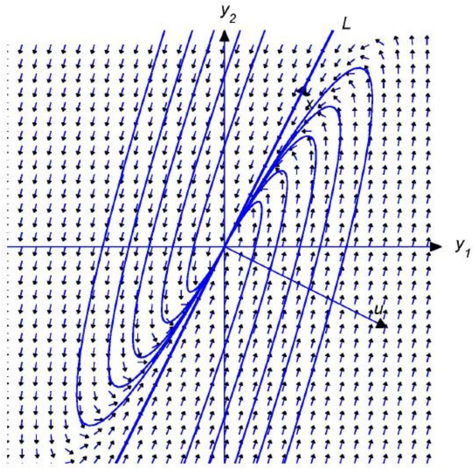


Figure 4.5.4

Negative eigenvalue; motion toward the origin



Negative eigenvalue; motion toward the origin



4.5E: Exercises

This page is a draft and is under active development.

In Exercises $(4.5E.1)$ to $(4.5E.12)$, find the general solution.

4.5E.1 Exercise $\mathbf{4.5E.1}$

$$\mathbf{y}' = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.2 Exercise $\mathbf{4.5E.2}$

$$\mathbf{y}' = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.3 Exercise $\mathbf{4.5E.3}$

$$\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -1 & -11 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.4 Exercise $\mathbf{4.5E.4}$

$$\mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.5 Exercise $\mathbf{4.5E.5}$

$$\mathbf{y}' = \begin{bmatrix} 4 & 12 \\ -3 & -8 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.6 Exercise $\mathbf{4.5E.6}$

$$\mathbf{y}' = \begin{bmatrix} -10 & 9 \\ -4 & 2 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.7 Exercise $\mathbf{4.5E.7}$

$$\mathbf{y}' = \begin{bmatrix} -13 & 16 \\ -9 & 11 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.8 Exercise $\mathbf{4.5E.8}$

$$\mathbf{y}' = \begin{bmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 0 & 4 & 2 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.9 Exercise $\mathbf{4.5E.9}$

$$\mathbf{y}' = \frac{1}{3}\begin{bmatrix} 1 & 1 & -3 \\ -4 & -4 & 3 \\ -2 & 1 & 0 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.10 Exercise $\mathbf{4.5E.10}$

$$\mathbf{y}' = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.11 Exercise $\mathbf{4.5E.11}$

$$\mathbf{y}' = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.12 Exercise $\mathbf{4.5}\boldsymbol{E}.\mathbf{12}$

$$\mathbf{y}' = \begin{bmatrix} 6 & -5 & 3 \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

In Exercises $(4.5E.13)$ to $(4.5E.23)$, solve the initial value problem.

4.5E.13 Exercise $\mathbf{4.5}\boldsymbol{E}.\mathbf{13}$

$$\mathbf{y}' = \begin{bmatrix} -11 & 8 \\ -2 & -3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.5E.14 Exercise $\mathbf{4.5}\boldsymbol{E}.\mathbf{14}$

$$\mathbf{y}' = \begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.5E.15 Exercise $\mathbf{4.5}\boldsymbol{E}.\mathbf{15}$

$$\mathbf{y}' = \begin{bmatrix} -3 & -4 \\ 1 & -7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.5E.16 Exercise $\mathbf{4.5}\boldsymbol{E}.\mathbf{16}$

$$\mathbf{y}' = \begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.5E.17 Exercise $\mathbf{4.5}\boldsymbol{E}.\mathbf{17}$

$$\mathbf{y}' = \begin{bmatrix} -7 & 3 \\ -3 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.5E.18 Exercise $\mathbf{4.5}\boldsymbol{E}\mathbf{.18}$

$$\mathbf{y}' = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 5 \\ -7 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.5E.19 Exercise $\mathbf{4.5}\boldsymbol{E}\mathbf{.19}$

$$\mathbf{y}' = \begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & 1 \\ -3 & 3 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -6 \\ -2 \\ 0 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.5E.20 Exercise $\mathbf{4.5}\boldsymbol{E}\mathbf{.20}$

$$\mathbf{y}' = \begin{bmatrix} -7 & -4 & 4 \\ -1 & 0 & 1 \\ -9 & -5 & 6 \end{bmatrix} \mathbf{y}, \quad \mathbf{y(0)} = \begin{bmatrix} -6 \\ 9 \\ -1 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.5E.21 Exercise $\mathbf{4.5}\boldsymbol{E}\mathbf{.21}$

$$\mathbf{y}' = \begin{bmatrix} -1 & -4 & -1 \\ 3 & 6 & 1 \\ -3 & -2 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y(0)} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.5E.22 Exercise $\mathbf{4.5}\boldsymbol{E}\mathbf{.22}$

$$\mathbf{y}' = \begin{bmatrix} 4 & -8 & -4 \\ -3 & -1 & -3 \\ 1 & -1 & 9 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.5E.23 Exercise $\mathbf{4.5E.23}$

$$\mathbf{y}' = \begin{bmatrix} -5 & -1 & 11 \\ -7 & 1 & 13 \\ -4 & 0 & 8 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

The coefficient matrices in Exercises $(4.5E.24)$ to $(4.5E.32)$ have eigenvalues of multiplicity 3. Find the general solution.

4.5E.24 Exercise $\mathbf{4.5E.24}$

$$\mathbf{y}' = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.25 Exercise $\mathbf{4.5E.25}$

$$\mathbf{y}' = \begin{bmatrix} 1 & 10 & -12 \\ 2 & 2 & 3 \\ 2 & -1 & 6 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.26 Exercise $\mathbf{4.5E.26}$

$$\mathbf{y}' = \begin{bmatrix} -6 & -4 & -4 \\ 2 & -1 & 1 \\ 2 & 3 & 1 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.27 Exercise $\mathbf{4.5E.27}$

$$\mathbf{y}' = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 5 & -3 \\ 1 & 1 & 1 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.28 Exercise $\mathbf{4.5E.28}$

$$\mathbf{y}' = \begin{bmatrix} -2 & -12 & 10 \\ 2 & -24 & 11 \\ 2 & -24 & 8 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.29 Exercise $\mathbf{4.5E.29}$

$$\mathbf{y}' = \begin{bmatrix} -1 & -12 & 8 \\ 1 & -9 & 4 \\ 1 & -6 & 1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.30 Exercise $\mathbf{4.5E.30}$

$$\mathbf{y}' = \begin{bmatrix} -4 & 0 & -1 \\ -1 & -3 & -1 \\ 1 & 0 & -2 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.31 Exercise $\mathbf{4.5E.31}$

$$\mathbf{y}' = \begin{bmatrix} -3 & -3 & 4 \\ 4 & 5 & -8 \\ 2 & 3 & -5 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.32 Exercise $\mathbf{4.5E.32}$

$$\mathbf{y}' = \begin{bmatrix} -3 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.33 Exercise $\mathbf{4.5E.33}$

Under the assumptions of Theorem $(4.5.1)$, suppose $\mathbf{u}$ and $\hat{\mathbf{u}}$ are vectors such that

$$(A-\lambda_1 I)\mathbf{u}=\mathbf{x} \quad \text{and} \quad (A-\lambda_1 I)\hat{\mathbf{u}}=\mathbf{x},$$

and let

$$\mathbf{y}_2=\mathbf{u}e^{\lambda_1 t}+\mathbf{x}te^{\lambda_1 t} \quad \text{and} \quad \hat{\mathbf{y}}_2=\hat{\mathbf{u}}e^{\lambda_1 t}+\mathbf{x}te^{\lambda_1 t}.$$

Show that $\mathbf{y}_2-\hat{\mathbf{y}}_2$ is a scalar multiple of $\mathbf{y}_1=\mathbf{x}e^{\lambda_1 t}$.

Answer

Add texts here. Do not delete this text first.

4.5E.34 Exercise $\mathbf{4.5E.34}$

Under the assumptions of Theorem $(4.5.2)$, let

$$\begin{aligned}
\mathbf{y}_1 &= \mathbf{x}e^{\lambda_1 t}, \\
\mathbf{y}_2 &= \mathbf{u}e^{\lambda_1 t}+\mathbf{x}te^{\lambda_1 t}, \text{ and} \\
\mathbf{y}_3 &= \mathbf{v}e^{\lambda_1 t}+\mathbf{u}te^{\lambda_1 t}+\mathbf{x}\frac{t^2e^{\lambda_1 t}}{2}.
\end{aligned}$$

Complete the proof of Theorem $(4.5.2)$ by showing that $\mathbf{y}_3$ is a solution of $\mathbf{y}'=A\mathbf{y}$ and that $\{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\}$ is linearly independent.

Answer

Add texts here. Do not delete this text first.

4.5E.35 Exercise $\mathbf{4.5E.35}$

Suppose the matrix

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

has a repeated eigenvalue λ_1 and the associated eigenspace is one-dimensional. Let $\mathbf{x}$ be a λ_1-eigenvector of A. Show that if $(A-\lambda_1 I)\mathbf{u}_1=\mathbf{x}$ and $(A-\lambda_1 I)\mathbf{u}_2=\mathbf{x}$, then $\mathbf{u}_2-\mathbf{u}_1$ is parallel to $\mathbf{x}$. Conclude from this that all vectors $\mathbf{u}$ such that $(A-\lambda_1 I)\mathbf{u}=\mathbf{x}$ define the same positive and negative half-planes with respect to the line L through the origin parallel to $\mathbf{x}$.

Answer

Add texts here. Do not delete this text first.

In Exercises $(4.5E.36)$ to $(4.5E.45)$, plot trajectories of the given system.

4.5E.36 Exercise $\mathbf{4.5E.36}$

$\mathbf{y}'=\begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix}\mathbf{y}$

Answer

Add texts here. Do not delete this text first.

4.5E.37 Exercise $\mathbf{4.5E.37}$

$$\mathbf{y}' = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.38 Exercise $\mathbf{4.5E.38}$

$$\mathbf{y}' = \begin{bmatrix} -1 & -3 \\ 3 & 5 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.39 Exercise $\mathbf{4.5E.39}$

$$\mathbf{y}' = \begin{bmatrix} -5 & 3 \\ -3 & 1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.40 Exercise $\mathbf{4.5E.40}$

$$\mathbf{y}' = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.41 Exercise $\mathbf{4.5E.41}$

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.42 Exercise $\mathbf{4.5E.42}$

$$\mathbf{y}' = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.43 Exercise $\mathbf{4.5E.43}$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.44 Exercise $\mathbf{4.5E.44}$

$$\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.5E.45 Exercise $\mathbf{4.5E.45}$

$$\mathbf{y}' = \begin{bmatrix} 0 & -4 \\ 1 & -4 \end{bmatrix}\mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6: Constant Coefficient Homogeneous Systems III

This page is a draft and is under active development.

4.6.1 Constant Coefficient Homogeneous Systems III

We now consider the system $\mathbf{y}' = A\mathbf{y}$, where A has a complex eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$. We continue to assume that A has real entries, so the characteristic polynomial of A has real coefficients. This implies that $overline\lambda = \alpha - i\beta$ is also an eigenvalue of A.

An eigenvector $\mathbf{x}$ of A associated with $\lambda = \alpha + i\beta$ will have complex entries, so we'll write

$$\mathbf{x} = \mathbf{u} + i\mathbf{v}$$

where $\mathbf{u}$ and $\mathbf{v}$ have real entries; that is, $\mathbf{u}$ and $\mathbf{v}$ are the real and imaginary parts of $\mathbf{x}$. Since $A\mathbf{x} = \lambda\mathbf{x}$,

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha + i\beta)(\mathbf{u} + i\mathbf{v}). \tag{4.6.1}$$

Taking complex conjugates here and recalling that A has real entries yields

$$A(\mathbf{u} - i\mathbf{v}) = (\alpha - i\beta)(\mathbf{u} - i\mathbf{v}),$$

which shows that $\mathbf{x} = \mathbf{u} - i\mathbf{v}$ is an eigenvector associated with $\overline{\lambda} = \alpha - i\beta$. The complex conjugate eigenvalues λ and $\overline{\lambda}$ can be separately associated with linearly independent solutions $\mathbf{y}' = A\mathbf{y}$; however, we won't pursue this approach, since solutions obtained in this way turn out to be complex-valued. Instead, we'll obtain solutions of $\mathbf{y}' = A\mathbf{y}$ in the form

$$\mathbf{y} = f_1\mathbf{u} + f_2\mathbf{v} \tag{4.6.2}$$

where f_1 and f_2 are real-valued scalar functions. The next theorem shows how to do this.

4.6.1.1 Theorem **4.6.1**

Let A be an $n \times n$ matrix with real entries. Let $\lambda = \alpha + i\beta$ $\beta \neq 0$ be a complex eigenvalue of A and let $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ be an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. Then
$\mathbf{u}$ and $\mathbf{v}$ are both nonzero and

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t),$$

which are the real and imaginary parts of

$$e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u} + i\mathbf{v}), \tag{4.6.3}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Proof

A function of the form (4.6.2) is a solution of $\mathbf{y}' = A\mathbf{y}$ if and only if

$$f_1'\mathbf{u} + f_2'\mathbf{v} = f_1A\mathbf{u} + f_2A\mathbf{v}. \tag{4.6.4}$$

Carrying out the multiplication indicated on the right side of (4.6.1) and collecting the real and imaginary parts of the result yields

$$A(\mathbf{u}+i\mathbf{v})=(\alpha\mathbf{u}-\beta\mathbf{v})+i(\alpha\mathbf{v}+\beta\mathbf{u}).$$

Equating real and imaginary parts on the two sides of this equation yields

$$\begin{array}{rcl} A\mathbf{u} & = & \alpha\mathbf{u}-\beta\mathbf{v} \\ A\mathbf{v} & = & \alpha\mathbf{v}+\beta\mathbf{u} \end{array}.$$

We leave it to you (Exercise $(4.6E.25)$) to show from this that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero. Substituting from these equations into (4.6.4) yields

$$\begin{aligned} f_1'\mathbf{u}+f_2'\mathbf{v} &= f_1(\alpha\mathbf{u}-\beta\mathbf{v})+f_2(\alpha\mathbf{v}+\beta\mathbf{u}) \\ &= (\alpha f_1+\beta f_2)\mathbf{u}+(-\beta f_1+\alpha f_2)\mathbf{v}. \end{aligned}$$

This is true if

$$\begin{array}{rcl} f_1' & = & \alpha f_1+\beta f_2 \\ f_2' & = & -\beta f_1+\alpha f_2, \end{array} \quad \text{or, equivalently,} \quad \begin{array}{rcl} {}_1'-\alpha f_1 & = & \beta f_2 \\ f_2'-\alpha f_2 & = & -\beta f_1. \end{array}$$

If we let $f_1=g_1e^{\alpha t}$ and $f_2=g_2e^{\alpha t}$, where g_1 and g_2 are to be determined, then the last two equations become

$$\begin{array}{rcl} g_1' & = & \beta g_2 \\ g_2' & = & -\beta g_1, \end{array}$$

which implies that

$$g_1''=\beta g_2'=-\beta^2 g_1,$$

so

$$g_1''+\beta^2 g_1=0$$

The general solution of this equation is

$$g_1=c_1\cos\beta t+c_2\sin\beta t.$$

Moreover, since $g_2=g_1'/\beta$,

$$g_2=-c_1\sin\beta t+c_2\cos\beta t.$$

Multiplying g_1 and g_2 by $e^{\alpha t}$ shows that

$$\begin{aligned} f_1 &= e^{\alpha t}(\ \ c_1\cos\beta t+c_2\sin\beta t), \\ f_2 &= e^{\alpha t}(-c_1\sin\beta t+c_2\cos\beta t). \end{aligned}$$

Substituting these into (4.6.2) shows that

$$\begin{aligned}\mathbf{y} &= e^{\alpha t}\left[(c_1\cos\beta t + c_2\sin\beta t)\mathbf{u} + (-c_1\sin\beta t + c_2\cos\beta t)\mathbf{v}\right]\\ &= c_1e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) + c_2e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t)\end{aligned} \tag{4.6.5}$$

is a solution of $\mathbf{y}' = A\mathbf{y}$ for any choice of the constants c_1 and c_2. In particular, by first taking $c_1 = 1$ and $c_2 = 0$ and then taking $c_1 = 0$ and $c_2 = 1$, we see that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of $\mathbf{y}' = A\mathbf{y}$. We leave it to you to verify that they are, respectively, the real and imaginary parts of (4.6.3) (Exercise $(4.6E.26)$), and that they are linearly independent (Exercise $(4.6E.27)$).

Example 4.6.1

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 4 & -5 \\ 5 & -2 \end{bmatrix}\mathbf{y}. \tag{4.6.6}$$

Answer

The characteristic polynomial of the coefficient matrix A in (4.6.6) is

$$\begin{vmatrix} 4-\lambda & -5 \\ 5 & -2-\lambda \end{vmatrix} = (\lambda-1)^2 + 16.$$

Hence, $\lambda = 1+4i$ is an eigenvalue of A. The associated eigenvectors satisfy $(A-(1+4i)I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\begin{bmatrix} 3-4i & -5 & \vdots & 0 \\ 5 & -3-4i & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & -\frac{3+4i}{5} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = (3+4i)x_2/5$. Taking $x_2 = 5$ yields $x_1 = 3+4i$, so

$$\mathbf{x} = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 4t + i\sin 4t)\begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t\begin{bmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t\begin{bmatrix} 3\sin 4t + 4\cos 4t \\ 5\sin 4t \end{bmatrix},$$

which are linearly independent solutions of (4.6.6). The general solution of (4.6.6) is

$$\mathbf{y} = c_1 e^t \begin{bmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{bmatrix} + c_2 e^t \begin{bmatrix} 3\sin 4t + 4\cos 4t \\ 5\sin 4t \end{bmatrix}.$$

Example 4.6.2

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} \mathbf{y}. \tag{4.6.7}$$

Answer

The characteristic polynomial of the coefficient matrix A in (4.6.7) is

$$\begin{vmatrix} -14-\lambda & 39 \\ -6 & 16-\lambda \end{vmatrix} = (\lambda-1)^2 + 9.$$

Hence, $\lambda = 1+3i$ is an eigenvalue of A. The associated eigenvectors satisfy $(A-(1+3i)I)\mathbf{x} = \mathbf{0}$. The augmented augmented matrix of this system is

$$\begin{bmatrix} -15-3i & 39 & \vdots & 0 \\ -6 & 15-3i & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & \frac{-5+i}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = (5-i)/2$. Taking $x_2 = 2$ yields $x_1 = 5-i$, so

$$\mathbf{x} = \begin{bmatrix} 5-i \\ 2 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 3t + i\sin 3t)\begin{bmatrix} 5-i \\ 2 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t \begin{bmatrix} \sin 3t + 5\cos 3t \\ 2\cos 3t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t \begin{bmatrix} -\cos 3t + 5\sin 3t \\ 2\sin 3t \end{bmatrix},$$

which are linearly independent solutions of (4.6.7). The general solution of (4.6.7) is

$$\mathbf{y} = c_1 e^t \begin{bmatrix} \sin 3t + 5\cos 3t \\ 2\cos 3t \end{bmatrix} + c_2 e^t \begin{bmatrix} -\cos 3t + 5\sin 3t \\ 2\sin 3t \end{bmatrix}.$$

Example 4.6.3

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -5 & 5 & 4 \\ -8 & 7 & 6 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{y}. \tag{4.6.8}$$

Answer

The characteristic polynomial of the coefficient matrix A in (4.6.8) is

$$\begin{vmatrix} -5-\lambda & 5 & 4 \\ -8 & 7-\lambda & 6 \\ 1 & 0 & -\lambda \end{vmatrix} = -(\lambda-2)(\lambda^2+1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = i$, and $\lambda_3 = -i$. The augmented matrix of $(A-2I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -7 & 5 & 4 & \vdots & 0 \\ -8 & 5 & 6 & \vdots & 0 \\ 1 & 0 & -2 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -2 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = x_2 = 2x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 2 \\ 2 \\ 1e^{2t} \end{bmatrix}$$

is a solution of (4.6.8).

The augmented matrix of $(A-iI)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{cccc} -5-i & 5 & 4 & \vdots & 0 \\ -8 & 7-i & 6 & \vdots & 0 \\ 1 & 0 & -i & \vdots & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccccc} 1 & 0 & -i & \vdots & 0 \\ 0 & 1 & 1-i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right].$$

Therefore $x_1 = ix_3$ and $x_2 = -(1-i)x_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} i \\ -1+i \\ 1 \end{bmatrix}.$$

The real and imaginary parts of

$$(\cos t + i \sin t)\begin{bmatrix} i \\ -1+i \\ 1 \end{bmatrix}$$

are

$$\mathbf{y}_2 = \begin{bmatrix} -\sin t \\ -\cos t - \sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} \cos t \\ \cos t - \sin t \\ \sin t \end{bmatrix},$$

which are solutions of (4.6.8). Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} 2 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (4.6.8). The general solution of (4.6.8) is

$$\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 2 \\ 1e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -\sin t \\ -\cos t - \sin t \\ \cos t \end{bmatrix} + c_3 \begin{bmatrix} \cos t \\ \cos t - \sin t \\ \sin t \end{bmatrix}.$$

Example4.6.4

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix} \mathbf{y}. \tag{4.6.9}$$

Answer

The characteristic polynomial of the coefficient matrix A in (4.6.9) is

$$\begin{vmatrix} 1-\lambda & -1 & -2 \\ 1 & 3-\lambda & 2 \\ 1 & -1 & 2-\lambda \end{vmatrix} = -(\lambda-2)\left((\lambda-2)^2+4\right).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 2+2i$, and $\lambda_3 = 2-2i$. The augmented matrix of $(A-2I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -1 & -1 & -2 & \vdots & 0 \\ 1 & 1 & 2 & \vdots & 0 \\ 1 & -1 & 0 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 1e^{2t} \end{bmatrix}$$

is a solution of (4.6.9).

The augmented matrix of $(A-(2+2i)I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -1-2i & -1 & -2 & \vdots & 0 \\ 1 & 1-2i & 2 & \vdots & 0 \\ 1 & -1 & -2 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\left[\begin{array}{ccccc}1 & 0 & -i & \vdots & 0 \\ 0 & 1 & i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0\end{array}\right].$$

Therefore $x_1 = ix_3$ and $x_2 = -ix_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \left[\begin{array}{l} i \\ -i \\ 1 \end{array}\right]$$

The real and imaginary parts of

$$e^{2t}(\cos 2t + i\sin 2t)\left[\begin{array}{l} i \\ -i \\ 1 \end{array}\right]$$

are

$$\mathbf{y}_2 = e^{2t}\left[\begin{array}{l} -\sin 2t \\ \sin 2t \\ \cos 2t \end{array}\right] \quad \text{and} \quad \mathbf{y}_2 = e^{2t}\left[\begin{array}{l} \cos 2t \\ -\cos 2t \\ \sin 2t \end{array}\right],$$

which are solutions of (4.6.9). Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\left|\begin{array}{lll} -1 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{array}\right| = -2,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (4.6.9). The general solution of (4.6.9) is

$$\mathbf{y} = c_1\left[\begin{array}{l} -1 \\ -1 \\ 1e^{2t} \end{array}\right] + c_2e^{2t}\left[\begin{array}{l} -\sin 2t \\ \sin 2t \\ \cos 2t \end{array}\right] + c_3e^{2t}\left[\begin{array}{l} \cos 2t \\ -\cos 2t \\ \sin 2t \end{array}\right].$$

4.6.2 Geometric Properties of Solutions when $n = 2$

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\left[\begin{array}{l} y_1' \\ y_2' \end{array}\right] = \left[\begin{array}{ll} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right]\left[\begin{array}{l} y_1 \\ y_2 \end{array}\right] \tag{4.6.10}$$

under the assumptions of this section; that is, when the matrix

$$A = \left[\begin{array}{ll} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right]$$

has a complex eigenvalue $\lambda=\alpha+i\beta\ \beta\neq 0\{\rm\})$ and $\mathbf{x}=\mathbf{u}+i\mathbf{v}$ is an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. To describe the trajectories accurately it's necessary to introduce a new rectangular coordinate system in the y_1-y_2 plane. This raises a point that hasn't come up before: It is always possible to choose $\mathbf{x}$ so that $(\mathbf{u},\mathbf{v})=0$. A special effort is required to do this, since not every eigenvector has this property. However, if we know an eigenvector that doesn't, we can multiply it by a suitable complex constant to obtain one that does. To see this, note that if $\mathbf{x}$ is a λ-eigenvector of A and k is an arbitrary real number, then

$$\mathbf{x}_1=(1+ik)\mathbf{x}=(1+ik)(\mathbf{u}+i\mathbf{v})=(\mathbf{u}-k\mathbf{v})+i(\mathbf{v}+k\mathbf{u})$$

is also a λ-eigenvector of A, since

$$A\mathbf{x}_1=A((1+ik)\mathbf{x})=(1+ik)A\mathbf{x}=(1+ik)\lambda\mathbf{x}=\lambda((1+ik)\mathbf{x})=\lambda\mathbf{x}_1.$$

The real and imaginary parts of $\mathbf{x}_1$ are

$$\mathbf{u}_1=\mathbf{u}-k\mathbf{v}\quad\text{and}\quad\mathbf{v}_1=\mathbf{v}+k\mathbf{u},\tag{4.6.11}$$

so

$$(\mathbf{u}_1,\mathbf{v}_1)=(\mathbf{u}-k\mathbf{v},\mathbf{v}+k\mathbf{u})=-\left[(\mathbf{u},\mathbf{v})k^2+(\|\mathbf{v}\|^2-\|\mathbf{u}\|^2k-(\mathbf{u},\mathbf{v})\right].$$

Therefore $(\mathbf{u}_1,\mathbf{v}_1)=0$ if

$$(\mathbf{u},\mathbf{v})k^2+(\|\mathbf{v}\|^2-\|\mathbf{u}\|^2)k-(\mathbf{u},\mathbf{v})=0.\tag{4.6.12}$$

If $(\mathbf{u},\mathbf{v})\neq 0$ we can use the quadratic formula to find two real values of k such that $(\mathbf{u}_1,\mathbf{v}_1)=0$ (Exercise $(4.6E.28)$).

4.6.2.1 Example $\mathbf{4.6.5}$:

In Example $(4.6.1)$, we found the eigenvector

$$\mathbf{x}-\begin{bmatrix}3+4i\\5\end{bmatrix}=\begin{bmatrix}3\\5\end{bmatrix}+i\begin{bmatrix}4\\0\end{bmatrix}$$

for the matrix of the system (4.6.6). Here $\mathbf{u}=\begin{bmatrix}3\\5\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}4\\0\end{bmatrix}$ are not orthogonal, since $(\mathbf{u},\mathbf{v})=12$. Since $\|\mathbf{v}\|^2-\|\mathbf{u}\|^2=-18$, (4.6.12) is equivalent to

$$2k^2-3k-1=0.$$

The zeros of this equation are $k_1=2$ and $k_2=-1/2$. Letting $k=2$ in (4.6.11) yields

$$\mathbf{u}_1=\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-5\\5\end{bmatrix}\quad\text{and}\quad\mathbf{v}_1=\mathbf{v}+2\mathbf{u}=\begin{bmatrix}10\\10\end{bmatrix},$$

and $(\mathbf{u}_1,\mathbf{v}_1)=0$. Letting $k=-1/2$ in (4.6.11) yields

$$\mathbf{u}_1=\mathbf{u}+\frac{\mathbf{v}}{2}=\begin{bmatrix}5\\5\end{bmatrix}\quad\text{and}\quad\mathbf{v}_1=\mathbf{v}-\frac{\mathbf{u}}{2}=\frac{1}{2}\begin{bmatrix}-5\\5\end{bmatrix},$$

and again $(\mathbf{u}_1, \mathbf{v}_1) = 0$.

(The numbers don't always work out as nicely as in this example. You'll need a calculator or computer to do Exercises $(4.6E.29)$ to $(4.6E.40)$.)

Henceforth, we'll assume that $(\mathbf{u}, \mathbf{v}) = 0$. Let $\mathbf{U}$ and $\mathbf{V}$ be unit vectors in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively; that is, $\mathbf{U} = \mathbf{u}/\|\mathbf{u}\|$ and $\mathbf{V} = \mathbf{v}/\|\mathbf{v}\|$. The new rectangular coordinate system will have the same origin as the y_1-y_2 system. The coordinates of a point in this system will be denoted by (z_1, z_2), where z_1 and z_2 are the displacements in the directions of $\mathbf{U}$ and $\mathbf{V}$, respectively.

From (4.6.5), the solutions of (4.6.10) are given by

$$\mathbf{y} = e^{\alpha t}\left[(c_1 \cos\beta t + c_2 \sin\beta t)\mathbf{u} + (-c_1 \sin\beta t + c_2 \cos\beta t)\mathbf{v}\right]. \tag{4.6.13}$$

For convenience, let's call the curve traversed by $e^{-\alpha t}\mathbf{y}(t)$ a shadow trajectory of (4.6.10). Multiplying (4.6.13) by $e^{-\alpha t}$ yields

$$e^{-\alpha t}\mathbf{y}(t) = z_1(t)\mathbf{U} + z_2(t)\mathbf{V},$$

where

$$\begin{aligned} z_1(t) &= \|\mathbf{u}\|(c_1 \cos\beta t + c_2 \sin\beta t) \\ z_2(t) &= \|\mathbf{v}\|(-c_1 \sin\beta t + c_2 \cos\beta t). \end{aligned}$$

Therefore

$$\frac{(z_1(t))^2}{\|\mathbf{u}\|^2} + \frac{(z_2(t))^2}{\|\mathbf{v}\|^2} = c_1^2 + c_2^2$$

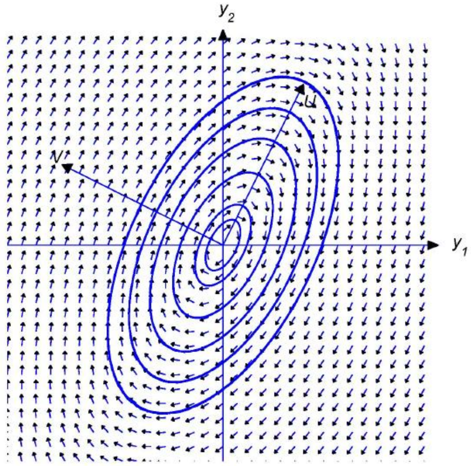
(verify!), which means that the shadow trajectories of (4.6.10) are ellipses centered at the origin, with axes of symmetry parallel to $\mathbf{U}$ and $\mathbf{V}$. Since

$$z_1' = \frac{\beta\|\mathbf{u}\|}{\|\mathbf{v}\|} z_2 \quad \text{and} \quad z_2' = -\frac{\beta\|\mathbf{v}\|}{\|\mathbf{u}\|} z_1,$$

the vector from the origin to a point on the shadow ellipse rotates in the same direction that $\mathbf{V}$ would have to be rotated by $\pi/2$ radians to bring it into coincidence with $\mathbf{U}$ (Figures 4.6.1 and 4.6.2).

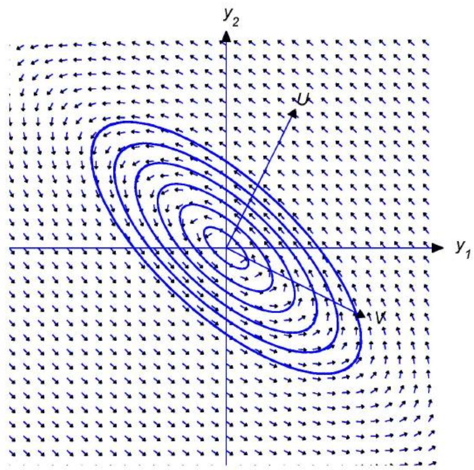
4.6.2.2 Figure **4.6.1**

Shadow trajectories traversed clockwise



4.6.2.3 Figure $\mathbf{4.6.2}$

Shadow trajectories traversed counterclockwise



Figures 4.6.1 and 4.6.2.

If $\alpha > 0$, then

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \mathbf{y}(t) = 0,$$

so the trajectory spirals away from the origin as t varies from $-\infty$ to ∞. The direction of the spiral depends upon the relative orientation of $\mathbf{U}$ and $\mathbf{V}$, as shown in Figures 4.6.3 and 4.6.4.

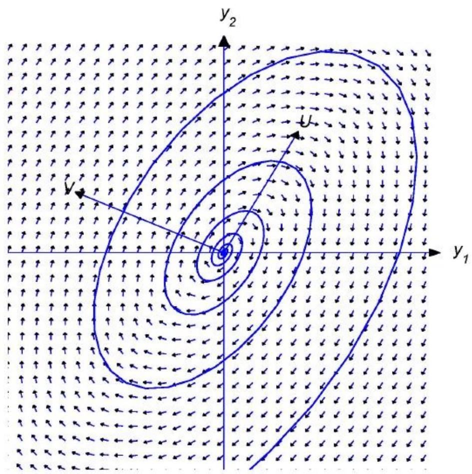
If $\alpha < 0$, then

$$\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \mathbf{y}(t) = 0,$$

so the trajectory spirals toward the origin as t varies from $-\infty$ to ∞. Again, the direction of the spiral depends upon the relative orientation of $\mathbf{U}$ and $\mathbf{V}$, as shown in Figures 4.6.5 and 4.6.6.

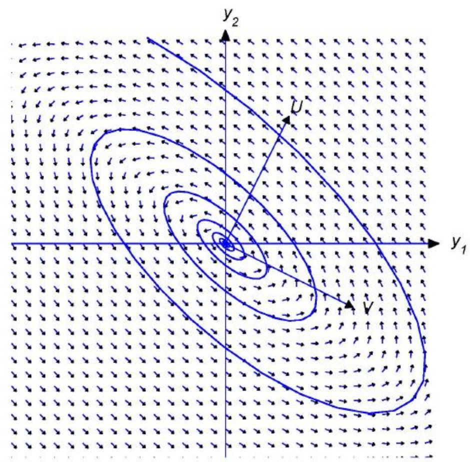
4.6.2.4 Figure $\mathbf{4.6.3}$

$\alpha > 0$; shadow trajectory spiraling outward



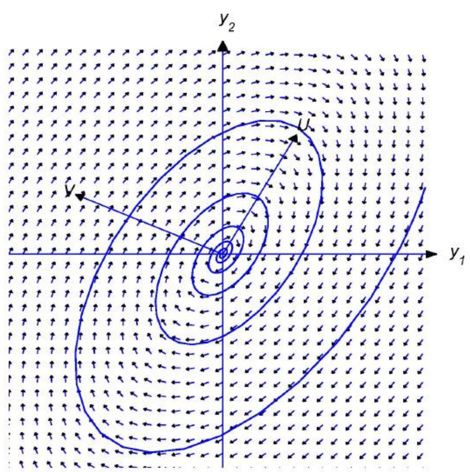
4.6.2.5 Figure $\mathbf{4.6.4}$

$\alpha > 0$; shadow trajectory spiraling outward



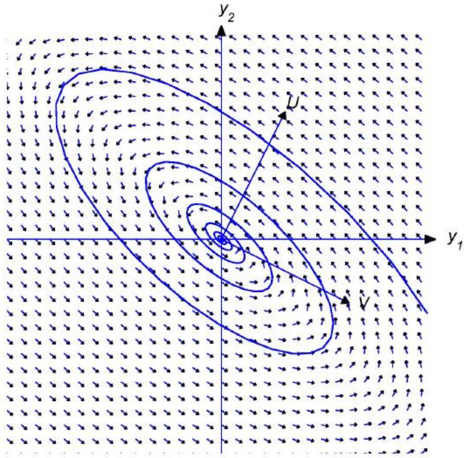
4.6.2.6 Figure $\mathbf{4.6.5}$

$\alpha < 0$; shadow trajectory spiraling inward



4.6.2.7 Figure $\PageIndex{6}$

$\alpha < 0$; shadow trajectory spiraling inward



4.6E: Exercises

This page is a draft and is under active development.

In Exercises $(4.6E.1)$ to $(4.6E.16)$, find the general solution.

4.6E.1 Exercise $\mathbf{4.6E.1}$

$$\mathbf{y}' = \begin{bmatrix} -1 & 2 \\ -5 & 5 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.2 Exercise $\mathbf{4.6E.2}$

$$\mathbf{y}' = \begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.3 Exercise $\mathbf{4.6E.3}$

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.4 Exercise $\mathbf{4.6E.4}$

$$\mathbf{y}' = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.5 Exercise $\mathbf{4.6E.5}$

$$\mathbf{y}' = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.6 Exercise $\mathbf{4.6E.6}$

$$\mathbf{y}' = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.7 Exercise $\mathbf{4.6E.7}$

$$\mathbf{y}' = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.8 Exercise $\mathbf{4.6E.8}$

$$\mathbf{y}' = \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.9 Exercise $\mathbf{4.6E.9}$

$$\mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 10 & 1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.10 Exercise $\mathbf{4.6E.10}$

$$\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 7 & -5 \\ 2 & 5 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.11 Exercise $\mathbf{4.6E.11}$

$$\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.12 Exercise $\mathbf{4.6E.12}$

$$\mathbf{y}' = \begin{bmatrix} 34 & 52 \\ -20 & -30 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.13 Exercise $\mathbf{4.6E.13}$

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -2 & -1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.14 Exercise $\mathbf{4.6E.14}$

$$\mathbf{y}' = \begin{bmatrix} 3 & -4 & -2 \\ -5 & 7 & -8 \\ -10 & 13 & -8 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.15 Exercise $\mathbf{4.6E.15}$

$$\mathbf{y}' = \begin{bmatrix} 6 & 0 & -3 \\ -3 & 3 & 3 \\ 1 & -2 & 6 \end{bmatrix} \mathbf{y}'$$

Answer

Add texts here. Do not delete this text first.

4.6E.16 Exercise $\mathbf{4.6E.16}$

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{y}'$$

Answer

Add texts here. Do not delete this text first.

In Exercises $(4.6E.17)$ to $(4.6E.24)$, solve the initial value problem.

4.6E.17 Exercise $\mathbf{4.6E.17}$

$$\mathbf{y}' = \begin{bmatrix} 4 & -6 \\ 3 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.6E.18 Exercise $\mathbf{4.6E.18}$

$$\mathbf{y}' = \begin{bmatrix} 7 & 15 \\ -3 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.6E.19 Exercise $\mathbf{4.6E.19}$

$$\mathbf{y}' = \begin{bmatrix} 7 & -15 \\ 3 & -5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 17 \\ 7 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.6E.20 Exercise $\mathbf{4.6E.20}$

$$\mathbf{y}' = \frac{1}{6} \begin{bmatrix} 4 & -2 \\ 5 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.6E.21 Exercise $\mathbf{4.6E.21}$

$$\mathbf{y}' = \begin{bmatrix} 5 & 2 & -1 \\ -3 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.6E.22 Exercise $\mathbf{4.6E.22}$

$$\mathbf{y}' = \begin{bmatrix} 4 & 4 & 0 \\ 8 & 10 & -20 \\ 2 & 3 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.6E.23 Exercise $\mathbf{4.6E.23}$

$$\mathbf{y}' = \begin{bmatrix} 1 & 15 & -15 \\ -6 & 18 & -22 \\ -3 & 11 & -15 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 15 \\ 17 \\ 10 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.6E.24 Exercise $\mathbf{4.6E.24}$

$$\mathbf{y}' = \begin{bmatrix} 4 & -4 & 4 \\ -10 & 3 & 15 \\ 2 & -3 & 1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 16 \\ 14 \\ 6 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.6E.25 Exercise $\mathbf{4.6E.25}$

Suppose an $n \times n$ matrix A with real entries has a complex eigenvalue $\lambda = \alpha + i\beta$ $(\beta \neq 0)$ with associated eigenvector $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ have real components. Show that $\mathbf{u}$ and $\mathbf{v}$ are both nonzero.

Answer

Add texts here. Do not delete this text first.

4.6E.26 Exercise $\mathbf{4.6E.26}$

Verify that

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t),$$

are the real and imaginary parts of

$$e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u}+i\mathbf{v}).$$

Answer

Add texts here. Do not delete this text first.

4.6E.27 Exercise $\mathbf{4.6}\boldsymbol{E}\mathbf{.27}$

Show that if the vectors $\mathbf{u}$ and $\mathbf{v}$ are not both $\mathbf{0}$ and $\beta \neq 0$ then the vector functions

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t)$$

are linearly independent on every interval.

Hint: There are two cases to consider: $\{\mathbf{u},\mathbf{v}\}$ linearly independent, and $\{\mathbf{u},\mathbf{v}\}$ linearly dependent. In either case, exploit the the linear independence of $\{\cos\beta t, \sin\beta t\}$ on every interval.

Answer

Add texts here. Do not delete this text first.

4.6E.28 Exercise $\mathbf{4.6}\boldsymbol{E}\mathbf{.28}$

Suppose $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are not orthogonal; that is, $(\mathbf{u},\mathbf{v}) \neq 0$.

(a) Show that the quadratic equation

$$(\mathbf{u},\mathbf{v})k^2 + (\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2)k - (\mathbf{u},\mathbf{v}) = 0$$

has a positive root k_1 and a negative root $k_2 = -1/k_1$.

(b) Let $\mathbf{u}_1^{(1)} = \mathbf{u} - k_1\mathbf{v}$, $\mathbf{v}_1^{(1)} = \mathbf{v} + k_1\mathbf{u}$, $\mathbf{u}_1^{(2)} = \mathbf{u} - k_2\mathbf{v}$, and $\mathbf{v}_1^{(2)} = \mathbf{v} + k_2\mathbf{u}$, so that $(\mathbf{u}_1^{(1)}, \mathbf{v}_1^{(1)}) = (\mathbf{u}_1^{(2)}, \mathbf{v}_1^{(2)}) = 0$, from the discussion given above. Show that

$$\mathbf{u}_1^{(2)} = \frac{\mathbf{v}_1^{(1)}}{k_1} \quad \text{and} \quad \mathbf{v}_1^{(2)} = -\frac{\mathbf{u}_1^{(1)}}{k_1}.$$

(c) Let $\mathbf{U}_1$, $\mathbf{V}_1$, $\mathbf{U}_2$, and $\mathbf{V}_2$ be unit vectors in the directions of $\mathbf{u}_1^{(1)}$, $\mathbf{v}_1^{(1)}$, $\mathbf{u}_1^{(2)}$, and $\mathbf{v}_1^{(2)}$, respectively. Conclude from part (a) that $\mathbf{U}_2 = \mathbf{V}_1$ and $\mathbf{V}_2 = -\mathbf{U}_1$, and that therefore the counterclockwise angles from $\mathbf{U}_1$ to $\mathbf{V}_1$ and from $\mathbf{U}_2$ to $\mathbf{V}_2$ are both $\pi/2$ or both $-\pi/2$.

Answer

Add texts here. Do not delete this text first.

In Exercises $(4.6E.29)$ to $(4.6E.32)$, find vectors $\mathbf{U}$ and $\mathbf{V}$ parallel to the axes of symmetry of the trajectories, and plot some typical trajectories.

4.6E.29 Exercise $\mathbf{4.6E.29}$

$$\mathbf{y}' = \begin{bmatrix} 3 & -5 \\ 5 & -3 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.30 Exercise $\mathbf{4.6E.30}$

$$\mathbf{y}' = \begin{bmatrix} -15 & 10 \\ -25 & 15 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.31 Exercise $\mathbf{4.6E.31}$

$$\mathbf{y}' = \begin{bmatrix} -4 & 8 \\ -4 & 4 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.32 Exercise $\mathbf{4.6E.32}$

$$\mathbf{y}' = \begin{bmatrix} -3 & -15 \\ 3 & 3 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

In Exercises $(4.6E.33)$ to $(4.6E.40)$, find vectors $\mathbf{U}$ and $\mathbf{V}$ parallel to the axes of symmetry of the shadow trajectories, and plot a typical trajectory.

4.6E.33 Exercise $\mathbf{4.6E.33}$

$$\mathbf{y}' = \begin{bmatrix} -5 & 6 \\ -12 & 7 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.34 Exercise $\mathbf{4.6E.34}$

$$\mathbf{y}' = \begin{bmatrix} 5 & -12 \\ 6 & -7 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.35 Exercise $\mathbf{4.6}\boldsymbol{E}\mathbf{.35}$

$$\mathbf{y}' = \begin{bmatrix} 4 & -5 \\ 9 & -2 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.36 Exercise $\mathbf{4.6}\boldsymbol{E}\mathbf{.36}$

$$\mathbf{y}' = \begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.37 Exercise $\mathbf{4.6}\boldsymbol{E}\mathbf{.37}$

$$\mathbf{y}' = \begin{bmatrix} -1 & 10 \\ -10 & -1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.38 Exercise $\mathbf{4.6}\boldsymbol{E}\mathbf{.38}$

$$\mathbf{y}' = \begin{bmatrix} -1 & -5 \\ 20 & -1 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.39 Exercise $\mathbf{4.6}\boldsymbol{E}\mathbf{.39}$

$$\mathbf{y}' = \begin{bmatrix} -7 & 10 \\ -10 & 9 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.6E.40 Exercise $\mathbf{4.6}\boldsymbol{E}\mathbf{.40}$

$$\mathbf{y}' = \begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix} \mathbf{y}$$

Answer

Add texts here. Do not delete this text first.

4.7: Variation of Parameters for Nonhomogeneous Linear Systems

This page is a draft and is under active development.

4.7.1 Variation of Parameters for Nonhomogeneous Linear Systems

We now consider the nonhomogeneous linear system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t),$$

where A is an $n \times n$ matrix function and $\mathbf{f}$ is an n-vector forcing function. Associated with this system is the complementary system $\mathbf{y}' = A(t)\mathbf{y}$.

The next theorem is analogous to Theorems $(2.3.2)$ and $(3.1.5)$. It shows how to find the general solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ if we know a particular solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ and a fundamental set of solutions of the complementary system. We leave the proof as an exercise (Exercise $(4.7E.21)$).

Theorem 4.7.1

Suppose the $n \times n$ matrix function A and the n-vector function $\mathbf{f}$ are continuous on (a, b). Let $\mathbf{y}_p$ be a particular solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ on (a, b), and let $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ be a fundamental set of solutions of the complementary equation $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b). Then $\mathbf{y}$ is a solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ on (a, b) if and only if

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n,$$

where c_1, c_2, …, c_n are constants.

Proof

Add proof here and it will automatically be hidden if you have a "AutoNum" template active on the page.

4.7.2 Finding a Particular Solution of a Nonhomogeneous System

We now discuss an extension of the method of variation of parameters to linear nonhomogeneous systems. This method will produce a particular solution of a nonhomogenous system $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ provided that we know a fundamental matrix for the complementary system. To derive the method, suppose Y is a fundamental matrix for the complementary system; that is,

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix},$$

where

$$\mathbf{y}_1=\begin{bmatrix} y_{11}\\ y_{21}\\ \vdots\\ y_{n1}\end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix} y_{12}\\ y_{22}\\ \vdots\\ y_{n2}\end{bmatrix}\quad \cdots,\quad \mathbf{y}_n=\begin{bmatrix} y_{1n}\\ y_{2n}\\ \vdots\\ v_{nn}\end{bmatrix}$$

is a fundamental set of solutions of the complementary system. In Section 4.3 we saw that $Y'=A(t)Y$. We seek a particular solution of

$$\mathbf{y}'=A(t)\mathbf{y}+\mathbf{f}(t) \tag{4.7.1}$$

of the form

$$\mathbf{y}_p=Y\mathbf{u}, \tag{4.7.2}$$

where $\mathbf{u}$ is to be determined. Differentiating (4.7.2) yields

$$\begin{aligned}\mathbf{y}_p' &= Y'\mathbf{u}+Y\mathbf{u}'\\ &= AY\mathbf{u}+Y\mathbf{u}' \text{ (since } Y'=AY)\\ &= A\mathbf{y}_p+Y\mathbf{u}' \text{ (since } Y\mathbf{u}=\mathbf{y}_p).\end{aligned}$$

Comparing this with (4.7.1) shows that $\mathbf{y}_p=Y\mathbf{u}$ is a solution of (4.7.1) if and only if

$$Y\mathbf{u}'=\mathbf{f}.$$

Thus, we can find a particular solution $\mathbf{y}_p$ by solving this equation for $\mathbf{u}'$, integrating to obtain $\mathbf{u}$, and computing $Y\mathbf{u}$. We can take all constants of integration to be zero, since any particular solution will suffice.

Exercise $(4.7E.22)$ sketches a proof that this method is analogous to the method of variation of parameters discussed in Sections 3.4 for scalar linear equations.

Example 4.7.1

(a) Find a particular solution of the system

$$\mathbf{y}'=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\mathbf{y}+\begin{bmatrix}2e^{4t}\\ e^{4t}\end{bmatrix}, \tag{4.7.3}$$

which we considered in Example (4.2.1).

(b)
Find the general solution of (4.7.3).

Answer

(a) The complementary system is

$$\mathbf{y}'=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\mathbf{y}. \tag{4.7.4}$$

The characteristic polynomial of the coefficient matrix is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & a-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3).$$

Using the method of Section 4.4, we find that

$$\mathbf{y}_1 = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

are linearly independent solutions of (4.7.4). Therefore

$$Y = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}$$

is a fundamental matrix for (4.7.4). We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (4.7.3), where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{vmatrix} = -2e^{2t}.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= -\frac{1}{2e^{2t}} \begin{vmatrix} 2e^{4t} & e^{-t} \\ e^{4t} & -e^{-t} \end{vmatrix} &= \frac{3e^{3t}}{2e^{2t}} &= \frac{3}{2}e^{t}, \\ u_2' &= -\frac{1}{2e^{2t}} \begin{vmatrix} e^{3t} & 2e^{4t} \\ e^{3t} & e^{4t} \end{vmatrix} &= \frac{e^{7t}}{2e^{2t}} &= \frac{1}{2}e^{5t}. \end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2} \begin{bmatrix} 3e^{t} \\ e^{5t} \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \frac{1}{10} \begin{bmatrix} 15e^{t} \\ e^{5t} \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \frac{1}{10} \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 15e^{t} \\ e^{5t} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix}$$

is a particular solution of (4.7.3).

(b) From Theorem $(4.7.1)$, the general solution of (4.7.3) is

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = \frac{1}{5}\begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix} + c_1\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + c_2\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \tag{4.7.5}$$

which can also be written as

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix} + \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}\mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

Writing (4.7.5) in terms of coordinates yields

$$\begin{aligned} y_1 &= \frac{8}{5}e^{4t} + c_1e^{3t} + c_2e^{-t} \\ y_2 &= \frac{7}{5}e^{4t} + c_1e^{3t} - c_2e^{-t}, \end{aligned}$$

so our result is consistent with Example $(4.2.1)$.

If A isn't a constant matrix, it's usually difficult to find a fundamental set of solutions for the system $\mathbf{y}' = A(t)\mathbf{y}$. It is beyond the scope of this text to discuss methods for doing this. Therefore, in the following examples and in the exercises involving systems with variable coefficient matrices we'll provide fundamental matrices for the complementary systems without explaining how they were obtained.

Example 4.7.2

Find a particular solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & 2e^{-2t} \\ 2e^{2t} & 4 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{4.7.6}$$

given that

$$Y = \begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix}$$

is a fundamental matrix for the complementary system.

Answer

We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (4.7.6) where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{vmatrix} = 2e^{6t}.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= \frac{1}{2e^{6t}}\begin{vmatrix} 1 & -1 \\ 1 & e^{2t} \end{vmatrix} &= \frac{e^{2t}+1}{2e^{6t}} &= \frac{e^{4t}+e^{-6t}}{2} \\ u_2' &= \frac{1}{2e^{6t}}\begin{vmatrix} e^{4t} & 1 \\ e^{6t} & 1 \end{vmatrix} &= \frac{e^{4t}-e^{6t}}{2e^{6t}} &= \frac{e^{-2t}-1}{2}. \end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2}\begin{bmatrix} e^{-4t}+e^{-6t} \\ e^{-2t}-1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = -\frac{1}{24}\begin{bmatrix} 3e^{-4t}+2e^{-6t} \\ 6e^{-2t}+12t \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = -\frac{1}{24}\begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix}\begin{bmatrix} 3e^{-4t}+2e^{-6t} \\ 6e^{-2t}+12t \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 4e^{-2t}+12t-3 \\ -3e^{2t}(4t+1)-8 \end{bmatrix}$$

is a particular solution of (4.7.6).

Example 4.7.3

Find a particular solution of

$$\mathbf{y}' = -\frac{2}{t^2}\begin{bmatrix} t & -3t^2 \\ 1 & -2t \end{bmatrix}\mathbf{y} + t^2\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{4.7.7}$$

given that

$$Y = \begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix}$$

is a fundamental matrix for the complementary system on $(-\infty, 0)$ and $(0, \infty)$.

Answer

We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (4.7.7) where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} t^2 \\ t^2 \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\left| 2t \quad 3t^2 \; 1 \quad 2t \right| = t^2.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= \frac{1}{t^2}\begin{vmatrix} t^2 & 3t^2 \\ t^2 & 2t \end{vmatrix} &= \frac{2t^3-3t^4}{t^2} &= 2t-3t^2, \\ u_2' &= \frac{1}{t^2}\begin{vmatrix} 2t & t^2 \\ 1 & t^2 \end{vmatrix} &= \frac{2t^3-t^2}{t^2} &= 2t-1. \end{aligned}$$

Therefore

$$\mathbf{u}' = \begin{bmatrix} 2t-3t^2 \\ 2t-1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \begin{bmatrix} t^2-t^3 \\ t^2-t \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix}\begin{bmatrix} t^2-t^3 \\ t^2-t \end{bmatrix} = \begin{bmatrix} t^3(t-1) \\ t^2(t-1) \end{bmatrix}$$

is a particular solution of (4.7.7).

Example 4.7.4

(a) Find a particular solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\mathbf{y} + \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}. \tag{4.7.8}$$

(b) Find the general solution of (4.7.8).

Answer

(a) The complementary system for (4.7.8) is

$$\mathbf{y}' = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\mathbf{y}. \tag{4.7.9}$$

The characteristic polynomial of the coefficient matrix is

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = -\lambda(\lambda-1)^2.$$

Using the method of Section 4.4, we find that

$$\mathbf{y}_1=\begin{bmatrix}1\\1\\1\end{bmatrix},\quad \mathbf{y}_2=\begin{bmatrix}e^t\\e^t\\0\end{bmatrix},\quad \text{and}\quad \mathbf{y}_3=\begin{bmatrix}e^t\\0\\e^t\end{bmatrix}$$

are linearly independent solutions of (4.7.9). Therefore

$$Y=\begin{bmatrix}1&e^t&e^t\\1&e^t&0\\1&0&e^t\end{bmatrix}$$

is a fundamental matrix for (4.7.9). We seek a particular solution $\mathbf{y}_p=Y\mathbf{u}$ of (4.7.8), where $Y\mathbf{u}'=\mathbf{f}$; that is,

$$\begin{bmatrix}1&e^t&e^t\\1&e^t&0\\e&0&e^t\end{bmatrix}\begin{bmatrix}u_1'\\u_2'\\u_3'\end{bmatrix}=\begin{bmatrix}e^t\\0\\e^{-t}\end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix}1&e^t&e^t\\1&e^t&0\\1&0&e^t\end{vmatrix}=-e^{2t}.$$

Thus, by Cramer's rule,

$$\begin{aligned}
u_1' &= -\frac{1}{e^{2t}}\begin{vmatrix}e^t&e^t&e^t\\0&e^t&0\\e^{-t}&0&e^t\end{vmatrix} &= -\frac{e^{3t}-e^t}{e^{2t}} &= e^{-t}-e^t\\
u_2' &= -\frac{1}{e^{2t}}\begin{vmatrix}1&e^t&e^t\\1&0&0\\1&e^{-t}&e^t\end{vmatrix} &= -\frac{1-e^{2t}}{e^{2t}} &= 1-e^{-2t}\\
u_3' &= -\frac{1}{e^{2t}}\begin{vmatrix}1&e^t&e^t\\1&e^t&0\\1&0&e^{-t}\end{vmatrix} &= \frac{e^{2t}}{e^{2t}} &= 1.
\end{aligned}$$

Therefore

$$\mathbf{u}'=\begin{bmatrix}e^{-t}-e^t\\1-e^{-2t}\\1\end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u}=\begin{bmatrix}-e^t-e^{-t}\\ \dfrac{e^{-2t}}{2}+t\\ t\end{bmatrix},$$

so

$$\mathbf{y}_p=Y\mathbf{u} \quad = \quad \begin{bmatrix}1 & e^t & e^t\\ 1 & e^t & 0\\ 1 & 0 & e^t\end{bmatrix}\begin{bmatrix}-e^t-e^{-t}\\ \dfrac{e^{-2t}}{2}+t\\ t\end{bmatrix} \quad = \quad \begin{bmatrix}e^t(2t-1)-\dfrac{e^{-t}}{2}\\ e^t(t-1)-\dfrac{e^{-t}}{2}\\ e^t(t-1)-e^{-t}\end{bmatrix}$$

is a particular solution of (4.7.8).

(b) From Theorem $(4.7.1)$ the general solution of (4.7.8) is

$$\mathbf{y}=\mathbf{y}_p+c_1\mathbf{y}_1+c_2\mathbf{y}_2+c_3\mathbf{y}_3=\begin{bmatrix}e^t(2t-1)-\dfrac{e^{-t}}{2}\\ e^t(t-1)-\dfrac{e^{-t}}{2}\\ e^t(t-1)-e^{-t}\end{bmatrix}+c_1\begin{bmatrix}1\\1\\1\end{bmatrix}+c_2\begin{bmatrix}e^t\\e^t\\0\end{bmatrix}+c_3\begin{bmatrix}e^t\\0\\e^t\end{bmatrix},$$

which can be written as

$$\mathbf{y}=\mathbf{y}_p+Y\mathbf{c}=\begin{bmatrix}e^t(2t-1)-\dfrac{e^{-t}}{2}\\ e^t(t-1)-\dfrac{e^{-t}}{2}\\ e^t(t-1)-e^{-t}\end{bmatrix}+\begin{bmatrix}1 & e^t & e^t\\ 1 & e^t & 0\\ 1 & 0 & e^t\end{bmatrix}\mathbf{c}$$

where $\mathbf{c}$ is an arbitrary constant vector.

Example 4.7.5

Find a particular solution of

$$\mathbf{y}'=\frac{1}{2}\begin{bmatrix}3 & e^{-t} & -e^{2t}\\ 0 & 6 & 0\\ -e^{-2t} & e^{-3t} & -1\end{bmatrix}\mathbf{y}+\begin{bmatrix}1\\ e^t\\ e^{-t}\end{bmatrix}, \tag{4.7.10}$$

given that

$$Y=\begin{bmatrix}e^t & 0 & e^{2t}\\ 0 & e^{3t} & e^{3t}\\ e^{-t} & 1 & 0\end{bmatrix}$$

is a fundamental matrix for the complementary system.

Answer

We seek a particular solution of (4.7.10) in the form $\mathbf{y}_p = Y\mathbf{u}$, where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 1 \\ e^t \\ e^{-t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} = -2e^{4t}.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= -\frac{1}{2e^{4t}}\begin{vmatrix} 1 & 0 & e^{2t} \\ e^t & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} &= \frac{e^{4t}}{2e^{4t}} &= \frac{1}{2} \\ u_2' &= -\frac{1}{2e^{4t}}\begin{vmatrix} e^t & 1 & e^{2t} \\ 0 & e^t & e^{3t} \\ e^{-t} & e^{-t} & 0 \end{vmatrix} &= \frac{e^{3t}}{2e^{4t}} &= \frac{1}{2}e^{-t} \\ u_3' &= -\frac{1}{2e^{4t}}\begin{vmatrix} e^t & 0 & 1 \\ 0 & e^{3t} & e^t \\ e^{-t} & 1 & e^{-t} \end{vmatrix} &= -\frac{e^{3t}-2e^{2t}}{2e^{4t}} &= \frac{2e^{-2t}-e^{-t}}{2} \end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2}\begin{bmatrix} 1 \\ e^{-t} \\ 2e^{-2t}-e^{-t} \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \frac{1}{2}\begin{bmatrix} t \\ -e^{-t} \\ e^{-t}-e^{-2t} \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \frac{1}{2}\begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix}\begin{bmatrix} t \\ -e^{-t} \\ e^{-t}-e^{-2t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t(t+1)-1 \\ -e^t \\ e^{-t}(t-1) \end{bmatrix}$$

is a particular solution of (4.7.10).

4.7E: Exercises

This page is a draft and is under active development.

In Exercises $(4.7E.1)$ to $(4.7E.10)$, find a particular solution.

4.7E.1 Exercise $4.7E.1$

$$\mathbf{y}' = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 21e^{4t} \\ 8e^{-3t} \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.2 Exercise $4.7E.2$

$$\mathbf{y}' = \frac{1}{5} \begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 50e^{3t} \\ 10e^{-3t} \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.3 Exercise $4.7E.3$

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ t \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.4 Exercise $4.7E.4$

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ -2e^{t} \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.5 Exercise $4.7E.5$

$$\mathbf{y}' = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 4e^{-3t} \\ 4e^{-5t} \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.6 Exercise $\mathbf{4.7}\boldsymbol{E}\mathbf{.6}$

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 \\ t \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.7 Exercise $\mathbf{4.7}\boldsymbol{E}\mathbf{.7}$

$$\mathbf{y}' = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.8 Exercise $\mathbf{4.7}\boldsymbol{E}\mathbf{.8}$

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 \\ e^t \\ e^t \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.9 Exercise $\mathbf{4.7}\boldsymbol{E}\mathbf{.9}$

$$\mathbf{y}' = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} e^t \\ e^{-5t} \\ e^t \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.10 Exercise $\mathbf{4.7}\boldsymbol{E}\mathbf{.10}$

$$\mathbf{y}' = \frac{1}{3}\begin{bmatrix} 1 & 1 & -3 \\ -4 & -4 & 3 \\ -2 & 1 & 0 \end{bmatrix}\mathbf{y} + \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

In Exercises $(4.7E.11)$ to $(4.7E.20)$, find a particular solution, given that Y is a fundamental matrix for the complementary system.

4.7E.11 Exercise $\mathbf{4.7E.11}$

$$\mathbf{y}' = \frac{1}{t}\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}\mathbf{y} + t\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}; \quad Y = t\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.12 Exercise $\mathbf{4.7E.12}$

$$\mathbf{y}' = \frac{1}{t}\begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} t \\ t^2 \end{bmatrix}; \quad Y = t\begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.13 Exercise $\mathbf{4.7E.13}$

$$\mathbf{y}' = \frac{1}{t^2-1}\begin{bmatrix} t & -1 \\ -1 & t \end{bmatrix}\mathbf{y} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad Y = \begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.14 Exercise $\mathbf{4.7E.14}$

$$\mathbf{y}' = \frac{1}{3}\begin{bmatrix} 1 & -2e^{-t} \\ 2e^t & -1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix}; \quad Y = \begin{bmatrix} 2 & e^{-t} \\ e^t & 2 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.15 Exercise $\mathbf{4.7E.15}$

$$\mathbf{y}' = \frac{1}{2t^4}\begin{bmatrix} 3t^3 & t^6 \\ 1 & -3t^3 \end{bmatrix}\mathbf{y} + \frac{1}{t}\begin{bmatrix} t^2 \\ 1 \end{bmatrix}; \quad Y = \frac{1}{t^2}\begin{bmatrix} t^3 & t^4 \\ -1 & t \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.16 Exercise $\mathbf{4.7E.16}$

$$\mathbf{y}' = \begin{bmatrix} \dfrac{1}{t-1} & -\dfrac{e^{-t}}{t-1} \\ \dfrac{e^t}{t+1} & \dfrac{1}{t+1} \end{bmatrix} \mathbf{y} + \begin{bmatrix} t^2-1 \\ t^2-1 \end{bmatrix}; \quad Y = \begin{bmatrix} t & e^{-t} \\ e^t & t \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.17 Exercise $\mathbf{4.7E.17}$

$$\mathbf{y}' = \frac{1}{t} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad Y = \begin{bmatrix} t^2 & t^3 & 1 \\ t^2 & 2t^3 & -1 \\ 0 & 2t^3 & 2 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.18 Exercise $\mathbf{4.7E.18}$

$$\mathbf{y}' = \begin{bmatrix} 3 & e^t & e^{2t} \\ e^{-t} & 2 & e^t \\ e^{-2t} & e^{-t} & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{3t} \\ 0 \\ 0 \end{bmatrix}; \quad Y = \begin{bmatrix} e^{5t} & e^{2t} & 0 \\ e^{4t} & 0 & e^t \\ e^{3t} & -1 & -1 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.19 Exercise $\mathbf{4.7E.19}$

$$\mathbf{y}' = \frac{1}{t} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & t \\ 0 & -t & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} t \\ t \\ t \end{bmatrix}; \quad Y = t \begin{bmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.20 Exercise $\mathbf{4.7E.20}$

$$\mathbf{y}' = -\frac{1}{t} \begin{bmatrix} e^{-t} & -t & 1-e^{-t} \\ e^{-t} & 1 & -t-e^{-t} \\ e^{-t} & -t & 1-e^{-t} \end{bmatrix} \mathbf{y} + \frac{1}{t} \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}; \quad Y = \frac{1}{t} \begin{bmatrix} e^t & e^{-t} & t \\ e^t & -e^{-t} & e^{-t} \\ e^t & e^{-t} & 0 \end{bmatrix}$$

Answer

Add texts here. Do not delete this text first.

4.7E.21 Exercise $\mathbf{4.7}\boldsymbol{E}\mathbf{.21}$

Prove Theorem (4.7.1).

Answer

Add texts here. Do not delete this text first.

4.7E.22 Exercise $\mathbf{4.7}\boldsymbol{E}\mathbf{.22}$

(a) Convert the scalar equation

$$P_0(t)y^{(n)}+P_1(t)y^{(n-1)}+\cdots+P_n(t)y=F(t) \tag{4.7E.1}$$

into an equivalent $n\times n$ system

$$\mathbf{y}'=A(t)\mathbf{y}+\mathbf{f}(t). \tag{4.7E.2}$$

(b) Suppose (4.7E.1) is normal on an interval (a,b) and $\{y_1,y_2,\ldots,y_n\}$ is a fundamental set of solutions of

$$P_0(t)y^{(n)}+P_1(t)y^{(n-1)}+\cdots+P_n(t)y=0 \tag{4.7E.3}$$

on (a,b). Find a corresponding fundamental matrix Y for

$$\mathbf{y}'=A(t)\mathbf{y} \tag{4.7E.4}$$

on (a,b) such that

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n$$

is a solution of (4.7E.3) if and only if $\mathbf{y}=Y\mathbf{c}$ with

$$\mathbf{c}=\begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$$

is a solution of (4.7E.4).

(c) Let $y_p=u_1y_1+u_1y_2+\cdots+u_ny_n$ be a particular solution of (4.7E.1), obtained by the method of variation of parameters for scalar equations as given in Section (3.4), and define

$$\mathbf{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}.$$

Show that $\mathbf{y}_p=Y\mathbf{u}$ is a solution of (4.7E.2).

(d) Let $\mathbf{y}_p = Y\mathbf{u}$ be a particular solution of (4.7E.2), obtained by the method of variation of parameters for systems as given in this section. Show that $y_p = u_1 y_1 + u_1 y_2 + \cdots + u_n y_n$ is a solution of (4.7E.1).

Answer

Add texts here. Do not delete this text first.

4.7E.23 Exercise $\PageIndex{23}$

Suppose the $n \times n$ matrix function A and the n-vector function $\mathbf{f}$ are continuous on (a, b). Let t_0 be in (a, b), let $\mathbf{k}$ be an arbitrary constant vector, and let Y be a fundamental matrix for the homogeneous system $\mathbf{y}' = A(t)\mathbf{y}$. Use variation of parameters to show that the solution of the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}$$

is

$$\mathbf{y}(t) = Y(t)\left(Y^{-1}(t_0)\mathbf{k} + \int_{t_0}^{t} Y^{-1}(s)\mathbf{f}(s)\,ds\right).$$

Answer

Add texts here. Do not delete this text first.

CHAPTER OVERVIEW

5: Vector-Valued Functions

In 1705, using Sir Isaac Newton's new laws of motion, the astronomer Edmond Halley made a prediction. He stated that comets that had appeared in 1531, 1607, and 1682 were actually the same comet and that it would reappear in 1758. Halley was proved to be correct, although he did not live to see it. However, the comet was later named in his honor. Halley's Comet follows an elliptical path through the solar system, with the Sun appearing at one focus of the ellipse. This motion is predicted by Johannes Kepler's first law of planetary motion, which we mentioned briefly previously. Kepler's third law of planetary motion can be used with the calculus of vector-valued functions to find the average distance of Halley's Comet from the Sun.

Figure 5.1: Halley's Comet appeared in view of Earth in 1986 and will appear again in 2061.

Vector-valued functions provide a useful method for studying various curves both in the plane and in three-dimensional space. We can apply this concept to calculate the velocity, acceleration, arc length, and curvature of an object's trajectory. In this chapter, we examine these methods and show how they are used.

5.1 Contributors

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors.

Topic hierarchy

5.1: Vector-Valued Functions and Space Curves

5.1E:

5.2: Calculus of Vector-Valued Functions

5.2E:

5.3: Arc Length and Curvature

5.3E: Exercises

5.4: Motion in Space

5.4E:Exercises

5E: Excercises

5.1: Vector-Valued Functions and Space Curves

This page is a draft and is under active development.

Our study of vector-valued functions combines ideas from our earlier examination of single-variable calculus with our description of vectors in three dimensions from the preceding chapter. In this section, we extend concepts from earlier chapters and also examine new ideas concerning curves in three-dimensional space. These definitions and theorems support the presentation of material in the rest of this chapter and also in the remaining chapters of the text.

5.1.1 Definition of a Vector-Valued Function

Our first step in studying the calculus of vector-valued functions is to define what exactly a vector-valued function is. We can then look at graphs of vector-valued functions and see how they define curves in both two and three dimensions.

Definition: Vector-valued Functions

A vector-valued function is a function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} \;\text{ or }\; \vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}, \tag{5.1.1}$$

where the component functions f, g, and h, are real-valued functions of the parameter t. Vector-valued functions are also written in the form

$$\vec{\mathbf{r}}(t) = \langle f(t), g(t)\rangle \;\text{ or }\; \vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle. \tag{5.1.2}$$

In both cases, the first form of the function defines a two-dimensional vector-valued function; the second form describes a three-dimensional vector-valued function.

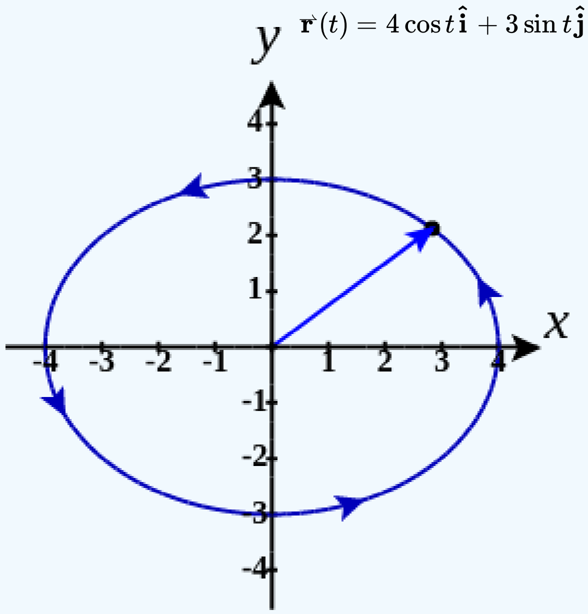
The parameter t can lie between two real numbers: $a \le t \le b$. Another possibility is that the value of t might take on all real numbers. Last, the component functions themselves may have domain restrictions that enforce restrictions on the value of t. We often use t as a parameter because t can represent time.

Example 5.1.1: Evaluating Vector-Valued Functions and Determining Domains

For each of the following vector-valued functions, evaluate $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}(\frac{\pi}{2})$, and $\vec{\mathbf{r}}(\frac{2\pi}{3})$. Do any of these functions have domain restrictions?

1. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$
2. $\vec{\mathbf{r}}(t) = 3\tan t\,\hat{\mathbf{i}} + 4\sec t\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}$

Solution



1. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned} \vec{\mathbf{r}}(0) &= 4\cos(0)\hat{\mathbf{i}} + 3\sin(0)\hat{\mathbf{j}} \\ &= 4\hat{\mathbf{i}} + 0\hat{\mathbf{j}} = 4\hat{\mathbf{i}} \\ \vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= 4\cos\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} \\ &= 0\hat{\mathbf{i}} + 3\hat{\mathbf{j}} = 3\hat{\mathbf{j}} \\ \vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) &= 4\cos\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} \\ &= 4(-\frac{1}{2})\hat{\mathbf{i}} + 3(\frac{\sqrt{3}}{2})\hat{\mathbf{j}} = -2\hat{\mathbf{i}} + \frac{3\sqrt{3}}{2}\hat{\mathbf{j}} \end{aligned}$$

   To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 4\cos t$ and the second component function is $g(t) = 3\sin t$. Neither of these functions has a domain restriction, so the domain of $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$ is all real numbers.
2. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned}
\vec{\mathbf{r}}(0) &= 3\tan(0)\hat{\mathbf{i}} + 4\sec(0)\hat{\mathbf{j}} + 5(0)\hat{\mathbf{k}} \\
&= 0\hat{\mathbf{i}} + 4j + 0\hat{\mathbf{k}} = 4\hat{\mathbf{j}} \\
\vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= 3\tan\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} + 5\left(\frac{\pi}{2}\right)\hat{\mathbf{k}}, \text{ which does not exist} \\
\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) &= 3\tan\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} + 5\left(\frac{2\pi}{3}\right)\hat{\mathbf{k}} \\
&= 3(-\sqrt{3})\hat{\mathbf{i}} + 4(-2)\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}} \\
&= -3\sqrt{3})\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}}
\end{aligned}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 3\tan t$, the second component function is $g(t) = 4\sec t$, and the third component function is $h(t) = 5t$. The first two functions are not defined for odd multiples of $\frac{\pi}{2}$, so the function is not defined for odd multiples of $\frac{\pi}{2}$. Therefore,

$$\mathrm{D}_{\vec{\mathbf{r}}} = \{t \mid t \neq \frac{(2n+1)\pi}{2}\},$$

where n is any integer.

Exercise 5.1.1

For the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (4t+1)\,\hat{\mathbf{j}}$, evaluate $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}(1)$, and $\vec{\mathbf{r}}(-4)$. Does this function have any domain restrictions?

Hint

Substitute the appropriate values of t into the function.

Answer

$\vec{\mathbf{r}}(0) = \hat{\mathbf{j}}$, $\vec{\mathbf{r}}(1) = -2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$, $\vec{\mathbf{r}}(-4) = 28\hat{\mathbf{i}} - 15\hat{\mathbf{j}}$

The domain of $\vec{\mathbf{r}}(t) = (t^2 - 3t)\hat{\mathbf{i}} + (4t+1)\hat{\mathbf{j}}$ is all real numbers.

Example 5.1.1 illustrates an important concept. The domain of a vector-valued function consists of real numbers. The domain can be all real numbers or a subset of the real numbers. The range of a vector-valued function consists of vectors. Each real number in the domain of a vector-valued function is mapped to either a two- or a three-dimensional vector.

5.1.2 Graphing Vector-Valued Functions

Recall that a plane vector consists of two quantities: direction and magnitude. Given any point in the plane (the *initial point*), if we move in a specific direction for a specific distance, we arrive at a second point. This represents the *terminal point* of the vector. We calculate the components of the vector by subtracting the coordinates of the initial point from the coordinates of the terminal point.

A vector is considered to be in *standard position* if the initial point is located at the origin. When graphing a vector-valued function, we typically graph the vectors in the domain of the function in standard position, because doing so guarantees the uniqueness of the graph. This convention applies to the graphs of three-dimensional vector-valued functions as well. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$$

consists of the set of all points $(f(t), g(t))$, and the path it traces is called a **plane curve**. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$$

consists of the set of all points $(f(t), g(t), h(t))$, and the path it traces is called a **space curve**. Any representation of a plane curve or space curve using a vector-valued function is called a **vector parameterization** of the curve.

Each plane curve and space curve has an **orientation**, indicated by arrows drawn in on the curve, that shows the direction of motion along the curve as the value of the parameter t increases.

Example $5.1.2$: Graphing a Vector-Valued Function

Create a graph of each of the following vector-valued functions:

1. The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$, $0 \leq t \leq 2\pi$
2. The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}}$, $0 \leq t \leq \sqrt[3]{2\pi}$
3. The space curve represented by $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$, $0 \leq t \leq 4\pi$

Solution

1. As with any graph, we start with a table of values. We then graph each of the vectors in the second column of the table in standard position and connect the terminal points of each vector to form a curve (Figure $5.1.1$). This curve turns out to be an ellipse centered at the origin.

Table $5.1.1$: Table of Values for $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$, $0 \leq t \leq 2\pi$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	π	$-4\hat{\mathbf{i}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\frac{\pi}{2}$	$3\hat{\mathbf{j}}$	$\frac{3\pi}{2}$	$-3\hat{\mathbf{j}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
2π	$4\hat{\mathbf{i}}$		

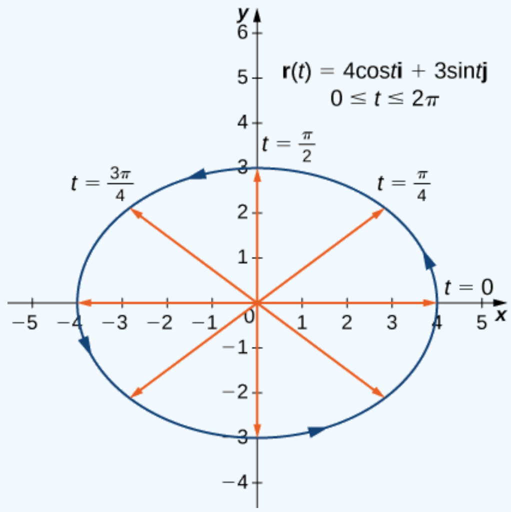


Figure 5.1.1: The graph of the first vector-valued function is an ellipse.

2. The table of values for $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}}\,, 0 \le t \le \sqrt[3]{2\pi}$ is as follows:

Table of Values for $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}}\,, 0 \le t \le \sqrt[3]{2\pi}$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	$\sqrt[3]{\pi}$	$-4\hat{\mathbf{i}}$
$\sqrt[3]{\frac{\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\sqrt[3]{\frac{5\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{\frac{\pi}{2}}$	$3\hat{\mathbf{j}}$	$\sqrt[3]{\frac{3\pi}{2}}$	$-3\hat{\mathbf{j}}$
$\sqrt[3]{\frac{3\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\sqrt[3]{\frac{7\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{2\pi}$	$4\hat{\mathbf{i}}$		

The graph of this curve is also an ellipse centered at the origin.

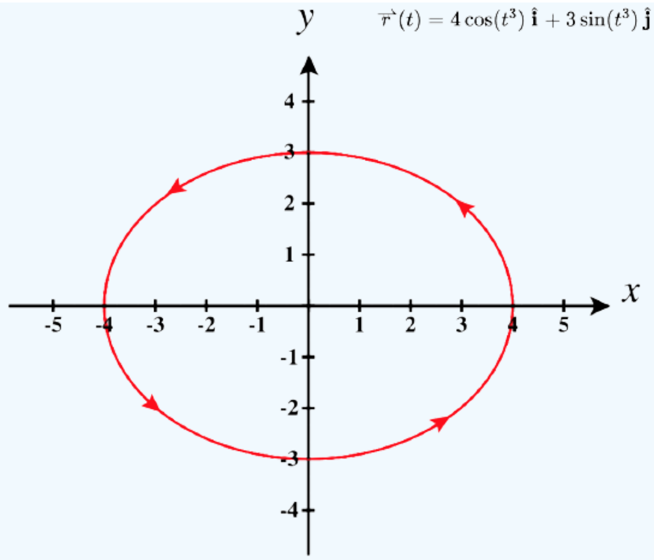


Figure 5.1.2: The graph of the second vector-valued function is also an ellipse.

3. We go through the same procedure for a three-dimensional vector function.

Table of Values for $\mathrm{r}(\mathrm{t}) = 4\cos \mathrm{t}\hat{\mathbf{i}} + 4\sin \mathrm{t}\hat{\mathbf{j}} + \mathrm{t}\hat{\mathbf{k}}$, $0 \le \mathrm{t} \le 4\pi$

t	$\vec{\mathbf{r}}\,(t)$	t	$\vec{\mathbf{r}}\,(t)$
0	$4\hat{\mathbf{i}}$	π	$-4\hat{\mathbf{i}} + \pi\hat{\mathbf{k}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} + 2\sqrt{2}\hat{\mathbf{j}} + \frac{\pi}{4}\hat{\mathbf{k}}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} - 2\sqrt{2}\hat{\mathbf{j}} + \frac{5\pi}{4}\hat{\mathbf{k}}$
$\frac{\pi}{2}$	$4\hat{\mathbf{j}} + \frac{\pi}{2}\hat{\mathbf{k}}$	$\frac{3\pi}{2}$	$-4\hat{\mathbf{j}} + \frac{3\pi}{2}\hat{\mathbf{k}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} + 2\sqrt{2}\hat{\mathbf{j}} + \frac{3\pi}{4}\hat{\mathbf{k}}$	$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} - 2\sqrt{2}\hat{\mathbf{j}} + \frac{7\pi}{4}\hat{\mathbf{k}}$
2π	$4\hat{\mathbf{j}} + 2\pi\hat{\mathbf{k}}$		

The values then repeat themselves, except for the fact that the coefficient of $\hat{\mathbf{k}}$ is always increasing (5.1.3). This curve is called a helix. Notice that if the $\hat{\mathbf{k}}$ component is eliminated, then the function becomes $\vec{\mathbf{r}}\,(t) = 4\cos t\hat{\mathbf{i}} + 4\sin t\hat{\mathbf{j}}$, which is a circle of radius 4 centered at the origin.

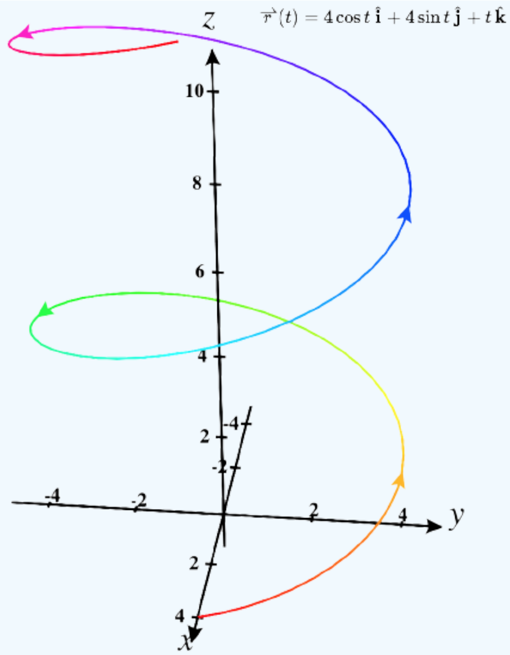


Figure 5.1.3: The graph of the third vector-valued function is a helix.

You may notice that the graphs in parts a. and b. are identical. This happens because the function describing curve b is a so-called reparameterization of the function describing curve a. In fact, any curve has an infinite number of reparameterizations; for example, we can replace t with $2t$ in any of the three previous curves without changing the shape of the curve. The interval over which t is defined may change, but that is all. We return to this idea later in this chapter when we study arc-length parameterization. As mentioned, the name of the shape of the curve of the graph in 5.1.3 is a *helix*. The curve resembles a spring, with a circular cross-section looking down along the z-axis. It is possible for a helix to be elliptical in cross-section as well. For example, the vector-valued function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ describes an elliptical helix. The projection of this helix into the xy-plane is an ellipse. Last, the arrows in the graph of this helix indicate the orientation of the curve as t progresses from 0 to 4π.

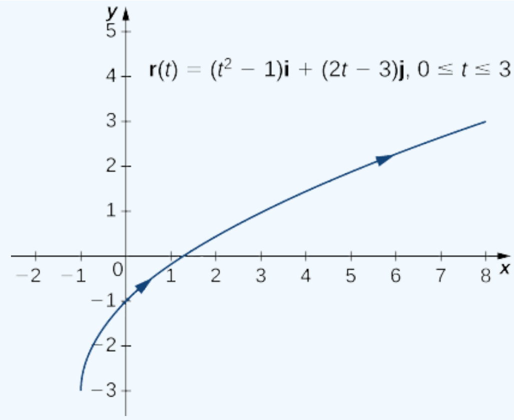
Exercise 5.1.2

Create a graph of the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 1)\hat{\mathbf{i}} + (2t - 3)\hat{\mathbf{j}}$, $0 \le t \le 3$.

Hint

Start by making a table of values, then graph the vectors for each value of t.

Answer



At this point, you may notice a similarity between vector-valued functions and parameterized curves. Indeed, given a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ we can define $x = f(t)$ and $y = g(t)$. If a restriction exists on the values of t (for example, t is restricted to the interval $[a, b]$ for some constants $a < b$, then this restriction is enforced on the parameter. The graph of the parameterized function would then agree with the graph of the vector-valued function, except that the vector-valued graph would represent vectors rather than points. Since we can parameterize a curve defined by a function $y = f(x)$, it is also possible to represent an arbitrary plane curve by a vector-valued function.

5.1.3 Limits and Continuity of a Vector-Valued Function

We now take a look at the limit of a vector-valued function. This is important to understand to study the calculus of vector-valued functions.

Definition: limit of a vector-valued function

A vector-valued function $\vec{\mathbf{r}}$ approaches the limit $\vec{\mathbf{L}}$ as t approaches a, written

$$\lim_{t\to a} \vec{\mathbf{r}}(t) = \vec{\mathbf{L}}, \tag{5.1.3}$$

provided

$$\lim_{t\to a} \left\| \vec{\mathbf{r}}(t) - \vec{\mathbf{L}} \right\| = 0. \tag{5.1.4}$$

This is a rigorous definition of the limit of a vector-valued function. In practice, we use the following theorem:

Theorem: Limit of a vector-valued function

Let f, g, and h be functions of t. Then the limit of the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ as t approaches a is given by

$$\lim_{t\to a} \vec{\mathbf{r}}(t) = [\lim_{t\to a} f(t)]\,\hat{\mathbf{i}} + [\lim_{t\to a} g(t)]\,\hat{\mathbf{j}}, \tag{5.1.5}$$

provided the limits $\lim_{t\to a} f(t)$ and $\lim_{t\to a} g(t)$ exist.

Similarly, the limit of the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ as t approaches a is given by

$$\lim_{t\to a} \vec{\mathbf{r}}(t) = [\lim_{t\to a} f(t)]\hat{\mathbf{i}} + [\lim_{t\to a} g(t)]\hat{\mathbf{j}} + [\lim_{t\to a} h(t)]\hat{\mathbf{k}}, \tag{5.1.6}$$

provided the limits $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$ and $\lim_{t\to a} h(t)$ exist.

In the following example, we show how to calculate the limit of a vector-valued function.

Example 5.1.3: Evaluating the Limit of a Vector-Valued Function

For each of the following vector-valued functions, calculate $\lim_{t\to 3} \vec{\mathbf{r}}(t)$ for

a. $\vec{\mathbf{r}}(t) = (t^2 - 3t + 4)\hat{\mathbf{i}} + (4t + 3)\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t) = \frac{2t-4}{t+1}\hat{\mathbf{i}} + \frac{t}{t^2+1}\hat{\mathbf{j}} + (4t - 3)\hat{\mathbf{k}}$

Solution

a. Use Equation 5.1.5 and substitute the value $t = 3$ into the two component expressions:

$$\begin{aligned} \lim_{t\to 3} \vec{\mathbf{r}}(t) &= \lim_{t\to 3}[(t^2 - 3t + 4)\hat{\mathbf{i}} + (4t + 3)\hat{\mathbf{j}}] \\ &= [\lim_{t\to 3}(t^2 - 3t + 4)]\hat{\mathbf{i}} + [\lim_{t\to 3}(4t + 3)]\hat{\mathbf{j}} \\ &= 4\hat{\mathbf{i}} + 15\hat{\mathbf{j}} \end{aligned}$$

b. Use Equation 5.1.6 and substitute the value $t = 3$ into the three component expressions:

$$\begin{aligned} \lim_{t\to 3} \vec{\mathbf{r}}(t) &= \lim_{t\to 3}(\frac{2t-4}{t+1}\hat{\mathbf{i}} + \frac{t}{t^2+1}\hat{\mathbf{j}} + (4t-3)\hat{\mathbf{k}}) \\ &= [\lim_{t\to 3}(\frac{2t-4}{t+1})]\hat{\mathbf{i}} + [\lim_{t\to 3}(\frac{t}{t^2+1})]\hat{\mathbf{j}}[\lim_{t\to 3}(4t-3)]\hat{\mathbf{k}} \\ &= \frac{1}{2}\hat{\mathbf{i}} + \frac{3}{10}\hat{\mathbf{j}} + 9\hat{\mathbf{k}} \end{aligned}$$

Exercise 5.1.3

Calculate $\lim_{t\to 2} \vec{\mathbf{r}}(t)$ for the function $\vec{\mathbf{r}}(t) = \sqrt{t^2 + 3t - 1}\,\hat{\mathbf{i}} - (4t - 3)\hat{\mathbf{j}} - \sin\frac{(t+1)\pi}{2}\hat{\mathbf{k}}$

Hint

Use Equation 5.1.6 from the preceding theorem.

Answer

$$\lim_{t\to 2} \vec{\mathbf{r}}(t) = 3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + \hat{\mathbf{k}} \tag{5.1.7}$$

Now that we know how to calculate the limit of a vector-valued function, we can define **continuity at a point** for such a function.

Definitions

Let f, g, and h be functions of t. Then, the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ is **continuous at point $t=a$** if the following three conditions hold:

1. $\vec{\mathbf{r}}(a)$ exists
2. $\lim_{t\to a}\vec{\mathbf{r}}(t)$ exists
3. $\lim_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{r}}(a)$

Similarly, the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ is **continuous at point $t=a$** if the following three conditions hold:

1. $\vec{\mathbf{r}}(a)$ exists
2. $\lim_{t\to a}\vec{\mathbf{r}}(t)$ exists
3. $\lim_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{r}}(a)$

5.1.3.0.1 Summary

- A vector-valued function is a function of the form $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$, where the component functions f, g, and h are real-valued functions of the parameter t.
- The graph of a vector-valued function of the form $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ is called a *plane curve*. The graph of a vector-valued function of the form $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ is called a *space curve*.
- It is possible to represent an arbitrary plane curve by a vector-valued function.
- To calculate the limit of a vector-valued function, calculate the limits of the component functions separately.

5.1.3.0.1 Key Equations

- **Vector-valued function**
  $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$,or $\vec{\mathbf{r}}(t)=\langle f(t),g(t)\rangle$ or $\vec{\mathbf{r}}(t)=\langle f(t),g(t),h(t)\rangle$
- **Limit of a vector-valued function**
  $\lim_{t\to a}\vec{\mathbf{r}}(t)=[\lim_{t\to a}f(t)]\hat{\mathbf{i}}+[\lim_{t\to a}g(t)]\hat{\mathbf{j}}$ or $\lim_{t\to a}\vec{\mathbf{r}}(t)=[\lim_{t\to a}f(t)]\hat{\mathbf{i}}+[\lim_{t\to a}g(t)]\hat{\mathbf{j}}+[\lim_{t\to a}h(t)]\hat{\mathbf{k}}$

5.1.3.0.1 Glossary

component functions

the component functions of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ are $f(t)$ and $g(t)$, and the component functions of the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ are $f(t)$, $g(t)$ and $h(t)$

helix

a three-dimensional curve in the shape of a spiral

limit of a vector-valued function

a vector-valued function $\vec{\mathbf{r}}(t)$ has a limit $\vec{\mathbf{L}}$ as t approaches a if $\lim t \to a \left|\vec{\mathbf{r}}(t) - \vec{\mathbf{L}}\right| = 0$

plane curve

the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations $x = f(t)$ and $y = g(t)$

reparameterization

an alternative parameterization of a given vector-valued function

space curve

the set of ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$

vector parameterization

any representation of a plane or space curve using a vector-valued function

vector-valued function

a function of the form $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$,where the component functions f, g, and h are real-valued functions of the parameter t.

5.1.4 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Edited by Paul Seeburger (Monroe Community College)

5.1E:

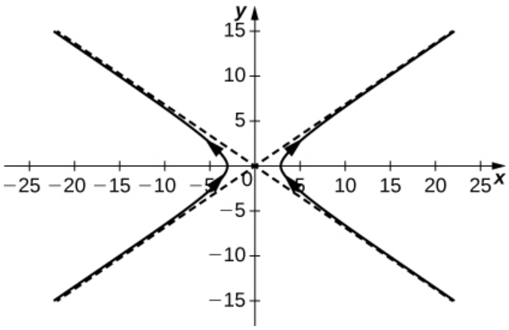
5.1E.1 Exercise $5.1E.1$

1) Give the component functions $x=f(t)$ and $y=g(t)$ for the vector-valued function $r(t)=3\sec t\mathbf{i}+2\tan t\mathbf{j}$.

Given $r(t)=3\sec t\mathbf{i}+2\tan t\mathbf{j}$, find the following values (if possible).

1. $r(\frac{\pi}{4})$
2. $r(\pi)$
3. $r(\frac{\pi}{2})$

Sketch the curve of the vector-valued function $r(t)=3\sec t\mathbf{i}+2\tan t\mathbf{j}$ and give the orientation of the curve. Sketch asymptotes as a guide to the graph.



2)

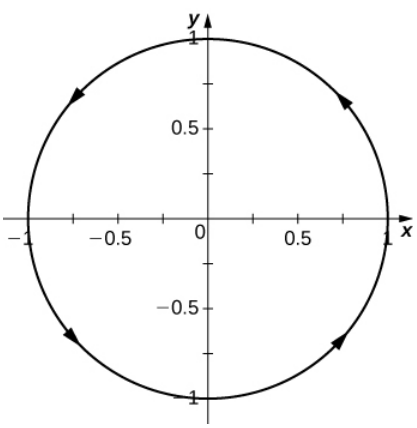
Evaluate $\lim_{t\to 0}\langle e^t\mathbf{i}+\frac{\sin t}{t}\mathbf{j}+e^{-t}\mathbf{k}\rangle$

3) Given the vector-valued function $r(t)=\langle\cos t,\sin t\rangle$ find the following values:

1. $\lim_{t\to\frac{\pi}{4}} r(t)$
2. $r(\frac{\pi}{3})$
3. Is $r(t)$ continuous at $t=\frac{\pi}{3}$?
4. Graph $r(t)$.

Answer

3) a. $\langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle$, b. $\langle\frac{1}{2},\frac{\sqrt{3}}{2}\rangle$, c. Yes, the limit as *t* approaches $\frac{\pi}{3}$ is equal to $r(\frac{\pi}{3})$, d.



5.1E.2 Exercise $5.1E.2$

1) Given the vector-valued function $r(t) = \langle t, t^2 + 1 \rangle$, find the following values:

1. $\lim\limits_{t \to -3} r(t)$
2. $r(-3)$
3. Is $r(t)$ continuous at $x = -3$?
4. $r(t+2) - r(t)$

2) Let $r(t) = e^t\mathbf{i} + \sin t\mathbf{j} + \ln t\mathbf{k}$. Find the following values:

1. $r(\frac{\pi}{4})$
2. $\lim\limits_{t \to \frac{\pi}{4}} r(t)$
3. Is $r(t)$ continuous at $t = t = \frac{\pi}{4}$?

Answer

a. $\langle e^{\frac{\pi}{4}}, \frac{\sqrt{2}}{2}, \ln(\frac{\pi}{4}) \rangle$; b. $\langle e^{\frac{\pi}{4}}, \frac{\sqrt{2}}{2}, \ln(\frac{\pi}{4}) \rangle$; c. Yes

5.1E.3 Exercise $5.1E.3$

Find the limit of the following vector-valued functions at the indicated value of *t*.

1) $\lim\limits_{t \to 4} \langle \sqrt{t-3}, \frac{\sqrt{t}-2}{t-4}, \tan(\frac{\pi}{t}) \rangle$

2) $\lim\limits_{t \to \frac{\pi}{2}} r(t)$ for $r(t) = e^t\mathbf{i} + \sin t\mathbf{j} + \ln t\mathbf{k}$

3) $\lim\limits_{t \to \infty} \langle e^{-2t}, \frac{2t+3}{3t-1}, \arctan(2t) \rangle$

4) $\lim\limits_{t \to e^2} \langle t\ln(t), \frac{\ln t}{t^2}, \sqrt{\ln(t^2)} \rangle$

5) $\lim\limits_{t \to \frac{\pi}{6}} \langle \cos 2t, \sin 2t, 1 \rangle$

6) $\lim\limits_{t \to \infty} r(t)$ for $r(t) = 2e^{-t}\mathbf{i} + e^{-t}\mathbf{j} + \ln(t-1)\mathbf{k}$

Answer

2) $\langle e^{\frac{\pi}{2}}, 1, \ln(\frac{\pi}{2}) \rangle$

4) $2e^2\mathbf{i} + \frac{2}{e^4}\mathbf{j} + 2\mathbf{k}$

6) The limit does not exist because the limit of $\ln(t-1)$ as *t* approaches infinity does not exist.

5.1E.4 Exercise $5.1E.4$

Describe the curve defined by the vector-valued function $r(t) = (1+t)\mathbf{i} + (2+5t)\mathbf{j} + (-1+6t)\mathbf{k}$.

Answer

Add texts here. Do not delete this text first.

5.1E.5 Exercise $5.1E.5$

Find the domain of the vector-valued functions.

1) $r(t) = \langle t^2, \tan t, \ln t \rangle$

2) $r(t) = \langle t^2, \sqrt{t-3}, \frac{3}{2t+1} \rangle$

3) $r(t)=\langle\csc(t),\frac{1}{\sqrt{t-3}},\ln(t-2)\rangle$

Answer

1) $t>0, t\neq(2k+1)\frac{\pi}{2}$, where k is an integer.

3) $t>3, t\neq n\pi$, where *n* is an integer.

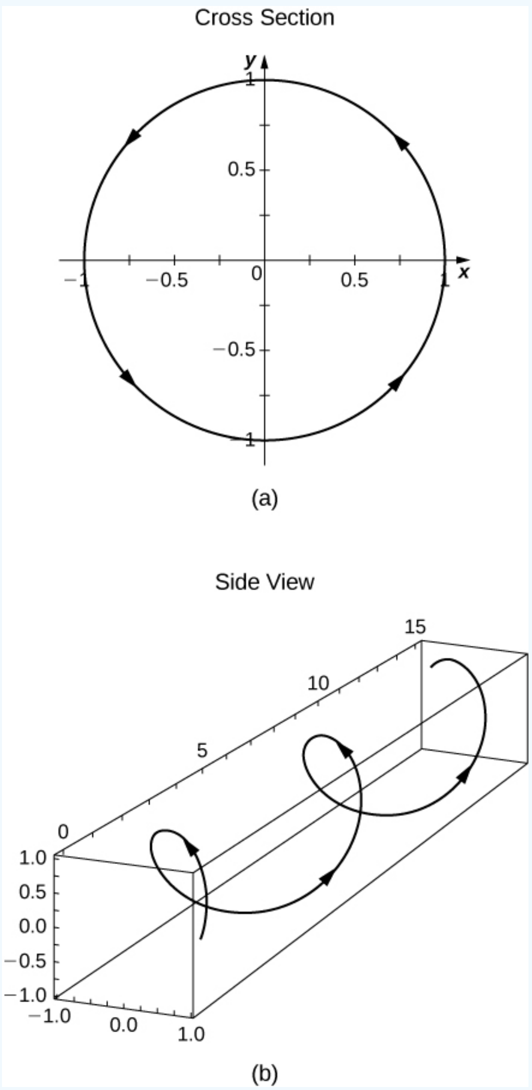
5.1E.6 Exercise $5.1E.6$

Let $r(t)=\langle\cos t, t, \sin t\rangle$ and use it to answer the following questions.

For what values of *t* is $r(t)$ continuous?

Sketch the graph of $r(t)$.

Answer



5.1E.7 Exercise $5.1E.7$

1) Find the domain of $r(t)=2e^{-t}\mathbf{i}+e^{-t}\mathbf{j}+\ln(t-1)\mathbf{k}$.

2) For what values of *t* is $r(t)=2e_S^{-t}\mathbf{i}+e^{-t}\mathbf{j}+\ln(t-1)\mathbf{k}$ continuous?

Answer

2) All *t* such that $t \in (1, \infty)$

5.1E.8 Exercise $5.1E.8$

Eliminate the parameter *t*, write the equation in Cartesian coordinates, then sketch the graphs of the vector-valued functions. (*Hint:* Let $x = 2t$ and $y = t^2$ Solve the first equation for *x* in terms of *t* and substitute this result into the second equation.)

1) $r(t) = 2t\mathbf{i} + t^2\mathbf{j}$

2) $r(t) = t^3\mathbf{i} + 2t\mathbf{j}$

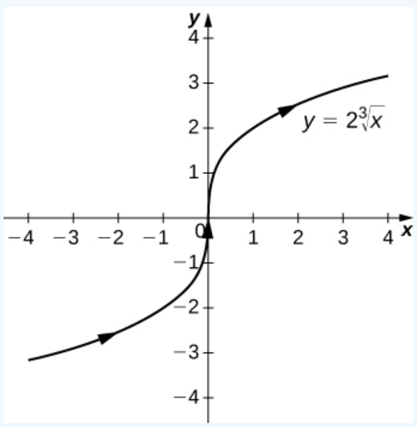
3) $r(t) = 2(\sinh t)\mathbf{i} + 2(\cosh t)\mathbf{j}, t > 0$

4) $r(t) = 3(\cos t)i + 3(\sin t)j$

5) $r(t) = \langle 3\sin t, 3\cos t\rangle$

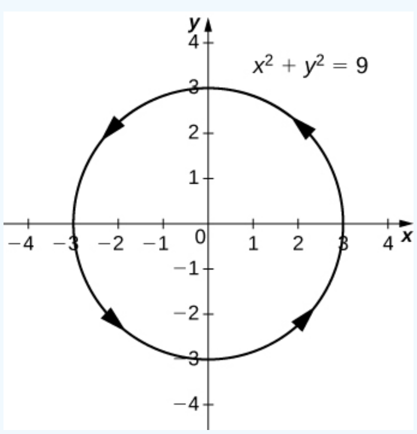
Answer

2) $y = 2\sqrt[3]{x}$, a variation of the cube-root function



4)

$x^2 + y^2 = 9$, a circle centered at $(0, 0)$ with radius 3, and a counterclockwise orientation



5.1E.9 Exercise $5.1E.9$

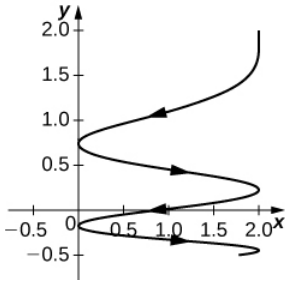
Use a graphing utility to sketch each of the following vector-valued functions:

1) **[T]** $r(t) = 2\cos t^2\mathbf{i} + (2 - \sqrt{t})\mathbf{j}$

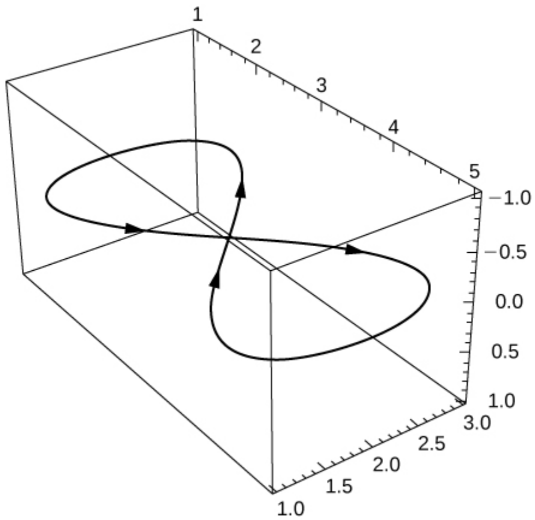
2) [T] $r(t) = \langle e^{\cos(3t)}, e^{-\sin(t)}\rangle$

3) [T] $r(t) = \langle 2 - \sin(2t), 3 + 2\cos t\rangle$

Answer

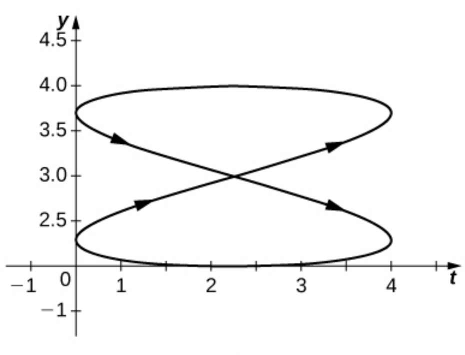
1) 

2)



(a)

View in the *yt*-plane



(b)

5.1E.10 Exercise $5.1E.10$

Find a vector-valued function that traces out the given curve in the indicated direction.

1) $4x^2 + 9y^2 = 36$; clockwise and counterclockwise

2) $r(t) = \langle t, t^2\rangle$; from left to right

Answer

2) For left to right, $y = x^2$, where t increases.

5.1E.11 Exercise $\mathbf{5.1}\boldsymbol{E}\mathbf{.11}$

Consider the curve described by the vector-valued function $\mathrm{r}(\mathrm{t}) = (50\mathrm{e}^{-\mathrm{t}}\cos\mathrm{t})\mathbf{i} + (50\mathrm{e}^{-\mathrm{t}}\sin\mathrm{t})\mathbf{j} + (5 - 5\mathrm{e}^{-\mathrm{t}})\mathbf{k}$.

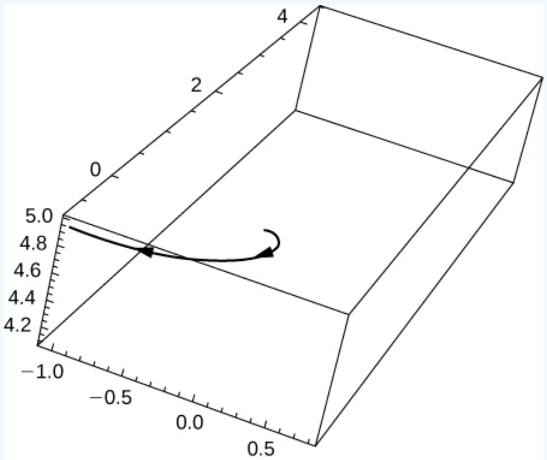
1) What is the initial point of the path corresponding to $\mathrm{r}(0)$?

2) What is $\lim\limits_{\mathrm{t}\to\infty} \mathrm{r}(\mathrm{t})$?

3) [T] Use technology to sketch the curve.

Answer

1) $(50, 0, 0)$

2) 

5.1E.12 Exercise $\mathbf{5.1}\boldsymbol{E}\mathbf{.11}$

Consider the curve described by the vector-valued function $\mathrm{r}(\mathrm{t}) = (50\mathrm{e}^{-\mathrm{t}}\cos\mathrm{t})\mathbf{i} + (50\mathrm{e}^{-\mathrm{t}}\sin\mathrm{t})\mathbf{j} + (5 - 5\mathrm{e}^{-\mathrm{t}})\mathbf{k}$.

1) What is the initial point of the path corresponding to $\mathrm{r}(0)$?

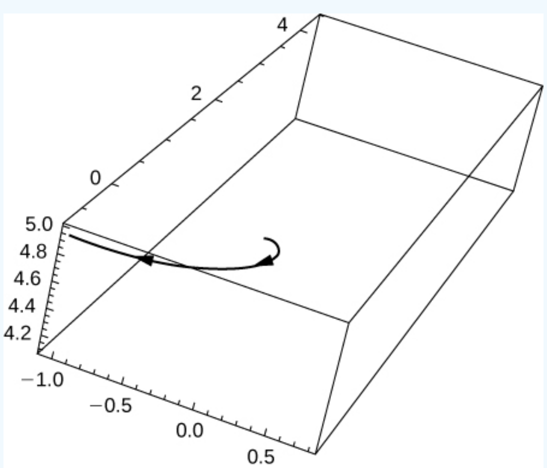
2) What is $\lim\limits_{\mathrm{t}\to\infty} \mathrm{r}(\mathrm{t})$?

3) **[T]** Use technology to sketch the curve.

4) Eliminate the parameter *t* to show that $\mathrm{z} = 5 - \frac{\mathrm{r}}{10}$ where $\mathrm{r} = \mathrm{x}^2 + \mathrm{y}^2$.

Answer

1) $(50, 0, 0)$

3) 

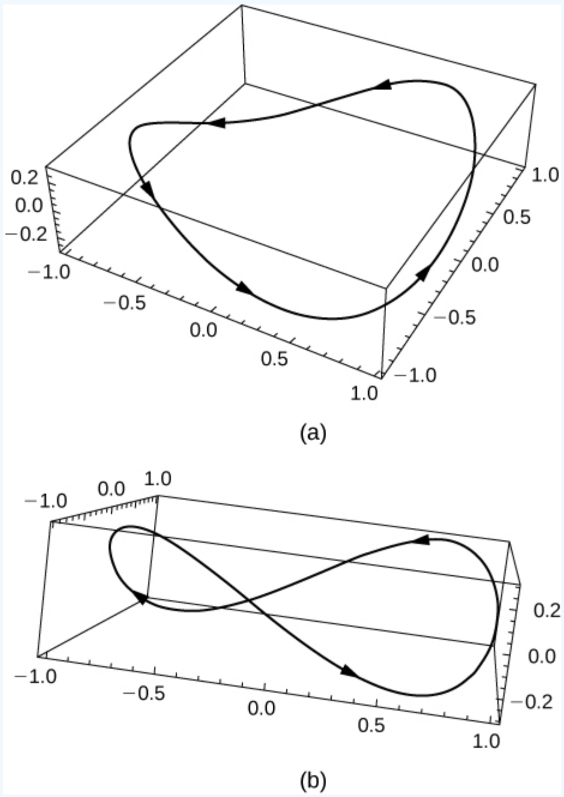
5.1E.13 Exercise $5.1E.12$

1) [T] Let $r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 0.3\sin(2t)\mathbf{k}$. Use technology to graph the curve (called the *roller-coaster curve*) over the interval $[0, 2\pi)$. Choose at least two views to determine the peaks and valleys.

2) **[T]** Use the result of the preceding problem to construct an equation of a roller coaster with a steep drop from the peak and steep incline from the "valley." Then, use technology to graph the equation.

3) Use the results of the preceding two problems to construct an equation of a path of a roller coaster with more than two turning points (peaks and valleys).

Answer



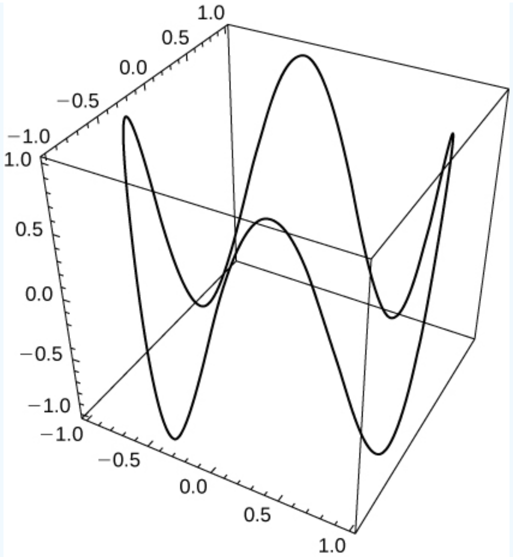
(a)

(b)

1)

3)

One possibility is $r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin(4t)\mathbf{k}$. By increasing the coefficient of *t* in the third component, the number of turning points will increase.



5.1E.14 Exercise $\mathbf{5.1E.13}$

1. Graph the curve $\mathrm{r(t)} = (4 + \cos(18\mathrm{t}))\cos(\mathrm{t})\mathbf{i} + (4 + \cos(18\mathrm{t})\sin(\mathrm{t}))\mathbf{j} + 0.3\sin(18\mathrm{t})\mathbf{k}$ using two viewing angles of your choice to see the overall shape of the curve.
2. Does the curve resemble a "slinky"?
3. What changes to the equation should be made to increase the number of coils of the slinky?

5.2: Calculus of Vector-Valued Functions

This page is a draft and is under active development.

To study the calculus of vector-valued functions, we follow a similar path to the one we took in studying real-valued functions. First, we define the derivative, then we examine applications of the derivative, then we move on to defining integrals. However, we will find some interesting new ideas along the way as a result of the vector nature of these functions and the properties of space curves.

5.2.0.0.1 Derivatives of Vector-Valued Functions

Now that we have seen what a vector-valued function is and how to take its limit, the next step is to learn how to differentiate a vector-valued function. The definition of the derivative of a vector-valued function is nearly identical to the definition of a real-valued function of one variable. However, because the range of a vector-valued function consists of vectors, the same is true for the range of the derivative of a vector-valued function.

Definition: Derivative of Vector-Valued Functions

The derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is

$$\vec{\mathbf{r}}\,'(t)=\lim_{\Delta t\to 0}\frac{\vec{\mathbf{r}}(t+\Delta t)-\vec{\mathbf{r}}(t)}{\Delta t} \tag{5.2.1}$$

provided the limit exists. If $\vec{\mathbf{r}}'(t)$ exists, then $\vec{\mathbf{r}}(t)$ is differentiable at t. If $\vec{\mathbf{r}}\,'(t)$ exists for all t in an open interval (a,b) then $\vec{\mathbf{r}}(t)$ is differentiable over the interval (a,b). For the function to be differentiable over the closed interval $[\mathrm{a},\mathrm{b}]$, the following two limits must exist as well:

$$\vec{\mathbf{r}}\,'(a)=\lim_{\Delta t\to 0^+}\frac{\vec{\mathbf{r}}(a+\Delta t)-\vec{\mathbf{r}}(a)}{\Delta t} \tag{5.2.2}$$

and

$$\vec{\mathbf{r}}\,'(b)=\lim_{\Delta t\to 0^-}\frac{\vec{\mathbf{r}}(b+\Delta t)-\vec{\mathbf{r}}(b)}{\Delta t} \tag{5.2.3}$$

Many of the rules for calculating derivatives of real-valued functions can be applied to calculating the derivatives of vector-valued functions as well. Recall that the derivative of a real-valued function can be interpreted as the slope of a tangent line or the instantaneous rate of change of the function. The derivative of a vector-valued function can be understood to be an instantaneous rate of change as well; for example, when the function represents the position of an object at a given point in time, the derivative represents its velocity at that same point in time.

We now demonstrate taking the derivative of a vector-valued function.

Example 5.2.1: Finding the Derivative of a Vector-Valued Function

Use the definition to calculate the derivative of the function

$$\vec{\mathbf{r}}(t)=(3t+4)\hat{\imath}+(t^2-4t+3)\hat{\jmath}.$$

Solution

Let's use Equation 5.2.1:

$$\begin{aligned}
\vec{\mathbf{r}}\,'(t) &= \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[(3(t+\Delta t)+4)\hat{i} + ((t+\Delta t)^2 - 4(t+\Delta t)+3)\hat{j}] - [(3t+4)\hat{i} + (t^2-4t+3)\hat{j}]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3t+3\Delta t+4)3\Delta t+4)\hat{i} - (3t+4)\hat{i} + (t^2+2t\Delta t+(\Delta t)^2 - 4t - 4\Delta t+3)\hat{j} - (t^2-4t+3)\hat{j}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3\Delta t)3\Delta t)\hat{i} + (2t\Delta t + (\Delta t)^2 - 4\Delta t)\hat{j}}{\Delta t} \\
&= \lim_{\Delta t \to 0} (3\hat{i} + (2t+\Delta t-4)\hat{j}) \\
&= 3\hat{i} + (2t-4)\hat{j}
\end{aligned}$$

Exercise 5.2.1

Use the definition to calculate the derivative of the function $\vec{\mathbf{r}}(t) = (2t^2+3)\hat{i} + (5t-6)\hat{j}$.

Hint

Use Equation 5.2.1.

Answer

$$\vec{\mathbf{r}}\,'(t) = 4t\hat{i} + 5\hat{j}$$

Notice that in the calculations in Example 5.2.1, we could also obtain the answer by first calculating the derivative of each component function, then putting these derivatives back into the vector-valued function. This is always true for calculating the derivative of a vector-valued function, whether it is in two or three dimensions. We state this in the following theorem. The proof of this theorem follows directly from the definitions of the limit of a vector-valued function and the derivative of a vector-valued function.

Theorem 5.2.1: Differentiation of Vector-Valued Functions

Let f, g, and h be differentiable functions of t.

1. If $\vec{\mathbf{r}}(t) = f(t)\hat{i} + g(t)\hat{j}$ then

$$\vec{\mathbf{r}}\,'(t) = f\,'(t)\hat{i} + g\,'(t)\hat{j}. \tag{5.2.4}$$

2. If $\vec{\mathbf{r}}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ then

$$\vec{\mathbf{r}}\,'(t) = f\,'(t)\hat{i} + g\,'(t)\hat{j} + h\,'(t)\hat{k}. \tag{5.2.5}$$

Example 5.2.2: Calculating the Derivative of Vector-Valued Functions

Use Theorem 5.2.1 to calculate the derivative of each of the following functions.

a. $\vec{\mathbf{r}}(t) = (6t+8)\hat{i} + (4t^2+2t-3)\hat{j}$
b. $\vec{\mathbf{r}}(t) = 3\cos t\hat{i} + 4\sin t\hat{j}$
c. $\vec{\mathbf{r}}(t) = e^t \sin t\hat{i} + e^t \cos t\hat{j} - e^{2t}\hat{k}$

Solution

We use Theorem 5.2.1 and what we know about differentiating functions of one variable.

a. The first component of

$$\vec{\mathbf{r}}(t) = (6t+8)\hat{i} + (4t^2+2t-3)\hat{j}$$

is $f(t)=6t+8$. The second component is $g(t)=4t^2+2t-3$. We have $f'(t)=6$ and $g'(t)=8t+2$, so the Theorem 5.2.1 gives $\vec{\mathbf{r}}\,'(t)=6\hat{i}+(8t+2)\hat{j}$.

b. The first component is $f(t)=3\cos t$ and the second component is $g(t)=4\sin t$. We have $f'(t)=-3\sin t$ and $g'(t)=4\cos t$, so we obtain $\vec{\mathbf{r}}\,'(t)=-3\sin t\hat{i}+4\cos t\hat{j}$.

c. The first component of $\vec{\mathbf{r}}(t)=e^t\sin t\hat{i}+e^t\cos t\hat{j}-e^{2t}\hat{k}$ is $f(t)=e^t\sin t$, the second component is $g(t)=e^t\cos t$, and the third component is $h(t)=-e^{2t}$. We have $f'(t)=e^t(\sin t+\cos t)$, $g'(t)=e^t(\cos t-\sin t)$, and $h'(t)=-2e^{2t}$, so the theorem gives $\vec{\mathbf{r}}\,'(t)=e^t(\sin t+\cos t)\hat{i}+e^t(\cos t-\sin t)\hat{j}-2e^{2t}\hat{k}$.

Exercise 5.2.2

Calculate the derivative of the function

$$\vec{\mathbf{r}}(t)=(t\ln t)\hat{i}+(5e^t)\hat{j}+(\cos t-\sin t)\hat{k}.$$

Hint

Identify the component functions and use Theorem 5.2.1.

Answer

$$\vec{\mathbf{r}}\,'(t)=(1+\ln t)\hat{i}+5e^t\hat{j}-(\sin t+\cos t)\hat{k}$$

We can extend to vector-valued functions the properties of the derivative that we presented previously. In particular, the constant multiple rule, the sum and difference rules, the product rule, and the chain rule all extend to vector-valued functions. However, in the case of the product rule, there are actually three extensions:

1. for a real-valued function multiplied by a vector-valued function,
2. for the dot product of two vector-valued functions, and
3. for the cross product of two vector-valued functions.

Theorem: Properties of the Derivative of Vector-Valued Functions

Let $\vec{\mathbf{r}}$ and $\vec{\mathbf{u}}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

i.	$\frac{d}{dt}[c\vec{\mathbf{r}}(t)]$	$=$	$c\vec{\mathbf{r}}\,'(t)$	Scalar multiple
ii.	$\frac{d}{dt}[\vec{\mathbf{r}}(t)\pm\vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}\,'(t)\pm\vec{\mathbf{u}}\,'(t)$	Sum and difference
iii.	$\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)]$	$=$	$f'(t)\vec{\mathbf{u}}(t)+f(t)\vec{\mathbf{u}}\,'(t)$	Scalar product
iv.	$\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}\,'(t)\cdot\vec{\mathbf{u}}(t)+\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}\,'(t)$	Dot product
v.	$\frac{d}{dt}[\vec{\mathbf{r}}(t)\times\vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}\,'(t)\times\vec{\mathbf{u}}(t)+\vec{\mathbf{r}}(t)\times\vec{\mathbf{u}}\,'(t)$	Cross product
vi.	$\frac{d}{dt}[\vec{\mathbf{r}}(f(t))]$	$=$	$\vec{\mathbf{r}}\,'(f(t))\cdot f'(t)$	Chain rule
vii.	If $\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)$	$=$	c, then $\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}\,'(t)=0$.	

Proof

The proofs of the first two properties follow directly from the definition of the derivative of a vector-valued function. The third property can be derived from the first two properties, along with the product rule. Let $\vec{\mathbf{u}}(t)=g(t)\hat{i}+h(t)\hat{j}$. Then

$$\begin{aligned}\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)] &= \frac{d}{dt}[f(t)(g(t)\hat{i}+h(t)\hat{j})] \\ &= \frac{d}{dt}[f(t)g(t)\hat{i}+f(t)h(t)\hat{j}] \\ &= \frac{d}{dt}[f(t)g(t)]\hat{i}+\frac{d}{dt}[f(t)h(t)]\hat{j} \\ &= (f'(t)g(t)+f(t)g'(t))\hat{i}+(f'(t)h(t)+f(t)h'(t))\hat{j} \\ &= f'(t)\vec{\mathbf{u}}(t)+f(t)\vec{\mathbf{u}}'(t).\end{aligned}$$

To prove property iv. let $\vec{\mathbf{r}}(t)=f_1(t)\hat{i}+g_1(t)\hat{j}$ and $\vec{\mathbf{u}}(t)=f_2(t)\hat{i}+g_2(t)\hat{j}$. Then

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)] &= \frac{d}{dt}[f_1(t)f_2(t)+g_1(t)g_2(t)] \\ &= f_1'(t)f_2(t)+f_1(t)f_2'(t)+g_1'(t)g_2(t)+g_1(t)g_2'(t) = f_1'(t)f_2(t)+g_1'(t)g_2(t)+f_1(t)f_2'(t)+g_1(t)g_2'(t) \\ &= (f_1'\hat{i}+g_1'\hat{j})\cdot(f_2\hat{i}+g_2\hat{j})+(f_1\hat{i}+g_1\hat{j})\cdot(f_2'\hat{i}+g_2'\hat{j}) \\ &= \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{u}}(t)+\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}'(t).\end{aligned}$$

The proof of property v. is similar to that of property iv. Property vi. can be proved using the chain rule. Last, property vii. follows from property iv:

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)] &= \frac{d}{dt}[c] \\ \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t)+\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0 \\ 2\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0 \\ \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0\end{aligned}$$

Now for some examples using these properties.

Example $5.2.3$: Using the Properties of Derivatives of Vector-Valued Functions

Given the vector-valued functions

$$\vec{\mathbf{r}}(t)=(6t+8)\hat{i}+(4t^2+2t-3)\hat{j}+5t\hat{k}$$

and

$$\vec{\mathbf{u}}(t)=(t^2-3)\hat{i}+(2t+4)\hat{j}+(t^3-3t)\hat{k},$$

calculate each of the following derivatives using the properties of the derivative of vector-valued functions.

a. $\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)]$
b. $\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}'(t)]$

Solution

We have $\vec{\mathbf{r}}'(t)=6\hat{i}+(8t+2)\hat{j}+5\hat{k}$ and $\vec{\mathbf{u}}'(t)=2t\hat{i}+2\hat{j}+(3t^2-3)\hat{k}$. Therefore, according to **property iv**:

1.

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)] &= \vec{\mathbf{r}}\,'(t)\cdot\vec{\mathbf{u}}(t)+\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}\,'(t)\\ &= (6\hat{i}+(8t+2)\hat{j}+5\hat{k})\cdot((t^2-3)\hat{i}+(2t+4)\hat{j}+(t^3-3t)\hat{k})\\ &\quad+((6t+8)\hat{i}+(4t^2+2t-3)\hat{j}+5t\hat{k})\cdot(2t\hat{i}+2\hat{j}+(3t^2-3)\hat{k})\\ &=6(t^2-3)+(8t+2)(2t+4)+5(t^3-3t)\\ &\quad+2t(6t+8)+2(4t^2+2t-3)+5t(3t^2-3)\\ &=20t^3+42t^2+26t-16.\end{aligned}$$

2. First, we need to adapt **property v** for this problem:

$$\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}\,'(t)]=\vec{\mathbf{u}}\,'(t)\times\vec{\mathbf{u}}\,'(t)+\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}\,''(t).$$

Recall that the cross product of any vector with itself is zero. Furthermore,$\vec{\mathbf{u}}\,''(t)$ represents the second derivative of $\vec{\mathbf{u}}(t)$:

$$\vec{\mathbf{u}}\,''(t)=\frac{d}{dt}[\vec{\mathbf{u}}\,'(t)]=\frac{d}{dt}[2t\hat{i}+2\hat{j}+(3t^2-3)\hat{k}]=2\hat{i}+6t\hat{k}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}\,'(t)] &=0+((t^2-3)\hat{i}+(2t+4)\hat{j}+(t^3-3t)\hat{k})\times(2\hat{i}+6t\hat{k})\\ &=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ t^2-3 & 2t+4 & t^3-3t\\ 2 & 0 & 6t\end{vmatrix}\\ &=6t(2t+4)\hat{i}-(6t(t^2-3)-2(t^3-3t))\hat{j}-2(2t+4)\hat{k}\\ &=(12t^2+24t)\hat{i}+(12t-4t^3)\hat{j}-(4t+8)\hat{k}.\end{aligned}$$

Exercise 5.2.3

Calculate $\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}\,'(t)]$ and $\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{r}}(t)]$ for the vector-valued functions:

- $\vec{\mathbf{r}}(t)=\cos t\hat{i}+\sin t\hat{j}-e^{2t}\hat{k}$
- $\vec{\mathbf{u}}(t)=t\hat{i}+\sin t\hat{j}+\cos t\hat{k}$,

Hint

Follow the same steps as in Example 5.2.3.

Answer

$\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}\,'(t)]=8e^{4t}$

$\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{r}}(t)]=-(e^{2t}(\cos t+2\sin t)+\cos 2t)\hat{i}+(e^{2t}(2t+1)-\sin 2t)\hat{j}+(t\cos t+\sin t-\cos 2t)\hat{k}$

5.2.1 Tangent Vectors and Unit Tangent Vectors

Recall that the derivative at a point can be interpreted as the slope of the tangent line to the graph at that point. In the case of a vector-valued function, the derivative provides a tangent vector to the curve represented by the function. Consider the vector-valued function

$$\vec{\mathbf{r}}(t)=\cos t\hat{i}+\sin t\hat{j} \tag{5.2.6}$$

The derivative of this function is

$$\vec{\mathbf{r}}\,'(t)=-\sin t\hat{i}+\cos t\hat{j}$$

If we substitute the value $t=\pi/6$ into both functions we get

$$\vec{\mathbf{r}}\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}\hat{i}+\frac{1}{2}\hat{j}$$

and

$$\vec{\mathbf{r}}\,'\left(\frac{\pi}{6}\right)=-\frac{1}{2}\hat{i}+\frac{\sqrt{3}}{2}\hat{j}.$$

The graph of this function appears in Figure $5.2.1$, along with the vectors $\vec{\mathbf{r}}\left(\frac{\pi}{6}\right)$ and $\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right)$.

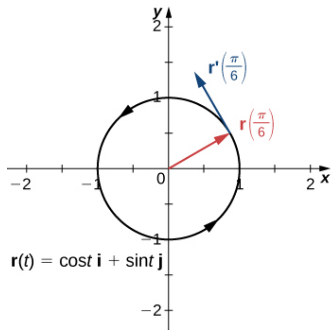


Figure $5.2.1$: The tangent line at a point is calculated from the derivative of the vector-valued function $\vec{\mathbf{r}}(t)$.

Notice that the vector $\vec{\mathbf{r}}\,'\left(\frac{\pi}{6}\right)$ is tangent to the circle at the point corresponding to $t=\frac{\pi}{6}$. This is an example of a tangent vector to the plane curve defined by Equation 5.2.6.

Definition: principal unit tangent vector

Let C be a curve defined by a vector-valued function $\vec{\mathbf{r}}$, and assume that $\vec{\mathbf{r}}\,'(t)$ exists when $\mathrm{t}=\mathrm{t}_0$ A tangent vector $\vec{\mathbf{r}}$ at $t=t_0$ is any vector such that, when the tail of the vector is placed at point $\vec{\mathbf{r}}(t_0)$ on the graph, vector $\vec{\mathbf{r}}$ is tangent to curve C. Vector $\vec{\mathbf{r}}\,'(t_0)$ is an example of a tangent vector at point $t=t_0$. Furthermore, assume that $\vec{\mathbf{r}}\,'(t)\neq 0$. The principal unit tangent vector at t is defined to be

$$\vec{\mathbf{T}}(t)=\frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|}, \tag{5.2.7}$$

provided $\|\vec{\mathbf{r}}\,'(t)\|\neq 0$.

The unit tangent vector is exactly what it sounds like: a unit vector that is tangent to the curve. To calculate a unit tangent vector, first find the derivative $\vec{\mathbf{r}}\,'(t)$. Second, calculate the magnitude of the derivative. The third step is to divide the derivative by its magnitude.

Example $5.2.4$: Finding a Unit Tangent Vector

Find the unit tangent vector for each of the following vector-valued functions:

a. $\vec{\mathbf{r}}(t)=\cos t\hat{i}+\sin t\hat{j}$
b. $\vec{\mathbf{u}}(t)=(3t^2+2t)\hat{i}+(2-4t^3)\hat{j}+(6t+5)\hat{k}$

Solution

a.
$$\begin{array}{llll}\text{First step:} & \vec{\mathbf{r}}\,'(t) & = & -\sin t\hat{i}+\cos t\hat{j}\\ \text{Second step:} & \|\vec{\mathbf{r}}\,'(t)\| & = & \sqrt{(-\sin t)^2+(\cos t)^2}=1\\ \text{Third step:} & \vec{\mathbf{T}}(t) & = & \dfrac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|}=\dfrac{-\sin t\hat{i}+\cos t\hat{j}}{1}=-\sin t\hat{i}+\cos t\hat{j}\end{array}$$

b.

$$\begin{aligned}
\text{First step:} \quad \vec{\mathbf{r}}\,'(t) &= (6t+2)\hat{i} - 12t^2\hat{j} + 6\hat{k} \\
\text{Second step:} \quad \|\vec{\mathbf{r}}\,'(t)\| &= \sqrt{(6t+2)^2 + (-12t^2)^2 + 6^2} \\
&= \sqrt{144t^4 + 36t^2 + 24t + 40} \\
&= 2\sqrt{36t^4 + 9t^2 + 6t + 10} \\
\text{Third step:} \quad \vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} = \frac{(6t+2)\hat{i} - 12t^2\hat{j} + 6\hat{k}}{2\sqrt{36t^4 + 9t^2 + 6t + 10}} \\
&= \frac{3t+1}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\hat{i} - \frac{6t^2}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\hat{j} + \frac{3}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\hat{k}
\end{aligned}$$

Exercise 5.2.4

Find the unit tangent vector for the vector-valued function

$$\vec{\mathbf{r}}(t) = (t^2 - 3)\hat{i} + (2t+1)\hat{j} + (t-2)\hat{k}.$$

Hint

Follow the same steps as in Example 5.2.4.

Answer

$$\vec{\mathbf{T}}(t) = \frac{2t}{\sqrt{4t^2+5}}\hat{i} + \frac{2}{\sqrt{4t^2+5}}\hat{j} + \frac{1}{\sqrt{4t^2+5}}\hat{k}$$

5.2.2 Integrals of Vector-Valued Functions

We introduced antiderivatives of real-valued functions in Antiderivatives and definite integrals of real-valued functions in The Definite Integral. Each of these concepts can be extended to vector-valued functions. Also, just as we can calculate the derivative of a vector-valued function by differentiating the component functions separately, we can calculate the antiderivative in the same manner. Furthermore, the Fundamental Theorem of Calculus applies to vector-valued functions as well.

The antiderivative of a vector-valued function appears in applications. For example, if a vector-valued function represents the velocity of an object at time *t*, then its antiderivative represents position. Or, if the function represents the acceleration of the object at a given time, then the antiderivative represents its velocity.

Definition: Definite and indefinite integrals of vector-valued functions

Let *f*, *g*, and *h* be integrable real-valued functions over the closed interval $[a, b]$.

1. The indefinite integral of a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{i} + g(t)\hat{j}$ is

$$\int [f(t)\hat{i} + g(t)\hat{j}]dt = \left[\int f(t)dt\right]\hat{i} + \left[\int g(t)dt\right]\hat{j}. \tag{5.2.8}$$

The definite integral of a vector-valued function is

$$\int_a^b [f(t)\hat{i} + g(t)\hat{j}]dt = \left[\int_a^b f(t)dt\right]\hat{i} + \left[\int_a^b g(t)dt\right]\hat{j}. \tag{5.2.9}$$

2. The indefinite integral of a vector-valued function $r(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ is

$$\int [f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}]dt = \left[\int f(t)dt\right]\hat{i} + \left[\int g(t)dt\right]\hat{j} + \left[\int h(t)dt\right]\hat{k}. \tag{5.2.10}$$

The definite integral of the vector-valued function is

$$\int_a^b [f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}]dt = \left[\int_a^b f(t)dt\right]\hat{i} + \left[\int_a^b g(t)dt\right]\hat{j} + \left[\int_a^b h(t)dt\right]\hat{k}. \tag{5.2.11}$$

Since the indefinite integral of a vector-valued function involves indefinite integrals of the component functions, each of these component integrals contains an integration constant. They can all be different. For example, in the two-dimensional case, we can have

$$\int \mathrm{f(t)dt = F(t) + C_1} \text{ and } \int \mathrm{g(t)dt = G(t) + C_2},$$

where *F* and *G* are antiderivatives of *f* and *g*, respectively. Then

$$\begin{aligned} \int [\mathrm{f(t)\hat{i} + g(t)\hat{j}]dt} &= \left[\int \mathrm{f(t)dt}\right]\mathrm{\hat{i}} + \left[\int \mathrm{g(t)dt}\right]\mathrm{\hat{j}} \\ &= \mathrm{(F(t) + C_1)\hat{i} + (G(t) + C_2)\hat{j}} \\ &= \mathrm{F(t)\hat{i} + G(t)\hat{j} + C_1\hat{i} + C_2\hat{j}} \\ &= \mathrm{F(t)\hat{i} + G(t)\hat{j}} + \vec{\mathbf{C}} \end{aligned}$$

where $\vec{\mathbf{C}} = C_1\hat{i} + C_2\hat{j}$. Therefore, the *integration constants* becomes a *constant vector*.

Example 5.2.5: Integrating Vector-Valued Functions

Calculate each of the following integrals:

a. $\int [(3t^2 + 2t)\hat{i} + (3t - 6)\hat{j} + (6t^3 + 5t^2 - 4)\hat{k}]dt$
b. $\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]dt$
c. $\int_0^{\frac{\pi}{3}} [\sin 2t\hat{i} + \tan t\hat{j} + e^{-2t}\hat{k}]dt$

Solution

a. We use the first part of the definition of the integral of a space curve:

$$\begin{aligned} \int [\mathrm{(3t^2 + 2t)\hat{i} + (3t - 6)\hat{j} + (6t^3 + 5t^2 - 4)\hat{k}]dt} &= \left[\int \mathrm{3t^2 + 2tdt}\right]\mathrm{\hat{i}} + \left[\int \mathrm{3t - 6dt}\right]\mathrm{\hat{j}} + \left[\int \mathrm{6t^3 + 5t^2 - 4dt}\right]\mathrm{\hat{k}} \\ &= \mathrm{(t^3 + t^2)\hat{i} + (\frac{3}{2}t^2 - 6t)\hat{j} + (\frac{3}{2}t^4 + \frac{5}{3}t^3 - 4t)\hat{k}} + \vec{\mathbf{C}}. \end{aligned}$$

b. First calculate $\mathrm{\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle}$:

$$\begin{aligned} \mathrm{\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \mathrm{t} & \mathrm{t^2} & \mathrm{t^3} \\ \mathrm{t^3} & \mathrm{t^2} & \mathrm{t} \end{vmatrix} \\ &= \mathrm{(t^2(t) - t^3(t^2))\hat{i} - (t^2 - t^3(t^3))\hat{j} + (t(t^2) - t^2(t^3))\hat{k}} \\ &= \mathrm{(t^3 - t^5)\hat{i} + (t^6 - t^2)\hat{j} + (t^3 - t^5)\hat{k}}. \end{aligned}$$

Next, substitute this back into the integral and integrate:

$$\begin{aligned} \int [\mathrm{\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]dt} &= \int \mathrm{(t^3 - t^5)\hat{i} + (t^6 - t^2)\hat{j} + (t^3 - t^5)\hat{k}dt} \\ &= \mathrm{(\frac{t^4}{4} - \frac{t^6}{6})\hat{i} + (\frac{t^7}{7} - \frac{t^3}{3})\hat{j} + (\frac{t^4}{4} - \frac{t^6}{6})\hat{k}} + \vec{\mathbf{C}}. \end{aligned}$$

c. Use the second part of the definition of the integral of a space curve:

$$\int_0^{\frac{\pi}{3}}[\sin 2\mathrm{t}\hat{\mathrm{i}}+\tan\mathrm{t}\hat{\mathrm{j}}+\mathrm{e}^{-2\mathrm{t}}\hat{\mathrm{k}}]\mathrm{dt} = \left[\int_0^{\frac{\pi}{3}}\sin 2\mathrm{tdt}\right]\hat{\mathrm{i}}+\left[\int_0^{\frac{\pi}{3}}\tan\mathrm{tdt}\right]\hat{\mathrm{j}}+\left[\int_0^{\frac{\pi}{3}}\mathrm{e}^{-2\mathrm{t}}\mathrm{dt}\right]\hat{\mathrm{k}}$$

$$=(-12\cos 2\mathrm{t})|_0^{\pi/3}\hat{\mathrm{i}}-(\ln(\cos\mathrm{t}))|_0^{\pi/3}\hat{\mathrm{j}}-(\frac{1}{2}\mathrm{e}^{-2\mathrm{t}})|_0^{\pi/3}\hat{\mathrm{k}}$$

$$=(-\frac{1}{2}\cos\frac{2\pi}{3}+\frac{1}{2}\cos 0)\hat{\mathrm{i}}-(\ln(\cos\frac{\pi}{3})-\ln(\cos 0))\hat{\mathrm{j}}-(\frac{1}{2}\mathrm{e}^{-2\pi/3}-\frac{1}{2}\mathrm{e}^{-2(0)})\hat{\mathrm{k}}$$

$$=(\frac{1}{4}+\frac{1}{2})\hat{\mathrm{i}}-(-\ln 2)\hat{\mathrm{j}}-(\frac{1}{2}\mathrm{e}^{-2\pi/3}-\frac{1}{2})\hat{\mathrm{k}}$$

$$=\frac{3}{4}\hat{\mathrm{i}}+(\ln 2)\hat{\mathrm{j}}+(\frac{1}{2}-\frac{1}{2}\mathrm{e}^{-2\pi/3})\hat{\mathrm{k}}.$$

Exercise 5.2.5

Calculate the following integral:

$$\int_1^3[(2t+4)\hat{i}+(3t^2-4t)\hat{j}]dt$$

Hint

Use the definition of the definite integral of a plane curve.

Answer

$$\int_1^3[(2t+4)\hat{i}+(3t^2-4t)\hat{j}]dt=16\hat{i}+10\hat{j}$$

5.2.3 Summary

- To calculate the derivative of a vector-valued function, calculate the derivatives of the component functions, then put them back into a new vector-valued function.
- Many of the properties of differentiation of scalar fundtions also apply to vector-valued functions.
- The derivative of a vector-valued function $r(t)$ is also a tangent vector to the curve. The unit tangent vector $T(t)$ is calculated by dividing the derivative of a vector-valued function by its magnitude.
- The antiderivative of a vector-valued function is found by finding the antiderivatives of the component functions, then putting them back together in a vector-valued function.
- The definite integral of a vector-valued function is found by finding the definite integrals of the component functions, then putting them back together in a vector-valued function.

5.2.4 Key Equations

- **Derivative of a vector-valued function**

$$\vec{\mathbf{r}}\,'(t)=\lim_{\Delta t\to 0}\frac{\vec{\mathbf{r}}(t+\Delta t)-\vec{\mathbf{r}}(t)}{\Delta t}$$

- **Principal unit tangent vector**

$$\vec{\mathbf{T}}(t)=\frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|}$$

- **Indefinite integral of a vector-valued function**

$$\int[f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}]dt=\left[\int f(t)dt\right]\hat{i}+\left[\int g(t)dt\right]\hat{j}+\left[\int h(t)dt\right]\hat{k}$$

- **Definite integral of a vector-valued function**

$$\int_a^b [f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}]dt = \left[\int_a^b f(t)dt\right]\hat{i} + \left[\int_a^b g(t)dt\right]\hat{j} + \left[\int_a^b h(t)dt\right]\hat{k}$$

definite integral of a vector-valued function

the vector obtained by calculating the definite integral of each of the component functions of a given vector-valued function, then using the results as the components of the resulting function

derivative of a vector-valued function

the derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is $\vec{\mathbf{r}}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t}$, provided the limit exists

indefinite integral of a vector-valued function

a vector-valued function with a derivative that is equal to a given vector-valued function

principal unit tangent vector

a unit vector tangent to a curve C

tangent vector

to $\vec{\mathbf{r}}(t)$ at $t = t_0$ any vector $\vec{\mathbf{v}}$ such that, when the tail of the vector is placed at point $\vec{\mathbf{r}}(t_0)$ on the graph, vector $\vec{\mathbf{v}}$ is tangent to curve C

5.2.4.0.1 Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

5.2E:

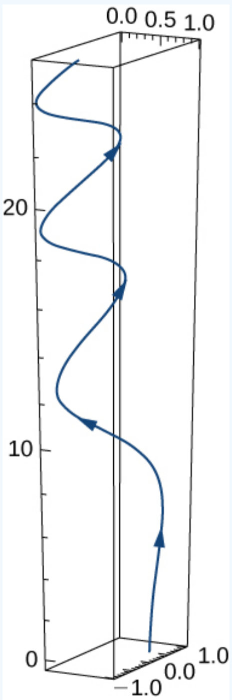
5.2E.1 Exercise $5.2E.1$

Compute the derivatives of the vector-valued functions.

1) $r(t)=t^3\mathbf{i}+3t^2\mathbf{j}+\frac{t^3}{6}\mathbf{k}$

2) $r(t)=\sin(t)\mathbf{i}+\cos(t)\mathbf{j}+e^t\mathbf{k}$

3) $r(t)=e^{-t}\mathbf{i}+\sin(3t)\mathbf{j}+10\sqrt{t}\mathbf{k}$. A sketch of the graph is shown here. Notice the varying periodic nature of the graph.



4) $r(t)=e^t\mathbf{i}+2e^t\mathbf{j}+\mathbf{k}$

5) $r(t)=\mathbf{i}+\mathbf{j}+\mathbf{k}$

6) $r(t)=te^t\mathbf{i}+t\ln(t)\mathbf{j}+\sin(3t)\mathbf{k}$

7) $r(t)=\frac{1}{t+1}\mathbf{i}+\arctan(t)\mathbf{j}+\ln t^3\mathbf{k}$

8) $r(t)=\tan(2t)\mathbf{i}+\sec(2t)\mathbf{j}+\sin^2(t)\mathbf{k}$

9) $r(t)=3\mathbf{i}+4\sin(3t)\mathbf{j}+t\cos(t)\mathbf{k}$

10) $r(t)=t^2\mathbf{i}+te^{-2t}\mathbf{j}-5e^{-4t}\mathbf{k}$

Answer

1) $\langle 3t^2, 6t, \frac{1}{2}t^2\rangle$

3) $\langle -e^{-t}, 3\cos(3t), 5t\rangle$

5) $\langle 0,0,0\rangle$

7) $\langle \frac{-1}{(t+1)^2}, \frac{1}{1+t^2}, \frac{3}{t}\rangle$

9) $\langle 0, 12\cos(3t), \cos t - t\sin t\rangle$

5.2E.2 Exercise $5.2E.2$

For the following problems, find a tangent vector at the indicated value of *t*.

1) $r(t) = t\mathbf{i} + \sin(2t)\mathbf{j} + \cos(3t)\mathbf{k}$; $t = \frac{\pi}{3}$

2) $r(t) = 3t^3\mathbf{i} + 2t^2\mathbf{j} + \frac{1}{t}\mathbf{k}; t = 1$

3) $r(t) = 3e^t\mathbf{i} + 2e^{-3t}\mathbf{j} + 4e^{2t}\mathbf{k}; t = \ln(2)$

4) $r(t) = \cos(2t)\mathbf{i} + 2\sin t\mathbf{j} + t^2\mathbf{k}; t = \frac{\pi}{2}$

Answer

1) $\frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle$

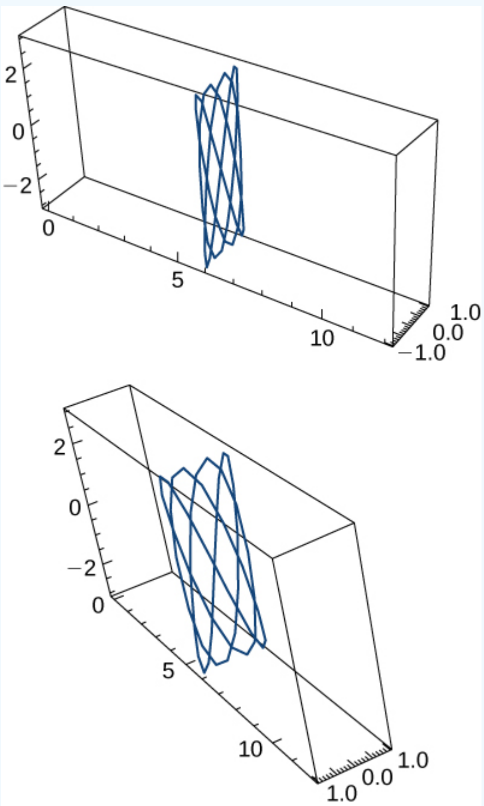
3) $\frac{1}{\sqrt{1060.5625}}\langle 6, -34, 32\rangle$

5.2E.3 Exercise $5.2E.3$

Find the unit tangent vector for the following parameterized curves.

1) $r(t) = 6\mathbf{i} + \cos(3t)\mathbf{j} + 3\sin(4t)\mathbf{k}, 0 \leq t < 2\pi$

2) $r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin t\mathbf{k}, 0 \leq t < 2\pi$. Two views of this curve are presented here:



3) $r(t) = 3\cos(4t)\mathbf{i} + 3\sin(4t)\mathbf{j} + 5t\mathbf{k}, 1 \leq t \leq 2$

4) $r(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$

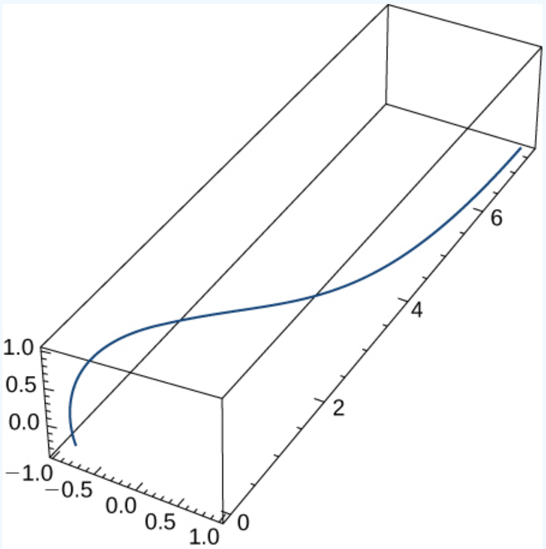
Answer

1) $\frac{1}{\sqrt{9\sin^2(3t)+144\cos^2(4t)}}\langle 0, -3\sin(3t), 12\cos(4t)\rangle$

3) $T(t) = -\frac{12}{13}\sin(4t)\mathbf{i} + \frac{12}{13}\cos(4t)\mathbf{j} + \frac{5}{13}\mathbf{k}$

5.2E.4 Exercise $\mathbf{5.2}\boldsymbol{E}.\mathbf{4}$

Let $r(t) = t\mathbf{i} + t^2\mathbf{j} - t^4\mathbf{k}$ and $s(t) = \sin(t)\mathbf{i} + e^t\mathbf{j} + \cos(t)\mathbf{k}$ Here is the graph of the function:



Find the following.

1) $\frac{d}{dt}[r(t^2)]$

2) $\frac{d}{dt}[t^2 \cdot s(t)]$

3) $\frac{d}{dt}[r(t) \cdot s(t)]$

Answer

1) $\langle 2t, 4t^3, -8t^7 \rangle$

3) $\sin(t) + 2te^t - 4t^3\cos(t) + t\cos(t) + t^2e^t + t^4\sin(t)$

5.2E.5 Exercise $\mathbf{5.2}\boldsymbol{E}.\mathbf{5}$

1) Compute the first, second, and third derivatives of $r(t) = 3t\mathbf{i} + 6\ln(t)\mathbf{j} + 5e^{-3t}\mathbf{k}$.

2) Find $r'(t) \cdot r''(t)$ for $r(t) = -3t^5\mathbf{i} + 5t\mathbf{j} + 2t^2\mathbf{k}$.

Answer

$900t^7 + 16t$

5.2E.6 Exercise $\mathbf{5.2}\boldsymbol{E}.\mathbf{6}$

1) The acceleration function, initial velocity, and initial position of a particle are

$a(t) = -5\cos t\mathbf{i} - 5\sin t\mathbf{j}$, $v(0) = 9\mathbf{i} + 2\mathbf{j}$, and $r(0) = 5\mathbf{i}$. Find $v(t)$ and $r(t)$.

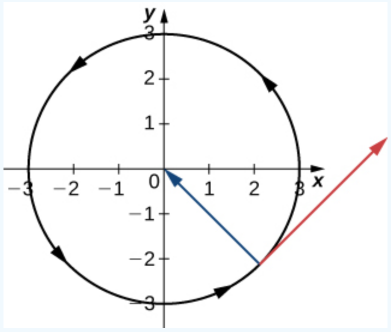
2) The position vector of a particle is $r(t) = 5\sec(2t)i - 4\tan(t)j + 7t^2k$.

a. Graph the position function and display a view of the graph that illustrates the asymptotic behavior of the function.

b. Find the velocity as *t* approaches but is not equal to $\frac{\pi}{4}$ (if it exists)

3) Find the velocity and the speed of a particle with the position function $r(t) = \left(\frac{2t-1}{2t+1}\right)\mathbf{i} + \ln(1-4t^2)\mathbf{j}$. The speed of a particle is the magnitude of the velocity and is represented by $\|r'(t)\|$.

4) A particle moves on a circular path of radius b according to the function $\mathrm{r}(\mathrm{t}) = \mathrm{b}\cos(\omega \mathrm{t})\mathbf{i} + \mathrm{b}\sin(\omega)\mathbf{j}$, where ω is the angular velocity, $\frac{d\theta}{dt}$.



Find the velocity function and show that $\mathrm{v}(\mathrm{t})$ is always orthogonal to $\mathrm{r}(\mathrm{t})$.

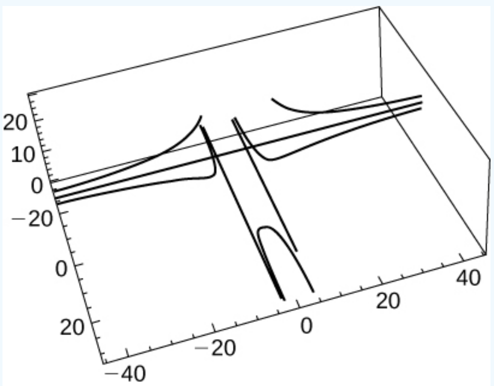
5) Show that the speed of the particle is proportional to the angular velocity.

6) Evaluate $\frac{d}{dt}[\mathrm{u}(\mathrm{t}) \times \mathrm{u}'(\mathrm{t})]$ given $\mathrm{u}(\mathrm{t}) = \mathrm{t}^2\mathbf{i} - 2\mathrm{t}\mathbf{j} + \mathbf{k}$.

7) Find the antiderivative of $\mathrm{r}'(\mathrm{t}) = \cos(2\mathrm{t})\mathbf{i} - 2\sin\mathrm{t}\mathbf{j} + \frac{1}{1+\mathrm{t}^2}\mathbf{k}$ that satisfies the initial condition $\mathrm{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

8) Evaluate $\int_0^3 \|\mathrm{ti} + \mathrm{t}^2\mathrm{j}\|\mathrm{dt}$.

Answer

2) 

, Undefined or infinite.

4) $\mathrm{r}'(\mathrm{t}) = -\mathrm{b}\omega\sin(\omega\mathrm{t})\mathbf{i} + \mathrm{b}\omega\cos(\omega\mathrm{t})\mathbf{j}$. To show orthogonality, note that $\mathrm{r}'(\mathrm{t}) \cdot \mathrm{r}(\mathrm{t}) = 0$.

6) $0\mathbf{i} + 2\mathbf{j} + 4\mathrm{t}\mathbf{j}$

8) $\frac{1}{3}(10^{\frac{3}{2}} - 1)$

5.2E.7 Exercise $5.2E.7$

1) An object starts from rest at point $\mathrm{P}(1, 2, 0)$ and moves with an acceleration of$\mathrm{a}(\mathrm{t}) = \mathbf{j} + 2\mathbf{k}$, where $\|\mathrm{a}(\mathrm{t})\|$ is measured in feet per second per second. Find the location of the object after $\mathrm{t} = 2$ sec.

2) Show that if the speed of a particle travelling along a curve represented by a vector-valued function is constant, then the velocity function is always perpendicular to the acceleration function.

3) Given $\mathrm{r}(\mathrm{t}) = \mathrm{t}\mathbf{i} + 3\mathrm{t}\mathbf{j} + \mathrm{t}^2\mathbf{k}$ and $\mathrm{u}(\mathrm{t}) = 4\mathrm{t}\mathbf{i} + \mathrm{t}^2\mathbf{j} + \mathrm{t}^3\mathbf{k}$, find $\frac{d}{dt}(\mathrm{r}(\mathrm{t}) \times \mathrm{u}(\mathrm{t}))$.

4) Given $\mathrm{r}(\mathrm{t}) = \langle \mathrm{t} + \cos\mathrm{t}, \mathrm{t} - \sin\mathrm{t} \rangle$, find the velocity and the speed at any time.

5) Find the velocity vector for the function $\mathrm{r}(\mathrm{t}) = \langle \mathrm{e}^{\mathrm{t}}, \mathrm{e}^{-\mathrm{t}}, 0 \rangle$.

6) Find the equation of the tangent line to the curve $\mathrm{r}(\mathrm{t}) = \langle \mathrm{e}^{\mathrm{t}}, \mathrm{e}^{-\mathrm{t}}, 0 \rangle$ at $\mathrm{t} = 0$.

7) Describe and sketch the curve represented by the vector-valued function $\mathrm{r}(\mathrm{t}) = \langle 6\mathrm{t}, 6\mathrm{t} - \mathrm{t}^2 \rangle$.

8) Locate the highest point on the curve $r(t)=\langle 6t, 6t-t^2\rangle$ and give the value of the function at this point.

Answer

2)

$$\begin{aligned} \|v(t)\| &= k && (5.2E.1)\\ v(t)\cdot v(t) &= k && (5.2E.2)\\ ddt(v(t)\cdot v(t)) &= \frac{d}{dt}k=0 && (5.2E.3)\\ v(t)\cdot v'(t)+v'(t)\cdot v(t) &= 0 && (5.2E.4)\\ 2v(t)\cdot v'(t) &= 0 && (5.2E.5)\\ v(t)\cdot v'(t) &= 0 && (5.2E.6)\end{aligned}$$

The last statement implies that the velocity and acceleration are perpendicular or orthogonal.

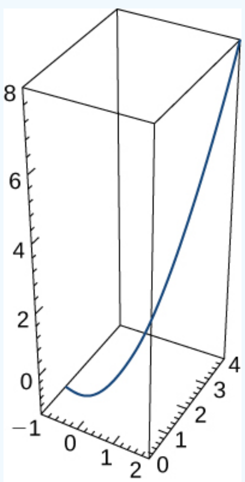
4) $v(t)=\langle 1-\sin t, 1-\cos t\rangle$, speed $=-v(t)\|=\sqrt{4-2(\sin t+\cos t)}$

6) $x-1=t, y-1=-t, z-0=0$

8) $r(t)=\langle 18,9\rangle$ at $t=3$

5.2E.8 Exercise $5.2E.8$

The position vector for a particle is $r(t)=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$ .The graph is shown here:



1) Find the velocity vector at any time.

2) Find the speed of the particle at time $t=2$ sec.

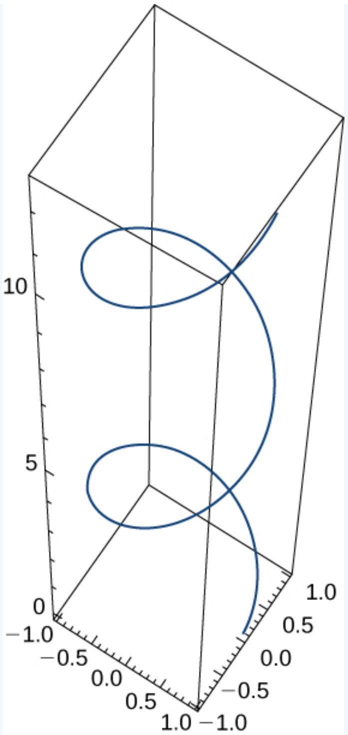
3) Find the acceleration at time $t=2$ sec.

Answer

2) $\sqrt{593}$

5.2E.9 Exercise $5.2E.9$

A particle travels along the path of a helix with the equation $r(t)=\cos(t)\mathbf{i}+\sin(t)\mathbf{j}+t\mathbf{k}$. See the graph presented here:



Find the following:

1) Velocity of the particle at any time

2) Speed of the particle at any time

3) Acceleration of the particle at any time

4) Find the unit tangent vector for the helix.

Answer

1) $v(t) = \langle -\sin t, \cos t, 1 \rangle$

3) $a(t) = -\cos t\mathbf{i} - \sin t\mathbf{j} + 0\mathbf{j}$

5.2E.10 Exercise $5.2E.10$

A particle travels along the path of an ellipse with the equation $r(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j} + 0\mathbf{k}$. Find the following:

1) Velocity of the particle

2) Speed of the particle at $t = \frac{\pi}{4}$

3) Acceleration of the particle at $t = \frac{\pi}{4}$

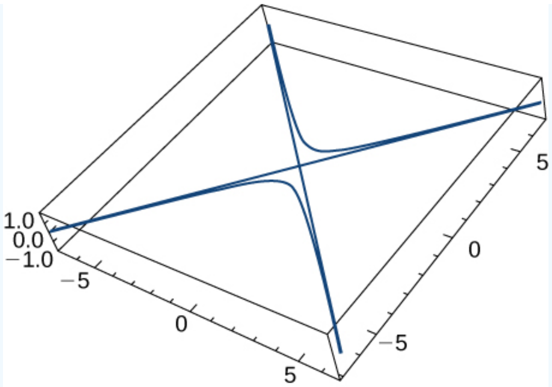
Answer

1) $v(t) = \langle -\sin t, 2\cos t, 0 \rangle$

3) $a(t) = \langle -\frac{\sqrt{2}}{2}, -\sqrt{2}, 0 \rangle$

5.2E.11 Exercise $5.2E.11$

Given the vector-valued function $r(t) = \langle \tan t, \sec t, 0 \rangle$ (graph is shown here), find the following:



1) Velocity

2) Speed

3) Acceleration

4) Find the minimum speed of a particle traveling along the curve $r(t)=\langle t+\cos t, t-\sin t\rangle$ \mathrm{t∈[0,2π)}\).

Answer

2) $\|v(t)\|=\sqrt{\sec^4 t+\sec^2 t\tan^2 t}=\sqrt{\sec^2 t(\sec^2 t+\tan^2 t)}$

4) 2

5.2E.12 Exercise $\mathbf{5.2E.12}$

Given $r(t)=t\mathbf{i}+2\sin t\mathbf{j}+2\cos t\mathbf{k}$ and $u(t)=\frac{1}{t}\mathbf{i}+2\sin t\mathbf{j}+2\cos t\mathbf{k}$, find the following:

1) $r(t)\times u(t)$

2) $\frac{d}{dt}(r(t)\times u(t))$

3) Now, use the product rule for the derivative of the cross product of two vectors and show this result is the same as the answer for the preceding problem.

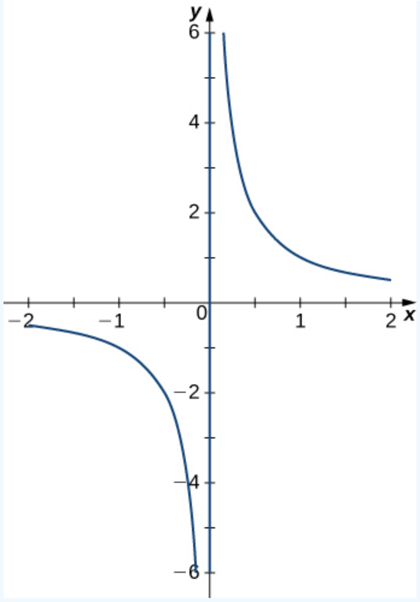
Answer

2) $\langle 0, 2\sin t(t-\frac{1}{t})-2\cos t(1+\frac{1}{t^2}), 2\sin t(1+\frac{1}{t^2})+2\cos t(t-\frac{2}{t})\rangle$

5.2E.13 Exercise $\mathbf{5.2E.13}$

Find the unit tangent vector $\mathbf{T}(t)$ for the following vector-valued functions.

1) $r(t)=\langle t,\frac{1}{t}\rangle$. The graph is shown here:



2) $r(t) = \langle t\cos t, t\sin t\rangle$

3) $r(t) = \langle t+1, 2t+1, 2t+2\rangle$

Answer

1) $T(t) = \langle \frac{t^2}{\sqrt{t^4+1}}, \frac{-1}{\sqrt{t^4+1}}\rangle$

3) $T(t) = \frac{1}{3}\langle 1, 2, 2\rangle$

5.2E.14 Exercise $5.2E.14$

Evaluate the following integrals:

1) $\int(e^t\mathbf{i} + \sin t\mathbf{j} + \frac{1}{2t-1}\mathbf{k})dt$

2) $\int_0^1 r(t)dt$, where $r(t) = \langle \sqrt[3]{t}, \frac{1}{t+1}, e^{-t}\rangle$

Answer

2) $\frac{3}{4}\mathbf{i} + \ln(2)\mathbf{j} + (1 - \frac{1}{e})\mathbf{j}$

5.3: Arc Length and Curvature

This page is a draft and is under active development.

In this section, we study formulas related to curves in both two and three dimensions, and see how they are related to various properties of the same curve. For example, suppose a vector-valued function describes the motion of a particle in space. We would like to determine how far the particle has traveled over a given time interval, which can be described by the arc length of the path it follows. Or, suppose that the vector-valued function describes a road we are building and we want to determine how sharply the road curves at a given point. This is described by the curvature of the function at that point. We explore each of these concepts in this section.

5.3.1 Arc Length for Vector Functions

We have seen how a vector-valued function describes a curve in either two or three dimensions. Recall that the formula for the arc length of a curve defined by the parametric functions $x = x(t), y = y(t), t_1 \leq t \leq t_2$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt. \tag{5.3.1}$$

In a similar fashion, if we define a smooth curve using a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$, where $a \leq t \leq b$, the arc length is given by the formula

$$s = \int_{a}^{b} \sqrt{(f'(t))^2 + (g'(t))^2}\, dt. \tag{5.3.2}$$

In three dimensions, if the vector-valued function is described by $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ over the same interval $a \leq t \leq b$, the arc length is given by

$$s = \int_{a}^{b} \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt. \tag{5.3.3}$$

Theorem: Arc-Length Formulas for Plane and Space curves

Plane curve: Given a smooth curve C defined by the function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_{a}^{b} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt \tag{5.3.4}$$

$$= \int_{a}^{b} \|\vec{\mathbf{r}}\,'(t)\| dt. \tag{5.3.5}$$

Space curve: Given a smooth curve C defined by the function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_{a}^{b} \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \tag{5.3.6}$$

$$= \int_{a}^{b} \|\vec{\mathbf{r}}\,'(t)\| dt. \tag{5.3.7}$$

The two formulas are very similar; they differ only in the fact that a space curve has three component functions instead of two. Note that the formulas are defined for smooth curves: curves where the vector-valued function $\vec{\mathbf{r}}(t)$ is differentiable with a non-zero derivative. The smoothness condition guarantees that the curve has no cusps (or corners) that could make the formula problematic.

Example 5.3.1: Finding the Arc Length

Calculate the arc length for each of the following vector-valued functions:

a. $\vec{\mathbf{r}}(t)=(3t-2)\,\hat{\mathbf{i}}+(4t+5)\,\hat{\mathbf{j}}, \quad 1\leq t\leq 5$
b. $\vec{\mathbf{r}}(t)=\langle t\cos t, t\sin t, 2t\rangle, 0\leq t\leq 2\pi$

Solution

a. Using Equation 5.3.5, $\vec{\mathbf{r}}\,'(t)=3\,\hat{\mathbf{i}}+4\,\hat{\mathbf{i}}$, so

$$\begin{aligned} s &=\int_a^b \|\vec{\mathbf{r}}\,'(t)\|dt \\ &=\int_a^5 \sqrt{3^2+4^2}dt \\ &=\int_1^5 5dt=5t\big|_1^5=20. \end{aligned}$$

b. Using Equation 5.3.7, $\vec{\mathbf{r}}\,'(t)=\langle\cos t-t\sin t, \sin t+t\cos t, 2\rangle$, so

$$\begin{aligned} s &=\int_a^b \|\,\vec{\mathbf{r}}\,'(t)\,\|\,dt \\ &=\int_0^{2\pi}\sqrt{(\cos t-t\sin t)^2+(\sin t+t\cos t)^2+2^2}\,dt \\ &=\int_0^{2\pi}\sqrt{(\cos^2 t-2t\sin t\cos t+t^2\sin^2 t)+(\sin^2 t+2t\sin t\cos t+t^2\cos^2 t)+4}\,dt \\ &=\int_0^{2\pi}\sqrt{\cos^2 t+\sin^2 t+t^2(\cos^2 t+\sin^2 t)+4}\,dt \\ &=\int_0^{2\pi}\sqrt{t^2+5}dt \end{aligned}$$

Here we can use a table integration formula

$$\int\sqrt{u^2+a^2}du=\frac{u}{2}\sqrt{u^2+a^2}+\frac{a^2}{2}\ln\left|u+\sqrt{u^2+a^2}\right|+C,$$

so we obtain

$$\begin{aligned} \int_0^{2\pi}\sqrt{t^2+5}dt &=\frac{1}{2}\left(t\sqrt{t^2+5}+5\ln\left|t+\sqrt{t^2+5}\right|\right)_0^{2\pi} \\ &=\frac{1}{2}\left(2\pi\sqrt{4\pi^2+5}+5\ln\left(2\pi+\sqrt{4\pi^2+5}\right)\right)-\frac{5}{2}\ln\sqrt{5} \\ &\approx 25.343\text{ units.} \end{aligned}$$

Exercise 5.3.1

Calculate the arc length of the parameterized curve

$$\vec{\mathbf{r}}(t) = \langle 2t^2+1, 2t^2-1, t^3\rangle, \quad 0 \le t \le 3.$$

Hint

Use Equation 5.3.7.

Answer

$\vec{\mathbf{r}}\,'(t) = \langle 4t, 4t, 3t^2\rangle$, so $s = \frac{1}{27}(113^{3/2} - 32^{3/2}) \approx 37.785$ units

We now return to the helix introduced earlier in this chapter. A vector-valued function that describes a helix can be written in the form

$$\vec{\mathbf{r}}(t) = R\cos\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{i}} + R\sin\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}, 0 \le t \le h, \tag{5.3.8}$$

where R represents the radius of the helix, h represents the height (distance between two consecutive turns), and the helix completes N turns. Let's derive a formula for the arc length of this helix using Equation 5.3.7. First of all,

$$\vec{\mathbf{r}}\,'(t) = -\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{i}} + \frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{j}} + \hat{\mathbf{k}}. \tag{5.3.9}$$

Therefore,

$$\begin{aligned}
s &= \int_a^b \|\vec{\mathbf{r}}\,'(t)\|dt & (5.3.10)\\
&= \int_0^h \sqrt{\left(-\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\right)^2 + \left(\frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\right)^2 + 1^2\,dt} & (5.3.11)\\
&= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2}\left(\sin^2\left(\frac{2\pi Nt}{h}\right) + \cos^2\left(\frac{2\pi Nt}{h}\right)\right) + 1\,dt} & (5.3.12)\\
&= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1dt} & (5.3.13)\\
&= \left[t\sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1}\right]_0^h & (5.3.14)\\
&= h\sqrt{\frac{4\pi^2N^2R^2 + h^2}{h^2}} & (5.3.15)\\
&= \sqrt{4\pi^2N^2R^2 + h^2}. & (5.3.16)
\end{aligned}$$

This gives a formula for the length of a wire needed to form a helix with N turns that has radius R and height h.

5.3.2 Arc-Length Parameterization

We now have a formula for the arc length of a curve defined by a vector-valued function. Let's take this one step further and examine what an **arc-length function** is.

If a vector-valued function represents the position of a particle in space as a function of time, then the arc-length function measures how far that particle travels as a function of time. The formula for the arc-length function follows directly from the formula for arc length:

$$s = \int_a^t \sqrt{(f\,'(u))^2 + (g\,'(u))^2 + (h\,'(u))^2}\,du. \tag{5.3.17}$$

If the curve is in two dimensions, then only two terms appear under the square root inside the integral. The reason for using the independent variable u is to distinguish between time and the variable of integration. Since $s(t)$ measures distance traveled as a function of time, $s'(t)$ measures the speed of the particle at any given time. Since we have a formula for $s(t)$ in Equation, we can differentiate both sides of the equation:

$$\begin{aligned} s'(t) &= \frac{d}{dt}\left[\int_a^t \sqrt{(f'(u))^2+(g'(u))^2+(h'(u))^2}\,du\right] && (5.3.18)\\ &= \frac{d}{dt}\left[\int_a^t \|\vec{\mathbf{r}}\,'(u)\| du\right] && (5.3.19)\\ &= \|\vec{\mathbf{r}}\,'(t)\|. && (5.3.20)\end{aligned}$$

If we assume that $\vec{\mathbf{r}}(t)$ defines a smooth curve, then the arc length is always increasing, so $s'(t)>0$ for $t>a$. Last, if $\vec{\mathbf{r}}(t)$ is a curve on which $\|\vec{\mathbf{r}}\,'(t)\|=1$ for all t, then

$$s(t)=\int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du=\int_a^t 1\,du=t-a, \tag{5.3.21}$$

which means that t represents the arc length as long as $a=0$.

Theorem: Arc-Length Function

Let $\vec{\mathbf{r}}(t)$ describe a smooth curve for $t\geq a$. Then the arc-length function is given by

$$s(t)=\int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du \tag{5.3.22}$$

Furthermore, $\frac{ds}{dt}=\|\vec{\mathbf{r}}\,'(t)\|>0$. If $\|\vec{\mathbf{r}}\,'(t)\|=1$ for all $t\geq a$, then the parameter t represents the arc length from the starting point at $t=a$.

A useful application of this theorem is to find an alternative parameterization of a given curve, called an **arc-length parameterization**. Recall that any vector-valued function can be reparameterized via a change of variables. For example, if we have a function $\vec{\mathbf{r}}(t)=\langle 3\cos t, 3\sin t\rangle, 0\leq t\leq 2\pi$ that parameterizes a circle of radius 3, we can change the parameter from t to $4t$, obtaining a new parameterization $\vec{\mathbf{r}}(t)=\langle 3\cos 4t, 3\sin 4t\rangle$. The new parameterization still defines a circle of radius 3, but now we need only use the values $0\leq t\leq \pi/2$ to traverse the circle once.

Suppose that we find the arc-length function $s(t)$ and are able to solve this function for t as a function of s. We can then reparameterize the original function $\vec{\mathbf{r}}(t)$ by substituting the expression for t back into $\vec{\mathbf{r}}(t)$. The vector-valued function is now written in terms of the parameter s. Since the variable s represents the arc length, we call this an *arc-length parameterization* of the original function $\vec{\mathbf{r}}(t)$. One advantage of finding the arc-length parameterization is that the distance traveled along the curve starting from $s=0$ is now equal to the parameter s. The arc-length parameterization also appears in the context of curvature (which we examine later in this section) and line integrals.

Example 5.3.2: Finding an Arc-Length Parameterization

Find the arc-length parameterization for each of the following curves:

a. $\vec{\mathbf{r}}(t)=4\cos t\,\hat{\mathbf{i}}+4\sin t\,\hat{\mathbf{j}}, \quad t\geq 0$
b. $\vec{\mathbf{r}}(t)=\langle t+3, 2t-4, 2t\rangle, \quad t\geq 3$

Solution

a. First we find the arc-length function using Equation 5.3.17:

$$\begin{aligned} s(t) &= \int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du \\ &= \int_0^t \|\langle -4\sin u, 4\cos u\rangle\|\,du \\ &= \int_0^t \sqrt{(-4\sin u)^2 + (4\cos u)^2}\,du \\ &= \int_0^t \sqrt{16\sin^2 u + 16\cos^2 u}\,du \\ &= \int_0^t 4\,du = 4t, \end{aligned}$$

b. which gives the relationship between the arc length s and the parameter t as $s = 4t$; so, $t = s/4$. Next we replace the variable t in the original function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$ with the expression $s/4$ to obtain

$$\vec{\mathbf{r}}(s) = 4\cos\left(\frac{s}{4}\right)\hat{\mathbf{i}} + 4\sin\left(\frac{s}{4}\right)\hat{\mathbf{j}}.$$

This is the arc-length parameterization of $\vec{\mathbf{r}}(t)$. Since the original restriction on t was given by $t \geq 0$, the restriction on s becomes $s/4 \geq 0$, or $s \geq 0$.

c. The arc-length function is given by Equation 5.3.17:

$$\begin{aligned} s(t) &= \int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du \\ &= \int_3^t \|\langle 1, 2, 2\rangle\|\,du \\ &= \int_3^t \sqrt{1^2 + 2^2 + 2^2}\,du \\ &= \int_3^t 3\,du \\ &= 3t - 9. \end{aligned}$$

Therefore, the relationship between the arc length s and the parameter t is $s = 3t - 9$, so $t = \frac{s}{3} + 3$. Substituting this into the original function $\vec{\mathbf{r}}(t) = \langle t + 3, 2t - 4, 2t\rangle$ yields

$$\vec{\mathbf{r}}(s) = \langle \left(\frac{s}{3} + 3\right) + 3,\ 2\left(\frac{s}{3} + 3\right) - 4,\ 2\left(\frac{s}{3} + 3\right)\rangle = \langle \frac{s}{3} + 6, \frac{2s}{3} + 2, \frac{2s}{3} + 6\rangle.$$

This is an arc-length parameterization of $\vec{\mathbf{r}}(t)$. The original restriction on the parameter t was $t \geq 3$, so the restriction on s is $(s/3) + 3 \geq 3$, or $s \geq 0$.

Exercise 5.3.2

Find the arc-length function for the helix

$$\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t, 4t\rangle, \quad t \geq 0.$$

Then, use the relationship between the arc length and the parameter t to find an arc-length parameterization of $\vec{\mathbf{r}}(t)$.

Hint

Start by finding the arc-length function.

Answer

$s=5t$, or $t=s/5$. Substituting this into $\vec{\mathbf{r}}(t)=\langle 3\cos t, 3\sin t, 4t\rangle$ gives

$\vec{\mathbf{r}}(s)=\langle 3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5}\rangle, \quad s\geq 0$.

5.3.3 Curvature

An important topic related to arc length is curvature. The concept of curvature provides a way to measure how sharply a smooth curve turns. A circle has constant curvature. The smaller the radius of the circle, the greater the curvature.

Think of driving down a road. Suppose the road lies on an arc of a large circle. In this case you would barely have to turn the wheel to stay on the road. Now suppose the radius is smaller. In this case you would need to turn more sharply to stay on the road. In the case of a curve other than a circle, it is often useful first to inscribe a circle to the curve at a given point so that it is tangent to the curve at that point and "hugs" the curve as closely as possible in a neighborhood of the point (Figure $5.3.1$). The curvature of the graph at that point is then defined to be the same as the curvature of the inscribed circle.

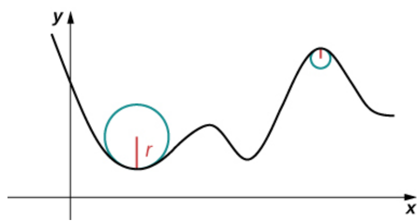


Figure $5.3.1$: The graph represents the curvature of a function $y=f(x)$. The sharper the turn in the graph, the greater the curvature, and the smaller the radius of the inscribed circle.

Definition: curvature

Let C be a smooth curve in the plane or in space given by $\vec{\mathbf{r}}(s)$, where s is the arc-length parameter. The curvature κ at s is

$$\kappa=\left\|\frac{d\vec{\mathbf{T}}}{ds}\right\|=\|\vec{\mathbf{T}}'(s)\|. \tag{5.3.23}$$

Visit this video for more information about the curvature of a space curve.

The formula in the definition of curvature is not very useful in terms of calculation. In particular, recall that $\vec{\mathbf{T}}(t)$ represents the unit tangent vector to a given vector-valued function $\vec{\mathbf{r}}(t)$, and the formula for $\vec{\mathbf{T}}(t)$ is

$$\vec{\mathbf{T}}(t)=\vec{\mathbf{r}}'(t)\ \|\ \vec{\mathbf{r}}'(t)\ \|. \tag{5.3.24}$$

To use the formula for curvature, it is first necessary to express $\vec{\mathbf{r}}(t)$ in terms of the arc-length parameter s, then find the unit tangent vector $\vec{\mathbf{T}}(s)$ for the function $\vec{\mathbf{r}}(s)$, then take the derivative of $\vec{\mathbf{T}}(s)$ with respect to s. This is a tedious process. Fortunately, there are equivalent formulas for curvature.

Theorem: Alternate Formulas of Curvature

If C is a smooth curve given by $\vec{\mathbf{r}}(t)$, then the curvature κ of C at t is given by

$$\kappa=\frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|}. \tag{5.3.25}$$

If C is a three-dimensional curve, then the curvature can be given by the formula

$$\kappa=\frac{\|\vec{\mathbf{r}}\,'(t)\times\vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3}. \tag{5.3.26}$$

If C is the graph of a function $y=f(x)$ and both y' and y'' exist, then the curvature κ at point (x,y) is given by

$$\kappa=\frac{|y''|}{[1+(y')^2]^{3/2}}. \tag{5.3.27}$$

Proof

The first formula follows directly from the chain rule:

$$\frac{d\vec{\mathbf{T}}}{dt}=\frac{d\vec{\mathbf{T}}}{ds}\frac{ds}{dt},$$

where s is the arc length along the curve C. Dividing both sides by ds/dt, and taking the magnitude of both sides gives

$$\left\|\frac{d\vec{\mathbf{T}}}{ds}\right\|=\left\|\frac{\vec{\mathbf{T}}\,'(t)}{\frac{ds}{dt}}\right\|.$$

Since $ds/dt=\|\vec{\mathbf{r}}\,'(t)\|$, this gives the formula for the curvature κ of a curve C in terms of any parameterization of C:

$$\kappa=\frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|}.$$

In the case of a three-dimensional curve, we start with the formulas $\vec{\mathbf{T}}(t)=(\vec{\mathbf{r}}\,'(t))/\|\vec{\mathbf{r}}\,'(t)\|$ and $ds/dt=\|\vec{\mathbf{r}}\,'(t)\|$. Therefore, $\vec{\mathbf{r}}\,'(t)=(ds/dt)\vec{\mathbf{T}}(t)$. We can take the derivative of this function using the scalar product formula:

$$\vec{\mathbf{r}}\,''(t)=\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t)+\frac{ds}{dt}\vec{\mathbf{T}}\,'(t).$$

Using these last two equations we get

$$\begin{aligned}\vec{\mathbf{r}}\,'(t)\times\vec{\mathbf{r}}\,''(t)&=\frac{ds}{dt}\vec{\mathbf{T}}(t)\times\left(\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t)+\frac{ds}{dt}\vec{\mathbf{T}}\,'(t)\right)\\&=\frac{ds}{dt}\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t)\times\vec{\mathbf{T}}(t)+(\frac{ds}{dt})^2\vec{\mathbf{T}}(t)\times\vec{\mathbf{T}}\,'(t).\end{aligned}$$

Since $\vec{\mathbf{T}}(t)\times\vec{\mathbf{T}}(t)=0$, this reduces to

$$\vec{\mathbf{r}}\,'(t)\times\vec{\mathbf{r}}\,''(t)=\left(\frac{ds}{dt}\right)^2\vec{\mathbf{T}}(t)\times\vec{\mathbf{T}}\,'(t).$$

Since $\vec{\mathbf{T}}\,'$ is parallel to $\vec{\mathbf{N}}$, and $\vec{\mathbf{T}}$ is orthogonal to $\vec{\mathbf{N}}$, it follows that $\vec{\mathbf{T}}$ and $\vec{\mathbf{T}}\,'$ are orthogonal. This means that $\|\vec{\mathbf{T}}\times\vec{\mathbf{T}}\,'\|=\|\vec{\mathbf{T}}\|\|\vec{\mathbf{T}}\,'\|\sin(\pi/2)=\|\vec{\mathbf{T}}\,'\|$, so

$$\vec{\mathbf{r}}\,'(t)\times\vec{\mathbf{r}}\,''(t)=\left(\frac{ds}{dt}\right)^2\|\vec{\mathbf{T}}\,'(t)\|.$$

Now we solve this equation for $\|\vec{\mathbf{T}}\,'(t)\|$ and use the fact that $ds/dt = \|\vec{\mathbf{r}}\,'(t)\|$:

$$\|\vec{\mathbf{T}}\,'(t)\| = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^2}.$$

Then, we divide both sides by $\|\vec{\mathbf{r}}\,'(t)\|$. This gives

$$\kappa = \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|} = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3}.$$

This proves 5.3.26. To prove 5.3.27, we start with the assumption that curve C is defined by the function $y = f(x)$. Then, we can define $\vec{\mathbf{r}}(t) = x\,\hat{\mathbf{i}} + f(x)\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Using the previous formula for curvature:

$$\begin{aligned}
\vec{\mathbf{r}}\,'(t) &= \hat{\mathbf{i}} + f'(x)\,\hat{\mathbf{j}} \\
\vec{\mathbf{r}}\,''(t) &= f''(x)\,\hat{\mathbf{j}} \\
\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x)\,\hat{\mathbf{k}}.
\end{aligned}$$

Therefore,

$$\kappa = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3} = \frac{|f''(x)|}{(1 + [f'(x)])^{3/2}}$$

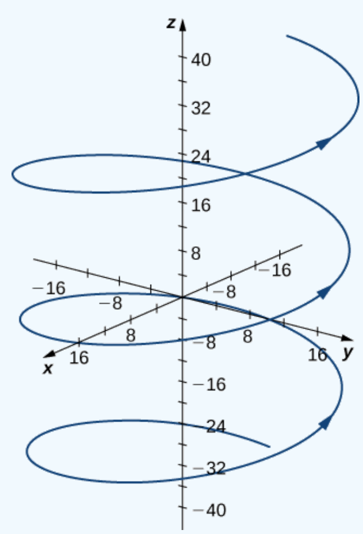
Example 5.3.3: Finding Curvature

Find the curvature for each of the following curves at the given point:

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}, \quad t = \dfrac{4\pi}{3}$
b. $\mathrm{f(x) = \sqrt{4x - x^2}, x = 2}$

Solution

a. This function describes a helix.



The curvature of the helix at $t = (4\pi)/3$ can be found by using 5.3.25. First, calculate T(t):

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} \\ &= \frac{\langle -4\sin t, 4\cos t, 3\rangle}{\sqrt{(-4\sin t)^2+(4\cos t)^2+3^2}} \\ &= \langle -\frac{4}{5}\sin t, \frac{4}{5}\cos t, \frac{3}{5}\rangle.\end{aligned}$$

Next, calculate $\vec{\mathbf{T}}\,'(t)$:

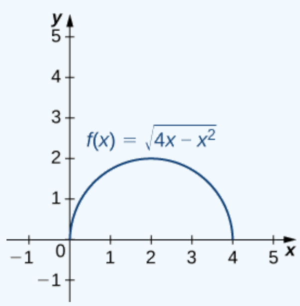
$$\vec{\mathbf{T}}\,'(t) = \langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle.$$

Last, apply 5.3.25 :

$$\begin{aligned}\kappa &= \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|} = \frac{\|\langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle\|}{\|\langle -4\sin t, 4\cos t, 3\rangle\|} \\ &= \frac{\sqrt{(-\frac{4}{5}\cos t)^2+(-\frac{4}{5}\sin t)^2+0^2}}{\sqrt{(-4\sin t)^2+(4\cos t)^2+3^2}} \\ &= \frac{4/5}{5} = \frac{4}{25}.\end{aligned}$$

The curvature of this helix is constant at all points on the helix.

2. This function describes a semicircle.



To find the curvature of this graph, we must use 5.3.27. First, we calculate y' and y'':

$$\begin{aligned}y &= \sqrt{4x-x^2} = (4x-x^2)^{1/2} \\ y' &= \frac{1}{2}(4x-x^2)^{-1/2}(4-2x) = (2-x)(4x-x^2)^{-1/2} \\ y'' &= -(4x-x^2)^{-1/2} + (2-x)(-\frac{1}{2})(4x-x^2)^{-3/2}(4-2x) \\ &= -\frac{4x-x^2}{(4x-x^2)^{3/2}} - \frac{(2-x)^2}{(4x-x^2)^{3/2}} \\ &= \frac{x^2-4x-(4-4x+x^2)}{(4x-x^2)^{3/2}} \\ &= -\frac{4}{(4x-x^2)^{3/2}}.\end{aligned}$$

Then, we apply 5.3.27:

$$\begin{aligned}\kappa &= \frac{|y''|}{[1+(y')^2]^{3/2}} \\ &= \frac{\left|-\dfrac{4}{(4x-x^2)^{3/2}}\right|}{\left[1+((2-x)(4x-x^2)^{-1/2})^2\right]^{3/2}} = \frac{\left|\dfrac{4}{(4x-x^2)^{3/2}}\right|}{\left[1+\dfrac{(2-x)^2}{4x-x^2}\right]^{3/2}} \\ &= \frac{\left|\dfrac{4}{(4x-x^2)^{3/2}}\right|}{\left[\dfrac{4x-x^2+x^2-4x+4}{4x-x^2}\right]^{3/2}} = \left|\frac{4}{(4x-x^2)^{3/2}}\right| \cdot \frac{(4x-x^2)^{3/2}}{8} \\ &= \frac{1}{2}.\end{aligned}$$

The curvature of this circle is equal to the reciprocal of its radius. There is a minor issue with the absolute value in 5.3.27 ; however, a closer look at the calculation reveals that the denominator is positive for any value of x.

Exercise 5.3.3

Find the curvature of the curve defined by the function

$$y = 3x^2 - 2x + 4$$

at the point $x = 2$.

Hint

Use 5.3.27.

Answer

$\kappa = \frac{6}{101^{3/2}} \approx 0.0059$

5.3.4 The Normal and Binormal Vectors

We have seen that the derivative $\vec{\mathbf{r}}\,'(t)$ of a vector-valued function is a tangent vector to the curve defined by $\vec{\mathbf{r}}(t)$, and the unit tangent vector $\vec{\mathbf{T}}(t)$ can be calculated by dividing $\vec{\mathbf{r}}\,'(t)$ by its magnitude. When studying motion in three dimensions, two other vectors are useful in describing the motion of a particle along a path in space: the principal unit normal vector and the **binormal vector**.

Definition: binormal vectors

Let C be a three-dimensional **smooth** curve represented by $\vec{\mathbf{r}}$ over an open interval I. If $\vec{\mathbf{T}}\,'(t) \neq \vec{\mathbf{0}}$, then the principal unit normal vector at t is defined to be

$$\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}\,'(t)}{\|\vec{\mathbf{T}}\,'(t)\|}. \tag{5.3.28}$$

The binormal vector at t is defined as

$$\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t), \tag{5.3.29}$$

where $\vec{\mathbf{T}}(t)$ is the unit tangent vector.

Note that, by definition, the binormal vector is orthogonal to both the unit tangent vector and the normal vector. Furthermore, $\vec{\mathbf{B}}(t)$ is always a unit vector. This can be shown using the formula for the magnitude of a cross product binormal vector is orthogonal to both the unit tangent vector and the normal vector.

$$\|\vec{\mathbf{B}}(t)\| = \|\vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)\| = \|\vec{\mathbf{T}}(t)\|\|\vec{\mathbf{N}}(t)\| \sin\theta, \tag{5.3.30}$$

where θ is the angle between $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$. Since $\vec{\mathbf{N}}(t)$ is the derivative of a unit vector, property (vii) of the derivative of a vector-valued function tells us that $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$ are orthogonal to each other, so $\theta = \pi/2$. Furthermore, they are both unit vectors, so their magnitude is 1. Therefore, $\|\vec{\mathbf{T}}(t)\|\|\vec{\mathbf{N}}(t)\| \sin\theta = (1)(1)\sin(\pi/2) = 1$ and $\vec{\mathbf{B}}(t)$ is a unit vector.

The principal unit normal vector can be challenging to calculate because the unit tangent vector involves a quotient, and this quotient often has a square root in the denominator. In the three-dimensional case, finding the cross product of the unit tangent vector and the unit normal vector can be even more cumbersome. Fortunately, we have alternative formulas for finding these two vectors, and they are presented in Motion in Space.

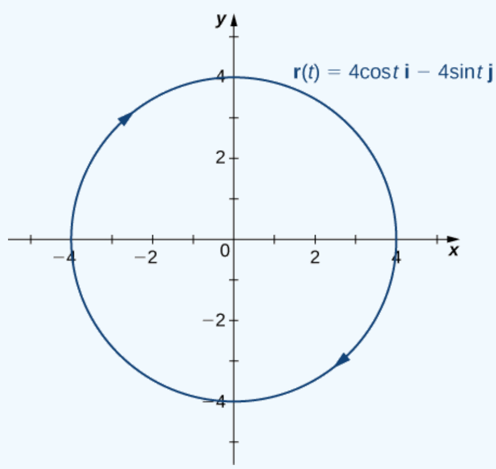
Example 5.3.4: Finding the Principal Unit Normal Vector and Binormal Vector

For each of the following vector-valued functions, find the principal unit normal vector. Then, if possible, find the binormal vector.binormal vector.binormal vector.binormal vector.

1. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} - 4\sin t\,\hat{\mathbf{j}}$
2. $\vec{\mathbf{r}}(t) = (6t+2)\,\hat{\mathbf{i}} + 5t^2\,\hat{\mathbf{j}} - 8t\,\hat{\mathbf{k}}$

Solution

1. This function describes a circle.



To find the principal unit normal vector, we first must find the unit tangent vector $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{(-4\sin t)^2 + (-4cost)^2}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{16\sin^2 t + 16\cos^2 t}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{16(\sin^2 t + \cos^2 t)}} \\ &= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{4} \\ &= -\sin t\,\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}}.\end{aligned}$$

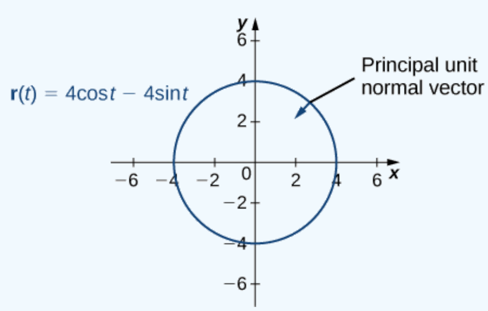
Next, we use 5.3.28 :

$$\begin{aligned}\vec{\mathbf{N}}(t) &= \frac{\vec{\mathbf{T}}\,'(t)}{\|\vec{\mathbf{T}}\,'(t)\|} \\ &= \frac{-\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}}{\sqrt{(-\cos t)^2 + (\sin t)^2}} \\ &= \frac{-\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}}{\sqrt{\cos^2 t + \sin^2 t}} \\ &= -\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}.\end{aligned}$$

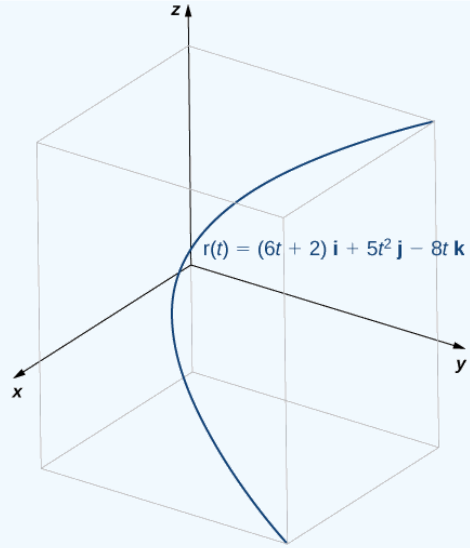
Notice that the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\vec{\mathbf{T}}(t)\cdot\vec{\mathbf{N}}(t) &= \langle -\sin t, -\cos t\rangle \cdot \langle -\cos t, \sin t\rangle \\ &= \sin t\cos t - \cos t\sin t \\ &= 0.\end{aligned}$$

Furthermore, the principal unit normal vector points toward the center of the circle from every point on the circle. Since $\vec{\mathbf{r}}(t)$ defines a curve in two dimensions, we cannot calculate the binormal vector.



2. This function looks like this:



To find the principal unit normal vector, we first find the unit tangent vector $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} \\ &= \frac{6\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}} - 8\,\hat{\mathbf{k}}}{\sqrt{6^2 + (10t)^2 + (-8)^2}} \\ &= \frac{6\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}} - 8\,\hat{\mathbf{k}}}{\sqrt{36 + 100t^2 + 64}} \\ &= \frac{6\,\hat{\mathbf{i}} + 10t\mathbf{j} - 8\mathbf{k}}{\sqrt{100(t^2+1)}} \\ &= \frac{3\,\hat{\mathbf{i}} - 5t\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}} \\ &= \frac{3}{5}(t^2+1)^{-1/2}\,\hat{\mathbf{i}} - t(t^2+1)^{-1/2}\,\hat{\mathbf{j}} - \frac{4}{5}(t^2+1)^{-1/2}\,\hat{\mathbf{k}}.\end{aligned}$$

Next, we calculate $\vec{\mathbf{T}}\,'(t)$ and $\|\vec{\mathbf{T}}\,'(t)\|$:

$$\begin{aligned}\vec{\mathbf{T}}'(t) &= \frac{3}{5}(-\frac{1}{2})(t^2+1)^{-3/2}(2t)\,\hat{\mathbf{i}} - ((t^2+1)^{-1/2} - t(\frac{1}{2})(t^2+1)^{-3/2}(2t))\,\hat{\mathbf{j}} - \frac{4}{5}(-\frac{1}{2})(t^2+1)^{-3/2}(2t)\,\hat{\mathbf{k}} \\ &= -\frac{3t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{i}} - \frac{1}{(t^2+1)^{3/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{k}} \\ \|\vec{\mathbf{T}}'(t)\| &= \sqrt{\left(-\frac{3t}{5(t^2+1)^{3/2}}\right)^2 + \left(-\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(\frac{4t}{5(t^2+1)^{3/2}}\right)^2} \\ &= \sqrt{\frac{9t^2}{25(t^2+1)^3} + \frac{1}{(t^2+1)^3} + \frac{16t^2}{25(t^2+1)^3}} \\ &= \sqrt{\frac{25t^2+25}{25(t^2+1)^3}} \\ &= \sqrt{\frac{1}{(t^2+1)^2}} \\ &= \frac{1}{t^2+1}.\end{aligned}$$

Therefore, according to 5.3.28 :

$$\begin{aligned}\vec{\mathbf{N}}(t) &= \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|} \\ &= \left(-\frac{3t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{i}} - \frac{1}{(t^2+1)^{3/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{k}}\right)(t^2+1) \\ &= -\frac{3t}{5(t^2+1)^{1/2}}\,\hat{\mathbf{i}} - \frac{5}{5(t^2+1)^{1/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{1/2}}\,\hat{\mathbf{k}} \\ &= -\frac{3t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} - 4t\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}.\end{aligned}$$

Once again, the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\vec{\mathbf{T}}(t)\cdot\vec{\mathbf{N}}(t) &= \left(\frac{3\,\hat{\mathbf{i}} - 5t\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}\right)\cdot\left(-\frac{3t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} - 4t\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}\right) \\ &= \frac{3(-3t) - 5t(-5) - 4(4t)}{5\sqrt{t^2+1}} \\ &= \frac{-9t + 25t - 16t}{5\sqrt{t^2+1}} \\ &= 0.\end{aligned}$$

Last, since $\vec{\mathbf{r}}(t)$ represents a three-dimensional curve, we can calculate the binormal vector using binormal vector using binormal vector using binormal vector using 5.3.29 :

$$\begin{aligned}
\vec{\mathbf{B}}(t) &= \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t) \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{3}{5\sqrt{t^2+1}} & -\frac{5t}{5\sqrt{t^2+1}} & -\frac{4}{5\sqrt{t^2+1}} \\ -\frac{3t}{5\sqrt{t^2+1}} & -\frac{5}{5\sqrt{t^2+1}} & \frac{4t}{5\sqrt{t^2+1}} \end{vmatrix} \\
&= \left(\left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right) - \left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{5}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{i}} \\
&\quad - \left(\left(-\frac{3}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right) - \left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{j}} \\
&\quad + \left(\left(-\frac{3}{5\sqrt{t^2+1}}\right)\left(\frac{5}{5\sqrt{t^2+1}}\right) - \left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{k}} \\
&= \left(\frac{-20t^2-20}{25(t^2+1)}\right)\hat{\mathbf{i}} + \left(\frac{-15-15t^2}{25(t^2+1)}\right)\hat{\mathbf{k}} \\
&= -20\left(\frac{t^2+1}{25(t^2+1)}\right)\hat{\mathbf{i}} - 15\left(\frac{t^2+1}{25(t^2+1)}\right)\hat{\mathbf{k}} \\
&= -\frac{4}{5}\hat{\mathbf{i}} - \frac{3}{5}\hat{\mathbf{k}}.
\end{aligned}$$

Exercise 5.3.4

Find the unit normal vector for the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (4t+1)\,\hat{\mathbf{j}}$ and evaluate it at $t = 2$.

Hint

First, find $\vec{\mathbf{T}}(t)$, then use 5.3.28.

Answer

$\vec{\mathbf{N}}(2) = \frac{\sqrt{2}}{2}(\,\hat{\mathbf{i}} - \hat{\mathbf{j}})$

For any smooth curve in three dimensions that is defined by a vector-valued function, we now have formulas for the unit tangent vector $\vec{\mathbf{T}}$, the unit normal vector $\vec{\mathbf{N}}$, and the binormal vector $\vec{\mathbf{B}}$. The unit normal vector and the binormal vector form a plane that is perpendicular to the curve at any point on the curve, called the normal plane. In addition, these three vectors form a frame of reference in three-dimensional space called the *Frenet frame of reference* (also called the **TNB** frame) (Figure 5.3.2). Last, the plane determined by the vectors $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ forms the osculating plane of C at any point P on the curve.

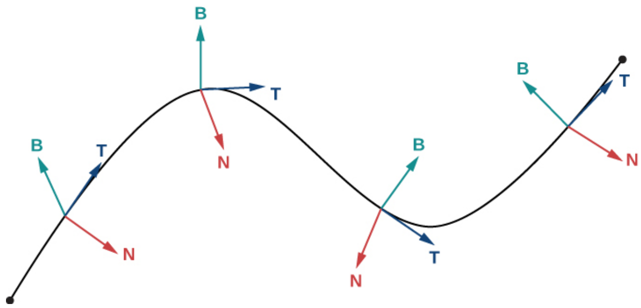


Figure 5.3.2: This figure depicts a Frenet frame of reference. At every point P on a three-dimensional curve, the unit tangent, unit normal, and binormal vectors form a three-dimensional frame of reference.

Suppose we form a circle in the osculating plane of C at point P on the curve. Assume that the circle has the same curvature as the curve does at point P and let the circle have radius r. Then, the curvature of the circle is given by $\frac{1}{r}$. We call r the radius of curvature of the curve, and it is equal to the reciprocal of the curvature. If this circle lies on the concave side of the curve and is tangent to the curve at point P, then this circle is called the *osculating circle* of C at P, as shown in Figure 5.3.3.

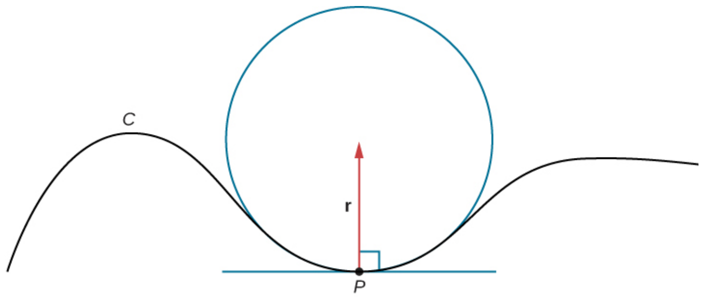


Figure 5.3.3: In this osculating circle, the circle is tangent to curve C at point P and shares the same curvature.

For more information on osculating circles, see this demonstration on curvature and torsion, this article on osculating circles, and this discussion of Serret formulas.

To find the equation of an osculating circle in two dimensions, we need find only the center and radius of the circle.

Example 5.3.5: Finding the Equation of an Osculating Circle

Find the equation of the osculating circle of the helix defined by the function $y = x^3 - 3x + 1$ at $t = 1$.

Solution

Figure 5.3.4 shows the graph of $y = x^3 - 3x + 1$.

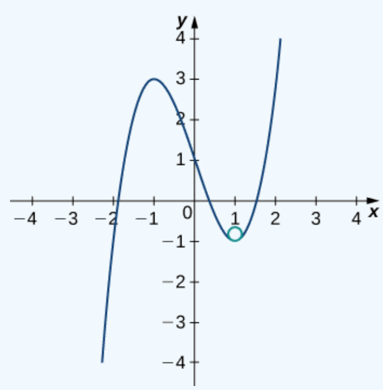


Figure 5.3.4: We want to find the osculating circle of this graph at the point where $t = 1$.

First, let's calculate the curvature at $x = 1$:

$$\kappa = \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}} = \frac{|6x|}{(1+[3x^2-3]^2)^{3/2}}. \tag{5.3.31}$$

This gives $\kappa = 6$. Therefore, the radius of the osculating circle is given by $R = \frac{1}{\kappa} = \frac{1}{6}$. Next, we then calculate the coordinates of the center of the circle. When $x = 1$, the slope of the tangent line is zero. Therefore, the center of the osculating circle is directly above the point on the graph with coordinates $(1, -1)$. The center is located at $(1, -\frac{5}{6})$. The formula for a circle with radius r and center (h, k) is given by $(x-h)^2 + (y-k)^2 = r^2$. Therefore, the equation of the osculating circle is $(x-1)^2 + (y+\frac{5}{6})^2 = \frac{1}{36}$. The graph and its osculating circle appears in the following graph.

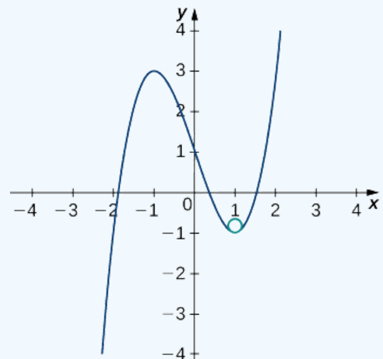


Figure 5.3.5: The osculating circle has radius $R = \frac{1}{6}$.

Exercise 5.3.5

Find the equation of the osculating circle of the curve defined by the vector-valued function $y = 2x^2 - 4x + 5$ at $x = 1$.

Hint

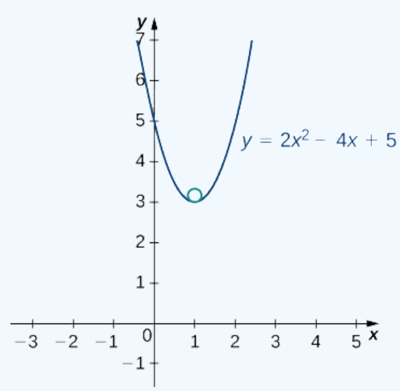
Use 5.3.27 to find the curvature of the graph, then draw a graph of the function around $x = 1$ to help visualize the circle in relation to the graph.

Answer

$\kappa = \frac{4}{[1+(4x-4)^2]^{3/2}}$

At the point $x = 1$, the curvature is equal to 4. Therefore, the radius of the osculating circle is $\frac{1}{4}$.

A graph of this function appears next:



The vertex of this parabola is located at the point $(1,3)$. Furthermore, the center of the osculating circle is directly above the vertex. Therefore, the coordinates of the center are $(1, \frac{13}{4})$. The equation of the osculating circle is

$(x-1)^2+(y-\frac{13}{4})^2=\frac{1}{16}$.

5.3.5 Key Concepts

- The arc-length function for a vector-valued function is calculated using the integral formula $\mathrm{s(t)=\int at\|r\,'(u)\|du}$. This formula is valid in both two and three dimensions.
- The curvature of a curve at a point in either two or three dimensions is defined to be the curvature of the inscribed circle at that point. The arc-length parameterization is used in the definition of curvature.
- There are several different formulas for curvature. The curvature of a circle is equal to the reciprocal of its radius.
- The principal unit normal vector at *t* is defined to be

$$\vec{\mathbf{N}}(t)=\frac{\vec{\mathbf{T}}\,'(t)}{\|\vec{\mathbf{T}}\,'(t)\|}. \tag{5.3.32}$$

- The binormal vector at t is defined as $\vec{\mathbf{B}}(t)=\vec{\mathbf{T}}(t)\times\vec{\mathbf{N}}(t)$, where $\vec{\mathbf{T}}(t)$ is the unit tangent vector.
- The Frenet frame of reference is formed by the unit tangent vector, the principal unit normal vector, and the Frenet frame of reference is formed by the unit tangent vector, the principal unit normal vector, and the binormal vector.
- The osculating circle is tangent to a curve at a point and has the same curvature as the tangent curve at that point.

5.3.5.0.1 Key Equations

- **Arc length of space curve**
  $s=\int_a^b\sqrt{[f\,'(t)]^2+[g\,'(t)]^2+[h\,'(t)]^2}\,dt=\int_a^b\|\vec{\mathbf{r}}\,'(t)\|\,dt$
- **Arc-length function**
  $s(t)=\int_a^t\sqrt{(f\,'(u))^2+(g\,'(u))^2+(h\,'(u))^2}\,du \text{ or } s(t)=\int_a^t\|\vec{\mathbf{r}}\,'(u)\|\,du$
- $\kappa=\frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|} \text{ or } \kappa=\frac{\|\vec{\mathbf{r}}\,'(t)\times\vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3} \text{ or } \kappa=\frac{|y\,''|}{[1+(y\,')^2]^{3/2}}$
- **Principal unit normal vector**
  $\vec{\mathbf{N}}(t)=\frac{\vec{\mathbf{T}}\,'(t)}{\|\vec{\mathbf{T}}\,'(t)\|}$
- **Binormal vector**
  $\vec{\mathbf{B}}(t)=\vec{\mathbf{T}}(t)\times\vec{\mathbf{N}}(t)$

Glossary

arc-length function
a function $s(t)$ that describes the arc length of curve C as a function of t

arc-length parameterization
a reparameterization of a vector-valued function in which the parameter is equal to the arc length

binormal vector

a unit vector orthogonal to the unit tangent vector and the unit normal vector

curvature

the derivative of the unit tangent vector with respect to the arc-length parameter

Frenet frame of reference

(TNB frame) a frame of reference in three-dimensional space formed by the unit tangent vector, the unit normal vector, and the binormal vector

normal plane

a plane that is perpendicular to a curve at any point on the curve

osculating circle

a circle that is tangent to a curve C at a point P and that shares the same curvature

osculating plane

the plane determined by the unit tangent and the unit normal vector

principal unit normal vector

a vector orthogonal to the unit tangent vector, given by the formula $\frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}$

radius of curvature

the reciprocal of the curvature

smooth

curves where the vector-valued function $\vec{\mathbf{r}}(t)$ is differentiable with a non-zero derivative

5.3.5.0.1 Contributors

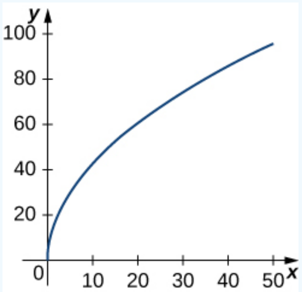
- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

5.3E: Exercises

5.3E.1 Exercise $5.3E.1$

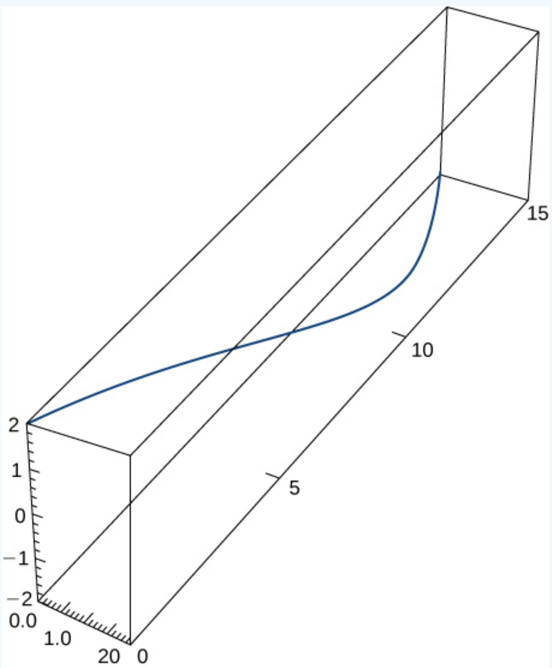
Find the arc length of the curve on the given interval.

a) $r(t) = t^2\mathbf{i} + 14t\mathbf{j}, 0 \le t \le 7$. This portion of the graph is shown here:



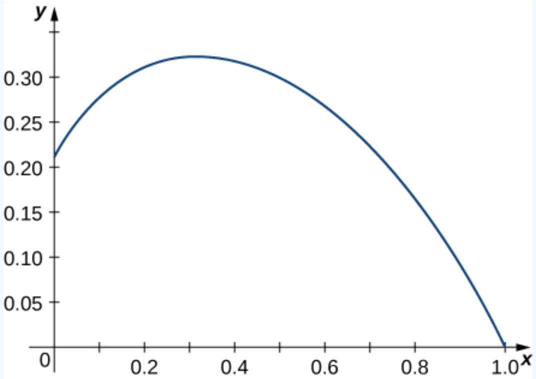
b) $r(t) = t^2\mathbf{i} + (2t^2 + 1)\mathbf{j}, 1 \le t \le 3$

c) $r(t) = \langle 2\sin t, 5t, 2\cos t\rangle, 0 \le t \le \pi$. This portion of the graph is shown here:



d) $r(t) = \langle t^2 + 1, 4t^3 + 3\rangle, -1 \le t \le 0$

e) $r(t) = \langle e^{-t\cos t}, e^{-t\sin t}\rangle$ over the interval $[0, \frac{\pi}{2}]$. Here is the portion of the graph on the indicated interval:



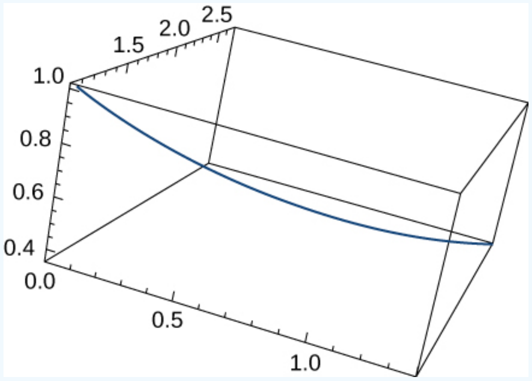
e) Find the length of one turn of the helix given by $r(t) = \frac{1}{2}\cos t\mathbf{i} + \frac{1}{2}\sin t\mathbf{j} + \sqrt{\frac{3}{4}}t\mathbf{k}$.

f) Find the arc length of the vector-valued function $r(t) = -t\mathbf{i} + 4t\mathbf{j} + 3t\mathbf{k}$ over $[0, 1]$.

g) A particle travels in a circle with the equation of motion $r(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 0\mathbf{k}$. Find the distance traveled around the circle by the particle.

h) Set up an integral to find the circumference of the ellipse with the equation $r(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j} + 0\mathbf{k}$.

i) Find the length of the curve $r(t) = \langle\sqrt{2}t, e^t, e^{-t}\rangle$ over the interval $0 \le t \le 1$. The graph is shown here:



j) Find the length of the curve $r(t) = \langle 2\sin t, 5t, 2\cos t\rangle$ for $t \in [-10, 10]$.

Answer

a) $8\sqrt{5}$

d) $\frac{1}{54}(37^{3/2} - 1)$

e) Length $= 2\pi$

g) 6π

i) $e - \frac{1}{e}$

5.3E.2 Exercise $\mathbf{5.3E.2}$

a) The position function for a particle is $r(t) = a\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$. Find the unit tangent vector and the unit normal vector at $t = 0$.

b) Given $r(t) = a\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$, find the binormal vector $B(0)$.

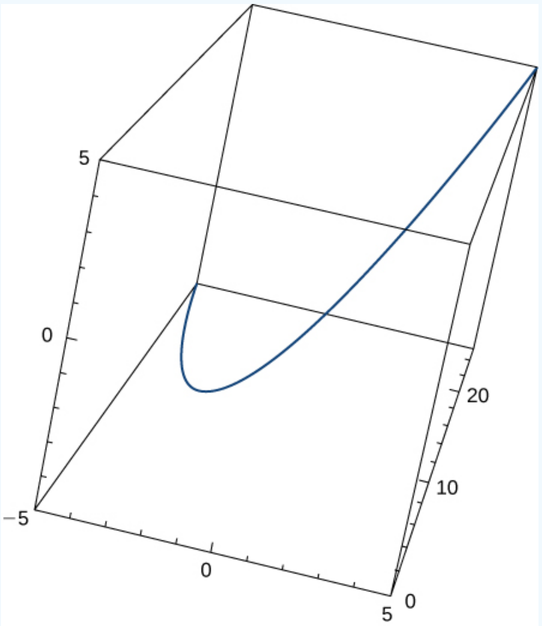
c) Given $r(t) = \langle 2e^t, e^t\cos t, e^t\sin t\rangle$, determine the tangent vector $T(t)$.

d) Given $r(t) = \langle 2e^t, e^t\cos t, e^t\sin t\rangle$, determine the unit tangent vector $T(t)$ evaluated at $t = 0$.

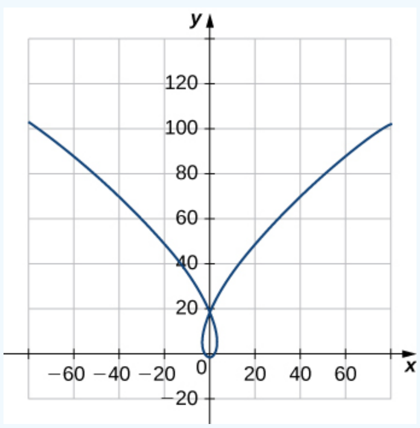
e) Given $r(t) = \langle 2e^t, e^t\cos t, e^t\sin t\rangle$, find the unit normal vector $N(t)$ evaluated at $t = 0$, $N(0)$.

f) Given $r(t)=\langle 2e^t, e^t\cos t, e^t\sin t\rangle$, find the unit normal vector evaluated at $t=0$.

g) Given $r(t)=t\mathbf{i}+t^2\mathbf{j}+t\mathbf{k}$, find the unit tangent vector $T(t)$. The graph is shown here:



h) Find the unit tangent vector $T(t)$ and unit normal vector $N(t)$ at $t=0$ for the plane curve $r(t)=\langle t^3-4t, 5t^2-2\rangle$. The graph is shown here:



i) Find the unit tangent vector $T(t)$ for $r(t)=3t\mathbf{i}+5t^2\mathbf{j}+2t\mathbf{k}$

j) Find the principal normal vector to the curve $r(t)=\langle 6\cos t, 6\sin t\rangle$ at the point determined by $t=\pi/3$.

k) Find $T(t)$ for the curve $r(t)=(t^3-4t)\mathbf{i}+(5t^2-2)\mathbf{j}$.

l) Find $N(t)$ for the curve $r(t)=(t^3-4t)\mathbf{i}+(5t^2-2)\mathbf{j}$.

m) Find the unit normal vector $N(t)$ for $r(t)=\langle 2\text{sint}, 5t, 2\text{cost}\rangle$.

n) Find the unit tangent vector $T(t)$ for $r(t)=\langle 2\sin t, 5t, 2\cos t\rangle$.

Answer

a) $T(0)=\mathbf{j}, N(0)=-\mathbf{i}$

c) $T(t)=\langle 2e^t, e^t\cos t-e^t\sin t, e^t\cos t+e^t\sin t\rangle$

e) $N(0)=\langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\rangle$

g) $T(t)=\frac{1}{\sqrt{4t^2+2}} < 1, 2t, 1 >$

i) $T(t)=\frac{1}{\sqrt{100t^2+13}}(3\mathbf{i}+10t\mathbf{j}+2\mathbf{k})$

k) $T(t)=\frac{1}{\sqrt{9t^4+76t^2+16}}([3t^2-4]\mathbf{i}+10t\mathbf{j})$

m) $N(t)=\langle -\sin t, 0, -\cos t\rangle$

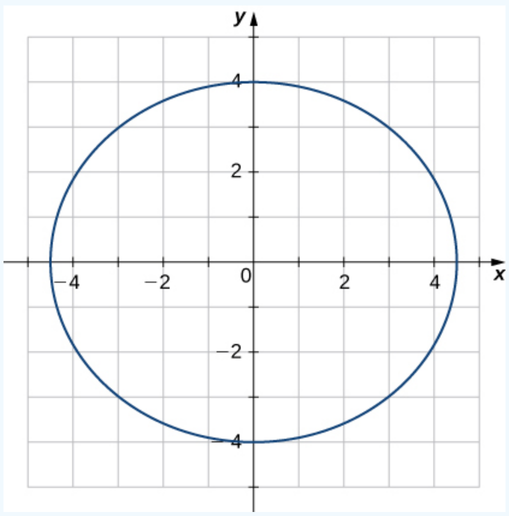
5.3E.3 Exercise $\mathbf{5.3E.3}$

a) Find the arc-length function $s(t)$ for the line segment given by $r(t)=\langle 3-3t, 4t\rangle$. Write *r* as a parameter of *s*.

b) Parameterize the helix $r(t)=\cos t\mathbf{i}+\sin t\mathbf{j}+t\mathbf{k}$ using the arc-length parameter *s*, from $t=0$.

c) Parameterize the curve using the arc-length parameter *s*, at the point at which $t=0$ for $r(t)=e^t\sin t\mathbf{i}+e^t\cos t\mathbf{j}$

d) Find the curvature of the curve $r(t)=5\cos t\mathbf{i}+4\sin t\mathbf{j}$ at $t=\pi/3$. (*Note:* The graph is an ellipse.)



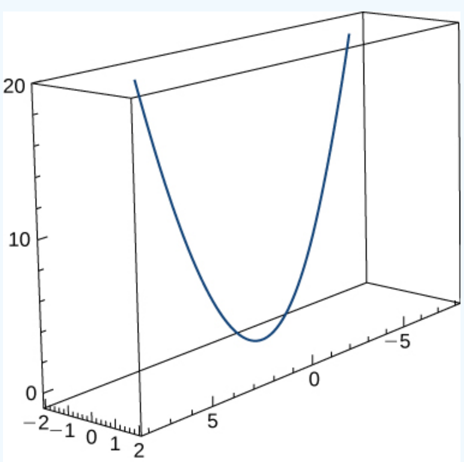
e) Find the *x*-coordinate at which the curvature of the curve $y=1/x$ is a maximum value.

f) Find the curvature of the curve $r(t)=5\cos t\mathbf{i}+5\sin t\mathbf{j}$. Does the curvature depend upon the parameter *t*?

h) Find the curvature κ for the curve $y=x-\frac{1}{4}x^2$ at the point $x=2$.

i) Find the curvature κ for the curve $y=\frac{1}{3}x^3$ at the point $x=1$.

j) Find the curvature κκ of the curve $r(t)=t\mathbf{i}+6t^2\mathbf{j}+4t\mathbf{k}$. The graph is shown here:



k) Find the curvature of $r(t)=\langle 2\sin t, 5t, 2\cos t\rangle$.

l) Find the curvature of $r(t)=\sqrt{2}t\mathbf{i}+e^t\mathbf{j}+e^{-t}\mathbf{k}$ at point $P(0,1,1)$.

m) At what point does the curve $y = e^x$ have maximum curvature?

n) What happens to the curvature as $x \to \infty$ for the curve $y = e^x$?

o) Find the point of maximum curvature on the curve $y = \ln x$.

p) Find the equations of the normal plane and the osculating plane of the curve $r(t) = \langle 2\sin(3t), t, 2\cos(3t)\rangle$ at point $(0, \pi, -2)$.

q) Find equations of the osculating circles of the ellipse $4y^2 + 9x^2 = 36$ at the points $(2, 0)$ and $(0, 3)$.

r) Find the equation for the osculating plane at point $t = \pi/4$ on the curve $r(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} + t$.

Answer

a) Arc-length function: $s(t) = 5t$; r as a parameter of *s*: $r(s) = (3 - \frac{3s}{5})\mathbf{i} + \frac{4s}{5}\mathbf{j}$

c) $(s) = (1 + \frac{s}{\sqrt{2}})\sin(\ln(1 + \frac{s}{\sqrt{2}}))\mathbf{i} + (1 + \frac{s}{\sqrt{2}})\cos[\ln(1 + \frac{s}{\sqrt{2}})]\mathbf{j}$

e) The maximum value of the curvature occurs at $x = \sqrt[4]{5}$.

h) $\frac{1}{2}$

j) $\kappa \approx \frac{49.477}{(17+144t^2)^{3/2}}$

l) $\frac{1}{2\sqrt{2}}$

n) The curvature approaches zero.

p) $y = 6x + \pi$ and $x + 6 = 6\pi$

r) $x + 2z = \frac{\pi}{2}$

5.3E.4 Exercise $5.3E.4$

a) Find the radius of curvature of $6y = x^3$ at the point $(2, \frac{4}{3})$.

b) Find the curvature at each point (x, y) on the hyperbola $r(t) = \langle a\cosh(t), b\sinh(t)\rangle$.

c) Calculate the curvature of the circular helix $r(t) = r\sin(t)\mathbf{i} + r\cos(t)\mathbf{j} + t\mathbf{k}$.

d) Find the radius of curvature of $y = \ln(x+1)$ at point $(2, \ln 3)$.

e) Calculate the curvature of the circular helix $r(t) = r\sin(t)\mathbf{i} + r\cos(t)\mathbf{j} + t\mathbf{k}$.

f) Find the radius of curvature of $y = \ln(x+1)$ at point $(2, \ln 3)$.

g) Find the curvature of the plane curve at $t = 0, 1, 2$.

h) Describe the curvature as *t* increases from $t = 0$ to $t = 2$.

Answer

a) $\frac{a^4b^4}{(b^4x^2+a^4y^2)^{3/2}}$

d) $\frac{10\sqrt{10}}{3}$

f) $\frac{38}{3}$

h) The curvature is decreasing over this interval.

5.3E.5 Exercise $5.3E.5$

The surface of a large cup is formed by revolving the graph of the function $y = 0.25x^{1.6}$ from $x = 0$ to $x = 5$ about the *y*-axis (measured in centimeters).

a) [T] Use technology to graph the surface.

b) Find the curvature κ of the generating curve as a function of *x*.

c) **[T]** Use technology to graph the curvature function.

Answer

b) $\kappa = \frac{6}{x^{2/5}(25+4x^{6/5})}$

5.4: Motion in Space

This page is a draft and is under active development.

We have now seen how to describe curves in the plane and in space, and how to determine their properties, such as arc length and curvature. All of this leads to the main goal of this chapter, which is the description of motion along plane curves and space curves. We now have all the tools we need; in this section, we put these ideas together and look at how to use them.

5.4.1 Motion Vectors in the Plane and in Space

Our starting point is using vector-valued functions to represent the position of an object as a function of time. All of the following material can be applied either to curves in the plane or to space curves. For example, when we look at the orbit of the planets, the curves defining these orbits all lie in a plane because they are elliptical. However, a particle traveling along a helix moves on a curve in three dimensions.

Definition: Speed, Velocity, and Acceleration

Let $\vec{r}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time.

The velocity vector $\vec{v}(t)$ of the object is given by

$$\text{Velocity } = \vec{v}(t) = \vec{r}\,'(t). \tag{5.4.1}$$

The acceleration vector $\vec{a}(t)$ is defined to be

$$\text{Acceleration } = \vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t). \tag{5.4.2}$$

The *speed* is defined to be

$$\text{Speed } = v(t) = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\| = \frac{ds}{dt}. \tag{5.4.3}$$

Since $\vec{\mathbf{r}}(t)$ can be in either two or three dimensions, these vector-valued functions can have either two or three components. In two dimensions, we define $\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}$ and in three dimensions $\vec{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$. Then the velocity, acceleration, and speed can be written as shown in the following table.

Table 5.4.1: Formulas for Position, Velocity, Acceleration, and Speed

Quantity	Two Dimensions	Three Dimensions
Position	$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}$	$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$
Velocity	$\vec{\mathbf{v}}(t) = x\,'(t)\,\hat{\mathbf{i}} + y\,'(t)\,\hat{\mathbf{j}}$	$\vec{\mathbf{v}}(t) = x\,'(t)\,\hat{\mathbf{i}} + y\,'(t)\,\hat{\mathbf{j}} + z\,'(t)\hat{\mathbf{k}}$
Acceleration	$\vec{\mathbf{a}}(t) = x\,''(t)\,\hat{\mathbf{i}} + y\,''(t)\,\hat{\mathbf{j}}$	$\vec{\mathbf{a}}(t) = x\,''(t)\,\hat{\mathbf{i}} + y\,''(t)\,\hat{\mathbf{j}} + z\,''(t)\hat{\mathbf{k}}$
Speed	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x\,'(t))^2 + (y\,'(t))^2}$	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x\,'(t))^2 + (y\,'(t))^2 + (z\,'(t$

Example 5.4.1: Studying Motion Along a Parabola

A particle moves in a parabolic path defined by the vector-valued function $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + \sqrt{5 - t^2}\,\hat{\mathbf{j}}$, where t measures time in seconds.

1. Find the velocity, acceleration, and speed as functions of time.
2. Sketch the curve along with the velocity vector at time $t = 1$.

Solution

1. We use Equations 5.4.1, 5.4.2, and 5.4.3:

$$\begin{aligned} \vec{\mathbf{v}}(t) &= \vec{\mathbf{r}}\,'(t) = 2t\,\hat{\mathbf{i}} - \frac{t}{\sqrt{5-t^2}}\hat{\mathbf{j}} \\ \vec{\mathbf{a}}(t) &= \vec{\mathbf{v}}\,'(t) = 2\hat{\mathbf{i}} - 5(5-t^2)^{-\frac{3}{2}}\hat{\mathbf{j}} \\ ||\vec{\mathbf{v}}(t)|| &= ||\vec{\mathbf{r}}\,'(t)|| \\ &= (2t)^2 + \left(-\frac{t}{\sqrt{5-t^2}}\right)^2 \\ &= \sqrt{4t^2 + \frac{t^2}{5-t^2}} \\ &= \sqrt{\frac{21t^2 - 4t^4}{5-t^2}}. \end{aligned}$$

2. The graph of $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + \sqrt{5-t^2}\,\hat{\mathbf{j}}$ is a portion of a parabola (Figure 5.4.1). The velocity vector at $t = 1$ is

$$\vec{\mathbf{v}}(1) = \vec{\mathbf{r}}\,'(1) = 2(1)\hat{\mathbf{i}} - \frac{1}{\sqrt{5-1^2}}\hat{\mathbf{j}} = 2\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} \tag{5.4.4}$$

and the acceleration vector at $t = 1$ is

$$\vec{\mathbf{a}}(1) = \vec{\mathbf{v}}\,'(1) = 2\hat{\mathbf{i}} - 5(5-1^2)^{-3/2}\hat{\mathbf{j}} = 2\hat{\mathbf{i}} - \frac{5}{8}\hat{\mathbf{j}}. \tag{5.4.5}$$

Notice that the velocity vector is tangent to the path, as is always the case.

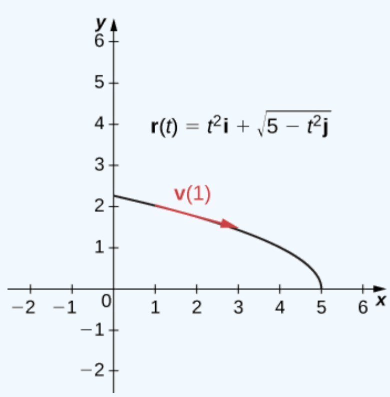


Figure 5.4.1: This graph depicts the velocity vector at time $t = 1$ for a particle moving in a parabolic path.

Exercise 5.4.1

A particle moves in a path defined by the vector-valued function $\vec{r}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (2t-4)\,\hat{\mathbf{j}} + (t+2)\,\hat{\mathbf{k}}$, where t measures time in seconds and where distance is measured in feet. Find the velocity, acceleration, and speed as functions of time.

Hint

Use Equations 5.4.1, 5.4.2, and 5.4.3.

Answer

$$\vec{v}(t) = \vec{r}'(t) = (2t-3)\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$$
$$\vec{a}(t) = \vec{v}\,'(t) = 2\,\hat{\mathbf{i}}$$
$$||\vec{r}\,'(t)|| = \sqrt{(2t-3)^2 + 2^2 + 1^2} = \sqrt{4t^2 - 12t + 14} \tag{5.4.6}$$

The units for velocity and speed are feet per second, and the units for acceleration are feet per second squared.

To gain a better understanding of the velocity and acceleration vectors, imagine you are driving along a curvy road. If you do not turn the steering wheel, you would continue in a straight line and run off the road. The speed at which you are traveling when you run off the road, coupled with the direction, gives a vector representing your velocity, as illustrated in Figure 5.4.2.

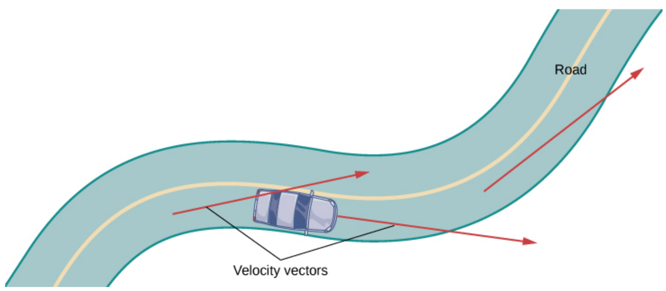


Figure 5.4.2: At each point along a road traveled by a car, the velocity vector of the car is tangent to the path traveled by the car.

However, the fact that you must turn the steering wheel to stay on the road indicates that your velocity is always changing (even if your speed is not) because your *direction* is constantly changing to keep you on the road. As you turn to the right, your acceleration vector also points to the right. As you turn to the left, your acceleration vector points to the left. This indicates that your velocity and acceleration vectors are constantly changing, regardless of whether your actual speed varies (Figure 5.4.3).

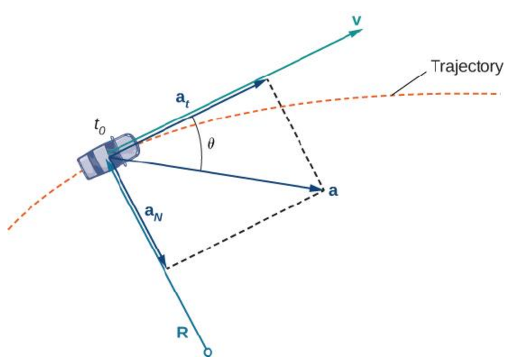


Figure 5.4.3: The dashed line represents the trajectory of an object (a car, for example). The acceleration vector points toward the inside of the turn at all times.

5.4.2 Components of the Acceleration Vector

We can combine some of the concepts discussed in Arc Length and Curvature with the acceleration vector to gain a deeper understanding of how this vector relates to motion in the plane and in space. Recall that the unit tangent vector $\vec{T}$ and the unit normal vector $\vec{N}$ form an osculating plane at any point P on the curve defined by a vector-valued function $\vec{r}(t)$. The following theorem shows that the acceleration vector $\vec{a}(t)$ lies in the osculating plane and can be written as a linear combination of the unit tangent and the unit normal vectors.

Theorem 5.4.1: The Plane of the Acceleration Vector

The acceleration vector $\vec{a}(t)$ of an object moving along a curve traced out by a twice-differentiable function $\vec{r}(t)$ lies in the plane formed by the unit tangent vector $\vec{T}(t)$ and the principal unit normal vector $\vec{N}(t)$ to C. Furthermore,

$$\vec{a}(t) = v'(t)\vec{T}(t) + [v(t)]^2 \kappa \vec{N}(t) \tag{5.4.7}$$

Here, $v(t) = \|\vec{v}(t)\|$ is the speed of the object and κ is the curvature of C traced out by $\vec{r}(t)$.

Proof

Because $\vec{v}(t) = \vec{r}\,'(t)$ and $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}$, we have $\vec{v}(t) = ||\vec{r}\,'(t)||\vec{T}(t) = v(t)\vec{T}(t)$.

Now we differentiate this equation:

$$\vec{a}(t) = \vec{v}\,'(t) = \frac{d}{dt}\left(v(t)\vec{T}(t)\right) = v\,'(t)\vec{T}(t) + v(t)\vec{T}\,'(t) \tag{5.4.8}$$

Since $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{||\vec{T}\,'(t)||}$, we know $\vec{T}\,'(t) = ||\vec{T}\,'(t)||\vec{N}(t)$, so

$$\vec{a}(t) = v\,'(t)\vec{T}(t) + v(t)||\vec{T}\,'(t)||\vec{N}(t). \tag{5.4.9}$$

A formula for curvature is $\kappa = \dfrac{||\vec{T}'(t)||}{||\vec{r}'(t)||}$, so $\vec{T}'(t) = \kappa||\vec{r}'(t)|| = \kappa v(t)$.

This gives $\vec{a}(t) = v\,'(t)\vec{T}(t) + \kappa(v(t))^2\vec{N}(t)$.

□

The coefficients of $\vec{T}(t)$ and $\vec{N}(t)$ are referred to as the **tangential component of acceleration** and the **normal component of acceleration**, respectively. We write $a_{\vec{T}}$ to denote the tangential component and $a_{\vec{N}}$ to denote the normal component.

Theroem 5.4.2: Tangential and Normal Components of Acceleration

Let $\vec{r}(t)$ be a vector-valued function that denotes the position of an object as a function of time. Then $\vec{a}(t) = \vec{r}\,''(t)$ is the acceleration vector. The tangential and normal components of acceleration $a_{\vec{T}}$ and $a_{\vec{N}}$ are given by the formulas

$$a_{\vec{T}} = \vec{a}\cdot\vec{T} = \frac{\vec{v}\cdot\vec{a}}{||\vec{v}||} \tag{5.4.10}$$

and

$$a_{\vec{N}} = \vec{a}\cdot\vec{N} = \frac{||\vec{v}\times\vec{a}||}{||\vec{v}||} = \sqrt{||\vec{a}||^2 - \left(a_{\vec{T}}\right)^2}. \tag{5.4.11}$$

These components are related by the formula

$$\vec{a}(t)=a_{\vec{T}}\vec{T}(t)+a_{\vec{N}}\vec{N}(t). \tag{5.4.12}$$

Here $\vec{T}(t)$ is the unit tangent vector to the curve defined by $\vec{r}(t)$, and $\vec{N}(t)$ is the unit normal vector to the curve defined by $\vec{r}(t)$.

The normal component of acceleration is also called the *centripetal component of acceleration* or sometimes the *radial component of acceleration*. To understand centripetal acceleration, suppose you are traveling in a car on a circular track at a constant speed. Then, as we saw earlier, the acceleration vector points toward the center of the track at all times. As a rider in the car, you feel a pull toward the *outside* of the track because you are constantly turning. This sensation acts in the opposite direction of centripetal acceleration. The same holds true for non-circular paths. The reason is that your body tends to travel in a straight line and resists the force resulting from acceleration that push it toward the side. Note that at point B in Figure $5.4.4$ the acceleration vector is pointing backward. This is because the car is decelerating as it goes into the curve.

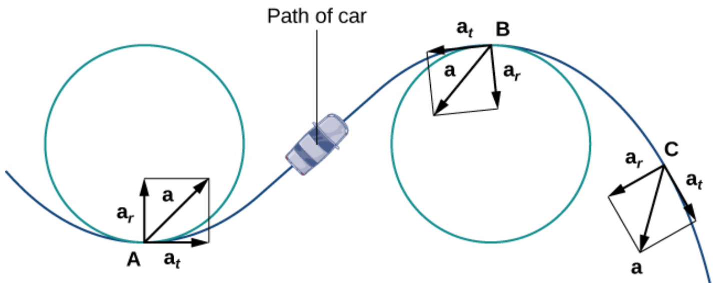


Figure $5.4.4$: The tangential and normal components of acceleration can be used to describe the acceleration vector.

The tangential and normal unit vectors at any given point on the curve provide a frame of reference at that point. The tangential and normal components of acceleration are the projections of the acceleration vector onto $\vec{T}$ and $\vec{N}$, respectively.

Example $5.4.2$: Finding Components of Acceleration

A particle moves in a path defined by the vector-valued function $\vec{r}(t)=t^2\,\hat{\mathbf{i}}+(2t-3)\,\hat{\mathbf{j}}+(3t^2-3t)\,\hat{\mathbf{k}}$, where t measures time in seconds and distance is measured in feet.

a. Find $a_{\vec{T}}$ and $a_{\vec{N}}$ as functions of t.
b. Find $a_{\vec{T}}$ and $a_{\vec{N}}$ at time $t=2$.

Solution

a.

$$\vec{v}(t)=\vec{r}(t)=2t\,\hat{\mathbf{i}}+2\,\hat{\mathbf{j}}+(6t-3)\,\hat{\mathbf{k}} \tag{5.4.13}$$

$$\vec{a}(t)=\vec{v}'(t)=2\,\hat{\mathbf{i}}+6\,\hat{\mathbf{k}} \tag{5.4.14}$$

$$
\begin{aligned}
a_{\vec{T}} &= \frac{\vec{v} \cdot \vec{a}}{||\vec{v}||} \\
&= \frac{(2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}) \cdot (2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}})}{||2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}||} \\
&= \frac{4t + 6(6t-3)}{\sqrt{(2t)^2 + 2^2 + (6t-3)^2}} \\
&= \frac{40t - 18}{40t^2 - 36t + 13}
\end{aligned}
$$

Then we apply Equation 5.4.11:

$$
\begin{aligned}
a_{\vec{N}} &= \sqrt{||\vec{a}||^2 - a_{\vec{T}}} \\
&= \sqrt{||2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}||^2 - \left(\frac{40t - 18}{\sqrt{40t^2 - 36 + 13}}\right)^2} \\
&= \sqrt{4 + 36 - \frac{(40t-18)^2}{40t^2 - 36t + 13}} \\
&= \sqrt{\frac{40(40t^2 - 36t + 13) - (1600t^2 - 1440t + 324)}{40t^2 - 36t + 13}} \\
&= \sqrt{\frac{196}{40t^2 - 36t + 13}} = \frac{14}{\sqrt{40t^2 - 36t + 13}}
\end{aligned}
$$

b.

$$
\begin{aligned}
a_{\vec{T}}(2) &= \frac{40(2) - 18}{\sqrt{40(2)^2 - 36(2) + 13}} \\
&= \frac{80 - 18}{\sqrt{160 - 72 + 13}} \\
&= \frac{62}{\sqrt{101}} \\
a_{\vec{N}}(2) &= \frac{14}{\sqrt{40(2)^2 - 36(2) + 13}} \\
&= \frac{14}{\sqrt{160 - 72 + 13}} = \frac{140}{\sqrt{101}}.
\end{aligned}
$$

The units of acceleration are feet per second squared, as are the units of the normal and tangential components of acceleration.

Exercise 5.4.2

An object moves in a path defined by the vector-valued function $\vec{r}(t) = 4t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$, where t measures time in seconds.

a. Find $a_{\vec{T}}$ and $a_{\vec{N}}$ as functions of t.
b. Find $a_{\vec{T}}$ and $a_{\vec{N}}$ at time $t = -3$.

Hint

Use Equations 5.4.10 and 5.4.11

Answer

a.

$$\begin{aligned} a_{\vec{T}} &= \frac{\vec{v}(t)\cdot\vec{a}(t)}{||\vec{v}(t)||} = \frac{\vec{r}'(t)\cdot\vec{r}''(t)}{||\vec{r}'(t)||} \\ &= \frac{(4\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}})\cdot(2\,\hat{\mathbf{j}})}{||4\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}||} \\ &= \frac{4t}{\sqrt{4^2 + (2t)^2}} \\ &= \frac{2t}{\sqrt{2+t^2}} \end{aligned}$$

$$\begin{aligned} a_{\vec{N}} &= \sqrt{||\vec{a}||^2 - a_{\vec{T}}^2} \\ &= \sqrt{||2\,\hat{\mathbf{j}}||^2 - \left(\frac{2t}{\sqrt{2+t^2}}\right)^2} \\ &= \sqrt{4 - \frac{4t^2}{2+t^2}} \end{aligned}$$

b.

$$\begin{aligned} a_{\vec{T}}(-3) &= \frac{2(-3)}{\sqrt{2+(-3)^2}} \\ &= \frac{-6}{\sqrt{11}} \end{aligned}$$

$$\begin{aligned} a_{\vec{N}}(-3) &= \sqrt{4 - \frac{4(-3)^2}{2+(-3)^2}} \\ &= \sqrt{4 - \frac{36}{11}} \\ &= \sqrt{\frac{8}{11}} \\ &= \frac{2\sqrt{2}}{\sqrt{11}} \end{aligned}$$

5.4.3 Projectile Motion

Now let's look at an application of vector functions. In particular, let's consider the effect of gravity on the motion of an object as it travels through the air, and how it determines the resulting trajectory of that object. In the following, we ignore the effect of air resistance. This situation, with an object moving with an initial velocity but with no forces acting on it other than gravity, is known as projectile motion. It describes the motion of objects from golf balls to baseballs, and from arrows to cannonballs.

First we need to choose a coordinate system. If we are standing at the origin of this coordinate system, then we choose the positive y-axis to be up, the negative y-axis to be down, and the positive x-axis to be forward (i.e., away from the thrower of the object). The effect of gravity is in a downward direction, so Newton's second law tells us that the force on the object resulting from gravity is equal to the mass of the object times the acceleration resulting from gravity, or $\vec{F}_g = m\vec{a}$, where $\vec{F}_g$ represents the force from gravity and $\vec{a} = -g\,\hat{\mathbf{j}}$ represents the acceleration resulting from gravity at Earth's surface. The value of g in the English system of measurement is approximately 32 ft/sec^2 and it is approximately 9.8 m/sec^2 in the metric system. This is the only force acting on the object. Since gravity acts in a downward direction, we can write the force resulting from gravity in the form $\vec{F}_g = -mg\,\hat{\mathbf{j}}$, as shown in Figure 5.4.5.

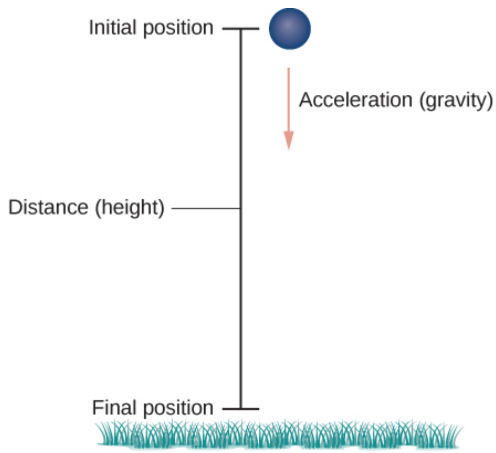


Figure 5.4.5: An object is falling under the influence of gravity.

Newton's second law also tells us that $F = m\vec{a}$, where $\vec{a}$ represents the acceleration vector of the object. This force must be equal to the force of gravity at all times, so we therefore know that

$$\begin{aligned}\vec{F} &= \vec{F}_g \\ m\vec{a} &= -mg\,\hat{\mathbf{j}} \\ \vec{a} &= -g\,\hat{\mathbf{j}}.\end{aligned}$$

Now we use the fact that the acceleration vector is the first derivative of the velocity vector. Therefore, we can rewrite the last equation in the form

$$\vec{v}\,'(t) = -g\,\hat{\mathbf{j}} \tag{5.4.15}$$

By taking the antiderivative of each side of this equation we obtain

$$\vec{v}(t) = \int -g\,\hat{\mathbf{j}}\;dt = -gt\,\hat{\mathbf{j}} + \vec{C}_1 \tag{5.4.16}$$

for some constant vector $\vec{C}_1$. To determine the value of this vector, we can use the velocity of the object at a fixed time, say at time $t = 0$. We call this velocity the *initial velocity*: $\vec{v}(0) = \vec{v}_0$. Therefore,

$\vec{v}(0) = -g(0)\,\hat{\mathbf{j}} + \vec{C}_1 = \vec{v}_0$ and $\vec{C}_1 = \vec{v}_0$. This gives the velocity vector as $\vec{v}(t) = -gt\,\hat{\mathbf{j}} + \vec{v}_0$.

Next we use the fact that velocity $\vec{v}(t)$ is the derivative of position $\vec{s}(t)$. This gives the equation

$$\vec{s}\,'(t) = -gt\,\hat{\mathbf{j}} + \vec{v}_0. \tag{5.4.17}$$

Taking the antiderivative of both sides of this equation leads to

$$\begin{aligned}\vec{s}(t) &= \int -gt\,\hat{\mathbf{j}} + \vec{v}_0\,dt \\ &= -\frac{1}{2}gt^2\,\hat{\mathbf{j}} + \vec{v}_0 + \vec{C}_2\end{aligned}$$

with another unknown constant vector $\vec{C}_2$. To determine the value of $\vec{C}_2$, we can use the position of the object at a given time, say at time $t = 0$. We call this position the *initial position*: $\vec{s}(0) = \vec{s}_0$. Therefore, $\vec{s}(0) = -(1/2)g(0)^2\,\hat{\mathbf{j}} + \vec{v}_0(0) + \vec{C}_2 = \vec{s}_0$. This gives the position of the object at any time as

$$\vec{s}(t) = -12gt^2\,\hat{\mathbf{j}} + \vec{v}_0 t + \vec{s}_0. \tag{5.4.18}$$

Let's take a closer look at the initial velocity and initial position. In particular, suppose the object is thrown upward from the origin at an angle θ to the horizontal, with initial speed $\vec{v}_0$. How can we modify the previous result to reflect this scenario? First, we can assume it is thrown from the origin. If not, then we can move the origin to the point from where it is thrown. Therefore, $\vec{s}_0 = \vec{0}$, as shown in Figure 5.4.6.

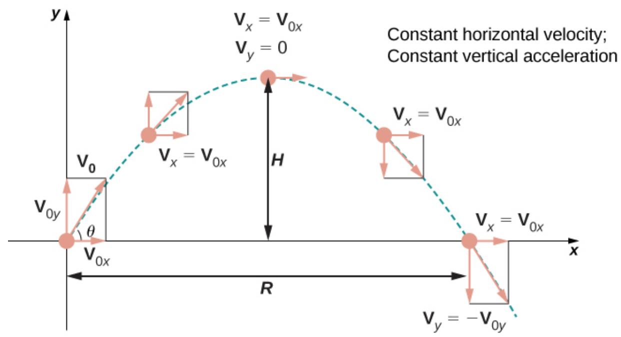


Figure 5.4.6: Projectile motion when the object is thrown upward at an angle θ. The horizontal motion is at constant velocity and the vertical motion is at constant acceleration.

We can rewrite the initial velocity vector in the form $\vec{v}_0 = v_0\cos\theta\,\hat{\mathbf{i}} + v_0\sin\theta\,\hat{\mathbf{j}}$. Then the equation for the position function $\vec{s}(t)$ becomes

$$\begin{aligned}\vec{s}(t) &= -\frac{1}{2}gt^2\,\hat{\mathbf{j}} + v_0 t\cos\theta\,\hat{\mathbf{i}} + v_0 t\sin\theta\,\hat{\mathbf{j}} \\ &= v_0 t\cos\theta\,\hat{\mathbf{i}} + v_0 t\sin\theta\,\hat{\mathbf{j}} - \frac{1}{2}gt^2\,\hat{\mathbf{j}} \\ &= v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}.\end{aligned}$$

The coefficient of $\hat{\mathbf{i}}$ represents the horizontal component of $\vec{s}(t)$ and is the horizontal distance of the object from the origin at time t. The maximum value of the horizontal distance (measured at the same initial and final altitude) is called the range R. The coefficient of $\hat{\mathbf{j}}$ represents the vertical component of $\vec{s}(t)$ and is the altitude of the object at time t. The maximum value of the vertical distance is the height H.

Example 5.4.3: Motion of a Cannonball

During an Independence Day celebration, a cannonball is fired from a cannon on a cliff toward the water. The cannon is aimed at an angle of 30° above horizontal and the initial speed of the cannonball is 600 ft/sec. The cliff is 100 ft above the water (Figure 5.4.7).

a. Find the maximum height of the cannonball.
b. How long will it take for the cannonball to splash into the sea?
c. How far out to sea will the cannonball hit the water?

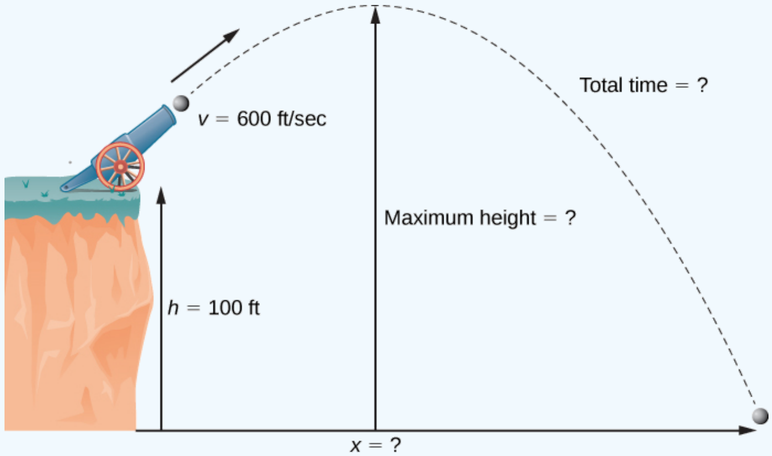


Figure 5.4.7: The flight of a cannonball (ignoring air resistance) is projectile motion.

Solution

We use the equation

$$\vec{s}(t) = v_0 t \cos\theta\,\hat{\mathbf{i}} + \left(v_0 t \sin\theta - \frac{1}{2}gt^2\right)\hat{\mathbf{j}} \tag{5.4.19}$$

with $\theta = 30^\circ$, $g = 32\dfrac{\text{ft}}{\text{sec}^2}$, and $v_0 = 600\dfrac{\text{ft}}{\text{sec}^2}$. Then the position equation becomes

$$\begin{aligned}\vec{s}(t) &= 600t(\cos 30^\circ)\,\hat{\mathbf{i}} + \left(600t\sin 30^\circ - \frac{1}{2}(32)t^2\right)\hat{\mathbf{j}} \\ &= 300t\sqrt{3}\,\hat{\mathbf{i}} + \left(300t - 16t^2\right)\hat{\mathbf{j}}\end{aligned}$$

a. The cannonball reaches its maximum height when the vertical component of its velocity is zero, because the cannonball is neither rising nor falling at that point. The velocity vector is

$$\begin{aligned}\vec{v}(t) &= \vec{s}\,'(t) \\ &= 300\sqrt{3}\,\hat{\mathbf{i}} + (300 - 32t)\,\hat{\mathbf{j}}\end{aligned}$$

Therefore, the vertical component of velocity is given by the expression $300 - 32t$. Setting this expression equal to zero and solving for t gives $t = 9.375$ sec. The height of the cannonball at this time is given by the vertical component of the position vector, evaluated at $t = 9.375$.

$$\vec{s}(9.375) = 300(9.375)\sqrt{3}\,\hat{\mathbf{i}} + (300(9.375) - 16(9.375)^2)\,\hat{\mathbf{j}} = 4871.39\,\hat{\mathbf{i}} + 1406.25\,\hat{\mathbf{j}}$$

Therefore, the maximum height of the cannonball is 1406.39 ft above the cannon, or 1506.39 ft above sea level.

b. When the cannonball lands in the water, it is 100 ft below the cannon. Therefore, the vertical component of the position vector is equal to −100. Setting the vertical component of $\vec{s}(t)$ equal to −100 and solving, we obtain

$$\begin{aligned} 300t - 16t^2 &= -100 \\ 16t^2 - 300t - 100 &= 0 \\ 4t^2 - 75 - 25 &= 0 \\ t &= \frac{75 \pm \sqrt{(-75)^2 - 4(4)(-25)}}{2(4)} \\ &= \frac{75 \pm \sqrt{6025}}{8} \\ &= \frac{75 \pm 5\sqrt{241}}{8} \end{aligned}$$

The positive value of t that solves this equation is approximately 19.08. Therefore, the cannonball hits the water after approximately 19.08 sec.

c. To find the distance out to sea, we simply substitute the answer from part (b) into $\vec{s}(t)$:

$$\begin{aligned} \vec{s}(19.08) &= 300(19.08)\sqrt{3}\,\hat{\mathbf{i}} + \left(300(19.08) - 16(19.08)^2\right)\,\hat{\mathbf{j}} \\ &= 9914.26\,\hat{\mathbf{i}} - 100.7424\,\hat{\mathbf{j}} \end{aligned}$$

Therefore, the ball hits the water about 9914.26 ft away from the base of the cliff. Notice that the vertical component of the position vector is very close to −100, which tells us that the ball just hit the water. Note that 9914.26 feet is not the true range of the cannon since the cannonball lands in the ocean at a location below the cannon. The range of the cannon would be determined by finding how far out the cannonball is when its height is 100 ft above the water (the same as the altitude of the cannon).

Exercise 5.4.3

An archer fires an arrow at an angle of 40° above the horizontal with an initial speed of 98 m/sec. The height of the archer is 171.5 cm. Find the horizontal distance the arrow travels before it hits the ground.

Hint

The equation for the position vector needs to account for the height of the archer in meters.

Answer

967.15 m

One final question remains: In general, what is the maximum distance a projectile can travel, given its initial speed? To determine this distance, we assume the projectile is fired from ground level and we wish it to return to ground level. In other words, we want to determine an equation for the range. In this case, the equation of projectile motion is

$$\vec{s} = v_0 t \cos\theta\,\hat{\mathbf{i}} + \left(v_0 t \sin\theta - \frac{1}{2}gt^2\right)\,\hat{\mathbf{j}}. \tag{5.4.20}$$

Setting the second component equal to zero and solving for t yields

$$\begin{aligned} v_0 t \sin\theta - \frac{1}{2} g t^2 &= 0 \\ t\left(v_0 \sin\theta - \frac{1}{2} g t\right) &= 0 \end{aligned}$$

Therefore, either $t = 0$ or $t = \dfrac{2v_0 \sin\theta}{g}$. We are interested in the second value of t, so we substitute this into $\vec{s}(t)$, which gives

$$\begin{aligned} \vec{s}\left(\frac{2v_0 \sin\theta}{g}\right) &= v_0\left(\frac{2v_0 \sin\theta}{g}\right)\cos\theta\,\hat{\mathbf{i}} + \left(v_0\left(\frac{2v_0 \sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\frac{2v_0 \sin\theta}{g}\right)^2\right)\hat{\mathbf{j}} \\ &= \left(\frac{2v_0^2 \sin\theta\cos\theta}{g}\right)\hat{\mathbf{i}} \\ &= \frac{v_0^2 \sin 2\theta}{g}\,\hat{\mathbf{i}}. \end{aligned}$$

Thus, the expression for the range of a projectile fired at an angle θ is

$$R = \frac{v_0^2 \sin 2\theta}{g}\,\hat{\mathbf{i}}. \tag{5.4.21}$$

The only variable in this expression is θ. To maximize the distance traveled, take the derivative of the coefficient of **i** with respect to θ and set it equal to zero:

$$\begin{aligned} \frac{d}{d\theta}\left(\frac{v_0^2 \sin 2\theta}{g}\right) &= 0 \\ \frac{2v_0^2 \cos 2\theta}{g} &= 0 \\ \theta &= 45^\circ \end{aligned}$$

This value of θ) is the smallest positive value that makes the derivative equal to zero. Therefore, in the absence of air resistance, the best angle to fire a projectile (to maximize the range) is at a 45° angle. The distance it travels is given by

$$\vec{s}\left(\frac{2v_0 \sin 45^\circ}{g}\right) = \frac{v_0^2 \sin 90^\circ}{g}\,\hat{\mathbf{i}} = \frac{v_0^2}{g}\,\hat{\mathbf{i}} \tag{5.4.22}$$

Therefore, the range for an angle of 45° is $\frac{v_0^2}{g}$ units.

5.4.4 Kepler's Laws

During the early 1600s, Johannes Kepler was able to use the amazingly accurate data from his mentor Tycho Brahe to formulate his three laws of planetary motion, now known as Kepler's laws of planetary motion. These laws also apply to other objects in the solar system in orbit around the Sun, such as comets (e.g., Halley's comet) and asteroids. Variations of these laws apply to satellites in orbit around Earth.

Theorem 5.4.2: Kepler's Laws of Planetary Motion

1. The path of any planet about the Sun is elliptical in shape, with the center of the Sun located at one focus of the ellipse (the law of ellipses).
2. A line drawn from the center of the Sun to the center of a planet sweeps out equal areas in equal time intervals (the law of equal areas) (Figure 5.4.8).
3. The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of the lengths of their semimajor orbital axes (the Law of Harmonies).

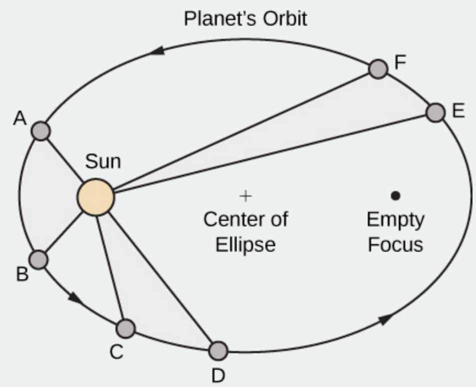


Figure 5.4.8: Kepler's first and second laws are pictured here. The Sun is located at a focus of the elliptical orbit of any planet. Furthermore, the shaded areas are all equal, assuming that the amount of time measured as the planet moves is the same for each region.

Kepler's third law is especially useful when using appropriate units. In particular, *1 astronomical unit* is defined to be the average distance from Earth to the Sun, and is now recognized to be 149,597,870,700 m or, approximately 93,000,000 mi. We therefore write 1 A.U. = 93,000,000 mi. Since the time it takes for Earth to orbit the Sun is 1 year, we use Earth years for units of time. Then, substituting 1 year for the period of Earth and 1 A.U. for the average distance to the Sun, Kepler's third law can be written as

$$T_p^2 = D_p^3 \tag{5.4.23}$$

for any planet in the solar system, where T_P is the period of that planet measured in Earth years and D_P is the average distance from that planet to the Sun measured in astronomical units. Therefore, if we know the average distance from a planet to the Sun (in astronomical units), we can then calculate the length of its year (in Earth years), and vice versa.

Kepler's laws were formulated based on observations from Brahe; however, they were not proved formally until Sir Isaac Newton was able to apply calculus. Furthermore, Newton was able to generalize Kepler's third law to other orbital systems, such as a moon orbiting around a planet. Kepler's original third law only applies to objects orbiting the Sun.

Proof

Let's now prove Kepler's first law using the calculus of vector-valued functions. First we need a coordinate system. Let's place the Sun at the origin of the coordinate system and let the vector-valued function $\vec{r}(t)$ represent the location of a planet as a function of time. Newton proved Kepler's law using his second law of motion and his law of universal gravitation. Newton's second law of motion can be written as $\vec{F} = m\vec{a}$, where $\vec{F}$ represents the net force acting on the planet. His law of universal gravitation can be written in the form

$\vec{F} = -\frac{GmM}{||\vec{r}||^2} \cdot \frac{\vec{r}}{||\vec{r}||}$, which indicates that the force resulting from the gravitational attraction of the Sun points back toward the Sun, and has magnitude $\frac{GmM}{||\vec{r}||^2}$ (Figure 5.4.9).

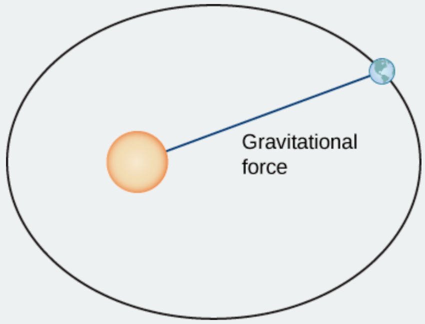


Figure 5.4.9: The gravitational force between Earth and the Sun is equal to the mass of the earth times its acceleration.

Setting these two forces equal to each other, and using the fact that $\vec{a}(t) = \vec{v}\,'(t)$, we obtain

$$m\vec{v}\,'(t) = -\frac{GmM}{||\vec{r}||^2} \cdot \frac{\vec{r}}{||\vec{r}||}, \tag{5.4.24}$$

which can be rewritten as

$$\frac{d\vec{v}}{dt} = -\frac{GM}{||\vec{r}||^3}\vec{r}. \tag{5.4.25}$$

This equation shows that the vectors $d\vec{v}/dt$ and $\vec{r}$ are parallel to each other, so $d\vec{v}/dt \times \vec{r} = \vec{0}$. Next, let's differentiate $\vec{r} \times \vec{v}$ with respect to time:

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = \vec{v} \times \vec{v} + \vec{0} = \vec{0}. \tag{5.4.26}$$

This proves that $\vec{r} \times \vec{v}$ is a constant vector, which we call $\vec{C}$. Since $\vec{r}$ and $\vec{v}$ are both perpendicular to $\vec{C}$ for all values of t, they must lie in a plane perpendicular to $\vec{C}$. Therefore, the motion of the planet lies in a plane.

Next we calculate the expression $d\vec{v}/dt \times \vec{C}$:

$$\frac{d\vec{v}}{dt} \times \vec{C} = -\frac{GM}{||\vec{r}||^3}\vec{r} \times (\vec{r} \times \vec{v}) = -\frac{GM}{||\vec{r}||^3}[(\vec{r} \cdot \vec{v})\vec{r} - (\vec{r} \cdot \vec{r})\vec{v}]. \tag{5.4.27}$$

The last equality in Equation 5.4.26 is from the triple cross product formula (Introduction to Vectors in Space). We need an expression for $\vec{r} \cdot \vec{v}$. To calculate this, we differentiate $\vec{r} \cdot \vec{r}$ with respect to time:

$$\frac{d}{dt}(\vec{r} \cdot \vec{r}) = \frac{d\vec{r}}{dt} \cdot \vec{r} + \vec{r} \cdot \frac{d\vec{r}}{dt} = 2\vec{r} \cdot \frac{d\vec{r}}{dt} = 2\vec{r} \cdot \vec{v}. \tag{5.4.28}$$

Since $\vec{r} \cdot \vec{r} = ||\vec{r}||^2$, we also have

$$\frac{d}{dt}(\vec{r} \cdot \vec{r}) = \frac{d}{dt}||\vec{r}||^2 = 2||\vec{r}||\frac{d}{dt}||\vec{r}||. \tag{5.4.29}$$

Combining Equation 5.4.28 and Equation 5.4.29, we get

$$\begin{aligned} 2\vec{r}\cdot\vec{v} &= 2||\vec{r}||\frac{d}{dt}||\vec{r}|| \\ \vec{r}\cdot\vec{v} &= ||\vec{r}||\frac{d}{dt}||\vec{r}||. \end{aligned}$$

Substituting this into Equation 5.4.27 gives us

$$\begin{aligned} \frac{d\vec{v}}{dt}\times\vec{C} &= -\frac{GM}{||\vec{r}||^3}[(\vec{r}\cdot\vec{v})\vec{r}-(\vec{r}\cdot\vec{r})\vec{v}] \\ &= -\frac{GM}{||\vec{r}||^3}\left[||\vec{r}\left(\frac{d}{dt}||\vec{r}||\right)\vec{r}-||\vec{r}||^2\vec{v}\right] \\ &= -GM\left[\frac{1}{||\vec{r}||^2}\left(\frac{d}{dt}||\vec{r}||\right)\vec{r}-\frac{1}{||\vec{r}||}\vec{v}\right] \\ &= GM\left[\frac{\vec{v}}{||\vec{r}||}-\frac{\vec{r}}{||\vec{r}||^2}\left(\frac{d}{dt}||\vec{r}||\right)\right]. \end{aligned} \tag{5.4.30}$$

However,

$$\begin{aligned} \frac{d}{dt}\frac{\vec{r}}{||\vec{r}||} &= \frac{\frac{d}{dt}(\vec{r})||\vec{r}||-\vec{r}\frac{d}{dt}||\vec{r}||}{||\vec{r}||^2} \\ &= \frac{\frac{d\vec{r}}{dt}}{||\vec{r}||}-\frac{\vec{r}}{||\vec{r}||^2}\frac{d}{dt}||\vec{r}|| \\ &= \frac{\vec{v}}{||\vec{r}||}-\frac{\vec{r}}{||\vec{r}||^2}\frac{d}{dt}||\vec{r}||. \end{aligned}$$

Therefore, Equation 5.4.30 becomes

$$\frac{d\vec{v}}{dt}\times\vec{C} = GM\left(\frac{d}{dt}\frac{\vec{r}}{||\vec{r}||}\right). \tag{5.4.31}$$

Since $\vec{C}$ is a constant vector, we can integrate both sides and obtain

$$\vec{v}\times\vec{C} = GM\frac{\vec{r}}{||\vec{r}||}+\vec{D}, \tag{5.4.32}$$

where $\vec{D}$ is a constant vector. Our goal is to solve for $||\vec{r}||$. Let’s start by calculating $\vec{r}\cdot(\vec{v}\times\vec{C}$:

$$\vec{r}\cdot(\vec{v}\times\vec{C} = GM\frac{||\vec{r}||^2}{||\vec{r}||}+\vec{r}\cdot\vec{D} = GM||\vec{r}||+\vec{r}\cdot\vec{D}. \tag{5.4.33}$$

However, $\vec{r}\cdot(\vec{v}\times\vec{C}) = (\vec{r}\times\vec{v})\cdot\vec{C}$, so

$$(\vec{r}\times\vec{v})\cdot\vec{C} = GM||\vec{r}||+\vec{r}\cdot\vec{D}. \tag{5.4.34}$$

Since $\vec{r}\times\vec{v}=\vec{C}$, we have

$$||\vec{C}||^2 = GM||\vec{r}||+\vec{r}\cdot\vec{D}. \tag{5.4.35}$$

Note that $\vec{r} \cdot \vec{D} = ||\vec{r}||||\vec{D}|| \cos\theta$, where θ is the angle between $\vec{r}$ and $\vec{D}$. Therefore,

$$||\vec{C}||^2 = GM||\vec{r}|| + ||\vec{r}||||\vec{D}|| \cos\theta \tag{5.4.36}$$

Solving for $||\vec{r}||$,

$$||\vec{r}|| = \frac{||\vec{C}||^2}{GM + ||\vec{D}|| \cos\theta} = \frac{||\vec{C}||^2}{GM}\left(\frac{1}{1 + e\cos\theta}\right). \tag{5.4.37}$$

where $e = ||\vec{D}||/GM$. This is the polar equation of a conic with a focus at the origin, which we set up to be the Sun. It is a hyperbola if $e > 1$, a parabola if $e = 1$, or an ellipse if $e < 1$. Since planets have closed orbits, the only possibility is an ellipse. However, at this point it should be mentioned that hyperbolic comets do exist. These are objects that are merely passing through the solar system at speeds too great to be trapped into orbit around the Sun. As they pass close enough to the Sun, the gravitational field of the Sun deflects the trajectory enough so the path becomes hyperbolic.

□

Kepler's third law of planetary motion can be modified to the case of one object in orbit around an object other than the Sun, such as the Moon around the Earth. In this case, Kepler's third law becomes

$$P^2 = \frac{4\pi^2 a^3}{G(m+M)}, \tag{5.4.38}$$

where m is the mass of the Moon and M is the mass of Earth, a represents the length of the major axis of the elliptical orbit, and P represents the period.

Example 5.4.4: Using Kepler's Third Law for Nonheliocentric Orbits

Given that the mass of the Moon is 7.35×10^{22} kg, the mass of Earth is 5.97×10^{24} kg, $G = 6.67 \times 10^{-11} \mathrm{m/kg \cdot sec^2}$, and the period of the moon is 27.3 days, let's find the length of the major axis of the orbit of the Moon around Earth.

Solution

It is important to be consistent with units. Since the universal gravitational constant contains seconds in the units, we need to use seconds for the period of the Moon as well:

$$27.3 \text{ days} \times \frac{24 \text{ hr}}{1 \text{ day}} \times \frac{3600 \text{ esc}}{1 \text{ hour}} = 2,358,720 \text{ sec} \tag{5.4.39}$$

Substitute all the data into Equation 5.4.38 and solve for a:

$$\begin{aligned}
(2,358,720sec)^2 &= \frac{4\pi^2a^3}{\left(6.67\times10^{-11}\frac{m}{\text{kg}\times\text{sec}^2}\right)(7.35\times10^{22}\text{kg}+5.97\times10^{24}\text{kg})} \\
5.563\times10^{12} &= \frac{4\pi^2a^3}{(6.67\times10^{-11}\text{m}^3)(6.04\times10^{24})} \\
(5.563\times10^{12})(6.67\times10^{-11}\text{m}^3)(6.04\times10^{24}) &= 4\pi^2a^3 \\
a^3 &= \frac{2.241\times10^{27}}{4\pi^2}\text{m}^3 \\
a &= 3.84\times10^8\text{m} \\
&\approx 384,000\,\text{km}.
\end{aligned}$$

Analysis

According to solarsystem.nasa.gov, the actual average distance from the Moon to Earth is 384,400 km. This is calculated using reflectors left on the Moon by Apollo astronauts back in the 1960s.

Exercise 5.4.4

Titan is the largest moon of Saturn. The mass of Titan is approximately $1.35\times10^{23}kg$. The mass of Saturn is approximately 5.68×10^{26} kg. Titan takes approximately 16 days to orbit Saturn. Use this information, along with the universal gravitation constant $G=6.67\times10^{-11}\text{m}/\text{kg}\cdot\text{sec}^2$ to estimate the distance from Titan to Saturn.

Hint

Make sure your units agree, then use Equation 5.4.38.

Answer

$$a\approx1.224\times10^9\text{m}=1,224,000\text{km} \tag{5.4.40}$$

Example 5.4.5: Halley's Comet

We now return to the chapter opener, which discusses the motion of Halley's comet around the Sun. Kepler's first law states that Halley's comet follows an elliptical path around the Sun, with the Sun as one focus of the ellipse. The period of Halley's comet is approximately 76.1 years, depending on how closely it passes by Jupiter and Saturn as it passes through the outer solar system. Let's use $T=76.1$ years. What is the average distance of Halley's comet from the Sun?

Solution

Using the equation $T^2=D^3$ with $T=76.1$, we obtain $D^3=5791.21$, so $D\approx17.96$ A.U. This comes out to approximately 1.67×10^9 mi.

A natural question to ask is: What are the maximum (aphelion) and minimum (perihelion) distances from Halley's Comet to the Sun? The eccentricity of the orbit of Halley's Comet is 0.967 (Source: http://nssdc.gsfc.nasa.gov/planetary...cometfact.html). Recall that the formula for the eccentricity of an ellipse is $e = c/a$, where a is the length of the semimajor axis and c is the distance from the center to either focus. Therefore, $0.967 = c/17.96$ and $c \approx 17.37$ A.U. Subtracting this from a gives the perihelion distance $p = a - c = 17.96 - 17.37 = 0.59$ A.U. According to the National Space Science Data Center (Source: http://nssdc.gsfc.nasa.gov/planetary...cometfact.html), the perihelion distance for Halley's comet is 0.587 A.U. To calculate the aphelion distance, we add

$$P = a + c = 17.96 + 17.37 = 35.33 \text{ A.U.} \tag{5.4.41}$$

This is approximately 3.3×10^9 mi. The average distance from Pluto to the Sun is 39.5 A.U. (Source: http://www.oarval.org/furthest.htm), so it would appear that Halley's Comet stays just within the orbit of Pluto.

NAVIGATING A BANKED TURN

How fast can a racecar travel through a circular turn without skidding and hitting the wall? The answer could depend on several factors:

- The weight of the car;
- The friction between the tires and the road;
- The radius of the circle;
- The "steepness" of the turn.

In this project we investigate this question for NASCAR racecars at the Bristol Motor Speedway in Tennessee. Before considering this track in particular, we use vector functions to develop the mathematics and physics necessary for answering questions such as this.

A car of mass m moves with constant angular speed ω around a circular curve of radius R (Figure 5.4.9). The curve is banked at an angle θ. If the height of the car off the ground is h, then the position of the car at time t is given by the function $\vec{r}(t) =< R\cos(\omega t), R\sin(\omega t), h >$.

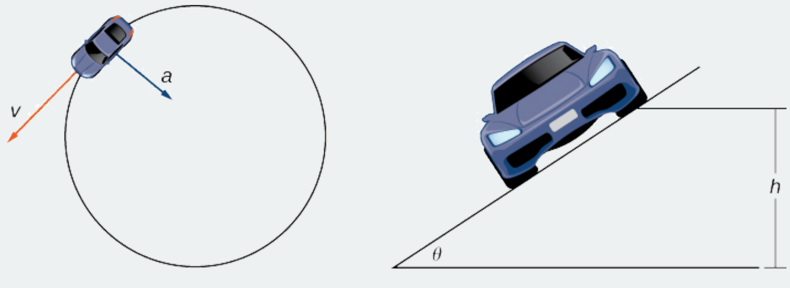


Figure 5.4.9: Views of a race car moving around a track.

1. Find the velocity function $\vec{v}(t)$ of the car. Show that $\vec{v}$ is tangent to the circular curve. This means that, without a force to keep the car on the curve, the car will shoot off of it.
2. Show that the speed of the car is ωR. Use this to show that $(2\pi 4)/\|\vec{v}\| = (2\pi)/\omega$.
3. Find the acceleration $\vec{a}$. Show that this vector points toward the center of the circle and that $\|\vec{a}\| = R\omega^2$.
4. The force required to produce this circular motion is called the *centripetal force*, and it is denoted $\vec{F}_{cent}$. This force points toward the center of the circle (not toward the ground). Show that $\|\vec{F}_{cent}\| = \left(m|\vec{v}|^2\right)/R$.

As the car moves around the curve, three forces act on it: gravity, the force exerted by the road (this force is perpendicular to the ground), and the friction force (Figure 5.4.10). Because describing the frictional force generated by the tires and the road is complex, we use a standard approximation for the frictional force. Assume that $\vec{f}=\mu\vec{N}$ for some positive constant μ. The constant μ is called the *coefficient of friction*.

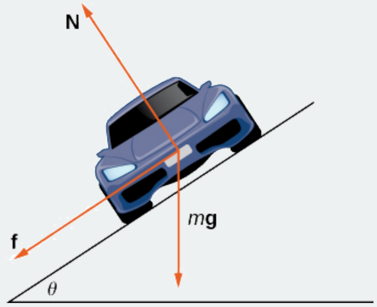


Figure 5.4.10: The car has three forces acting on it: gravity (denoted by $m\vec{g}$), the friction force $\vec{f}$, and the force exerted by the road $\vec{N}$.

Let v_{max} denote the maximum speed the car can attain through the curve without skidding. In other words, v_{max} is the fastest speed at which the car can navigate the turn. When the car is traveling at this speed, the magnitude of the centripetal force is

$$\|\vec{F}_{cent}\|=\frac{m(v_{max})^2}{R}. \tag{5.4.42}$$

The next three questions deal with developing a formula that relates the speed v_{max} to the banking angle θ.

5. Show that $\vec{N}\cos\theta=m\vec{g}+\vec{f}\sin\theta$. Conclude that $\vec{N}=(m\vec{g})/(\cos\theta-\mu\sin\theta)$.
6. The centripetal force is the sum of the forces in the horizontal direction, since the centripetal force points toward the center of the circular curve. Show that

$$\vec{F}_{cent}=\vec{N}\sin\theta+\vec{f}\cos\theta. \tag{5.4.43}$$

Conclude that

$$\vec{F}_{cent}=\frac{\sin\theta+\mu\cos\theta}{cos\theta-\mu\sin\theta}m\vec{g}. \tag{5.4.44}$$

7. Show that $(v_{\max})^2=((\sin\theta+\mu\, cos\theta)/(\cos\theta-\mu\sin\theta))gR$. Conclude that the maximum speed does not actually depend on the mass of the car.

   Now that we have a formula relating the maximum speed of the car and the banking angle, we are in a position to answer the questions like the one posed at the beginning of the project.

   The Bristol Motor Speedway is a NASCAR short track in Bristol, Tennessee. The track has the approximate shape shown in Figure 5.4.11. Each end of the track is approximately semicircular, so when cars make turns they are traveling along an approximately circular curve. If a car takes the inside track and speeds along the bottom of turn 1, the car travels along a semicircle of radius approximately 211 ft with a banking angle of 24°. If the car decides to take the outside track and speeds along the top of turn 1, then the car travels along a semicircle with a banking angle of 28°. (The track has variable angle banking.)

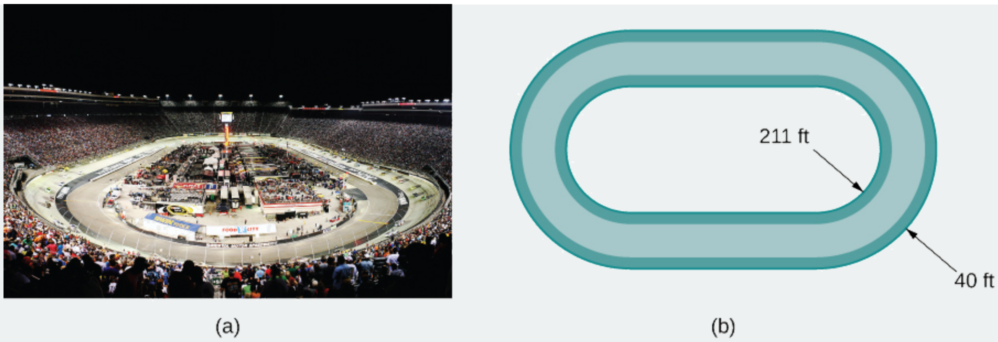


Figure 5.4.11: At the Bristol Motor Speedway, Bristol, Tennessee (a), the turns have an inner radius of about 211 ft and a width of 40 ft (b). (credit: part (a) photo by Raniel Diaz, Flickr)

The coefficient of friction for a normal tire in dry conditions is approximately 0.7. Therefore, we assume the coefficient for a NASCAR tire in dry conditions is approximately 0.98.

Before answering the following questions, note that it is easier to do computations in terms of feet and seconds, and then convert the answers to miles per hour as a final step.

8. In dry conditions, how fast can the car travel through the bottom of the turn without skidding?
9. In dry conditions, how fast can the car travel through the top of the turn without skidding?
10. In wet conditions, the coefficient of friction can become as low as 0.1. If this is the case, how fast can the car travel through the bottom of the turn without skidding?
11. Suppose the measured speed of a car going along the outside edge of the turn is 105 mph. Estimate the coefficient of friction for the car's tires.

5.4.5 Key Concepts

- If $\vec{r}(t)$ represents the position of an object at time t, then $\vec{r}'(t)$ represents the velocity and $\vec{r}\,''(t)$ represents the acceleration of the object at time t. The magnitude of the velocity vector is speed.
- The acceleration vector always points toward the concave side of the curve defined by $\vec{r}(t)$. The tangential and normal components of acceleration $a_{\vec{T}}$ and $a_{\vec{N}}$ are the projections of the acceleration vector onto the unit tangent and unit normal vectors to the curve.
- Kepler's three laws of planetary motion describe the motion of objects in orbit around the Sun. His third law can be modified to describe motion of objects in orbit around other celestial objects as well.
- Newton was able to use his law of universal gravitation in conjunction with his second law of motion and calculus to prove Kepler's three laws.

5.4.6 Key Equations

- **Velocity**

$$\vec{v}(t) = \vec{r}\,'(t)$$

- **Acceleration**

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

- **Speed**

$$v(t) = ||\vec{v}(t)|| = ||\vec{r}\,'(t)|| = \frac{ds}{dt}$$

- **Tangential component of acceleration**

$$a_{\vec{T}} = \vec{a} \cdot \vec{T} = \frac{\vec{v} \cdot \vec{a}}{||\vec{v}||}$$

- **Normal component of acceleration**

$$a_{\vec{N}} = \vec{a} \cdot \vec{N} = \frac{||\vec{v} \times \vec{a}||}{||\vec{v}||} = \sqrt{||\vec{a}||^2 - a_{\vec{T}}}$$

acceleration vector
the second derivative of the position vector

Kepler's laws of planetary motion
three laws governing the motion of planets, asteroids, and comets in orbit around the Sun

normal component of acceleration
the coefficient of the unit normal vector $\vec{N}$ when the acceleration vector is written as a linear combination of $\vec{T}$ and $\vec{N}$

projectile motion
motion of an object with an initial velocity but no force acting on it other than gravity

tangential component of acceleration
the coefficient of the unit tangent vector $\vec{T}$ when the acceleration vector is written as a linear combination of $\vec{T}$ and $\vec{N}$

velocity vector
the derivative of the position vector

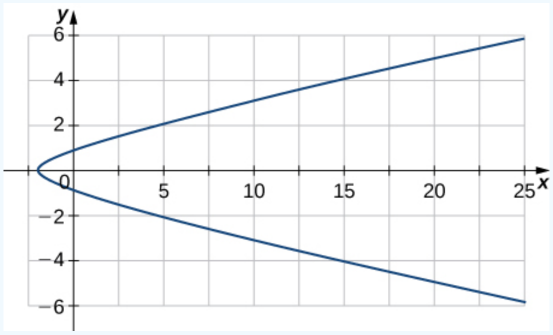
5.4.6.0.1 Contributors

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Edited by Paul Seeburger

5.4E:Exercises

5.4E.1 Exercise $\mathbf{5.4E.1}$

1) Given $\overrightarrow{r(t)}=(3t^2-2)\hat{\mathbf{i}}+(2t-sin(t))\hat{\mathbf{j}}$, find the velocity of a particle moving along this curve.



2) Given $\overrightarrow{r(t)}=(3t^2-2)\hat{\mathbf{i}}+(2t-sin(t))\hat{\mathbf{j}}$, find the acceleration vector of a particle moving along the curve in the preceding exercise.

Answer

1) $\overrightarrow{v(t)}=(6t)\hat{\mathbf{i}}+(2-cos(t))\hat{\mathbf{j}}$

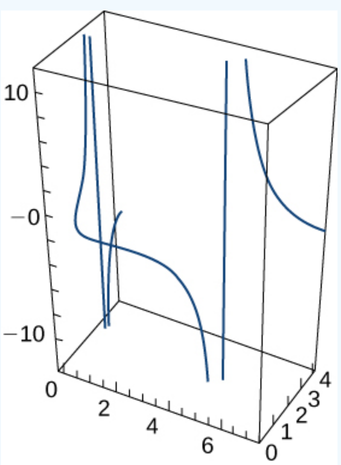
5.4E.2 Exercise $\mathbf{5.4E.2}$

Given the following position functions, find the velocity, acceleration, and speed in terms of the parameter \(t.\)

1) $\overrightarrow{r(t)}=\langle 3cost, 3sint, t^2\rangle$

2) $\overrightarrow{r(t)}=e^{-t}\hat{\mathbf{i}}+t^2\hat{\mathbf{j}}+tan(t)\hat{\mathbf{k}}$

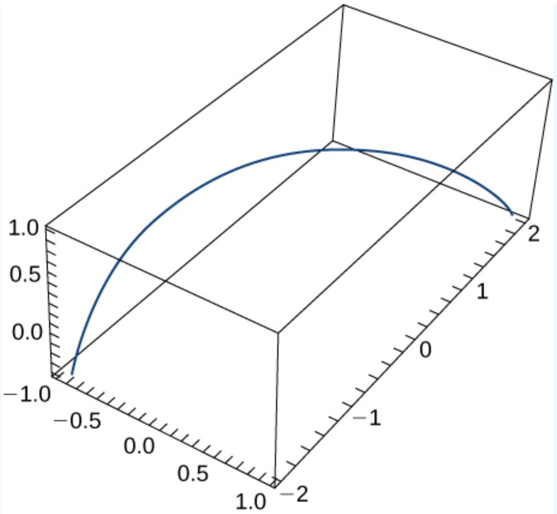
3) $\overrightarrow{r(t)}=2cost\hat{\mathbf{j}}+3sint\hat{\mathbf{k}}.$ The graph is shown here:



4) $\overrightarrow{r(t)}=\langle t^2-1, t\rangle$

5) $\overrightarrow{r(t)}=\langle e^t, e^{-t}\rangle$

6) $\overrightarrow{r(t)}=\langle sint, t, cost\rangle.$ The graph is shown here:



Answer

1) $\overrightarrow{v(t)} = \langle -3sint, 3cost, 2t\rangle$,$a(t) = \langle -3cost, -3sint, 2\rangle$, speed $= \sqrt{9+4t^2}$

3)$\overrightarrow{v(t)} = -2sint\hat{\mathbf{j}} + 3costk, a(t) = -2cost\hat{\mathbf{j}} - 3sint\hat{\mathbf{k}}$, speed \(= \sqrt{4sin^2(t)+9cos^2(t)\)

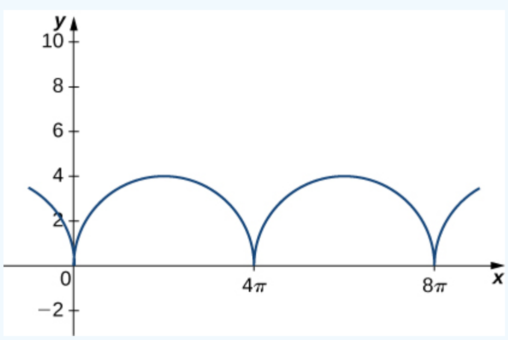
5) $\overrightarrow{v(t)} = e^t\,\hat{\mathbf{i}} - e^{-t}\,\hat{\mathbf{j}}, a(t) = e^t\,\hat{\mathbf{i}} + e^{-t}\,\hat{\mathbf{j}}$, speed $= \sqrt{e^{2t}+e^{-2t}}$

5.4E.3 Exercise $\mathbf{5.4E.3}$

1) The position function of an object is given by $\overrightarrow{r(t)} = \langle t^2, 5t, t^2-16t\rangle$. At what time is the speed a minimum?

2) Let $\overrightarrow{r(t)} = rcosh(\omega t)\,\hat{\mathbf{i}} + rsinh(\omega t)\,\hat{\mathbf{j}}$. Find the velocity and acceleration vectors. Further, show that the acceleration is proportional to $\overrightarrow{r(t)}$.

3) Consider the motion of a point on the circumference of a rolling circle. As the circle rolls, it generates the cycloid $\overrightarrow{r(t)} = (\omega t - sin(\omega t))\,\hat{\mathbf{i}} + (1-cos(\omega t))\,\hat{\mathbf{j}}$, where ω is the angular velocity of the circle and b is the radius of the circle.



Find the equations for the velocity, acceleration, and speed of the particle at any time.

Answer

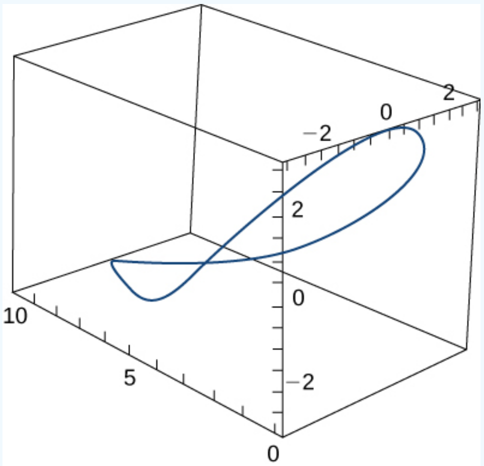
1) $t = 4$

3) $\overrightarrow{v(t)} = (\omega - \omega cos(\omega t))\,\hat{\mathbf{i}} + (\omega sin(\omega t))\,\hat{\mathbf{j}}, a(t) = (\omega^2 sin(\omega t))i + (\omega^2 cos(\omega t))\,\hat{\mathbf{j}},$

speed=$\sqrt{\omega^2 - 2\omega^2 cos(\omega t) + \omega^2 cos^2(\omega t) + \omega^2 sin^2(\omega t)} = \sqrt{2\omega^2(1-cos(\omega t)}$.

5.4E.4 Exercise $\mathbf{5.4E.4}$

A person on a hang glider is spiralling upward as a result of the rapidly rising air on a path having position vector $\vec{r}(t) = (3cost)\hat{\mathbf{i}} + (3sint)\hat{\mathbf{j}} + t^2\hat{\mathbf{k}}$. The path is similar to that of a helix, although it is not a helix. The graph is shown here:



Find

a) the velocity and acceleration vectors

b) the glider's speed at any time

c) the times, if any, at which the glider's acceleration is orthogonal to its velocity

Answer

b) speed= $\sqrt{9+4t^2}$

5.4E.5 Exercise $\mathbf{5.4E.5}$

Given that $\vec{r}(t) = \langle e^{-5t}sint, e^{-5t}cost, 4e^{-5t}\rangle$ is the position vector of a moving particle, find the following quantities:

a) The velocity of the particle

b) The speed of the particle

c) The acceleration of the particle

Answer

a) $\vec{v}(t) = \langle e^{-5t}(cost-5sint), -e^{-5t}(sint+5cost), -20e^{-5t}\rangle$

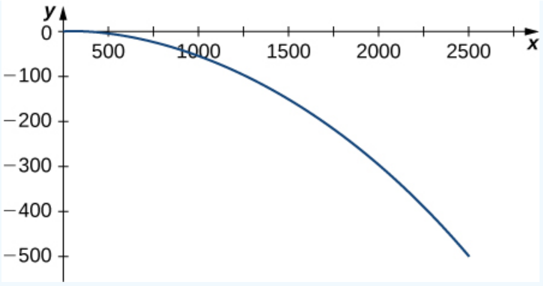
c) $\vec{a}(t) = \langle e^{-5t}(-sint-5cost) - 5e^{-5t}(cost-5sint), -e^{-5t}(cost-5sint) + 5e^{-5t}(sint+5cost), 100e^{-5t}\rangle$

5.4E.6 Exercise $\mathbf{5.4E.6}$

Find the maximum speed of a point on the circumference of an automobile tire of radius 1 ft when the automobile is travelling at 55 mph.

5.4E.7 Exercise $\mathbf{5.4E.7}$

A projectile is shot in the air from ground level with an initial velocity of 500 m/sec at an angle of 60° with the horizontal. The graph is shown here:



a) At what time does the projectile reach maximum height?

b) What is the approximate maximum height of the projectile?

c) At what time is the maximum range of the projectile attained?

d) What is the maximum range?

e) What is the total flight time of the projectile?

Answer

a) 44.185sec, c) $t = 88.37$ sec, e) 88.37 sec

5.4E.8 Exercise $\mathbf{5.4}\boldsymbol{E}\mathbf{.8}$

1) A projectile is fired at a height of 1.5 m above the ground with an initial velocity of 100 m/sec and at an angle of 30° above the horizontal. Use this information to answer the following questions:

a) Determine the maximum height of the projectile.

b) Determine the range of the projectile.

2) A golf ball is hit in a horizontal direction off the top edge of a building that is 100 ft tall. How fast must the ball be launched to land 450 ft away?

3) A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 m. Find the minimum velocity necessary to achieve this range.

4) Prove that an object moving in a straight line at constant speed has an acceleration of zero.

Answer

1b) The range is approximately 886.29 m.

3) 42.16 m/sec

5.4E.9 Exercise $\mathbf{5.4}\boldsymbol{E}\mathbf{.9}$

1) The acceleration of an object is given by $\overrightarrow{a(t)} = t\hat{\mathbf{j}} + t\hat{\mathbf{k}}$. The velocity at $t = 1$ sec is $\overrightarrow{v(1)} = 5\hat{\mathbf{j}}$, and the position of the object at $t = 1$ sec is $\overrightarrow{r(1)} = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$ Find the object's position at any time.

2) Find $\overrightarrow{r(t)}$ given that $\overrightarrow{a(t)} = -32\hat{\mathbf{j}}$, $\overrightarrow{v(0)} = 6003\hat{\mathbf{i}} + 600\hat{\mathbf{j}}$, and $\overrightarrow{r(0)} = 0$.

Answer

1) $\overrightarrow{r(t)} = 0\hat{\mathbf{i}} + (\frac{1}{6}t^3 + 4.5t - \frac{14}{3})\hat{\mathbf{j}} + (\frac{1}{6}t^3 - \frac{1}{2}t - \frac{1}{3})\hat{\mathbf{k}}$

5.4E.10 Exercise $5.4E.10$

1) Find the tangential and normal components of acceleration for $\vec{r}(t) = acos(\omega t)\hat{\mathbf{i}} + bsin(\omega t)\hat{\mathbf{j}}$ at $t = 0$.

2) Given $\vec{r}(t) = t^2\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}$ and find the tangential and normal components of acceleration at $t = 1$.

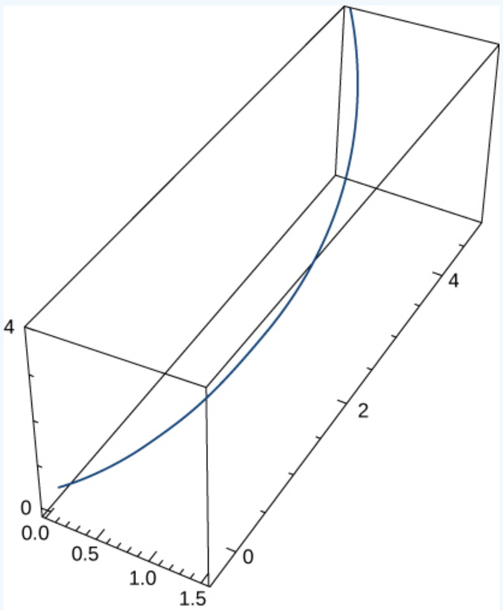
Answer

1) $0, a\omega^2$

5.4E.11 Exercise $5.4E.11$

For each of the following problems, find the tangential and normal components of acceleration.

1) $\vec{r}(t) = \langle e^t cost, e^t sint, e^t \rangle$. The graph is shown here:



2) $\vec{r}(t) = \langle cos(2t), sin(2t), 1 \rangle$

3) $\vec{r}(t) = \langle 2t, t^2, t^3 \rangle$

4) $\vec{r}(t) = t^2\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}$

5) $\vec{r}(t) = 3cos(2\pi t)\hat{\mathbf{i}} + 3sin(2\pi t)\hat{\mathbf{j}}$

Answer

1) $3e^t, 2e^t$

3) $2t, 4 + 2t^2$

5) $0, 23\pi$

5.4E.12 Exercise $5.4E.12$

1) Find the position vector-valued function $\vec{r}(t)$ given that $\vec{a}(t) = \hat{\mathbf{i}} + e^t\hat{\mathbf{j}}, \vec{v}(0) = 2\hat{\mathbf{j}}, and \vec{r}(0) = 2i$.

2) The force on a particle is given by $\vec{f}(t) = (cost)\hat{\mathbf{i}} + (sint)\hat{\mathbf{j}}$. The particle is located at point $(c, 0)$ at $t = 0$. The initial velocity of the particle is given by $\vec{v}(0) = v_0 j$. Find the path of the particle of mass **m**. (Recall, $F = m \cdot a$.)

3) An automobile that weighs 2700 lb makes a turn on a flat road while travelling at 56 ft/sec. If the radius of the turn is 70 ft, what is the required frictional force to keep the car from skidding?

4) Using Kepler's laws, it can be shown that $v_0 = 2GMr$ is the minimum speed needed when $\theta = 0$ so that an object will escape from the pull of a central force resulting from mass M. Use this result to find the minimum speed when $\theta = 0$ for a space capsule to escape from the gravitational pull of Earth if the probe is at an altitude of 300 km above Earth's surface.

5) Find the time in years it takes the dwarf planet Pluto to make one orbit about the Sun given that $a = 39.5 A.U.$

6) Suppose that the position function for an object in three dimensions is given by the equation $\vec{r(t)} = tcos(t)\hat{\mathbf{i}} + tsin(t)\hat{\mathbf{j}} + 3t\hat{\mathbf{k}}$.

a) Show that the particle moves on a circular cone.

b) Find the angle between the velocity and acceleration vectors when $t = 1.5$.

c) Find the tangential and normal components of acceleration when $t = 1.5$.

Answer

2)$\vec{r(t)} = (-1mcost + c + 1m)\hat{\mathbf{i}} + (-sintm + (v_0 + 1m)t)\hat{\mathbf{j}}$

4)$10.94 km/sec$

6) $0.43 m/sec^2, 2.46 m/sec^2$

LibreTexts

5E: Excercises

5E.1 Exercise $\mathbf{5E.1}$

True or False? Justify your answer with a proof or a counterexample.

1. A parametric equation that passes through points P and Q can be given by $\vec{r}(t) = \langle t^2, 3t+1, t-2\rangle\rangle$, where $P(1,4,-1)$ and $Q(16,11,2)$

2. $\frac{d}{dt}[\vec{u}(t) \times \vec{u}(t)] = 2\vec{u}'(t) \times \vec{u}(t)$

Answer

False, $\frac{d}{dt}[\vec{u}(t) \times \vec{u}(t)] = 0$

3. The curvature of a circle of radius r is constant everywhere. Furthermore, the curvature is equal to $\frac{1}{r}$.

4. The speed of a particle with a position function$\vec{r}(t)$ is $\frac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

Answer

False, it is $|\vec{r}'(t)|$.

Exercise $5E.2$

Find the domains of the vector-valued functions.

1. $\vec{r}(t) = \langle sin(t), ln(t), t\rangle$

2. $\vec{r}(t) = \langle e^t, 14-t, sec(t)\rangle$

Answer

$t < 4, t \neq n\pi^2$

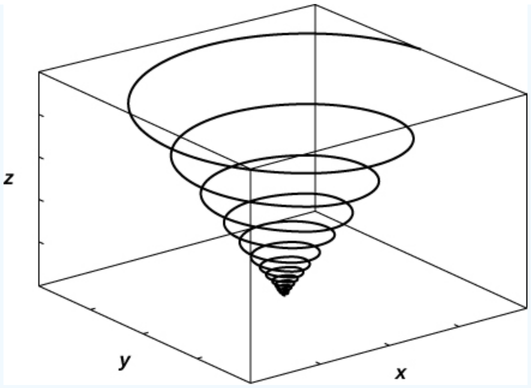
Exercise $5E.3$

Sketch the curves for the following vector equations. Use a calculator if needed.

[T] 1. $\vec{r}(t) = \langle t^2, t^3\rangle$

[T] 2. $\vec{r}(t) = \langle sin(20t)e^{-t}, cos(20t)e^{-t}, e^{-t}\rangle$

Answer



Exercise $5E.4$

Find a vector function that describes the following curves.

1. The intersection of the cylinder $x^2+y^2=4$ with the plane $x+z=6$.

2. The intersection of the cone $z=x^2+y^2$ and plane $z=y-4$.

Answer

$\vec{r}(t)=\langle t,2-t,-2-t\rangle$

Exercise $5E.5$

Find the derivatives of $\vec{u}(t), \vec{u}'(t), \vec{u}'(t)\times\vec{u}(t), \vec{u}(t)\times\vec{u}'(t)$, and $\vec{u}(t)\cdot\vec{u}'(t)$. Also find the unit tangent vector.

1. $\vec{u}(t)=\langle e^t, e-t\rangle$

2. $\vec{u}(t)=\langle t^2, 2t+6, 4t^5-12\rangle$

Answer

\(\vec{u'(t)}=⟨2t,2,20t^4⟩, \vec{{u′′(t)}=⟨2,0,80t^3⟩, \dfrac{d}{dt}[\vec{u′(t)}×\vec{u(t)}]=⟨−480t^3−160t^4,24+75t^2,12+4t⟩, \dfrac{d}{dt}[\vec{u(t)}×\vec{u′(t)}]=⟨480t^3+160t^4,−24−75t^2,−12−4t⟩, \dfrac{d}{dt}[\vec{u(t)}⋅\vec{u′(t)}]=720t^8−9600t^3+6t^2+4,\) unit tangent vector: $\vec{T}(t) =< 2t400t^8+4t^2+4, 2400t^8+4t^2+4, 20t4400t^8+4t^2+4 >$

Exercise $5E.6$

Evaluate the following integrals.

1) $\int (tan(t)sec(t)\hat{\mathbf{i}}-te^{3t}\hat{\mathbf{j}})dt$

2) $\int 14\vec{u}(t)dt$, where $\vec{u}(t)=\langle ln(t)t, t, sin(\pi t)\rangle$

Answer

TBA

Exercise $5E.7$

Find the length for the following curves.

1) $\overrightarrow{r(t)} = \langle 3(t), 4cos(t), 4sin(t) \rangle$ for $1 \leq t \leq 4$

2) $\overrightarrow{r(t)} = 2\hat{\mathbf{i}} + t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}$ for $0 \leq t \leq 1$

Answer

TBA

Exercise $5E.8$

Reparameterize the following functions with respect to their arc length measured from $t=0$ in direction of increasing t.

1) $\overrightarrow{r(t)} = 2t\hat{\mathbf{i}} + (4t-5)\hat{\mathbf{j}} + (1-3t)\hat{\mathbf{k}}$

2) $\overrightarrow{r(t)} = cos(2t)\hat{\mathbf{i}} + 8t\hat{\mathbf{j}} + sin(2t)\hat{\mathbf{k}}$

3) $\overrightarrow{r((s))} = cos(2s)\hat{\mathbf{i}} + tan(2s)\hat{\mathbf{j}} + sin(3s)\hat{\mathbf{k}}$

Answer

Add texts here. Do not delete this text first.

Exercise $5E.9$

Find the curvature for the following vector functions.

1) $\overrightarrow{r(t)} = 2sin(t)\hat{\mathbf{i}} - 4t\hat{\mathbf{j}} + 2cos(t)\hat{\mathbf{k}}$

2) $\overrightarrow{r(t)} = 2e^t\hat{\mathbf{i}} + 2e^{-t}\hat{\mathbf{j}} + 2t\hat{\mathbf{k}}$

Answer

Add texts here. Do not delete this text first.

Exercise $5E.10$

1) Find the unit tangent vector, the unit normal vector, and the binormal vector for $\overrightarrow{r(t)} = 2cos(t)\hat{\mathbf{i}} - +3t\hat{\mathbf{j}} + 2sin(t)\hat{\mathbf{k}}$.

2) Find the tangential and normal acceleration components with the position vector $\overrightarrow{r(t)} = \langle cost, sint, e^t \rangle$.

3) A Ferris wheel car is moving at a constant speed v and has a constant radius r. Find the tangential and normal acceleration of the Ferris wheel car.

4) The position of a particle is given by $\overrightarrow{r(t)} = \langle t^2, ln(t), sin(\pi t) \rangle$, where t is measured in seconds and r is measured in meters. Find the velocity, acceleration, and speed functions. What are the position, velocity, speed, and acceleration of the particle at 1 sec?

Answer

$\overrightarrow{v(t)} = \langle 2t, \frac{1}{t}, \pi cos(\pi t) \rangle$, $\overrightarrow{a(t)} = \langle 2, -\frac{1}{t^2}, -\pi^2 sin(\pi t) \rangle$, speed=$\sqrt{4t^2 + \frac{1}{t^2} + \pi^2 cos(\pi t)}$, $andat\backslash(t=1,$ $\overrightarrow{r(1)} = \langle 1, 0, 0 \rangle$, $\overrightarrow{v(t)} = \langle 2, 1, -\pi \rangle$, $\overrightarrow{a(t)} = \langle 2, -1, 0 \rangle$, speed=$\sqrt{5+\pi^2}$.

Exercise $5E.11$

The following problems consider launching a cannonball out of a cannon. The cannonball is shot out of the cannon with an angle θ and initial velocity v_0. The only force acting on the cannonball is gravity, so we begin with a constant acceleration $\overrightarrow{a(t)} = -g\hat{\mathbf{j}}$

a) Find the velocity vector function $\overrightarrow{v(t)}$

b) Find the position vector $\overrightarrow{r(t)}$ and the parametric representation for the position.

c) At what angle do you need to fire the cannonball for the horizontal distance to be greatest? What is the total distance it would travel?

CHAPTER OVERVIEW

6: Differentiation of Functions of Several Variables

6.1: Functions of Several Variables
6.1E:
6.2: Limits and Continuity
6.2E:
6.3: Partial Derivatives
6.3E:
6.4: Tangent Planes and Linear Approximations
6.4E:Exercises
6.5: The Chain Rule for Multivariable Functions
6.5E: Exercises
6.6: Directional Derivatives and the Gradient
6.6E: Exercises
6.7: Maxima/Minima Problems
6.7E:
6.8: Lagrange Multipliers
6.8E:
6E: Chapter Review Excersies

6.1: Functions of Several Variables

This page is a draft and is under active development.

Our first step is to explain what a function of more than one variable is, starting with functions of two independent variables. This step includes identifying the domain and range of such functions and learning how to graph them. We also examine ways to relate the graphs of functions in three dimensions to graphs of more familiar planar functions.

6.1.1 Functions of Two Variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

Definition: function of two variables

A *function of two variables* $z = (x, y)$ maps each ordered pair (x, y) in a subset D of the real plane R^2 to a unique real number *z*. The set D is called the domain of the function. The range of f is the set of all real numbers *z* that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in Figure $6.1.1$.

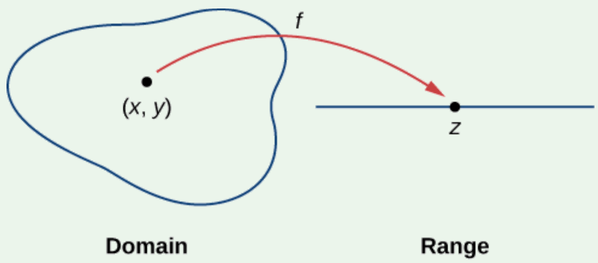


Figure $6.1.1$: The domain of a function of two variables consists of ordered pairs (x, y).

Determining the domain of a function of two variables involves taking into account any domain restrictions that may exist. Let's take a look.

Example 6.1.1: Domains and Ranges for Functions of Two Variables

Find the domain and range of each of the following functions:

a. $f(x, y) = 3x + 5y + 2$
b. $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution

a. This is an example of a linear function in two variables. There are no values or combinations of x and y that cause $f(x, y)$ to be undefined, so the domain of f is R^2. To determine the range, first pick a value for *z*. We need to find a solution to the equation $f(x, y) = z$, or $3x - 5y + 2 = z$. One such solution can be obtained by first setting $y = 0$, which yields the equation $3x + 2 = z$. The solution to this equation is $x = \frac{z-2}{3}$, which gives the ordered pair $(\frac{z-2}{3}, 0)$ as a solution to the equation $f(x, y) = z$ for any value of z. Therefore, the range of the function is all real numbers, or R.

b. For the function $g(x, y)$ to have a real value, the quantity under the square root must be nonnegative:

$$9 - x^2 - y^2 \geq 0.$$

This inequality can be written in the form

$$x^2 + y^2 \leq 9.$$

Therefore, the domain of $g(x, y)$ is $\{(x, y) \in R^2 \mid x^2 + y^2 \leq 9\}$. The graph of this set of points can be described as a disk of radius 3 centered at the origin. The domain includes the boundary circle as shown in the following graph.

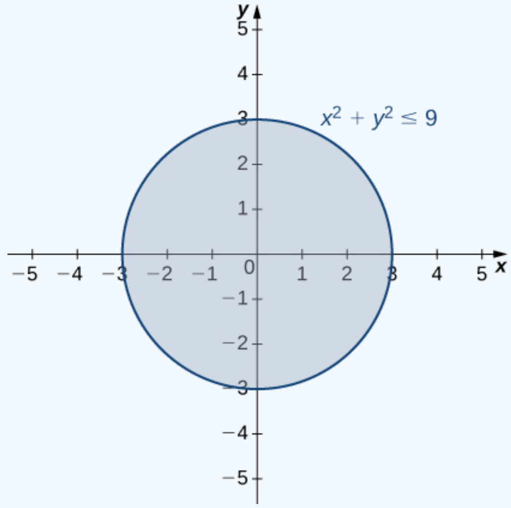


Figure 6.1.2: The domain of the function $g(x,y)=\sqrt{9-x^2-y^2}$ is a closed disk of radius 3.

To determine the range of $g(x,y)=\sqrt{9-x^2-y^2}$ we start with a point (x_0,y_0) on the boundary of the domain, which is defined by the relation $x^2+y^2=9$. It follows that $x_0^2+y_0^2=9$ and

$$\begin{aligned} g(x_0,y_0) &= \sqrt{9-x_0^2-y_0^2} \\ &= \sqrt{9-(x_0^2+y_0^2)} \\ &= \sqrt{9-9} \\ &= 0. \end{aligned}$$

If $x_0^2+y_0^2=0$ (in other words, $x_0=y_0=0$), then

$$\begin{aligned} g(x_0,y_0) &= \sqrt{9-x_0^2-y_0^2} \\ &= \sqrt{9-(x_0^2+y_0^2)} \\ &= \sqrt{9-0}=3. \end{aligned}$$

This is the maximum value of the function. Given any value c between 0 and 3, we can find an entire set of points inside the domain of g such that $g(x,y)=c$:

$$\begin{aligned} \sqrt{9-x^2-y^2} &= c \\ 9-x^2-y^2 &= c^2 \\ x^2+y^2 &= 9-c^2. \end{aligned}$$

Since $9-c^2>0$, this describes a circle of radius $\sqrt{9-c^2}$ centered at the origin. Any point on this circle satisfies the equation $g(x,y)=c$. Therefore, the range of this function can be written in interval notation as $[0,3]$.

Exercise 6.1.1

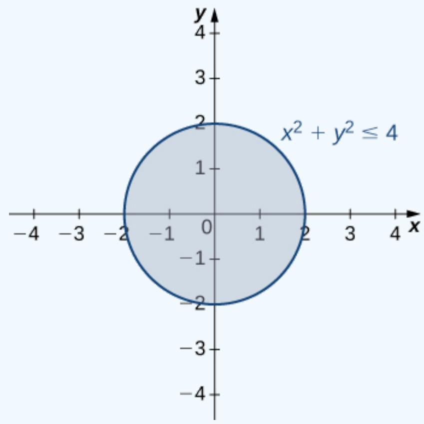
Find the domain and range of the function $f(x,y)=\sqrt{36-9x^2-9y^2}$.

Hint

Determine the set of ordered pairs that do not make the radicand negative.

Solution

The domain is $\{(x,y)|x^2+y^2\leq 4\}$ the shaded circle defined by the inequality $x^2+y^2\leq 4$, which has a circle of radius 2 as its boundary. The range is $[0,6]$.



6.1.2 Graphing Functions of Two Variables

Suppose we wish to graph the function $z=(x,y)$. This function has two independent variables (x and y) and one dependent variable (z). When graphing a function $y=f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x,y) in the plane, and every point in the plane has an ordered pair (x,y) associated with it. With a function of two variables, each ordered pair (x,y) in the domain of the function is mapped to a real number z. Therefore, the graph of the function f consists of ordered triples (x,y,z). The graph of a function $z=(x,y)$ of two variables is called a surface.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x,y) coordinate system laying flat. Then, every point in the domain of the function f has a unique *z*-value associated with it. If *z* is positive, then the graphed point is located above the *xy*-plane, if *z* is negative, then the graphed point is located below the *xy*-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f.

Example 6.1.2: Graphing Functions of Two Variables

Create a graph of each of the following functions:

a. $g(x,y)=\sqrt{9-x^2-y^2}$
b. $f(x,y)=x^2+y^2$

Solution

a. In Example $6.1.2$, we determined that the domain of $g(x,y)=\sqrt{9-x^2-y^2}$ is $\{(x,y)\in R^2 \mid x^2+y^2\leq 9\}$ and the range is $\{z\in R^2 \mid 0\leq z\leq 3\}$. When $x^2+y^2=9$ we have $g(x,y)=0$. Therefore any point on the circle of radius 3 centered at the origin in the xy-plane maps to $z=0$ in R^3. If $x^2+y^2=8$, then $g(x,y)=1$, so any point on the circle of radius $2\sqrt{2}$ centered at the origin in the xy-plane maps to $z=1$ in R^3. As x^2+y^2 gets closer to zero, the value of z approaches 3. When $x^2+y^2=0$, then $g(x,y)=3$. This is the origin in the xy-plane If x^2+y^2 is equal to any other value between 0 and 9, then $g(x,y)$ equals some other constant between 0 and 3. The surface described by this function is a hemisphere centered at the origin with radius 3 as shown in the following graph.

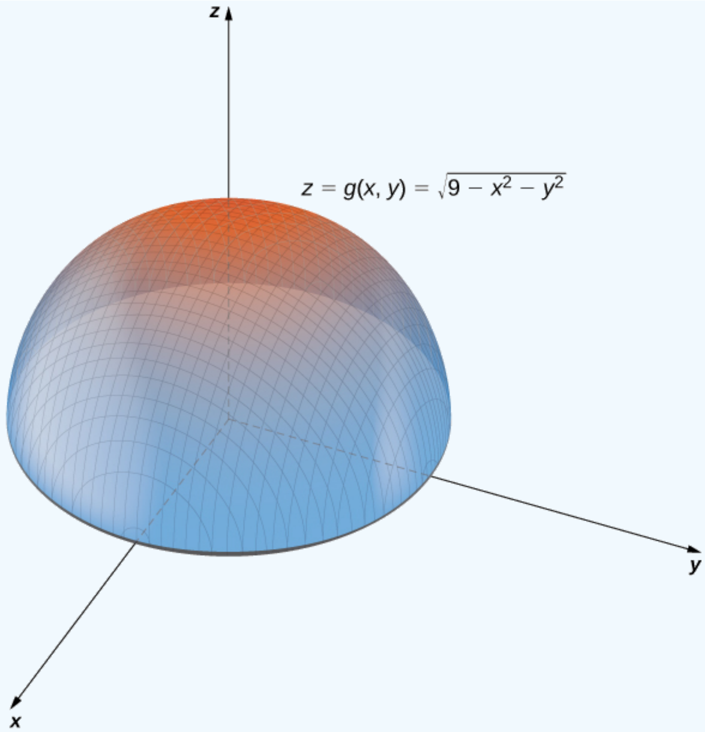


Figure 6.1.3: Graph of the hemisphere represented by the given function of two variables.

b. This function also contains the expression $x^2 + y^2$. Setting this expression equal to various values starting at zero, we obtain circles of increasing radius. The minimum value of $f(x, y) = x^2 + y^2$ is zero (attained when $x = y = 0.$. When $x = 0$, the function becomes $z = y^2$, and when $y = 0$, then the function becomes $z = x^2$. These are cross-sections of the graph, and are parabolas. Recall from Introduction to Vectors in Space that the name of the graph of $f(x, y) = x^2 + y^2$ is a **paraboloid**. The graph of f appears in the following graph.

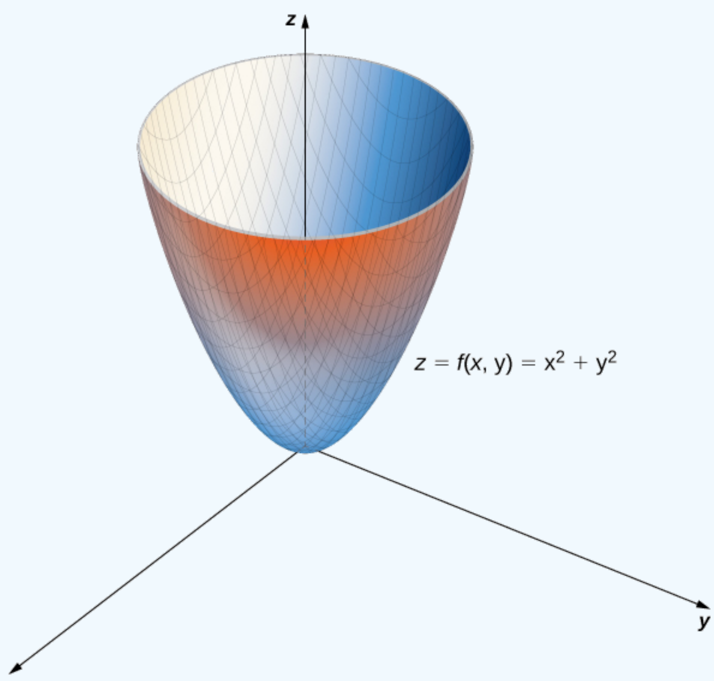


Figure 6.1.4: A paraboloid is the graph of the given function of two variables.

Use the following app to visualize graphs.

Example 6.1.3: Nuts and Bolts

A profit function for a hardware manufacturer is given by

$$f(x,y) = 16 - (x-3)^2 - (y-2)^2,$$

where x is the number of nuts sold per month (measured in thousands) and y represents the number of bolts sold per month (measured in thousands). Profit is measured in thousands of dollars. Sketch a graph of this function.

Solution

This function is a polynomial function in two variables. The domain of f consists of (x,y) coordinate pairs that yield a nonnegative profit:

$$\begin{aligned} 16 - (x-3)^2 - (y-2)^2 &\geq 0 \\ (x-3)^2 + (y-2)^2 &\leq 16. \end{aligned}$$

This is a disk of radius 4 centered at $(3,2)$. A further restriction is that both x and y must be nonnegative. When $x=3$ and $y=2$, $f(x,y)=16$. Note that it is possible for either value to be a noninteger; for example, it is possible to sell 2.5 thousand nuts in a month. The domain, therefore, contains thousands of points, so we can consider all points within the disk. For any $z<16$, we can solve the equation $f(x,y)=16$:

$$\begin{aligned} 16 - (x-3)^2 - (y-2)^2 &= z \\ (x-3)^2 + (y-2)^2 &= 16 - z. \end{aligned}$$

Since $z<16,$ we know that $16-z>0,$ so the previous equation describes a circle with radius $\sqrt{16-z}$ centered at the point $(3,2)$. Therefore. the range of $f(x,y)$ is $\{z\in\mathbb{R} | z\leq 16\}$. The graph of $f(x,y)$ is also a paraboloid, and this paraboloid points downward as shown.

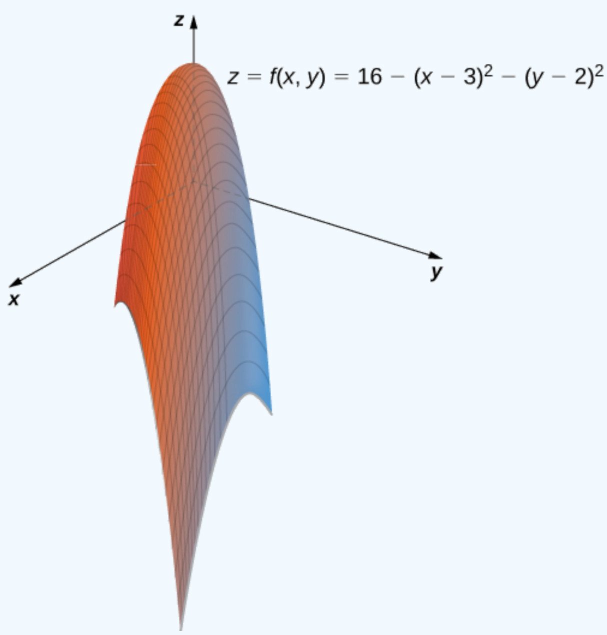
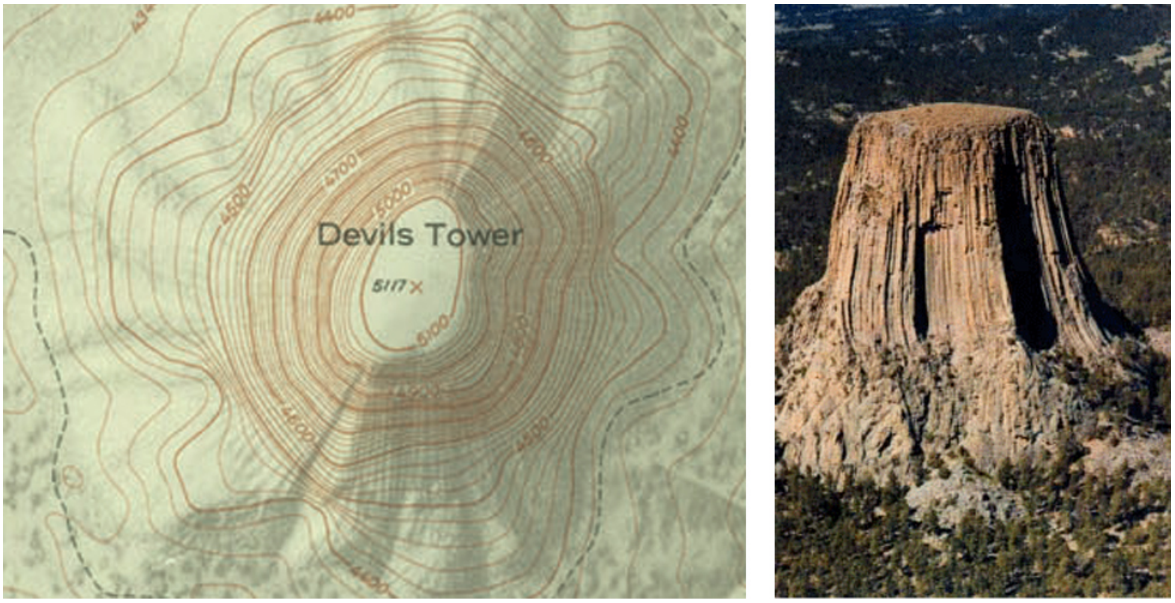


Figure 6.1.5: The graph of the given function of two variables is also a paraboloid.

6.1.3 Level Curves

If hikers walk along rugged trails, they might use a topographical map that shows how steeply the trails change. A *topographical map* contains curved lines called contour lines. Each contour line corresponds to the points on the map that have equal elevation (Figure 6.1.6). A level curve of a function of two variables $f(x,y)$ is completely analogous to a contour line on a topographical map.



(a) (b)

Figure 6.1.6: (a) A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (b) A perspective photo of Devil's Tower shows just how steep its sides are. Notice the top of the tower has the same shape as the center of the topographical map.

Definition: level curves

Given a function $f(x, y)$ and a number c in the range of f, a level curve of a function of two variables for the value c is defined to be the set of points satisfying the equation $f(x, y) = c$.

Returning to the function $g(x, y) = \sqrt{9 - x^2 - y^2}$, we can determine the level curves of this function. The range of g is the closed interval $[0, 3]$. First, we choose any number in this closed interval—say, $c = 2$. The level curve corresponding to $c = 2$ is described by the equation

$$\sqrt{9 - x^2 - y^2} = 2. \tag{6.1.1}$$

To simplify, square both sides of this equation:

$$9 - x^2 - y^2 = 4. \tag{6.1.2}$$

Now, multiply both sides of the equation by -1 and add $9$to each side:

$$x^2 + y^2 = 5. \tag{6.1.3}$$

This equation describes a circle centered at the origin with radius $\sqrt{5}$. Using values of c between 0 and 3 yields other circles also centered at the origin. If $c = 3$, then the circle has radius 0, so it consists solely of the origin. Figure $6.1.7$ is a graph of the level curves of this function corresponding to $c = 0, 1, 2,$ and 3. Note that in the previous derivation it may be possible that we introduced extra solutions by squaring both sides. This is not the case here because the range of the square root function is nonnegative.

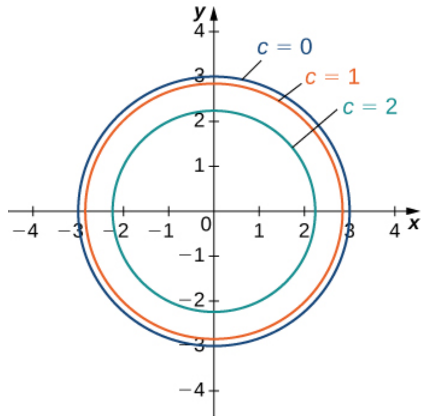


Figure 6.1.7: Level curves of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$, using $c = 0, 1, 2,$ and 3 ($c = 3$ corresponds to the origin).

A graph of the various level curves of a function is called a *contour map*.

Example 6.1.4: Making a Contour Map

Given the function $f(x, y) = \sqrt{8 + 8x - 4y - 4x^2 - y^2}$, find the level curve corresponding to $c = 0$. Then create a contour map for this function. What are the domain and range of f?

Solution

To find the level curve for $c = 0$, we set $f(x, y) = 0$ and solve. This gives

$$0 = \sqrt{8 + 8x - 4y - 4x^2 - y^2}.$$

We then square both sides and multiply both sides of the equation by -1:

$$4x^2 + y^2 - 8x + 4y - 8 = 0.$$

Now, we rearrange the terms, putting the x terms together and the y terms together, and add 8 to each side:

$$4x^2 - 8x + y^2 + 4y = 8.$$

Next, we group the pairs of terms containing the same variable in parentheses, and factor 4 from the first pair:

$$4(x^2 - 2x) + (y^2 + 4y) = 8.$$

Then we complete the square in each pair of parentheses and add the correct value to the right-hand side:

$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 4(1) + 4.$$

Next, we factor the left-hand side and simplify the right-hand side:

$$4(x - 1)^2 + (y + 2)^2 = 16.$$

Last, we divide both sides by 16 :

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1.$$

This equation describes an ellipse centered at $(1, -2)$. The graph of this ellipse appears in the following graph.

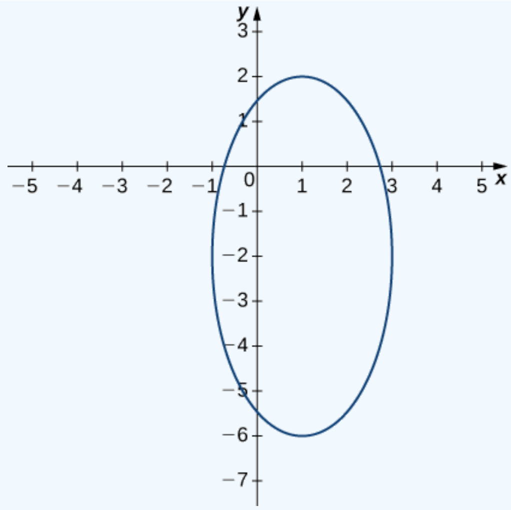


Figure 6.1.8: Level curve of the function $f(x,y) = \sqrt{8+8x-4y-4x^2-y^2}$ corresponding to $c=0$

We can repeat the same derivation for values of c less than 4. Then, Equation becomes

$$\frac{4(x-1)^2}{16-c^2} + \frac{(y+2)^2}{16-c^2} = 1$$

for an arbitrary value of c. Figure $6.1.9$ shows a contour map for $f(x,y)$ using the values $c=0,1,2,$ and 3. When $c=4$, the level curve is the point $(-1,2)$.

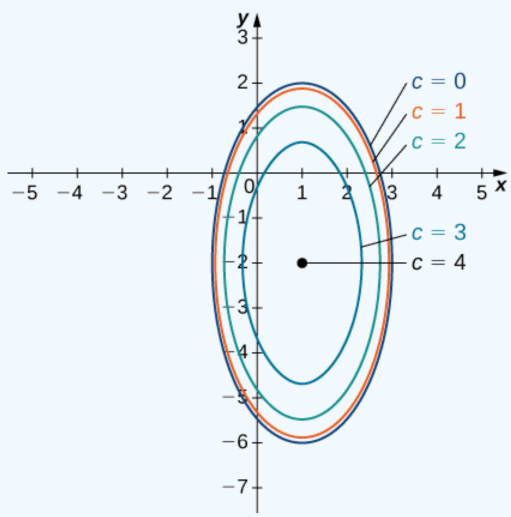


Figure 6.1.9: Contour map for the function $f(x,y) = \sqrt{8+8x-4y-4x^2-y^2}$ using the values $c=0,1,2,3,$ and 4.

Exercise 6.1.4

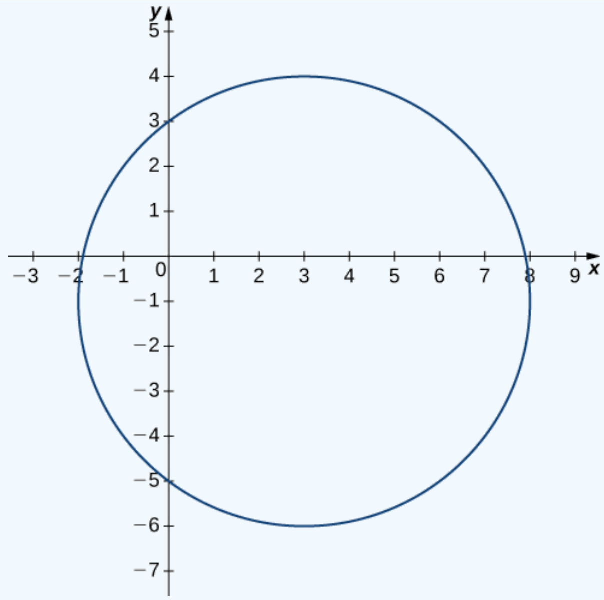
Find and graph the level curve of the function $g(x,y) = x^2+y^2-6x+2y$ corresponding to $c=15$.

Hint

First, set $g(x,y)=15$ and then complete the square.

Solution

The equation of the level curve can be written as $(x-3)^2+(y+1)^2=25,$ which is a circle with radius 5 centered at $(3,-1)$.



Another useful tool for understanding the **graph of a function of two variables** is called a vertical trace. Level curves are always graphed in the $xy-plane$, but as their name implies, vertical traces are graphed in the $xz-$ or $yz-$ planes.

Definition: vertical traces

Consider a function $z = f(x, y)$ with domain $D \subseteq \mathbb{R}^2$. A *vertical trace* of the function can be either the set of points that solves the equation $f(a, y) = z$ for a given constant $x = a$ or $f(x, b) = z$ for a given constant $y = b$.

Example 6.1.5: Finding Vertical Traces

Find vertical traces for the function $f(x, y) = \sin x \cos y$ corresponding to $x = -\frac{\pi}{4}, 0,$ and $\frac{\pi}{4}$, and $y = -\frac{\pi}{4}, 0,$ and $\frac{\pi}{4}$.

Solution

First set $x = -\frac{\pi}{4}$ in the equation $z = \sin x \cos y$:

$$z = \sin(-\frac{\pi}{4}) \cos y = -\frac{\sqrt{2} \cos y}{2} \approx -0.7071 \cos y.$$

This describes a cosine graph in the plane $x = -\frac{\pi}{4}$. The other values of z appear in the following table.

Vertical Traces Parallel to the $xz-Plane$ for the Function $f(x, y) = \sin x \cos y$

c	Vertical Trace for $x = c$
$-\frac{\pi}{4}$	$z = -\frac{\sqrt{2}\cos y}{2}$
0	$z = 0$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2}\cos y}{2}$

In a similar fashion, we can substitute the $y-values$ in the equation $f(x, y)$ to obtain the traces in the $yz-plane,$ as listed in the following table.

Vertical Traces Parallel to the $yz-Plane$ for the Function $f(x, y) = \sin x \cos y$

d	Vertical Trace for $y = d$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2}\sin x}{2}$

d	Vertical Trace for $y = d$
0	$z = \sin x$
$-\frac{\pi}{4}$	$z = \frac{\sqrt{2}\sin x}{2}$

The three traces in the $xz-plane$ are cosine functions; the three traces in the $yz-plane$ are sine functions. These curves appear in the intersections of the surface with the planes $x = -\frac{\pi}{4}, x = 0, x = \frac{\pi}{4}$ and $y = -\frac{\pi}{4}, y = 0, y = \frac{\pi}{4}$ as shown in the following figure.

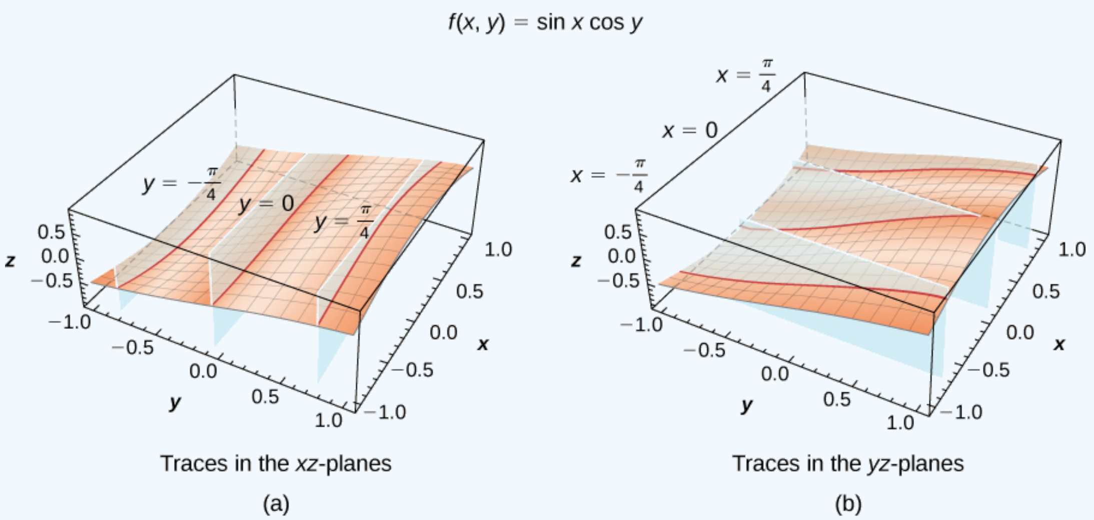


Figure 6.1.10: Vertical traces of the function $f(x,y)$ are cosine curves in the $xz-planes$ (a) and sine curves in the $yz-planes$ (b).

Exercise 6.1.5

Determine the equation of the vertical trace of the function $g(x,y) = -x^2 - y^2 + 2x + 4y - 1$ corresponding to $y = 3$, and describe its graph.

Hint

Set $y = 3$ in the equation $z = -x^2 - y^2 + 2x + 4y - 1$ and complete the square.

Solution

$z = 3 - (x-1)^2$. This function describes a parabola opening downward in the plane $y = 3$.

Functions of two variables can produce some striking-looking surfaces. Figure $6.1.11$ shows two examples.

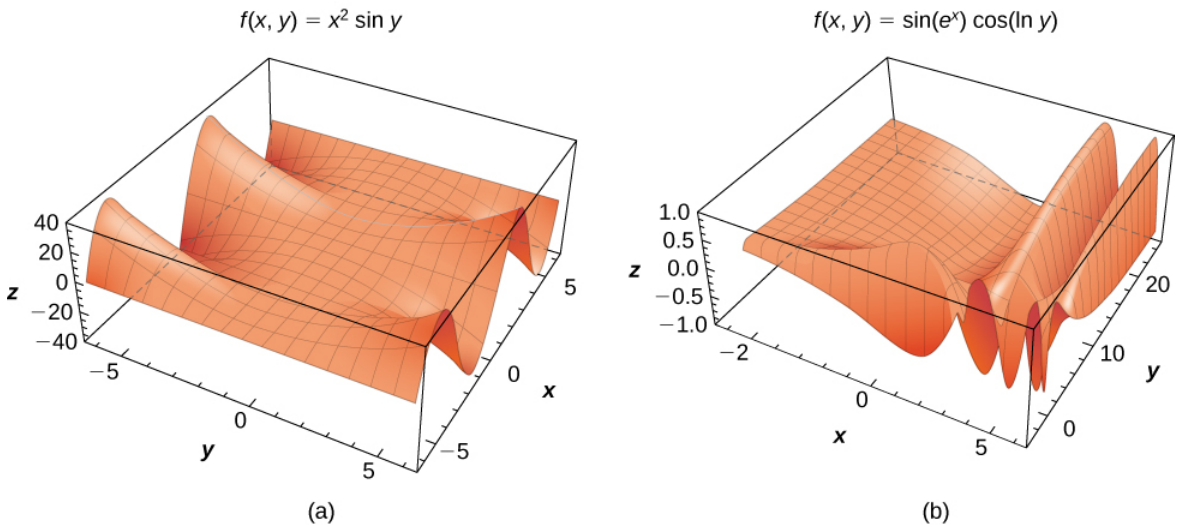


Figure 6.1.11: Examples of surfaces representing functions of two variables: (a) a combination of a power function and a sine function and (b) a combination of trigonometric, exponential, and logarithmic functions.

6.1.4 Functions of More Than Two Variables

So far, we have examined only functions of two variables. However, it is useful to take a brief look at functions of more than two variables. Two such examples are

$$\underbrace{f(x,y,z)=x^2-2xy+y^2+3yz-z^2+4x-2y+3x-6}_{\text{a polynomial in three variables}} \tag{6.1.4}$$

and

$$g(x,y,t)=(x^2-4xy+y^2)\sin t-(3x+5y)\cos t. \tag{6.1.5}$$

In the first function, (x,y,z) represents a point in space, and the function f maps each point in space to a fourth quantity, such as temperature or wind speed. In the second function, (x,y) can represent a point in the plane, and t can represent time. The function might map a point in the plane to a third quantity (for example, pressure) at a given time t. The method for finding the domain of a function of more than two variables is analogous to the method for functions of one or two variables.

Example 6.1.6: Domains for Functions of Three Variables

Find the domain of each of the following functions:

a. $f(x,y,z)=\dfrac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$

b. $g(x,y,t)=\dfrac{\sqrt{2t-4}}{x^2-y^2}$

Solution:

a. For the function $f(x,y,z)=\dfrac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$ to be defined (and be a real value), two conditions must hold:

1. The denominator **cannot** be zero.
2. The radicand **cannot** be negative.

Combining these conditions leads to the inequality

$$9-x^2-y^2-z^2>0.$$

Moving the variables to the other side and reversing the inequality gives the domain as

$$domain(f)=\{(x,y,z)\in R^3 \mid x^2+y^2+z^2<9\},$$

which describes a ball of radius 3 centered at the origin. (Note: The surface of the ball is not included in this domain.)

b. For the function $g(x, y, t) = \frac{\sqrt{2t-4}}{x^2-y^2}$ to be defined (and be a real value), two conditions must hold:

1. The radicand cannot be negative.
2. The denominator cannot be zero.

Since the radicand cannot be negative, this implies $2t-4 \geq 0$, and therefore that $t \geq 2$. Since the denominator cannot be zero, $x^2-y^2 \neq 0$, or $x^2 \neq y^2$, Which can be rewritten as $y = \pm x$, which are the equations of two lines passing through the origin. Therefore, the domain of g is

$$domain(g) = \{(x, y, t) | y \neq \pm x, t \geq 2\}.$$

Exercise 6.1.6

Find the domain of the function $h(x, y, t) = (3t-6)\sqrt{y-4x^2+4}$.

Hint

Check for values that make radicands negative or denominators equal to zero.

Solution

$$domain(h) = \{(x, y, t) \in \mathbb{R}^3 \mid y \geq 4x^2-4\}$$

Functions of two variables have level curves, which are shown as curves in the $xy-plane$. However, when the function has three variables, the curves become surfaces, so we can define level surfaces for functions of three variables.

Definition: level surface of a function of three variables

Given a function $f(x, y, z)$ and a number c in the range of f, a *level surface of a function of three variables* is defined to be the set of points satisfying the equation $f(x, y, z) = c$.

Example 6.1.7: Finding a Level Surface

Find the level surface for the function $f(x, y, z) = 4x^2 + 9y^2 - z^2$ corresponding to $c = 1$.

Solution

The level surface is defined by the equation $4x^2 + 9y^2 - z^2 = 1$. This equation describes a hyperboloid of one sheet as shown in Figure $6.1.12$.

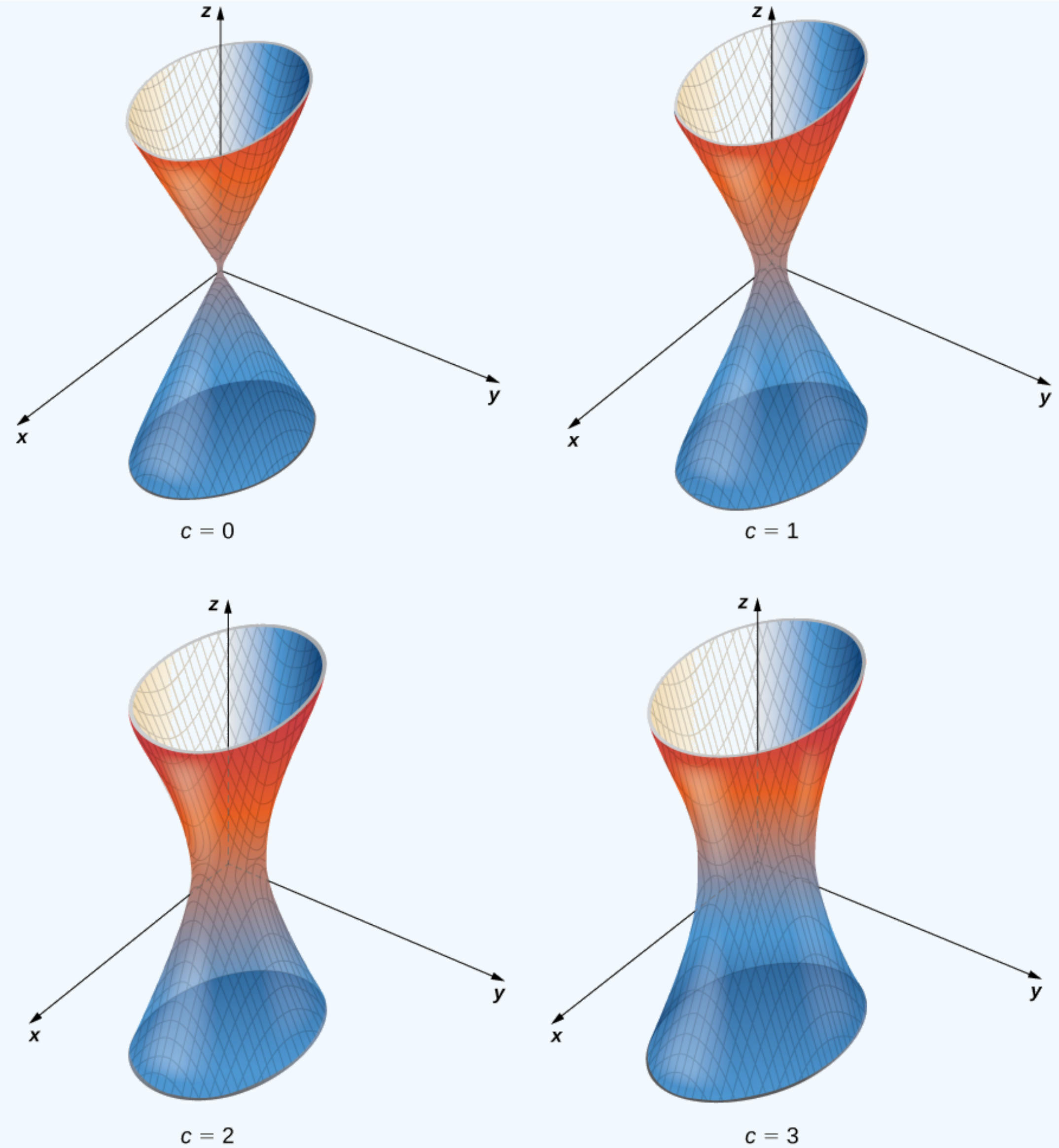


Figure 6.1.12: A hyperboloid of one sheet with some of its level surfaces.

Exercise 6.1.5

Find the equation of the level surface of the function

$$g(x,y,z)=x^2+y^2+z^2-2x+4y-6z$$

corresponding to $c=2$, and describe the surface, if possible.

Hint

Set $g(x,y,z)=c$ and complete the square.

Solution

((x−1)^2+(y+2)^2+(z−3)^2=16\) describes a sphere of radius 4 centered at the point $(1,-2,3)$.

6.1.5 Summary

- The graph of a function of two variables is a surface in $\mathbb{R}^3$ and can be studied using level curves and vertical traces.
- A set of level curves is called a contour map.

6.1.6 Key Equations

- **Vertical trace**

  $f(a, y) = z$ for $x = a$ or $f(x, b) = z$ for $y = b$
- **Level surface of a function of three variables**

  $f(x, y, z) = c$

6.1.7 Glossary

contour map

a plot of the various level curves of a given function $f(x, y)$

function of two variables

a function $z = f(x, y)$ that maps each ordered pair (x, y) in a subset D of R^2 to a unique real number z

graph of a function of two variables

a set of ordered triples (x, y, z) that satisfies the equation $z = f(x, y)$ plotted in three-dimensional Cartesian space

level curve of a function of two variables

the set of points satisfying the equation $f(x, y) = c$ for some real number c in the range of f

level surface of a function of three variables

the set of points satisfying the equation $f(x, y, z) = c$ for some real number c in the range of f

surface

the graph of a function of two variables, $z = f(x, y)$

vertical trace

the set of ordered triples (c, y, z) that solves the equation $f(c, y) = z$ for a given constant $x = c$ or the set of ordered triples (x, d, z) that solves the equation $f(x, d) = z$ for a given constant $y = d$

6.1.8 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

6.1E:

6.1E.1 Exercise $6.1E.1$ functional value

For the following exercises, evaluate each function at the indicated values.

1) $W(x,y)=4x^2+y^2$. Find $W(2,-1), W(-3,6)$.

Answer

Solution:$17,72$

2) $W(x,y)=4x^2+y^2$. Find $W(2+h,3+h)$.

3) The volume of a right circular cylinder is calculated by a function of two variables, $V(x,y)=\pi x^2 y$, where x is the radius of the right circular cylinder and y represents the height of the cylinder. Evaluate $V(2,5)$ and explain what this means.

Answer

Solution:$20\pi.$ This is the volume when the radius is 2 and the height is 5.

4) An oxygen tank is constructed of a right cylinder of height y and radius x with two hemispheres of radius x mounted on the top and bottom of the cylinder. Express the volume of the cylinder as a function of two variables, x and y, find $V(10,2)$, and explain what this means.

6.1E.2 Exercise $6.1E.2$ Domain

For the following exercises, find the domain of the function.

1) $V(x,y)=4x^2+y^2$

Answer

Solution:All points in the $xy-plane$

2) $f(x,y)=\sqrt{x^2+y^2-4}$

3) $f(x,y)=4ln(y^2-x)$

Answer

Solution:$x<y^2$

4) $g(x,y)=\sqrt{16-4x^2-y^2}$

5) $z(x,y)=y^2-x^2$

Answer

Solution:All real ordered pairs in the $xy-plane$ of the form (a,b)

6) $f(x,y)=\frac{y+2}{x^2}$

6.1E.3 Exercise $6.1E.3$ range

Find the range of the functions.

1) $g(x,y)=\sqrt{16-4x^2-y^2}$

Answer

Solution:$\{z|0\leq z\leq 4\}$

2) $V(x,y) = 4x^2 + y^2$

3) $z = y^2 - x^2$

Answer

Solution:The set R.

6.1E.4 Exercise $6.1E.4$ Level Curves

For the following exercises, find the level curves of each function at the indicated value of c to visualize the given function.

1) $z(x,y) = y^2 - x^2, c = 1$

2) $z(x,y) = y^2 - x^2, c = 4$

Answer

Solution:$y^2 - x^2 = 4,$ a hyperbola

3) $g(x,y) = x^2 + y^2; c = 4, c = 9$

4) $g(x,y) = 4 - x - y; c = 0, 4$

Answer

Solution:$4 = x + y,$ a line; $x + y = 0,$ line through the origin

5) $h(x,y) = 2x - y; c = 0, -2, 2$

Answer

Solution:$2x - y = 0, 2x - y = -2, 2x - y = 2;$ three lines

6) $f(x,y) = x^2 - y; c = 1, 2$

7) $g(x,y) = \frac{x}{x+y}; c = -1, 0, 2$

Answer

Solution:$\frac{x}{x+y} = -1, \frac{x}{x+y} = 0, \frac{x}{x+y} = 2$

8) $g(x,y) = x^3 - y; c = -1, 0, 2$

9) $g(x,y) = e^{xy}; c = \frac{1}{2}, 3$

Answer

Solution:$e^{xy} = \frac{1}{2}, e^{xy} = 3$

10) $f(x,y) = x^2; c = 4, 9$

12) $f(x,y) = xy - x; c = -2, 0, 2$

Answer

Solution:$xy - x = -2, xy - x = 0, xy - x = 2$

13) $h(x,y) = ln(x^2 + y^2); c = -1, 0, 1$

14) $g(x,y) = ln(\frac{y}{x^2}); c = -2, 0, 2$

Answer

Solution:$e^{-2}x^2 = y, y = x^2, y = e^2x^2$

15) $z = f(x,y) = \sqrt{x^2+y^2}, c = 3$

16) $f(x,y) = \frac{y+2}{x^2}, c =$any constant

Answer

Solution:The level curves are parabolas of the form $y = cx^2 - 2$.

6.1E.5 Exercise $6.1E.5$ Vertical Traces

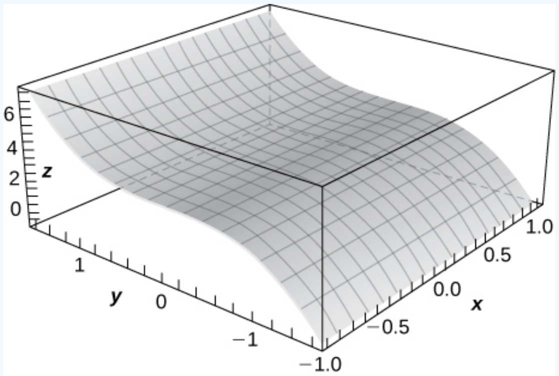
For the following exercises, find the vertical traces of the functions at the indicated values of x and y, and plot the traces.

1) $z = 4 - x - y; x = 2$

2) $f(x,y) = 3x + y^3, x = 1$

Answer

Solution:$z = 3 + y^3,$ a curve in the **zy-plane** with rulings parallel to the $x - axis$



3) $z = cos\sqrt{x^2+y^2}\ x = 1$

6.1E.6 Exercise $6.1E.6$ Domain

Find the domain of the following functions.

1) $z = \sqrt{100 - 4x^2 - 25y^2}$

Answer

Solution:$\frac{x^2}{25} + \frac{y^2}{4} \leq 1$

2) $z = ln(x - y^2)$

3) $f(x,y,z) = \frac{1}{\sqrt{36 - 4x^2 - 9y^2 - z^2}}$

Answer

Solution:$\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{36} < 1$

4) $f(x,y,z) = \sqrt{49 - x^2 - y^2 - z^2}$

5) $f(x,y,z)=\sqrt[3]{16-x^2-y^2-z^2}$

Answer

Solution:All points in $xyz-space$

6) $f(x,y)=cos\sqrt{x^2+y^2}$

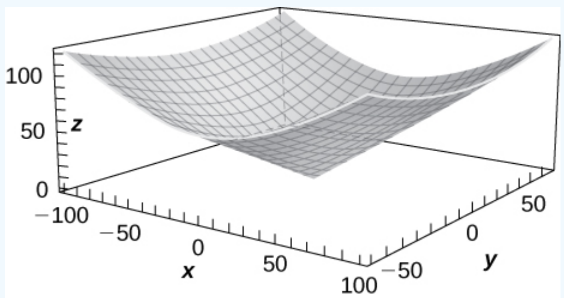
6.1E.7 Exercise $\mathbf{6.1}\boldsymbol{E}\mathbf{.7}$ Graph

For the following exercises, plot a graph of the function.

1) $z=f(x,y)=\sqrt{x^2+y^2}$

Answer

Solution:

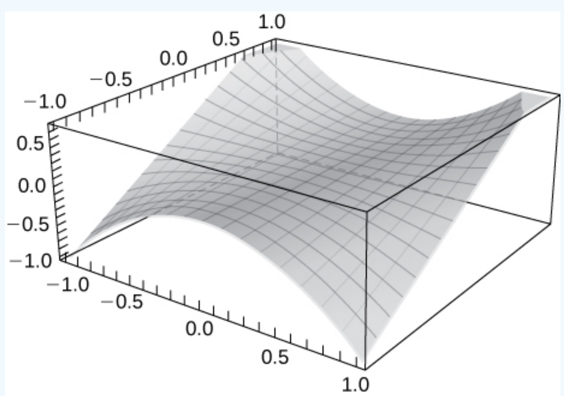


2) $z=x^2+y^2$

3) Use technology to graph $z=x^2y$.

Answer

Solution:



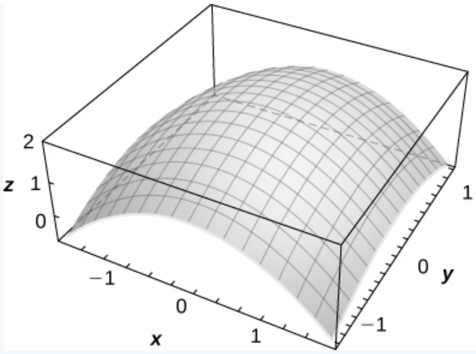
6.1E.8 Exercise $\mathbf{6.1}\boldsymbol{E}\mathbf{.8}$ Level curves

Sketch the following by finding the level curves. Verify the graph using technology.

1) $f(x,y)=\sqrt{4-x^2-y^2}$

2) $f(x,y)=2-\sqrt{x^2+y^2}$

Answer

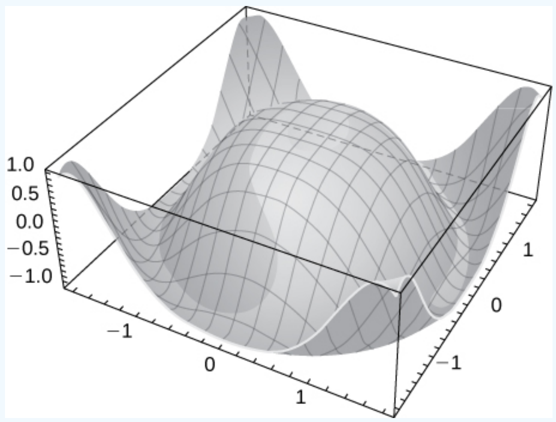


3) $z=1+e^{-x^2-y^2}$

4) $z=cos\sqrt{x^2+y^2}$

Answer

Solution:



5) $z=y^2-x^2$

6.1E.9 Exercise $6.1E.9$ Contour lines

1) Describe the contour lines for several values of c for $z=x^2+y^2-2x-2y$.

Answer

Solution:The contour lines are circles.

6.1E.10 Exercise $6.1E.10$ level surface

Find the level surface for the functions of three variables and describe it.

1) $w(x,y,z)=x-2y+z, c=4$

2) $w(x,y,z)=x^2+y^2+z^2, c=9$

Answer

Solution:$x^2+y^2+z^2=9$, a sphere of radius 3

3) $w(x,y,z)=x^2+y^2-z^2, c=-4$

4) $w(x,y,z)=x^2+y^2-z^2, c=4$

Answer

Solution:$x^2+y^2-z^2=4, \quad$ a hyperboloid of one sheet

5) $w(x,y,z)=9x^2-4y^2+36z^2, c=0$

6.1E.11 Exercise $\mathbf{6.1E.11}$ level curve at a given point

For the following exercises, find an equation of the level curve of f that contains the point P.

1) $f(x,y)=1-4x^2-y^2, P(0,1)$

Answer

Solution:$4x^2+y^2=1,$

2) $g(x,y)=y^2 arctan x, P(1,2)$

3) $g(x,y)=e^{xy}(x^2+y^2), P(1,0)$

Answer

Solution:$1=e^{xy}(x^2+y^2)$

6.1E.12 Exercise $\mathbf{6.1E.12}$ Applications

1) The strength E of an electric field at point (x,y,z) resulting from an infinitely long charged wire lying along the $y-axis$ is given by $E(x,y,z)=k/\sqrt{x^2+y^2}$, where k is a positive constant. For simplicity, let $k=1$ and find the equations of the level surfaces for $E=10$ and $E=100$.

2) A thin plate made of iron is located in the $xy-plane$. The temperature T in degrees Celsius at a point $P(x,y)$ is inversely proportional to the square of its distance from the origin. Express T as a function of x and y.

Answer

Solution:$T(x,y)=\dfrac{k}{x^2+y^2}$

3) Refer to the preceding problem. Using the temperature function found there, determine the proportionality constant if the temperature at point $P(1,2)$ is $50^\circ C$. Use this constant to determine the temperature at point $Q(3,4)$.

4) Refer to the preceding problem. Find the level curves for $T=40^\circ C$ and $T=100^\circ C$, and describe what the level curves represent.

Answer

Solution:$x^2+y^2=\dfrac{k}{40}, x^2+y^2=\dfrac{k}{100}$. The level curves represent circles of radii $\sqrt{10k}/20$ and $\sqrt{k}/10$

6.2: Limits and Continuity

This page is a draft and is under active development.

We have now examined functions of more than one variable and seen how to graph them. In this section, we see how to take the limit of a function of more than one variable, and what it means for a function of more than one variable to be continuous at a point in its domain. It turns out these concepts have aspects that just don't occur with functions of one variable.

6.2.1 Limit of a Function of Two Variables

Recall from Section 2.2 that the definition of a limit of a function of one variable:

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L \tag{6.2.1}$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x-a| < \delta$ for all x in the domain of f, then

$$|f(x) - L| > \varepsilon. \tag{6.2.2}$$

Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

Definition: δ Disks

Consider a point $(a,b) \in \mathbb{R}^2$. A δ **disk** centered at point (a,b) is defined to be an open disk of radius δ centered at point (a,b) —that is,

$$\{(x,y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 < \delta^2\} \tag{6.2.3}$$

as shown in Figure $6.2.1$.

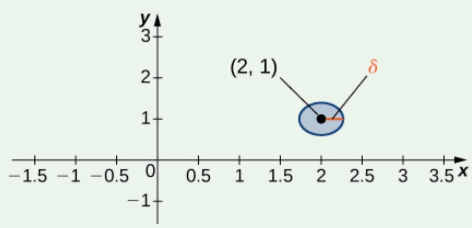


Figure $6.2.1$: A δ disk centered around the point $(2,1)$.

The idea of a δ disk appears in the definition of the limit of a function of two variables. If δ is small, then all the points (x,y) in the δ disk are close to (a,b). This is completely analogous to x being close to a in the definition of a limit of a function of one variable. In one dimension, we express this restriction as

$$a - \delta < x < a + \delta. \tag{6.2.4}$$

In more than one dimension, we use a δ disk.

Definition: limit of a function of two variables

Let f be a function of two variables, x and y. The limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y) \to (a,b)} f(x,y) = L \tag{6.2.5}$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x,y) in a δ disk around (a,b), except possibly for (a,b) itself, the value of $f(x,y)$ is no more than ε away from L (Figure $6.2.2$). Using symbols, we write the following: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \tag{6.2.6}$$

whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta. \tag{6.2.7}$$

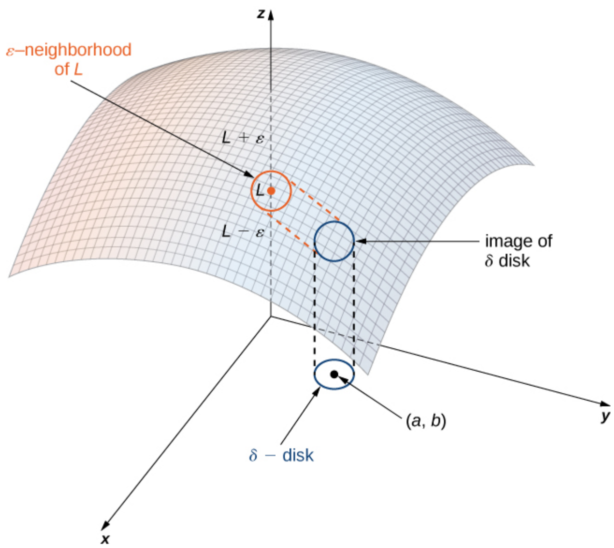


Figure 6.2.2: The limit of a function involving two variables requires that $f(x,y)$ be within ε of L whenever (x,y) is within δ of (a,b). The smaller the value of ε, the smaller the value of δ.

Proving that a limit exists using the definition of a **limit of a function of two variables** can be challenging. Instead, we use the following theorem, which gives us shortcuts to finding limits. The formulas in this theorem are an extension of the formulas in the limit laws theorem in The Limit Laws.

Limit laws for functions of two variables

Let $f(x,y)$ and $g(x,y)$ be defined for all $(x,y) \neq (a,b)$ in a neighborhood around (a,b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \tag{6.2.8}$$

and

$$\lim_{(x,y)\to(a,b)} g(x,y) = M, \tag{6.2.9}$$

and let c be a constant. Then each of the following statements holds:

Constant Law:

$$\lim_{(x,y)\to(a,b)} c = c \tag{6.2.10}$$

Identity Laws:

$$\lim_{(x,y)\to(a,b)} x = a \tag{6.2.11}$$

$$\lim_{(x,y)\to(a,b)} y = b \tag{6.2.12}$$

Sum Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y)+g(x,y)) = L+M \tag{6.2.13}$$

Difference Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y)-g(x,y)) = L-M \tag{6.2.14}$$

Constant Multiple Law:

$$\lim_{(x,y)\to(a,b)} (cf(x,y)) = cL \tag{6.2.15}$$

Product Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y)g(x,y)) = LM \tag{6.2.16}$$

Quotient Law:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \text{ for } M \neq 0 \tag{6.2.17}$$

Power Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n \tag{6.2.18}$$

for any positive integer n.

Root Law:

$$\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} \tag{6.2.19}$$

for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive.

The proofs of these properties are similar to those for the limits of functions of one variable. We can apply these laws to finding limits of various functions.

Example 6.2.1: Finding the Limit of a Function of Two Variables

Find each of the following limits:

a. $\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6)$
b. $\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y}$

Solution

a. First use the sum and difference laws to separate the terms:

$$\begin{aligned} &\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6) \\ &= \left(\lim_{(x,y)\to(2,-1)} x^2\right) - \left(\lim_{(x,y)\to(2,-1)} 2xy\right) + \left(\lim_{(x,y)\to(2,-1)} 3y^2\right) - \left(\lim_{(x,y)\to(2,-1)} 4x\right) \\ &+ \left(\lim_{(x,y)\to(2,-1)} 3y\right) - \left(\lim_{(x,y)\to(2,-1)} 6\right). \end{aligned}$$

Next, use the constant multiple law on the second, third, fourth, and fifth limits:

$$\begin{aligned} &= (\lim_{(x,y)\to(2,-1)} x^2) - 2(\lim_{(x,y)\to(2,-1)} xy) + 3(\lim_{(x,y)\to(2,-1)} y^2) - 4(\lim_{(x,y)\to(2,-1)} x) \\ &+ 3(\lim_{(x,y)\to(2,-1)} y) - \lim_{(x,y)\to(2,-1)} 6. \end{aligned}$$

Now, use the power law on the first and third limits, and the product law on the second limit:

$$\begin{aligned} &\left(\lim_{(x,y)\to(2,-1)} x\right)^2 - 2\left(\lim_{(x,y)\to(2,-1)} x\right)\left(\lim_{(x,y)\to(2,-1)} y\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right)^2 \\ &\qquad - 4\left(\lim_{(x,y)\to(2,-1)} x\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right) - \lim_{(x,y)\to(2,-1)} 6. \end{aligned}$$

Last, use the identity laws on the first six limits and the constant law on the last limit:

$$\begin{aligned} \lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6) &= (2)^2 - 2(2)(-1) + 3(-1)^2 - 4(2) + 3(-1) - 6 \\ &= -6. \end{aligned}$$

b. Before applying the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, constant multiple law, and identity law,

$$\begin{aligned} \lim_{(x,y)\to(2,-1)} (4x - 3y) &= \lim_{(x,y)\to(2,-1)} 4x - \lim_{(x,y)\to(2,-1)} 3y \\ &= 4(\lim_{(x,y)\to(2,-1)} x) - 3(\lim_{(x,y)\to(2,-1)} y) \\ &= 4(2) - 3(-1) = 11. \end{aligned}$$

Since the limit of the denominator is nonzero, the quotient law applies. We now calculate the limit of the numerator using the difference law, constant multiple law, and identity law:

$$\begin{aligned}\lim_{(x,y)\to(2,-1)}(2x+3y) &= \lim_{(x,y)\to(2,-1)} 2x+\lim_{(x,y)\to(2,-1)} 3y \\ &= 2(\lim_{(x,y)\to(2,-1)} x)+3(\lim_{(x,y)\to(2,-1)} y) \\ &= 2(2)+3(-1)=1.\end{aligned}$$

Therefore, according to the quotient law we have

$$\begin{aligned}\lim_{(x,y)\to(2,-1)}\frac{2x+3y}{4x-3y} &= \frac{\lim_{(x,y)\to(2,-1)}(2x+3y)}{\lim_{(x,y)\to(2,-1)}(4x-3y)} \\ &= \frac{1}{11}.\end{aligned}$$

Exercise 6.2.1:

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)}\sqrt[3]{\frac{x^2-y}{y^2+x-1}}.$$

Hint

Use the limit laws.

Answer

$$\lim_{(x,y)\to(5,-2)}\sqrt[3]{\frac{x^2-y}{y^2+x-1}}=\frac{3}{2}$$

Since we are taking the limit of a function of two variables, the point (a,b) is in $\mathbb{R}^2$, and it is possible to approach this point from an infinite number of directions. Sometimes when calculating a limit, the answer varies depending on the path taken toward (a,b). If this is the case, then the limit fails to exist. In other words, the limit must be unique, regardless of path taken.

Example 6.2.2: Limits That Fail to Exist

Show that neither of the following limits exist:

a. $\lim_{(x,y)\to(0,0)}\frac{2xy}{3x^2+y^2}$
b. $\lim_{(x,y)\to(0,0)}\frac{4xy^2}{x^2+3y^4}$

Solution

a. The domain of the function $f(x,y)=\frac{2xy}{3x^2+y^2}$ consists of all points in the xy-plane except for the point $(0,0)$ (Figure $6.2.1$). To show that the limit does not exist as (x,y) approaches $(0,0)$, we note that it is impossible to satisfy the definition of a limit of a function of two variables because of the fact that the function takes different values along different lines passing through point $(0,0)$. First, consider the line $y=0$ in the $xy-plane$. Substituting $y=0$ into $f(x,y)$ gives

$$f(x,0)=\frac{2x(0)}{3x^2+0^2}=0$$

for any value of x. Therefore the value of f remains constant for any point on the x-axis, and as y approaches zero, the function remains fixed at zero.

Next, consider the line $y=x$. Substituting $y=x$ into $f(x,y)$ gives

$$f(x,x)=\frac{2x(x)}{3x^2+x^2}=\frac{2x^2}{4x^2}=\frac{1}{2}.$$

This is true for any point on the line $y = x$. If we let x approach zero while staying on this line, the value of the function remains fixed at $\frac{1}{2}$, regardless of how small x is.

Choose a value for ε that is less than $1/2$—say, $1/4$. Then, no matter how small a δ disk we draw around $(0, 0)$, the values of $f(x, y)$ for points inside that δ disk will include both 0 and $\frac{1}{2}$. Therefore, the definition of limit at a point is never satisfied and the limit fails to exist.

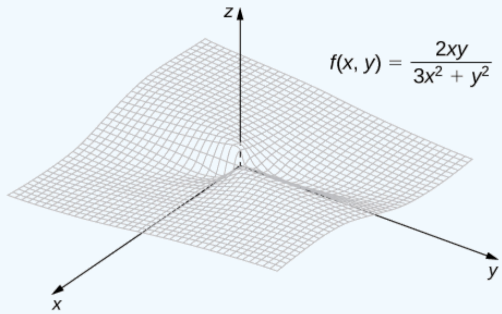


Figure 6.2.3: Graph of the function $f(x, y) = (2xy)/(3x^2 + y^2)$. Along the line $y = 0$, the function is equal to zero; along the line $y = x$, the function is equal to $\frac{1}{2}$.

Figure 6.2.3: Graph of the function $f(x, y) = \frac{2xy}{3x^2 + y^2}$. Along the line $y = 0$, the function is equal to zero; along the line $y = x$, the function is equal to $\frac{1}{2}$.

In a similar fashion to a., we can approach the origin along any straight line passing through the origin. If we try the x-axis (i.e., $y = 0$), then the function remains fixed at zero. The same is true for the y-axis. Suppose we approach the origin along a straight line of slope k. The equation of this line is $y = kx$. Then the limit becomes

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + 3y^4} &= \lim_{(x,y)\to(0,0)} \frac{4x(kx)^2}{x^2 + 3(kx)^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4k^2x^3}{x^2 + 3k^4x^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4k^2x}{1 + 3k^4x^2} \\ &= \frac{\lim_{(x,y)\to(0,0)} (4k^2x)}{\lim_{(x,y)\to(0,0)} (1 + 3k^4x^2)} \\ &= 0. \end{aligned}$$

regardless of the value of k. It would seem that the limit is equal to zero. What if we chose a curve passing through the origin instead? For example, we can consider the parabola given by the equation $x = y^2$. Substituting y^2 in place of x in $f(x, y)$ gives

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + 3y^4} &= \lim_{(x,y)\to(0,0)} \frac{4(y^2)y^2}{(y^2)^2 + 3y^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4y^4}{y^4 + 3y^4} \\ &= \lim_{(x,y)\to(0,0)} 1 \\ &= 1. \end{aligned}$$

By the same logic in part a, it is impossible to find a δ disk around the origin that satisfies the definition of the limit for any value of $\varepsilon < 1$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + 3y^4}$$

does **not** exist.

Exercise 6.2.2:

Show that

$$\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$$

does not exist.

Hint

Pick a line with slope k passing through point $(2,1)$.

Answer

If $y = k(x-2)+1$, then $\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2} = \frac{k}{1+k^2}$. Since the answer depends on k, the limit fails to exist.

6.2.2 Interior Points and Boundary Points

To study continuity and differentiability of a function of two or more variables, we first need to learn some new terminology.

Definition: interior and boundary points

Let S be a subset of $\mathbb{R}^2$ (Figure 6.2.4).

1. A point P_0 is called an *interior point* of S if there is a δ disk centered around P_0 contained completely in S.
2. A point P_0 is called a *boundary point* of S if every δ disk centered around P_0 contains points both inside and outside S.

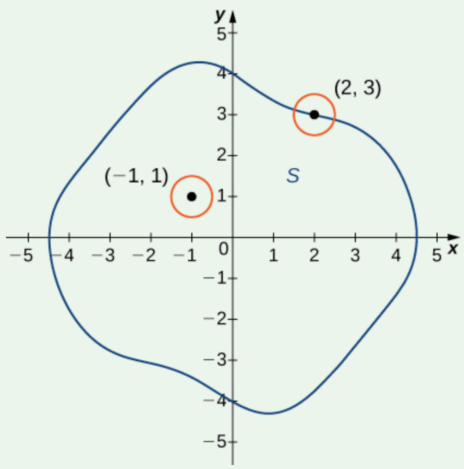


Figure 6.2.4: In the set S shown, $(-1,1)$ is an interior point and $(2,3)$ is a boundary point.

Definition: Open and closed sets

Let S be a subset of $\mathbb{R}^2$ (Figure 6.2.4).

1. S is called an *open set* if every point of S is an interior point.
2. S is called a *closed set* if it contains all its boundary points.

An example of an open set is a δ disk. If we include the boundary of the disk, then it becomes a closed set. A set that contains some, but not all, of its boundary points is neither open nor closed. For example if we include half the boundary of a δ disk but not the other half, then the set is neither open nor closed.

Definition: connected sets and Regions

Let S be a subset of $\mathbb{R}^2$ (Figure 6.2.4).

1. An open set S is a *connected set* if it cannot be represented as the union of two or more disjoint, nonempty open subsets.
2. A set S is a *region* if it is open, connected, and nonempty.

The definition of a limit of a function of two variables requires the δ disk to be contained inside the domain of the function. However, if we wish to find the limit of a function at a boundary point of the domain, the δ **disk** is not contained inside the domain. By definition, some of the points of the δ **disk** are inside the domain and some are outside. Therefore, we need only consider points that are inside both the δ disk and the domain of the function. This leads to the definition of the limit of a function at a boundary point.

Definition

Let f be a function of two variables, x and y, and suppose (a,b) is on the boundary of the domain of f. Then, the limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L, \tag{6.2.20}$$

if for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that for any point (x,y) inside the domain of f and within a suitably small distance positive δ of (a,b), the value of $f(x,y)$ is no more than ε away from L (Figure $6.2.2$). Using symbols, we can write: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta. \tag{6.2.21}$$

Example 6.2.3: Limit of a Function at a Boundary Point

Prove

$$\lim_{(x,y)\to(4,3)} \sqrt{25 - x^2 - y^2} = 0.$$

Solution

The domain of the function $f(x,y) = \sqrt{25 - x^2 - y^2}$ is $(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 25$, which is a circle of radius 5 centered at the origin, along with its interior as shown in Figure $6.2.5$.

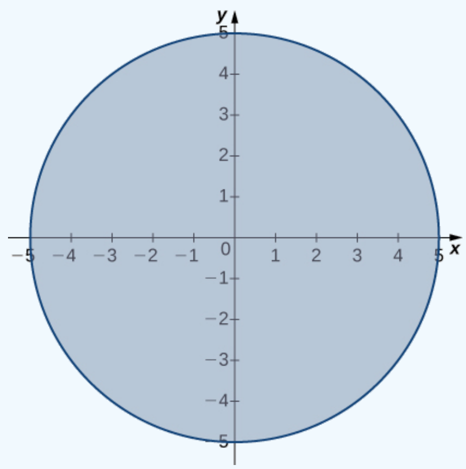


Figure $6.2.5$: Domain of the function $f(x,y) = \sqrt{25 - x^2 - y^2}$.

We can use the limit laws, which apply to limits at the boundary of domains as well as interior points:

$$\begin{aligned}
\lim_{(x,y)\to(4,3)} \sqrt{25 - x^2 - y^2} &= \sqrt{\lim_{(x,y)\to(4,3)} (25 - x^2 - y^2)} \\
&= \sqrt{\lim_{(x,y)\to(4,3)} 25 - \lim_{(x,y)\to(4,3)} x^2 - \lim_{(x,y)\to(4,3)} y^2} \\
&= \sqrt{25 - 4^2 - 3^2} \\
&= 0
\end{aligned}$$

See the following graph.

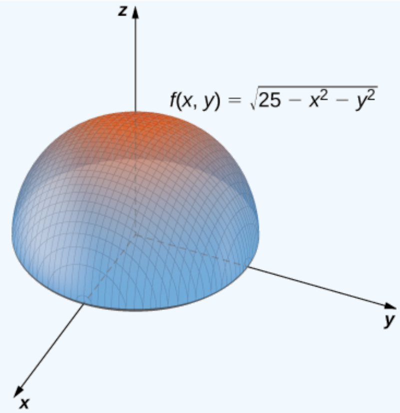


Figure 6.2.6: Graph of the function $f(x,y) = \sqrt{25 - x^2 - y^2}$.

Exercise 6.2.3

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)} \sqrt{29 - x^2 - y^2}.$$

Hint

Determine the domain of $f(x,y) = \sqrt{29 - x^2 - y^2}$.

Answer

$$\lim_{(x,y)\to(5,-2)} \sqrt{29 - x^2 - y^2}$$

6.2.3 Continuity of Functions of Two Variables

In Continuity, we defined the continuity of a function of one variable and saw how it relied on the limit of a function of one variable. In particular, three conditions are necessary for $f(x)$ to be continuous at point $x = a$

1. $f(a)$ exists.
2. $\lim\limits_{x\to a} f(x)$ exists.
3. $\lim\limits_{x\to a} f(x) = f(a)$.

These three conditions are necessary for continuity of a function of two variables as well.

Definition: continuous Functions

A function $f(x,y)$ is continuous at a point (a,b) in its domain if the following conditions are satisfied:

1. $f(a,b)$ exists.
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Example 6.2.4: Demonstrating Continuity for a Function of Two Variables

Show that the function

$$f(x,y) = \frac{3x+2y}{x+y+1}$$

is continuous at point $(5,-3)$.

Solution

There are three conditions to be satisfied, per the definition of continuity. In this example, $a = 5$ and $b = -3$.

1. $f(a,b)$ exists. This is true because the domain of the function f consists of those ordered pairs for which the denominator is nonzero (i.e., $x+y+1 \neq 0$). Point $(5,-3)$ satisfies this condition. Furthermore,

$$f(a,b)=f(5,-3)=\frac{3(5)+2(-3)}{5+(-3)+1}=\frac{15-6}{2+1}=3.$$

2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists. This is also true:

$$\begin{aligned}\lim_{(x,y)\to(a,b)} f(x,y) &= \lim_{(x,y)\to(5,-3)} \frac{3x+2y}{x+y+1}\\ &= \frac{\lim_{(x,y)\to(5,-3)}(3x+2y)}{\lim_{(x,y)\to(5,-3)}(x+y+1)}\\ &= \frac{15-6}{5-3+1}\\ &= 3.\end{aligned}$$

3. $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$. This is true because we have just shown that both sides of this equation equal three.

Exercise 6.2.4

Show that the function

$$f(x,y)=\sqrt{26-2x^2-y^2}$$

is continuous at point $(2,-3)$.

Hint

Use the three-part definition of continuity.

Answer

1. The domain of f contains the ordered pair $(2,-3)$ because $f(a,b)=f(2,-3)=\sqrt{16-2(2)^2-(-3)^2}=3$
2. $\lim_{(x,y)\to(a,b)} f(x,y)=3$
3. $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)=3$

Continuity of a function of any number of variables can also be defined in terms of delta and epsilon. A function of two variables is continuous at a point (x_0,y_0) in its domain if for every $\varepsilon>0$ there exists a $\delta>0$ such that, whenever $\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$ it is true, $|f(x,y)-f(a,b)|<\varepsilon$. This definition can be combined with the formal definition (that is, the *epsilon–delta definition*) of continuity of a function of one variable to prove the following theorems:

The Sum of Continuous Functions Is Continuous

If $f(x,y)$ is continuous at (x_0,y_0), and $g(x,y)$ is continuous at $x_0,y_0)$, then $f(x,y)+g(x,y)$ is continuous at (x_0,y_0).

The Product of Continuous Functions Is Continuous

If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y)=g(x)h(y)$ is continuous at (x_0,y_0).

The Composition of Continuous Functions Is Continuous

Let g be a function of two variables from a domain $D\subseteq\mathbb{R}^2$ to a range $R\subseteq R$. Suppose g is continuous at some point $(x_0,y_0)\in D$ and define $z_0=g(x_0,y_0)$. Let f be a function that maps R to R such that z_0 is in the domain of f. Last, assume f is continuous at z_0. Then $f\circ g$ is continuous at (x_0,y_0) as shown in Figure 6.2.7.

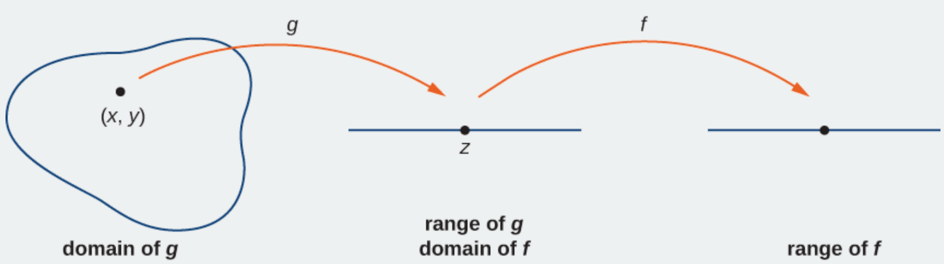


Figure 6.2.7: The composition of two continuous functions is continuous.

Let's now use the previous theorems to show continuity of functions in the following examples.

Example 6.2.5: More Examples of Continuity of a Function of Two Variables

Show that the functions $f(x,y)=4x^3y^2$ and $g(x,y)=\cos(4x^3y^2)$ are continuous everywhere.

Solution

The polynomials $g(x)=4x^3$ and $h(y)=y^2$ are continuous at every real number, and therefore by the product of continuous functions theorem, $f(x,y)=4x^3y^2$ is continuous at every point (x,y) in the xy-plane. Since $f(x,y)=4x^3y^2$ is continuous at every point (x,y) in the xy-plane and $g(x)=\cos x$ is continuous at every real number x, the continuity of the composition of functions tells us that $g(x,y)=\cos(4x^3y^2)$ is continuous at every point (x,y) in the $xy-plane$.

Exercise 6.2.5

Show that the functions $f(x,y)=2x^2y^3+3$ and $g(x,y)=(2x^2y^3+3)^4$ are continuous everywhere.

Hint

Use the continuity of the sum, product, and composition of two functions.

Answer

The polynomials $g(x)=2x^2$ and $h(y)=y^3$ are continuous at every real number; therefore, by the product of continuous functions theorem, $f(x,y)=2x^2y^3$ is continuous at every point (x,y) in the xy-plane. Furthermore, any constant function is continuous everywhere, so $g(x,y)=3$ is continuous at every point (x,y) in the xy-plane. Therefore, $f(x,y)=2x^2y^3+3$ is continuous at every point (x,y) in the xy-plane. Last, $h(x)=x^4$ is continuous at every real number x, so by the continuity of composite functions theorem $g(x,y)=(2x^2y^3+3)^4$ is continuous at every point (x,y) in the xy-plane.

6.2.4 Functions of Three or More Variables

The limit of a function of three or more variables occurs readily in applications. For example, suppose we have a function $f(x,y,z)$ that gives the temperature at a physical location (x,y,z) in three dimensions. Or perhaps a function $g(x,y,z,t)$ can indicate air pressure at a location (x,y,z) at time t. How can we take a limit at a point in $\mathbb{R}^3$? What does it mean to be continuous at a point in four dimensions?

The answers to these questions rely on extending the concept of a δ disk into more than two dimensions. Then, the ideas of the limit of a function of three or more variables and the continuity of a function of three or more variables are very similar to the definitions given earlier for a function of two variables.

Definition: δ-balls

Let (x_0,y_0,z_0) be a point in $\mathbb{R}^3$. Then, a δ-ball in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0,y_0,z_0) —that is,

$$(x,y,z)\in\mathbb{R}^3 \mid \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}<\delta. \tag{6.2.22}$$

To define a δ-ball in higher dimensions, add additional terms under the radical to correspond to each additional dimension. For example, given a point $P=(w_0,x_0,y_0,z_0)$ in $\mathbb{R}^4$, a δ ball around P can be described by

$$(w,x,y,z)\in\mathbb{R}^4 \mid \sqrt{(w-w_0)^2+(x-x_0)^2+(y-y+0)^2+(z-z_0)^2<\delta}. \tag{6.2.23}$$

To show that a limit of a function of three variables exists at a point (x_0,y_0,z_0), it suffices to show that for any point in a δ ball centered at (x_0,y_0,z_0), the value of the function at that point is arbitrarily close to a fixed value (the limit value). All the limit laws for functions of two variables hold for functions of more than two variables as well.

Example 6.2.6: Finding the Limit of a Function of Three Variables

Find

$$\lim_{(x,y,z)\to(4,1,-3)}\frac{x^2y-3z}{2x+5y-z}.$$

Solution

Before we can apply the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, the identity law, and the constant law,

$$\begin{aligned} \lim(x,y,z) \to (4,1,-3)(2x+5y-z) &= 2(\lim_{(x,y,z)\to(4,1,-3)} x)+5(\lim_{(x,y,z)\to(4,1,-3)} y)-(\lim_{(x,y,z)\to(4,1,-3)} z) \\ &= 2(4)+5(1)-(-3) \\ &= 16. \end{aligned}$$

Since this is nonzero, we next find the limit of the numerator. Using the product law, difference law, constant multiple law, and identity law,

$$\begin{aligned} \lim_{(x,y,z)\to(4,1,-3)}(x^2y-3z) &= (\lim_{(x,y,z)\to(4,1,-3)} x)^2(\lim_{(x,y,z)\to(4,1,-3)} y)-3\lim_{(x,y,z)\to(4,1,-3)} z \\ &= (4^2)(1)-3(-3) \\ &= 16+9 \\ &= 25 \end{aligned}$$

Last, applying the quotient law:

$$\lim_{(x,y,z)\to(4,1,-3)} \frac{x^2y-3z}{2x+5y-z} = \frac{\lim_{(x,y,z)\to(4,1,-3)}(x^2y-3z)}{\lim_{(x,y,z)\to(4,1,-3)}(2x+5y-z)} = \frac{25}{16} \tag{6.2.24}$$

Exercise 6.2.6

Find

$$\lim_{(x,y,z)\to(4,-1,3)} \sqrt{13-x^2-2y^2+z^2}$$

Hint

Use the limit laws and the continuity of the composition of functions.

Answer

$$\lim_{(x,y,z)\to(4,-1,3)} \sqrt{13-x^2-2y^2+z^2} = 2$$

6.2.5 Key Concepts

- To study limits and continuity for functions of two variables, we use a δ disk centered around a given point.
- A function of several variables has a limit if for any point in a δ ball centered at a point P, the value of the function at that point is arbitrarily close to a fixed value (the limit value).
- The limit laws established for a function of one variable have natural extensions to functions of more than one variable.
- A function of two variables is continuous at a point if the limit exists at that point, the function exists at that point, and the limit and function are equal at that point.

6.2.6 Glossary

boundary point

a point P_0 of R is a boundary point if every δ disk centered around P_0 contains points both inside and outside R

closed set

a set S that contains all its boundary points

connected set

an open set S that cannot be represented as the union of two or more disjoint, nonempty open subsets

δ disk

an open disk of radius δ centered at point (a,b)

δ ball

all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0,y_0,z_0)

interior point

a point P_0 of $\mathbb{R}$ is a boundary point if there is a δ disk centered around P_0 contained completely in $\mathbb{R}$

open set

a set S that contains none of its boundary points

6.2.7 Contributors and Attributions

region

an open, connected, nonempty subset of $\mathbb{R}^2$

6.2E:

6.2E.1 Exercise $6.2E.1$

For the following exercises, find the limit of the function.

1) $\lim\limits_{(x,y)\to(1,2)} x$

2) $\lim\limits_{(x,y)\to(1,2)} \dfrac{5x^2y}{x^2+y^2}$

Answer

Solution:2.0

6.2E.2 Exercise $6.2E.2$

1) Show that the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{5x^2y}{x^2+y^2}$ exists and is the same along the paths: $y-axis$ and $x-axis$, and along $y=x$.

6.2E.3 Exercise $6.2E.3$

For the following exercises, evaluate the limits at the indicated values of x and y. If the limit does not exist, state this and explain why the limit does not exist.

1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x^2+10y^2+4}{4x^2-10y^2+6}$

Answer

Solution:$\dfrac{2}{3}$

2) $\lim\limits_{(x,y)\to(11,13)} \sqrt{\dfrac{1}{xy}}$

3) $\lim\limits_{(x,y)\to(0,1)} \dfrac{y^2 sinx}{x}$

Answer

Solution:1

4) $\lim\limits_{(x,y)\to(0,0)} sin(\dfrac{x^8+y^7}{x-y+10})$

5) $\lim\limits_{(x,y)\to(\pi/4,1)} \dfrac{ytanx}{y+1}$

Answer

Solution:$\dfrac{1}{2}$

6) $\lim\limits_{(x,y)\to(0,\pi/4)} \dfrac{secx+2}{3x-tany}$

7) $\lim\limits_{(x,y)\to(2,5)} (\dfrac{1}{x}-\dfrac{5}{y})$

Answer

Solution:$-\frac{1}{2}$

8) $\lim_{(x,y)\to(4,4)} xlny$

9) $\lim_{(x,y)\to(4,4)} e^{-x^2-y^2}$

Answer

Solution:e^{-32}

10) $\lim_{(x,y)\to(0,0)} \sqrt{9-x^2-y^2}$

11) $\lim_{(x,y)\to(1,2)} (x^2y^3-x^3y^2+3x+2y)$

Answer

Solution:11.0

12) $\lim_{(x,y)\to(\pi,\pi)} xsin(\frac{x+y}{4})$

13) $\lim_{(x,y)\to(0,0)} \frac{xy+1}{x^2+y^2+1}$

Answer

Solution:1.0

14) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

15) $\lim_{(x,y)\to(0,0)} ln(x^2+y^2)$

Answer

Solution:The limit does not exist because when x and y both approach zero, the function approaches $ln0$, which is undefined (approaches negative infinity).

6.2E.4 Exercise $6.2E.4$

For the following exercises, complete the statement.

1) A point (x_0, y_0) in a plane region R is an interior point of R if _________________.

2) A point (x_0, y_0) in a plane region R is called a boundary point of R if ___________.

Answer

Solution:every open disk centered at (x_0, y_0) contains points inside R and outside R

6.2E.5 Exercise $6.2E.5$

For the following exercises, use algebraic techniques to evaluate the limit.

1) $\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1}$

2) $\lim_{(x,y)\to(0,0)} \frac{x^4-4y^4}{x^2+2y^2}$

Answer

Solution:0.0

3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x-y}$

4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Answer

Solution:0.00

6.2E.6 Exercise $\mathbf{6.2E.6}$

For the following exercises, evaluate the limits of the functions of three variables.

1) $lim_{(x,y,z)\to(1,2,3)}\dfrac{xz^2-y^2z}{xyz-1}$

2) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2-y^2-z^2}{x^2+y^2-z^2}$

Answer

Solution:The limit does not exist.

6.2E.7 Exercise $\mathbf{6.2E.7}$

For the following exercises, evaluate the limit of the function by determining the value the function approaches along the indicated paths. If the limit does not exist, explain why not.

1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+y^3}{x^2+y^2}$

a. Along the $x-axis\ (y=0)$

b. Along the $y-axis\ (x=0)$

c. Along the path $y=2x$

2) Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy+y^3}{x^2+y^2}$ using the results of previous problem.

Answer

Solution: The limit does not exist. The function approaches two different values along different paths.

3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

a. Along the $x-axis\ (y=0)$

b. Along the $y-axis\ (x=0)$

c. Along the path $y=x^2$

4) Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ using the results of previous problem.

Answer

Solution: The limit does not exist because the function approaches two different values along the paths.

6.2E.8 Exercise $6.2E.8$

Discuss the continuity of the following functions. Find the largest region in the $xy-plane$ in which the following functions are continuous.

1) $f(x,y)=sin(xy)$

2) $f(x,y)=ln(x+y)$

Answer

Solution:The function f is continuous in the region $y>-x$.

3) $f(x,y)=e^{3xy}$

4) $f(x,y)=\frac{1}{xy}$

Answer

Solution:The function f is continuous at all points in the $xy-plane$ except at $(0,0)$.

6.2E.9 Exercise $6.2E.9$

Discuss the continuity of the following functions. Find the largest region in the $xy-plane$ in which the following functions are continuous.

31) $f(x,y)=sin(xy)$

32) $f(x,y)=ln(x+y)$

Answer

Solution:The function f is continuous in the region $y>-x$.

33) $f(x,y)=e^{3xy}$

34) $f(x,y)=\frac{1}{xy}$

Answer

Solution:The function f is continuous at all points in the $xy-plane$ except at \(\displaystyle (0,0).\

6.2E.10 Exercise $6.2E.10$

For the following exercises, determine the region in which the function is continuous. Explain your answer.

2) $f(x,y)=\frac{x^2y}{x^2+y^2}$

2) $f(x,y)=\begin{cases} \frac{x^2y}{x^2+y^2} & if(x,y)\neq(0,0) \\ 0 & if(x,y)=(0,0) \end{cases}$

(Hint: Show that the function approaches different values along two different paths.)

Answer

Solution:The function is continuous at $(0,0)$ since the limit of the function at $(0,0)$ is 0, the same value of $f(0,0)$.

3) $f(x,y)=\frac{sin(x^2+y^2)}{x^2+y^2}$

6.2E.11 Exercise $6.2E.11$

1) Determine whether $g(x, y)=\frac{x^2-y^2}{x^2+y^2}$ is continuous at $(0, 0)$.

Answer

Solution:The function is discontinuous at $(0, 0)$. The limit at $(0, 0)$ fails to exist and $g(0, 0)$ does not exist.

6.2E.12 Exercise $6.2E.11$

39) Create a plot using graphing software to determine where the limit does not exist. Determine the region of the coordinate plane in which $f(x, y)=\frac{1}{x^2-y}$ is continuous.

40) Determine the region of the $xy-plane$ in which the composite function $g(x, y)=arctan(\frac{xy^2}{x+y})$ is continuous. Use technology to support your conclusion.

Answer

Solution:Since the function $arctanx$ is continuous over $(-\infty, \infty), g(x, y)=arctan(\frac{xy^2}{x+y})$ is continuous where $z=\frac{xy^2}{x+y}$ is continuous. The inner function z is continuous on all points of the $xy-plane$ except where $y=-x$. Thus, $g(x, y)=arctan(\frac{xy^2}{x+y})$ is continuous on all points of the coordinate plane except at points at which $y=-x$.

41) Determine the region of the $xy-plane$ in which $f(x, y)=ln(x^2+y^2-1)$ is continuous. Use technology to support your conclusion. (Hint: Choose the range of values for x and y carefully!)

42) At what points in space is $g(x, y, z)=x^2+y^2-2z^2$ continuous?

Answer

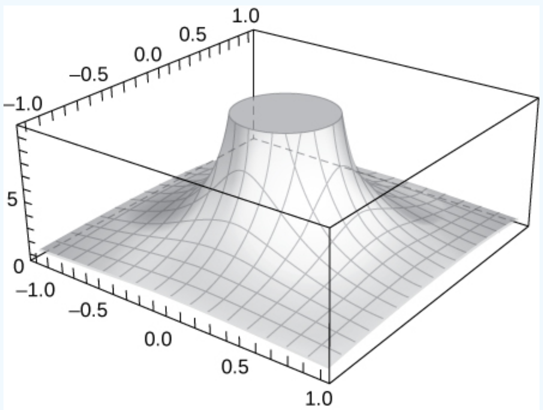
Solution:All points $P(x, y, z)$ in space.

43) At what points in space is $g(x, y, z)=\frac{1}{x^2+z^2-1}$ continuous?

44) Show that $\lim\limits_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$ does not exist at $(0, 0)$ by plotting the graph of the function.

Answer

Solution:The graph increases without bound as x and y both approach zero.



45) [T] Evaluate $\lim_{(x,y)\to(0,0)} \frac{-xy^2}{x^2+y^4}$ by plotting the function using a CAS. Determine analytically the limit along the path $x = y^2$.

6.2E.13 Exercise $6.2E.12$

1) [T]

a. Use a CAS to draw a contour map of $z = \sqrt{9-x^2-y^2}$.

b. What is the name of the geometric shape of the level curves?

c. Give the general equation of the level curves.

d. What is the maximum value of z?

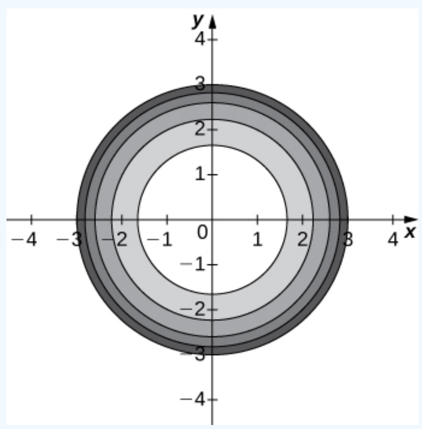
e. What is the domain of the function?

f. What is the range of the function?

Answer

Solution:

a.



b. The level curves are circles centered at $(0,0)$ with radius $9-c$. c. $x^2+y^2 = 9-c$ d. $z = 3$ e. $(x,y) \in R^2 \mid x^2+y^2 \leq 9$ f. $z|0 \leq z \leq 3$

6.2E.14 Exercise $6.2E.13$

1) True or False: If we evaluate $\lim_{(x,y)\to(0,0)} f(x)$ along several paths and each time the limit is 1, we can conclude that $\lim_{(x,y)\to(0,0)} f(x) = 1$.

6.2E.15 Exercise $6.2E.14$

1) Use polar coordinates to find $\lim_{(x,y)\to(0,0)} \frac{sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$. You can also find the limit using L'Hôpital's rule.

Answer

Solution:1.0

2) Use polar coordinates to find $\lim_{(x,y)\to(0,0)} cos(x^2+y^2)$.

3) Discuss the continuity of $f(g(x,y))$ where $f(t) = 1/t$ and $g(x,y) = 2x-5y$.

Answer

Solution: $f(g(x,y))$ is continuous at all points (x,y) that are not on the line $2x-5y=0$.

4) Given $f(x,y)=x^2-4y,$ find $\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$.

5) Given $f(x,y)=x^2-4y,$ find $\lim_{h\to 0}\frac{f(1+h,y)-f(1,y)}{h}$.

Answer

Solution: 2.0

6.3: Partial Derivatives

This page is a draft and is under active development.

Now that we have examined limits and continuity of functions of two variables, we can proceed to study derivatives. Finding derivatives of functions of two variables is the key concept in this chapter, with as many applications in mathematics, science, and engineering as differentiation of single-variable functions. However, we have already seen that limits and continuity of multivariable functions have new issues and require new terminology and ideas to deal with them. This carries over into differentiation as well.

6.3.1 Derivatives of a Function of Two Variables

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Definition: partial derivatives

Let $f(x, y)$ be a function of two variables. Then the *partial derivative* of f with respect to x, written as $\partial f/\partial x$,, or f_x, is defined as

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} \tag{6.3.1}$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined as

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k}. \tag{6.3.2}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f/\partial x$ is spoken as the "partial of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

Example 6.3.1: Calculating Partial Derivatives from the Definition

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12.$$

Solution

First, calculate $f(x+h, y)$.

$$\begin{aligned} f(x+h, y) &= (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h) + 5y - 12 \\ &= x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12. \end{aligned}$$

Next, substitute this into Equation 6.3.1 and simplify:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} \\ &= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12) - (x^2 - 3xy + 2y^2 - 4x + 5y - 12)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12 - x^2 + 3xy - 2y^2 + 4x - 5y + 12}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 - 3hy - 4h}{h} \\ &= \lim_{h\to 0} \frac{h(2x + h - 3y - 4)}{h} \\ &= \lim_{h\to 0} (2x + h - 3y - 4) \\ &= 2x - 3y - 4. \end{aligned}$$

To calculate $\frac{\partial f}{\partial y}$, first calculate $f(x, y+h)$:

$$\begin{aligned} f(x+h, y) &= x^2 - 3x(y+h) + 2(y+h)^2 - 4x + 5(y+h) - 12 \\ &= x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12. \end{aligned}$$

Next, substitute this into Equation 6.3.2 and simplify:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} \\ &= \lim_{h\to 0} \frac{(x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12) - (x^2 - 3xy + 2y^2 - 4x + 5y - 12)}{h} \\ &= \lim_{h\to 0} \frac{x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12 - x^2 + 3xy - 2y^2 + 4x - 5y + 12}{h} \\ &= \lim_{h\to 0} \frac{-3xh + 4yh + 2h^2 + 5h}{h} \\ &= \lim_{h\to 0} \frac{h(-3x + 4y + 2h + 5)}{h} \\ &= \lim_{h\to 0} (-3x + 4y + 2h + 5) \\ &= -3x + 4y + 5 \end{aligned}$$

Exercise 6.3.1

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y)=4x^2+2xy-y^2+3x-2y+5.$$

Hint

Use Equations 6.3.1and 6.3.2from the definition of partial derivatives.

Answer

$\frac{\partial f}{\partial x}=8x+2y+3$

$\frac{\partial f}{\partial y}=2x-2y-2$

The idea to keep in mind when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable. To see why this is true, first fix y and define $g(x)=f(x,y)$ as a function of x. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \\ &= \frac{\partial f}{\partial x}. \end{aligned} \tag{6.3.3}$$

The same is true for calculating the partial derivative of f with respect to y. This time, fix x and define $h(y)=f(x,y)$ as a function of y. Then

$$\begin{aligned} h'(x) &= \lim_{k\to 0}\frac{h(x+k)-h(x)}{k} \\ &= \lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k} \\ &= \frac{\partial f}{\partial y}. \end{aligned} \tag{6.3.4}$$

All differentiation rules apply.

Example 6.3.2: Calculating Partial Derivatives

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the following functions by holding the opposite variable constant then differentiating:

a. $f(x,y)=x^2-3xy+2y^2-4x+5y-12$
b. $g(x,y)=\sin(x^2y-2x+4)$

Solution:

a. To calculate $\partial f/\partial x$, treat the variable y as a constant. Then differentiate $f(x,y)$ with respect to x using the sum, difference, and power rules:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[x^2-3xy+2y^2-4x+5y-12\right] \\ &= \frac{\partial}{\partial x}[x^2]-\frac{\partial}{\partial x}[3xy]+\frac{\partial}{\partial x}[2y^2]-\frac{\partial}{\partial x}[4x]+\frac{\partial}{\partial x}[5y]-\frac{\partial}{\partial x}[12] \\ &= 2x-3y+0-4+0-0 \\ &= 2x-3y-4. \end{aligned}$$

The derivatives of the third, fifth, and sixth terms are all zero because they do not contain the variable x, so they are treated as constant terms. The derivative of the second term is equal to the coefficient of x, which is $-3y$. Calculating $\partial f/\partial y$:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[x^2-3xy+2y^2-4x+5y-12\right] \\ &= \frac{\partial}{\partial y}[x^2]-\frac{\partial}{\partial y}[3xy]+\frac{\partial}{\partial y}[2y^2]-\frac{\partial}{\partial y}[4x]+\frac{\partial}{\partial y}[5y]-\frac{\partial}{\partial y}[12] \\ &= -3x+4y-0+5-0 \\ &= -3x+4y+5. \end{aligned}$$

These are the same answers obtained in Example 6.3.1.

b. To calculate $\partial g/\partial x$, treat the variable y as a constant. Then differentiate $g(x,y)$ with respect to x using the chain rule and power rule:

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}\left[\sin(x^2y-2x+4)\right] \\ &= \cos(x^2y-2x+4)\frac{\partial}{\partial x}[x^2y-2x+4] \\ &= (2xy-2)\cos(x^2y-2x+4). \end{aligned}$$

To calculate $\partial g/\partial y$, treat the variable x as a constant. Then differentiate $g(x,y)$ with respect to y using the chain rule and power rule:

$$\begin{aligned} \frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}\left[\sin(x^2y-2x+4)\right] \\ &= \cos(x^2y-2x+4)\frac{\partial}{\partial y}[x^2y-2x+4] \\ &= x^2\cos(x^2y-2x+4). \end{aligned}$$

Exercise 6.3.2

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y)=\tan(x^3-3x^2y^2+2y^4)$$

by holding the opposite variable constant, then differentiating.

Hint

Use Equations 6.3.1and 6.3.1from the definition of partial derivatives.

Answer

$$\frac{\partial f}{\partial x}=(3x^2-6xy^2)\sec^2(x^3-3x^2y^2+2y^4)$$
$$\frac{\partial f}{\partial y}=(-6x^2y+8y^3)\sec^2(x^3-3x^2y^2+2y^4)$$

How can we interpret these partial derivatives? Recall that the graph of a function of two variables is a surface in R^3. If we remove the limit from the definition of the partial derivative with respect to x, the difference quotient remains:

$$\frac{f(x+h,y)-f(x,y)}{h}. \tag{6.3.5}$$

This resembles the difference quotient for the derivative of a function of one variable, except for the presence of the y variable. Figure 6.3.1 illustrates a surface described by an arbitrary function $z=f(x,y)$.

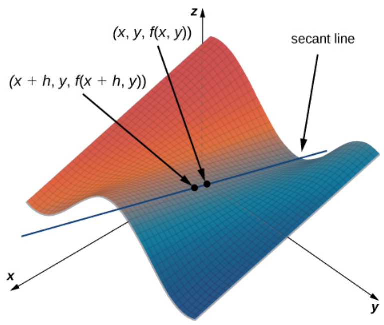


Figure 6.3.1: Secant line passing through the points $(x,y,f(x,y))$ and $(x+h,y,f(x+h,y))$.

In Figure 6.3.1, the value of h is positive. If we graph $f(x,y)$ and $f(x+h,y)$ for an arbitrary point (x,y), then the slope of the secant line passing through these two points is given by

$$\frac{f(x+h,y)-f(x,y)}{h}. \tag{6.3.6}$$

This line is parallel to the x-axis. Therefore, the slope of the secant line represents an average rate of change of the function f as we travel parallel to the x-axis. As h approaches zero, the slope of the secant line approaches the slope of the tangent line.

If we choose to change y instead of x by the same incremental value h, then the secant line is parallel to the y-axis and so is the tangent line. Therefore, $\partial f/\partial x$ represents the slope of the tangent line passing through the point $(x,y,f(x,y))$parallel to the x-axis and $\partial f/\partial y$ represents the slope of the tangent line passing through the point $(x,y,f(x,y))$parallel to the y-axis. If we wish to find the slope of a tangent line passing through the same point in any other direction, then we need what are called directional derivatives.

We now return to the idea of contour maps, which we introduced in Functions of Several Variables. We can use a **contour map** to estimate partial derivatives of a function $g(x,y)$.

Example 6.3.3: Partial Derivatives from a Contour Map

Use a contour map to estimate $\partial g/\partial x$ at the point $(\sqrt{5},0)$ for the function

$$g(x,y)=\sqrt{9-x^2-y^2}.$$

Solution

Figure 6.3.2 represents a contour map for the function $g(x,y)$.

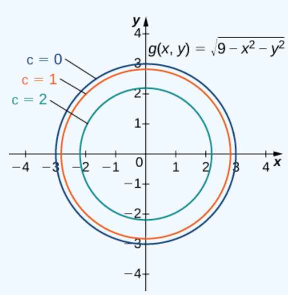


Figure 6.3.2: Contour map for the function $g(x,y)=\sqrt{9-x^2-y^2}$, using $c=0,1,2,$ and 3($c=3$ corresponds to the origin).

The inner circle on the contour map corresponds to $c=2$ and the next circle out corresponds to $c=1$. The first circle is given by the equation $2=\sqrt{9-x^2-y^2}$; the second circle is given by the equation $1=\sqrt{9-x^2-y^2}$. The first equation simplifies to $x^2+y^2=5$ and the second equation simplifies to $x^2+y^2=8.$ The x-intercept of the first circle is $(\sqrt{5},0)$ and the x-intercept of the second circle is $(2\sqrt{2},0)$. We can estimate the value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5},0)$ using the slope formula:

$$\left.\frac{\partial g}{\partial x}\right|_{(x,y)=(\sqrt{5},0)}\approx\frac{g(\sqrt{5},0)-g(2\sqrt{2},0)}{\sqrt{5}-2\sqrt{2}}=\frac{2-1}{\sqrt{5}-2\sqrt{2}}=\frac{1}{\sqrt{5}-2\sqrt{2}}\approx-1.688.$$

To calculate the exact value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5},0)$, we start by finding $\partial g/\partial x$ using the chain rule. First, we rewrite the function as

$$g(x,y)=\sqrt{9-x^2-y^2}=(9-x^2-y^2)^{1/2} \tag{6.3.7}$$

and then differentiate with respect to x while holding y constant:

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{1}{2}(9-x^2-y^2)^{-1/2}(-2x)\\ &= -\frac{x}{\sqrt{9-x^2-y^2}}.\end{aligned}$$

Next, we evaluate this expression using $x=\sqrt{5}$ and $y=0$:

$$\frac{\partial g}{\partial x}\left.\right|_{(x,y)=(\sqrt{5},0)} = -\frac{\sqrt{5}}{\sqrt{9-(\sqrt{5})^2-(0)^2}} = -\frac{\sqrt{5}}{\sqrt{4}} = -\frac{\sqrt{5}}{2} \approx -1.118. \tag{6.3.8}$$

The estimate for the partial derivative corresponds to the slope of the secant line passing through the points $(\sqrt{5},0,g(\sqrt{5},0))$ and $(2\sqrt{2},0,g(2\sqrt{2},0))$. It represents an approximation to the slope of the tangent line to the surface through the point $(\sqrt{5},0,g(\sqrt{5},0))$, which is parallel to the x-axis.

Exercise 6.3.3

Use a contour map to estimate $\partial f/\partial y$ at point $(0,\sqrt{2})$ for the function

$$f(x,y)=x^2-y^2.$$

Compare this with the exact answer.

Hint

Create a contour map for f using values of c from -3 to 3. Which of these curves passes through point $(0,\sqrt{2})$?

Answer

Using the curves corresponding to $c=-2$ and $c=-3$, we obtain

`\left. \dfrac{∂f}{∂y}\right|_{(x,y)=(0,\sqrt{2})}≈\dfrac{f(0,\sqrt{3})−f(0,\sqrt{2})}{\sqrt{3}−\sqrt{2}}=\dfrac{−3−(−2)}{\sqrt{3}−\sqrt{2}}⋅\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=−\sqrt{3}−\sqrt{2}≈−3.14`

The exact answer is

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,\sqrt{2})} = (-2y|_{(x,y)=(0,\sqrt{2})} = -2\sqrt{2} \approx -2.828.$$

6.3.2 Functions of More Than Two Variables

Suppose we have a function of three variables, such as $w=f(x,y,z)$. We can calculate partial derivatives of w with respect to any of the independent variables, simply as extensions of the definitions for partial derivatives of functions of two variables.

Definition: Partial Derivatives

Let $f(x,y,z)$ be a function of three variables. Then, the partial derivative of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined to be

$$\frac{\partial f}{\partial x}=f_x(x,y,z)=\lim_{h\to 0}\frac{f(x+h,y,z)-f(x,y,z)}{h}. \tag{6.3.9}$$

The *partial derivative* of f with *respect* to y, written as $\partial f/\partial y$, or f_y, is defined to be

$$\frac{\partial f}{\partial y}=f_y(x,y,z)=\lim_{k\to 0}\frac{f(x,y+k,z)-f(x,y,z)}{k}. \tag{6.3.10}$$

The *partial derivative* of f with respect to z, written as $\partial f/\partial z$, or f_z, is defined to be

$$\frac{\partial f}{\partial z}=f_z(x,y,z)=\lim_{m\to 0}\frac{f(x,y,z+m)-f(x,y,z)}{m}. \tag{6.3.11}$$

We can calculate a partial derivative of a function of three variables using the same idea we used for a function of two variables. For example, if we have a function f of x,y, and z, and we wish to calculate $\partial f/\partial x$, then we treat the other two independent variables as if they are constants, then differentiate with respect to x.

Example 6.3.4: Calculating Partial Derivatives for a Function of Three Variables

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x,y,z)=x^2-3xy+2y^2-4xz+5yz^2-12x+4y-3z.$$

Then, find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Solution:

We first calculate $\partial f/\partial x$ using Equation 6.3.9, then we calculate the other two partial derivatives by holding the remaining variables constant. To use the equation to find $\partial f/\partial x$, we first need to calculate $f(x+h,y,z)$:

$$\begin{aligned}f(x+h,y,z) &= (x+h)^2-3(x+h)y+2y^2-4(x+h)z+5yz^2-12(x+h)+4y-3z\\ &= x^2+2xh+h^2-3xy-3xh+2y^2-4xz-4hz+5yz^2-12x-12h+4y-3z\end{aligned}$$

and recall that $f(x,y,z)=x^2-3xy+2y^2-4zx+5yz^2-12x+4y-3z.$ Next, we substitute these two expressions into the equation:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\left[\frac{x^2+2xh+h^2-3xy-3hy+2y^2-4xz-4hz+5yz^2-12x-12h+4y-3zh-x^2-3xy+2y^2-4xz+5yz^2-12x+4y-3z}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{2xh+h^2-3hy-4hz-12h}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{h(2x+h-3y-4z-12)}{h}\right]$$
$$= \lim_{h\to 0}(2x+h-3y-4z-12)$$
$$= 2x-3y-4z-12.$$

Then we find $\partial f/\partial y$ by holding x and z constant. Therefore, any term that does not include the variable y is constant, and its derivative is zero. We can apply the sum, difference, and power rules for functions of one variable:

$$\begin{aligned}&\frac{\partial}{\partial y}\left[x^2-3xy+2y^2-4xz+5yz^2-12x+4y-3z\right]\\ &=\frac{\partial}{\partial y}[x^2]-\frac{\partial}{\partial y}[3xy]+\frac{\partial}{\partial y}[2y^2]-\frac{\partial}{\partial y}[4xz]+\frac{\partial}{\partial y}[5yz^2]-\frac{\partial}{\partial y}[12x]+\frac{\partial}{\partial y}[4y]-\frac{\partial}{\partial z}[3z]\\ &=0-3x+4y-0+5z^2-0+4-0\\ &=-3x+4y+5z^2+4.\end{aligned}$$

To calculate $\partial f/\partial z$, we hold x and y constant and apply the sum, difference, and power rules for functions of one variable:

$$\begin{aligned}&\frac{\partial}{\partial z}[x^2-3xy+2y^2-4xz+5yz^2-12x+4y-3z]\\ &=\frac{\partial}{\partial z}[x^2]-\frac{\partial}{\partial z}[3xy]+\frac{\partial}{\partial z}[2y^2]-\frac{\partial}{\partial z}[4xz]+\frac{\partial}{\partial z}[5yz^2]-\frac{\partial}{\partial z}[12x]+\frac{\partial}{\partial z}[4y]-\frac{\partial}{\partial z}[3z]\\ &=0-0+0-4x+10yz-0+0-3\\ &=-4x+10yz-3\end{aligned}$$

Exercise 6.3.4

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x,y,z)=2x^2-4x^2y+2y^2+5xz^2-6x+3z-8.$$

Then find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Hint

Use the strategy in the preceding example.

Answer

$\frac{\partial f}{\partial x}=4x-8xy+5z^2-6$, $\frac{\partial f}{\partial y}=-4x^2+4y$, $\frac{\partial f}{\partial z}=10xz+3$

Example 6.3.5: Calculating Partial Derivatives for a Function of Three Variables

Calculate the three partial derivatives of the following functions.

a. $f(x,y,z)=x^2y-4xz+y^2x-3yz$
b. $g(x,y,z)=\sin(x^2y-z)+\cos(x^2-yz)$

Solution

In each case, treat all variables as constants except the one whose partial derivative you are calculating.

a.

$$\begin{aligned}\frac{\partial f}{\partial x}&=\frac{\partial}{\partial x}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right]\\ &=\frac{\frac{\partial}{\partial x}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial x}(x-3yz)}{(x-3yz)^2}\\ &=\frac{(2xy-4z)(x-3yz)-(x^2y-4xz+y^2)(1)}{(x-3yz)^2}\\ &=\frac{2x^2y-6xy^2z-4xz+12yz^2-x^2y+4xz-y^2}{(x-3yz)^2}\\ &=\frac{x^2y-6xy^2z-4xz+12yz^2+4xz-y^2}{(x-3yz)^2}\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right] \\
&= \frac{\frac{\partial}{\partial y}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial y}(x-3yz)}{(x-3yz)^2} \\
&= \frac{(x^2+2y)(x-3yz)-(x^2y-4xz+y^2)(-3z)}{(x-3yz)^2} \\
&= \frac{x^3-3x^2yz+2xy-6y^2z+3x^2yz-12xz^2+3y^2z}{(x-3yz)^2} \\
&= \frac{x^3+2xy-3y^2z-12xz^2}{(x-3yz)^2} \\
\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right] \\
&= \frac{\frac{\partial}{\partial z}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial z}(x-3yz)}{(x-3yz)^2} \\
&= \frac{(-4x)(x-3yz)-(x^2y-4xz+y^2)(-3y)}{(x-3yz)^2} \\
&= \frac{-4x^2+12xyz+3x^2y^2-12xyz+3y^3}{(x-3yz)^2} \\
&= \frac{-4x^2+3x^2y^2+3y^3}{(x-3yz)^2}
\end{aligned}$$

b.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[\sin(x^2y-z)+\cos(x^2-yz)\right] \\
&= (\cos(x^2y-z))\frac{\partial}{\partial x}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial x}(x^2-yz) \\
&= 2xy\cos(x^2y-z)-2x\sin(x^2-yz) \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}[\sin(x^2y-z)+\cos(x^2-yz)] \\
&= (\cos(x^2y-z))\frac{\partial}{\partial y}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial y}(x^2-yz) \\
&= x^2\cos(x^2y-z)+z\sin(x^2-yz) \\
\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}[\sin(x^2y-z)+\cos(x^2-yz)] \\
&= (\cos(x^2y-z))\frac{\partial}{\partial z}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial z}(x^2-yz) \\
&= -\cos(x^2y-z)+y\sin(x^2-yz)
\end{aligned}$$

Exercise 6.3.5

Calculate $\partial f/\partial x, \partial f/\partial y,$ and $\partial f/\partial z$ for the function

$$f(x,y,z)=\sec(x^2y)-\tan(x^3yz^2).$$

Hint

Use the strategy in the preceding example.

Answer

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2xy\sec(x^2y)\tan(x^2y)-3x^2yz^2\sec^2(x^3yz^2) \\
\frac{\partial f}{\partial y} &= x^2\sec(x^2y)\tan(x^2y)-x^3z^2\sec^2(x^3yz^2) \\
\frac{\partial f}{\partial z} &= -2x^3yz\sec^2(x^3yz^2)
\end{aligned}$$

6.3.3 Higher-Order Partial Derivatives

Consider the function

$$f(x,y)=2x^3-4xy^2+5y^3-6xy+5x-4y+12. \tag{6.3.12}$$

Its partial derivatives are

$$\frac{\partial f}{\partial x}=6x^2-4y^2-6y+5 \tag{6.3.13}$$

and

$$\frac{\partial f}{\partial y}=-8xy+15y^2-6x-4. \tag{6.3.14}$$

Each of these partial derivatives is a function of two variables, so we can calculate partial derivatives of these functions. Just as with derivatives of single-variable functions, we can call these *second-order derivatives*, third-order derivatives, and so on. In general, they are referred to as **higher-order partial derivatives.** There are four second-order partial derivatives for any function (provided they all exist):

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]\\ \frac{\partial^2 f}{\partial x\,\partial y} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]\\ \frac{\partial^2 f}{\partial y\,\partial x} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]\\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right].\end{aligned}$$

An alternative notation for each is f_{xx}, f_{xy}, f_{yx}, and f_{yy}, respectively. Higher-order partial derivatives calculated with respect to different variables, such as f_{xy} and f_{yx}, are commonly called **mixed partial derivatives.**

Example 6.3.6: Calculating Second Partial Derivatives

Calculate all four second partial derivatives for the function

$$f(x,y) = xe^{-3y} + \sin(2x - 5y). \tag{6.3.15}$$

Solution:

To calculate $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$, we first calculate $\partial f/\partial x$:

$$\frac{\partial f}{\partial x} = e^{-3y} + 2\cos(2x - 5y). \tag{6.3.16}$$

To calculate $\frac{\partial^2 f}{\partial x^2}$, differentiate $\partial f/\partial x$ (Equation 6.3.16) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]\\ &= \frac{\partial}{\partial x}[e^{-3y} + 2\cos(2x-5y)]\\ &= -4\sin(2x-5y).\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial x \partial y}$, differentiate $\partial f/\partial x$ (Equation 6.3.16) with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial x\,\partial y} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]\\ &= \frac{\partial}{\partial y}[e^{-3y} + 2\cos(2x-5y)]\\ &= -3e^{-3y} + 10\sin(2x-5y).\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$, first calculate $\partial f/\partial y$:

$$\frac{\partial f}{\partial y} = -3xe^{-3y} - 5\cos(2x - 5y). \tag{6.3.17}$$

To calculate $\frac{\partial^2 f}{\partial y \partial x}$, differentiate $\partial f/\partial y$ (Equation 6.3.17) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial y\,\partial x} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]\\ &= \frac{\partial}{\partial x}[-3xe^{-3y} - 5\cos(2x-5y)]\\ &= -3e^{-3y} + 10\sin(2x-5y).\end{aligned}$$

To calculate $\frac{\partial^2 f}{\partial y^2}$, differentiate $\partial f/\partial y$ (Equation 6.3.17) with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]\\ &= \frac{\partial}{\partial y}[-3xe^{-3y} - 5\cos(2x-5y)]\\ &= 9xe^{-3y} - 25\sin(2x-5y).\end{aligned}$$

Exercise 6.3.6

Calculate all four second partial derivatives for the function

$$f(x,y) = \sin(3x - 2y) + \cos(x + 4y).$$

Hint

Follow the same steps as in the previous example.

Answer

$$\frac{\partial^2 f}{\partial x^2} = -9\sin(3x-2y) - \cos(x+4y)$$
$$\frac{\partial^2 f}{\partial x\,\partial y} = 6\sin(3x-2y) - 4\cos(x+4y)$$
$$\frac{\partial^2 f}{\partial y\,\partial x} = 6\sin(3x-2y) - 4\cos(x+4y)$$
$$\frac{\partial^2 f}{\partial y^2} = -4\sin(3x-2y) - 16\cos(x+4y)$$

Example 6.3.7

Calculate all four-second partial derivatives for the function

$f(x,y) = (3+2x^2+5y^2)^{1/2}$.

Solution

To calculate $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$, we first calculate $\partial f/\partial x$:

$$\frac{\partial f}{\partial x} = \frac{1}{2}(3+2x^2+5y^2)^{-1/2}(4x) = \frac{2x}{(3+2x^2+5y^2)^{1/2}} \tag{6.3.18}$$

To calculate $\frac{\partial^2 f}{\partial x^2}$, differentiate $\partial f/\partial x$ (Equation 6.3.18) with respect to x:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \\
&= \frac{\partial}{\partial x}\left(\frac{2x}{(3+2x^2+5y^2)^{1/2}}\right) \\
&= \frac{(3+2x^2+5y^2)^{1/2}(2) - (2x)(1/2)(4+2x^2+5y^2)^{-1/2}(4x)}{((3+2x^2+5y^2)^{1/2})^2} \\
&= \frac{2(3+2x^2+5y^2)^{1/2} - \frac{4x^2}{(3+2x^2+5y^2)^{1/2}}}{(3+2x^2+5y^2)} \\
&= \frac{\left(\frac{2(3+2x^2+5y^2)-4x^2}{(3+2x^2+5y^2)^{1/2}}\right)}{(3+2x^2+5y^2)} \\
&= \frac{6+4x^2+10y^2-4x^2}{(3+2x^2+5y^2)^{3/2}} \\
&= \frac{6+10y^2}{(3+2x^2+5y^2)^{3/2}}
\end{aligned}$$

At this point, we should notice that, in both Example and the checkpoint, it was true that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. Under certain conditions, this is always true. In fact, it is a direct consequence of the following theorem.

Equality of Mixed Partial Derivatives (Clairaut's Theorem)

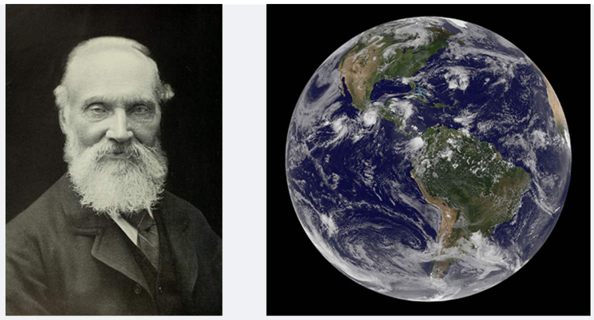
Suppose that $f(x,y)$ is defined on an open disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy} = f_{yx}$.

Clairaut's theorem guarantees that as long as mixed second-order derivatives are continuous, the order in which we choose to differentiate the functions (i.e., which variable goes first, then second, and so on) does not matter. It can be extended to higher-order derivatives as well. The proof of Clairaut's theorem can be found in most advanced calculus books.

Two other second-order partial derivatives can be calculated for any function $f(x,y)$. The partial derivative f_{xx} is equal to the partial derivative of f_x with respect to x, and f_{yy} is equal to the partial derivative of f_y with respect to y.

Lord Kelvin and the Age of Earth

During the late 1800s, the scientists of the new field of geology were coming to the conclusion that Earth must be "millions and millions" of years old. At about the same time, Charles Darwin had published his treatise on evolution. Darwin's view was that evolution needed many millions of years to take place, and he made a bold claim that the Weald chalk fields, where important fossils were found, were the result of 300 million years of erosion.



(a) (b)

Figure 6.3.5: (a) William Thomson (Lord Kelvin), 1824-1907, was a British physicist and electrical engineer; (b) Kelvin used the heat diffusion equation to estimate the age of Earth (credit: modification of work by NASA).

At that time, eminent physicist William Thomson (Lord Kelvin) used an important partial differential equation, known as the heat diffusion equation, to estimate the age of Earth by determining how long it would take Earth to cool from molten rock to what we had at that time. His conclusion was a range of **20** to **400** million years, but most likely about **50** million years. For many decades, the proclamations of this irrefutable icon of science did not sit well with geologists or with Darwin.

- Read Kelvin's paper on estimating the age of the Earth.

Kelvin made reasonable assumptions based on what was known in his time, but he also made several assumptions that turned out to be wrong. One incorrect assumption was that Earth is solid and that the cooling was therefore via conduction only, hence justifying the use of the diffusion equation. But the most serious error was a forgivable one—omission of the fact that Earth contains radioactive elements that continually supply heat beneath Earth's mantle. The discovery of radioactivity came near the end of Kelvin's life and he acknowledged that his calculation would have to be modified.

Kelvin used the simple one-dimensional model applied only to Earth's outer shell, and derived the age from graphs and the roughly known temperature gradient near Earth's surface. Let's take a look at a more appropriate version of the diffusion equation in radial coordinates, which has the form

$$\frac{\partial T}{\partial t} = K\left[\frac{\partial^2 T}{\partial^2 r} + \frac{2}{r}\frac{\partial T}{\partial r}\right] \tag{6.3.19}$$

.

Here, $T(r,t)$ is temperature as a function of r (measured from the center of Earth) and time t. K is the heat conductivity—for molten rock, in this case. The standard method of solving such a partial differential equation is by separation of variables, where we express the solution as the product of functions containing each variable separately. In this case, we would write the temperature as

$$T(r,t) = R(r)f(t). \tag{6.3.20}$$

1. Substitute this form into Equation 6.3.19 and, noting that $f(t)$ is constant with respect to distance (r) and $R(r)$ is constant with respect to time (t), show that

$$\frac{1}{f}\frac{\partial f}{\partial t} = \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right]. \tag{6.3.21}$$

2. This equation represents the separation of variables we want. The left-hand side is only a function of t and the right-hand side is only a function of r, and they must be equal for all values of r and t. Therefore, they both must be equal to a constant. Let's call that constant $-\lambda^2$. (The convenience of this choice is seen on substitution.) So, we have

$$\frac{1}{f}\frac{\partial f}{\partial t} = -\lambda^2 \text{and} \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right] = -\lambda^2. \tag{6.3.22}$$

3. Now, we can verify through direct substitution for each equation that the solutions are $f(t) = Ae^{-\lambda^2 t}$ and $R(r) = B\left(\frac{\sin \alpha r}{r}\right) + C\left(\frac{\cos \alpha r}{r}\right)$, where $\alpha = \lambda/\sqrt{K}$. Note that $f(t) = Ae^{+\lambda n^2 t}$ is also a valid solution, so we could have chosen $+\lambda^2$ for our constant. Can you see why it would not be valid for this case as time increases?
4. Let's now apply boundary conditions.
   a. The temperature must be finite at the center of Earth, $r = 0$. Which of the two constants, B or C, must therefore be zero to keep R finite at $r = 0$? (Recall that $\sin(\alpha r)/r \to \alpha =$ as $r \to 0$, but $\cos(\alpha r)/r$ behaves very differently.)
   b. Kelvin argued that when magma reaches Earth's surface, it cools very rapidly. A person can often touch the surface within weeks of the flow. Therefore, the surface reached a moderate temperature very early and remained nearly constant at a surface temperature T_s. For simplicity, let's set $T = 0$ at $r = R_E$ and find α such that this is the temperature there for all time t. (Kelvin took the value to be $300K \approx 80^\circ F$. We can add this $300K$ constant to our solution later.) For this to be true, the sine argument must be zero at $r = R_E$. Note that α has an infinite series of values that satisfies this condition. Each value of α represents a valid solution (each with its own value for A). The total or general solution is the sum of all these solutions.
   c. At $t = 0$, we assume that all of Earth was at an initial hot temperature T_0 (Kelvin took this to be about $7000K$.) The application of this boundary condition involves the more advanced application of Fourier coefficients. As noted in part b. each value of α_n represents a valid solution, and the general solution is a sum of all these solutions. This results in a series solution:

$$T(r,t) = \left(\frac{T_0 R_E}{\pi}\right)\sum_n \frac{(-1)^{n-1}}{n} e^{-\lambda n^2 t}\frac{\sin(\alpha_n r)}{r}, \text{where } \alpha_n = n\pi/R_E \tag{6.3.23}$$

.

Note how the values of α_n come from the boundary condition applied in part b. The term $\frac{-1^{n-1}}{n}$ is the constant A_n for each term in the series, determined from applying the Fourier method. Letting $\beta = \frac{\pi}{R_E}$, examine the first few terms of this solution shown here and note how λ^2 in the exponential causes the higher terms to decrease quickly as time progresses:

$$T(r,t) \tag{6.3.24}$$

$$= \frac{T_0 R_E}{\pi r}\left(e^{-K\beta^2 t}(\sin\beta r) - \frac{1}{2}e^{-4K\beta^2 t}(\sin 2\beta r) + \frac{1}{3}e^{-9K\beta^2 t}(\sin 3\beta r) - \frac{1}{4}e^{-16K\beta^2 t}(\sin 4\beta r) + \frac{1}{5}e^{-25K\beta^2 t}(\sin 5\beta r)...\right).$$

Near time $t = 0$, many terms of the solution are needed for accuracy. Inserting values for the conductivity K and $\beta = \pi/R_E$ for time approaching merely thousands of years, only the first few terms make a significant contribution. Kelvin only needed to look at the solution near Earth's surface (Figure 6.3.6) and, after a long time, determine what time best yielded the estimated temperature gradient known during his era ($1^\circ F$ increase per $50ft$). He simply chose a range of times with a gradient close to this value. In Figure 6.3.6, the solutions are plotted and scaled, with the $300 - K$ surface temperature added. Note that the center of Earth would be relatively cool. At the time, it was thought Earth must be solid.

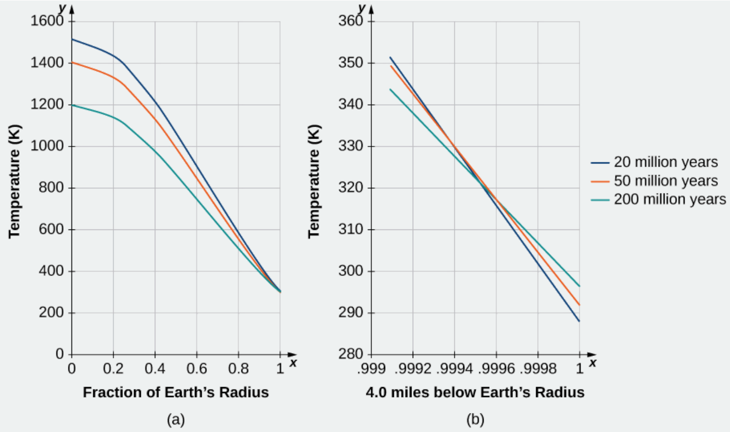


Figure 6.3.6: Temperature versus radial distance from the center of Earth. (a) Kelvin's results, plotted to scale. (b) A close-up of the results at a depth of 4.0 miles below Earth's surface.

Epilog

On May **20, 1904,** physicist **Ernest Rutherford** spoke at the Royal Institution to announce a revised calculation that included the contribution of radioactivity as a source of Earth's heat. In Rutherford's own words:

> "I came into the room, which was half-dark, and presently spotted Lord Kelvin in the audience, and realized that I was in for trouble at the last part of my speech dealing with the age of the Earth, where my views conflicted with his. To my relief, Kelvin fell fast asleep, but as I came to the important point, I saw the old bird sit up, open an eye and cock a baleful glance at me.
>
> Then a sudden inspiration came, and I said Lord Kelvin had limited the age of the Earth, provided no new source [of heat] was discovered. That prophetic utterance referred to what we are now considering tonight, radium! Behold! The old boy beamed upon me."

Rutherford calculated an age for Earth of about **500** million years. Today's accepted value of Earth's age is about **4.6** billion years.

6.3.4 Key Concepts

- A partial derivative is a derivative involving a function of more than one independent variable.
- To calculate a partial derivative with respect to a given variable, treat all the other variables as constants and use the usual differentiation rules.
- Higher-order partial derivatives can be calculated in the same way as higher-order derivatives.

6.3.5 Key Equations

- **Partial derivative of f with respect to x**

$$\frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

- **Partial derivative of f with respect to y**

$$\frac{\partial f}{\partial y}=\lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}$$

6.3.6 Glossary

higher-order partial derivatives
second-order or higher partial derivatives, regardless of whether they are mixed partial derivatives

mixed partial derivatives
second-order or higher partial derivatives, in which at least two of the differentiation are with respect to different variables

partial derivative
a derivative of a function of more than one independent variable in which all the variables but one are held constant

6.3.7 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

6.3E:

6.3E.1 Exercise $6.3E.1$

For the following exercises, calculate the partial derivative using the limit definitions only.

1) $\dfrac{\partial z}{\partial x}$ for $z = x^2 - 3xy + y^2$

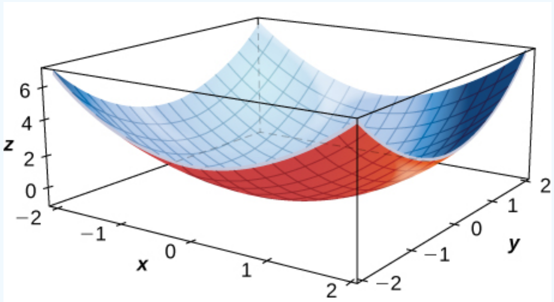
2) $\dfrac{\partial z}{\partial y}$ for $z = x^2 - 3xy + y^2$

Answer

Solution: $\dfrac{\partial z}{\partial y} = -3x + 2y$

6.3E.2 Exercise $6.3E.2$

For the following exercises, calculate the sign of the partial derivative using the graph of the surface.



1) $f_x(1,1)$

2) $f_x(-1,1)$

Answer

Solution:The sign is negative.

3) $f_y(1,1)$

4) $f_x(0,0)$

Answer

Solution:The partial derivative is zero at the origin.

6.3E.3 Exercise $6.3E.3$

For the following exercises, calculate the partial derivatives.

1) $\dfrac{\partial z}{\partial x}$ for $z = sin(3x)cos(3y)$

2) $\dfrac{\partial z}{\partial y}$ for $z = sin(3x)cos(3y)$

Answer

Solution: $\dfrac{\partial z}{\partial y} = -3sin(3x)sin(3y)$

3) $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = x^8 e^3 y$

4) $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = ln(x^6 + y^4)$

Answer

Solution: $\dfrac{\partial z}{\partial x} = \dfrac{6x^5}{x^6+y^4}; \dfrac{\partial z}{\partial y} = \dfrac{4y^3}{x^6+y^4}$

5) Find $f_y(x,y)$ for $f(x,y) = e^{xy} cos(x) sin(y)$.

6) Let $z = e^{xy}$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Answer

Solution:\(\displaystyle \frac{∂z}{∂x}=ye^{xy};\frac{∂z}{∂y}=xe^{xy}\

7) Let $z = ln(\dfrac{x}{y})$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

8) Let $z = tan(2x - y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Answer

Solution: $\dfrac{\partial z}{\partial x} = 2sec^2(2x-y), \dfrac{\partial z}{\partial y} = -sec^2(2x-y)$

9) Let $z = sinh(2x+3y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

10) Let $f(x,y) = arctan(\dfrac{y}{x})$. Evaluate $f_x(2,-2)$ and $f_y(2,-2)$.

Answer

Solution: $f_x(2,-2) = \dfrac{1}{4} = f_y(2,-2)$

11) Let $f(x,y) = \dfrac{xy}{x-y}$. Find $f_x(2,-2)$ and $f_y(2,-2)$. Evaluate the partial derivatives at point $P(0,1)$.

6.3E.4 Exercise $6.3E.4$

1) Find $\dfrac{\partial z}{\partial x}$ at $(0,1)$ for $z = e^{-x} cos(y)$.

Answer

Solution: $\dfrac{\partial z}{\partial x} = -cos(1)$

2) Given $f(x,y,z) = x^3yz^2$, find $\dfrac{\partial^2 f}{\partial x \partial y}$ and $f_z(1,1,1)$.

3) Given $f(x,y,z) = 2sin(x+y)$, find $f_x(0,\dfrac{\pi}{2},-4)$, $f_y(0,\dfrac{\pi}{2},-4)$,and $f_z(0,\dfrac{\pi}{2},-4)$.

Answer

Solution: $f_x = 0, f_y = 0, f_z = 0$

6.3E.5 Exercise $\mathbf{6.3E.5}$

1) The area of a parallelogram with adjacent side lengths that are a and b, and in which the angle between these two sides is θ, is given by the function $A(a,b,\theta) = basin(\theta)$. Find the rate of change of the area of the parallelogram with respect to the following:

a. Side a

b. Side b

c. $Angle\theta$

2) Express the volume of a right circular cylinder as a function of two variables:

a. its radius r and its height h.

b. Show that the rate of change of the volume of the cylinder with respect to its radius is the product of its circumference multiplied by its height.

c. Show that the rate of change of the volume of the cylinder with respect to its height is equal to the area of the circular base.

Answer

Solution:$a.\ V(r,h) = \pi r^2 h\ \ b.\ \dfrac{\partial V}{\partial r} = 2\pi rh\ \ c.\ \dfrac{\partial V}{\partial h} = \pi r^2$

3) Calculate $\dfrac{\partial w}{\partial z}$ for $w = zsin(xy^2 + 2z)$.

6.3E.6 Exercise $\mathbf{6.3E.6}$

Find the indicated higher-order partial derivatives.

1) f_{xy} for $z = ln(x-y)$

Answer

Solution:$f_{xy} = \dfrac{1}{(x-y)^2}$

2) f_{yx} for $z = ln(x-y)$

3) Let $z = x^2 + 3xy + 2y^2$. Find $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$.

Answer

Solution:$\dfrac{\partial^2 z}{\partial x^2} = 2, \dfrac{\partial^2 z}{\partial y^2} = 4$

4) Given $z = e^x tany$, find $\dfrac{\partial^2 z}{\partial x \partial y}$ and $\dfrac{\partial^2 z}{\partial y \partial x}$.

5) Given $f(x,y,z) = xyz$, find f_{xyy}, f_{yxy}, and f_{yyx}.

Answer

Solution:$f_{xyy} = f_{yxy} = f_{yyx} = 0$

6) Given $f(x,y,z) = e^{-2x} sin(z^2 y)$, show that $f_{xyy} = f_{yxy}$.

7) Show that $z = \dfrac{1}{2}(e^y - e^{-y}) sinx$ is a solution of the differential equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

Answer

Solution:

$$\frac{d^2z}{dx^2} = -\frac{1}{2}(e^y - e^{-y})sinx$$
$$\frac{d^2z}{dy^2} = \frac{1}{2}(e^y - e^{-y})sinx$$
$$\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = 0$$

8) Find $f_{xx}(x,y)$ for $f(x,y) = \frac{4x^2}{y} + \frac{y^2}{2x}$.

9) Let $f(x,y,z) = x^2y^3z - 3xy^2z^3 + 5x^2z - y^3z$. Find f_{xyz}.

Answer

Solution: $f_{xyz} = 6y^2x - 18yz^2$

10) Let $F(x,y,z) = x^3yz^2 - 2x^2yz + 3xz - 2y^3z$. Find F_{xyz}.

11) Given $f(x,y) = x^2 + x - 3xy + y^3 - 5$, find all points at which $f_x = f_y = 0$ simultaneously.

Answer

Solution: $(\frac{1}{4}, \frac{1}{2}), (1,1)$

12) Given $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$, find all points at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously.

13) Given $f(x,y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$, find all points on f at which $f_x = f_y = 0$ simultaneously.

Answer

Solution: $(0,0), (0,2), (\sqrt{3}, -1), (-\sqrt{3}, -1)$

14) Given $f(x,y) = 15x^3 - 3xy + 15y^3$, find all points at which $f_x(x,y) = f_y(x,y) = 0$ simultaneously.

15) Show that $z = e^x siny$ satisfies the equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

Answer

Solution: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x sin(y) - e^x siny = 0$

16) Show that $f(x,y) = ln(x^2 + y^2)$ solves Laplace’s equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

17) Show that $z = e^{-t}cos(\frac{x}{c})$ satisfies the heat equation $\frac{\partial z}{\partial t} = -e^{-t}cos(\frac{x}{c})$.

Answer

Solution: $c^2\frac{\partial^2 z}{\partial x^2} = e^{-t}cos(\frac{x}{c})$

18) Find $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x,y)}{\Delta x}$ for $f(x,y) = -7x - 2xy + 7y$.

19) Find $\lim_{\Delta y \to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}$ for $f(x,y) = -7x - 2xy + 7y$.

Answer

Solution: $\frac{\partial f}{\partial y} = -2x + 7$

20) Find $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ for $f(x, y) = x^2 y^2 + xy + y$.

21) Find $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ for $f(x, y) = sin(xy)$.

Answer

Solution: $\frac{\partial f}{\partial x} = ycosxy$

6.3E.7 Exercise $\mathbf{6.3E.7}$

1) The function $P(T, V) = \frac{nRT}{V}$ gives the pressure at a point in a gas as a function of temperature T and volume V. The letters n and R are constants. Find $\frac{\partial P}{\partial V}$ and $\frac{\partial P}{\partial T}$, and explain what these quantities represent.

2) The equation for heat flow in the $xy - plane$ is $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. Show that $f(x, y, t) = e^{-2t} sinx siny$ is a solution.

3) The basic wave equation is $f_{tt} = f_{xx}$. Verify that $f(x, t) = sin(x + t)$ and $f(x, t) = sin(x - t)$ are solutions.

4) The law of cosines can be thought of as a function of three variables. Let x, y, and θ be two sides of any triangle where the angle θ is the included angle between the two sides. Then, $F(x, y, \theta) = x^2 + y^2 - 2xycos\theta$ gives the square of the third side of the triangle. Find $\frac{\partial F}{\partial \theta}$ and $\frac{\partial F}{\partial x}$ when $x = 2, y = 3$, and $\theta = \frac{\pi}{6}$.

Answer

Solution:\(\displaystyle \frac{∂F}{∂θ}=6,\frac{∂F}{∂x}=4−3\sqrt{3}\

5) Suppose the sides of a rectangle are changing with respect to time. The first side is changing at a rate of 2in./sec whereas the second side is changing at the rate of 4 in/sec. How fast is the diagonal of the rectangle changing when the first side measures 16 in. and the second side measures 20 in.? (Round answer to three decimal places.)

6) A **Cobb-Douglas production function** is $f(x, y) = 200x^{0.7} y^{0.3}$, where x and y represent the amount of labor and capital available. Let $x = 500$ and $y = 1000$. Find $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$ at these values, which represent the marginal productivity of labor and capital, respectively.

Answer

Solution: $\frac{\delta f}{\delta x}$ at $(500, 1000) = 172.36$, $\frac{\delta f}{\delta y}$ at $(500, 1000) = 36.93$

7) The apparent temperature index is a measure of how the temperature feels, and it is based on two variables: h, which is relative humidity, and t, which is the air temperature. $A = 0.885t - 22.4h + 1.20th - 0.544$. Find $\frac{\partial A}{\partial t}$ and $\frac{\partial A}{\partial h}$ when $t = 20^\circ F$ and $h = 0.90$.

6.4: Tangent Planes and Linear Approximations

This page is a draft and is under active development.

In this section, we consider the problem of finding the tangent plane to a surface, which is analogous to finding the equation of a tangent line to a curve when the curve is defined by the graph of a function of one variable, $y=f(x)$. The slope of the tangent line at the point $x=a$ is given by $m=f'(a)$; what is the slope of a tangent plane? We learned about the equation of a plane in Equations of Lines and Planes in Space; in this section, we see how it can be applied to the problem at hand.

6.4.1 Tangent Planes

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly. Therefore, in a small-enough neighborhood around the point, a tangent plane touches the surface at that point only.

Definition: tangent lines

Let $P_0=(x_0,y_0,z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the *tangent plane* to S at P_0 (Figure 6.4.1).

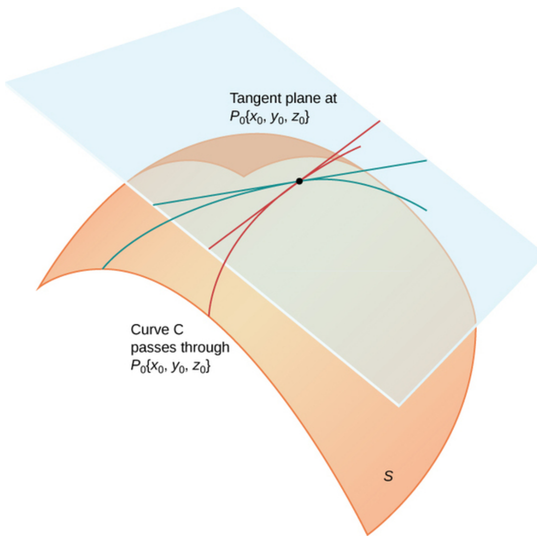


Figure 6.4.1: The tangent plane to a surface S at a point P_0 contains all the tangent lines to curves in S that pass through P_0.

For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition: tangent planes

Let S be a surface defined by a differentiable function $z=f(x,y)$, and let $P_0=(x_0,y_0)$ be a point in the domain of f. Then, the equation of the tangent plane to S at P_0 is given by

$$z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0). \tag{6.4.1}$$

To see why this formula is correct, let's first find two tangent lines to the surface S. The equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $x=x_0$ is $z=f(x_0,y_0)+f_y(x_0,y_0)(y-y_0)$. Similarly, the equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $y=y_0$ is $z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)$. A parallel vector to the first tangent line is $\vec{\mathbf{a}}=\hat{\mathbf{j}}+f_y(x_0,y_0)\hat{\mathbf{k}}$; a parallel vector to the second tangent line is $\vec{\mathbf{b}}=\hat{\mathbf{i}}+f_x(x_0,y_0)\hat{\mathbf{k}}$. We can take the cross product of these two vectors:

$$\begin{aligned}\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= (\,\hat{\mathbf{j}} + f_y(x_0, y_0)\,\hat{\mathbf{k}}) \times (\,\hat{\mathbf{i}} + f_x(x_0, y_0)\,\hat{\mathbf{k}}) \\ &= \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{bmatrix} \\ &= f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}.\end{aligned}$$

This vector is perpendicular to both lines and is therefore perpendicular to the tangent plane. We can use this vector as a normal vector to the tangent plane, along with the point $P_0 = (x_0, y_0, f(x_0, y_0))$ in the equation for a plane:

$$\begin{aligned}\vec{\mathbf{n}} \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\ (f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}) \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\ f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) &= 0.\end{aligned}$$

Solving this equation for z gives Equation 6.4.1.

Example 6.4.1: Finding a Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at point $(2, -1)$.

Solution

First, we must calculate $f_x(x, y)$ and $f_y(x, y)$, then use Equation with $x_0 = 2$ and $y_0 = -1$:

$$\begin{aligned}f_x(x, y) &= 4x - 3y + 2 \\ f_y(x, y) &= -3x + 16y - 4 \\ f(2, -1) &= 2(2)^2 - 3(2)(-1) + 8(-1)^2 + 2(2) - 4(-1) + 4 = 34 \\ f_x(2, -1) &= 4(2) - 3(-1) + 2 = 13 \\ f_y(2, -1) &= -3(2) + 16(-1) - 4 = -26.\end{aligned}$$

Then Equation 6.4.1 becomes

$$\begin{aligned}z &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ z &= 34 + 13(x - 2) - 26(y - (-1)) \\ z &= 34 + 13x - 26 - 26y - 26 \\ z &= 13x - 26y - 18.\end{aligned}$$

(See the following figure).

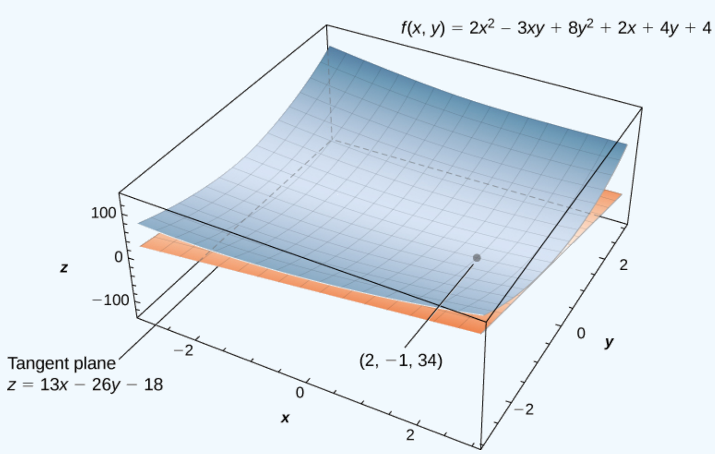


Figure 6.4.2: Calculating the equation of a tangent plane to a given surface at a given point.

Exercise 6.4.1

Find the equation of the tangent plane to the surface defined by the function $f(x,y)=x^3-x^2y+y^2-2x+3y-2$ at point $(-1,3)$.

Hint

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 6.4.1.

Answer

$z=7x+8y-3$

Example 6.4.2: Finding Another Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x,y)=\sin(2x)\cos(3y)$ at the point $(\pi/3,\pi/4)$.

Solution

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 6.4.1 with $x_0=\pi/3$ and $y_0=\pi/4$:

$$\begin{aligned} f_x(x,y) &= 2\cos(2x)\cos(3y) \\ f_y(x,y) &= -3\sin(2x)\sin(3y) \\ f\left(\frac{\pi}{3},\frac{\pi}{4}\right) &= \sin\left(2\left(\frac{\pi}{3}\right)\right)\cos(3(\frac{\pi}{4})) = (\frac{\sqrt{3}}{2})(-\frac{\sqrt{2}}{2}) = -\frac{\sqrt{6}}{4} \\ f_x\left(\frac{\pi}{3},\frac{\pi}{4}\right) &= 2\cos(2(\frac{\pi}{3}))\cos\left(3\left(\frac{\pi}{4}\right)\right) = 2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} \\ f_y\left(\frac{\pi}{3},\frac{\pi}{4}\right) &= -3\sin\left(2\left(\frac{\pi}{3}\right)\right)\sin\left(3\left(\frac{\pi}{4}\right)\right) = -3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{6}}{4}. \end{aligned}$$

Then Equation 6.4.1 becomes

$$\begin{aligned} z &= f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \\ &= -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{3}\right)-\frac{3\sqrt{6}}{4}(y-\frac{\pi}{4}) \\ &= \frac{\sqrt{2}}{2}x-\frac{3\sqrt{6}}{4}y-\frac{\sqrt{6}}{4}-\frac{\pi\sqrt{2}}{6}+\frac{3\pi\sqrt{6}}{16} \end{aligned}$$

A tangent plane to a surface does not always exist at every point on the surface. Consider the piecewise function

$$f(x,y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}. \tag{6.4.2}$$

The graph of this function follows.

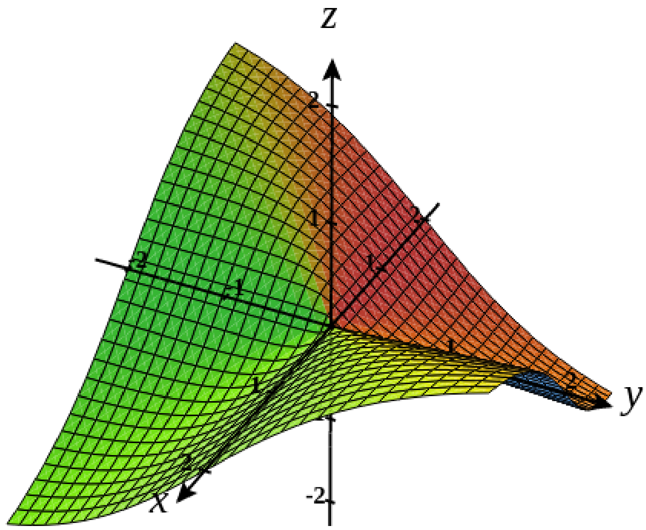


***Figure* $6.4.3$*: Graph of a function that does not have a tangent plane at the origin. Dynamic figure powered by CalcPlot3D.*

If either $x=0$ or $y=0$, then $f(x,y)=0$, so the value of the function does not change on either the x- or y-axis. Therefore, $f_x(x,0)=f_y(0,y)=0$, so as either x or y approach zero, these partial derivatives stay equal to zero. Substituting them into Equation gives $z=0$ as the equation of the tangent line. However, if we approach the origin from a different direction, we get a different story. For example, suppose we approach the origin along the line $y=x$. If we put $y=x$ into the original function, it becomes

$$f(x,x)=\frac{x(x)}{\sqrt{x^2+(x)^2}}=\frac{x^2}{\sqrt{2x^2}}=\frac{|x|}{\sqrt{2}}. \tag{6.4.3}$$

When $x>0$, the slope of this curve is equal to $\sqrt{2}/2$; when $x<0$, the slope of this curve is equal to $-(\sqrt{2}/2)$. This presents a problem. In the definition of tangent plane, we presumed that all tangent lines through point P (in this case, the origin) lay in the same plane. This is clearly not the case here. When we study differentiable functions, we will see that this function is not differentiable at the origin.

6.4.2 Linear Approximations

Recall from Linear Approximations and Differentials that the formula for the linear approximation of a function $f(x)$ at the point $x=a$ is given by

$$y\approx f(a)+f'(a)(x-a). \tag{6.4.4}$$

The diagram for the linear approximation of a function of one variable appears in the following graph.

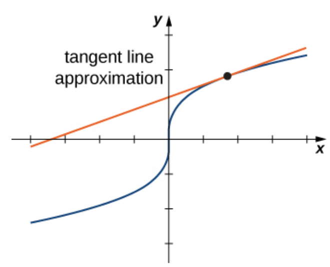


Figure 6.4.4: Linear approximation of a function in one variable.

The tangent line can be used as an approximation to the function $f(x)$ for values of x reasonably close to $x=a$. When working with a function of two variables, the tangent line is replaced by a tangent plane, but the approximation idea is much the same.

Definition: Linear Approximation

Given a function $z=f(x,y)$ with continuous partial derivatives that exist at the point (x_0,y_0), the linear approximation of f at the point (x_0,y_0) is given by the equation

$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0). \tag{6.4.5}$$

Notice that this equation also represents the tangent plane to the surface defined by $z=f(x,y)$ at the point (x_0,y_0). The idea behind using a linear approximation is that, if there is a point (x_0,y_0) at which the precise value of $f(x,y)$ is known, then for values of (x,y) reasonably close to (x_0,y_0), the linear approximation (i.e., tangent plane) yields a value that is also reasonably close to the exact value of $f(x,y)$ (Figure). Furthermore the plane that is used to find the linear approximation is also the tangent plane to the surface at the point (x_0,y_0).

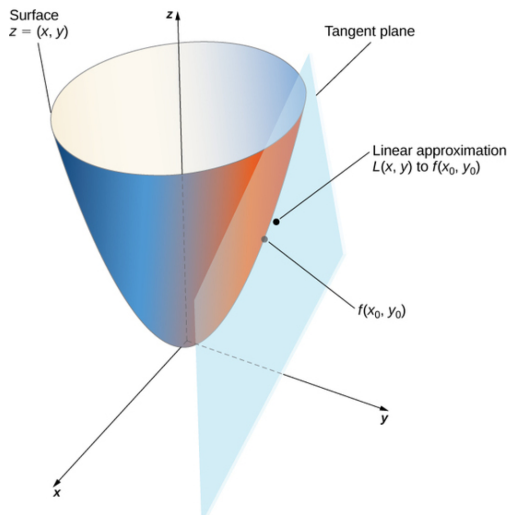


Figure 6.4.5: Using a tangent plane for linear approximation at a point.

Example 6.4.3: Using a Tangent Plane Approximation

Given the function $f(x,y)=\sqrt{41-4x^2-y^2}$, approximate $f(2.1,2.9)$ using point $(2,3)$ for (x_0,y_0). What is the approximate value of $f(2.1,2.9)$ to four decimal places?

Solution

To apply Equation 6.4.5, we first must calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0=2$ and $y_0=3$:

$$\begin{aligned} f(x_0,y_0) &= f(2,3)=\sqrt{41-4(2)^2-(3)^2}=\sqrt{41-16-9}=\sqrt{16}=\sqrt{4} \\ f_x(x,y) &= -\frac{4x}{\sqrt{41-4x^2-y^2}} \text{so } f_x(x_0,y_0)=-\frac{4(2)}{\sqrt{41-4(2)^2-(3)^2}}=-2 \\ f_y(x,y) &= -\frac{y}{\sqrt{41-4x^2-y^2}} \text{so } f_y(x_0,y_0)=-\frac{3}{\sqrt{41-4(2)^2-(3)^2}}=-\frac{3}{4}. \end{aligned}$$

Now we substitute these values into Equation 6.4.5:

$$\begin{aligned} L(x,y) &= f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \\ &= 4-2(x-2)-\frac{3}{4}(y-3) \\ &= \frac{41}{4}-2x-\frac{3}{4}y. \end{aligned}$$

Last, we substitute $x=2.1$ and $y=2.9$ into $L(x,y)$:

$$L(2.1,2.9)=\frac{41}{4}-2(2.1)-\frac{3}{4}(2.9)=10.25-4.2-2.175=3.875.$$

The approximate value of $f(2.1,2.9)$ to four decimal places is

$$f(2.1, 2.9) = \sqrt{41 - 4(2.1)^2 - (2.9)^2} = \sqrt{14.95} \approx 3.8665,$$

which corresponds to a 0.2 error in approximation.

Exercise 6.4.2

Given the function $f(x, y) = e^{5-2x+3y}$, approximate $f(4.1, 0.9)$ using point $(4, 1)$ for (x_0, y_0). What is the approximate value of $f(4.1, 0.9)$ to four decimal places?

Hint

First calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 4$ and $y_0 = 1$, then use Equation 6.4.5.

Answer

$L(x, y) = 6 - 2x + 3y$, so $L(4.1, 0.9) = 6 - 2(4.1) + 3(0.9) = 0.5$ $f(4.1, 0.9) = e^{5-2(4.1)+3(0.9)} = e^{-0.5} \approx 0.6065.$

6.4.3 Differentiability

When working with a function $y = f(x)$ of one variable, the function is said to be differentiable at a point $x = a$ if $f'(a)$ exists. Furthermore, if a function of one variable is differentiable at a point, the graph is “smooth” at that point (i.e., no corners exist) and a tangent line is well-defined at that point.

The idea behind differentiability of a function of two variables is connected to the idea of smoothness at that point. In this case, a surface is considered to be smooth at point P if a tangent plane to the surface exists at that point. If a function is differentiable at a point, then a tangent plane to the surface exists at that point. Recall the formula (Equation 6.4.1) for a tangent plane at a point (x_0, y_0) is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

For a tangent plane to exist at the point (x_0, y_0), the partial derivatives must therefore exist at that point. However, this is not a sufficient condition for smoothness, as was illustrated in Figure. In that case, the partial derivatives existed at the origin, but the function also had a corner on the graph at the origin.

Definition: differentiable Functions

A function $f(x, y)$ is *differentiable* at a point $P(x_0, y_0)$ if, for all points (x, y) in a δ disk around P, we can write

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y), \tag{6.4.6}$$

where the error term E satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0. \tag{6.4.7}$$

The last term in Equation 6.4.6 is to as the **error term** and it represents how closely the tangent plane comes to the surface in a small neighborhood (δ disk) of point P. For the function f to be differentiable at P, the function must be smooth—that is, the graph of f must be close to the tangent plane for points near P.

Example 6.4.4: Demonstrating Differentiability

Show that the function $f(x, y) = 2x^2 - 4y$ is differentiable at point $(2, -3)$.

Solution

First, we calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = -3$, then we use Equation 6.4.6:

$$\begin{aligned} f(2, -3) &= 2(2)^2 - 4(-3) = 8 + 12 = 20 \\ f_x(2, -3) &= 4(2) = 8 \\ f_y(2, -3) &= -4. \end{aligned}$$

Therefore $m_1 = 8$ and $m_2 = -4$, and Equation 6.4.6 becomes

$$\begin{aligned}
f(x,y) &= f(2,-3)+f_x(2,-3)(x-2)+f_y(2,-3)(y+3)+E(x,y)\\
2x^2-4y &= 20+8(x-2)-4(y+3)+E(x,y)\\
2x^2-4y &= 20+8x-16-4y-12+E(x,y)\\
2x^2-4y &= 8x-4y-8+E(x,y)\\
E(x,y) &= 2x^2-8x+8.
\end{aligned}$$

Next, we calculate the limit in Equation 6.4.7:

$$\begin{aligned}
\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x+0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(2,-3)} \frac{2x^2-8x+8}{\sqrt{(x-2)^2+(y+3)^2}}\\
&= \lim_{(x,y)\to(2,-3)} \frac{2(x^2-4x+4)}{\sqrt{(x-2)^2+(y+3)^2}}\\
&= \lim_{(x,y)\to(2,-3)} \frac{2(x-2)^2}{\sqrt{(x-2)^2+(y+3)^2}}\\
&= \lim_{(x,y)\to(2,-3)} \frac{2((x-2)^2+(y+3)^2)}{\sqrt{(x-2)^2+(y+3)^2}}\\
&= \lim_{(x,y)\to(2,-3)} 2\sqrt{(x-2)^2+(y+3)^2}\\
&= 0.
\end{aligned}$$

Since $E(x,y)\geq 0$ for any value of x or y, the original limit must be equal to zero. Therefore, $f(x,y)=2x^2-4y$ is differentiable at point $(2,-3)$.

Exercise 6.4.3

Show that the function $f(x,y)=3x-4y^2$ is differentiable at point $(-1,2)$.

Hint

First, calculate $f(x_0,y_0), f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0=-1$ and $y_0=2$, then use Equation 6.4.7 to find $E(x,y)$. Last, calculate the limit.

Answer

$$f(-1,2) = -19, f_x(-1,2)=3, f_y(-1,2)=-16, E(x,y)=-4(y-2)^2.$$

$$\begin{aligned}
\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(-1,2)} \frac{-4(y-2)^2}{\sqrt{(x+1)^2+(y-2)^2}}\\
&\leq \lim_{(x,y)\to(-1,2)} \frac{-4((x+1)^2+(y-2)^2)}{\sqrt{(x+1)^2+(y-2)^2}}\\
&= \lim_{(x,y)\to(2,-3)} -4\sqrt{(x+1)^2+(y-2)^2}\\
&= 0.
\end{aligned}$$

This function from (Equation 6.4.2)

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

is not differentiable at the origin (Figure 6.4.3). We can see this by calculating the partial derivatives. This function appeared earlier in the section, where we showed that $f_x(0,0)=f_y(0,0)=0$. Substituting this information into Equations 6.4.6 and 6.4.7 using $x_0=0$ and $y_0=0$, we get

$$\begin{aligned} f(x,y) &= f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)+E(x,y) \\ E(x,y) &= \frac{xy}{\sqrt{x^2+y^2}}. \end{aligned}$$

Calculating

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \tag{6.4.8}$$

gives

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \\ &= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}. \end{aligned}$$

Depending on the path taken toward the origin, this limit takes different values. Therefore, the limit does not exist and the function f is not differentiable at the origin as shown in the following figure.

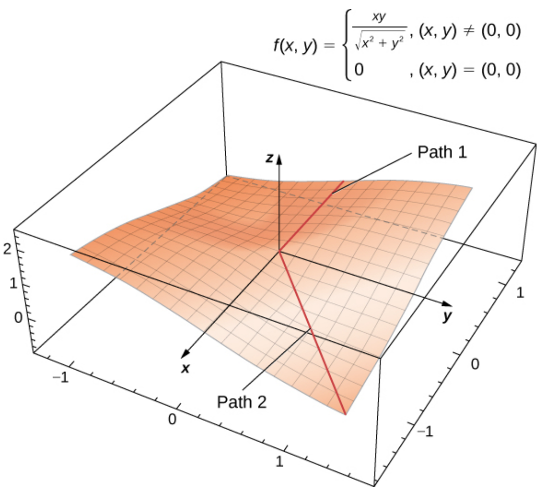


Figure 6.4.6: This function $f(x,y)$ (Equation 6.4.2) is not differentiable at the origin.

Differentiability and continuity for functions of two or more variables are connected, the same as for functions of one variable. In fact, with some adjustments of notation, the basic theorem is the same.

THEOREM: Differentiability Implies Continuity

Let $z=f(x,y)$ be a function of two variables with (x_0,y_0) in the domain of f. If $f(x,y)$ is differentiable at (x_0,y_0), then $f(x,y)$ is continuous at (x_0,y_0).

Note shows that if a function is differentiable at a point, then it is continuous there. However, if a function is continuous at a point, then it is not necessarily differentiable at that point. For example, the function discussed above (Equation 6.4.2)

$$f(x,y)=\begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

is *continuous* at the origin, but it is *not differentiable* at the origin. This observation is also similar to the situation in single-variable calculus.

We can further explores the connection between continuity and differentiability at a point. This next theorem says that if the function and its partial derivatives are continuous at a point, the function is differentiable.

Theorem: Continuity of First Partials Implies Differentiability

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f. If $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ all exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0), then $f(x, y)$ is differentiable there.

Recall that earlier we showed that the function in Equation 6.4.2 was not differentiable at the origin. Let's calculate the partial derivatives f_x and f_y:

$$\frac{\partial f}{\partial x} = \frac{y^3}{(x^2+y^2)^{3/2}} \tag{6.4.9}$$

and

$$\frac{\partial f}{\partial y} = \frac{x^3}{(x^2+y^2)^{3/2}}. \tag{6.4.10}$$

The contrapositive of the preceding theorem states that if a function is not differentiable, then at least one of the hypotheses must be false. Let's explore the condition that $f_x(0, 0)$ must be continuous. For this to be true, it must be true that

$$\lim_{(x,y)\to(0,0)} f_x(x, y) = f_x(0, 0) \tag{6.4.11}$$

therefor

$$\lim_{(x,y)\to(0,0)} f_x(x, y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{3/2}}. \tag{6.4.12}$$

Let $x = ky$. Then

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{y^3}{(x^2+y^2)^{3/2}} &= \lim_{y\to 0} \frac{y^3}{((ky)^2+y^2)^{3/2}} \\ &= \lim_{y\to 0} \frac{y^3}{(k^2y^2+y^2)^{3/2}} \\ &= \lim_{y\to 0} \frac{y^3}{|y|^3(k^2+1)^{3/2}} \\ &= \frac{1}{(k^2+1)^{3/2}} \lim_{y\to 0} \frac{|y|}{y}. \end{aligned}$$

If $y > 0$, then this expression equals $1/(k^2+1)^{3/2}$; if $y < 0$, then it equals $-(1/(k^2+1)^{3/2})$. In either case, the value depends on k, so the limit fails to exist.

6.4.4 Differentials

In Linear Approximations and Differentials we first studied the concept of differentials. The differential of y, written dy, is defined as $f'(x)dx$. The differential is used to approximate $\Delta y = f(x+\Delta x) - f(x)$, where $\Delta x = dx$. Extending this idea to the linear approximation of a function of two variables at the point (x_0, y_0) yields the formula for the total differential for a function of two variables.

Definition: Total Differential

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f, and let Δx and Δy be chosen so that $(x_0 + \Delta x, y_0 + \Delta y)$ is also in the domain of f. If f is differentiable at the point (x_0, y_0), then the differentials dx and dy are defined as

$$dx = \Delta x \tag{6.4.13}$$

and

$$dy = \Delta y. \tag{6.4.14}$$

The differential dz, also called the ***total differential*** of $z = f(x, y)$ at (x_0, y_0), is defined as

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy. \tag{6.4.15}$$

Notice that the symbol ∂ is not used to denote the total differential; rather, d appears in front of z. Now, let's define $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$. We use dz to approximate Δz, so

$$\Delta z \approx dz = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy. \tag{6.4.16}$$

Therefore, the differential is used to approximate the change in the function $z = f(x_0,y_0)$ at the point (x_0,y_0) for given values of Δx and Δy. Since $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$, this can be used further to approximate $f(x+\Delta x, y+\Delta y)$:

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \Delta z \approx f(x,y) + fx(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y. \tag{6.4.17}$$

See the following figure.

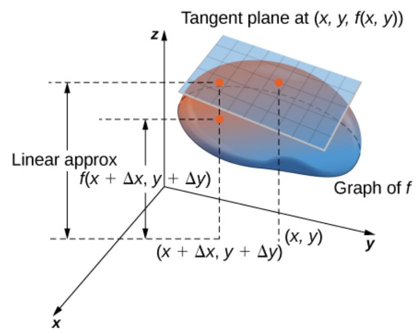


Figure 6.4.7: The linear approximation is calculated via the formula $f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$.

One such application of this idea is to determine error propagation. For example, if we are manufacturing a gadget and are off by a certain amount in measuring a given quantity, the differential can be used to estimate the error in the total volume of the gadget.

Example 6.4.5: Approximation by Differentials

Find the differential dz of the function $f(x,y) = 3x^2 - 2xy + y^2$ and use it to approximate Δz at point $(2,-3)$. Use $\Delta x = 0.1$ and $\Delta y = -0.05$. What is the exact value of Δz?

Solution

First, we must calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0 = 2$ and $y_0 = -3$:

$$f(x_0,y_0) \ = f(2,-3) = 3(2)^2 - 2(2)(-3) + (-3)^2 = 12 + 12 + 9 = 33$$

$$f_x(x,y = 6x - 2y$$

$$f_y(x,y) = -2x + 2y$$

$$\begin{aligned} f_x(x_0,y_0) &= fx(2,-3) \\ &= 6(2) - 2(-3) = 12 + 6 = 18 \end{aligned}$$

$$\begin{aligned} f_y(x_0,y_0) &= f_y(2,-3) \\ &= -2(2) + 2(-3) \\ &= -4 - 6 = -10. \end{aligned}$$

Then, we substitute these quantities into Equation 6.4.15:

$$\begin{aligned} dz &= f_x(x_0,y_0)dx + f_y(x_0,y_0)dy \\ dz &= 18(0.1) - 10(-0.05) = 1.8 + 0.5 = 2.3. \end{aligned}$$

This is the approximation to $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$. The exact value of Δz is given by

$$\begin{aligned} \Delta z &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \\ &= f(2+0.1, -3-0.05) - f(2,-3) \\ &= f(2.1, -3.05) - f(2,-3) \\ &= 2.3425. \end{aligned}$$

Exercise 6.4.4

Find the differential dz of the function $f(x,y)=4y^2+x^2y-2xy$ and use it to approximate Δz at point $(1,-1)$. Use $\Delta x=0.03$ and $\Delta y=-0.02$. What is the exact value of Δz?

Hint

First, calculate $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ using $x_0=1$ and $y_0=-1$, then use Equation 6.4.15.

Answer

$dz=0.18$

$\Delta z=f(1.03,-1.02)-f(1,-1)=0.180682$

6.4.5 Differentiability of a Function of Three Variables

All of the preceding results for differentiability of functions of two variables can be generalized to functions of three variables. First, the definition:

Definition: Differentiability at a point

A function $f(x,y,z)$ is differentiable at a point $P(x_0,y_0,z_0)$ if for all points (x,y,z) in a δ disk around P we can write

$$f(x,y)=f(x_0,y_0,z_0)+f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)+E(x,y,z), \tag{6.4.18}$$

where the error term E satisfies

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x,y,z)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}=0. \tag{6.4.19}$$

If a function of three variables is differentiable at a point (x_0,y_0,z_0), then it is continuous there. Furthermore, continuity of first partial derivatives at that point guarantees differentiability.

6.4.6 Key Concepts

- The analog of a tangent line to a curve is a tangent plane to a surface for functions of two variables.
- Tangent planes can be used to approximate values of functions near known values.
- A function is differentiable at a point if it is "smooth" at that point (i.e., no corners or discontinuities exist at that point).
- The total differential can be used to approximate the change in a function $z=f(x_0,y_0)$ at the point (x_0,y_0) for given values of Δx and Δy.

6.4.7 Key Equations

- **Tangent plane**

  $z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

- **Linear approximation**

  $L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

- **Total differential**

  $dz=f_x(x_0,y_0)dx+f_y(x_0,y_0)dy$.

- **Differentiability (two variables)**

  $f(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+E(x,y),$

  where the error term E satisfies

  $$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0.$$

- **Differentiability (three variables)**

  $f(x,y)=f(x_0,y_0,z_0)+f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)+E(x,y,z),$

  where the error term E satisfies

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x,y,z)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = 0.$$

6.4.8 Glossary

differentiable

a function $f(x,y)$ is differentiable at (x_0,y_0) if $f(x,y)$ can be expressed in the form $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + E(x,y),$

where the error term $E(x,y)$ satisfies $\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$

linear approximation

given a function $f(x,y)$ and a tangent plane to the function at a point (x_0,y_0), we can approximate $f(x,y)$ for points near (x_0,y_0) using the tangent plane formula

tangent plane

given a function $f(x,y)$ that is differentiable at a point (x_0,y_0), the equation of the tangent plane to the surface $z=f(x,y)$ is given by $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

total differential

the total differential of the function $f(x,y)$ at (x_0,y_0) is given by the formula $dz = f_x(x_0,y_0)dx + fy(x_0,y_0)dy$

-

6.4E:Exercises

6.4E.1 Exercise $6.4E.1$

For the following exercises, find a unit normal vector to the surface at the indicated point.

1) $f(x,y)=x^3$, $(2,-1,8)$

Answer

\(\displaystyle (\frac{\sqrt{145}}{145})(12 \hat{\mathbf{i}− \hat{\mathbf{k})\)

2) $ln(\frac{x}{y-z})=0$ when $x=y=1$

6.4E.2 Exercise $6.4E.2$

For the following exercises, as a useful review for techniques used in this section, find a normal vector and a tangent vector at point P.

3) $x^2+xy+y^2=3, P(-1,-1)$

Answer

Normal vector: \(\displaystyle \hat{\mathbf{i}+ \hat{\mathbf{j}\), tangent vector: \(\displaystyle − \hat{\mathbf{j}\)

4) $(x^2+y^2)^2=9(x^2-y^2), d(\sqrt{2},1)$

5) $xy^2-2x^2+y+5x=6, P(4,2)$

Answer

Normal vector: \(\displaystyle 7 \hat{\mathbf{i}−17 \hat{\mathbf{j}\), tangent vector: \(\displaystyle 7 \hat{\mathbf{i}+7 \hat{\mathbf{j}\)

6) $2x^3-x^2y^2=3x-y-7, P(1,-2)$

7) $ze^{x^2-y^2}-3=0, P(2,2,3)$

Answer

\(\displaystyle −1.094 \hat{\mathbf{i}−0.18238 \hat{\mathbf{j}\)

6.4E.3 Exercise $6.4E.3$

For the following exercises, find the equation for the tangent plane to the surface at the indicated point. (Hint: Solve for z in terms of x and y.)

8) $-8x-3y-7z=-19, P(1,-1,2)$

9) $z=-9x^2-3y^2, P(2,1,-39)$

Answer

$-36x-6y-z=-39$

10) $x^2+10xyz+y^2+8z^2=0, P(-1,-1,-1)$

11) $z=ln(10x^2+2y^2+1), P(0,0,0)$

Answer

$z=0$

12) $z=e^{7x^2+4y^2}, P(0,0,1)$

13) $xy+yz+zx=11, P(1,2,3)$

Answer

$5x+4y+3z-22=0$

14) $x^2+4y^2=z^2, P(3,2,5)$

15) $x^3+y^3=3xyz, P(1,2,\frac{3}{2})$

Answer

$4x-5y+4z=0$

16) $z=axy, P(1,\frac{1}{a},1)$

17) $z=sinx+siny+sin(x+y), P(0,0,0)$

Answer

$2x+2y-z=0$

18) $h(x,y)=ln\sqrt{x^2+y^2}, P(3,4)$

19) $z=x^2-2xy+y^2, P(1,2,1)$

Answer

$-2(x-1)+2(y-2)-(z-1)=0$

6.4E.4 Exercise $6.4E.4$

For the following exercises, find parametric equations for the normal line to the surface at the indicated point. (Recall that to find the equation of a line in space, you need a point on the line, $P_0(x_0,y_0,z_0)$, and a vector $n=\langle a,b,c\rangle$ that is parallel to the line. Then the equation of the line is $x-x_0=at, y-y_0=bt, z-z_0=ct.$)

20) $-3x+9y+4z=-4, P(1,-1,2)$

21) $z=5x^2-2y^2, P(2,1,18)$

Answer

$x=20t+2, y=-4t+1, z=-t+18$

22) $x^2-8xyz+y^2+6z^2=0, P(1,1,1)$

23) $z=ln(3x^2+7y^2+1), P(0,0,0)$

Answer

$x=0, y=0, z=t$

24) $z=e^{4x^2+6y^2}, P(0,0,1)$

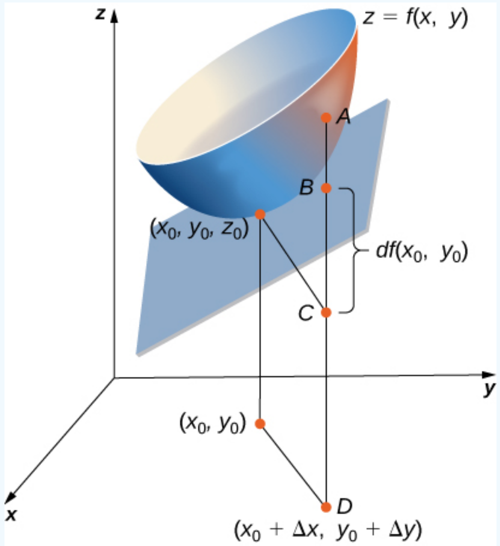
25) $z=x^2-2xy+y^2$ at point $P(1,2,1)$

Answer

Solution:$x-1=2t; y-2=-2t; z-1=t$

6.4E.5 Exercise $6.4E.5$

For the following exercises, use the figure shown here.



26) The length of line segment AC is equal to what mathematical expression?

27) The length of line segment BC is equal to what mathematical expression?

Answer

The differential of the function $z(x,y)$ is $dz = f_x dx + f_y dy$

28) Using the figure, explain what the length of line segment AB represents.

6.4E.6 Exercise $6.4E.6$

For the following exercises, complete each task.

29) Show that $f(x,y) = e^{xy}x$ is differentiable at point $(1,0)$.

Answer

Using the definition of differentiability, we have $e^{xy}x \approx x + y$.

30) Find the total differential of the function $w = e^y cos(x) + z^2$.

31) Show that $f(x,y) = x^2 + 3y$ is differentiable at every point. In other words, show that $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where both ε_1 and ε_2 approach zero as $(\Delta x, \Delta y)$ approaches $(0,0)$.

Answer

$\Delta z = 2x\Delta x + 3\Delta y + (\Delta x)^2$. $(\Delta x)^2 \to 0$ for small Δx and z satisfies the definition of differentiability.

32) Find the total differential of the function $z = \dfrac{xy}{y+x}$ where x changes from 10 to 10.5 and y changes from 15 to 13.

33) Let $z = f(x,y) = xe^y$. Compute Δz from $P(1,2)$ to $Q(1.05, 2.1)$and then find the approximate change in z from point P to point Q. Recall $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$, and dz and Δz are approximately equal.

Answer

$\Delta z \approx 1.185422$and $dz \approx 1.108$. They are relatively close.

34) The volume of a right circular cylinder is given by $V(r,h) = \pi r^2 h$. Find the differential dV. Interpret the formula geometrically.

35) See the preceding problem. Use differentials to estimate the amount of aluminum in an enclosed aluminum can with diameter $8.0 cm$ and height $12 cm$ if the aluminum is 0.04 cm thick.

Answer

$16 cm^3$

36) Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x,y) moves from point $(1,1)$ to point $(1.01, 0.97)$. Compare this approximation with the actual change in the function.

37) Let $z = f(x,y) = x^2 + 3xy - y^2$. Find the exact change in the function and the approximate change in the function as x changes from 2.00 to 92.05 and y changes from 3.00 to 2.96.

Answer

$\Delta z =$ exact change $= 0.6449$, approximate change is $dz = 0.65$. The two values are close.

38) The centripetal acceleration of a particle moving in a circle is given by $a(r,v) = \frac{v^2}{r}$, where v is the velocity and r is the radius of the circle. Approximate the maximum percent error in measuring the acceleration resulting from errors of 3 in v and 2 in r. (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in a is given by $\frac{da}{a}$.)

39) The radius r and height h of a right circular cylinder are measured with possible errors of 4 and 5, respectively. Approximate the maximum possible percentage error in measuring the volume (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in V is given by $\frac{dV}{V}$.)

Answer

13 or 0.13

40) The base radius and height of a right circular cone are measured as 10 in. and 25 in., respectively, with a possible error in measurement of as much as 0.1 in. each. Use differentials to estimate the maximum error in the calculated volume of the cone.

41) The **electrical resistance** R produced by wiring resistors R_1 and R_2 in parallel can be calculated from the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 and R_2 are measured to be 7Ω and 6Ω, respectively, and if these measurements are accurate to within 0.05Ω, estimate the maximum possible error in computing R. (The symbol Ω represents an ohm, the unit of electrical resistance.)

Answer

0.025

42) The area of an ellipse with axes of length $2a$ and $2b$ is given by the formula $A = \pi ab$. Approximate the percent change in the area when a increases by 2 and b increases by 1.5

43) The period T of a **simple pendulum** with small oscillations is calculated from the formula $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the length of the pendulum and g is the acceleration resulting from gravity. Suppose that L and g have errors of, at most, 0.5 and 0.1, respectively. Use differentials to approximate the maximum percentage error in the calculated value of T.

Answer

0.3

44) **Electrical power** P is given by $P=\frac{V^2}{R}$, where V is the voltage and R is the resistance. Approximate the maximum percentage error in calculating power if $120V$ is applied to a $2000-\Omega$ resistor and the possible percent errors in measuring V and R are 3 and 4, respectively.

6.4E.7 Exercise $\mathbf{6.4E.7}$

For the following exercises, find the linear approximation of each function at the indicated point.

45) $f(x,y)=x\sqrt{y}, P(1,4)$

Answer

$$2x+\frac{1}{4}y-1$$

46) $f(x,y)=e^x cosy; P(0,0)$

47) $f(x,y)=arctan(x+2y), P(1,0)$

Answer

$$\frac{1}{2}x+y+\frac{1}{4}\pi-\frac{1}{2}$$

48) $f(x,y)=\sqrt{20-x^2-7y^2}, P(2,1)$

49) $f(x,y,z)=\sqrt{x^2+y^2+z^2}, P(3,2,6)$

Answer

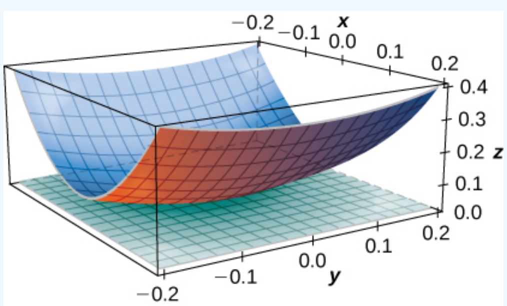
$$\frac{3}{7}x+\frac{2}{7}y+\frac{6}{7}z$$

50) [T] Find the equation of the tangent plane to the surface $f(x,y)=x^2+y^2$ at point $(1,2,5)$, and graph the surface and the tangent plane at the point.

51) [T] Find the equation for the tangent plane to the surface at the indicated point, and graph the surface and the tangent plane: $z=ln(10x^2+2y^2+1), P(0,0,0)$.

Answer

$$z=0$$



52) [T] Find the equation of the tangent plane to the surface $z=f(x,y)=sin(x+y^2)$ at point $(\frac{\pi}{4},0,\frac{\sqrt{2}}{2})$, and graph the surface and the tangent plane.

6.5: The Chain Rule for Multivariable Functions

This page is a draft and is under active development.

In single-variable calculus, we found that one of the most useful differentiation rules is the chain rule, which allows us to find the derivative of the composition of two functions. The same thing is true for multivariable calculus, but this time we have to deal with more than one form of the chain rule. In this section, we study extensions of the chain rule and learn how to take derivatives of compositions of functions of more than one variable.

6.5.1 Chain Rules for One or Two Independent Variables

Recall that the chain rule for the derivative of a composite of two functions can be written in the form

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x). \tag{6.5.1}$$

In this equation, both $f(x)$ and $g(x)$ are functions of one variable. Now suppose that f is a function of two variables and g is a function of one variable. Or perhaps they are both functions of two variables, or even more. How would we calculate the derivative in these cases? The following theorem gives us the answer for the case of one independent variable.

Chain Rule for One Independent Variable

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}, \tag{6.5.2}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Proof

The proof of this theorem uses the definition of differentiability of a function of two variables. Suppose that f is differentiable at the point $P(x_0, y_0)$, where $x_0 = g(t_0)$ and $y_0 = h(t_0)$ for a fixed value of t_0. We wish to prove that $z = f(x(t), y(t))$ is differentiable at $t = t_0$ and that Equation holds at that point as well.

Since f is differentiable at P, we know that

$$z(t) = f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

We then subtract $z_0 = f(x_0, y_0)$ from both sides of this equation:

$$\begin{aligned} z(t) - z(t_0) &= f(x(t), y(t)) - f(x(t_0), y(t_0)) \\ &= f_x(x_0, y_0)(x(t) - x(t_0)) + f_y(x_0, y_0)(y(t) - y(t_0)) + E(x(t), y(t)). \end{aligned}$$

Next, we divide both sides by $t - t_0$:

$$z(t) - z(t_0)t - t_0 = fx(x_0, y_0)(x(t) - x(t_0)t - t_0) + f_y(x_0, y_0)(y(t) - y(t_0)t - t_0) + E(x(t), y(t))t - t_0.$$

Then we take the limit as t approaches t_0:

$$\begin{aligned}\lim_{t\to t_0} \frac{z(t) - z(t_0)}{t - t_0} &= f_x(x_0, y_0) \lim_{t\to t_0} \left(\frac{x(t) - x(t_0)}{t - t_0}\right) + f_y(x_0, y_0) \lim_{t\to t_0} \left(\frac{y(t) - y(t_0)}{t - t_0}\right) \\ &+ \lim_{t\to t_0} \frac{E(x(t), y(t))}{t - t_0}.\end{aligned}$$

The left-hand side of this equation is equal to dz/dt, which leads to

$$\frac{dz}{dt} = f_x(x_0, y_0)\frac{dx}{dt} + f_y(x_0, y_0)\frac{dy}{dt} + \lim_{t\to t_0} \frac{E(x(t), y(t))}{t - t_0}.$$

The last term can be rewritten as

$$\begin{aligned}\lim_{t\to t_0} \frac{E(x(t), y(t))}{t - t_0} &= \lim_{t\to t_0} \frac{(E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0}) \\ &= \lim_{t\to t_0}(\frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}) \lim_{t\to t_0}(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0}).\end{aligned}$$

As t approaches t_0, $(x(t), y(t))$ approaches $(x(t_0), y(t_0))$, so we can rewrite the last product as

$$\lim_{(x,y)\to(x_0,y_0)} \frac{(E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \lim_{(x,y)\to(x_0,y_0)}(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0}).$$

Since the first limit is equal to zero, we need only show that the second limit is finite:

$$\begin{aligned}\lim_{(x,y)\to(x_0,y_0)} \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t + 0} &= \lim_{(x,y)\to(x_0,y_0)} \sqrt{\frac{(x - x_0)^2 + (y - y_0)^2}{(t - t_0)^2}} \\ &= \lim_{(x,y)\to(x_0,y_0)} \sqrt{\left(\frac{x - x_0}{t - t_0}\right)^2 + \left(\frac{y - y_0}{t - t_0}\right)^2} \\ &= \sqrt{\left[\lim_{(x,y)\to(x_0,y_0)} \left(\frac{x - x_0}{t - t_0}\right)\right]^2 + \left[\lim_{(x,y)\to(x_0,y_0)} \left(\frac{y - y_0}{t - t_0}\right)\right]^2}.\end{aligned}$$

Since $x(t)$ and $y(t)$ are both differentiable functions of t, both limits inside the last radical exist. Therefore, this value is finite. This proves the chain rule at $t = t_0$; the rest of the theorem follows from the assumption that all functions are differentiable over their entire domains.

□

Closer examination of Equation reveals an interesting pattern. The first term in the equation is $\frac{\partial f}{\partial x} \cdot \frac{dx}{dt}$ and the second term is $\frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$. Recall that when multiplying fractions, cancelation can be used. If we treat these derivatives as fractions, then each product "simplifies" to something resembling $\partial f/dt$. The variables x and y that disappear in this simplification are often called **intermediate variables**: they are independent variables for the function f, but are dependent variables for the variable t. Two terms appear on the right-hand side of the formula, and f is a function of two variables. This pattern works with functions of more than two variables as well, as we see later in this section.

Example 6.5.1: Using the Chain Rule

Calculate dz/dt for each of the following functions:

a. $z=f(x,y)=4x^2+3y^2, x=x(t)=\sin t, y=y(t)=\cos t$
b. $z=f(x,y)=\sqrt{x^2-y^2}, x=x(t)=e^{2t}, y=y(t)=e^{-t}$

Solution

a. To use the chain rule, we need four quantities—$\partial z/\partial x, \partial z/\partial y, dx/dt$, and dy/dt:

- $\dfrac{\partial z}{\partial x}=8x$
- $\dfrac{dx}{dt}=\cos t$
- $\dfrac{\partial z}{\partial y}=6y$
- $\dfrac{dy}{dt}=-\sin t$

Now, we substitute each of these into Equation:

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}=(8x)(\cos t)+(6y)(-\sin t)=8x\cos t-6y\sin t.$$

This answer has three variables in it. To reduce it to one variable, use the fact that $x(t)=\sin t \text{and} y(t)=\cos t.$ We obtain

$$\frac{dz}{dt}=8x\cos t-6y\sin t=8(\sin t)\cos t-6(\cos t)\sin t=2\sin t\cos t.$$

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x,y)$, then differentiating with respect to t:

$$z=f(x,y)=f(x(t),y(t))=4(x(t))^2+3(y(t))^2=4\sin^2 t+3\cos^2 t.$$

Then

$$\frac{dz}{dt}=2(4\sin t)(\cos t)+2(3\cos t)(-\sin t)=8\sin t\cos t-6\sin t\cos t=2\sin t\cos t, \tag{6.5.3}$$

which is the same solution. However, it may not always be this easy to differentiate in this form.

b. To use the chain rule, we again need four quantities—$\partial z/\partial x, \partial z/dy, dx/dt,$ and dy/dt :

- $\dfrac{\partial z}{\partial x}=\dfrac{x}{\sqrt{x^2-y^2}}$
- $\dfrac{dx}{dt}=2e^{2t}$
- $\dfrac{\partial z}{\partial y}=\dfrac{-y}{\sqrt{x^2-y^2}}$

- $\dfrac{dx}{dt} = -e^{-t}$.

We substitute each of these into Equation:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \\ &= \left(\frac{x}{\sqrt{x^2-y^2}}\right)(2e^{2t}) + \left(\frac{-y}{\sqrt{x^2-y^2}}\right)(-e^{-t}) \\ &= \frac{2xe^{2t}-ye^{-t}}{\sqrt{x^2-y^2}}.\end{aligned}$$

To reduce this to one variable, we use the fact that $x(t) = e^{2t}$ and $y(t) = e^{-t}$. Therefore,

$$\begin{aligned}\frac{dz}{dt} &= \frac{2xe^2t + ye^{-t}}{\sqrt{x^2-y^2}} \\ &= \frac{2(e^{2t})e^{2t} + (e^{-t})e^{-t}}{\sqrt{e^{4t}-e^{-2t}}} \\ &= \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}.\end{aligned}$$

To eliminate negative exponents, we multiply the top by e^{2t} and the bottom by $\sqrt{e^{4t}}$:

$$\begin{aligned}\frac{dz}{dt} &= \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}\cdot\frac{e^{2t}}{\sqrt{e^{4t}}} \\ &= \frac{2e^{6t}+1}{\sqrt{e^{8t}-e^{2t}}} \\ &= \frac{2e^{6t}+1}{\sqrt{e^{2t}(e^{6t}-1)}} \\ &= \frac{2e^{6t}+1}{e^t\sqrt{e^{6t}-1}}.\end{aligned}$$

Again, this derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x,y)$, then differentiating with respect to t:

$$\begin{aligned}z &= f(x,y) \\ &= f(x(t), y(t)) \\ &= \sqrt{(x(t))^2 - (y(t))^2} \\ &= \sqrt{e^{4t}-e^{-2t}} \\ &= (e^{4t}-e^{-2t})^{1/2}.\end{aligned}$$

Then

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{2}(e^{4t} - e^{-2t})^{-1/2}\left(4e^{4t} + 2e^{-2t}\right) \\ &= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}. \end{aligned}$$

This is the same solution.

Exercise 6.5.1

Calculate dz/dt given the following functions. Express the final answer in terms of t.

$$z = f(x, y) = x^2 - 3xy + 2y^2 \tag{6.5.4}$$

$$x = x(t) = 3\sin 2t, y = y(t) = 4\cos 2t \tag{6.5.5}$$

Hint

Calculate $\partial z/\partial x, \partial z/dy, dx/dt,$ and $dy/dt,$ then use Equation.

Answer

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (2x - 3y)(6\cos 2t) + (-3x + 4y)(-8\sin 2t) \\ &= -92\sin 2t\cos 2t - 72(\cos^2 2t - \sin^2 2t) \\ &= -46\sin 4t - 72\cos 4t. \end{aligned}$$

It is often useful to create a visual representation of Equation for the chain rule. This is called a **tree diagram** for the chain rule for functions of one variable and it provides a way to remember the formula (Figure 6.5.1). This diagram can be expanded for functions of more than one variable, as we shall see very shortly.

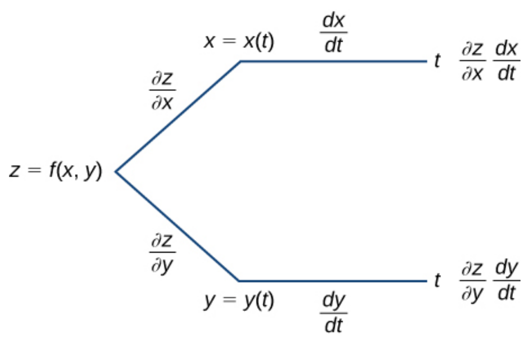


Figure 6.5.1: Tree diagram for the case $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

In this diagram, the leftmost corner corresponds to $z = f(x, y)$. Since f has two **independent variables**, there are two lines coming from this corner. The upper branch corresponds to the variable x and the lower branch corresponds to the variable y. Since each of these variables is then dependent on one variable t, one branch then comes from x and one branch comes from y. Last, each of the branches on the far right has a label that represents the path traveled to reach that branch. The top branch is reached by following the x branch, then the t branch; therefore, it is labeled $(\partial z/\partial x) \times (dx/dt)$. The bottom branch is similar: first the y branch, then the t

branch. This branch is labeled $(\partial z/\partial y) \times (dy/dt)$. To get the formula for dz/dt, add all the terms that appear on the rightmost side of the diagram. This gives us Equation.

In Note, $z = f(x, y)$ is a function of x and y, and both $x = g(u, v)$ and $y = h(u, v)$ are functions of the independent variables u and v.

Chain Rule for Two Independent Variables

Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then, $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial x}{\partial u} \tag{6.5.6}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \tag{6.5.7}$$

We can draw a tree diagram for each of these formulas as well as follows.

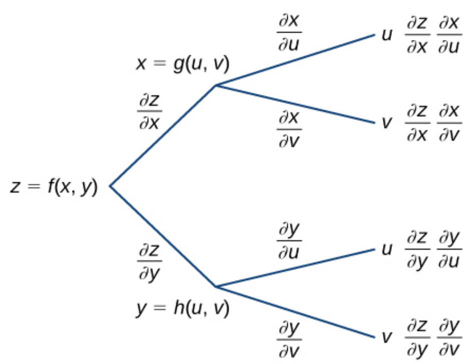


Figure 6.5.2: Tree diagram for $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$.

To derive the formula for $\partial z/\partial u$, start from the left side of the diagram, then follow only the branches that end with u and add the terms that appear at the end of those branches. For the formula for $\partial z/\partial v$, follow only the branches that end with v and add the terms that appear at the end of those branches.

There is an important difference between these two chain rule theorems. In Note, the left-hand side of the formula for the derivative is not a partial derivative, but in Note it is. The reason is that, in Note, z is ultimately a function of t alone, whereas in Note, z is a function of both u and v.

Example 6.5.2: Using the Chain Rule for Two Variables

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ using the following functions:

$$z = f(x, y) = 3x^2 - 2xy + y^2,\ x = x(u, v) = 3u + 2v,\ y = y(u, v) = 4u - v. \tag{6.5.8}$$

Solution

To implement the chain rule for two variables, we need six partial derivatives—$\partial z/\partial x$, $\partial z/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$:

$$\begin{aligned} \frac{\partial z}{\partial x} &= 6x - 2y & \frac{\partial z}{\partial y} &= -2x + 2y \\ \frac{\partial x}{\partial u} &= 3 & \frac{\partial x}{\partial v} &= 2 \\ \frac{\partial y}{\partial u} &= 4 & \frac{\partial y}{\partial v} &= -1. \end{aligned}$$

To find $\partial z/\partial u$, we use Equation:

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \\ &= 3(6x - 2y) + 4(-2x + 2y) \\ &= 10x + 2y. \end{aligned}$$

Next, we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\begin{aligned} \frac{\partial z}{\partial u} &= 10x + 2y \\ &= 10(3u + 2v) + 2(4u - v) \\ &= 38u + 18v. \end{aligned}$$

To find $\partial z/\partial v$, we use Equation:

$$\begin{aligned} \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \\ &= 2(6x - 2y) + (-1)(-2x + 2y) \\ &= 14x - 6y. \end{aligned}$$

Then we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\begin{aligned} \frac{\partial z}{\partial v} &= 14x - 6y \\ &= 14(3u + 2v) - 6(4u - v) \\ &= 18u + 34v \end{aligned}$$

Exercise 6.5.2

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x, y) = \frac{2x - y}{x + 3y},\ x(u, v) = e^{2u} \cos 3v,\ y(u, v) = e^{2u} \sin 3v. \tag{6.5.9}$$

Hint

Calculate $\partial z/\partial x,\ \partial z/\partial y,\ \partial x/\partial u,\ \partial x/\partial v,\ \partial y/\partial u,$ and $\partial y/\partial v$, then use Equation and Equation.

Answer

$\frac{\partial z}{\partial u} = 0, \frac{\partial z}{\partial v} = \frac{-21}{(3 \sin 3v + \cos 3v)^2}$

6.5.2 The Generalized Chain Rule

Now that we've see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the *generalized chain rule* states.

Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m independent variables, and for each $i \in 1, \ldots, m,$ let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j} \tag{6.5.10}$$

for any $j \in 1, 2, \ldots, n$.

In the next example we calculate the derivative of a function of three independent variables in which each of the three variables is dependent on two other variables.

Example 6.5.3: Using the Generalized Chain Rule

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ using the following functions:

$$\begin{aligned} w &= f(x,y,z) = 3x^2 - 2xy + 4z^2 \\ x &= x(u,v) = e^u \sin v \\ y &= y(u,v) = e^u \cos v \\ z &= z(u,v) = e^u. \end{aligned}$$

Solution

The formulas for $\partial w/\partial u$ and $\partial w/\partial v$ are

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}. \end{aligned}$$

Therefore, there are nine different partial derivatives that need to be calculated and substituted. We need to calculate each of them:

$$\begin{array}{lll} \frac{\partial w}{\partial x} = 6x - 2y & \frac{\partial w}{\partial y} = -2x & \frac{\partial w}{\partial z} = 8z \\ \frac{\partial x}{\partial u} = e^u \sin v & \frac{\partial y}{\partial u} = e^u \cos v & \frac{\partial z}{\partial u} = e^u \\ dfrac\partial x\partial v = e^u \cos v & \frac{\partial y}{\partial v} = -e^u \sin v & \frac{\partial z}{\partial v} = 0. \end{array}$$

Now, we substitute each of them into the first formula to calculate $\partial w/\partial u$:

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \\ &= (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u, \end{aligned}$$

then substitute $x(u,v) = e^u \sin v, y(u,v) = e^u \cos v,$ and $z(u,v) = e^u$ into this equation:

$$\begin{aligned}\frac{\partial w}{\partial u} &= (6x-2y)e^u\sin v - 2xe^u\cos v + 8ze^u \\ &= (6e^u\sin v - 2eu\cos v)e^u\sin v - 2(e^u\sin v)e^u\cos v + 8e^{2u} \\ &= 6e^{2u}\sin^2 v - 4e^{2u}\sin v\cos v + 8e^{2u} \\ &= 2e^{2u}(3\sin^2 v - 2\sin v\cos v + 4).\end{aligned}$$

Next, we calculate $\partial w/\partial v$:

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v} \\ &= (6x-2y)e^u\cos v - 2x(-e^u\sin v) + 8z(0),\end{aligned}$$

then we substitute $x(u,v)=e^u\sin v, y(u,v)=e^u\cos v,\;$ and $z(u,v)=e^u\;$ into this equation:

$$\begin{aligned}\frac{\partial w}{\partial v} &= (6x-2y)e^u\cos v - 2x(-e^u\sin v) \\ &= (6e^u\sin v - 2e^u\cos v)e^u\cos v + 2(e^u\sin v)(e^u\sin v) \\ &= 2e^{2u}\sin^2 v + 6e^{2u}\sin v\cos v - 2e^{2u}\cos^2 v \\ &= 2e^{2u}(\sin^2 v + \sin v\cos v - \cos^2 v).\end{aligned}$$

Exercise 6.5.3

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ given the following functions:

$$\begin{aligned}w &= f(x,y,z) = \frac{x+2y-4z}{2x-y+3z} \\ x &= x(u,v) = e^{2u}\cos 3v \\ y &= y(u,v) = e^{2u}\sin 3v \\ z &= z(u,v) = e^{2u}.\end{aligned}$$

Hint

Calculate nine partial derivatives, then use the same formulas from Example.

Answer

$$\frac{\partial w}{\partial u} = 0$$
$$\frac{\partial w}{\partial v} = \frac{15 - 33\sin 3v + 6\cos 3v}{(3+2\cos 3v - \sin 3v)^2}$$

Example 6.5.4: Drawing a Tree Diagram

Create a tree diagram for the case when

$$w = f(x,y,z), x = x(t,u,v), y = y(t,u,v), z = z(t,u,v)$$

and write out the formulas for the three partial derivatives of w.

Solution

Starting from the left, the function f has three independent variables: x, y, and z. Therefore, three branches must be emanating from the first node. Each of these three branches also has three branches, for each of the variables t, u, and v.

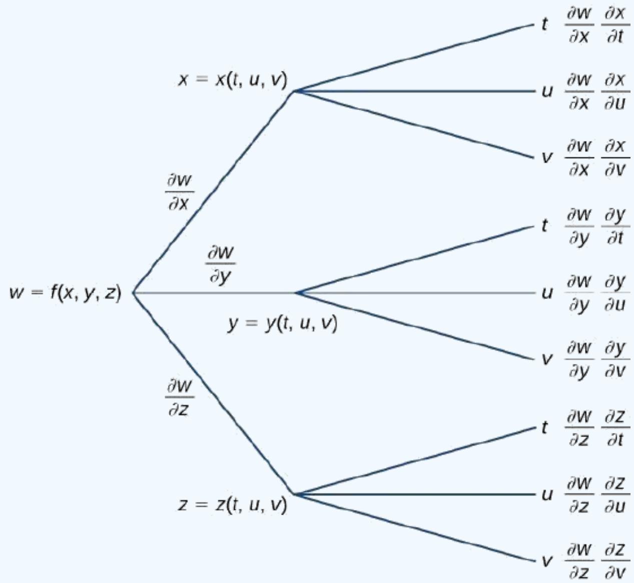


Figure $6.5.3$: Tree diagram for a function of three variables, each of which is a function of three independent variables.

The three formulas are

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\ \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}. \end{aligned}$$

Exercise $6.5.4$

Create a tree diagram for the case when

$$w = f(x, y), x = x(t, u, v), y = y(t, u, v)$$

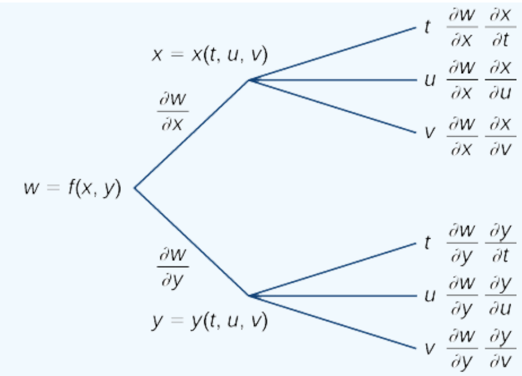
and write out the formulas for the three partial derivatives of w.

Hint

Determine the number of branches that emanate from each node in the tree.

Answer

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} \\ \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} \end{aligned}$$



6.5.3 Implicit Differentiation

Recall from implicit differentiation provides a method for finding dy/dx when y is defined implicitly as a function of x. The method involves differentiating both sides of the equation defining the function with respect to x, then solving for dy/dx. Partial derivatives provide an alternative to this method.

Consider the ellipse defined by the equation $x^2+3y^2+4y-4=0$ as follows.

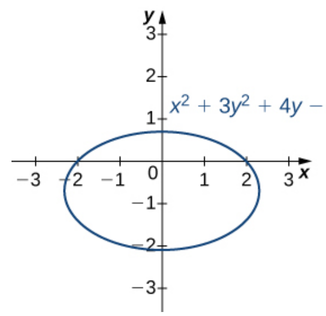


Figure 6.5.4: Graph of the ellipse defined by $x^2+3y^2+4y-4=0$.

This equation implicitly defines y as a function of x. As such, we can find the derivative dy/dx using the method of implicit differentiation:

$$\begin{aligned}\frac{d}{dx}(x^2+3y^2+4y-4) &= \frac{d}{dx}(0)\\ 2x+6y\frac{dy}{dx}+4\frac{dy}{dx} &= 0\\ (6y+4)\frac{dy}{dx} &= -2x\\ \frac{dy}{dx} &= -\frac{x}{3y+2}\end{aligned}$$

We can also define a function $z=f(x,y)$ by using the left-hand side of the equation defining the ellipse. Then $f(x,y)=x^2+3y^2+4y-4.$ The ellipse $x^2+3y^2+4y-4=0$ can then be described by the equation $f(x,y)=0$. Using this function and the following theorem gives us an alternative approach to calculating dy/dx.

Theorem: Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z=f(x,y)$ defines y implicitly as a function $y=g(x)$ of x via the equation $f(x,y)=0.$ Then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \tag{6.5.11}$$

provided $f_y(x,y) \neq 0$.

If the equation $f(x,y,z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{dz}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial z} \text{ and } \frac{dz}{dy} = -\frac{\partial f/\partial y}{\partial f/\partial z} \tag{6.5.12}$$

as long as $f_z(x,y,z) \neq 0$.

Equation is a direct consequence of Equation. In particular, if we assume that y is defined implicitly as a function of x via the equation $f(x,y) = 0$, we can apply the chain rule to find dy/dx :

$$\begin{aligned} \frac{d}{dx} f(x,y) &= \frac{d}{dx}(0) \\ \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} &= 0 \\ \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} &= 0. \end{aligned}$$

Solving this equation for dy/dx gives Equation. Equation can be derived in a similar fashion.

Let's now return to the problem that we started before the previous theorem. Using Note and the function $f(x,y) = x^2 + 3y^2 + 4y - 4,$ we obtain

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x \\ \frac{\partial f}{\partial y} &= 6y + 4. \end{aligned}$$

Then Equation gives

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x}{6y+4} = -\frac{x}{3y+2}, \tag{6.5.13}$$

which is the same result obtained by the earlier use of implicit differentiation.

Example 6.5.5: Implicit Differentiation by Partial Derivatives

a. Calculate dy/dx if y is defined implicitly as a function of x via the equation $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$. What is the equation of the tangent line to the graph of this curve at point $(2,1)$?
b. Calculate $\partial z/\partial x$ and $\partial z/\partial y$, given $x^2 e^y - yze^x = 0$.

Solution

a. Set $f(x,y) = 3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0,$ then calculate f_x and f_y : $f_x = 6x - 2y + 4$ $f_y = -2x + 2y - 6.$

The derivative is given by

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{6x-2y+4}{-2x+2y-6} = \frac{3x-y+2}{x-y+3}. \tag{6.5.14}$$

The slope of the tangent line at point $(2,1)$ is given by

$$\frac{dy}{dx}\ |_{(x,y)=(2,1)} = \frac{3(2)-1+2}{2-1+3} = \frac{7}{4} \tag{6.5.15}$$

To find the equation of the tangent line, we use the point-slope form (Figure):

$$\begin{aligned} y-y_0 &= m(x-x_0) \\ y-1 &= \frac{7}{4}(x-2) \\ y &= \frac{7}{4}x-\frac{7}{2}+1 \\ y &= \frac{7}{4}x-\frac{5}{2}. \end{aligned}$$

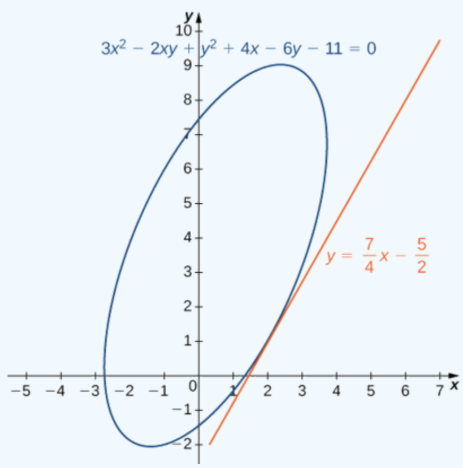


Figure 6.5.5: Graph of the rotated ellipse defined by $3x^2-2xy+y^2+4x-6y-11=0$.

b. We have $f(x,y,z) = x^2e^y - yze^x$. Therefore,

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xe^y - yze^x \\ \frac{\partial f}{\partial y} &= x^2e^y - ze^x \\ \frac{\partial f}{\partial z} &= -ye^x \end{aligned}$$

Using Equation,

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{\partial f/\partial x}{\partial f/\partial y} \\ &= -\frac{2xe^y - yze^x}{-ye^x} \\ &= \frac{2xe^y - yze^x}{ye^x} \end{aligned} \quad \text{and} \quad \begin{aligned} \frac{\partial z}{\partial y} &= -\frac{\partial f/\partial y}{\partial f/\partial z} \\ &= -\frac{x^2e^y - ze^x}{-ye^x} \\ &= \frac{x^2e^y - ze^x}{ye^x} \end{aligned}$$

Exercise 6.5.5

Find dy/dx if y is defined implicitly as a function of x by the equation $x^2+xy-y^2+7x-3y-26=0$. What is the equation of the tangent line to the graph of this curve at point $(3,-2)$?

Hint

Calculate $\partial f/dx$ and $\partial f/dy$, then use Equation.

Solution

Equation of the tangent line: $y=-\dfrac{11}{4}x+\dfrac{25}{4}$

6.5.4 Key Concepts

- The chain rule for functions of more than one variable involves the partial derivatives with respect to all the independent variables.
- Tree diagrams are useful for deriving formulas for the chain rule for functions of more than one variable, where each independent variable also depends on other variables.

6.5.5 Key Equations

- **Chain rule, one independent variable**

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

- **Chain rule, two independent variables**

$$\frac{dz}{du}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\,\frac{dz}{dv}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

- **Generalized chain rule**

$$\frac{\partial w}{\partial t_j}=\frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j}+\frac{\partial w}{\partial x_2}\frac{\partial x_1}{\partial t_j}+\cdots+\frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

6.5.6 Glossary

generalized chain rule

the chain rule extended to functions of more than one independent variable, in which each independent variable may depend on one or more other variables

intermediate variable

given a composition of functions (e.g., $f(x(t),y(t))$) , the intermediate variables are the variables that are independent in the outer function but dependent on other variables as well; in the function $f(x(t),y(t))$, the variables x and y are examples of intermediate variables

tree diagram

illustrates and derives formulas for the generalized chain rule, in which each independent variable is accounted for

6.5.7 Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

6.5E: Exercises

6.5E.1 Exercise $6.5E.1$

For the following exercises, use the information provided to solve the problem.

1) Let $w(x, y, z) = xycosz$, where $x = t, y = t^2,$ and $z = arcsint.$ Find $\frac{dw}{dt}$.

Answer

$$\frac{dw}{dt} = ycosz + xcosz(2t) - \frac{xysinz}{\sqrt{1-t^2}}$$

2) Let $w(t, v) = e^{tv}$ where $t = r + s$ and $v = rs$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.

3) If $w = 5x^2 + 2y^2, x = -3s + t,$ and $y = s - 4t,$ find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

Answer

$$\frac{\partial w}{\partial s} = -30x + 4y, \frac{\partial w}{\partial t} = 10x - 16y$$

4) If $w = xy^2, x = 5cos(2t),$ and $y = 5sin(2t)$, find $\frac{\partial w}{\partial t}$.

5) If $f(x, y) = xy, x = rcos\theta,$ and $y = rsin\theta$, find ∂f∂r and express the answer in terms of r and θ.

Answer

$$\frac{\partial f}{\partial r} = rsin(2\theta)$$

6) Suppose $f(x, y) = x + y, u = e^x siny, x = t^2$ and $y = \pi t$, where $x = rcos\theta$ and $y = rsin\theta$. Find $\frac{\partial f}{\partial \theta}$.

6.5E.2 Exercise $6.5E.2$

For the following exercises, find $\frac{df}{dt}$ using the chain rule and direct substitution.

7) $f(x, y) = x^2 + y^2, x = t, y = t^2$

Answer

$$\frac{df}{dt} = 2t + 4t^3$$

8) $f(x, y) = \sqrt{x^2 + y^2}, y = t^2, x = t$

9) $f(x, y) = xy, x = 1 - \sqrt{t}, y = 1 + \sqrt{t}$

Answer

$$\frac{df}{dt} = -1$$

10) $f(x, y) = \frac{x}{y}, x = e^t, y = 2e^t$

11) $f(x, y) = ln(x + y), x = e^t, y = e^t$

Answer

$$\frac{df}{dt} = 1$$

12) $f(x, y) = x^4, x = t, y = t$

6.5E.3 Exercise $\mathbf{6.5E.3}$

13) Let $w(x, y, z) = x^2 + y^2 + z^2, x = cost, y = sint,$ and $z = e^t$. Express w as a function of t and find $\frac{dw}{dt}$ directly. Then, find $\frac{dw}{dt}$ using the chain rule.

Answer

$\frac{dw}{dt} = 2e^{2t}$ in both cases

14) Let $z = x^2y,$ where $x = t^2$ and $y = t^3$. Find $\frac{dz}{dt}$.

15) Let $u = e^x siny$, where $x = t^2$ and $y = \pi t$. Find $\frac{du}{dt}$ when $x = ln2$ and $y = \frac{\pi}{4}$.

Answer

$$2\sqrt{2}t + \sqrt{2}\pi = \frac{du}{dt}$$

6.5E.4 Exercise $\mathbf{6.5E.4}$

For the following exercises, find $\frac{dy}{dx}$ using partial derivatives.

16) $sin(6x) + tan(8y) + 5 = 0$

17) $x^3 + y^2x - 3 = 0$

Answer

$$\frac{dy}{dx} = -\frac{3x^2 + y^2}{2xy}$$

18) $sin(x + y) + cos(x - y) = 4$

19) $x^2 - 2xy + y^4 = 4$

Answer

$$\frac{dy}{dx} = \frac{y - x}{-x + 2y^3}$$

20) $xe^y + ye^x - 2x^2y = 0$

21) $x^{2/3} + y^{2/3} = a^{2/3}$

Answer

$$\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}$$

22) $xcos(xy) + ycosx = 2$

23) $e^{xy} + ye^y = 1$

Answer

$$\frac{dy}{dx} = -\frac{ye^{xy}}{xe^{xy} + e^{y}(1+y)}$$

24) $x^2y^3 + cosy = 0$

6.5E.5 Exercise $6.5E.5$

25) Find $\frac{dz}{dt}$ using the chain rule where $z = 3x^2y^3$, $x = t^4$, and $y = t^2$.

Answer

$$\frac{dz}{dt} = 42t^{13}$$

26) Let $z = 3cosx - sin(xy)$, $x = \frac{1}{t}$, and $y = 3t$. Find $\frac{dz}{dt}$.

27) Let $z = e^{1-xy}$, $x = t^{1/3}$, and $y = t^3$. Find $\frac{dz}{dt}$.

Answer

$$\frac{dz}{dt} = -\frac{10}{3}t^{7/3} \times e^{1-t^{10/3}}$$

28) Find $\frac{dz}{dt}$ by the chain rule where $z = cosh^2(xy)$, $x = \frac{1}{2}t$, and $y = e^t$.

29) Let $z = \frac{x}{y}$, $x = 2cosu$, and $y = 3sinv$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Answer

$$\frac{\partial z}{\partial u} = \frac{-2sinu3}{sinv} \text{ and } \frac{\partial z}{\partial v} = \frac{-2cosucosv3}{sin^2v}$$

30) Let $z = e^{x^2y}$, where $x = \sqrt{uv}$ and $y = \frac{1}{v}$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

31) If $z = xye^{x/y}$, $x = rcos\theta$, and $y = rsin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ when $r = 2$ and $\theta = \frac{\pi}{6}$.

Answer

$$\frac{\partial z}{\partial r} = \sqrt{3}e^{\sqrt{3}}, \frac{\partial z}{\partial \theta} = (2 - 4\sqrt{3})e^{\sqrt{3}}$$

32) Find $\frac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3$, $x = e^{rs^2}$, $y = ln(\frac{r+s}{t})$, and $z = rst^2$.

33) If $w = sin(xyz)$, $x = 1 - 3t$, $y = e^{1-t}$, and $z = 4t$, find $\frac{\partial w}{\partial t}$.

Answer

$$\frac{\partial w}{\partial t} = cos(xyz) \times yz \times (-3) - cos(xyz)xze^{1-t} + cos(xyz)xy \times 4$$

6.5E.6 Exercise $6.5E.6$

For the following exercises, use this information: A function $f(x,y)$ is said to be homogeneous of degree n if $f(tx,ty) = t^n f(x,y)$. For all **homogeneous functions** of degree n, the following equation is true: $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y)$.

Show that the given function is homogeneous and verify that $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)$.

34) $f(x,y)=3x^2+y^2$

35) $f(x,y)=\sqrt{x^2+y^2}$

Answer

$$f(tx,ty)=\sqrt{t^2x^2+t^2y^2}=t^1f(x,y),\ \frac{\partial f}{\partial y}=x\frac{1}{2}(x^2+y^2)^{-1/2}\times 2x+y\frac{1}{2}(x^2+y^2)^{-1/2}\times 2y=1f(x,y)$$

36) $f(x,y)=x^2y-2y^3$

6.5E.7 Exercise $\mathbf{6.5E.7}$

37) The volume of a right circular cylinder is given by $V(x,y)=\pi x^2y$, where x is the radius of the cylinder and y is the cylinder height. Suppose x and y are functions of t given by $x=\frac{1}{2}t$ and $y=\frac{1}{3}t$ so that x and y are both increasing with time. How fast is the volume increasing when $x=2$ and $y=5$?

Answer

$$\frac{34\pi}{3}$$

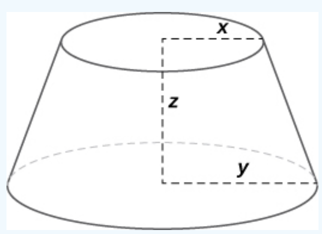
38) The pressure P of a gas is related to the volume and temperature by the formula $PV=kT$, where temperature is expressed in kelvins. Express the pressure of the gas as a function of both V and T. Find $\frac{dP}{dt}$ when $k=1$, $\frac{dV}{dt}=2$ cm^3/min, $\frac{dT}{dt}=12$ K/min, $V=20cm^3$, and $T=20^\circ F$.

39) The radius of a right circular cone is increasing at 3 cm/min whereas the height of the cone is decreasing at 2 cm/min. Find the rate of change of the volume of the cone when the radius is 13 cm and the height is 18 cm.

Answer

$$\frac{dV}{dt}=\frac{1066\pi}{3}cm^3/min$$

40) The volume of a frustum of a cone is given by the formula $V=\frac{1}{3}\pi z(x^2+y^2+xy)$, where x is the radius of the smaller circle, y is the radius of the larger circle, and z is the height of the frustum (see figure). Find the rate of change of the volume of this frustum when $x=10in.$, $y=12in.$, and $z=18in.$



41) A closed box is in the shape of a rectangular solid with dimensions x, y, and z. (Dimensions are in inches.) Suppose each dimension is changing at the rate of 0.5 in./min. Find the rate of change of the total surface area of the box when $x=2in.$, $y=3in.$, and $z=1in.$

Answer

$$\frac{dA}{dt}=12in.^2/min$$

42) The total resistance in a circuit that has three individual resistances represented by $x, y,$ and z is given by the formula $R(x,y,z) = \frac{xyz}{yz+xz+xy}$. Suppose at a given time the x resistance is 100Ω, the y resistance is 200Ω, and the z resistance is 300Ω. Also, suppose the x resistance is changing at a rate of $2\Omega/min$, the y resistance is changing at the rate of $1\Omega/min$, and the z resistance has no change. Find the rate of change of the total resistance in this circuit at this time.

43) The temperature T at a point (x,y) is $T(x,y)$ and is measured using the Celsius scale. A fly crawls so that its position after t seconds is given by $x=\sqrt{1+t}$ and $y=2+\frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2,3)=4$ and $T_y(2,3)=3$. How fast is the temperature increasing on the fly's path after 3 sec?

Answer

$2^\circ C/sec$

44) The x and y components of a fluid moving in two dimensions are given by the following functions: $u(x,y)=2y$ and $v(x,y)=-2x; x\geq 0; y\geq 0$. The speed of the fluid at the point (x,y) is $s(x,y)=\sqrt{u(x,y)^2+v(x,y)^2}$. Find $\frac{\partial s}{\partial x}$ and $\frac{\partial s}{\partial y}$ using the chain rule.

45) Let $u=u(x,y,z)$, where $x=x(w,t), y=y(w,t), z=z(w,t), w=w(r,s),$ and $t=t(r,s)$. Use a tree diagram and the chain rule to find an expression for $\frac{\partial u}{\partial r}$.

Answer

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}(\frac{\partial x}{\partial w}\frac{\partial w}{\partial r}+\frac{\partial x}{\partial t}\frac{\partial t}{\partial r})+\frac{\partial u}{\partial y}(\frac{\partial y}{\partial w}\frac{\partial w}{\partial r}+\frac{\partial y}{\partial t}\frac{\partial t}{\partial r})+\frac{\partial u}{\partial z}(\frac{\partial z}{\partial w}\frac{\partial w}{\partial r}+\frac{\partial z}{\partial t}\frac{\partial t}{\partial r})$$

6.6: Directional Derivatives and the Gradient

This page is a draft and is under active development.

A function $z = f(x,y)$ has two partial derivatives: $\partial z/\partial x$ and $\partial z/\partial y$. These derivatives correspond to each of the independent variables and can be interpreted as instantaneous rates of change (that is, as slopes of a tangent line). For example, $\partial z/\partial x$ represents the slope of a tangent line passing through a given point on the surface defined by $z = f(x,y)$, assuming the tangent line is parallel to the $\boldsymbol{x}$**-axis**. Similarly, $\partial z/\partial y$ represents the slope of the tangent line parallel to the $\boldsymbol{y}$**-axis.** Now we consider the possibility of a tangent line parallel to neither axis.

6.6.1 Directional Derivatives

We start with the graph of a surface defined by the equation $z = f(x,y)$. Given a point (a,b) in the domain of f, we choose a direction to travel from that point. We measure the direction using an angle θ, which is measured counterclockwise in the xy-plane, starting at zero from the positive x-axis (Figure 6.6.1). The distance we travel is h and the direction we travel is given by the unit vector $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Therefore, the z-coordinate of the second point on the graph is given by $z = f(a + h\cos\theta, b + h\sin\theta)$.

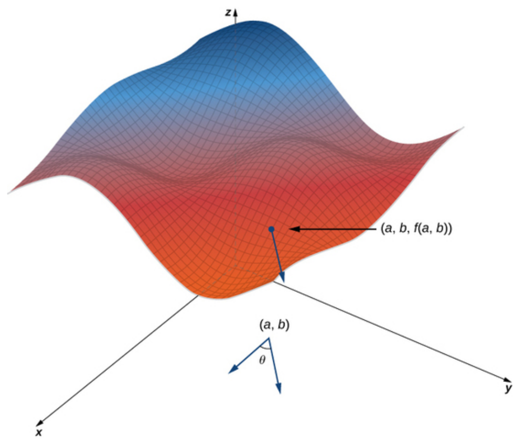


Figure 6.6.1: Finding the directional derivative at a point on the graph of $z = f(x,y)$. The slope of the black arrow on the graph indicates the value of the directional derivative at that point.

We can calculate the slope of the secant line by dividing the difference in $\boldsymbol{z}$**-values** by the length of the line segment connecting the two points in the domain. The length of the line segment is h. Therefore, the slope of the secant line is

$$m_{sec} = \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h} \tag{6.6.1}$$

To find the slope of the tangent line in the same direction, we take the limit as h approaches zero.

Definition: Directional Derivatives

Suppose $z = f(x,y)$ is a function of two variables with a domain of D. Let $(a,b) \in D$ and define $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Then the directional derivative of f in the direction of $\vec{\mathbf{u}}$ is given by

$$D_{\vec{\mathbf{u}}} f(a,b) = \lim_{h\to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h} \tag{6.6.2}$$

provided the limit exists.

Equation 6.6.2 provides a formal definition of the directional derivative that can be used in many cases to calculate a directional derivative.

Note that since the point (a,b) is chosen randomly from the domain D of the function f, we can use this definition to find the directional derivative as a function of x and y.

That is,

$$D_{\vec{\mathbf{u}}} f(x,y) = \lim_{h\to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h} \tag{6.6.3}$$

Example 6.6.1: Finding a Directional Derivative from the Definition

Let $\theta = \arccos(3/5)$. Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = x^2 - xy + 3y^2$ in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$.

Then determine $D_{\vec{\mathbf{u}}} f(-1,2)$.

Solution

First of all, since $\cos\theta = 3/5$ and θ is acute, this implies

$$\sin\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Using $f(x,y) = x^2 - xy + 3y^2$, we first calculate $f(x + h\cos\theta, y + h\sin\theta)$:

$$\begin{aligned} f(x + h\cos\theta, y + h\sin\theta) &= (x + h\cos\theta)^2 - (x + h\cos\theta)(y + h\sin\theta) + 3(y + h\sin\theta)^2 \\ &= x^2 + 2xh\cos\theta + h^2\cos^2\theta - xy - xh\sin\theta - yh\cos\theta - h^2\sin\theta\cos\theta + 3y^2 + 6yh\sin\theta + 3h^2\sin^2\theta \\ &= x^2 + 2xh(\frac{3}{5}) + \frac{9h^2}{25} - xy - \frac{4xh}{5} - \frac{3yh}{5} - \frac{12h^2}{25} + 3y^2 + 6yh(\frac{4}{5}) + 3h^2(\frac{16}{25}) \\ &= x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}. \end{aligned}$$

We substitute this expression into Equation 6.6.2 with $a = x$ and $b = y$:

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= \lim_{h\to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}) - (x^2 - xy + 3y^2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}}{h} \\ &= \lim_{h\to 0} \frac{2x}{5} + \frac{9h}{5} + \frac{21y}{5} \\ &= \frac{2x + 21y}{5}. \end{aligned}$$

To calculate $D_{\vec{\mathbf{u}}} f(-1,2)$, we substitute $x = -1$ and $y = 2$ into this answer (Figure 6.6.2):

$$D_{\vec{\mathbf{u}}} f(-1,2) = \frac{2(-1) + 21(2)}{5} = \frac{-2 + 42}{5} = 8.$$

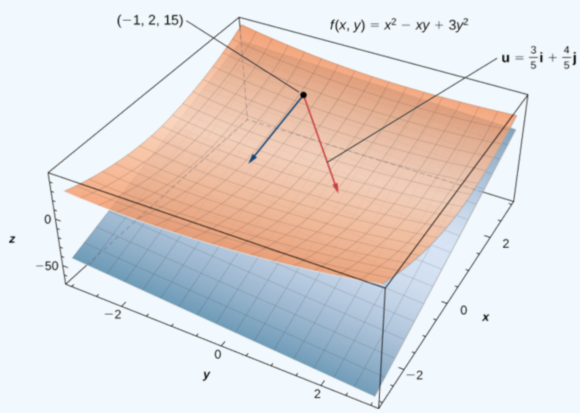


Figure 6.6.2: Finding the directional derivative in a given direction $\vec{\mathbf{u}}$ at a given point on a surface. The plane is tangent to the surface at the given point $(-1,2,15)$.

An easier approach to calculating directional derivatives that involves partial derivatives is outlined in the following theorem.

Directional Derivative of a Function of Two Variables

Let $z = f(x,y)$ be a function of two variables x and y, and assume that f_x and f_y exist. Then the directional derivative of f in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$ is given by

$$D_{\vec{\mathbf{u}}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta. \tag{6.6.4}$$

Proof

Applying the definition of a directional derivative stated above in Equation 6.6.2, the directional derivative of f in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$ at a point (x_0,y_0) in the domain of f can be written

$$D_{\vec{\mathbf{u}}} f((x_0,y_0)) = \lim_{t\to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0,y_0)}{t}. \tag{6.6.5}$$

Let $x = x_0 + t\cos\theta$ and $y = y_0 + t\sin\theta$, and define $g(t) = f(x,y)$. Since f_x and f_y both exist, we can use the chain rule for functions of two variables to calculate $g'(t)$:

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta. \tag{6.6.6}$$

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0,y_0)\cos\theta + f_y(x_0,y_0)\sin\theta \tag{6.6.7}$$

By the definition of $g'(t)$, it is also true that

$$g'(0) = \lim_{t\to 0}\frac{g(t) - g(0)}{t} = \lim_{t\to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0,y_0)}{t}. \tag{6.6.8}$$

Therefore, $D_{\vec{\mathbf{u}}} f(x_0,y_0) = f_x(x_0,y_0)\cos\theta + f_y(x_0,y_0)\sin\theta$.

Since the point (x_0,y_0) is an arbitrary point from the domain of f, this result holds for all points in the domain of f for which the partials f_x and f_y exist.

Therefore,

$$D_{\vec{\mathbf{u}}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta. \tag{6.6.9}$$

□

Example 6.6.2: Finding a Directional Derivative: Alternative Method

Let $\theta = \arccos(3/5)$. Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = x^2 - xy + 3y^2$ in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$.

Then determine $D_{\vec{\mathbf{u}}} f(-1,2)$.

Solution

First, we must calculate the partial derivatives of f:

$$\begin{aligned} f_x(x,y) &= 2x - y \\ f_y(x,y) &= -x + 6y, \end{aligned}$$

Then we use Equation 6.6.4 with $\theta = \arccos(3/5)$:

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= (2x - y)\frac{3}{5} + (-x + 6y)\frac{4}{5} \\ &= \frac{6x}{5} - \frac{3y}{5} - \frac{4x}{5} + \frac{24y}{5} \\ &= \frac{2x + 21y}{5}. \end{aligned}$$

To calculate $D_{\vec{\mathbf{u}}} f(-1,2)$, let $x = -1$ and $y = 2$:

$$D_{\vec{\mathbf{u}}} f(-1,2) = \frac{2(-1) + 21(2)}{5} = \frac{-2 + 42}{5} = 8.$$

This is the same answer obtained in Example 6.6.1.

Exercise 6.6.1:

Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y)=3x^2y-4xy^3+3y^2-4x$ in the direction of $\vec{\mathbf{u}}=(\cos\frac{\pi}{3})\,\hat{\mathbf{i}}+(\sin\frac{\pi}{3})\,\hat{\mathbf{j}}$ using Equation 6.6.4.

What is $D_{\vec{\mathbf{u}}} f(3,4)$?

Hint

Calculate the partial derivatives and determine the value of θ.

Answer

$$D_{\vec{\mathbf{u}}} f(x,y)=\frac{(6xy-4y^3-4)(1)}{2}+\frac{(3x^2-12xy^2+6y)\sqrt{3}}{2}$$

$$D_{\vec{\mathbf{u}}} f(3,4)=\frac{72-256-4}{2}+\frac{(27-576+24)\sqrt{3}}{2}=-94-\frac{525\sqrt{3}}{2}$$

If the vector that is given for the direction of the derivative is not a unit vector, then it is only necessary to divide by the norm of the vector. For example, if we wished to find the directional derivative of the function in Example 6.6.2 in the direction of the vector $\langle -5,12\rangle$, we would first divide by its magnitude to get $\vec{\mathbf{u}}$. This gives us $\vec{\mathbf{u}}=\langle -\frac{5}{13},\frac{12}{13}\rangle$.

Then

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= f_x(x,y)\cos\theta+f_y(x,y)\sin\theta \\ &= -\frac{5}{13}(2x-y)+\frac{12}{13}(-x+6y) \\ &= -\frac{22}{13}x+\frac{17}{13}y \end{aligned}$$

6.6.2 Gradient

The right-hand side of Equation 6.6.4 is equal to $f_x(x,y)\cos\theta+f_y(x,y)\sin\theta$, which can be written as the dot product of two vectors. Define the first vector as $\vec{\nabla} f(x,y)=f_x(x,y)\,\hat{\mathbf{i}}+f_y(x,y)\,\hat{\mathbf{j}}$ and the second vector as $\vec{\mathbf{u}}=(\cos\theta)\,\hat{\mathbf{i}}+(\sin\theta)\,\hat{\mathbf{j}}$. Then the right-hand side of the equation can be written as the dot product of these two vectors:

$$D_{\vec{\mathbf{u}}} f(x,y)=\vec{\nabla} f(x,y)\cdot\vec{\mathbf{u}}. \tag{6.6.10}$$

The first vector in Equation 6.6.10 has a special name: the gradient of the function f. The symbol ∇ is called **nabla** and the vector $\vec{\nabla} f$ is read "**del** f."

Definition: The Gradient

Let $z=f(x,y)$ be a function of x and y such that f_x and f_y exist. The vector $\vec{\nabla} f(x,y)$ is called the **gradient** of f and is defined as

$$\vec{\nabla} f(x,y)=f_x(x,y)\,\hat{\mathbf{i}}+f_y(x,y)\,\hat{\mathbf{j}}. \tag{6.6.11}$$

The vector $\vec{\nabla} f(x,y)$ is also written as "**grad** f."

Example 6.6.3: Finding Gradients

Find the gradient $\vec{\nabla} f(x,y)$ of each of the following functions:

a. $f(x,y)=x^2-xy+3y^2$
b. $f(x,y)=\sin 3x\cos 3y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x and f_y, then use Equation 6.6.11.

a. $f_x(x,y)=2x-y$ and $f_y(x,y)=-x+6y$, so

$$\begin{aligned} \vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}}+f_y(x,y)\,\hat{\mathbf{j}} \\ &= (2x-y)\,\hat{\mathbf{i}}+(-x+6y)\,\hat{\mathbf{j}}. \end{aligned}$$

b. $f_x(x,y) = 3\cos 3x \cos 3y$ and $f_y(x,y) = -3\sin 3x \sin 3y$, so

$$\begin{aligned}\vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} \\ &= (3\cos 3x \cos 3y)\,\hat{\mathbf{i}} - (3\sin 3x \sin 3y)\,\hat{\mathbf{j}}.\end{aligned}$$

Exercise 6.6.2

Find the gradient $\vec{\nabla} f(x,y)$ of $f(x,y) = \dfrac{x^2-3y^2}{2x+y}$.

Hint

Calculate the partial derivatives, then use Equation 6.6.11.

Answer

$$\vec{\nabla} f(x,y) = \frac{2x^2+2xy+6y^2}{(2x+y)^2}\,\hat{\mathbf{i}} - \frac{x^2+12xy+3y^2}{(2x+y)^2}\,\hat{\mathbf{j}}$$

The gradient has some important properties. We have already seen one formula that uses the gradient: the formula for the directional derivative. Recall from The Dot Product that if the angle between two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is φ, then $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\| \cos\varphi$. Therefore, if the angle between $\vec{\nabla} f(x_0,y_0)$ and $\vec{\mathbf{u}} = (cos\theta)\,\hat{\mathbf{i}} + (sin\theta)\,\hat{\mathbf{j}}$ is φ, we have

$$D_{\vec{\mathbf{u}}} f(x_0,y_0) = \vec{\nabla} f(x_0,y_0) \cdot \vec{\mathbf{u}} = \|\vec{\nabla} f(x_0,y_0)\|\|\vec{\mathbf{u}}\| \cos\varphi = \|\vec{\nabla} f(x_0,y_0)\| \cos\varphi. \tag{6.6.12}$$

The $\|\vec{\mathbf{u}}\|$ disappears because $\vec{\mathbf{u}}$ is a unit vector. Therefore, the directional derivative is equal to the magnitude of the gradient evaluated at (x_0,y_0) multiplied by $\cos\varphi$. Recall that $\cos\varphi$ ranges from -1 to 1.

If $\varphi = 0$, then $\cos\varphi = 1$ and $\vec{\nabla} f(x_0,y_0)$ and $\vec{\mathbf{u}}$ both point in the same direction.

If $\varphi = \pi$, then $\cos\varphi = -1$ and $\vec{\nabla} f(x_0,y_0)$ and $\vec{\mathbf{u}}$ point in opposite directions.

In the first case, the value of $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is maximized; in the second case, the value of $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is minimized.

We can also see that if $\vec{\nabla} f(x_0,y_0) = \vec{\mathbf{0}}$, then

$$D_{\vec{\mathbf{u}}} f(x_0,y_0) = \vec{\nabla} f(x_0,y_0) \cdot \vec{\mathbf{u}} = 0 \tag{6.6.13}$$

for any vector $\vec{\mathbf{u}}$. These three cases are outlined in the following theorem.

Properties of the Gradient

Suppose the function $z = f(x,y)$ is differentiable at (x_0,y_0) (Figure 6.6.3).

i. If $\vec{\nabla} f(x_0,y_0) = \vec{\mathbf{0}}$, then $D_{\vec{\mathbf{u}}} f(x_0,y_0) = 0$ for any unit vector $\vec{\mathbf{u}}$.
ii. If $\vec{\nabla} f(x_0,y_0) \neq 0$, then $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is **maximized** when $\vec{\mathbf{u}}$ points in the same direction as $\vec{\nabla} f(x_0,y_0)$. The maximum value of $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is $\|\vec{\nabla} f(x_0,y_0)\|$.
iii. If $\vec{\nabla} f(x_0,y_0) \neq 0$, then $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is **minimized** when $\vec{\mathbf{u}}$ points in the opposite direction from $\vec{\nabla} f(x_0,y_0)$. The minimum value of $D_{\vec{\mathbf{u}}} f(x_0,y_0)$ is $-\|\vec{\nabla} f(x_0,y_0)\|$.

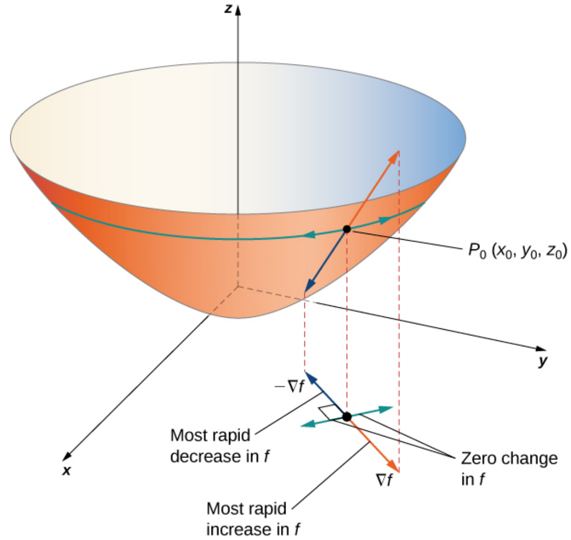


Figure 6.6.3: The gradient indicates the maximum and minimum values of the directional derivative at a point.

Example 6.6.4: Finding a Maximum Directional Derivative

Find the direction for which the directional derivative of $f(x, y) = 3x^2 - 4xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Solution:

The maximum value of the directional derivative occurs when $\vec{\nabla} f$ and the unit vector point in the same direction. Therefore, we start by calculating $\vec{\nabla} f(x, y)$:

$$f_x(x, y) = 6x - 4y \text{ and } f_y(x, y) = -4x + 4y$$

so

$$\vec{\nabla} f(x, y) = f_x(x, y)\,\hat{\mathbf{i}} + f_y(x, y)\,\hat{\mathbf{j}} = (6x - 4y)\,\hat{\mathbf{i}} + (-4x + 4y)\,\hat{\mathbf{j}}.$$

Next, we evaluate the gradient at $(-2, 3)$:

$$\vec{\nabla} f(-2, 3) = (6(-2) - 4(3))\,\hat{\mathbf{i}} + (-4(-2) + 4(3))\,\hat{\mathbf{j}} = -24\,\hat{\mathbf{i}} + 20\,\hat{\mathbf{j}}.$$

We need to find a unit vector that points in the same direction as $\vec{\nabla} f(-2, 3)$, so the next step is to divide $\vec{\nabla} f(-2, 3)$ by its magnitude, which is $\sqrt{(-24)^2 + (20)^2} = \sqrt{976} = 4\sqrt{61}$. Therefore,

$$\frac{\vec{\nabla} f(-2, 3)}{\|\vec{\nabla} f(-2, 3)\|} = \frac{-24}{4\sqrt{61}} i + \frac{20}{4\sqrt{61}} j = -\frac{6\sqrt{61}}{61}\,\hat{\mathbf{i}} + \frac{5\sqrt{61}}{61}\,\hat{\mathbf{j}}.$$

This is the unit vector that points in the same direction as $\vec{\nabla} f(-2, 3)$. To find the angle corresponding to this unit vector, we solve the equations

$$\cos\theta = \frac{-6\sqrt{61}}{61} \text{ and } \sin\theta = \frac{5\sqrt{61}}{61}$$

for θ. Since cosine is negative and sine is positive, the angle must be in the second quadrant. Therefore, $\theta = \pi - \arcsin((5\sqrt{61})/61) \approx 2.45$ rad.

The maximum value of the directional derivative at $(-2, 3)$ is $\|\vec{\nabla} f(-2, 3)\| = 4\sqrt{61}$ (Figure 6.6.4).

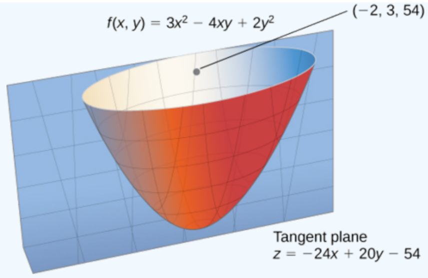


Figure 6.6.4: The maximum value of the directional derivative at $(-2, 3)$ is in the direction of the gradient.

Exercise 6.6.3

Find the direction for which the directional derivative of $g(x, y) = 4x - xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Hint

Evaluate the gradient of g at point $(-2, 3)$.

Answer

The gradient of g at $(-2, 3)$ is $\vec{\nabla} g(-2, 3) = \hat{\mathbf{i}} + 14\hat{\mathbf{j}}$. The unit vector that points in the same direction as $\vec{\nabla} g(-2, 3)$ is

$$\frac{\vec{\nabla} g(-2,3)}{\|\vec{\nabla} g(-2,3)\|} = \frac{1}{\sqrt{197}}\hat{\mathbf{i}} + \frac{14}{\sqrt{197}}\hat{\mathbf{j}} = \frac{\sqrt{197}}{197}\hat{\mathbf{i}} + \frac{14\sqrt{197}}{197}\hat{\mathbf{j}},$$

which gives an angle of $\theta = \arcsin((14\sqrt{197})/197) \approx 1.499$ rad.

The maximum value of the directional derivative is $\|\vec{\nabla} g(-2, 3)\| = \sqrt{197}$.

Figure 6.6.5 shows a portion of the graph of the function $f(x, y) = 3 + \sin x \sin y$. Given a point (a, b) in the domain of f, the maximum value of the gradient at that point is given by $\|\vec{\nabla} f(a, b)\|$. This would equal the rate of greatest ascent if the surface represented a topographical map. If we went in the opposite direction, it would be the rate of greatest descent.

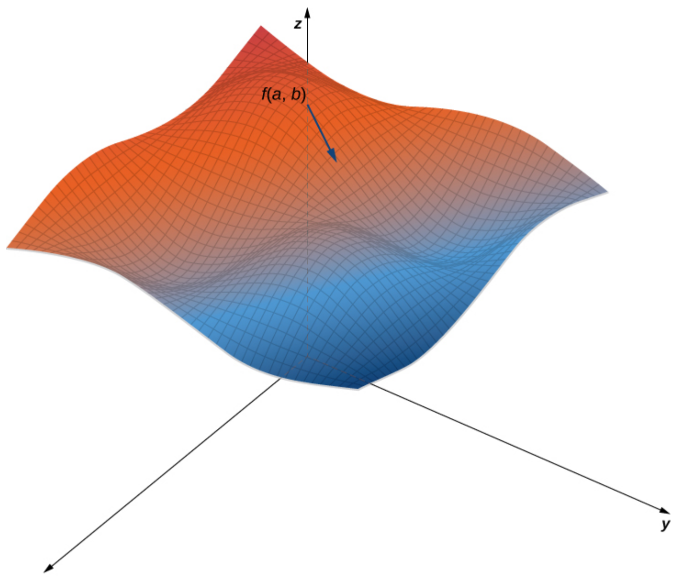


Figure 6.6.5: A typical surface in $\mathbb{R}^3$. Given a point on the surface, the directional derivative can be calculated using the gradient.

When using a topographical map, the steepest slope is always in the direction where the contour lines are closest together (Figure 6.6.6). This is analogous to the contour map of a function, assuming the level curves are obtained for equally spaced values throughout the range of that function.

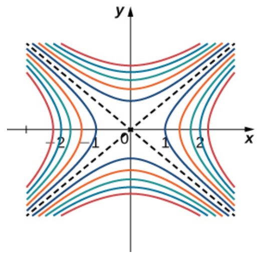


Figure 6.6.6: Contour map for the function $f(x,y)=x^2-y^2$ using level values between -5 and 5.

6.6.3 Gradients and Level Curves

Recall that if a curve is defined parametrically by the function pair $(x(t),y(t))$, then the vector $x'(t)\,\hat{\mathbf{i}}+y'(t)\,\hat{\mathbf{j}}$ is tangent to the curve for every value of t in the domain. Now let's assume $z=f(x,y)$ is a differentiable function of x and y, and (x_0,y_0) is in its domain. Let's suppose further that $x_0=x(t_0)$ and $y_0=y(t_0)$ for some value of t, and consider the level curve $f(x,y)=k.$ Define $g(t)=f(x(t),y(t))$ and calculate $g'(t)$ on the level curve. By the chain Rule,

$$g'(t)=f_x(x(t),y(t))x'(t)+f_y(x(t),y(t))y'(t). \tag{6.6.14}$$

But $g'(t)=0$ because $g(t)=k$ for all t. Therefore, on the one hand,

$$f_x(x(t),y(t))x'(t)+f_y(x(t),y(t))y'(t)=0; \tag{6.6.15}$$

on the other hand,

$$f_x(x(t),y(t))x'(t)+f_y(x(t),y(t))y'(t)=\vec{\nabla}f(x,y)\cdot\langle x'(t),y'(t)\rangle. \tag{6.6.16}$$

Therefore,

$$\vec{\nabla}f(x,y)\cdot\langle x'(t),y'(t)\rangle=0. \tag{6.6.17}$$

Thus, the dot product of these vectors is equal to zero, which implies they are orthogonal. However, the second vector is tangent to the level curve, which implies the gradient must be normal to the level curve, which gives rise to the following theorem.

Gradient Is Normal to the Level Curve

Suppose the function $z=f(x,y)$ has continuous first-order partial derivatives in an open disk centered at a point (x_0,y_0). If $\vec{\nabla}f(x_0,y_0)\neq 0$, then $\vec{\nabla}f(x_0,y_0)$ is normal to the level curve of f at (x_0,y_0).

We can use this theorem to find tangent and normal vectors to level curves of a function.

Example 6.6.5: Finding Tangents to Level Curves

For the function $f(x,y)=2x^2-3xy+8y^2+2x-4y+4,$ find a tangent vector to the level curve at point $(-2,1)$. Graph the level curve corresponding to $f(x,y)=18$ and draw in $\vec{\nabla}f(-2,1)$ and a tangent vector.

Solution:

First, we must calculate $\vec{\nabla}f(x,y)$:

$$f_x(x,y)=4x-3y+2 \text{ and } f_y=-3x+16y-4 \text{ so } \vec{\nabla}f(x,y)=(4x-3y+2)\,\hat{\mathbf{i}}+(-3x+16y-4)\,\hat{\mathbf{j}}.$$

Next, we evaluate $\vec{\nabla}f(x,y)$ at $(-2,1)$:

$$\vec{\nabla}f(-2,1)=(4(-2)-3(1)+2)\,\hat{\mathbf{i}}+(-3(-2)+16(1)-4)\,\hat{\mathbf{j}}=-9\,\hat{\mathbf{i}}+18\,\hat{\mathbf{j}}.$$

This vector is orthogonal to the curve at point $(-2,1)$. We can obtain a tangent vector by reversing the components and multiplying either one by -1. Thus, for example, $-18\,\hat{\mathbf{i}}-9\,\hat{\mathbf{j}}$ is a tangent vector (Figure 6.6.7).

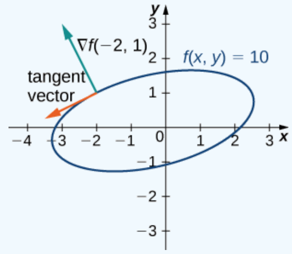


Figure 6.6.7: Tangent and normal vectors to $2x^2 - 3xy + 8y^2 + 2x - 4y + 4 = 18$ at point $(-2, 1)$.

Exercise 6.6.4

For the function $f(x, y) = x^2 - 2xy + 5y^2 + 3x - 2y + 4$, find the tangent to the level curve at point $(1, 1)$. Draw the graph of the level curve corresponding to $f(x, y) = 8$ and draw $\vec{\nabla} f(1, 1)$ and a tangent vector.

Hint

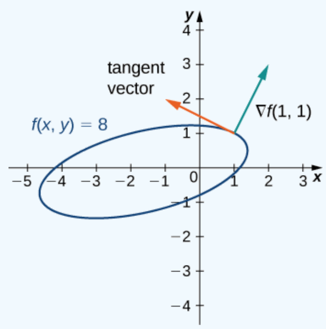
Calculate the gradient at point $(1, 1)$.

Answer

$\vec{\nabla} f(x, y) = (2x - 2y + 3)\,\hat{\mathbf{i}} + (-2x + 10y - 2)\,\hat{\mathbf{j}}$

$\vec{\nabla} f(1, 1) = 3\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}}$

Tangent vector: $6\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}}$ or $-6\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}}$



6.6.4 Three-Dimensional Gradients and Directional Derivatives

The definition of a gradient can be extended to functions of more than two variables.

Definition: Gradients in 3D

Let $w = f(x, y, z)$ be a function of three variables such that f_x, f_y, and f_z exist. The vector $\vec{\nabla} f(x, y, z)$ is called the gradient of f and is defined as

$$\vec{\nabla} f(x, y, z) = f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}}. \tag{6.6.18}$$

$\vec{\nabla} f(x, y, z)$ can also be written as **grad** $f(x, y, z)$.

Calculating the gradient of a function in three variables is very similar to calculating the gradient of a function in two variables. First, we calculate the partial derivatives f_x, f_y, and f_z, and then we use Equation 6.6.10.

Example 6.6.6: Finding Gradients in Three Dimensions

Find the gradient $\vec{\nabla} f(x, y, z)$ of each of the following functions:

a. $f(x, y, z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz$
b. $f(x, y, z) = e^{-2z} \sin 2x \cos 2y$

Solution:

For both parts a. and b., we first calculate the partial derivatives f_x, f_y, and f_z, then use Equation 6.6.10.

a. $f_z(x, y, z) = 10x - 2y + 3z$, $f_y(x, y, z) = -2x + 2y - 4z$, and $f_z(x, y, z) = 3x - 4y + 2z$, so

$$\begin{aligned}\vec{\nabla} f(x,y,z) &= f_x(x,y,z)\,\hat{\mathbf{i}} + f_y(x,y,z)\,\hat{\mathbf{j}} + f_z(x,y,z)\,\hat{\mathbf{k}} \\ &= (10x - 2y + 3z)\,\hat{\mathbf{i}} + (-2x + 2y - 4z)\,\hat{\mathbf{j}} + (-4x + 3y + 2z)\,\hat{\mathbf{k}}.\end{aligned}$$

b. $f_x(x,y,z) = -2e^{-2z}\cos 2x \cos 2y$, $f_y(x,y,z) = -2e^{-2z}\sin 2x \sin 2y$, and $f_z(x,y,z) = -2e^{-2z}\sin 2x \cos 2y$, so

$$\begin{aligned}\vec{\nabla} f(x,y,z) &= f_x(x,y,z)\,\hat{\mathbf{i}} + f_y(x,y,z)\,\hat{\mathbf{j}} + f_z(x,y,z)\,\hat{\mathbf{k}} \\ &= (2e^{-2z}\cos 2x \cos 2y)\,\hat{\mathbf{i}} + (-2e^{-2z})\,\hat{\mathbf{j}} + (-2e^{-2z})\,\hat{\mathbf{k}} \\ &= 2e^{-2z}(\cos 2x \cos 2y\,\hat{\mathbf{i}} - \sin 2x \sin 2y\,\hat{\mathbf{j}} - \sin 2x \cos 2y\,\hat{\mathbf{k}}).\end{aligned}$$

Exercise 6.6.5:

Find the gradient $\vec{\nabla} f(x,y,z)$ of $f(x,y,z) = \dfrac{x^2 - 3y^2 + z^2}{2x + y - 4z.}$

Answer

$$\vec{\nabla} f(x,y,z) = \frac{2x^2 + 2xy + 6y^2 - 8xz - 2z^2}{(2x+y-4z)^2}\,\hat{\mathbf{i}} - \frac{x^2 + 12xy + 3y^2 - 24yz + z^2}{(2x+y-4z)^2}\,\hat{\mathbf{j}} + \frac{4x^2 - 12y^2 - 4z^2 + 4xz + 2yz}{(2x+y-4z)^2}\,\hat{\mathbf{k}}$$

The directional derivative can also be generalized to functions of three variables. To determine a direction in three dimensions, a vector with three components is needed. This vector is a unit vector, and the components of the unit vector are called *directional cosines***.** Given a three-dimensional unit vector $\vec{\mathbf{u}}$ in standard form (i.e., the initial point is at the origin), this vector forms three different angles with the positive x-, y-, and z-axes. Let's call these angles $\alpha, \beta,$ and γ. Then the directional cosines are given by $\cos\alpha, \cos\beta,$ and $\cos\gamma$. These are the components of the unit vector $\vec{\mathbf{u}}$; since $\vec{\mathbf{u}}$ is a unit vector, it is true that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Definition: Directional Derivative of a Function of Three variables

Suppose $w = f(x,y,z)$ is a function of three variables with a domain of D. Let $(x_0, y_0, z_0) \in D$ and let $\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}$ be a unit vector. Then, the directional derivative of f in the direction of u is given by

$$D_{\vec{\mathbf{u}}} f(x_0, y_0, z_0) = \lim_{t \to 0} \frac{f(x_0 + t\cos\alpha, y_0 + t\cos\beta, z_0 + t\cos\gamma) - f(x_0, y_0, z_0)}{t} \tag{6.6.19}$$

provided the limit exists.

We can calculate the directional derivative of a function of three variables by using the gradient, leading to a formula that is analogous to Equation 6.6.4.

Directional Derivative of a Function of Three Variables

Let $f(x,y,z)$ be a differentiable function of three variables and let $\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}$ be a unit vector. Then, the directional derivative of f in the direction of $\vec{\mathbf{u}}$ is given by

$$D_{\vec{\mathbf{u}}} f(x,y,z) = \vec{\nabla} f(x,y,z) \cdot \vec{\mathbf{u}} = f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma. \tag{6.6.20}$$

The three angles $\alpha, \beta,$ and γ determine the unit vector $\vec{\mathbf{u}}$. In practice, we can use an arbitrary (nonunit) vector, then divide by its magnitude to obtain a unit vector in the desired direction.

Example 6.6.7: Finding a Directional Derivative in Three Dimensions

Calculate $D_{\vec{\mathbf{v}}} f(1, -2, 3)$ in the direction of $v = -\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$ for the function

$$f(x,y,z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz.$$

Solution:

First, we find the magnitude of v:

$$\|\vec{\mathbf{v}}\| = \sqrt{(-1)^2 + (2)^2} = 3.$$

Therefore, $\frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \frac{-\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}}{3} = -\frac{1}{3}\,\hat{\mathbf{i}} + \frac{2}{3}\,\hat{\mathbf{j}} + \frac{2}{3}\,\hat{\mathbf{k}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$, so $\cos\alpha = -\frac{1}{3}, \cos\beta = \frac{2}{3}$, and $\cos\gamma = \frac{2}{3}$. Next, we calculate the partial derivatives of f:

$$\begin{aligned} f_x(x,y,z) &= 10x - 2y + 3z \\ f_y(x,y,z) &= -2x + 2y - 4z \\ f_z(x,y,z) &= -4y + 2z + 3x, \end{aligned}$$

then substitute them into Equation 6.6.20:

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(x,y,z) &= f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma \\ &= (10x - 2y + 3z)(-\frac{1}{3}) + (-2x + 2y - 4z)(\frac{2}{3}) + (-4y + 2z + 3x)(\frac{2}{3}) \\ &= -\frac{10x}{3} + \frac{2y}{3} - \frac{3z}{3} - \frac{4x}{3} + \frac{4y}{3} - \frac{8z}{3} - \frac{8y}{3} + \frac{4z}{3} + \frac{6x}{3} \\ &= -\frac{8x}{3} - \frac{2y}{3} - \frac{7z}{3}. \end{aligned}$$

Last, to find $D_{\vec{\mathbf{v}}} f(1,-2,3)$, we substitute $x = 1$, $y = -2$, and $z = 3$:

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(1,-2,3) &= -\frac{8(1)}{3} - \frac{2(-2)}{3} - \frac{7(3)}{3} \\ &= -\frac{8}{3} + \frac{4}{3} - \frac{21}{3} \\ &= -\frac{25}{3}. \end{aligned}$$

Exercise 6.6.6:

Calculate $D_{\vec{\mathbf{v}}} f(x,y,z)$ and $D_{\vec{\mathbf{v}}} f(0,-2,5)$ in the direction of $\vec{\mathbf{v}} = -3\,\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$ for the function

$$f(x,y,z) = 3x^2 + xy - 2y^2 + 4yz - z^2 + 2xz.$$

Hint

First, divide $\vec{\mathbf{v}}$ by its magnitude, calculate the partial derivatives of f, then use Equation 6.6.20.

Answer

$D_{\vec{\mathbf{v}}} f(x,y,z) = -\frac{3}{13}(6x + y + 2z) + \frac{12}{13}(x - 4y + 4z) - \frac{4}{13}(2x + 4y - 2z)$

$D_{\vec{\mathbf{v}}} f(0,-2,5) = \frac{384}{13}$

6.6.5 Summary

- A directional derivative represents a rate of change of a function in any given direction.
- The gradient can be used in a formula to calculate the directional derivative.
- The gradient indicates the direction of greatest change of a function of more than one variable.

- **directional derivative (two dimensions)**

$$D_{\vec{\mathbf{u}}} f(a,b) = \lim_{h\to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h}$$

or

$$D_{\vec{\mathbf{u}}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

- **gradient (two dimensions)**

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}$$

- **gradient (three dimensions)**

$$\vec{\nabla} f(x,y,z) = f_x(x,y,z)\,\hat{\mathbf{i}} + f_y(x,y,z)\,\hat{\mathbf{j}} + f_z(x,y,z)\,\hat{\mathbf{k}}$$

- **directional derivative (three dimensions)**

$$D_{\vec{\mathbf{u}}} f(x,y,z) = \vec{\nabla} f(x,y,z) \cdot \vec{\mathbf{u}} = f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_x(x,y,z)\cos\gamma$$

directional derivative

the derivative of a function in the direction of a given unit vector

gradient

the gradient of the function $f(x,y)$ is defined to be $\vec{\nabla} f(x,y) = (\partial f/\partial x)\,\hat{\mathbf{i}} + (\partial f/\partial y)\,\hat{\mathbf{j}},$ which can be generalized to a function of any number of independent variables

6.6.6 Contributors

-

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

6.6E: Exercises

6.6E.1 Exercise $6.6E.1$

1) $f(x,y)=5-2x^2-\frac{1}{2}y^2$ at point $P(3,4)$ in the direction of $u=(cos\frac{\pi}{4})\hat{\mathbf{i}}+(sin\frac{\pi}{4})\hat{\mathbf{j}}$

2) $f(x,y)=y^2cos(2x)$ at point $P(\frac{\pi}{3},2)$ in the direction of $u=(cos\frac{\pi}{4})\hat{\mathbf{i}}+(sin\frac{\pi}{4})\hat{\mathbf{j}}$

Answer

$-3\sqrt{3}$

3) Find the directional derivative of $f(x,y)=y^2sin(2x)$ at point $P(\frac{\pi}{4},2)$ in the direction of $u=5\hat{\mathbf{i}}+12\hat{\mathbf{j}}$.

6.6E.2 Exercise $6.6E.2$

For the following exercises, find the directional derivative of the function at point P in the direction of v.

4) $f(x,y)=xy, P(0,-2), v=\frac{1}{2}\hat{\mathbf{i}}+\frac{\sqrt{3}}{2}\hat{\mathbf{j}}$

Answer

-1

5) $h(x,y)=e^x siny, P(1,\frac{\pi}{2}), v=-\hat{\mathbf{i}}$

6) $h(x,y,z)=xyz, P(2,1,1), v=2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$

Answer

$\frac{2}{\sqrt{6}}$

7) $f(x,y)=xy, P(1,1), u=\langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle$

8) $f(x,y)=x^2-y^2, u=\langle\frac{\sqrt{3}}{2},\frac{1}{2}\rangle, P(1,0)$

Answer

$\sqrt{3}$

9) $f(x,y)=3x+4y+7, u=\langle\frac{3}{5},\frac{4}{5}\rangle, P(0,\frac{\pi}{2})$

10) $f(x,y)=e^x cosy, u=\langle 0,1\rangle, P=(0,\frac{\pi}{2})$

Answer

-1.0

11) $f(x,y)=y^{10}, u=\langle 0,-1\rangle, P=(1,-1)$

12) $f(x,y)=ln(x^2+y^2), u=\langle\frac{3}{5},\frac{4}{5}\rangle, P(1,2)$

Answer

$\frac{22}{25}$

13) $f(x,y)=x^2y, P(-5,5), v=3\hat{\mathbf{i}}-4\hat{\mathbf{j}}$

14) $f(x,y)=y^2+xz, P(1,2,2), v=\langle 2,-1,2\rangle$

Answer

$$\frac{2}{3}$$

6.6E.3 Exercise $\mathbf{6.6E.3}$

For the following exercises, find the directional derivative of the function in the direction of the unit vector $u=cos\theta\hat{\mathbf{i}}+sin\theta\hat{\mathbf{j}}$.

15) $f(x,y)=x^2+2y^2, \theta=\frac{\pi}{6}$

16) $f(x,y)=\frac{y}{x+2y}, \theta=-\frac{\pi}{4}$

Answer

$$\frac{-\sqrt{2}(x+y)}{2(x+2y)^2}$$

17) $f(x,y)=cos(3x+y), \theta=\frac{\pi}{4}$

18) $w(x,y)=ye^x, \theta=\frac{\pi}{3}$

Answer

$$\frac{e^x(y+\sqrt{3})}{2}$$

19) $f(x,y)=xarctan(y), \theta=\frac{\pi}{2}$

20) $f(x,y)=ln(x+2y), \theta=\frac{\pi}{3}$

Answer

$$\frac{1+2\sqrt{3}}{2(x+2y)}$$

6.6E.4 Exercise $\mathbf{6.6E.4}$

For the following exercises, find the gradient.

21) Find the gradient of $f(x,y)=\frac{14-x^2-y^2}{3}$. Then, find the gradient at point $P(1,2)$.

22) Find the gradient of $f(x,y,z)=xy+yz+xz$ at point $P(1,2,3)$.

Answer

$$\langle 5,4,3\rangle$$

23) Find the gradient of $f(x,y,z))$ at P and in the direction of $u: f(x,y,z)=ln(x^2+2y^2+3z^2), P(2,1,4), u=\frac{-3}{13}\hat{\mathbf{i}}-\frac{4}{13}\hat{\mathbf{j}}-\frac{12}{13}\hat{\mathbf{k}}$.

24) $f(x,y,z)=4x^5y^2z^3, P(2,-1,1), u=\frac{1}{3}\hat{\mathbf{i}}+\frac{2}{3}\hat{\mathbf{j}}-\frac{2}{3}\hat{\mathbf{k}}$

Answer

-320

6.6E.5 Exercise $\mathbf{6.6E.5}$

For the following exercises, find the directional derivative of the function at point P in the direction of Q.

25) $f(x,y)=x^2+3y^2, P(1,1), Q(4,5)$

26) $f(x,y,z)=\dfrac{y}{x+z}, P(2,1,-1), Q(-1,2,0)$

Answer

$\dfrac{3}{\sqrt{11}}$

6.6E.6 Exercise $\mathbf{6.6E.6}$

For the following exercises, find the derivative of the function at P in the direction of u.

27) $f(x,y)=-7x+2y, P(2,-4), u=4\hat{\mathbf{i}}-3\hat{\mathbf{j}}$

28) $f(x,y)=ln(5x+4y), P(3,9), u=6\hat{\mathbf{i}}+8\hat{\mathbf{j}}$

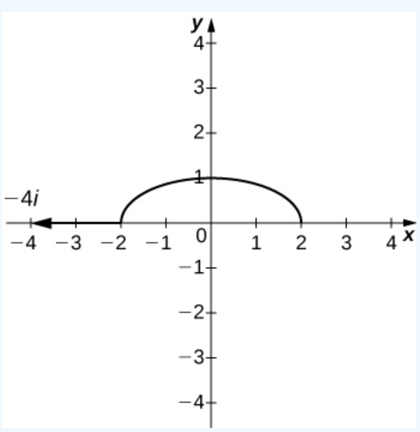
Answer

$\dfrac{31}{255}$

29) [T] Use technology to sketch the level curve of $f(x,y)=4x-2y+3$ that passes through $P(1,2)$ and draw the gradient vector at P.

30) [T] Use technology to sketch the level curve of $f(x,y)=x^2+4y^2$ that passes through $P(-2,0)$ and draw the gradient vector at P.

Answer



6.6E.7 Exercise $\mathbf{6.6E.7}$

For the following exercises, find the gradient vector at the indicated point.

31) $f(x,y)=xy^2-yx^2, P(-1,1)$

32) $f(x,y)=xe^y-ln(x), P(-3,0)$

Answer

$\frac{4}{3}\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$

33) $f(x,y,z) = xy - ln(z), P(2,-2,2)$

34) $f(x,y,z) = x\sqrt{y^2+z^2}, P(-2,-1,-1)$

Answer

$\sqrt{2}\hat{\mathbf{i}} + \sqrt{2}\hat{\mathbf{j}} + \sqrt{2}\hat{\mathbf{k}}$

6.6E.8 Exercise $\mathbf{6.6E.8}$

For the following exercises, find the derivative of the function.

35) $f(x,y) = x^2 + xy + y^2$ at point $(-5,-4)$ in the direction the function increases most rapidly

36) $f(x,y) = e^{xy}$ at point $(6,7)$ in the direction the function increases most rapidly

Answer

$1.6(10^{19})$

37) $f(x,y) = arctan(\frac{y}{x})$ at point $(-9,9)$ in the direction the function increases most rapidly

38) $f(x,y,z) = ln(xy + yz + zx)$ at point $(-9,-18,-27)$ in the direction the function increases most rapidly

Answer

$\frac{5\sqrt{2}}{99}$

39) $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ at point $(5,-5,5)$ in the direction the function increases most rapidly

6.6E.9 Exercise $\mathbf{6.6E.9}$

For the following exercises, find the maximum rate of change of f at the given point and the direction in which it occurs.

40) $f(x,y) = xe^{-y}, (1,0)$

Answer

$\sqrt{5}, \langle 1,2\rangle$

41) $f(x,y) = \sqrt{x^2+2y}, (4,10)$

42) $f(x,y) = cos(3x+2y), (\frac{\pi}{6}, -\frac{\pi}{8})$

Answer

$\sqrt{\frac{13}{2}}, \langle -3,-2\rangle$

6.6E.10 Exercise $\mathbf{6.6E.10}$

For the following exercises, find equations of

a. the tangent plane and

b. the normal line to the given surface at the given point.

43) The level curve $f(x,y,z) = 12$ for $f(x,y,z) = 4x^2 - 2y^2 + z^2$ at point $(2,2,2)$.

44) $f(x,y,z)=xy+yz+xz=3$ at point $(1,1,1)$

Answer

$a.\,x+y+z=3, b.\,x-1=y-1=z-1$

45) $f(x,y,z)=xyz=6$ at point $(1,2,3)$

46) $f(x,y,z)=xe^{y}cosz-z=1$ at point $(1,0,0)$

Answer

$a.\,x+y-z=1, b.\,x-1=y=-z$

6.6E.11 Exercise $\mathbf{6.6E.11}$

For the following exercises, solve the problem.

47) The temperature T in a metal sphere is inversely proportional to the distance from the center of the sphere (the origin: $(0,0,0)$). The temperature at point $(1,2,2)$ is $120^\circ C$.

a. Find the rate of change of the temperature at point $(1,2,2)$ in the direction toward point $(2,1,3)$.

b. Show that, at any point in the sphere, the direction of greatest increase in temperature is given by a vector that points toward the origin.

48) The **electrical potential** (voltage) in a certain region of space is given by the function $V(x,y,z)=5x^2-3xy+xyz$.

a. Find the rate of change of the voltage at point $(3,4,5)$ in the direction of the vector $\langle 1,1,-1\rangle$.

b. In which direction does the voltage change most rapidly at point $(3,4,5)$?

c. What is the maximum rate of change of the voltage at point $(3,4,5)$?

Answer

$a.\,\dfrac{32}{\sqrt{3}}, b.\,\langle 38,6,12\rangle, c.\,2\sqrt{406}$

49) If the electric potential at a point (x,y) in the xy-plane is $V(x,y)=e^{-2x}cos(2y)$, then the electric intensity vector at (x,y) is $E=-\nabla V(x,y)$.

a. Find the electric intensity vector at $(\frac{\pi}{4},0)$.

b. Show that, at each point in the plane, the electric potential decreases most rapidly in the direction of the vector E.

50) In two dimensions, the motion of an ideal fluid is governed by a velocity potential φ. The velocity components of the fluid u in the *x*-direction and v in the y-direction, are given by $\langle u,v\rangle=\nabla\varphi$. Find the velocity components associated with the velocity potential $\varphi(x,y)=sin(\pi x)\,sin(2\pi y)$.

Answer

$\langle u,v\rangle=\langle \pi cos(\pi x)sin(2\pi y), 2\pi sin(\pi x)cos(2\pi y)\rangle$

6.7: Maxima/Minima Problems

This page is a draft and is under active development.

One of the most useful applications for derivatives of a function of one variable is the determination of maximum and/or minimum values. This application is also important for functions of two or more variables, but as we have seen in earlier sections of this chapter, the introduction of more independent variables leads to more possible outcomes for the calculations. The main ideas of finding critical points and using derivative tests are still valid, but new wrinkles appear when assessing the results.

6.7.1 Critical Points

For functions of a single variable, we defined critical points as the values of the function when the derivative equals zero or does not exist. For functions of two or more variables, the concept is essentially the same, except for the fact that we are now working with partial derivatives.

Definition: Critical Points

Let $z = f(x, y)$ be a function of two variables that is differentiable on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a critical point of a function of two variables f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Example 6.7.1: Finding Critical Points

Find the critical points of each of the following functions:

a. $f(x, y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$
b. $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

Solution:

a. First, we calculate $f_x(x, y)$ and $f_y(x, y)$:

$$\begin{aligned} f_x(x, y) &= \frac{1}{2}(-18x + 36)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2} \\ &= \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= \frac{1}{2}(8y + 24)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2} \\ &= \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} \end{aligned}.$$

Next, we set each of these expressions equal to zero:

$$\frac{-9x+18}{\sqrt{4y^2-9x^2+24y+36x+36}}=0$$
$$\frac{4y+12}{\sqrt{4y^2-9x^2+24y+36x+36}}=0.$$

Then, multiply each equation by its common denominator:

$$-9x+18=0$$
$$4y+12=0.$$

Therefore, $x=2$ and $y=-3$, so $(2,-3)$ is a critical point of f.

We must also check for the possibility that the denominator of each partial derivative can equal zero, thus causing the partial derivative not to exist. Since the denominator is the same in each partial derivative, we need only do this once:

$$4y^2-9x^2+24y+36x+36=0. \tag{6.7.1}$$

This equation represents a hyperbola. We should also note that the domain of f consists of points satisfying the inequality

$$4y^2-9x^2+24y+36x+36\geq 0. \tag{6.7.2}$$

Therefore, any points on the hyperbola are not only critical points, they are also on the boundary of the domain. To put the hyperbola in standard form, we use the method of completing the square:

$$\begin{aligned} 4y^2-9x^2+24y+36x+36 &= 0 \\ 4y^2-9x^2+24y+36x &= -36 \\ 4y^2+24y-9x^2+36x &= -36 \\ 4(y^2+6y)-9(x^2-4x) &= -36 \\ 4(y^2+6y+9)-9(x^2-4x+4) &= -36-36+36 \\ 4(y+3)^2-9(x-2)^2 &= -36. \end{aligned}$$

Dividing both sides by -36 puts the equation in standard form:

$$\frac{4(y+3)^2}{-36}-\frac{9(x-2)^2}{-36}=1$$
$$\frac{(x-2)^2}{4}-\frac{(y+3)^2}{9}=1.$$

Notice that point $(2,-3)$ is the center of the hyperbola.

Thus, the critical points of the function f are $(2,-3)$ and all points on the hyperbola, $\dfrac{(x-2)^2}{4}-\dfrac{(y+3)^2}{9}=1$.

b. First, we calculate $g_x(x,y)$ and $g_y(x,y)$:

$$g_x(x,y) = 2x + 2y + 4$$
$$g_y(x,y) = 2x - 8y - 6.$$

Next, we set each of these expressions equal to zero, which gives a system of equations in x and y:

$$2x + 2y + 4 = 0$$
$$2x - 8y - 6 = 0.$$

Subtracting the second equation from the first gives $10y + 10 = 0$, so $y = -1$. Substituting this into the first equation gives $2x + 2(-1) + 4 = 0$, so $x = -1$.

Therefore $(-1,-1)$ is a critical point of g. There are no points in $\mathbb{R}^2$ that make either partial derivative not exist.

Figure $6.7.1$ shows the behavior of the surface at the critical point.

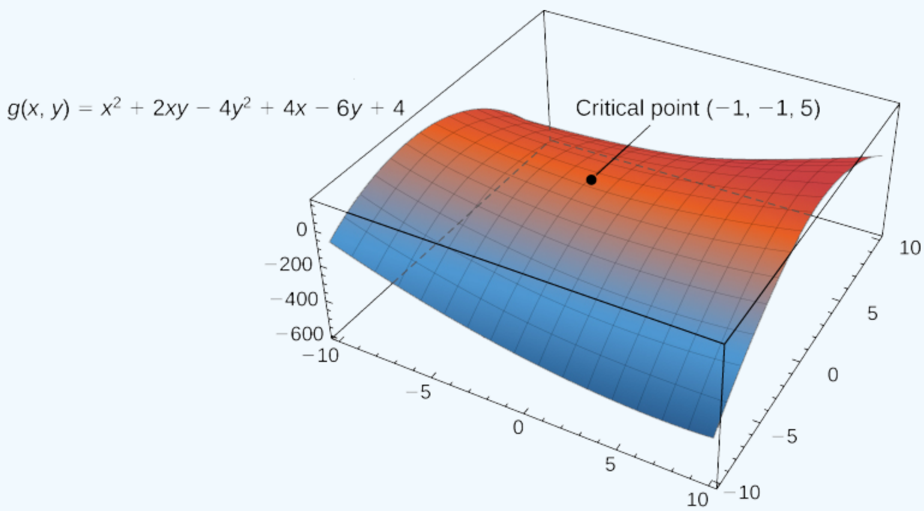


Figure $6.7.1$: The function $g(x,y)$ has a critical point at $(-1,-1,6)$.

Exercise $6.7.1$:

Find the critical point of the function $f(x,y) = x^3 + 2xy - 2x - 4y$.

Hint

Calculate $f_x(x,y)$ and $f_y(x,y)$, then set them equal to zero.

Answer

The only critical point of f is $(2,-5)$.

The main purpose for determining critical points is to locate relative maxima and minima, as in single-variable calculus. When working with a function of one variable, the definition of a local extremum involves finding an interval around the critical point such that the function value is either greater than or less than all the other function values in that interval. When working with a function of two or more variables, we work with an open disk around the point.

Definition: Global and Local Extrema

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Then f has a **local maximum** at $(x_0, y_0$ if

$$f(x_0, y_0) \geq f(x, y) \tag{6.7.3}$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local maximum value. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a **global maximum** (also called an absolute maximum) at (x_0, y_0).

The function f has a local minimum at (x_0, y_0) if

$$f(x_0, y_0) \leq f(x, y) \tag{6.7.4}$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a local minimum value. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a global minimum (also called an absolute minimum) at (x_0, y_0).

If $f(x_0, y_0)$ is either a local maximum or local minimum value, then it is called a **local extremum** (see the following figure).

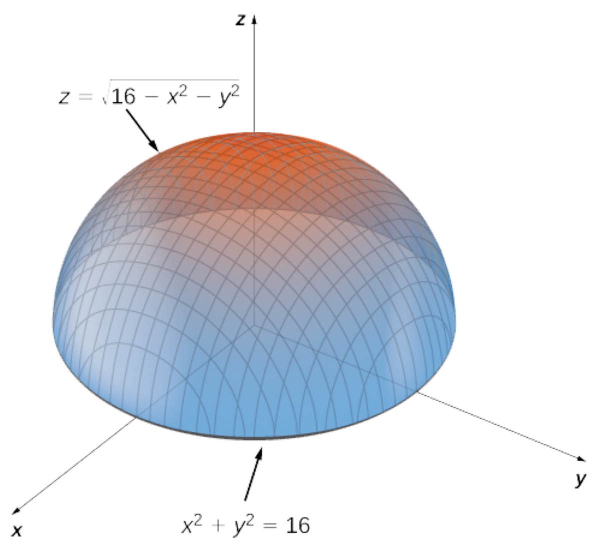


Figure 6.7.2: The graph of $z = \sqrt{16 - x^2 - y^2}$ has a maximum value when $(x, y) = (0, 0)$. It attains its minimum value at the boundary of its domain, which is the circle $x^2 + y^2 = 16$.

In Calculus 1, we showed that extrema of functions of one variable occur at critical points. The same is true for functions of more than one variable, as stated in the following theorem.

Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y each exists at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

6.7.2 Second Derivative Test

Consider the function $f(x) = x^3$. This function has a critical point at $x = 0$, since $f'(0) = 3(0)^2 = 0$. However, f does not have an extreme value at $x = 0$. Therefore, the existence of a critical value at $x = x_0$ does not guarantee a local extremum at $x = x_0$. The same is true for a function of two or more variables. One way this can happen is at a **saddle point**. An example of a saddle point appears in the following figure.

$z = x^2 - y^2$

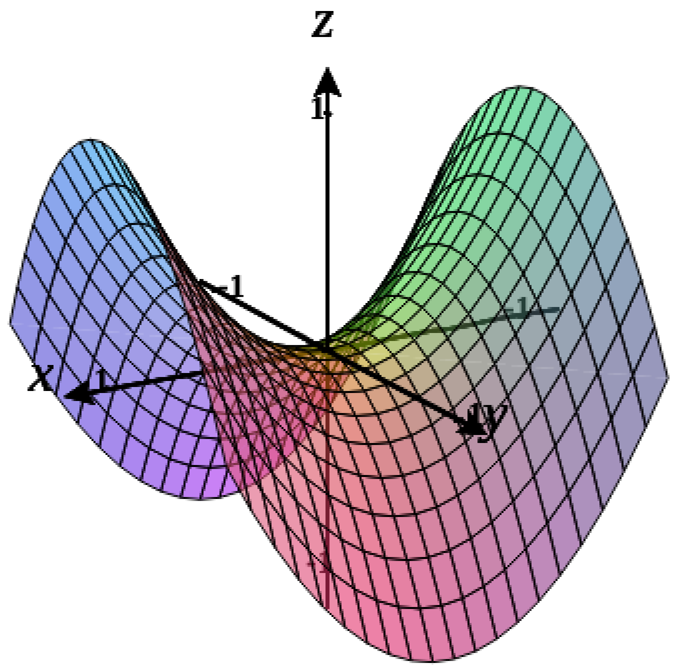


Figure 6.7.3: *Graph of the function $z = x^2 - y^2$. This graph has a saddle point at the origin.*

In this graph, the origin is a saddle point. This is because the first partial derivatives of f $(x, y) = x^2 - y^2$ are both equal to zero at this point, but it is neither a maximum nor a minimum for the function. Furthermore the vertical trace corresponding to $y = 0$ is $z = x^2$ (a parabola opening upward), but the vertical trace corresponding to $x = 0$ is $z = -y^2$ (a parabola opening downward). Therefore, it is both a global maximum for one trace and a global minimum for another.

Definition: Saddle Point

Given the function $z = f(x, y)$, the point $(x_0, y_0, f(x_0, y_0))$ is a saddle point if both $f_0(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0).

The second derivative test for a function of one variable provides a method for determining whether an extremum occurs at a critical point of a function. When extending this result to a function of two variables, an issue arises related to the fact that there are, in fact, four different second-order partial derivatives, although equality of mixed partials reduces this to three. The second derivative test for a function of two variables, stated in the following theorem, uses a **discriminant** D that replaces $f''(x_0)$ in the second derivative test for a function of one variable.

Second Derivative Test

Let $z=f(x,y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0,y_0). Suppose $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$. Define the quantity

$$D=f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-\left(f_{xy}(x_0,y_0)\right)^2. \tag{6.7.5}$$

Then:

i. If $D>0$ and $f_{xx}(x_0,y_0)>0$, then f has a local minimum at (x_0,y_0).
ii. If $D>0$ and $f_{xx}(x_0,y_0)<0$, then f has a local maximum at (x_0,y_0).
iii. If $D<0$, then f has a saddle point at (x_0,y_0).
iv. If $D=0$, then the test is inconclusive.

See Figure 6.7.4.

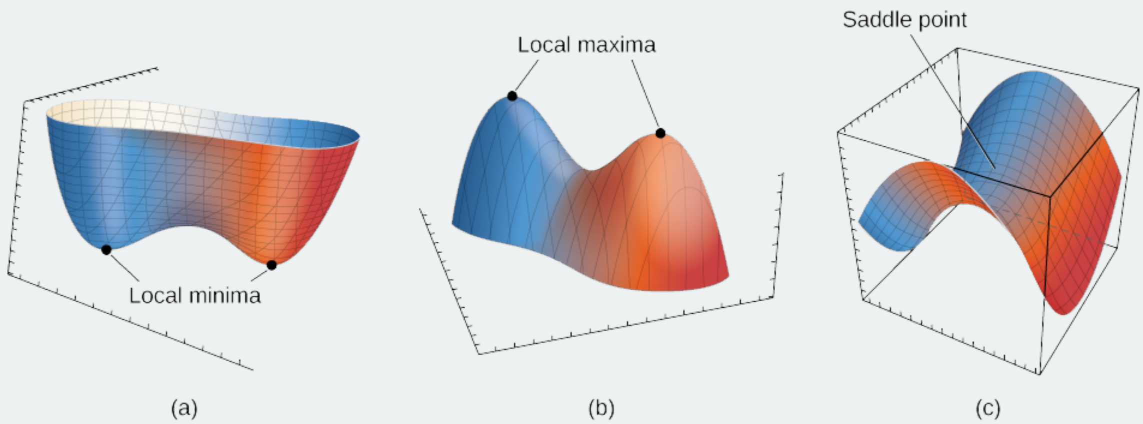


Figure 6.7.4: The second derivative test can often determine whether a function of two variables has a local minima (a), a local maxima (b), or a saddle point (c).

To apply the second derivative test, it is necessary that we first find the critical points of the function. There are several steps involved in the entire procedure, which are outlined in a problem-solving strategy.

Problem-Solving Strategy: Using the Second Derivative Test for Functions of Two Variables

Let $z=f(x,y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0,y_0). To apply the second derivative test to find local extrema, use the following steps:

1. Determine the critical points (x_0,y_0) of the function f where $f_x(x_0,y_0)=f_y(x_0,y_0)=0$. Discard any points where at least one of the partial derivatives does not exist.
2. Calculate the discriminant $D=f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-\left(f_{xy}(x_0,y_0)\right)^2$ for each critical point of f.

3. Apply the four cases of the test to determine whether each critical point is a local maximum, local minimum, or saddle point, or whether the theorem is inconclusive.

Example 6.7.2: Using the Second Derivative Test

Find the critical points for each of the following functions, and use the second derivative test to find the local extrema:

a. $f(x,y) = 4x^2 + 9y^2 + 8x - 36y + 24$
b. $g(x,y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

Solution:

a. Step **1** of the problem-solving strategy involves finding the critical points of f. To do this, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set each of them equal to zero:

$$\begin{aligned} f_x(x,y) &= 8x + 8 \\ f_y(x,y) &= 18y - 36. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 8x + 8 &= 0 \\ 18y - 36 &= 0. \end{aligned}$$

The solution to this system is $x = -1$ and $y = 2$. Therefore $(-1, 2)$ is a critical point of f.

Step 2 of the problem-solving strategy involves calculating D. To do this, we first calculate the second partial derivatives of f :

$$\begin{aligned} f_{xx}(x,y) &= 8 \\ f_{xy}(x,y) &= 0 \\ f_{yy}(x,y) &= 18. \end{aligned}$$

Therefore, $D = f_{xx}(-1,2) f_{yy}(-1,2) - \left(f_{xy}(-1,2)\right)^2 = (8)(18) - (0)^2 = 144.$

Step 3 states to check Note. Since $D > 0$ and $f_{xx}(-1,2) > 0,$ this corresponds to case 1. Therefore, f has a local minimum at $(-1,2)$ as shown in the following figure.

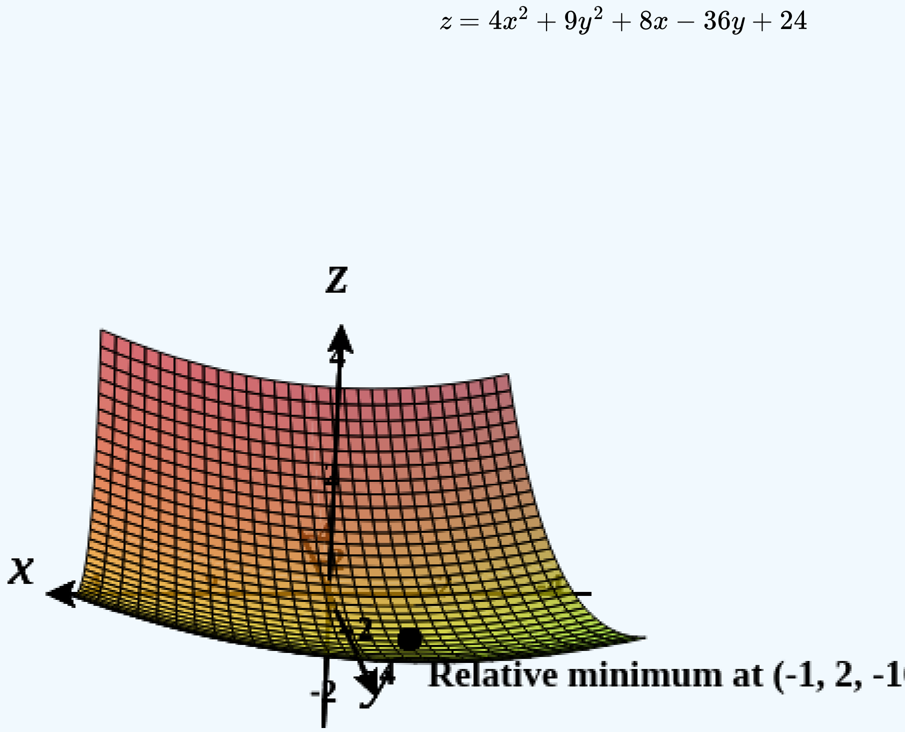


Figure 6.7.5: *The function $f(x,y)$ has a local minimum at $(-1,2,-16)$. Note the scale on the y-axis in this plot is in thousands.*

b. For step 1, we first calculate $g_x(x,y)$ and $g_y(x,y)$, then set each of them equal to zero:

$$\begin{aligned} g_x(x,y) &= x^2 + 2y - 6 \\ g_y(x,y) &= 2y + 2x - 3. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} x^2 + 2y - 6 &= 0 \\ 2y + 2x - 3 &= 0. \end{aligned}$$

To solve this system, first solve the second equation for y. This gives $y = \dfrac{3-2x}{2}$. Substituting this into the first equation gives

$$\begin{aligned} x^2 + 3 - 2x - 6 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0. \end{aligned}$$

Therefore, $x = -1$ or $x = 3$. Substituting these values into the equation $y = \dfrac{3-2x}{2}$ yields the critical points $\left(-1, \frac{5}{2}\right)$ and $\left(3, -\frac{3}{2}\right)$.

Step 2 involves calculating the second partial derivatives of g:

$$\begin{aligned} g_{xx}(x,y) &= 2x \\ g_{xy}(x,y) &= 2 \\ g_{yy}(x,y) &= 2. \end{aligned}$$

Then, we find a general formula for D:

$$\begin{aligned} D(x_0,y_0) &= g_{xx}(x_0,y_0)g_{yy}(x_0,y_0) - \left(g_{xy}(x_0,y_0)\right)^2 \\ &= (2x_0)(2) - 2^2 \\ &= 4x_0 - 4. \end{aligned}$$

Next, we substitute each critical point into this formula:

$$\begin{aligned} D\left(-1,\tfrac{5}{2}\right) &= (2(-1))(2) - (2)^2 = -4 - 4 = -8 \\ D\left(3,-\tfrac{3}{2}\right) &= (2(3))(2) - (2)^2 = 12 - 4 = 8. \end{aligned}$$

In step 3, we note that, applying Note to point $\left(-1,\frac{5}{2}\right)$ leads to case 3, which means that $\left(-1,\frac{5}{2}\right)$ is a saddle point. Applying the theorem to point $\left(3,-\frac{3}{2}\right)$ leads to case 1, which means that $\left(3,-\frac{3}{2}\right)$ corresponds to a local minimum as shown in the following figure.

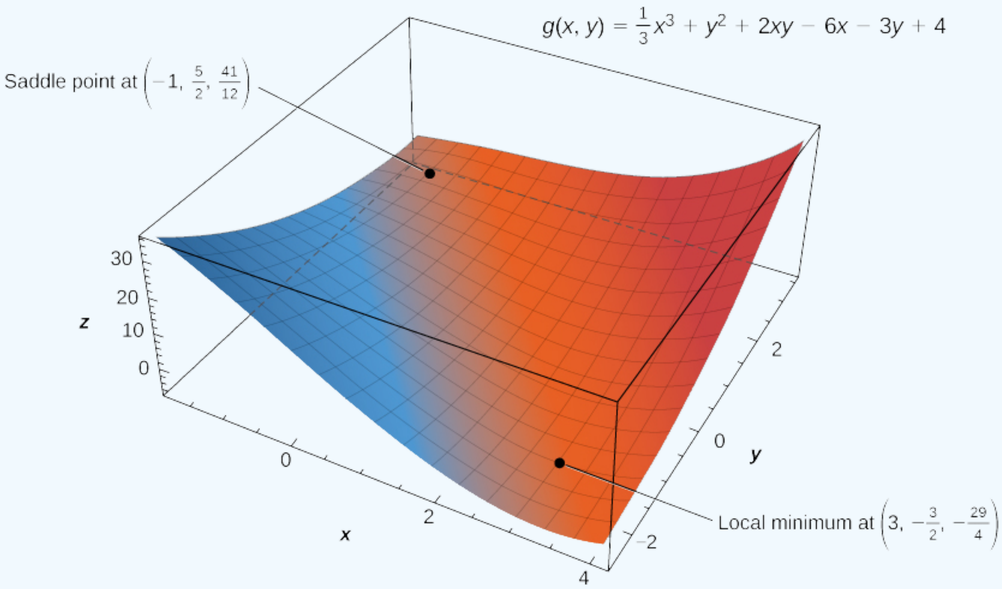


Figure 6.7.6: The function $g(x,y)$ has a local minimum and a saddle point.

Exercise 6.7.2

Use the second derivative to find the local extrema of the function

$$f(x,y) = x^3 + 2xy - 6x - 4y^2.$$

Hint

Follow the problem-solving strategy for applying the second derivative test.

Answer

$\left(\frac{4}{3}, \frac{1}{3}\right)$ is a saddle point, $\left(-\frac{3}{2}, -\frac{3}{8}\right)$ is a local maximum.

6.7.3 Absolute Maxima and Minima

When finding global extrema of functions of one variable on a closed interval, we start by checking the critical values over that interval and then evaluate the function at the endpoints of the interval. When working with a function of two variables, the closed interval is replaced by a closed, bounded set. A set is bounded if all the points in that set can be contained within a ball (or disk) of finite radius. First, we need to find the critical points inside the set and calculate the corresponding critical values. Then, it is necessary to find the maximum and minimum value of the function on the boundary of the set. When we have all these values, the largest function value corresponds to the global maximum and the smallest function value corresponds to the absolute minimum. First, however, we need to be assured that such values exist. The following theorem does this.

Extreme Value Theorem

A continuous function $f(x, y)$ on a closed and bounded set D in the plane attains an absolute maximum value at some point of D and an absolute minimum value at some point of D.

Now that we know any continuous function f defined on a closed, bounded set attains its extreme values, we need to know how to find them.

Finding Extreme Values of a Function of Two Variables

Assume $z = f(x, y)$ is a differentiable function of two variables defined on a closed, bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

1. The values of f at the critical points of f in D.
2. The values of f on the boundary of D.

The proof of this theorem is a direct consequence of the extreme value theorem and Fermat's theorem. In particular, if either extremum is not located on the boundary of D, then it is located at an interior point of D. But an interior point (x_0, y_0) of D that's an absolute extremum is also a local extremum; hence, (x_0, y_0) is a critical point of f by Fermat's theorem. Therefore the only possible values for the global extrema of f on D are the extreme values of f on the interior or boundary of D.

Problem-Solving Strategy: Finding Absolute Maximum and Minimum Values

Let $z = f(x, y)$ be a continuous function of two variables defined on a closed, bounded set D, and assume f is differentiable on D. To find the absolute maximum and minimum values of f on D, do the following:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary of its domain.
4. The maximum and minimum values of f will occur at one of the values obtained in steps 2 and 3.

Finding the maximum and minimum values of f on the boundary of D can be challenging. If the boundary is a rectangle or set of straight lines, then it is possible to parameterize the line segments and determine the maxima on each of these segments, as seen in Example. The same approach can be used for other shapes such as circles and ellipses.

If the boundary of the set D is a more complicated curve defined by a function $g(x,y)=c$ for some constant c, and the first-order partial derivatives of g exist, then the method of Lagrange multipliers can prove useful for determining the extrema of f on the boundary which is introduced in Lagrange Multipliers.

Example 6.7.3: Finding Absolute Extrema

Use the problem-solving strategy for finding absolute extrema of a function to determine the absolute extrema of each of the following functions:

a. $f(x,y)=x^2-2xy+4y^2-4x-2y+24$ on the domain defined by $0\leq x\leq 4$ and $0\leq y\leq 2$
b. $g(x,y)=x^2+y^2+4x-6y$ on the domain defined by $x^2+y^2\leq 16$

Solution:

a. Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set them each equal to zero:

$$\begin{aligned}f_x(x,y)&=2x-2y-4\\f_y(x,y)&=-2x+8y-2.\end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned}2x-2y-4&=0\\-2x+8y-2&=0.\end{aligned}$$

The solution to this system is $x=3$ and $y=1$. Therefore $(3,1)$ is a critical point of f. Calculating $f(3,1)$ gives $f(3,1)=17$.

The next step involves finding the extrema of f on the boundary of its domain. The boundary of its domain consists of four line segments as shown in the following graph:

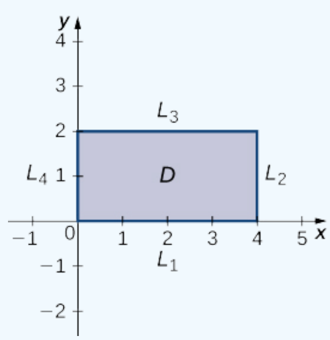


Figure 6.7.7: Graph of the domain of the function $f(x,y)=x^2-2xy+4y^2-4x-2y+24$.

L_1 is the line segment connecting $(0,0)$ and $(4,0)$, and it can be parameterized by the equations $x(t)=t, y(t)=0$ for $0\leq t\leq 4$. Define $g(t)=f\big(x(t),y(t)\big)$. This gives $g(t)=t^2-4t+24$.

Differentiating g leads to $g'(t) = 2t - 4$. Therefore, g has a critical value at $t = 2$, which corresponds to the point $(2, 0)$. Calculating $f(2, 0)$ gives the z-value 20.

L_2 is the line segment connecting $(4, 0)$ and $(4, 2)$, and it can be parameterized by the equations $x(t) = 4, y(t) = t$ for $0 \leq t \leq 2$. Again, define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = 4t^2 - 10t + 24$. Then, $g'(t) = 8t - 10$. g has a critical value at $t = \frac{5}{4}$, which corresponds to the point $\left(0, \frac{5}{4}\right)$. Calculating $f\left(0, \frac{5}{4}\right)$ gives the z-value 27.75.

L_3 is the line segment connecting $(0, 2)$ and $(4, 2)$, and it can be parameterized by the equations $x(t) = t, y(t) = 2$ for $0 \leq t \leq 4$. Again, define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = t^2 - 8t + 36$. The critical value corresponds to the point $(4, 2)$. So, calculating $f(4, 2)$ gives the z-value 20.

L_4 is the line segment connecting $(0, 0)$ and $(0, 2)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \leq t \leq 2$. This time, $g(t) = 4t^2 - 2t + 24$ and the critical value $t = \frac{1}{4}$ correspond to the point $\left(0, \frac{1}{4}\right)$. Calculating $f\left(0, \frac{1}{4}\right)$ gives the z-value 23.75.

We also need to find the values of $f(x, y)$ at the corners of its domain. These corners are located at $(0, 0), (4, 0), (4, 2)$ and $(0, 2)$:

$$\begin{aligned}
f(0,0) &= (0)^2 - 2(0)(0) + 4(0)^2 - 4(0) - 2(0) + 24 = 24 \\
f(4,0) &= (4)^2 - 2(4)(0) + 4(0)^2 - 4(4) - 2(0) + 24 = 24 \\
f(4,2) &= (4)^2 - 2(4)(2) + 4(2)^2 - 4(4) - 2(2) + 24 = 20 \\
f(0,2) &= (0)^2 - 2(0)(2) + 4(2)^2 - 4(0) - 2(2) + 24 = 36.
\end{aligned}$$

The absolute maximum value is 36, which occurs at $(0, 2)$, and the global minimum value is 20, which occurs at both $(4, 2)$ and $(2, 0)$ as shown in the following figure.

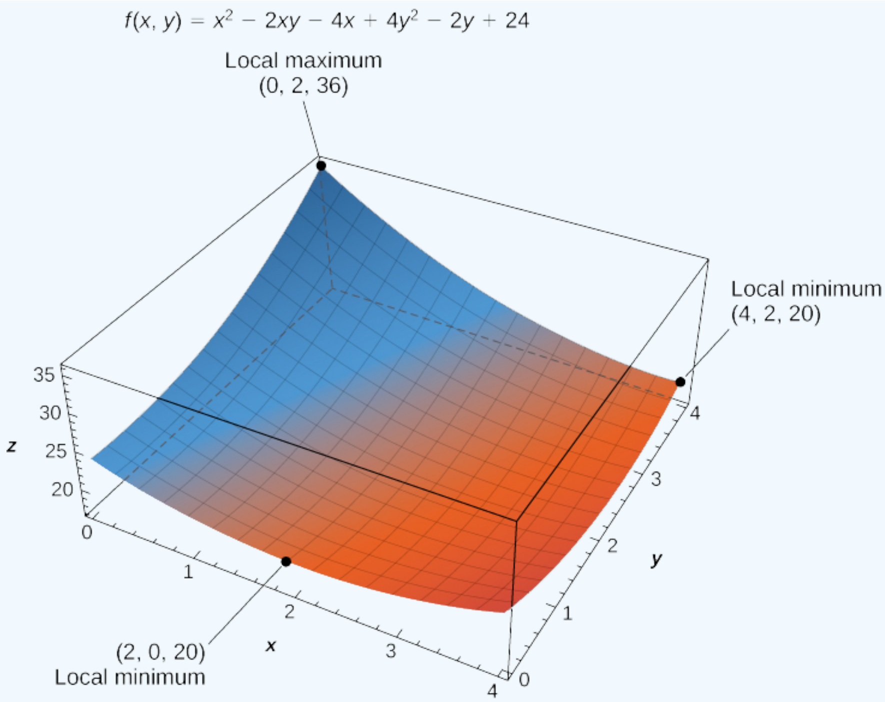


Figure 6.7.8: The function $f(x,y)$ has two global minima and one global maximum over its domain.

b. Using the problem-solving strategy, step 1 involves finding the critical points of g on its domain. Therefore, we first calculate $g_x(x,y)$ and $g_y(x,y)$, then set them each equal to zero:

$$g_x(x,y) = 2x + 4$$
$$g_y(x,y) = 2y - 6.$$

Setting them equal to zero yields the system of equations

$$2x + 4 = 0$$
$$2y - 6 = 0.$$

The solution to this system is $x = -2$ and $y = 3$. Therefore, $(-2,3)$ is a critical point of g. Calculating $g(-2,3)$, we get

$$g(-2,3) = (-2)^2 + 3^2 + 4(-2) - 6(3) = 4 + 9 - 8 - 18 = -13. \tag{6.7.6}$$

The next step involves finding the extrema of g on the boundary of its domain. The boundary of its domain consists of a circle of radius 4 centered at the origin as shown in the following graph.

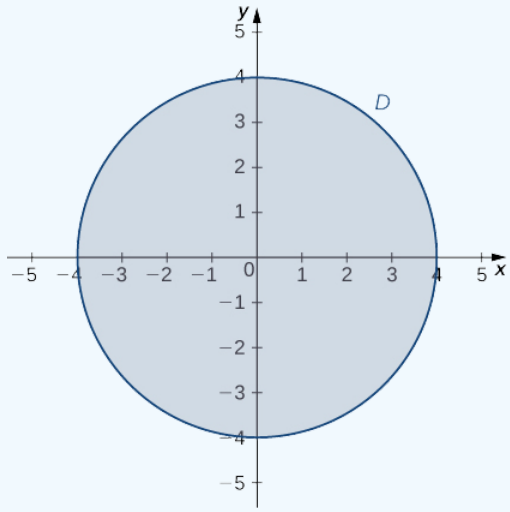


Figure 6.7.9: Graph of the domain of the function $g(x,y)=x^2+y^2+4x-6y$.

The boundary of the domain of g can be parameterized using the functions $x(t)=4\cos t,\; y(t)=4\sin t$ for $0\leq t\leq 2\pi$. Define $h(t)=g\big(x(t),y(t)\big)$:

$$\begin{aligned} h(t) &= g\big(x(t),y(t)\big) \\ &= (4\cos t)^2+(4\sin t)^2+4(4\cos t)-6(4\sin t) \\ &= 16\cos^2 t+16\sin^2 t+16\cos t-24\sin t \\ &= 16+16\cos t-24\sin t. \end{aligned}$$

Setting $h'(t)=0$ leads to

$$\begin{aligned} -16\sin t-24\cos t &= 0 \\ -16\sin t &= 24\cos t \\ \frac{-16\sin t}{-16\cos t} &= \frac{24\cos t}{-16\cos t} \\ \tan t &= -\frac{4}{3}. \end{aligned}$$

$$\begin{aligned} -16\sin t-24\cos t &= 0 \\ -16\sin t &= 24\cos t \\ \frac{-16\sin t}{-16\cos t} &= \frac{24\cos t}{-16\cos t} \\ \tan t &= -\frac{3}{2}. \end{aligned}$$

This equation has two solutions over the interval $0\leq t\leq 2\pi$. One is $t=\pi-\arctan(\frac{3}{2})$ and the other is $t=2\pi-\arctan(\frac{3}{2})$. For the first angle,

$$\begin{aligned} \sin t &= \sin(\pi-\arctan(\tfrac{3}{2}))=\sin(\arctan(\tfrac{3}{2}))=\frac{3\sqrt{13}}{13} \\ \cos t &= \cos(\pi-\arctan(\tfrac{3}{2}))=-\cos(\arctan(\tfrac{3}{2}))=-\frac{2\sqrt{13}}{13}. \end{aligned}$$

Therefore, $x(t) = 4\cos t = -\frac{8\sqrt{13}}{13}$ and $y(t) = 4\sin t = \frac{12\sqrt{13}}{13}$, so $\left(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13}\right) &= \left(-\frac{8\sqrt{13}}{13}\right)^2 + \left(\frac{12\sqrt{13}}{13}\right)^2 + 4\left(-\frac{8\sqrt{13}}{13}\right) - 6\left(\frac{12}{\sqrt{13}}13\right) \\ &= \frac{144}{13} + \frac{64}{13} - \frac{32\sqrt{13}}{13} - \frac{72\sqrt{13}}{13} \\ &= \frac{208 - 104\sqrt{13}}{13} \approx -12.844. \end{aligned}$$

For the second angle,

$$\begin{aligned} \sin t &= \sin(2\pi - \arctan(\tfrac{3}{2})) = -\sin(\arctan(\tfrac{3}{2})) = -\frac{3\sqrt{13}}{13} \\ \cos t &= \cos(2\pi - \arctan(\tfrac{3}{2})) = \cos(\arctan(\tfrac{3}{2})) = \frac{2\sqrt{13}}{13}. \end{aligned}$$

Therefore, $x(t) = 4\cos t = \frac{8\sqrt{13}}{13}$ and $y(t) = 4\sin t = -\frac{12\sqrt{13}}{13}$, so $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right) &= \left(\frac{8\sqrt{13}}{13}\right)^2 + \left(-\frac{12\sqrt{13}}{13}\right)^2 + 4\left(\frac{8\sqrt{13}}{13}\right) - 6\left(-\frac{12\sqrt{13}}{13}\right) \\ &= \frac{144}{13} + \frac{64}{13} + \frac{32\sqrt{13}}{13} + \frac{72\sqrt{13}}{13} \\ &= \frac{208 + 104\sqrt{13}}{13} \approx 44.844. \end{aligned}$$

The absolute minimum of g is -13, which is attained at the point $(-2, 3)$, which is an interior point of D. The absolute maximum of g is approximately equal to 44.844, which is attained at the boundary point $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$. These are the absolute extrema of g on D as shown in the following figure.

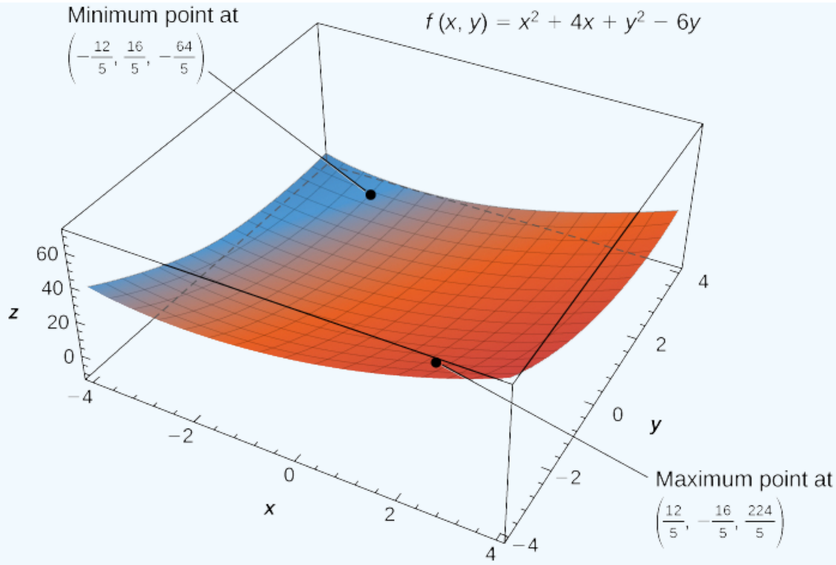


Figure 6.7.10: The function $f(x, y)$ has a local minimum and a local maximum.

Exercise 6.7.3:

Use the problem-solving strategy for finding absolute extrema of a function to find the absolute extrema of the function

$$f(x, y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$$

on the domain defined by $0 \leq x \leq 2$ and $-1 \leq y \leq 3$.

Hint

Calculate $f_x(x, y)$ and $f_y(x, y)$, and set them equal to zero. Then, calculate f for each critical point and find the extrema of f on the boundary of D.

Answer

The absolute minimum occurs at $(1, 0) : f(1, 0) = -1$.

The absolute maximum occurs at $(0, 3) : f(0, 3) = 63$.

Example 6.7.4: Profitable Golf Balls

Pro-T company has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2, \tag{6.7.7}$$

where z is measured in thousands of dollars. The maximum number of golf balls that can be produced and sold is $50,000$, and the maximum number of hours of advertising that can be purchased is 25. Find the values of x and y that maximize profit, and find the maximum profit.

Figure 6.7.11: (credit: modification of work by oatsy40, Flickr)

Solution

Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set them each equal to zero:

$$\begin{aligned} f_x(x,y) &= 48-2x-2y \\ f_y(x,y) &= 96-2x-18y. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 48-2x-2y &= 0 \\ 96-2x-18y &= 0. \end{aligned}$$

The solution to this system is $x=21$ and $y=3$. Therefore $(21,3)$ is a critical point of f. Calculating $f(21,3)$ gives $f(21,3)=48(21)+96(3)-21^2-2(21)(3)-9(3)^2=648$.

The domain of this function is $0\le x\le 50$ and $0\le y\le 25$ as shown in the following graph.

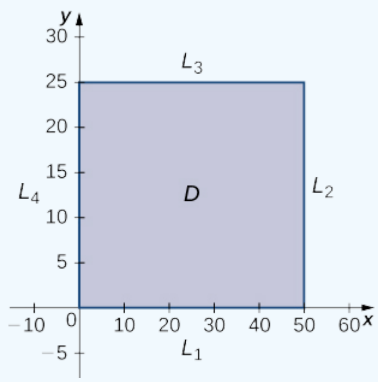


Figure 6.7.12: Graph of the domain of the function $f(x,y)=48x+96y-x^2-2xy-9y^2$.

L_1 is the line segment connecting $(0,0)$ and $(50,0)$, and it can be parameterized by the equations $x(t)=t, y(t)=0$ for $0\le t\le 50$. We then define $g(t)=f(x(t),y(t))$:

$$\begin{aligned} g(t) &= f(x(t),y(t)) \\ &= f(t,0) \\ &= 48t+96(0)-y^2-2(t)(0)-9(0)^2 \\ &= 48t-t^2. \end{aligned}$$

Setting $g'(t)=0$ yields the critical point $t=24$, which corresponds to the point $(24,0)$ in the domain of f. Calculating $f(24,0)$ gives 576.

L_2 is the line segment connecting and $(50, 25)$, and it can be parameterized by the equations $x(t) = 50, y(t) = t$ for $0 \leq t \leq 25$. Once again, we define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(50, t) \\ &= 48(50) + 96t - 50^2 - 2(50)t - 9t^2 \\ &= -9t^2 - 4t - 100. \end{aligned}$$

This function has a critical point at $t = -\frac{2}{9}$, which corresponds to the point $(50, -29)$. This point is not in the domain of f.

L_3 is the line segment connecting $(0, 25)$ and $(50, 25)$, and it can be parameterized by the equations $x(t) = t, y(t) = 25$ for $0 \leq t \leq 50$. We define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(t, 25) \\ &= 48t + 96(25) - t^2 - 2t(25) - 9(25^2) \\ &= -t^2 - 2t - 3225. \end{aligned}$$

This function has a critical point at $t = -1$, which corresponds to the point $(-1, 25)$, which is not in the domain.

L_4 is the line segment connecting $(0, 0)$ to $(0, 25)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \leq t \leq 25$. We define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(0, t) \\ &= 48(0) + 96t - (0)^2 - 2(0)t - 9t^2 \\ &= 96t - t^2. \end{aligned}$$

This function has a critical point at $t = \frac{16}{3}$, which corresponds to the point $\left(0, \frac{16}{3}\right)$, which is on the boundary of the domain. Calculating $f\left(0, \frac{16}{3}\right)$ gives 256.

We also need to find the values of $f(x, y)$ at the corners of its domain. These corners are located at $(0, 0), (50, 0), (50, 25)$ and $(0, 25)$:

$$\begin{aligned} f(0,0) &= 48(0) + 96(0) - (0)^2 - 2(0)(0) - 9(0)^2 = 0 \\ f(50,0) &= 48(50) + 96(0) - (50)^2 - 2(50)(0) - 9(0)^2 = -100 \\ f(50,25) &= 48(50) + 96(25) - (50)^2 - 2(50)(25) - 9(25)^2 = -5825 \\ f(0,25) &= 48(0) + 96(25) - (0)^2 - 2(0)(25) - 9(25)^2 = -3225. \end{aligned}$$

The maximum critical value is 648, which occurs at $(21, 3)$. Therefore, a maximum profit of $\$648,000$ is realized when $21,000$ golf balls are sold and 3 hours of advertising are purchased per month as shown in the following figure.

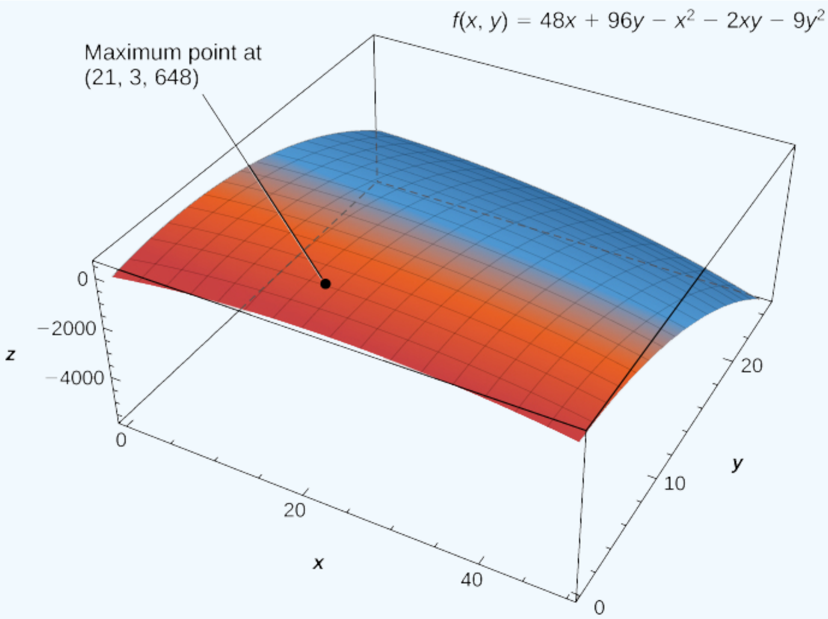


Figure 6.7.13: The profit function $f(x,y)$ has a maximum at $(21,3,648)$.

6.7.4 Key Concepts

- A critical point of the function $f(x,y)$ is any point (x_0,y_0) where either $f_x(x_0,y_0)=f_y(x_0,y_0)=0$, or at least one of $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ do not exist.
- A saddle point is a point (x_0,y_0) where $f_x(x_0,y_0)=f_y(x_0,y_0)=0$, but (x_0,y_0) is neither a maximum nor a minimum at that point.
- To find extrema of functions of two variables, first find the critical points, then calculate the discriminant and apply the second derivative test.

6.7.5 Key Equations

- **Discriminant**

$$D=f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-(f_{xy}(x_0,y_0))^2$$

6.7.6 Glossary

critical point of a function of two variables

the point (x_0,y_0) is called a critical point of $f(x,y)$ if one of the two following conditions holds:

1. $f_x(x_0,y_0)=f_y(x_0,y_0)=0$
2. At least one of $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ do not exist

discriminant

the discriminant of the function $f(x,y)$ is given by the formula $D=f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-(f_{xy}(x_0,y_0))^2$

saddle point

given the function $z=f(x,y)$, the point $(x_0,y_0,f(x_0,y_0))$ is a saddle point if both $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$, but f does not have a local extremum at (x_0,y_0)

6.7.7 Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

6.7E:

6.7E.1 Exercise $6.7E.1$

For the following exercises, find all critical points.

1) $f(x,y)=1+x^2+y^2$

2) $f(x,y)=(3x-2)^2+(y-4)^2$

Answer

$(\frac{2}{3},4)$

3) $f(x,y)=x^4+y^4-16xy$

4) $f(x,y)=15x^3-3xy+15y^3$

Answer

$(0,0)(\frac{1}{15},\frac{1}{15})$

6.7E.2 Exercise $6.7E.2$

For the following exercises, find the critical points of the function by using algebraic techniques (completing the square) or by examining the form of the equation. Verify your results using the partial derivatives test.

5) $f(x,y)=\sqrt{x^2+y^2+1}$

6) $f(x,y)=-x^2-5y^2+8x-10y-13$

Answer

Maximum at $(4,-1,8)$

7) $f(x,y)=x^2+y^2+2x-6y+6$

8) $f(x,y)=\sqrt{x^2+y^2}+1$

Answer

Relative minimum at $(0,0,1)$

6.7E.3 Exercise $6.7E.3$

For the following exercises, use the second derivative test to identify any critical points and determine whether each critical point is a maximum, minimum, saddle point, or none of these.

9) $f(x,y)=-x^3+4xy-2y^2+1$

10) $f(x,y)=x^2y^2$

Answer

The second derivative test fails. Since $x^2y^2>0$ for all x and y different from zero, and $x^2y^2=0$ when either x or y equals zero (or both), then the absolute minimum occurs at $(0,0)$.

11) $f(x,y)=x^2-6x+y^2+4y-8$

12) $f(x,y)=2xy+3x+4y$

Answer

$f(-2,-\frac{3}{2})=-6$ is a saddle point.

13) $f(x,y)=8xy(x+y)+7$

14) $f(x,y)=x^2+4xy+y^2$

Answer

$f(0,0)=0$; $(0,0,0)$is a saddle point.

15) $f(x,y)=x^3+y^3-300x-75y-3$

16) $f(x,y)=9-x^4y^4$

Answer

$f(0,0)=9$ is a local maximum.

20) $f(x,y)=7x^2y+9xy^2$

21) $f(x,y)=3x^2-2xy+y^2-8y$

Answer

Relative minimum located at $(2,6)$.

22) $f(x,y)=3x^2+2xy+y^2$

23) $f(x,y)=y2+xy+3y+2x+3$

Answer

$(1,-2)$ is a saddle point.

24) $f(x,y)=x^2+xy+y^2-3x$

25) $f(x,y)=x^2+2y^2-x^2y$

Answer

$(2,1)$ and $(-2,1)$ are saddle points; $(0,0)$ is a relative minimum.

26) $f(x,y)=x^2+y-e^y$

27) $f(x,y)=e^{-(x^2+y^2+2x)}$

Answer

$(-1,0)$ is a relative maximum.

28) $f(x,y)=x^2+xy+y^2-x-y+1$

29) $f(x,y)=x^2+10xy+y^2$

Answer

$(0,0)$ is a saddle point.

30) $f(x,y)=-x^2-5y^2+10x-30y-62$

31) $f(x,y)=120x+120y-xy-x^2-y^2$

Answer

The relative maximum is at $(40,40)$.

32) $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

33) $f(x, y) = x^2 + x - 3xy + y^3 - 5$

Answer

$(\frac{1}{4}, \frac{1}{2})$ is a saddle point and $(1, 1)$ is the relative minimum.

34) $f(x, y) = 2xye^{-x^2-y^2}$

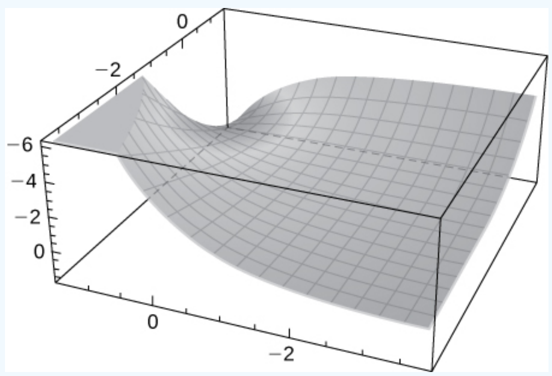
6.7E.4 Exercise $\mathbf{6.7E.4}$

For the following exercises, determine the extreme values and the saddle points. Use a CAS to graph the function.

35) [T] $f(x, y) = ye^x - e^y$

Answer

A saddle point is located at $(0, 0)$.

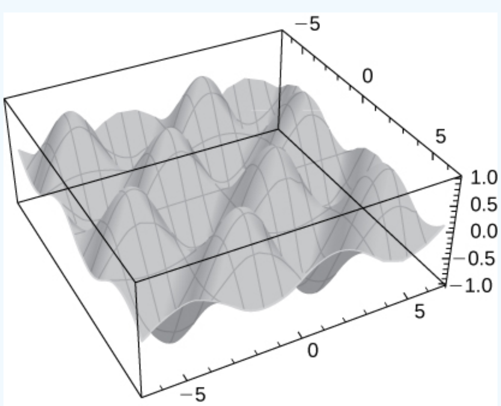


36) [T] $f(x, y) = x sin(y)$

37) [T] $f(x, y) = sin(x)sin(y), x \in (0, 2\pi), y \in (0, 2\pi)$

Answer

There is a saddle point at (π, π), local maxima at $(\frac{\pi}{2}, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, \frac{3\pi}{2})$, and local minima at $(\frac{\pi}{2}, \frac{3\pi}{2})$ and $(\frac{3\pi}{2}, \frac{\pi}{2})$.



6.7E.5 Exercise $\mathbf{6.7E.5}$

Find the absolute extrema of the given function on the indicated closed and bounded set R.

38) $f(x, y) = xy - x - 3y$; R is the triangular region with vertices $(0, 0)$, $(0, 4)$, and $(5, 0)$.

39) Find the absolute maximum and minimum values of $f(x,y)=x^2+y^2-2y+1$ on the region $R=(x,y) \mid x^2+y^2 \leq 4$.

Answer

$(0,1,0)$ is the absolute minimum and $(0,-2,9)$ is the absolute maximum.

40) $f(x,y)=x^3-3xy-y^3$ on $R=(x,y):-2\leq x\leq 2, -2\leq y\leq 2$

41) $f(x,y)=\dfrac{-2y}{x^2+y^2+1}$ on $R=(x,y):x^2+y^2\leq 4$

Answer

There is an absolute minimum at $(0,1,-1)$ and an absolute maximum at $(0,-1,1)$.

6.7E.6 Exercise $\boldsymbol{6.7E.6}$

42) Find three positive numbers the sum of which is 27, such that the sum of their squares is as small as possible.

43) Find the points on the surface $x^2-yz=5$ that are closest to the origin.

Answer

$(\sqrt{5},0,0), (-\sqrt{5},0,0)$

44) Find the maximum volume of a rectangular box with three faces in the coordinate planes and a vertex in the first octant on the line $x+y+z=1$.

45) The sum of the length and the girth (perimeter of a cross-section) of a package carried by a delivery service cannot exceed 108 in. Find the dimensions of the rectangular package of largest volume that can be sent.

Answer

18 by 36 by 18 in.

46) A cardboard box without a lid is to be made with a volume of 4 ft^3. Find the dimensions of the box that requires the least amount of cardboard.

47) Find the point on the surface $f(x,y)=x^2+y^2+10$ nearest the plane $x+2y-z=0.$ Identify the point on the plane.

Answer

$(\frac{47}{24},\frac{47}{12},\frac{235}{24})$

48) Find the point in the plane $2x-y+2z=16$ that is closest to the origin.

49) A company manufactures two types of athletic shoes: jogging shoes and cross-trainers. The total revenue from x units of jogging shoes and y units of cross-trainers is given by $R(x,y)=-5x^2-8y^2-2xy+42x+102y,$ where x and y are in thousands of units. Find the values of x and y to maximize the total revenue.

Answer

$x=3$ and $y=6$

50) A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96in. Find the dimensions of the box that meets this condition and has the largest volume.

51) Find the maximum volume of a cylindrical soda can such that the sum of its height and circumference is 120 cm.

Answer

$V=\dfrac{64,000}{\pi}\approx 20,372cm^3$

6.8: Lagrange Multipliers

This page is a draft and is under active development.

Solving optimization problems for functions of two or more variables can be similar to solving such problems in single-variable calculus. However, techniques for dealing with multiple variables allow us to solve more varied optimization problems for which we need to deal with additional conditions or constraints. In this section, we examine one of the more common and useful methods for solving optimization problems with constraints.

In the previous section, an applied situation was explored involving maximizing a profit function, subject to certain **constraints.** In that example, the constraints involved a maximum number of golf balls that could be produced and sold in 1 month (x), and a maximum number of advertising hours that could be purchased per month (y). Suppose these were combined into a single budgetary constraint, such as $20x+4y\leq 216$, that took into account both the cost of producing the golf balls and the number of advertising hours purchased per month. The goal is still to maximize profit, but now there is a different type of constraint on the values of x and y. This constraint and the corresponding profit function

$$f(x,y)=48x+96y-x^2-2xy-9y^2 \tag{6.8.1}$$

is an example of an **optimization problem**, and the function $f(x,y)$ is called the **objective function**. A graph of various level curves of the function $f(x,y)$ follows.

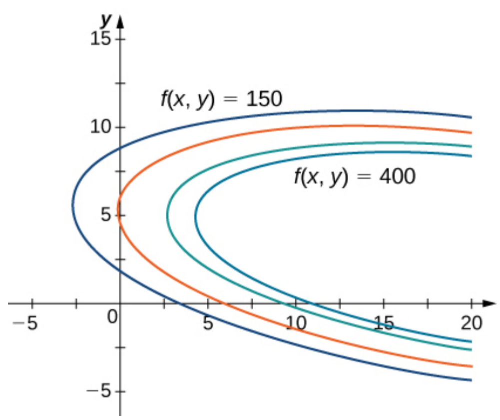


Figure 6.8.1: Graph showing level curves of the function $f(x,y)=48x+96y-x^2-2xy-9y^2$ corresponding to $c=150,250,350,$ and 400.

In Figure 6.8.1, the value c represents different profit levels (i.e., values of the function f). As the value of c increases, the curve shifts to the right. Since our goal is to maximize profit, we want to choose a curve as far to the right as possible. If there were no restrictions on the number of golf balls the company could produce or the number of units of advertising available, then we could produce as many golf balls as we want, and advertise as much as we want, and there would be not be a maximum profit for the company. Unfortunately, we have a budgetary constraint that is modeled by the inequality $20x+4y\leq 216.$ To see how this constraint interacts with the profit function, Figure shows the graph of the line $20x+4y=216$ superimposed on the previous graph.

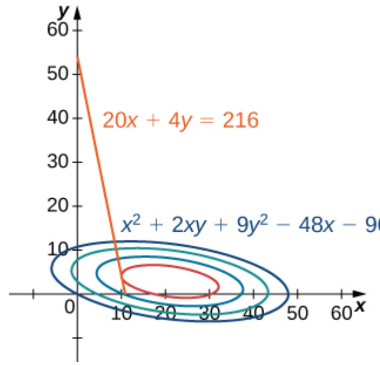


Figure 6.8.2: Graph of level curves of the function $f(x,y)=48x+96y-x^2-2xy-9y^2$ corresponding to $c=150,250,350,$ and 395. The red graph is the constraint function.

As mentioned previously, the maximum profit occurs when the level curve is as far to the right as possible. However, the level of production corresponding to this maximum profit must also satisfy the budgetary constraint, so the point at which this profit occurs must also lie on (or to the left of) the red line in Figure 6.8.2. Inspection of this graph reveals that this point exists where the line is tangent to the level curve of f. Trial and error reveals that this profit level seems to be around 395, when x and y are both just less than 5. We return to the solution of this problem later in this section. From a theoretical standpoint, at the point where the profit curve is tangent to the constraint line, the gradient of both of the functions evaluated at that point must point in the same (or opposite) direction. Recall that the gradient of a function of more than one variable is a vector. If two vectors point in the same (or opposite) directions, then one must be a constant multiple of the other. This idea is the basis of the **method of Lagrange multipliers.**

Method of Lagrange Multipliers: One Constraint

Theorem 6.8.1: Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x,y)=0$. Suppose that f, when restricted to points on the curve $g(x,y)=0$, has a local extremum at the point (x_0,y_0) and that $\vec{\nabla} g(x_0,y_0)\neq 0$. Then there is a number λ called a **Lagrange multiplier,** for which

$$\vec{\nabla} f(x_0,y_0)=\lambda \vec{\nabla} g(x_0,y_0). \tag{6.8.2}$$

Proof

Assume that a constrained extremum occurs at the point (x_0,y_0). Furthermore, we assume that the equation $g(x,y)=0$ can be smoothly parameterized as

$$x=x(s) \text{ and } y=y(s)$$

where s is an arc length parameter with reference point (x_0,y_0) at $s=0$. Therefore, the quantity $z=f(x(s),y(s))$ has a relative maximum or relative minimum at $s=0$, and this implies that $\frac{dz}{ds}=0$ at that point. From the chain rule,

$$\begin{aligned}\frac{dz}{ds} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}\\ &= \left(\frac{\partial f}{\partial x}\hat{\mathbf{i}}+\frac{\partial f}{\partial y}\hat{\mathbf{j}}\right)\cdot\left(\frac{\partial x}{\partial s}\hat{\mathbf{i}}+\frac{\partial y}{\partial s}\hat{\mathbf{j}}\right)\\ &= 0,\end{aligned}$$

where the derivatives are all evaluated at $s=0$. However, the first factor in the dot product is the gradient of f, and the second factor is the unit tangent vector $\vec{\mathbf{T}}(0)$ to the constraint curve. Since the point (x_0,y_0) corresponds to $s=0$, it follows from this equation that

$$\vec{\nabla} f(x_0,y_0)\cdot\vec{\mathbf{T}}(0)=0,$$

which implies that the gradient is either the zero vector $\vec{\mathbf{0}}$ or it is normal to the constraint curve at a constrained relative extremum. However, the constraint curve $g(x,y)=0$ is a level curve for the function $g(x,y)$ so that if $\vec{\nabla} g(x_0,y_0)\neq 0$ then $\vec{\nabla} g(x_0,y_0)$ is normal to this curve at (x_0,y_0) It follows, then, that there is some scalar λ such that

$$\vec{\nabla} f(x_0,y_0)=\lambda\vec{\nabla} g(x_0,y_0)$$

□

To apply **Theorem 6.8.1** to an optimization problem similar to that for the golf ball manufacturer, we need a problem-solving strategy.

Problem-Solving Strategy: Steps for Using Lagrange Multipliers

1. Determine the objective function $f(x,y)$ and the constraint function $g(x,y)$. Does the optimization problem involve maximizing or minimizing the objective function?
2. Set up a system of equations using the following template:

$$\vec{\nabla} f(x_0,y_0)=\lambda\vec{\nabla} g(x_0,y_0) \tag{6.8.3}$$
$$g(x_0,y_0)=0 \tag{6.8.4}$$

3. Solve for x_0 and y_0.
4. The largest of the values of f at the solutions found in step 3 maximizes f; the smallest of those values minimizes f.

Example 6.8.1: Using Lagrange Multipliers

Use the method of Lagrange multipliers to find the minimum value of $f(x,y)=x^2+4y^2-2x+8y$ subject to the constraint $x+2y=7$.

Solution

Let's follow the problem-solving strategy:

1. The objective function is $f(x,y)=x^2+4y^2-2x+8y$. To determine the constraint function, we must first subtract 7 from both sides of the constraint. This gives $x+2y-7=0$. The constraint

function is equal to the left-hand side, so $g(x, y) = x + 2y - 7$. The problem asks us to solve for the minimum value of f, subject to the constraint (Figure 6.8.3).

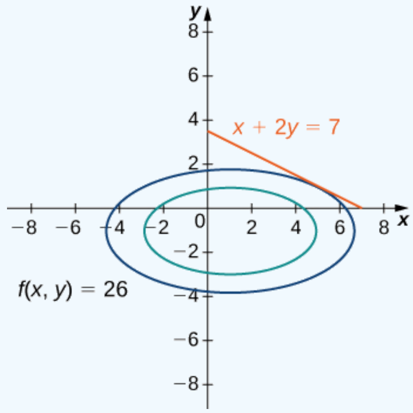


Figure 6.8.3: Graph of level curves of the function $f(x, y) = x^2 + 4y^2 - 2x + 8y$ corresponding to $c = 10$ and 26. The red graph is the constraint function.

2. We then must calculate the gradients of both f and g:

$$\vec{\nabla} f(x, y) = (2x - 2)\,\hat{\mathbf{i}} + (8y + 7)\,\hat{\mathbf{j}} \tag{6.8.5}$$
$$\vec{\nabla} g(x, y) = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}.$$

The equation $\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$ becomes

$$(2x_0 - 2)\,\hat{\mathbf{i}} + (8y_0 + 9)\,\hat{\mathbf{j}} = \lambda\left(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\right), \tag{6.8.6}$$

which can be rewritten as

$$(2x_0 - 2)\,\hat{\mathbf{i}} + (8y_0 + 8)\,\hat{\mathbf{j}} = \lambda\hat{\mathbf{i}} + 2\lambda\hat{\mathbf{j}}. \tag{6.8.7}$$

Next, we set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$2x_0 - 2 = \lambda \tag{6.8.8}$$
$$8y_0 + 8 = 2\lambda. \tag{6.8.9}$$

The equation $g(x_0, y_0) = 0$ becomes $x_0 + 2y_0 - 7 = 0$. Therefore, the system of equations that needs to be solved is

$$2x_0 - 2 = \lambda \tag{6.8.10}$$
$$8y_0 + 8 = 2\lambda \tag{6.8.11}$$
$$x_0 + 2y_0 - 7 = 0. \tag{6.8.12}$$

3. This is a linear system of three equations in three variables. We start by solving the second equation for λ and substituting it into the first equation. This gives $\lambda = 4y_0 + 4$, so substituting this into the first equation gives

$$2x_0 - 2 = 4y_0 + 4.$$

Solving this equation for x_0 gives $x_0 = 2y_0 + 3$. We then substitute this into the third equation:

$$\begin{aligned}(2y_0+3)+2y_0-7&=0\\4y_0-4&=0\\y_0&=1.\end{aligned}$$

Since $x_0 = 2y_0 + 3,$ this gives $x_0 = 5.$

4. Next, we evaluate $f(x,y) = x^2 + 4y^2 - 2x + 8y$ at the point $(5,1)$,

$$f(5,1) = 5^2 + 4(1)^2 - 2(5) + 8(1) = 27. \tag{6.8.13}$$

To ensure this corresponds to a minimum value on the constraint function, let's try some other points on the constraint from either side of the point $(5,1)$, such as the intercepts of $g(x,y)=0$, Which are $(7,0)$ and $(0,3.5)$.

We get $f(7,0)=35>27$ and $f(0,3.5)=77>27$.

So it appears that f has a relative minimum of 27 at $(5,1)$, subject to the given constraint.

Exercise $6.8.1$

Use the method of Lagrange multipliers to find the maximum value of

$$f(x,y) = 9x^2 + 36xy - 4y^2 - 18x - 8y$$

subject to the constraint $3x + 4y = 32.$

Hint

Use the problem-solving strategy for the method of Lagrange multipliers.

Answer

Subject to the given constraint, f has a maximum value of 976 at the point $(8,2)$.

Let's now return to the problem posed at the beginning of the section.

Example $6.8.2$: Golf Balls and Lagrange Multipliers

The golf ball manufacturer, Pro-T, has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2,$$

where z is measured in thousands of dollars. The budgetary constraint function relating the cost of the production of thousands golf balls and advertising units is given by $20x + 4y = 216.$ Find the values of x and y that maximize profit, and find the maximum profit.

Solution:

Again, we follow the problem-solving strategy:

1. The objective function is $f(x,y)=48x+96y-x^2-2xy-9y^2.$ To determine the constraint function, we first subtract 216 from both sides of the constraint, then divide both sides by 4, which gives $5x+y-54=0.$ The constraint function is equal to the left-hand side, so $g(x,y)=5x+y-54.$ The problem asks us to solve for the maximum value of f, subject to this constraint.
2. So, we calculate the gradients of both f and g:

$$\begin{aligned}\vec{\nabla} f(x,y) &= (48-2x-2y)\hat{\mathbf{i}}+(96-2x-18y)\hat{\mathbf{j}}\\ \vec{\nabla} g(x,y) &= 5\hat{\mathbf{i}}+\hat{\mathbf{j}}.\end{aligned}$$

   The equation $\vec{\nabla} f(x_0,y_0)=\lambda\vec{\nabla} g(x_0,y_0)$ becomes

$$(48-2x_0-2y_0)\hat{i}+(96-2x_0-18y_0)\hat{\mathbf{j}}=\lambda(5\hat{\mathbf{i}}+\hat{\mathbf{j}}),$$

   which can be rewritten as

$$(48-2x_0-2y_0)\hat{\mathbf{i}}+(96-2x_0-18y_0)\hat{\mathbf{j}}=\lambda 5\hat{\mathbf{i}}+\lambda\hat{\mathbf{j}}.$$

   We then set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$\begin{aligned}48-2x_0-2y_0 &= 5\lambda\\ 96-2x_0-18y_0 &= \lambda.\end{aligned}$$

   The equation $g(x_0,y_0)=0$ becomes $5x_0+y_0-54=0$. Therefore, the system of equations that needs to be solved is

$$\begin{aligned}48-2x_0-2y_0 &= 5\lambda\\ 96-2x_0-18y_0 &= \lambda\\ 5x_0+y_0-54 &= 0.\end{aligned}$$

3. We use the left-hand side of the second equation to replace λ in the first equation:

$$\begin{aligned}48-2x_0-2y_0 &= 5(96-2x_0-18y_0)\\ 48-2x_0-2y_0 &= 480-10x_0-90y_0\\ 8x_0 &= 432-88y_0\\ x_0 &= 54-11y_0.\end{aligned}$$

   Then we substitute this into the third equation:

$$\begin{aligned}5(54-11y_0)+y_0-54 &= 0\\ 270-55y_0+y_0-54 &= 0\\ 216-54y_0 &= 0\\ y_0 &= 4.\end{aligned}$$

Since $x_0 = 54 - 11y_0,$ this gives $x_0 = 10.$

4. We then substitute $(10, 4)$ into $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2,$ which gives

$$\begin{aligned} f(10,4) &= 48(10) + 96(4) - (10)^2 - 2(10)(4) - 9(4)^2 \\ &= 480 + 384 - 100 - 80 - 144 = 540. \end{aligned}$$

Therefore the maximum profit that can be attained, subject to budgetary constraints, is $\$540,000$ with a production level of $10,000$ golf balls and 4 hours of advertising bought per month. Let's check to make sure this truly is a maximum. The endpoints of the line that defines the constraint are $(10.8, 0)$ and $(0, 54)$ Let's evaluate f at both of these points:

$$\begin{aligned} f(10.8,0) &= 48(10.8) + 96(0) - 10.8^2 - 2(10.8)(0) - 9(0^2) \\ &= 401.76 \\ f(0,54) &= 48(0) + 96(54) - 0^2 - 2(0)(54) - 9(54^2) \\ &= -21,060. \end{aligned}$$

The second value represents a loss, since no golf balls are produced. Neither of these values exceed 540, so it seems that our extremum is a maximum value of f, subject to the given constraint.

Exercise 6.8.2: Optimizing the Cobb-Douglas function

A company has determined that its production level is given by the Cobb-Douglas function $f(x, y) = 2.5x^{0.45}y^{0.55}$ where x represents the total number of labor hours in 1 year and y represents the total capital input for the company. Suppose 1 unit of labor costs \$40 and 1 unit of capital costs \$50. Use the method of Lagrange multipliers to find the maximum value of $f(x, y) = 2.5x^{0.45}y^{0.55}$ subject to a budgetary constraint of $\$500,000$ per year.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers.

Answer

Subject to the given constraint, a maximum production level of 13890 occurs with 5625 labor hours and \$5500 of total capital input.

In the case of an objective function with three variables and a single constraint function, it is possible to use the method of Lagrange multipliers to solve an optimization problem as well. An example of an objective function with three variables could be the **Cobb-Douglas function** in Exercise 6.8.2: $f(x, y, z) = x^{0.2}y^{0.4}z^{0.4},$ where x represents the cost of labor, y represents capital input, and z represents the cost of advertising. The method is the same as for the method with a function of two variables; the equations to be solved are

$$\begin{aligned} \vec{\nabla} f(x,y,z) &= \lambda \vec{\nabla} g(x,y,z) \\ g(x,y,z) &= 0. \end{aligned}$$

Example 6.8.3: Lagrange Multipliers with a Three-Variable objective function

Maximize the function $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $x + y + z = 1$.

Solution:

1. The objective function is $f(x,y,z) = x^2 + y^2 + z^2$. To determine the constraint function, we subtract 1 from each side of the constraint: $x + y + z - 1 = 0$ which gives the constraint function as $g(x,y,z) = x + y + z - 1$.

2. Next, we calculate $\vec{\nabla} f(x,y,z)$ and $\vec{\nabla} g(x,y,z)$:

$$\begin{aligned} \vec{\nabla} f(x,y,z) &= \langle 2x, 2y, 2z\rangle \\ \vec{\nabla} g(x,y,z) &= \langle 1,1,1\rangle. \end{aligned}$$

This leads to the equations

$$\begin{aligned} \langle 2x_0, 2y_0, 2z_0\rangle &= \lambda\langle 1,1,1\rangle \\ x_0 + y_0 + z_0 - 1 &= 0 \end{aligned}$$

which can be rewritten in the following form:

$$\begin{aligned} 2x_0 &= \lambda \\ 2y_0 &= \lambda \\ 2z_0 &= \lambda \\ x_0 + y_0 + z_0 - 1 &= 0. \end{aligned}$$

3. Since each of the first three equations has λ on the right-hand side, we know that $2x_0 = 2y_0 = 2z_0$ and all three variables are equal to each other. Substituting $y_0 = x_0$ and $z_0 = x_0$ into the last equation yields $3x_0 - 1 = 0$, so $x_0 = \frac{1}{3}$ and $y_0 = \frac{1}{3}$ and $z_0 = \frac{1}{3}$ which corresponds to a critical point on the constraint curve.

4. Then, we evaluate f at the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$:

$$f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = (\frac{1}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2 = \frac{3}{9} = \frac{1}{3} \tag{6.8.14}$$

Therefore, a possible extremum of the function is $\frac{1}{3}$. To verify it is a minimum, choose other points that satisfy the constraint from either side of the point we obtained above and calculate f at those points. For example,

$$\begin{aligned} f(1,0,0) &= 1^2 + 0^2 + 0^2 = 1 \\ f(0,-2,3) &= 0^2 + +(-2)^2 + 3^2 = 13. \end{aligned}$$

Both of these values are greater than $\frac{1}{3}$, leading us to believe the extremum is a minimum, subject to the given constraint.

Exercise 6.8.3:

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x,y,z)=x+y+z$$

subject to the constraint $x^2+y^2+z^2=1.$

Hint

Use the problem-solving strategy for the method of Lagrange multipliers with an objective function of three variables.

Answer

Evaluating f at both points we obtained, gives us,

$$f\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)=\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{3}=\sqrt{3}$$
$$f\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)=-\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}=-\sqrt{3}$$

Since the constraint is continuous, we compare these values and conclude that f has a relative minimum of $-\sqrt{3}$ at the point $\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)$, subject to the given constraint.

The method of Lagrange multipliers can be applied to problems with more than one constraint. In this case the objective function, w is a function of three variables:

$$w=f(x,y,z) \tag{6.8.15}$$

and it is subject to two constraints:

$$g(x,y,z)=0 \text{ and } h(x,y,z)=0. \tag{6.8.16}$$

There are two Lagrange multipliers, λ_1 and λ_2, and the system of equations becomes

$$\begin{aligned}\vec{\nabla} f(x_0,y_0,z_0) &=\lambda_1\vec{\nabla} g(x_0,y_0,z_0)+\lambda_2\vec{\nabla} h(x_0,y_0,z_0)\\ g(x_0,y_0,z_0)&=0\\ h(x_0,y_0,z_0)&=0\end{aligned}$$

Example 6.8.4: Lagrange Multipliers with Two Constraints

Find the maximum and minimum values of the function

$$f(x,y,z)=x^2+y^2+z^2$$

subject to the constraints $z^2=x^2+y^2$ and $x+y-z+1=0$.

Solution:

Let's follow the problem-solving strategy:

1. The objective function is $f(x,y,z)=x^2+y^2+z^2$. To determine the constraint functions, we first subtract z^2 from both sides of the first constraint, which gives $x^2+y^2-z^2=0$, so $g(x,y,z)=x^2+y^2-z^2$. The second constraint function is $h(x,y,z)=x+y-z+1$.
2. We then calculate the gradients of f, g, and h:

$$\begin{aligned}\vec{\nabla} f(x,y,z) &= 2x\,\hat{\mathbf{i}}+2y\hat{\mathbf{j}}+2z\hat{\mathbf{k}}\\ \vec{\nabla} g(x,y,z) &= 2x\,\hat{\mathbf{i}}+2y\hat{\mathbf{j}}-2z\hat{\mathbf{k}}\\ \vec{\nabla} h(x,y,z) &= \hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}.\end{aligned}$$

The equation $\vec{\nabla} f(x_0,y_0,z_0)=\lambda_1\vec{\nabla} g(x_0,y_0,z_0)+\lambda_2\vec{\nabla} h(x_0,y_0,z_0)$ becomes

$$2x_0\,\hat{\mathbf{i}}+2y_0\hat{\mathbf{j}}+2z_0\hat{\mathbf{k}}=\lambda_1(2x_0\,\hat{\mathbf{i}}+2y_0\hat{\mathbf{j}}-2z_0\hat{\mathbf{k}})+\lambda_2(\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}), \tag{6.8.17}$$

which can be rewritten as

$$2x_0\,\hat{\mathbf{i}}+2y_0\hat{\mathbf{j}}+2z_0\hat{\mathbf{k}}=(2\lambda_1x_0+\lambda_2)\hat{\mathbf{i}}+(2\lambda_1y_0+\lambda_2)\hat{\mathbf{j}}-(2\lambda_1z_0+\lambda_2)\hat{\mathbf{k}}. \tag{6.8.18}$$

Next, we set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$\begin{aligned}2x_0 &= 2\lambda_1x_0+\lambda_2\\ 2y_0 &= 2\lambda_1y_0+\lambda_2\\ 2z_0 &= -2\lambda_1z_0-\lambda_2.\end{aligned}$$

The two equations that arise from the constraints are $z_0^2=x_0^2+y_0^2$ and $x_0+y_0-z_0+1=0$. Combining these equations with the previous three equations gives

$$\begin{aligned}2x_0 &= 2\lambda_1x_0+\lambda_2\\ 2y_0 &= 2\lambda_1y_0+\lambda_2\\ 2z_0 &= -2\lambda_1z_0-\lambda_2\\ z_0^2 &= x_0^2+y_0^2\\ x_0+y_0-z_0+1 &= 0.\end{aligned}$$

3. The first three equations contain the variable λ_2. Solving the third equation for λ_2 and replacing into the first and second equations reduces the number of equations to four:

$$\begin{aligned} 2x_0 &= 2\lambda_1 x_0 - 2\lambda_1 z_0 - 2z_0 \\ 2y_0 &= 2\lambda_1 y_0 - 2\lambda_1 z_0 - 2z_0 \\ z_0^2 &= x_0^2 + y_0^2 \\ x_0 + y_0 - z_0 + 1 &= 0. \end{aligned}$$

Next, we solve the first and second equation for λ_1. The first equation gives $\lambda_1 = \dfrac{x_0 + z_0}{x_0 - z_0}$, the second equation gives $\lambda_1 = \dfrac{y_0 + z_0}{y_0 - z_0}$. We set the right-hand side of each equation equal to each other and cross-multiply:

$$\begin{aligned} \frac{x_0 + z_0}{x_0 - z_0} &= \frac{y_0 + z_0}{y_0 - z_0} \\ (x_0 + z_0)(y_0 - z_0) &= (x_0 - z_0)(y_0 + z_0) \\ x_0 y_0 - x_0 z_0 + y_0 z_0 - z_0^2 &= x_0 y_0 + x_0 z_0 - y_0 z_0 - z_0^2 \\ 2y_0 z_0 - 2x_0 z_0 &= 0 \\ 2z_0(y_0 - x_0) &= 0. \end{aligned}$$

Therefore, either $z_0 = 0$ or $y_0 = x_0$. If $z_0 = 0$, then the first constraint becomes $0 = x_0^2 + y_0^2$. The only real solution to this equation is $x_0 = 0$ and $y_0 = 0$, which gives the ordered triple $(0, 0, 0)$. This point does not satisfy the second constraint, so it is not a solution. Next, we consider $y_0 = x_0$, which reduces the number of equations to three:

$$\begin{aligned} y_0 &= x_0 \\ z_0^2 &= x_0^2 + y_0^2 \\ x_0 + y_0 - z_0 + 1 &= 0. \end{aligned}$$

We substitute the first equation into the second and third equations:

$$\begin{aligned} z_0^2 &= x_0^2 + x_0^2 \\ &= x_0 + x_0 - z_0 + 1 = 0. \end{aligned}$$

Then, we solve the second equation for z_0, which gives $z_0 = 2x_0 + 1$. We then substitute this into the first equation,

$$\begin{aligned} z_0^2 &= 2x_0^2 \\ (2x_0^2 + 1)^2 &= 2x_0^2 \\ 4x_0^2 + 4x_0 + 1 &= 2x_0^2 \\ 2x_0^2 + 4x_0 + 1 &= 0, \end{aligned}$$

and use the quadratic formula to solve for x_0:

$$x_0 = \frac{-4 \pm \sqrt{4^2 - 4(2)(1)}}{2(2)} = \frac{-4 \pm \sqrt{8}}{4} = \frac{-4 \pm 2\sqrt{2}}{4} = -1 \pm \frac{\sqrt{2}}{2}. \tag{6.8.19}$$

Recall $y_0 = x_0$, so this solves for y_0 as well. Then, $z_0 = 2x_0 + 1$, so

$$z_0 = 2x_0 + 1 = 2\left(-1 \pm \frac{\sqrt{2}}{2}\right) + 1 = -2 + 1 \pm \sqrt{2} = -1 \pm \sqrt{2}. \tag{6.8.20}$$

Therefore, there are two ordered triplet solutions:

$$\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right) \text{ and } \left(-1 - \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}, -1 - \sqrt{2}\right). \tag{6.8.21}$$

4. We substitute $\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right)$ into $f(x, y, z) = x^2 + y^2 + z^2$, which gives

$$\begin{aligned} f\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right) &= \left(-1 + \frac{\sqrt{2}}{2}\right)^2 + \left(-1 + \frac{\sqrt{2}}{2}\right)^2 + (-1 + \sqrt{2})^2 \\ &= \left(1 - \sqrt{2} + \frac{1}{2}\right) + \left(1 - \sqrt{2} + \frac{1}{2}\right) + (1 - 2\sqrt{2} + 2) \\ &= 6 - 4\sqrt{2}. \end{aligned}$$

Then, we substitute $\left(-1 - \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right)$ into $f(x, y, z) = x^2 + y^2 + z^2$, which gives

$$\begin{aligned} f\left(-1 - \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right) &= \left(-1 - \frac{\sqrt{2}}{2}\right)^2 + \left(-1 - \frac{\sqrt{2}}{2}\right)^2 + (-1 - \sqrt{2})^2 \\ &= \left(1 + \sqrt{2} + \frac{1}{2}\right) + \left(1 + \sqrt{2} + \frac{1}{2}\right) + (1 + 2\sqrt{2} + 2) \\ &= 6 + 4\sqrt{2}. \end{aligned}$$

$6 + 4\sqrt{2}$ is the maximum value and $6 - 4\sqrt{2}$ is the minimum value of $f(x, y, z)$, subject to the given constraints.

Exercise 6.8.4

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x, y, z) = x^2 + y^2 + z^2 \tag{6.8.22}$$

subject to the constraints $2x + y + 2z = 9$ and $5x + 5y + 7z = 29$.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers with two constraints.

Answer

$f(2,1,2)=9$ is a minimum value of f, subject to the given constraints.

6.8.1 Key Concepts

- An objective function combined with one or more constraints is an example of an optimization problem.
- To solve optimization problems, we apply the method of Lagrange multipliers using a four-step problem-solving strategy.

- **Method of Lagrange multipliers, one constraint**

  $\vec{\nabla} f(x_0,y_0)=\lambda\vec{\nabla} g(x_0,y_0)$

  $g(x_0,y_0)=0$

- **Method of Lagrange multipliers, two constraints**

  $\vec{\nabla} f(x_0,y_0,z_0)=\lambda_1\vec{\nabla} g(x_0,y_0,z_0)+\lambda_2\vec{\nabla} h(x_0,y_0,z_0)$

  $g(x_0,y_0,z_0)=0$

  $h(x_0,y_0,z_0)=0$

constraint

an inequality or equation involving one or more variables that is used in an optimization problem; the constraint enforces a limit on the possible solutions for the problem

Lagrange multiplier

the constant (or constants) used in the method of Lagrange multipliers; in the case of one constant, it is represented by the variable λ

method of Lagrange multipliers

a method of solving an optimization problem subject to one or more constraints

objective function

the function that is to be maximized or minimized in an optimization problem

optimization problem

calculation of a maximum or minimum value of a function of several variables, often using Lagrange multipliers

-

6.8E:

6.8E.1 Exercise $6.8E.1$

For the following exercises, use the method of Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraints.

1) $f(x,y)=x^2y; x^2+2y=6$

2) $f(x,y,z)=xyz; x^2+2y^2+3z^2=6$

Answer

maximum: \(\frac{2}{\sqrt{3}\), minimum: \(-\frac{2}{\sqrt{3}\)

3) $f(x,y)=xy; 4x^2+8y^2=16$

4) $f(x,y)=4x^3+y^2; 2x^2+y^2=1$

Answer

maximum: $\sqrt{2}$, minimum: $-sqrt2$

5) $f(x,y,z)=x^2+y^2+z^2, x^4+y^4+z^4=1$

6) $f(x,y,z)=yz+xy, xy=1, y^2+z^2=1$

Answer

maximum: 32, minimum = 11

7) $f(x,y)=x^2+y^2, (x-1)^2+4y^2=4$

8) $f(x,y)=4xy, x^2+y^2=1$

Answer

maximum: 2, minimum = -2

9) $f(x,y,z)=x+y+z, x+y+z=1$

10) $f(x,y,z)=x+3y-z, x^2+y^2+z^2=4$

11) $f(x,y,z)=x^2+y^2+z^2, xyz=4$

6.8E.2 Exercise $6.8E.2$

1) Minimize $f(x,y)=x^2+y^2$ on the hyperbola $xy=1$.

Answer

2

2) Minimize $f(x,y)=xy$ on the ellipse $b^2x^2+a^2y^2=a^2b^2$.

3) Maximize $f(x,y,z)=2x+3y+5z$ on the sphere $x^2+y^2+z^2=19$

Answer

$19\sqrt{2}$

4) Maximize $f(x,y)=x^2-y^2; x>0, y>0; g(x,y)=y-x^2=0$.

5) The curve $x^3-y^3=1$ is asymptotic to the line $y=x$. Find the point(s) on the curve $x^3-y^3=1$ farthest from the line $y=x$.

Answer

(12√3,−12√3)

6) Maximize $U(x,y)=8x^4/5y; 4x+2y=12$

7) Minimize $f(x,y)=x^2+y^2, x+2y-5=0.$

Answer

$f(1,2)=5$

8) Maximize $f(x,y)=6-x^2-y^2; x+y-2=0.$

9) Minimize $f(x,y,z)=x^2+y^2+z^2; x+y+z=1.$

Answer

13

10) Minimize $f(x,y)=x^2-y^2$ subject to the constraint $x-2y+6=0.$

11) Minimize $f(x,y,z)=x^2+y^2+z^2$ when$x+y+z=9$ and $x+2y+3z=20.$

Answer

minimum: f(2,3,4)=29

6.8E.3 Exercise $\mathbf{6.8E.3}$

use the method of Lagrange multipliers to solve the following applied problems.

1) A pentagon is formed by placing an isosceles triangle on a rectangle, as shown in the diagram. If the perimeter of the pentagon is 1010 in., find the lengths of the sides of the pentagon that will maximize the area of the pentagon.

A rectangle with an isosceles triangle on top. The side of the isosceles triangle with the two equal angles of size θ overlaps the top length of the rectangle.

2) A rectangular box without a top (a topless box) is to be made from 1212 ft^2 of cardboard. Find the maximum volume of such a box.

Answer

The maximum volume is 44 ft^3. The dimensions are 1×2×21×2×2 ft.

3) Find the minimum and maximum distances between the ellipse $x^2+xy+2y^2=1$ and the origin.

4) Find the point on the surface $x^2-2xy+y^2-x+y=0$ closest to the point $(1,2,-3)$.

Answer

(1,12,−3)

5) Show that, of all the triangles inscribed in a circle of radius R (see diagram), the equilateral triangle has the largest perimeter.

A circle with an equilateral triangle drawn inside of it such that each vertex of the triangle touches the circle.

6) Find the minimum distance from point $(0,1)$ to the parabola $x^2=4y$.

Answer

1.0

7) Find the minimum distance from the parabola $y=x^2$ to point $(0,3)$.

8) Find the minimum distance from the plane $x+y+z=1$ to point $(2,1,1)$.

Answer

3–√3

9) A large container in the shape of a rectangular solid must have a volume of 480480 m^3. The bottom of the container costs \$5/m^2 to construct whereas the top and sides cost \$3/m^2 to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has the minimum cost.

10) Find the point on the line $y = 2x + 3$ that is closest to point (4,2).

Answer

(25,195)

---

110 Find the point on the plane $4x + 3y + z = 2$ that is closest to the point (1,−1,1).

12) Find the maximum value of $f(x, y) = sinx siny$, where x and y denote the acute angles of a right triangle. Draw the contours of the function using a CAS.

Answer

12 An alternating series of hills and holes of amplitude 1 across xyz space.

---

13) A rectangular solid is contained within a tetrahedron with vertices at (1,0,0),(0,1,0),(0,0,1),(1,0,0),(0,1,0),(0,0,1), and the origin. The base of the box has dimensions x, y and the height of the box is z. If the sum of x, y, and z is 1.0, find the dimensions that maximize the volume of the rectangular solid.

14) [T] By investing *x* units of labour and *y* units of capital, a watch manufacturer can produce $P(x, y) = 50x0.4y$ watches. Find the maximum number of watches that can be produced on a budget of $\$20,000$ if labour costs \$100/unit and capital costs \$200/unit. Use a CAS to sketch a contour plot of the function.

Answer

Roughly 3365 watches at the critical point (80,60)(80,60)

A series of curves in the first quadrant, with the first starting near (2, 120), decreasing sharply to near (20, 20), and then decreasing slowly to (120, 5). The next curve starts near (10, 120), decreases sharply to near (40, 40), and then decreases slowly to (120, 20). The next curve starts near (20, 120), decreases sharply to near (60, 60), and then decreases slowly to (120, 40). The next curve starts near (40, 120), decreases to near (80, 80), and then decreases a little slowly to (120, 60). The last curve starts near (60, 120) and decreases rather evenly through (100, 100) to (120, 90).

6.8E.4

6E: Chapter Review Excersies

6E.1 Exercise $6E.1$

Evaluate the indicated limit or explain why it does not exist.

1) $\lim_{(x,y)\to(0,0)} 2\sqrt{x^2+y^2}$

2) $\lim_{(x,y)\to(0,0)} \frac{3x}{x^2+y^2}$

3) $\lim_{(x,y)\to(0,0)} \frac{2\sin(xy)}{x^2+y^2}$

4) $\lim_{(x,y)\to(0,0)} \frac{4x^2y^2}{x^2+y^4}$

Answer

Add texts here. Do not delete this text first.

6E.2 Exercise $6E.2$

Define $f(0,0)$ in a way that extends \[f(x,y)=2xy \frac{x^2-y^2}{x^2+y^2}\[to be continuous at the origin.

Answer

Add texts here. Do not delete this text first.

6E.3 Exercise $6E.3$

Find the first partial derivative of \[f(x,y,z)=3x^{(y\ln z)}\[at $(e,2,e)$.

Answer

Add texts here. Do not delete this text first.

6E.4 Exercise $6E.5$

1) Find the equations of the tangent plane and normal line to the graph of \[f(x,y)=\tan^{-1}(y/x)\[at $(1,-1)$.

2) Given $f(x,y)=\ln(x^2+y^2)$.

a) Find an equation of the plane tangent to the graph of f at $(1,-2)$.

b) Find an equation of the tangent line at $(1,-2)$ to the level curve of f that passes through $(1,-2)$.

Answer

Add texts here. Do not delete this text first.

6E.5 Exercise $6E.6$

1) Find and classify all the critical points of $g(x,y)=2xye^{-x+y}$.

2) Find the maximum and minimum values of $f(x,y)=3xy$ on the closed disk $x^2+y^2\leq 9$.

Answer

1) $(0,0),(1-1) local minimum$

6E.6 Exercise $6E.7$

Find the Jacobian matrix $D\mathbf{f}(x, y, z)$ for the transformation of $\Re^2$ to $\Re^3$

given by \[{\bf f}(x,y)=(xe^y+cos(\pi y),x^2 z,x-e^y).\[Use it to find an approximate value for $\mathbf{f}(1.02, 0.01)$.

Answer

Add texts here. Do not delete this text first.

6E.7 Exercise $6E.8$

The temperature at position (x, y) in a region of the $xy-$plane is $T^\circ C$, where $T(x, y) = x^2 - x + y + 2y^2$.

1) In what direction should an ant at position $(3, -2)$ move if it wishes to cool off as quickly as possible?

2) If the ant moves in that direction at speed k (units distance per unit time), at what rate does it experience a decrease of temperature?

3) At what direction should an ant move from $(1, -1)$ to experience zero temperature change

Answer

Add texts here. Do not delete this text first.

6E.8 Exercise $6E.9$

By using Lagrange multipliers solve the following:

A rectangular box having no top and having a prescribed volume $V\, m^3$ is to be constructed using two different materials. The material used for the bottom and front of the box is five times as costly (per square metre) as the material used for the back and the other two sides. what should be the dimensions of the box to minimize the cost of materials?

Answer

Add texts here. Do not delete this text first.

6E.9 Exercise $6E.10$

By using Least square approximation to approximate $g(x) = x^3$ over the interval $[0, 1]$ by a linear function $f(x) = px^2 + q$.

Answer

Add texts here. Do not delete this text first.

CHAPTER OVERVIEW

7: Multiple Integration

This page is a draft and is under active development.

In this chapter we extend the concept of a definite integral of a single variable to double and triple integrals of functions of two and three variables, respectively. We examine applications involving integration to compute volumes, masses, and centroids of more general regions. We will also see how the use of other coordinate systems (such as polar, cylindrical, and spherical coordinates) makes it simpler to compute multiple integrals over some types of regions and functions. As an example, we will use polar coordinates to find the volume of structures such as l'Hemisfèric (Figure 7.1).

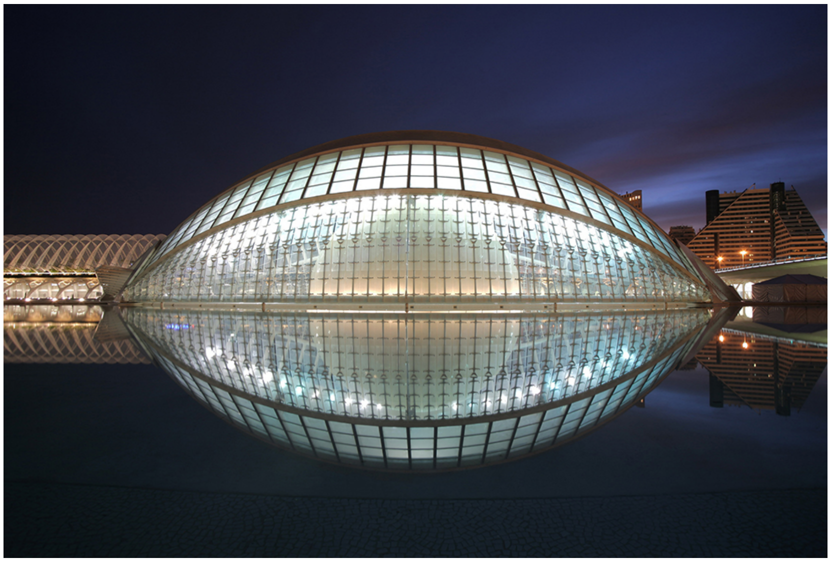


Figure 7.1: The City of Arts and Sciences in Valencia, Spain, has a unique structure along an axis of just two kilometers that was formerly the bed of the River Turia. The l'Hemisfèric has an IMAX cinema with three systems of modern digital projections onto a concave screen of 900 square meters. An oval roof over 100 meters long has been made to look like a huge human eye that comes alive and opens up to the world as the "Eye of Wisdom." (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

In the preceding chapter, we discussed differential calculus with multiple independent variables. Now we examine integral calculus in multiple dimensions. Just as a partial derivative allows us to differentiate a function with respect to one variable while holding the other variables constant, we will see that an iterated integral allows us to integrate a function with respect to one variable while holding the other variables constant.

7.1 Contributors

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

Topic hierarchy

7.1 :Double Integrals over Rectangular Regions
7.1E: Exercises
7.2: Double Integrals over General Regions
7.2E: Exercises
7.3: Double Integrals in Polar Coordinates
7.3E:
7.4: Triple Integrals
7.4E:
7.5: Triple Integrals in Cylindrical and Spherical Coordinates
7.5E:
7.6: Calculating Centers of Mass and Moments of Inertia
7.6E
7.7: Change of Variables in Multiple Integrals
7.7E:
7E: Chapter Review Excersies

7.1 :Double Integrals over Rectangular Regions

This page is a draft and is under active development.

Learning Objectives

- Recognize when a function of two variables is integrable over a rectangular region.
- Recognize and use some of the properties of double integrals.
- Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

7.1 .1 Volumes and Double Integrals

We begin by considering the space above a rectangular region R. Consider a continuous function $f(x,y) \geq 0$ of two variables defined on the closed rectangle R:

$$R = [a,b] \times [c,d] = \left\{ (x,y) \in \mathbb{R}^2 \,|\, a \leq x \leq b,\; c \leq y \leq d \right\} \tag{7.1 .1}$$

Here $[a,b] \times [c,d]$ denotes the Cartesian product of the two closed intervals $[a,b]$ and $[c,d]$. It consists of rectangular pairs (x,y) such that $a \leq x \leq b$ and $c \leq y \leq d$. The graph of f represents a surface above the xy-plane with equation $z = f(x,y)$ where z is the height of the surface at the point (x,y). Let S be the solid that lies above R and under the graph of f (Figure $7.1.1$). The base of the solid is the rectangle R in the xy-plane. We want to find the volume V of the solid S.

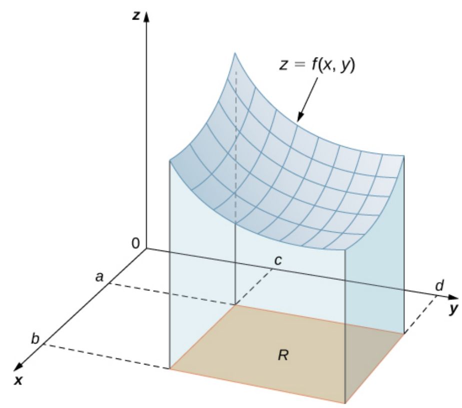


Figure $7.1.1$: The graph of $f(x,y)$ over the rectangle R in the xy-plane is a curved surface.

We divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy (Figure $7.1.2$). We do this by dividing the interval $[a,b]$ into m subintervals and dividing the interval $[c,d]$ into n subintervals. Hence $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, and $\Delta A = \Delta x \Delta y$.

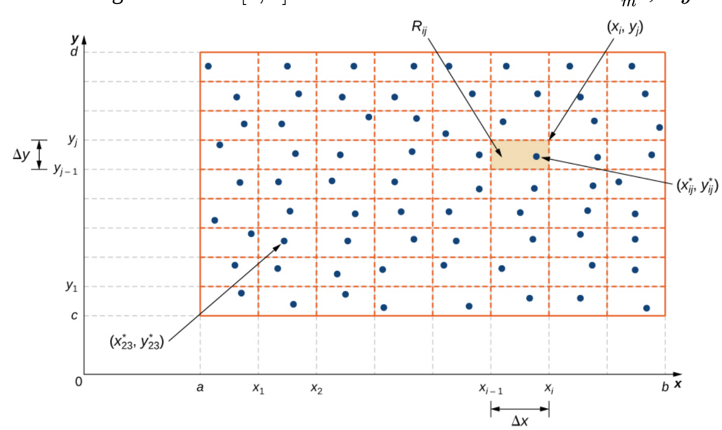


Figure $7.1.2$: Rectangle R is divided into small rectangles R_{ij} each with area ΔA.

The volume of a thin rectangular box above R_{ij} is $f(x^*_{ij}, y^*_{ij})\,\Delta A$, where (x^*_{ij}, y^*_{ij}) is an arbitrary sample point in each R_{ij} as shown in the following figure, $f(x^*_{ij}, y^*_{ij})$ is the height of the corresponding thin rectangular box, and ΔA is the area of each rectangle R_{ij}.

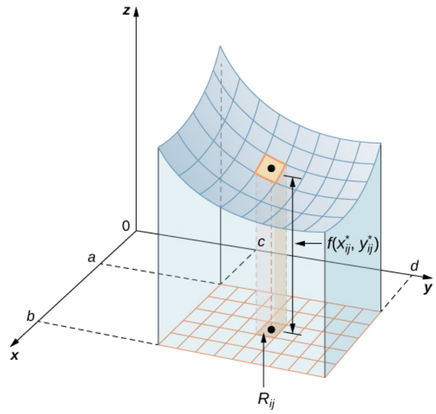


Figure 7.1.3: A thin rectangular box above R_{ij} with height $f(x^*_{ij},\, y^*_{ij})$.

Using the same idea for all the subrectangles, we obtain an approximate volume of the solid S as

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij},\, y^*_{ij})\Delta A. \tag{7.1 .2}$$

This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if *m* and *n* become larger.

$$V = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij},\, y^*_{ij})\Delta A \tag{7.1 .3}$$

or

$$V = \lim_{\Delta x,\, \Delta y\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij},\, y^*_{ij})\Delta A. \tag{7.1 .4}$$

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base R. Now we are ready to define the double integral.

Definition

The double integral of the function $f(x,\, y)$ over the rectangular region R in the xy-plane is defined as

$$\iint_R f(x,\, y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij},\, y^*_{ij})\Delta A. \tag{7.1 .5}$$

If $f(x,y) \geq 0$, then the volume V of the solid S, which lies above R in the xy-plane and under the graph of f, is the double integral of the function $f(x,y)$ over the rectangle R. If the function is ever negative, then the double integral can be considered a “signed” volume in a manner similar to the way we defined net signed area in The Definite Integral.

Example 7.1.1: Setting up a Double Integral and Approximating It by Double Sums

Consider the function $z = f(x,\, y) = 3x^2 - y$ over the rectangular region $R = [0,2] \times [0,2]$ (Figure 7.1.4).

a. Set up a double integral for finding the value of the signed volume of the solid S that lies above R and “under” the graph of f.
b. Divide R into four squares with $m = n = 2$, and choose the sample point as the upper right corner point of each square (1,1),(2,1),(1,2), and (2,2) (Figure 7.1.4) to approximate the signed volume of the solid S that lies above R and “under” the graph of f.
c. Divide R into four squares with $m = n = 2$, and choose the sample point as the midpoint of each square: (1/2, 1/2), (3/2, 1/2), (1/2,3/2), and (3/2, 3/2) to approximate the signed volume.

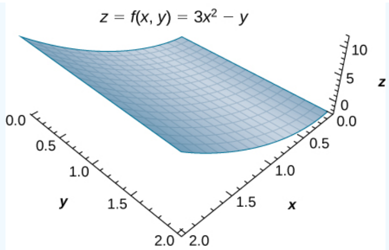


Figure 7.1.4: The function $z = f(x, y)$ graphed over the rectangular region $R = [0, 2] \times [0, 2]$.

Solution

a. As we can see, the function $z = f(x, y) = 3x^2$ is above the plane. To find the signed volume of S, we need to divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy, and choose (x^*_{ij}, y^*_{ij}) as sample points in each R_{ij}. Hence, a double integral is set up as

$$V = \iint_R (3x^2 - y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} [3(x^*_{ij})^2 - y^*_{ij}]\Delta A.$$

b. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. Also, the sample points are (1, 1), (2, 1), (1, 2), and (2, 2) as shown in the following figure.

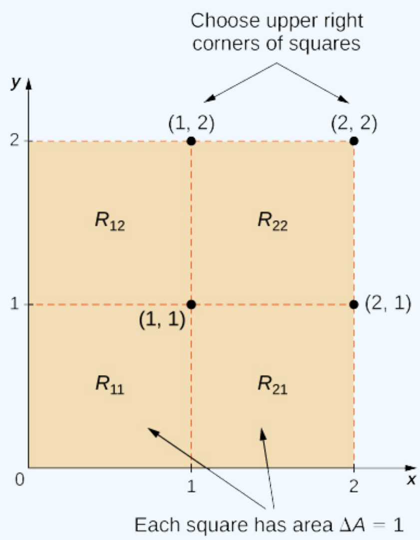


Figure 7.1.5: Subrectangles for the rectangular region R=[0,2]×[0,2].R=[0,2]×[0,2].

Hence,

$$V = \sum_{i=1}^{2}\sum_{j=1}^{2} f(x^*_{ij}, y^*_{ij})\Delta A$$

$$= \sum_{i=1}^{2}(f(x^*_{i1}, y^*_{i1}) + f(x^*_{i2}, y^*_{i2}))\Delta A$$

$$= f(x^*_{11}, y^*_{11})\Delta A + f(x^*_{21}, y^*_{21})\Delta A + f(x^*_{12}, y^*_{12})\Delta A + f(x^*_{22}, y^*_{22})\Delta A$$

$$= f(1,1)(1) + f(2,1)(1) + f(1,2)(1) + f(2,2)(1)$$

$$= (3-1)(1) + (12-1)(1) + (3-2)(1) + (12-2)(1)$$

$$= 2 + 11 + 1 + 10 = 24.$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. In this case the sample points are (1/2, 1/2), (3/2, 1/2), (1/2, 3/2), and (3/2, 3/2).
Hence,

$$\begin{aligned} V &= \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1/2, 1/2)(1) + f(3/2, 1/2)(1) + f(1/2, 3/2)(1) + f(3/2, 3/2)(1) \\ &= \left(\frac{3}{4} - \frac{1}{4}\right)(1) + \left(\frac{27}{4} - \frac{1}{2}\right)(1) + \left(\frac{3}{4} - \frac{3}{2}\right)(1) + \left(\frac{27}{4} - \frac{3}{2}\right)(1) \\ &= \frac{2}{4} + \frac{25}{4} + \left(-\frac{3}{4}\right) + \frac{21}{4} = \frac{45}{4} = 11. \end{aligned}$$

Analysis

Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

Exercise $7.1.1$

Use the same function$z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$.

Divide R into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square (0,1), (1,1), (0,2), and (1,2) (Figure $7.1.5$) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

Hint

Follow the steps of the previous example.

Answer

$$V = \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\,\Delta A = 0$$

Note that we developed the concept of double integral using a rectangular region R. This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into R, particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region R. Also, the heights may not be exact if the surface $z = f(x, y)$ is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid S approach 0 as m and n approach infinity. Also, the double integral of the function $z = f(x, y)$ exists provided that the function f is not too discontinuous. If the function is bounded and continuous over R except on a finite number of smooth curves, then the double integral exists and we say that ff is integrable over R.

Since $\Delta A = \Delta x \Delta y = \Delta y \Delta x$, we can express dA as $dx\,dy$ or $dy\,dx$. This means that, when we are using rectangular coordinates, the double integral over a region R denoted by

$$\iint_R f(x, y)\,dA \tag{7.1.6}$$

can be written as

$$\iint_R f(x, y)\,dx\,dy \tag{7.1.7}$$

or

$$\iint_R f(x, y)\,dy\,dx. \tag{7.1.8}$$

Now let's list some of the properties that can be helpful to compute double integrals.

7.1 .2 Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if $f(x, y)$ is a product of two functions $g(x)$ and $h(y)$.

Theorem: PROPERTIES OF DOUBLE INTEGRALS

Assume that the functions $f(x,y)$ and $g(x,y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

i. The sum $f(x,y)+g(x,y)$ is integrable and

$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA. \tag{7.1 .9}$$

ii. If c is a constant, then $cf(x,y)$ is integrable and

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA. \tag{7.1 .10}$$

iii. If $R = S\cup T$ and $S\cap T=\emptyset$ except an overlap on the boundaries, then

$$\iint_R f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA. \tag{7.1 .11}$$

iv. If $f(x,y)\geq g(x,y)$ for (x,y) in R, then

$$\iint_R f(x,y)\,dA = \iint_R g(x,y)\,dA. \tag{7.1 .12}$$

v. If $m\leq f(x,y)\leq M$ and $A(R)=$ the area of R, then

$$m\cdot A(R)\leq \iint_R f(x,y)\,dA \leq M\cdot A(R). \tag{7.1 .13}$$

vi. In the case where $f(x,y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R=\{(x,y)\mid a\leq x\leq b,\ c\leq y\leq d\}$, the double integral can be written as

$$\iint_R f(x,y)\,dA = \left(\int_a^a g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right). \tag{7.1 .14}$$

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let’s get to that now.

7.1 .3 Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for m and n.Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Definitions: iterated integrals

Assume a, b, c, and d are real numbers. We define an *iterated integral* for a function $f(x,y)$ over the rectangular region $R=[a,b]\times[c,d]$ as

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx \tag{7.1 .15}$$

or

$$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy. \tag{7.1 .16}$$

The notation $\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$ means that we integrate $f(x,y)$ with respect to y while holding x constant. Similarly, the notation $\int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$ means that we integrate $f(x,y)$ with respect to x while holding y constant. The fact that double integrals can be split into iterated integrals is expressed in Fubini’s theorem. Think of this theorem as an essential tool for evaluating double integrals.

Theorem: FUBINI'S THEOREM

Suppose that $f(x, y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$. Then we see from Figure 7.1.6 that the double integral of f over the region equals an iterated integral,

$$\iint_R f(x, y)\, dA = \iint_R f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy. \tag{7.1 .17}$$

More generally, Fubini's theorem is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

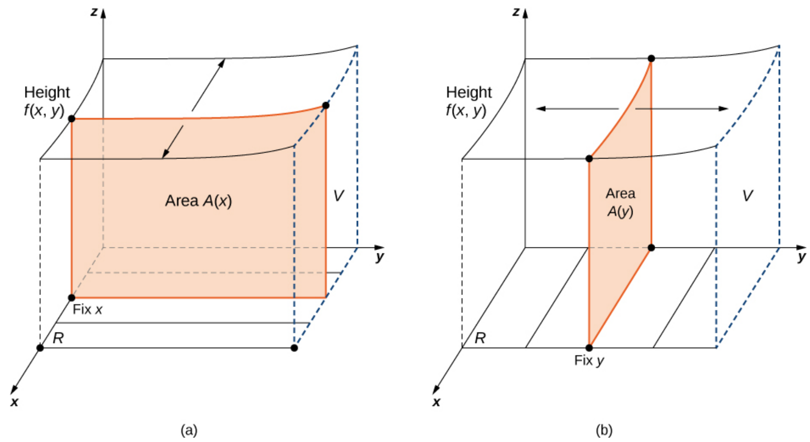


***Figure 7.1.6**: (a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V.*

Example 7.1.2: Using Fubini's Theorem

Use Fubini's theorem to compute the double integral $\iint_R f(x, y)\, dA$ where $f(x, y) = x$ and $R = [0, 2] \times [0, 1]$.

Solution

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

$$\begin{aligned} \iint_R f(x, y)\, dA &= \iint_R f(x, y)\, dx\, dy \\ &= \int_{y=0}^{y=1} \int_{x=0}^{x=2} x\, dx\, dy \\ &= \int_{y=0}^{y=1} \left[\frac{x^2}{2} \Big|_{x=0}^{x=2} \right] dy \\ &= \int_{y=0}^{y=1} 2\, dy = 2y \Big|_{y=0}^{y=1} = 2 \end{aligned}$$

The double integration in this example is simple enough to use Fubini's theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function $f(x, y)$ is more complex. Note that the order of integration can be changed (see Example 7).

Example 7.1.3: Illustrating Properties i and ii

Evaluate the double integral

$$\iint_R (xy - 3xy^2)\, dA, \text{ where } R = \{(x, y) \,|\, 0 \leq x \leq 2, 1 \leq y \leq 2\}. \tag{7.1 .18}$$

Solution

This function has two pieces: one piece is xy and the other is $3xy^2$. Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.

$$\begin{aligned}
\iint_R (xy - 3xy^2)\,dA &= \iint_R xy\,dA + \iint_R (-3xy^2)\,dA && \text{Property i: Integral of a sum is the sum of the integrals.}\\
&= \int_{y=1}^{y=2}\int_{x=0}^{x=2} xy\,dx\,dy - \int_{y=1}^{y=2}\int_{x=0}^{x=2} 3xy^2\,dx\,dy && \text{Convert double integrals to iterated integrals.}\\
&= \int_{y=1}^{y=2}\left(\frac{x^2}{2}y\right)\Big|_{x=0}^{x=2} dy - 3\int_{y=1}^{y=2}\left(\frac{x^2}{2}y^2\right)\Big|_{x=0}^{x=2} dy && \text{Integrate with respect to } x\text{,holding } y \text{ constant.}\\
&= \int_{y=1}^{y=2} 2y\,dy \int_{y=1}^{y=2} 6y^2 dy && \text{Property ii: Placing the constant before the integral.}\\
&= \int_1^2 y\,dy - 6\int_1^2 y^2\,dy && \text{Integrate with respect to y.}\\
&= 2\frac{y^2}{2}\Big|_1^2 - 6\frac{y^2}{2}\Big|_1^2\\
&= y^2\Big|_1^2 - 2y^3\Big|_1^2\\
&= (4-1) - 2(8-1) = 3 - 2(7) = 3 - 14 = -11.
\end{aligned}$$

Example 7.1.4: Illustrating Property v.

Over the region $R = \{(x,y) \mid 1 \le x \le 3,\ 1 \le y \le 2\}$, we have $2 \le x^2 + y^2 \le 13$. Find a lower and an upper bound for the integral $\iint_R (x^2 + y^2)\,dA$.

Solution

For a lower bound, integrate the constant function 2 over the region R. For an upper bound, integrate the constant function 13 over the region R.

$$\begin{aligned}
\int_1^2\int_1^3 2\,dx\,dy &= \int_1^2 [2x\Big|_1^3]\,dy = \int_1^2 2(2)dy = 4y\Big|_1^2 = 4(2-1) = 4\\
\int_1^2\int_1^3 13dx\,dy &= \int_1^2 [13x\Big|_1^3]\,dy = \int_1^2 13(2)\,dy = 26y\Big|_1^2 = 26(2-1) = 26.
\end{aligned}$$

Hence, we obtain $4 \le \iint_R (x^2 + y^2)\,dA \le 26$.

Example 7.1.5: Illustrating Property vi

Evaluate the integral $\iint_R e^y \cos x\,dA$ over the region $R = \{(x,y) \mid 0 \le x \le \frac{x}{2},\ 0 \le y \le 1\}$.

Solution

This is a great example for property vi because the function $f(x,y)$ is clearly the product of two single-variable functions e^y and $\cos x$. Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem. Try redoing Example 3 using this method.

$$\begin{aligned}
\iint_R e^y \cos x\,dA &= \int_0^1\int_0^{x/2} e^y \cos x\,dx\,dy\\
&= \left(\int_0^1 e^y dy\right)\left(\int_0^{\pi/2} \cos x\,dx\right)\\
&= (e^y\Big|_0^1)(\sin x\Big|_0^{\pi/2})\\
&= e - 1.
\end{aligned}$$

Exercise 7.1.2

a. Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3 - x + 4y)\, dy\, dx.$$

b. Show that

$$0 \leq \iint_R \sin \pi x \, \cos \pi y \, dA \leq \frac{1}{32} \tag{7.1 .19}$$

where $R = [0, \frac{1}{4}] \times [\frac{1}{4}, \frac{1}{2}]$.

Hint

Use properties i. and ii. and evaluate the iterated integral, and then use property v.

Answer

a. 26

b. Answers may vary.

As we mentioned before, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint_R f(x,y)\,dA$ can be written as $\iint_R f(x,y)\,dx\,dy$ or $\iint_R f(x,y)\,dy\,dx$. The next example shows that the results are the same regardless of which order of integration we choose.

Example 7.1.6: Evaluating an Iterated Integral in Two Ways

Let's return to the function $f(x,y) = 3x^2 - y$ from Example 1, this time over the rectangular region $R = [0,2] \times [0,3]$. Use Fubini's theorem to evaluate $\iint_R f(x,y)\,dA$ in two different ways:

a. First integrate with respect to y and then with respect to x;
b. First integrate with respect to x and then with respect to y.

Solution

Figure 7.1.6 shows how the calculation works in two different ways.

a. First integrate with respect to y and then integrate with respect to x:

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=2} \int_{y=0}^{y=3} (3x^2 - y)\,dy\,dx \tag{7.1 .20}$$

$$\int_{x=0}^{x=2} \left(\int_{y=0}^{y=3} (3x^2 - y)\,dy \right) dx = \int_{x=0}^{x=2} \left[3x^2 y - \frac{y^2}{2} \Big|_{y=0}^{y=3} \right] dx \tag{7.1 .21}$$

$$= \int_{x=0}^{x=2} \left(9x^2 - \frac{9}{2} \right) dx = 3x^3 - \frac{9}{2}x \Big|_{x=0}^{x=2} = 15. \tag{7.1 .22}$$

b. First integrate with respect to x and then integrate with respect to y:

$$\iint_R f(x,y)\,dA = \int_{y=0}^{y=3} \int_{x=0}^{x=2} (3x^2 - y)\,dx\,dy \tag{7.1 .23}$$

$$\int_{y=0}^{y=3} \left(\int_{x=0}^{x=2} (3x^2 - y)\,dx \right) dy = \int_{y=0}^{y=3} \left[x^3 - xy \Big|_{x=0}^{x=2} \right] dy \tag{7.1 .24}$$

$$= \int_{y=0}^{y=3} (8 - 2y)\,dy = 8y - y^2 \Big|_{y=0}^{y=3} = 15. \tag{7.1 .25}$$

Analysis

With either order of integration, the double integral gives us an answer of 15. We might wish to interpret this answer as a volume in cubic units of the solid S below the function $f(x,y) = 3x^2 - y$ over the region $R = [0,2] \times [0,3]$. However, remember that the interpretation of a double integral as a (non-signed) volume works only when the integrand f is a nonnegative function over the base region R.

Exercise 7.1.3

Evaluate

$$\int_{y=-3}^{y=2} \int_{x=3}^{x=5} (2-3x^2+y^2)\,dx\,dy.$$

Hint

Use Fubini’s theorem.

Answer

$-\frac{1340}{3}$

In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

Example 7.1.7: Switching the Order of Integration

Consider the double integral $\iint_R x\ \sin(xy)\,dA$ over the region $R=\{(x,y)\,|\,0\le x\le \pi,\ 1\le y\le 2\}$ (Figure 7.1.7).

a. Express the double integral in two different ways.
b. Analyze whether evaluating the double integral in one way is easier than the other and why.
c. Evaluate the integral.

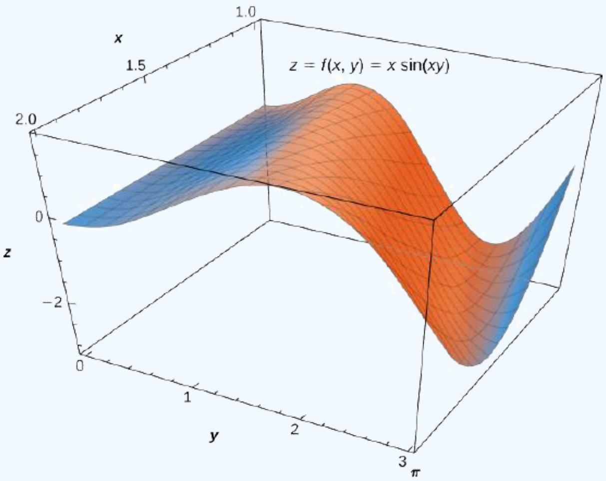


***Figure 7.1.7:** The function $z=f(x,y)=x\ \sin(xy)$ over the rectangular region $R=[0,\pi]\times[1,2]$.*

a. We can express $\iint_R x\ \sin(xy)\,dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.

$$\iint_R x\ \sin(xy)\,dA \tag{7.1 .26}$$

$$=\int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\ \sin(xy)\,dy\,dx \tag{7.1 .27}$$

Integrate first with respect to y.

$$=\int_{y=1}^{y=2}\int_{x=0}^{x=\pi} x\ \sin(xy)\,dx\,dy \tag{7.1 .28}$$

Integrate first with respect to x.

b. If we want to integrate with respect to y first and then integrate with respect to x, we see that we can use the substitution $u=xy$, which gives $du=x\,dy$. Hence the inner integral is simply $\int \sin u\,du$ and we can change the limits to be functions of x,

$$\iint_R x\ \sin(xy)\,dA=\int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\ \sin(xy)\,dy\,dx=\int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx. \tag{7.1 .29}$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u = x$ and $dv = \sin(xy)dx$

Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so

$$\iint_R x\sin(xy)\,dA = \int_{y=1}^{y=2}\int_{x=0}^{x=\pi} x\sin(xy)\,dx\,dy = \int_{y=1}^{y=2}\left[-\frac{x\cos(xy)}{y}\Big|_{x=0}^{x=\pi} + \frac{1}{y}\int_{x=0}^{x=\pi}\cos(xy)\,dx\right]dy. \tag{7.1 .30}$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.

$$\iint_R x\,\sin(xy)\,dA = \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\,\sin(xy)\,dy\,dx \tag{7.1 .31}$$

$$= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u\Big|_{u=x}^{u=2x}\right]dx = \int_{x=0}^{x=\pi}(-\cos 2x + \cos x)\,dx \tag{7.1 .32}$$

$$= \left(-\frac{1}{2}\sin 2x + \sin x\right)\Big|_{x=0}^{x=\pi} = 0. \tag{7.1 .33}$$

Exercise 7.1.4

Evaluate the integral $\iint_R xe^{xy}\,dA$ where $R = [0,1]\times[0,\ln 5]$.

Hint

Integrate with respect to y first.

Answer

$\frac{4-\ln 5}{\ln 5}$

7.1 .4 Applications of Double Integrals

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by integrating the constant function $f(x,y) = 1$ over the region R.

Definition

The area of the region R is given by $A(R) = \iint_R 1\,dA$.

This definition makes sense because using $f(x,y) = 1$ and evaluating the integral make it a product of length and width. Let's check this formula with an example and see how this works.

Example 7.1.8: Finding Area Using a Double Integral

Find the area of the region $R = \{\,(x,y)\,|\,0 \le x \le 3,\ 0 \le y \le 2\}$ by using a double integral, that is, by integrating 1 over the region R.

Solution

The region is rectangular with length 3 and width 2, so we know that the area is 6. We get the same answer when we use a double integral:

$$A(R) = \int_0^2\int_0^3 1\,dx\,dy = \int_0^2 \left[x|_0^3\right]\,dy = \int_0^2 3dy = 3\int_0^2 dy = 3y\Big|_0^2 = 3(2) = 6\text{ units}^2. \tag{7.1 .34}$$

We have already seen how double integrals can be used to find the volume of a solid bounded above by a function $f(x,y) \ge 0$ over a region R provided $f(x,y) \ge 0$ for all (x,y) in R. Here is another example to illustrate this concept.

Example 7.1.9: Volume of an Elliptic Paraboloid

Find the volume V of the solid S that is bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

Solution

First notice the graph of the surface $z=27-2x^2-y^2$ in Figure 7.1.8(a) and above the square region $R_1=[-3,3]\times[-3,3]$. However, we need the volume of the solid bounded by the elliptic paraboloid $2x^2+y^2+z=27$, the planes $x=3$ and $y=3$, and the three coordinate planes.

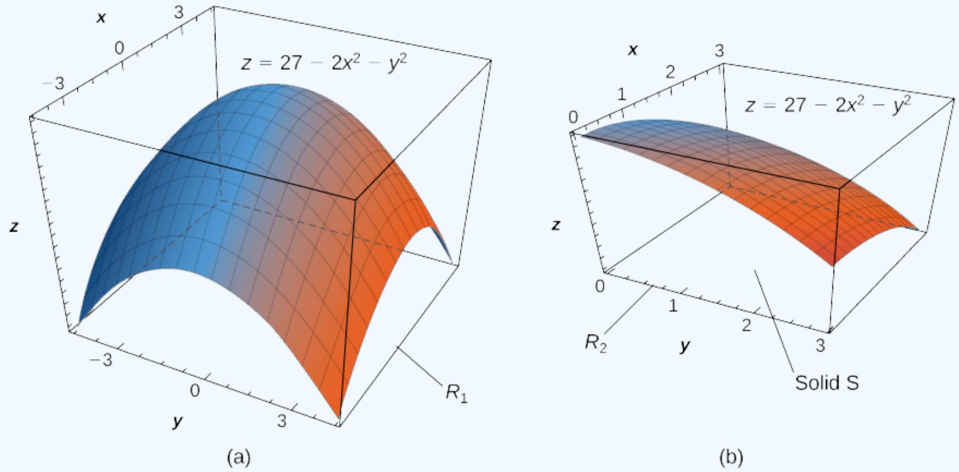


Figure 7.1.8: (a) The surface $z=27-2x^2-y^2$ above the square region $R1=[-3,3]\times[-3,3]$. (b) The solid S lies under the surface $z=27-2x^2-y^2$ above the square region $R1=[0,3]\times[0,3]$.

Now let's look at the graph of the surface in Figure 7.1.8(b). We determine the volume V by evaluating the double integral over R_2:

$$V=\iint_R z\,dA=\iint_R (27-2x^2-y^2)\,dA \tag{7.1.35}$$

$$=\int_{y=0}^{y=3}\int_{x=0}^{x=3}(27-2x^2-y^2)\,dx\,dy \quad \text{Convert to literal integral.} \tag{7.1.36}$$

$$=\int_{y=0}^{y=3}\left[27x-\frac{2}{3}x^3-y^2x\right]\Big|_{x=0}^{x=3}dy \quad \text{Integrate with respect to } x. \tag{7.1.37}$$

$$=\int_{y=0}^{y=3}(63-3y^2)dy=63y-y^3\Big|_{y=0}^{y=3}=162. \tag{7.1.38}$$

Exercise 7.1.5

Find the volume of the solid bounded above by the graph of $f(x,y)=xy\sin(x^2y)$ and below by the xy-plane on the rectangular region $R=[0,1]\times[0,\pi]$.

Hint

Graph the function, set up the integral, and use an iterated integral.

Answer

$\frac{\pi}{2}$

Recall that we defined the average value of a function of one variable on an interval $[a,b]$ as

$$f_{ave}=\frac{1}{b-a}\int_a^b f(x)\,dx. \tag{7.1.39}$$

Similarly, we can define the average value of a function of two variables over a region R. The main difference is that we divide by an area instead of the width of an interval.

Definition

The average value of a function of two variables over a region R is

$$F_{ave}=\frac{1}{\text{Area of } R}\iint_R f(x,y)\,dx\,dy. \tag{7.1.40}$$

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

Example 7.1.10: Calculating Average Storm Rainfall

The weather map in Figure 7.1.9 shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest of the USA on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.

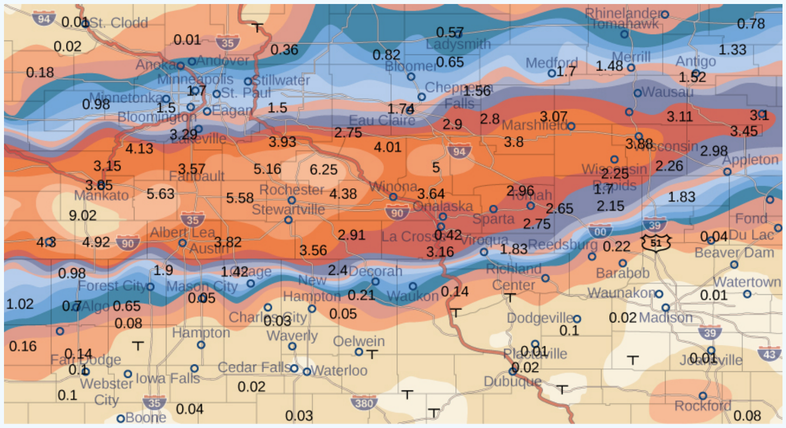


Figure 7.1.9: Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south.

Solution

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles ($m = 2$ and $n = 3$), as shown in Figure 7.1.9. Assume $f(x, y)$ denotes the storm rainfall in inches at a point approximately x miles to the east of the origin and y miles to the north of the origin. Let R represent the entire area of $300 \times 250 = 75000$ square miles. Then the area of each subrectangle is

$$\Delta A = \frac{1}{6}(75000) = 12500. \tag{7.1 .41}$$

Assume $(x_{ij}*, y_{ij}*)$ are approximately the midpoints of each subrectangle R_{ij}. Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

At (x_{11}, y_{11}) the rainfall is 0.08.

At (x_{12}, y_{12}) the rainfall is 0.08.

At (x_{13}, y_{13}) the rainfall is 0.01.

At (x_{21}, y_{21}) the rainfall is 1.70.

At (x_{22}, y_{22}) the rainfall is 1.74.

At (x_{23}, y_{23}) the rainfall is 3.00.

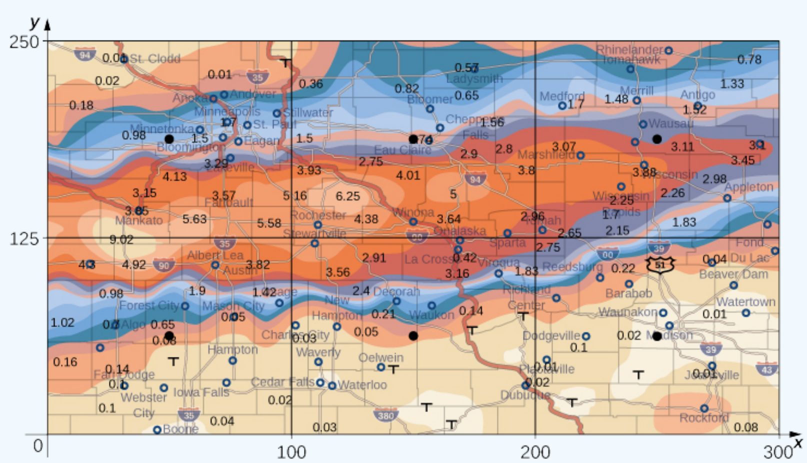


Figure 7.1.10: Storm rainfall with rectangular axes and showing the midpoints of each subrectangle.

According to our definition, the average storm rainfall in the entire area during those two days was

$$f_{ave} = \frac{1}{Area\ R} \iint_R = f(x,y)\,dx\,dy = \frac{1}{75000} \iint_R f(x,y)\,dx\,dy \tag{7.1 .42}$$

$$\cong \frac{1}{75000} \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \tag{7.1 .43}$$

$$\cong \frac{1}{75000} [f(x_{11}^*, y_{11}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A \tag{7.1 .44}$$

$$+ f(x_{13}^*, y_{13}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A + f(x_{23}^*, y_{23}^*)\Delta A] \tag{7.1 .45}$$

$$\cong \frac{1}{75000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]\Delta A \tag{7.1 .46}$$

$$\cong \frac{1}{75000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]12500 \tag{7.1 .47}$$

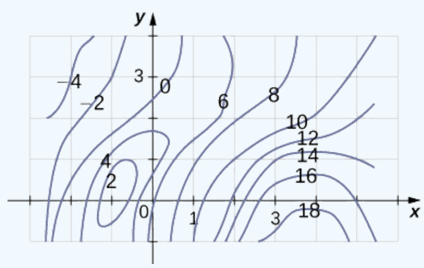
$$\cong \frac{1}{6}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00] \tag{7.1 .48}$$

$$\cong 1.10. \tag{7.1 .49}$$

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

Exercise 7.1.6

A contour map is shown for a function $f(x,y)$ on the rectangle $R = [-3,6] \times [-1,4]$.



a. Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of

$$\iint_R f(x,y)\,dA. \tag{7.1 .50}$$

b. Estimate the average value of the function $f(x,y)$.

Hint

Divide the region into six rectangles, and use the contour lines to estimate the values for $f(x,y)$.

Answer

Answers to both parts a. and b. may vary.

7.1 .4.1 Key Concepts

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
- Properties of double integral are useful to simplify computation and find bounds on their values.
- We can use Fubini’s theorem to write and evaluate a double integral as an iterated integral.
- Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

7.1 .4.1 Key Equations

- $$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i j*, y_i j*)\,\Delta A \tag{7.1 .51}$$

- $$\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx \tag{7.1 .52}$$

  or

  $$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy \tag{7.1 .53}$$

- $$f_{ave} = \frac{1}{\text{Area of } R} \iint_R f(x,y)\,dx\,dy \tag{7.1 .54}$$

7.1 .4.1 Glossary

double integral

of the function $f(x,y)$ over the region R in the xy-plane is defined as the limit of a double Riemann sum,

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\,\Delta A. \tag{7.1 .55}$$

double Riemann sum

of the function $f(x,y)$ over a rectangular region R is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\,\Delta A, \tag{7.1 .56}$$

where R is divided into smaller subrectangles R_{ij} and (x^*_{ij}, y^*_{ij}) is an arbitrary point in R_{ij}

Fubini's theorem

if $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$, then the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy \tag{7.1 .57}$$

iterated integral

for a function $f(x,y)$ over the region R is

a.

$$\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx, \tag{7.1 .58}$$

b.

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy, \tag{7.1 .59}$$

where a, b, c, and d are any real numbers and $R = [a,b] \times [c,d]$

7.1 .5 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

7.1E: Exercises

7.1E.1 Exercise $7.1E.1-2$

In the following exercises, use the midpoint rule with $m=4$ and $n=2$ to estimate the volume of the solid bounded by the surface $z=f(x,y)$, the vertical planes $x=1$, $x=2$, $y=1$, and $y=2$, and the horizontal plane $x=0$.

1. $f(x,y)=4x+2y+8xy$
2. $f(x,y)=16x^2+\frac{y}{2}$

Answer

1) 27 units^3

2) 37.6 units^3

7.1E.2 Exercise $7.1E.3-4$

In the following exercises, estimate the volume of the solid under the surface $z=f(x,y)$ and above the rectangular region R by using a Riemann sum with $m=n=2$ and the sample points to be the lower left corners of the subrectangles of the partition.

3. $f(x,y)=\sin(x)-\cos(y)$, $R=[0,\pi]\times[0,\pi]$
4. $f(x,y)=\cos(x)+\cos(y)$, $R=[0,\pi]\times[0,\frac{\pi}{2}]$

Answer

3) 0 units^3

4) $\left(\frac{6+3\sqrt{2}}{8}\right)\pi^2\text{ units}^3$

7.1E.3 Exercise $7.1E.5-8$

5. Use the midpoint rule with $m=n=2$ to estimate $\iint_R f(x,y)dA$, where the values of the function f on $R=[8,10]\times[9,11]$ are given in the following table.

	y				
x	9	9.5	10	10.5	11
8	9.8	5	6.7	5	5.6
8.5	9.4	4.5	8	5.4	3.4
9	8.7	4.6	6	5.5	3.4
9.5	6.7	6	4.5	5.4	6.7
10	6.8	6.4	5.5	5.7	6.8

Answer

5) 21.3 units^3.

6) 35.42 units^3.

6. The values of the function f on the rectangle $R=[0,2]\times[7,9]$ are given in the following table. Estimate the double integral $\iint_R f(x,y)dA$ by using a Riemann sum with $m=n=2$. Select the sample points to be the upper right corners of the subsquares of R.

	$y_0=7$	$y_1=8$	$y_2=9$

	$y_0 = 7$	$y_1 = 8$	$y_2 = 9$
$x_0 = 0$	10.22	10.21	9.85
$x_1 = 1$	6.73	9.75	9.63
$x_2 = 2$	5.62	7.83	8.21

7. The depth of a children's 4-ft by 4-ft swimming pool, measured at 1-ft intervals, is given in the following table.

1. Estimate the volume of water in the swimming pool by using a Riemann sum with $m = n = 2$. Select the sample points using the midpoint rule on $R = [0, 4] \times [0, 4]$.
2. Find the average depth of the swimming pool.

	y				
x	0	1	2	3	4
0	1	1.5	2	2.5	3
1	1	1.5	2	2.5	3
2	1	1.5	1.5	2.5	3
3	1	1	1.5	2	2.5
4	1	1	1	1.5	2

Answer

1. 28 ft^3
2. 1.75 ft.

8. The depth of a 3-ft by 3-ft hole in the ground, measured at 1-ft intervals, is given in the following table.

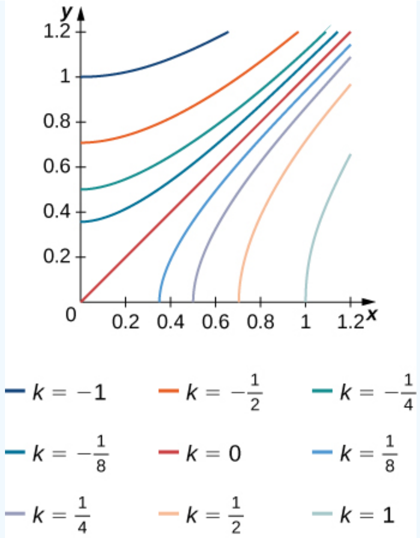
1. Estimate the volume of the hole by using a Riemann sum with $m = n = 3$ and the sample points to be the upper left corners of the subsquares of R.
2. Find the average depth of the hole.

	y			
x	0	1	2	3
0	6	6.5	6.4	6
1	6.5	7	7.5	6.5
2	6.5	6.7	6.5	6
3	6	6.5	5	5.6

7.1E.4 Exercise $\mathbf{7.1E.9-10}$

9. The level curves $f(X, Y) = K$ of the function *f* are given in the following graph, where *k* is a constant.

1. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)dA$, where $R = [0.2, 1] \times [0, 0.8]$.
2. Estimate the average value of the function *f* on R.

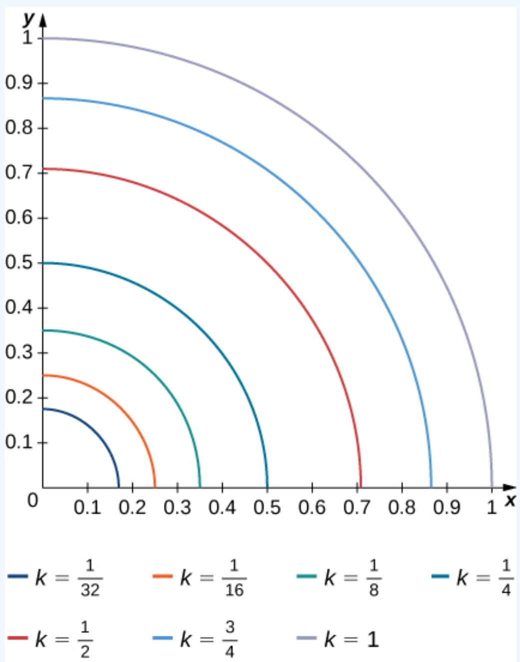


Answer

a. 0.112 b. $f_{ave} \simeq 0.175$; here $f(0.4, 0.2) \simeq 0.1$, $f(0.2, 0.6) \simeq -0.2$, $f(0.8, 0.2) \simeq 0.6$, and $f(0.8, 0.6) \simeq 0.2$

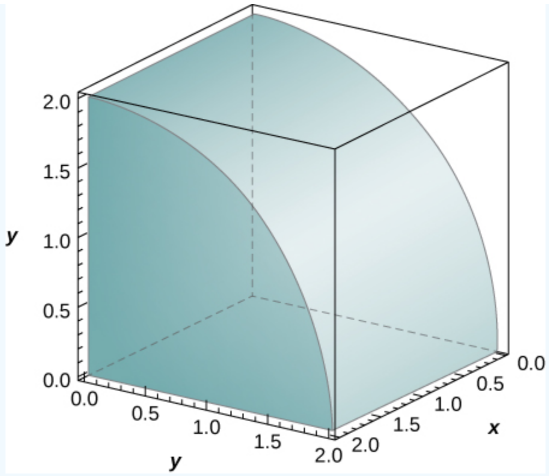
10. The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.

1. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)dA$, where $R = [0.1, 0.5] \times [0.1, 0.5]$.
2. Estimate the average value of the function f on R.



7.1E.5 Exercise $7.1E.11 - 12$

11. The solid lying under the surface $z = \sqrt{4 - y^2}$ and above the rectangular region $R = [0, 2] \times [0, 2]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)$, where $f(x, y) = \sqrt{4 - y^2}$ by finding the volume of the corresponding solid.

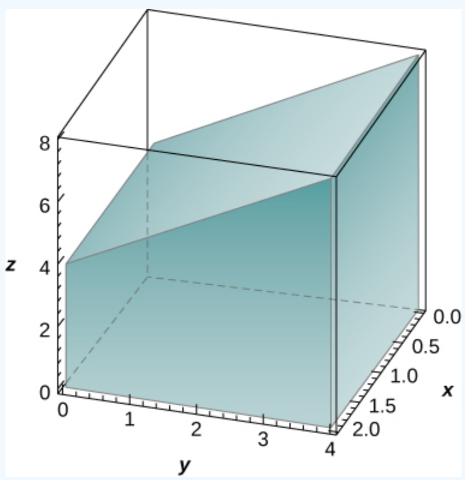


Answer

11) 2π units^3

12) 48 units^3

12. The solid lying under the plane $z = y + 4$ and above the rectangular region $R = [0, 2] \times [0, 4]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y) dA$, where $f(x, y) = y + 4$, by finding the volume of the corresponding solid.



7.1E.6 Exercise $7.1E.13-20$

In the following exercises, calculate the integrals by interchanging the order of integration.

13.

$$\int_{-1}^{1}\left(\int_{-2}^{2}(2x+3y+5)dx\right)\, dy \tag{7.1E.1}$$

Answer

40

14.

$$\int_{0}^{2}\left(\int_{0}^{1}(x+2e^{y}+3)dx\right)\, dy \tag{7.1E.2}$$

15.

$$\int_1^{27}\left(\int_1^2 (\sqrt[3]{x}+\sqrt[3]{y})dy\right)\,dx \tag{7.1E.3}$$

Answer

$\frac{81}{2}+39\sqrt[3]{2}$.

16.

$$\int_1^{16}\left(\int_1^8 (\sqrt[4]{x}+2\sqrt[3]{y})dy\right)\,dx \tag{7.1E.4}$$

17.

$$\int_{\ln 2}^{\ln 3}\left(\int_0^1 e^{x+y}dy\right)\,dx \tag{7.1E.5}$$

Answer

$e-1$.

18.

$$\int_0^2\left(\int_0^1 3^{x+y}dy\right)\,dx \tag{7.1E.6}$$

19.

$$\int_1^6\left(\int_2^9 \frac{\sqrt{y}}{y^2}dy\right)\,dx \tag{7.1E.7}$$

Answer

$15-\frac{10\sqrt{2}}{9}$.

20.

$$\int_1^9\left(\int_4^2 \frac{\sqrt{x}}{y^2}dy\right)\,dx \tag{7.1E.8}$$

7.1E.7 Exercise $7.1E.21-34$

In the following exercises, evaluate the iterated integrals by choosing the order of integration.

21.

$$\int_0^{\pi}\int_0^{\pi/2}\sin(2x)\cos(3y)dx\,dy \tag{7.1E.9}$$

Answer

0.

22.

$$\int_{\pi/12}^{\pi/8}\int_{\pi/4}^{\pi/3}[\cot(x)+\tan(2y)]dx\,dy \tag{7.1E.10}$$

23.

$$\int_1^e \int_1^e \left[\frac{1}{x}\sin(\ln\ x) + \frac{1}{y}\cos(\ln\ y)\right] dx\ dy \tag{7.1E.11}$$

Answer

$(e-1)(1+\sin(1)-\cos(1))$

24.

$$\int_1^e \int_1^e \frac{\sin(\ln\ x)\cos(\ln\ y)}{xy} dx\ dy \tag{7.1E.12}$$

25.

$$\int_1^2 \int_1^2 \left(\frac{\ln\ y}{x} + \frac{x}{2y+1}\right) dy\ dx \tag{7.1E.13}$$

Answer

$\frac{3}{4}\ln\left(\frac{5}{3}\right) + 2b\ \ln^2 2 - \ln\ 2$

26.

$$\int_1^e \int_1^2 x^2 \ln(x) dy\ dx \tag{7.1E.14}$$

27.

$$\int_1^{\sqrt{3}} \int_1^2 y\ \arctan\left(\frac{1}{x}\right) dy\ dx \tag{7.1E.15}$$

Answer

$\frac{1}{8}[(2\sqrt{3}-3)\pi + 6\ \ln\ 2]$.

28.

$$\int_0^1 \int_0^{1/2} (\arcsin\ x + \arcsin\ y) dy\ dx \tag{7.1E.16}$$

29.

$$\int_0^1 \int_0^2 xe^{x+4y} dy\ dx \tag{7.1E.17}$$

Answer

$\frac{1}{4}e^4(e^4-1)$.

30.

$$\int_1^2 \int_0^1 xe^{x-y} dy\ dx \tag{7.1E.18}$$

31.

$$\int_1^e \int_1^e \left(\frac{\ln\ y}{\sqrt{y}} + \frac{\ln\ x}{\sqrt{x}}\right) dy\ dx \tag{7.1E.19}$$

Answer

$4(e-1)(2-\sqrt{e})$.

32.

$$\int_1^e \int_1^e \left(\frac{x\ \ln\ y}{\sqrt{y}} + \frac{y\ \ln\ x}{\sqrt{x}}\right) dy\ dx \tag{7.1E.20}$$

33.

$$\int_0^1 \int_1^2 \left(\frac{x}{x^2+y^2}\right) dy\ dx \tag{7.1E.21}$$

Answer

$-\frac{\pi}{4} + \ln\left(\frac{5}{4}\right) - \frac{1}{2}\ln\ 2 + \arctan\ 2$.

34.

$$\int_0^1 \int_1^2 \frac{y}{x+y^2} dy\ dx \tag{7.1E.22}$$

7.1E.8 Exercise $7.1E.35-38$

In the following exercises, find the average value of the function over the given rectangles.

35. $f(x,y) = -x+2y$, $R = [0,1] \times [0,1]$

Answer

$\frac{1}{2}$.

36. $f(x,y) = x^4 + 2y^3$, $R = [1,2] \times [2,3]$

37. $f(x,y) = \sinh\ x + \sinh\ y$, $R = [0,1] \times [0,2]$

Answer

$\frac{1}{2}(2\ \cosh\ 1 + \cosh\ 2 - 3)$.

38. $f(x,y) = \arctan(xy)$, $R = [0,1] \times [0,1]$

7.1E.9 Exercise $7.1E.39$

39. Let f and g be two continuous functions such that $0 \leq m_1 \leq f(x) \leq M_1$ for any $x \in [a,b]$ and $0 \leq m_2 \leq g(y) \leq M_2$ for any $y \in [c,d]$. Show that the following inequality is true:

$$m_1 m_2 (b-a)(c-d) \leq \int_a^b \int_c^d f(x)g(y)dydx \leq M_1 M_2 (b-a)(c-d). \tag{7.1E.23}$$

7.1E.10 Exercise $7.1E.40-43$

In the following exercises, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

40. $\frac{1}{e^2} \leq \iint_R e^{-x^2-y^2}\ dA \leq 1$, where $R = [0,1] \times [0,1]$

41. $\frac{\pi^2}{144} \leq \iint_R \sin(x)\ \cos(y) dA \leq \frac{\pi^2}{48}$, where $R = \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \times \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

42. $0 \leq \iint_R e^{-y}\ \cos(x)\ dA \leq \frac{\pi}{2}$, where $R = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$

43. $0 \leq \iint_R (\ln\ x)(\ln\ y) dA \leq (e-1)^2$, where $R = [1,e] \times [1,e]$

7.1E.11 Exercise $\mathbf{7.1E.44}$

44. Let *f* and *g* be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a, b]$ and $0 \le m_2 \le g(y) \le M_2$ for any $y \in [c, d]$. Show that the following inequality is true:

$(m_1 + m_2)(b-a)(c-d) \le \int_a^b \int_c^d |f(x) + g(y)|\, dy\, dx \le (M_1 + M_2)(b-a)(c-d)$.

7.1E.12 Exercise $\mathbf{7.1E.45-48}$

In the following exercises, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

45. $\frac{2}{e} \le \iint_R (e^{-x^2} + e^{-y^2}) dA \le 2$, where $R = [0,1] \times [0,1]$

46. $\frac{\pi^2}{36} \iint_R (\sin(x) + \cos(y)) dA \le \frac{\pi^2\sqrt{3}}{36}$, where $R = [\frac{\pi}{6}, \frac{\pi}{3}] \times [\frac{\pi}{6}, \frac{\pi}{3}]$

47. $\frac{\pi}{2} e^{-\pi/2} \le \iint_R (\cos(x) + e^{-y}) dA \le \pi$, where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

48. $\frac{1}{e} \le \iint_R (e^{-y} - \ln x) dA \le 2$, where $R = [0,1] \times [0,1]$

7.1E.13 Exercise $\mathbf{7.1E.49-50}$

In the following exercises, the function *f* is given in terms of double integrals.

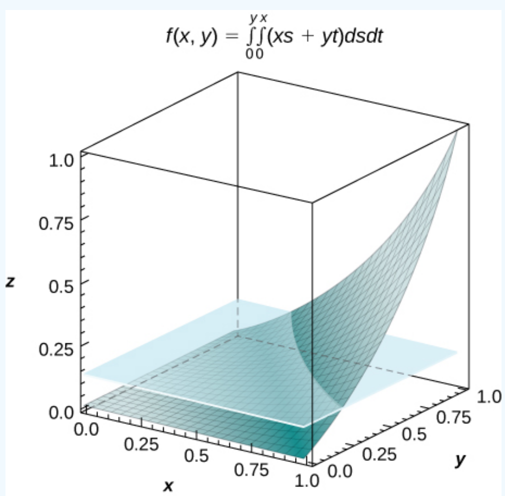
1. Determine the explicit form of the function *f*.
2. Find the volume of the solid under the surface $z = f(x, y)$ and above the region *R*.
3. Find the average value of the function *f* on *R*.
4. Use a computer algebra system (CAS) to plot $z = f(x, y)$ and $z = f_{ave}$ in the same system of coordinates.

49. [T] $f(x, y) = \int_0^y \int_0^x (xs + yt) ds\, dt$, where $(x, y) \in R = [0,1] \times [0,1]$

Answer

a. $f(x,y) = \frac{1}{2} xy(x^2 + y^2)$; b. $V = \int_0^1 \int_0^1 f(x,y) dx\, dy = \frac{1}{8}$; c. $f_{ave} = \frac{1}{8}$;

d.



50. [T] $f(x, y) = \int_0^x \int_0^y [\cos(s) + \cos(t)] dt\, ds$, where $(x, y) \in R = [0,3] \times [0,3]$

7.1E.14 Exercise $\mathbf{7.1E.51-52}$

51. Show that if *f* and *g* are continuous on $[a, b]$ and $[c, d]$, respectively, then

$\int_a^b \int_c^d |f(x) + g(y)| dy\, dx = (d-c) \int_a^b f(x) dx$

$+\int_a^b \int_c^d g(y)dy\,dx = (b-a)\int_c^d g(y)dy + \int_c^d \int_a^b f(x)dx\,dy$.

52. Show that $\int_a^b \int_c^d yf(x) + xg(y)dy\,dx = \frac{1}{2}(d^2-c^2)\left(\int_a^b f(x)dx\right) + \frac{1}{2}(b^2-a^2)\left(\int_c^d g(y)dy\right)$.

7.1E.15 Exercise $7.1E.53-54$

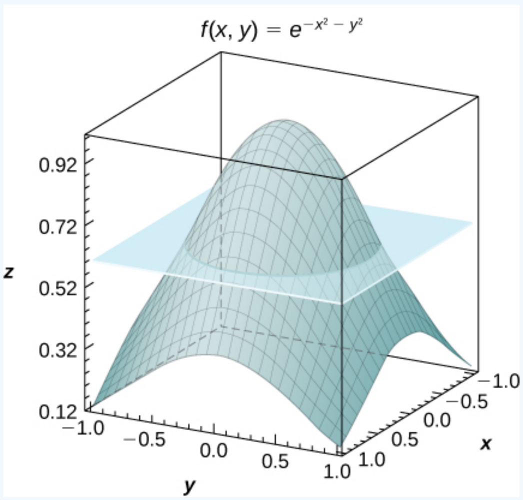
53. [T] Consider the function $f(x,y) = e^{-x^2-y^2}$, where $(x,y) \in R = [-1,1] \times [-1,1]$.

1. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R e^{-x^2-y^2}\,dA$. Round your answers to the nearest hundredths.
2. For $m = n = 2$, find the average value of f over the region R. Round your answer to the nearest hundredths.
3. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R e^{-x^2-y^2}\,dA$ and the plane $z = f_{ave}$.

Answer

a. For $m = n = 2$, $I = 4e^{-0.5} \approx 2.43$ b. $f_{ave} = e^{-0.5} \simeq 0.61$;

c.



54. [T] Consider the function $f(x,y) = \sin(x^2)\,\cos(y^2)$, where $(x, y \in R = [-1,1] \times [-1,1]$.

1. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R \sin(x^2)\,\cos(y^2)\,dA$. Round your answers to the nearest hundredths.
2. For $m = n = 2$, find the average value of f over the region R. Round your answer to the nearest hundredths.
3. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R \sin(x^2)\,\cos(y^2)\,dA$ and the plane $z = f_{ave}$.

7.1E.16 Exercise $7.1E.55-56$

In the following exercises, the functions fnfn are given, where $n \geq 1$ is a natural number.

1. Find the volume of the solids S_n under the surfaces $z = f_n(x,y)$ and above the region R.
2. Determine the limit of the volumes of the solids S_n as n increases without bound.

55. $f(x,y) = x^n + y^n + xy,\ (x,y) \in R = [0,1] \times [0,1]$

Answer

a. $\frac{2}{n+1} + \frac{1}{4}$ b. $\frac{1}{4}$

56. $f(x,y) = \frac{1}{x^n} + \frac{1}{y^n},\ (x,y) \in R = [1,2] \times [1,2]$

7.1E.17 Exercise $\PageIndex{57}$

57. Show that the average value of a function f on a rectangular region $R=[a,b]\times[c,d]$ is $f_{ave}\approx\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}f(x_{ij}^*,y_{ij}^*)$,where (x_{ij}^*,y_{ij}^*) are the sample points of the partition of R, where $1\le i\le m$ and $1\le j\le n$.

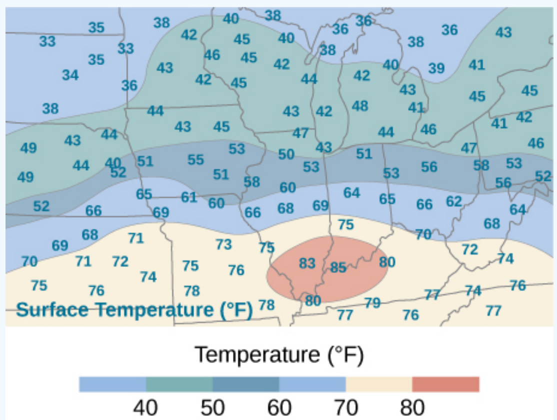
7.1E.18 Exercise $\PageIndex{58}$

58. Use the midpoint rule with $m=n$ to show that the average value of a function f on a rectangular region $R=[a,b]\times[c,d]$ is approximated by

$$f_{ave}\approx\frac{1}{n^2}\sum_{i,j=1}^{n}f\left(\frac{1}{2}(x_{i=1}+x_i),\ \frac{1}{2}(y_{j=1}+y_j)\right). \tag{7.1E.24}$$

7.1E.19 Exercise $\PageIndex{59}$

59. An isotherm map is a chart connecting points having the same temperature at a given time for a given period of time. Use the preceding exercise and apply the midpoint rule with $m=n=2$ to find the average temperature over the region given in the following figure.



Answer

56.5° F; here $f(x_1^*,y_1^*)=71,\ f(x_2^*,y_1^*)=72,\ f(x_2^*,y_1^*)=40,\ f(x_2^*,y_2^*)=43$, where x_i^* and y_j^* are the midpoints of the subintervals of the partitions of [a,b] and [c,d], respectively

7.2: Double Integrals over General Regions

This page is a draft and is under active development.

Learning objectives

- Recognize when a function of two variables is integrable over a general region.
- Evaluate a double integral by computing an iterated integral over a region bounded by two vertical lines and two functions of x, or two horizontal lines and two functions of y.
- Simplify the calculation of an iterated integral by changing the order of integration.
- Use double integrals to calculate the volume of a region between two surfaces or the area of a plane region.
- Solve problems involving double improper integrals.

Previously, we studied the concept of double integrals and examined the tools needed to compute them. We learned techniques and properties to integrate functions of two variables over rectangular regions. We also discussed several applications, such as finding the volume bounded above by a function over a rectangular region, finding area by integration, and calculating the average value of a function of two variables.

In this section we consider double integrals of functions defined over a general bounded region D on the plane. Most of the previous results hold in this situation as well, but some techniques need to be extended to cover this more general case.

7.2.1 General Regions of Integration

An example of a general bounded region D on a plane is shown in Figure 7.2.1. Since D is bounded on the plane, there must exist a rectangular region R on the same plane that encloses the region D that is, a rectangular region R exists such that D is a subset of $R(D \subseteq R)$.

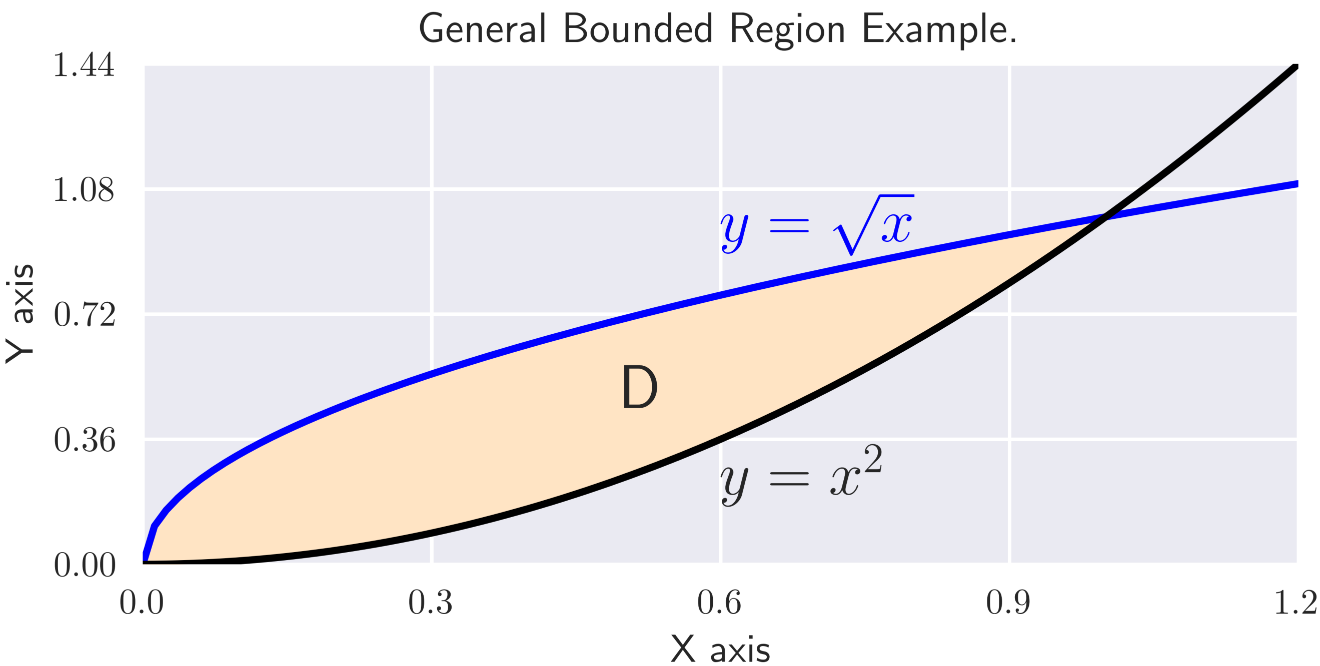


Figure 7.2.1. For a region D that is a subset of R, we can define a function $g(x,y)$ to equal $f(x,y)$ at every point in D and 0 at every point of R not in D.

Suppose $z = f(x,y)$ is defined on a general planar bounded region D as in Figure 7.2.1. In order to develop double integrals of f over D we extend the definition of the function to include all points on the rectangular region R and then use the concepts and tools from the preceding section. But how do we extend the definition of f to include all the points on R? We do this by defining a new function $g(x,y)$ on R as follows:

$$g(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases} \tag{7.2.1}$$

Note that we might have some technical difficulties if the boundary of D is complicated. So we assume the boundary to be a piecewise smooth and continuous simple closed curve. Also, since all the results developed in the section on Double Integrals over Rectangular Regions used an integrable function $f(x,y)$ we must be careful about $g(x,y)$ and verify that $g(x,y)$ is an integrable function over the rectangular region R. This happens as long as the region D is bounded by simple closed curves. For now we will concentrate on the descriptions of the regions rather than the function and extend our theory appropriately for integration.

We consider two types of planar bounded regions.

Definition: Type I and Type II regions

A region D in the (x,y)-plane is of Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is (Figure 7.2.2),

$$D = \{(x,y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}. \tag{7.2.2}$$

A region D in the xy-plane is of Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is (Figure 7.2.3),

$$D = \{(x,y) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}. \tag{7.2.3}$$

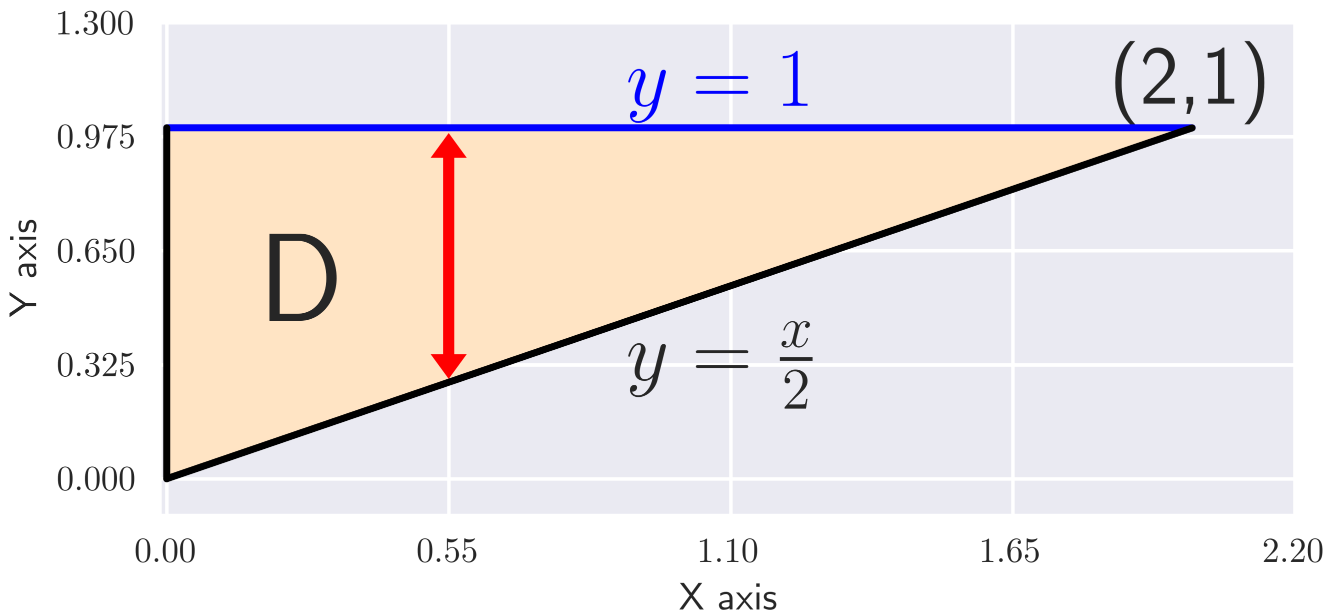


Figure 7.2.2. A Type I region lies between two vertical lines and the graphs of two functions of x.

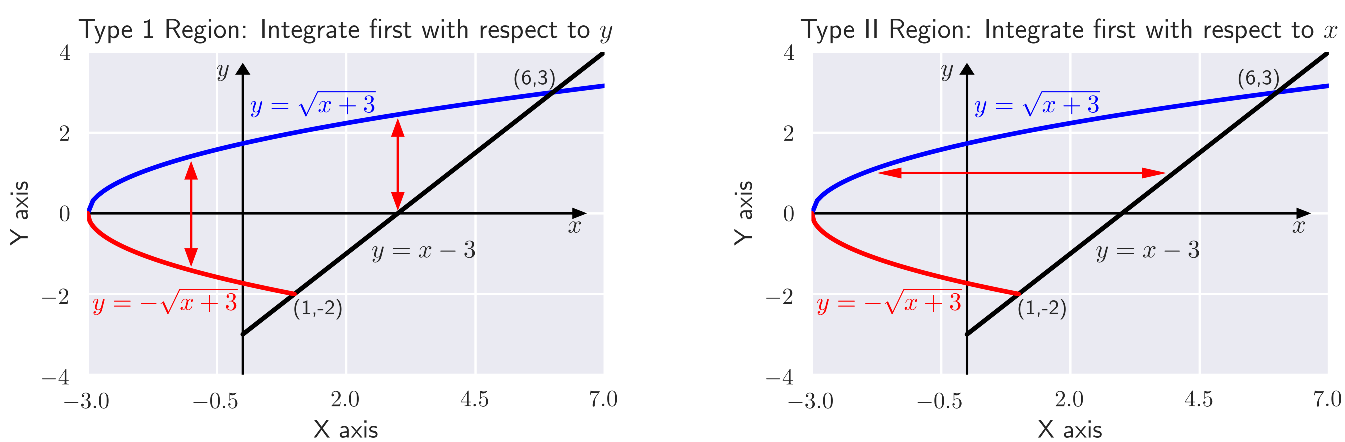


Figure 7.2.3. A Type II region lies between two horizontal lines and the graphs of two functions of y.

Example 7.2.1: Describing a Region as Type I and Also as Type II

Consider the region in the first quadrant between the functions $y = \sqrt{x}$ and $y = x^3$ (Figure 7.2.4). Describe the region first as Type I and then as Type II.

When describing a region as Type I, we need to identify the function that lies above the region and the function that lies below the region. Here, region D is bounded above by $y = \sqrt{x}$ and below by $y = x^3$ in the interval for x in $[0, 1]$. Hence, as Type I, D is described as the set $\{(x, y) \mid 0 \le x \le 1, \ x^3 \le y \le \sqrt[3]{x}\}$.

However, when describing a region as Type II, we need to identify the function that lies on the left of the region and the function that lies on the right of the region. Here, the region D is bounded on the left by $x = y^2$ and on the right by $x = \sqrt[3]{y}$ in the interval for y in $[0, 1]$. Hence, as Type II, D is described as the set $\{(x, y) \mid 0 \le y \le 1, \ y^2 \le x \le \sqrt[3]{y}\}$.

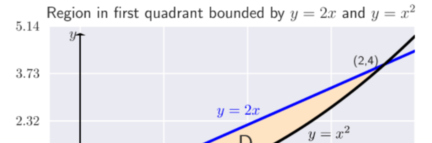
Exercise 7.2.1

Consider the region in the first quadrant between the functions $y = 2x$ and $y = x^2$. Describe the region first as Type I and then as Type II.

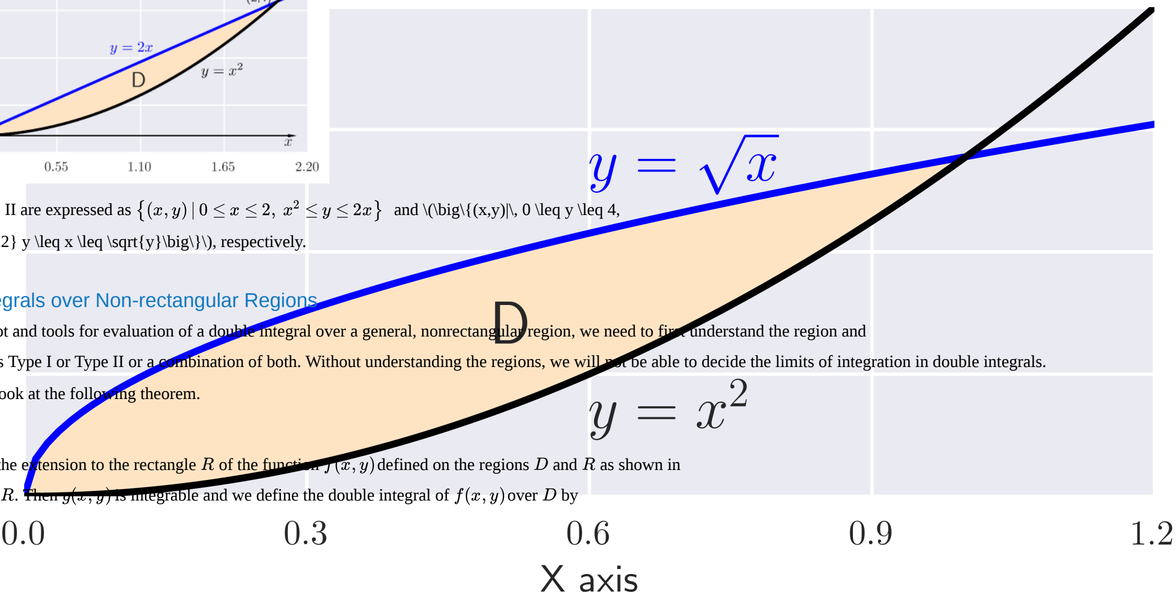
Hint

Graph the functions, and draw vertical and horizontal lines.

Answer



...eneral Bounded Region Example.



Type I and Type II are expressed as $\{(x,y)\,|\,0 \leq x \leq 2,\ x^2 \leq y \leq 2x\}$ and \(\big\{(x,y)|\, 0 \leq y \leq 4, \space \frac{1}{2} y \leq x \leq \sqrt{y}\big\}\), respectively.

7.2.2 Double Integrals over Non-rectangular Regions

To develop the concept and tools for evaluation of a double integral over a general, nonrectangular region, we need to first understand the region and be able to express it as Type I or Type II or a combination of both. Without understanding the regions, we will not be able to decide the limits of integration in double integrals. As a first step, let us look at the following theorem.

Suppose $g(x,y)$ is the extension to the rectangle R of the function $f(x,y)$ defined on the regions D and R as shown in Figure 7.2.1 inside R. Then $g(x,y)$ is integrable and we define the double integral of $f(x,y)$ over D by

Figure 7.2.4: Region D can be described as Type I or as Type II.

$$\iint_D f(x,y)\,dA = \iint_R g(x,y)\,dA. \tag{7.2.4}$$

The right-hand side of this equation is what we have seen before, so this theorem is reasonable because R is a rectangle and $\iint_R g(x,y)dA$ has been discussed in the preceding section. Also, the equality works because the values of $g(x,y)$ are 0 for any point (x,y) that lies outside D and hence these points do not add anything to the integral. However, it is important that the rectangle R contains the region D.

As a matter of fact, if the region D is bounded by smooth curves on a plane and we are able to describe it as Type I or Type II or a mix of both, then we can use the following theorem and not have to find a rectangle R containing the region.

Theorem: Fubini's Theorem (Strong Form)

For a function $f(x,y)$ that is continuous on a region D of Type I, we have

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx. \tag{7.2.5}$$

Similarly, for a function $f(x,y)$ that is continuous on a region D of Type II, we have

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy. \tag{7.2.6}$$

The integral in each of these expressions is an iterated integral, similar to those we have seen before. Notice that, in the inner integral in the first expression, we integrate $f(x,y)$ with x being held constant and the limits of integration being $g_1(x)$ and $g_2(x)$. In the inner integral in the second expression, we integrate $f(x,y)$ with y being held constant and the limits of integration are $h_1(y)$ and $h_2(y)$.

Example 7.2.2: Evaluating an Iterated Integral over a Type I Region

Evaluate the integral $\iint_D x^2 e^{xy}\,dA$ where D is shown in Figure 7.2.5.

Solution

First construct the region as a Type I region (Figure 7.2.5). Here $D = \{(x,y)\,|\,0 \leq x \leq 2,\ \frac{1}{2}x \leq y \leq 1\}$.

Then we have

$$\iint_D x^2 e^{xy}\,dA = \int_{x=0}^{x=2} \int_{y=1/2x}^{y=1} x^2 e^{xy}\,dy\,dx. \tag{7.2.7}$$

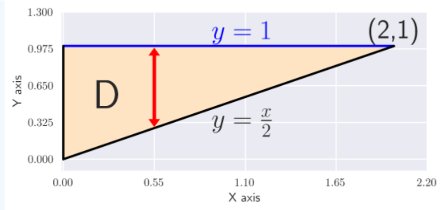


Figure 7.2.5. We can express region D as a Type I region and integrate from $y = \frac{1}{2}x$ to $y = 1$ between the lines $x = 0$ and $x = 2$.

Therefore, we have

$$\begin{aligned} \int_{x=0}^{x=2}\int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy}\,dy\,dx &= \int_{x=0}^{x=2}\left[\int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy}\,dy\right]dx && \text{Iterated integral for a Type I region.} \\ &= \int_{x=0}^{x=2}\left[x^2\frac{e^{xy}}{x}\right]\Big|_{y=1/2x}^{y=1} dx && \text{Integrate with respect to } y \\ &= \int_{x=0}^{x=2}\left[xe^x - xe^{x^2/2}\right]dx && \text{Integrate with respect to } x \\ &= \left[xe^x - e^x - e^{\frac{1}{2}x^2}\right]\Big|_{x=0}^{x=2} = 2. \end{aligned}$$

In Example 7.2.2, we could have looked at the region in another way, such as $D = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le 2y\}$ (Figure 7.2.6).

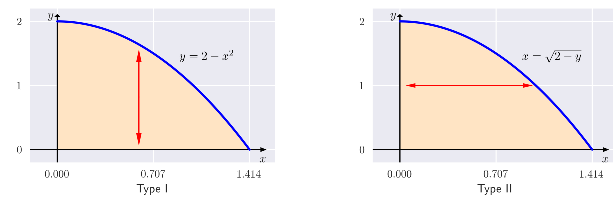


Figure 7.2.6.

This is a Type II region and the integral would then look like

$$\iint_D x^2 e^{xy}\,dA = \int_{y=0}^{y=1}\int_{x=0}^{x=2y} x^2 e^{xy}\,dx\,dy. \tag{7.2.8}$$

However, if we integrate first with respect to x this integral is lengthy to compute because we have to use integration by parts twice.

Example 7.2.3: Evaluating an Iterated Integral over a Type II Region

Evaluate the integral

$$\iint_D (3x^2 + y^2)\,dA$$

where $D = \{(x,y) \mid -2 \le y \le 3,\ y^2 - 3 \le x \le y + 3\}$.

Solution

Notice that D can be seen as either a Type I or a Type II region, as shown in Figure 7.2.7. However, in this case describing D as Type I is more complicated than describing it as Type II. Therefore, we use D as a Type II region for the integration.

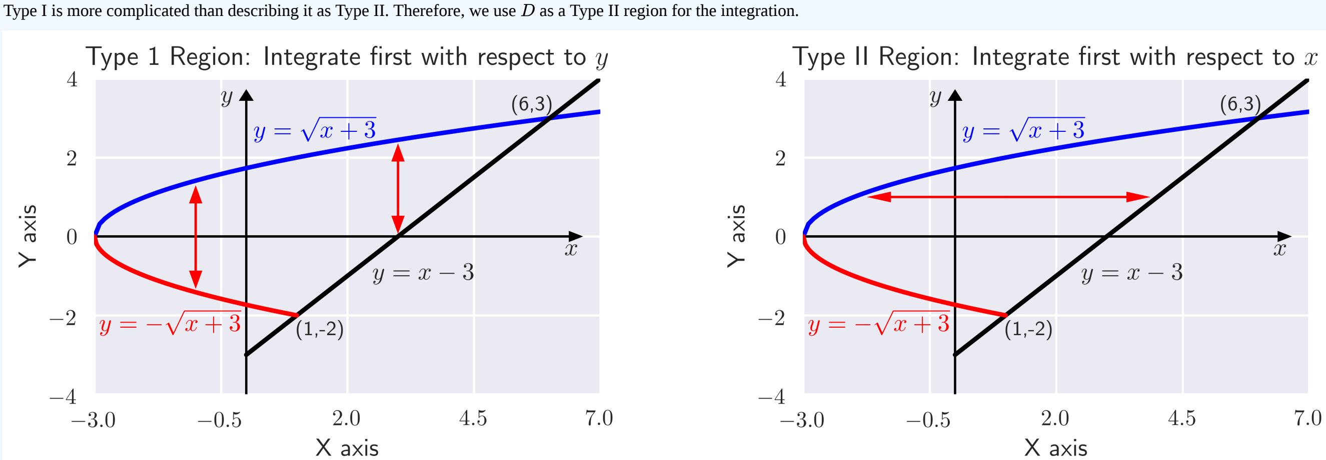


Figure 7.2.7. The region D in this example can be either (a) Type I or (b) Type II.

Choosing this order of integration, we have

$$
\begin{aligned}
\iint\limits_D (3x^2+y^2)\,dA &= \int_{y=-2}^{y=3}\int_{x=y^2-3}^{x=y+3}(3x^2+y^2)\,dx\,dy \\
&= \int_{y=-2}^{y=3}\left[(x^3+xy^2)\right]\Big|_{y^2-3}^{y+3}\,dy && \text{Iterated integral, Type II region} \\
&= \int_{y=-2}^{y=3}\left((y+3)^3+(y+3)y^2-(y^2-3)y^2\right)\,dy \\
&= \int_{-2}^{3}(54+27y-12y^2+2y^3+8y^4-y^6)\,dy && \text{Integrate with repsect to } x. \\
&= \left[54y+\frac{27y^2}{2}-4y^3+\frac{y^4}{2}+\frac{8y^5}{5}-\frac{y^7}{7}\right]_{-2}^{3} \\
&= \frac{2375}{7}.
\end{aligned}
$$

Exercise 7.2.2

Sketch the region D and evaluate the iterated integral

$$\iint\limits_D xy\,dy\,dx \tag{7.2.9}$$

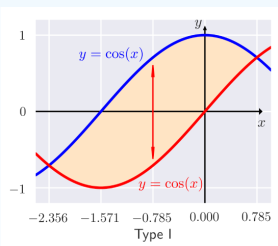
where D is the region bounded by the curves

$y=\cos\,x$ and $y=\sin\,x$ in the interval $[-3\pi/4,\ \pi/4]$

Hint

Express D as a Type I region, and integrate with respect to y first.

Answer



$\frac{\pi}{4}$

Recall from Double Integrals over Rectangular Regions the properties of double integrals. As we have seen from the examples here, all these properties are also valid for a function defined on a non-rectangular bounded region on a plane. In particular, property 3 states:

If $R=S\cup T$ and $S\cap T=0$ except at their boundaries, then

$$\iint\limits_R f(x,y)\,dA=\iint\limits_S f(x,y)\,dA+\iint_T f(x,y)\,dA. \tag{7.2.10}$$

Similarly, we have the following property of double integrals over a non-rectangular bounded region on a plane.

Theorem: Decomposing Regions into Smaller Regions

Suppose the region D can be expressed as $D=D_1\cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint\limits_D f(x,y)\,dA=\iint\limits_{D_1} f(x,y)\,dA+\iint\limits_{D_2} f(x,y)\,dA. \tag{7.2.11}$$

This theorem is particularly useful for non-rectangular regions because it allows us to split a region into a union of regions of Type I and Type II. Then we can compute the double integral on each piece in a convenient way, as in the next example.

Example 7.2.4: Decomposing Regions

Express the region D shown in Figure 7.2.8 as a union of regions of Type I or Type II, and evaluate the integral

$$\iint\limits_D (2x+5y)\,dA.$$

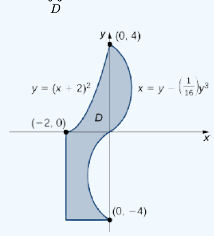


Figure 7.2.8. This region can be decomposed into a union of three regions of Type I or Type II.

Solution

The region D is not easy to decompose into any one type; it is actually a combination of different types. So we can write it as a union of three regions D_1, D_2, and D_3 where, $D_1 = \{(x,y) \mid -2 \le x \le 0,\ 0 \le y \le (x+2)^2\}$,
$D_2 = \{(x,y) \mid 0 \le y \le 4,\ 0 \le x \le (y - \frac{1}{16}y^3)\}$, and
$D_3 = \{(x,y) \mid -4 \le y \le 0,\ -2 \le x \le (y - \frac{1}{16}y^3)\}$.
These regions are illustrated more clearly in Figure 7.2.9.

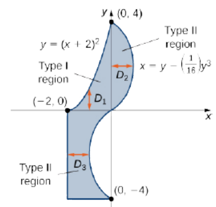


Figure 7.2.9. Breaking the region into three subregions makes it easier to set up the integration.

Here D_1 is Type I and D_2 and D_3 are both of Type II. Hence,

$$\begin{aligned}
\iint\limits_D (2x+5y)\,dA &= \iint\limits_{D_1} (2x+5y)\,dA + \iint\limits_{D_2} (2x+5y)\,dA + \iint\limits_{D_3} (2x+5y)\,dA \\
&= \int_{x=-2}^{x=0}\int_{y=0}^{y=(x+2)^2} (2x+5y)\,dy\,dx + \int_{y=0}^{y=4}\int_{x=0}^{x=y-(1/16)y^3} (2+5y)\,dx\,dy + \int_{y=-4}^{y=0}\int_{x=-2}^{x=y-(1/16)y^3} (2x+5y)\,dx\,dy \\
&= \int_{x=-2}^{x=0} \left[\frac{1}{2}(2+x)^2(20+24x+5x^2)\right] dx + \int_{y=0}^{y=4} \left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2\right] dy \\
&+ \int_{y=-4}^{y=0} \left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2 + 10y - 4\right] dy \\
&= \frac{40}{3} + \frac{1664}{35} - \frac{1696}{35} = \frac{1304}{105}.
\end{aligned}$$

Now we could redo this example using a union of two Type II regions (see the Checkpoint).

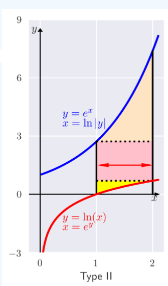
Exercise 7.2.3

Consider the region bounded by the curves $y = \ln x$ and $y = e^x$ in the interval $[1,2]$. Decompose the region into smaller regions of Type II.

Hint

Sketch the region, and split it into three regions to set it up.

Answer



$$\{(x,y) \mid 0 \le y \le \ln(2),\ 1 \le x \le e^y\} \cup \{(x,y) \mid \ln(2) \le y \le e,\ 1 \le x \le 2\} \cup \{(x,y) \mid e \le y \le e^2,\ \ln y \le x \le 2\}$$

Exercise 7.2.4

Redo Example 7.2.4 using a union of two Type II regions.

Hint

$$\{(x,y) \mid 0 \le y \le 4,\ 2 + \sqrt{y} \le x \le (y - \frac{1}{16}y^3)\} \cup \{(x,y) \mid -4 \le y \le 0,\ -2 \le x \le (y - \frac{1}{16}y^{13})\}$$

Answer

Same as in the example shown.

7.2.3 Changing the Order of Integration

As we have already seen when we evaluate an iterated integral, sometimes one order of integration leads to a computation that is significantly simpler than the other order of integration. Sometimes the order of integration does not matter, but it is important to learn to recognize when a change in order will simplify our work.

Example 7.2.5: Changing the Order of Integration

Reverse the order of integration in the iterated integral

$$\int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=2-x^2} x e^{x^2}\,dy\,dx.$$

Then evaluate the new iterated integral.

Solution

The region as presented is of Type I. To reverse the order of integration, we must first express the region as Type II. Refer to Figure 7.2.10.

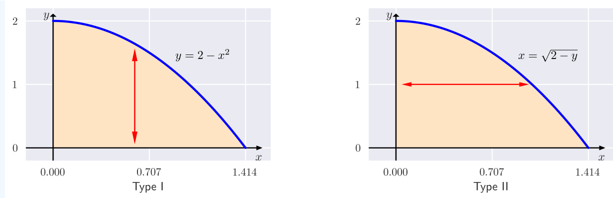


Figure 7.2.10. Converting a region from Type I to Type II.

We can see from the limits of integration that the region is bounded above by $y = 2 - x^2$ and below by $y = 0$ where x is in the interval $[0, \sqrt{2}]$. By reversing the order, we have the region bounded on the left by $x = 0$ and on the right by $x = \sqrt{2-y}$ where y is in the interval $[0, 2]$. We solved $y = 2 - x^2$ in terms of x to obtain $x = \sqrt{2-y}$.

Hence

$$\begin{aligned} \int_0^{\sqrt{2}} \int_0^{2-x^2} xe^{x^2}\, dy\, dx &= \int_0^2 \int_0^{\sqrt{2-y}} xe^{x^2}\, dx\, dy && \text{Reverse the order of integration then use substitution.} \\ &= \int_0^2 \left[\frac{1}{2} e^{x^2} \Big|_0^{\sqrt{2-y}} \right] dy = \int_0^2 \frac{1}{2}(e^{2-y} - 1)\, dy \\ &= -\frac{1}{2}(e^{2-y} + y)\Big|_0^2 = \frac{1}{2}(e^2 - 3). \end{aligned}$$

Example 7.2.6: Evaluating an Iterated Integral by Reversing the Order of Integration

Consider the iterated integral

$$\iint_R f(x, y)\, dx\, dy \tag{7.2.12}$$

where $z = f(x, y) = x - 2y$ over a triangular region R that has sides on $x = 0,\ y = 0$, and the line $x + y = 1$. Sketch the region, and then evaluate the iterated integral by

a. integrating first with respect to y and then
b. integrating first with respect to x.

Solution

A sketch of the region appears in Figure 7.2.11.

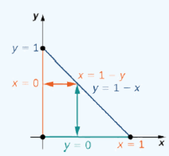


Figure 7.2.11. A triangular region R for integrating in two ways.

We can complete this integration in two different ways.

a. One way to look at it is by first integrating y from $y = 0$ to $y = 1 - x$ vertically and then integrating x from $x = 0$ to $x = 1$:

$$\iint_R f(x, y)\, dx\, dy = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (x - 2y)\, dy\, dx = \int_{x=0}^{x=1} \left[xy - 2y^2 \right]_{y=0}^{y=1-x} dx \tag{7.2.13}$$

$$\int_{x=0}^{x=1} \left[x(1-x) - (1-x)^2 \right] dx = \int_{x=0}^{x=1} [-1 + 3x - 2x^2] dx = \left[-x + \frac{3}{2}x^2 - \frac{2}{3}x^3 \right]_{x=0}^{x=1} = -\frac{1}{6}. \tag{7.2.14}$$

b. The other way to do this problem is by first integrating x from $x = 0$ to $x = 1 - y$ horizontally and then integrating y from $y = 0$ to $y = 1$:

$$\begin{aligned} \iint_R f(x, y)\, dx\, dy &= \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} (x - 2y)\, dx\, dy = \int_{y=0}^{y=1} \left[\frac{1}{2}x^2 - 2xy \right]_{x=0}^{x=1-y} dy \\ &= \int_{y=0}^{y=1} \left[\frac{1}{2}(1-y)^2 - 2y(1-y) \right] dy = \int_{y=0}^{y=1} \left[\frac{1}{2} - 3y + \frac{5}{2}y^2 \right] dy \\ &= \left[\frac{1}{2}y - \frac{3}{2}y^2 + \frac{5}{6}y^3 \right]_{y=0}^{y=1} = -\frac{1}{6}. \end{aligned}$$

Exercise 7.2.5

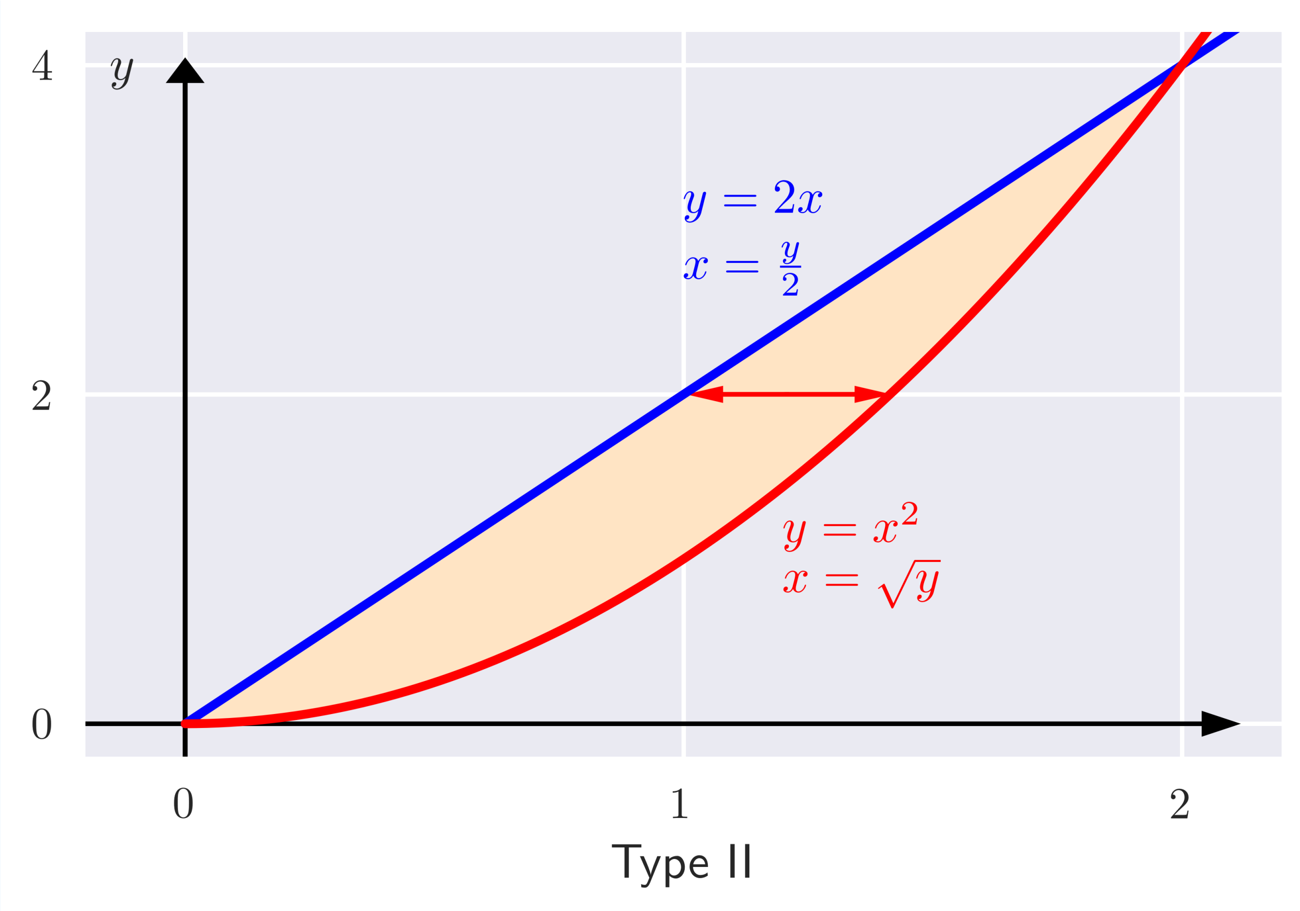
Evaluate the iterated integral $\iint_D (x^2 + y^2)\, dA$ over the region D in the first quadrant between the functions $y = 2x$ and $y = x^2$.

Evaluate the iterated integral by integrating first with respect to y and then integrating first with resect to x.

Hint

Sketch the region and follow Example 7.2.6

Answer



$\frac{216}{35}$

7.2.4 Calculating Volumes, Areas, and Average Values

We can use double integrals over general regions to compute volumes, areas, and average values. The methods are the same as those in Double Integrals over Rectangular Regions, but without the restriction to a rectangular region, we can now solve a wider variety of problems.

Example 7.2.7: Finding the Volume of a Tetrahedron

Find the volume of the solid bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $2x + 3y + z = 6$.

Solution

The solid is a tetrahedron with the base on the xy-plane and a height $z = 6 - 2x - 3y$. The base is the region D bounded by the lines, $x = 0$, $y = 0$ and $2x + 3y = 6$ where $z = 0$ (Figure 7.2.12). Note that we can consider the region D as Type I or as Type II, and we can integrate in both ways.

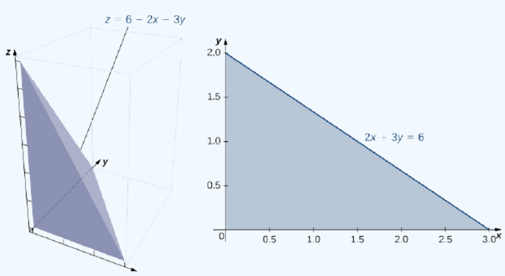


Figure 7.2.12. A tetrahedron consisting of the three coordinate planes and the plane $z = 6 - 2x - 3y$, with the base bound by $x = 0,\ y = 0$, and $2x + 3y = 6$.

First, consider D as a Type I region, and hence $D = \left\{(x, y) \mid 0 \leq x \leq 3,\ 0 \leq y \leq 2 - \frac{2}{3}x\right\}$.

Therefore, the volume is

$$\begin{aligned} V &= \int_{x=0}^{x=3} \int_{y=0}^{y=2-(2x/3)} (6-2x-3y)\,dy\,dx = \int_{x=0}^{x=3} \left[\left(6y - 2xy - \frac{3}{2}y^2\right)\Big|_{y=0}^{y=2-(2x/3)} \right] dx \\ &= \int_{x=0}^{x=3} \left[\frac{2}{3}(x-3)^2\right] dx = 6. \end{aligned}$$

Now consider D as a Type II region, so $D = \left\{(x,y) \mid 0 \le y \le 2,\ 0 \le x \le 3 - \frac{3}{2}y\right\}$. In this calculation, the volume is

$$\begin{aligned} V &= \int_{y=0}^{y=2} \int_{x=0}^{x=3-(3y/2)} (6-2x-3y)\,dx\,dy = \int_{y=0}^{y=2} \left[(6x - x^2 - 3xy)\big|_{x=0}^{x=3-(3y/2)} \right] dy \\ &= \int_{y=0}^{y=2} \left[\frac{9}{4}(y-2)^2\right] dy = 6. \end{aligned}$$

Therefore, the volume is 6 cubic units.

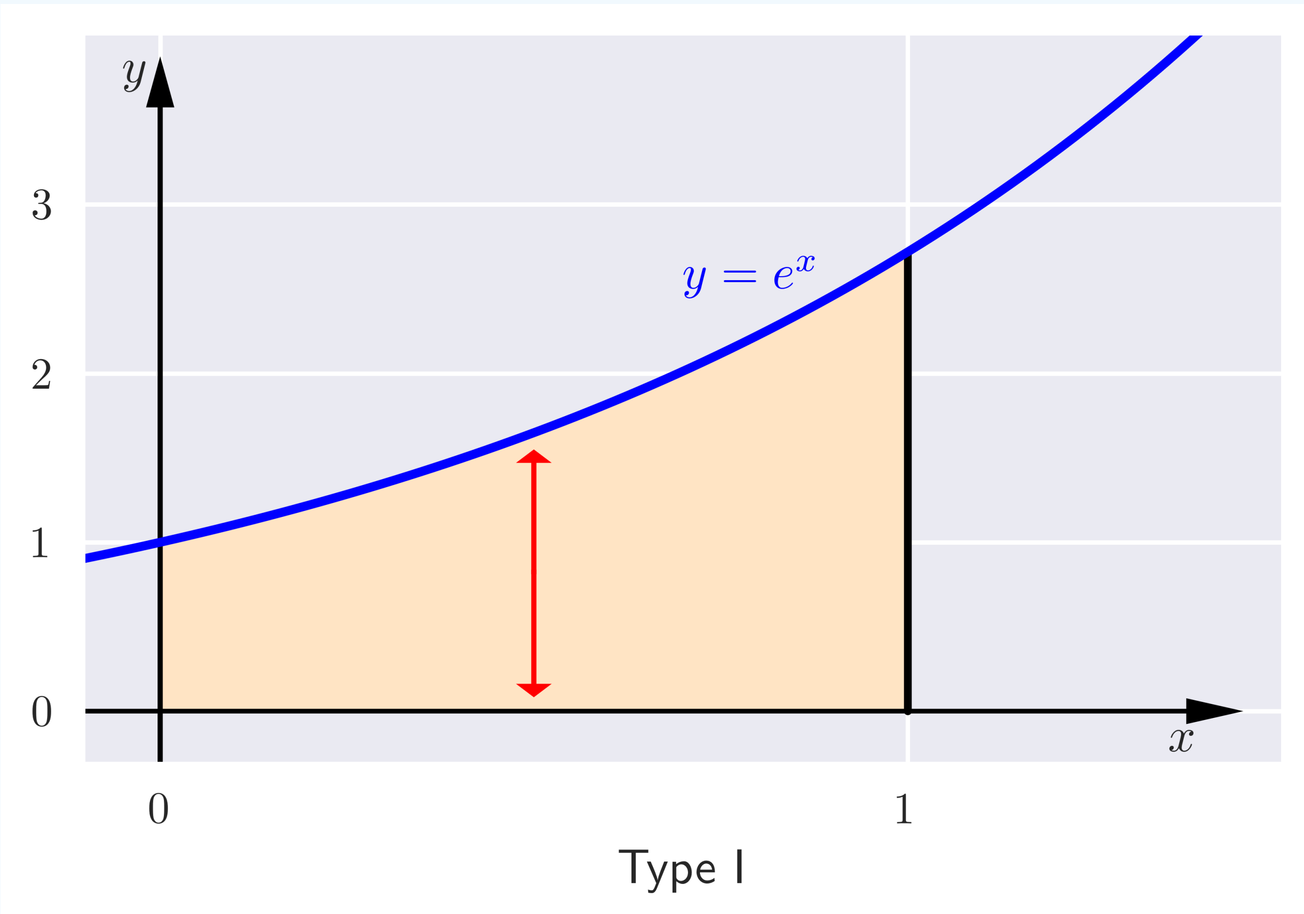
Exercise 7.2.6

Find the volume of the solid bounded above by $f(x,y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$ where x is in the interval $[0,1]$.

Hint

Sketch the region, and describe it as Type I.

Answer



$\frac{e^2}{4} + 10e - \frac{49}{4}$ cubic units

Finding the area of a rectangular region is easy, but finding the area of a non-rectangular region is not so easy. As we have seen, we can use double integrals to find a rectangular area. As a matter of fact, this comes in very handy for finding the area of a general non-rectangular region, as stated in the next definition.

Definition: double integrals

The area of a plane-bounded region D is defined as the double integral

$$\iint_D 1\,dA. \tag{7.2.15}$$

We have already seen how to find areas in terms of single integration. Here we are seeing another way of finding areas by using double integrals, which can be very useful, as we will see in the later sections of this chapter.

Example 7.2.8: Finding the Area of a Region

Find the area of the region bounded below by the curve $y = x^2$ and above by the line $y = 2x$ in the first quadrant (Figure 7.2.13).

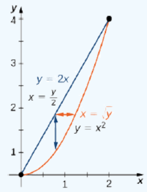


Figure 7.2.13. The region bounded by $y = x^2$ and $y = 2x$.

Solution

We just have to integrate the constant function $f(x, y) = 1$ over the region. Thus, the area A of the bounded region is $\int_{x=0}^{x=2}\int_{y=x^2}^{y=2x} dy\,dx$ *or* $\int_{y=0}^{y=4}\int_{x=y/2}^{x=\sqrt{y}} dx\,dy$:

$$A = \iint\limits_D 1\,dx\,dy = \int_{x=0}^{x=2}\int_{y=x^2}^{y=2x} 1\,dy\,dx = \int_{x=0}^{x=2}\left[y\big|_{y=x^2}^{y=2x}\right]dx = \int_{x=0}^{x=2}(2x - x^2)\,dx = x^2 - \frac{x^3}{3}\bigg|_0^2 = \frac{4}{3}. \tag{7.2.16}$$

Exercise 7.2.7

Find the area of a region bounded above by the curve $y = x^3$ and below by $y = 0$ over the interval $[0, 3]$.

Hint

Sketch the region.

Answer

$\frac{81}{4}$ square units

We can also use a double integral to find the average value of a function over a general region. The definition is a direct extension of the earlier formula.

Definition

If $f(x, y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is

$$f_{ave} = \frac{1}{A(D)}\iint\limits_D f(x, y)\,dA. \tag{7.2.17}$$

Note that the area is $A(D) = \iint\limits_D 1\,dA$.

Example 7.2.9: Finding an Average Value

Find the average value of the function $f(x, y) = 7xy^2$ on the region bounded by the line $x = y$ and the curve $x = \sqrt{y}$ (Figure 7.2.14).

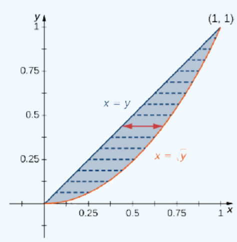


Figure 7.2.14. The region bounded by $x = y$ and $x = \sqrt{y}$.

Solution

First find the area $A(D)$ where the region D is given by the figure. We have

$$A(D) = \iint\limits_D 1\,dA = \int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} 1\,dx\,dy = \int_{y=0}^{y=1}\left[x\big|_{x=y}^{x=\sqrt{y}}\right]dy = \int_{y=0}^{y=1}(\sqrt{y} - y)\,dy = \frac{2}{3}y^{2/3} - \frac{y^2}{2}\bigg|_0^1 = \frac{1}{6} \tag{7.2.18}$$

Then the average value of the given function over this region is

$$\begin{aligned} f_{ave} &= \frac{1}{A(D)}\iint\limits_D f(x, y)\,dA = \frac{1}{A(D)}\int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} 7xy^2\,dx\,dy = \frac{1}{1/6}\int_{y=0}^{y=1}\left[\frac{7}{2}x^2y^2\bigg|_{x=y}^{x=\sqrt{y}}\right]dy \\ &= 6\int_{y=0}^{y=1}\left[\frac{7}{2}y^2(y - y^2)\right]dy = 6\int_{y=0}^{y=1}\left[\frac{7}{2}(y^3 - y^4)\right]dy = \frac{42}{2}\left(\frac{y^4}{4} - \frac{y^5}{5}\right)\bigg|_0^1 = \frac{42}{40} = \frac{21}{20}. \end{aligned}$$

Exercise 7.2.8

Find the average value of the function $f(x, y) = xy$ over the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 3)$.

Hint

Express the line joining $(0,0)$ and $(1,3)$ as a function $y=g(x)$.

Answer

$\frac{3}{4}$

7.2.5 Improper Double Integrals

An **improper double integral** is an integral $\iint_D f\,dA$ where either D is an unbounded region or f is an unbounded function.

For example, $D=\{(x,y)\mid |x-y|\geq 2\}$ is an unbounded region, and the function $f(x,y)=1/(1-x^2-2y^2)$ over the ellipse $x^2+3y^2\geq 1$ is an unbounded function. Hence, both of the following integrals are improper integrals:

i.

$$\iint_D xy\,dA \text{ where } D=\{(x,y)|\,|x-y|\geq 2\}; \tag{7.2.19}$$

ii.

$$\iint_D \frac{1}{1-x^2-2y^2}\,dA \text{ where } D=\{(x,y)|\,x^2+3y^2\leq 1\}. \tag{7.2.20}$$

In this section we would like to deal with improper integrals of functions over rectangles or simple regions such that *f* has only finitely many discontinuities.

Not all such improper integrals can be evaluated; however, a form of Fubini's theorem does apply for some types of improper integrals.

Theorem: Fubini's Theorem for Improper Integrals

If D is a bounded rectangle or simple region in the plane defined by

$\{(x,y): a\leq x\leq b,\ g(x)\leq y\leq h(x)\}$ and also by

$\{(x,y): c\leq y\leq d,\ j(y)\leq x\leq k(y)\}$ and f is a nonnegative function on D with finitely many discontinuities in the interior of D then

$$\iint_D f\,dA=\int_{x=a}^{x=b}\int_{y=g(x)}^{y=h(x)} f(x,y)\,dy\,dx=\int_{y=c}^{y=d}\int_{x=j(y)}^{x=k(y)} f(x,y)\,dx\,dy \tag{7.2.21}$$

It is very important to note that we required that the function be nonnegative on D for the theorem to work. We consider only the case where the function has finitely many discontinuities inside D.

Example 7.2.10: Evaluating a Double Improper Integral

Consider the function $f(x,y)=\frac{e^y}{y}$ over the region $D=\{(x,y): 0\leq x\leq 1,\ x\leq y\leq\sqrt{x}\}$.

Notice that the function is nonnegative and continuous at all points on D except $(0,0)$. Use Fubini's theorem to evaluate the improper integral.

Solution

First we plot the region D (Figure $7.2.15$); then we express it in another way.

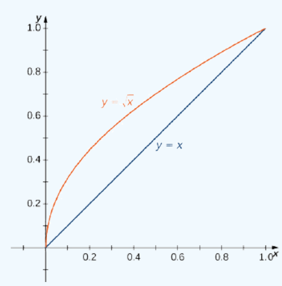


Figure $7.2.15$. The function f is continuous at all points of the region D except $(0,0)$.

The other way to express the same region D is

$$D=\{(x,y):\ 0\leq y\leq 1,\ y^2\leq x\leq y\}.$$

Thus we can use Fubini's theorem for improper integrals and evaluate the integral as

$$\int_{y=0}^{y=1}\int_{x=y^2}^{x=y}\frac{e^y}{y}\,dx\,dy.$$

Therefore, we have

$$\int_{y=0}^{y=1}\int_{x=y^2}^{x=y}\frac{e^y}{y}\,dx\,dy=\int_{y=0}^{y=1}\frac{e^y}{y}x\Big|_{x=y^2}^{x=y}\,dy=\int_{y=0}^{y=1}\frac{e^y}{y}(y-y^2)\,dy=\int_0^1(e^y-ye^y)\,dy=e-2. \tag{7.2.22}$$

As mentioned before, we also have an improper integral if the region of integration is unbounded. Suppose now that the function f is continuous in an unbounded rectangle R.

Theorem: Improper Integrals on an Unbounded Region

If R is an unbounded rectangle such as $R = \{(x,y) : a \le x \le \infty,\ c \le y \le \infty\}$, then when the limit exists, we have

$$\iint_R f(x,y)\,dA = \lim_{(b,d)\to(\infty,\infty)} \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx = \lim_{(b,d)\to(\infty,\infty)} \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy. \tag{7.2.23}$$

The following example shows how this theorem can be used in certain cases of improper integrals.

Example 7.2.11

Evaluate the integral $\iint_R xye^{-x^2-y^2}\,dA$ where R is the first quadrant of the plane.

Solution

The region R is the first quadrant of the plane, which is unbounded. So

$$\begin{aligned}\iint_R xye^{-x^2-y^2}\,dA &= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} \left(\int_{y=0}^{y=d} xye^{-x^2-y^2}\,dy\right) dx \\ &= \lim_{(b,d)\to(\infty,\infty)} \int_{y=0}^{x=b} xye^{-x^2-y^2}\,dy \\ &= \lim_{(b,d)\to(\infty,\infty)} \frac{1}{4}\left(1-e^{-b^2}\right)\left(1-e^{-d^2}\right) = \frac{1}{4}\end{aligned}$$

Thus,

$$\iint_R xye^{-x^2-y^2}\,dA$$

is convergent and the value is $\frac{1}{4}$.

Exercise 7.2.9

$$\iint_D \frac{y}{\sqrt{1-x^2-y^2}}\,dA \tag{7.2.24}$$

where $D = \{(x,y) : x \ge 0,\ y \ge 0,\ x^2+y^2 \le 1\}$.

Hint

Notice that the integral is nonnegative and discontinuous on $x^2+y^2=1$. Express the region D as $D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}$ and integrate using the method of substitution.

Answer

$\frac{\pi}{4}$

In some situations in probability theory, we can gain insight into a problem when we are able to use double integrals over general regions. Before we go over an example with a double integral, we need to set a few definitions and become familiar with some important properties.

Definition

Consider a pair of continuous random variables X and Y such as the birthdays of two people or the number of sunny and rainy days in a month. The **joint density function** $\boldsymbol{f}$ of X and Y satisfies the probability that (X,Y) lies in a certain region D:

$$P((X,Y)\in D) = \iint_D f(x,y)\,dA. \tag{7.2.25}$$

Since the probabilities can never be negative and must lie between 0 and 1 the joint density function satisfies the following inequality and equation:

$$f(x,y) \ge 0 \text{ and } \iint_R f(x,y)\,dA = 1. \tag{7.2.26}$$

Definition

The variables X and Y are said to be independent random variables if their joint density function is the product of their individual density functions:

$$f(x,y) = f_1(x)f_2(y). \tag{7.2.27}$$

Example 7.2.12: Application to Probability

At Sydney's Restaurant, customers must wait an average of 15 minutes for a table. From the time they are seated until they have finished their meal requires an additional 40 minutes, on average. What is the probability that a customer spends less than an hour and a half at the diner, assuming that waiting for a table and completing the meal are independent events?

Solution

Waiting times are mathematically modeled by exponential density functions, with m being the average waiting time, as

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{m}e^{-t/m} & \text{if } t \ge 0. \end{cases} \tag{7.2.28}$$

if X and Y are random variables for 'waiting for a table' and 'completing the meal,' then the probability density functions are, respectively,

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0. \\ \frac{1}{15}e^{-x/15} & \text{if } x \geq 0. \end{cases} \quad \text{and} \quad f_2(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{40}e^{-y/40} & \text{if } y \geq 0. \end{cases} \tag{7.2.29}$$

Clearly, the events are independent and hence the joint density function is the product of the individual functions

$$f(x,y) = f_1(x)f_2(y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0, \\ \frac{1}{600}e^{-x/15} & \text{if } x, y \geq 0 \end{cases} \tag{7.2.30}$$

We want to find the probability that the combined time $X+Y$ is less than 90 minutes. In terms of geometry, it means that the region D is in the first quadrant bounded by the line $x+y=90$ (Figure 7.2.16).

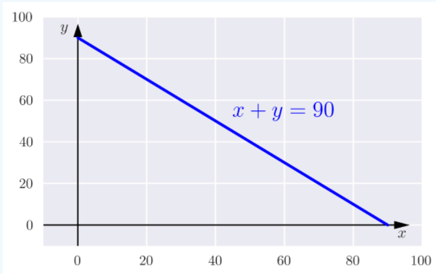


Figure 7.2.16. The region of integration for a joint probability density function.

Hence, the probability that (X,Y) is in the region D is

$$P(X+Y \leq 90) = P((X,Y) \in D) = \iint_D f(x,y)\,dA = \iint_D \frac{1}{600}e^{-x/15}e^{-y/40}\,dA. \tag{7.2.31}$$

Since $x+y=90$ is the same as $y=90-x$, we have a region of Type I, so

$$D = \{(x,y) \mid 0 \leq x \leq 90,\ 0 \leq y \leq 90-x\}, \tag{7.2.32}$$

$$P(X+Y \leq 90) = \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-(/15}e^{-y/40}dx\,dy = \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-x/15}e^{-y/40}dx\,dy \tag{7.2.33}$$

$$= \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-(x/15+y/40)}dx\,dy = 0.8328 \tag{7.2.34}$$

Thus, there is an 83.2% chance that a customer spends less than an hour and a half at the restaurant.

Another important application in probability that can involve improper double integrals is the calculation of expected values. First we define this concept and then show an example of a calculation.

Definition

In probability theory, we denote the expected values $E(X)$ and $E(Y)$ respectively, as the most likely outcomes of the events.

The expected values $E(X)$ and $E(Y)$ are given by

$$E(X) = \iint_S x\,f(x,y)\,dA \text{ and } E(Y) = \iint_S y\,f(x,y)\,dA, \tag{7.2.35}$$

where S is the sample space of the random variables X and Y.

Example 7.2.13: Finding Expected Value

Find the expected time for the events 'waiting for a table' and 'completing the meal' in Example 7.2.12

Solution

Using the first quadrant of the rectangular coordinate plane as the sample space, we have improper integrals for $E(X)$ and $E(Y)$.

The expected time for a table is

$$E(X) = \iint_S x\frac{1}{600}e^{-x/15}e^{-y/40}dA = \frac{1}{600}\int_{x=0}^{x=\infty}\int_{y=0}^{y=\infty} xe^{-x/15}e^{-y/40}dA \tag{7.2.36}$$

$$= \frac{1}{600}\lim_{(a,b)\to(\infty,\infty)}\int_{x=0}^{x=a}\int_{y=0}^{y=b} xe^{-x/15}e^{-y/40}dx\,dy \tag{7.2.37}$$

$$= \frac{1}{600}\left(\lim_{a\to\infty}\int_{x=0}^{x=a} xe^{-x/15}dx\right)\left(\lim_{b\to\infty}\int_{y=0}^{y=b} e^{-y/40}dy\right) \tag{7.2.38}$$

$$\frac{1}{600}\left(\left(\lim_{a\to\infty}(-15e^{-x/15}(x+15))\right)\Big|_{x=0}^{x=a}\right)\left(\left(\lim_{b\to\infty}(-40e^{-y/40})\right)\Big|_{y=0}^{y=b}\right) \tag{7.2.39}$$

$$= \frac{1}{600}\left(\lim_{a\to\infty}(-15e^{-a/15}(x+15)+225)\right)\left(\lim_{b\to\infty}(-40e^{-b/40}+40)\right) \tag{7.2.40}$$

$$= \frac{1}{600}(225)(40) = 15. \tag{7.2.41}$$

A similar calculation shows that $E(Y) = 40$. This means that the expected values of the two random events are the average waiting time and the average dining time, respectively.

Exercise 7.2.10

The joint density function for two random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{600}(x^2+y^2) & \text{if } \leq x \leq 15,\ 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases} \tag{7.2.42}$$

Find the probability that X is at most 10 and Y is at least 5.

Hint

Compute the probability

$$P(X \leq 10,\ Y \geq 5) = \int_{x=-\infty}^{10} \int_{y=5}^{y=10} \frac{1}{6000}(x^2+y^2)dy\,dx. \tag{7.2.43}$$

Answer

$\frac{55}{72} \approx 0.7638$

7.2.6 Key Concepts

- A general bounded region D on the plane is a region that can be enclosed inside a rectangular region. We can use this idea to define a double integral over a

general bounded region.

- To evaluate an iterated integral of a function over a general non-rectangular region, we sketch the region and express it as a Type I or as a Type II region or
- as a union of several Type I or Type II regions that overlap only on their boundaries.
- We can use double integrals to find volumes, areas, and average values of a function over general regions, similarly to calculations over rectangular regions.
- We can use Fubini's theorem for improper integrals to evaluate some types of improper integrals.

7.2.7 Key Equations

- **Iterated integral over a Type I region**

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx \tag{7.2.44}$$

- **Iterated integral over a Type II region**

$$\iint_D f(x,y)\,dA = \iint_D (x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy \tag{7.2.45}$$

7.2.7.0.1 Glossary

improper double integral

a double integral over an unbounded region or of an unbounded function

Type I

a region D in the xy- plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

Type II

a region D in the xy-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(h)$

7.2.8 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://c

7.2E: Exercises

7.2E.1 Exercises $7.2E.1-14$

In the following exercises, specify whether the region is of Type I or Type II.

1. The region D bounded by $y=x^3,\ y=x^3+1,\ x=0,$ and $x=1$ as given in the following figure.

A region is bounded by y = 1 + x cubed, y = x cubed, x = 0, and x = 1.

2. Find the average value of the function $f(x,y)=3xy$ on the region graphed in the previous exercise.

Answer

$\frac{27}{20}$

3. Find the area of the region D given in the previous exercise.

4. The region D bounded by $y=sin\,x,\ y=1+sin\,x,\ x=0$, and $x=\frac{\pi}{2}$ as given in the following figure.

A region is bounded by y = 1 + sin x, y = sin x, x = 0, and x = pi/2.

Answer

Type I but not Type II

5. Find the average value of the function $f(x,y)=cos\,x$ on the region graphed in the previous exercise.

6. Find the area of the region D given in the previous exercise.

Answer

$\frac{\pi}{2}$

7. The region D bounded by $x=y^2-1$ and $x=\sqrt{1-y^2}$ as given in the following figure.

A region is bounded by x = negative 1 + y squared and x = the square root of the quantity (1 minus y squared).

8. Find the volume of the solid under the graph of the function $f(x,y)=xy+1$ and above the region in the figure in the previous exercise.

Answer

$\frac{1}{6}(8+3\pi)$

9. The region D bounded by $y=0,\ x=-10+y,$ and $x=10-y$ as given in the following figure.

A region is bounded by x = negative 10 + y, x = 10 minus y, and y = 0.

10. Find the volume of the solid under the graph of the function $f(x,y)=x+y$ and above the region in the figure from the previous exercise.

Answer

$\frac{1000}{3}$

11. The region D bounded by $y=0,\ x=y-1,\ x=\frac{\pi}{2}$ as given in the following figure.

A region is bounded by x = pi/2, y = 0, and x = negative 1 + y.

12. The region D bounded by $y=0$ and $y=x^2-1$ as given in the following figure.

A region is bounded by y = 0 and y = negative 1 + x squared.

Answer

Type I and Type II

13. Let D be the region bounded by the curves of equations $y=x,\ y=-x$ and $y=2-x^2$. Explain why D is neither of Type I nor II.

14. Let D be the region bounded by the curves of equations $y=cos\,x$ and $y=4-x^2$ and the x-axis. Explain why D is neither of Type I nor II.

Answer

The region D is not of Type I: it does not lie between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. The region is not of Type II: it does not lie between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$.

7.2E.2 Exercises $7.2E.15-20$

In the following exercises, evaluate the double integral $\iint_D f(x,y)dA$ over the region D.

15. $f(x,y)=2x+4y$ and

$D=\{(x,y)|\,0\le x\le 1,\ x^3\le y\le x^3+1\}$

16. $f(x,y)=1$ and

$D=\{(x,y)|\,0\le x\le \frac{\pi}{2},\ \sin x\le y\le 1+\sin x\}$

Answer

$\frac{\pi}{2}$

17. $f(x,y)=2$ and

$D=\{(x,y)|\,0\le y\le 1,\ y-1\le x\le \arccos y\}$

18. $f(x,y)=xy$ and

$D=\{(x,y)|\,-1\le y\le 1,\ y^2-1\le x\le \sqrt{1-y^2}\}$

Answer

0

19. $f(x,y)=sin\,y$ and D is the triangular region with vertices $(0,0),\ (0,3)$, and $(3,0)$

20. $f(x,y)=-x+1$ and D is the triangular region with vertices $(0,0),\ (0,2)$, and $(2,2)$

Answer

$\frac{2}{3}$

7.2E.3 Exercises $\mathbf{7.2E.21-26}$

Evaluate the iterated integrals.

21.

$$\int_0^3\int_{2x}^{3x}(x+y^2)dy\,dx \tag{7.2E.1}$$

22.

$$\int_0^1\int_{2\sqrt{x}}^{2\sqrt{x}+1}(xy+1)dy\,dx \tag{7.2E.2}$$

Answer

$\frac{41}{20}$

23.

$$\int_e^{e^2}\int_{ln\,u}^{2}(v+ln\,u)dv\,du \tag{7.2E.3}$$

24.

$$\int_1^2\int_{-u^2-1}^{-u}(8uv)dv\,du \tag{7.2E.4}$$

Answer

-63

25.

$$\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(2x+4y^3)dx\,dy \tag{7.2E.5}$$

26.

$$\int_0^1\int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}}4dx\,dy \tag{7.2E.6}$$

Answer

π

7.2E.4 Exercises $\mathbf{7.2E.27-29}$

27. Let D be the region bounded by $y=1-x^2,\ y=4-x^2$, and the x- and y-axes.

a. Show that

$$\iint_D x dA=\int_0^1\int_{1-x^2}^{4-x^2}x\,dy\,dx+\int_1^2\int_0^{4-x^2}x\,dy\,dx \tag{7.2E.7}$$

by dividing the region D into two regions of Type I.

b. Evaluate the integral

$$\iint_D s\,dA. \tag{7.2E.8}$$

28. Let D be the region bounded by $y=1,\ y=x,\ y=ln\,x$, and the x-axis.

a. Show that

$$\iint_D y^2 dA=\int_{-1}^0\int_{-x}^{2-x^2}y^2dy\,dx+\int_0^1\int_x^{2-x^2}y^2dy\,dx \tag{7.2E.9}$$

by dividing the region D into two regions of Type I, where $D=(x,y)|y\geq x, y\geq -x,\ y\leq 2-x^2$.

b. Evaluate the integral

$$\iint_D y^2 dA. \tag{7.2E.10}$$

29. Let D be the region bounded by $y=x^2$, $y=x+2$, and $y=-x$.

a. Show that

$$\iint_D x\,dA=\int_0^1\int_{-y}^{\sqrt{y}}x\,dx\,dy+\int_1^2\int_{y-2}^{\sqrt{y}}x\,dx\,dy \tag{7.2E.11}$$

by dividing the region D into two regions of Type II, where $D=(x,y)|y\geq x^2,\ y\geq -x,\ y\leq x+2$.

b. Evaluate the integral

$$\iint_D x\,dA. \tag{7.2E.12}$$

Answer

a. Answers may vary; b. $\frac{8}{12}$

7.2E.5 Exercises $\mathbf{7.2E.30-33}$

30. The region D bounded by $x=0, y=x^5+1$, and $y=3-x^2$ is shown in the following figure. Find the area $A(D)$ of the region D.

A region is bounded by y = 1 + x to the fifth power, y = 3 minus x squared, and x = 0.

31. The region D bounded by $y=cos\ x,\ y=4\ cos\ x$, and $x=\pm\frac{\pi}{3}$ is shown in the following figure. Find the area $A(D)$ of the region D.

A region is bounded by y = cos x, y = 4 + cos x, x = negative 1, and x = 1.

Answer

$\frac{8\pi}{3}$

32. Find the area $A(D)$ of the region $D=\left\{(x,y)|\ y\geq 1-x^2, y\leq 4-x^2,\ y\geq 0,\ x\geq 0\right\}$.

33. Let D be the region bounded by $y=1,\ y=x,\ y=ln\ x$, and the x-axis. Find the area $A(D)$ of the region D.

Answer

$e-\frac{3}{2}$

7.2E.6 Exercises $\mathbf{7.2E.34-35}$

34. Find the average value of the function $f(x,y)=sin\ y$ on the triangular region with vertices $(0,0),\ (0,3)$, and $(3,0)$.

35. Find the average value of the function $f(x,y)=-x+1$ on the triangular region with vertices $(0,0),\ (0,2)$, and $(2,2)$.

Answer

$\frac{2}{3}$

7.2E.7 Exercises $\mathbf{7.2E.36-39}$

In the following exercises, change the order of integration and evaluate the integral.

36.

$$\int_{-1}^{\pi/2}\int_0^{x+1} sin\ x\ dy\ dx \tag{7.2E.13}$$

37.

$$\int_0^1\int_{x-1}^{1-x} x\ dy\ dx \tag{7.2E.14}$$

Answer

$$\int_0^1\int_{x-1}^{1-x} x\ dy\ dx=\int_{-1}^0\int_0^{y+1} x\ dx\ dy+\int_0^1\int_{-}^{1-y} x\ dx\ dy=\frac{1}{3} \tag{7.2E.15}$$

38.

$$\int_{-1}^0\int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2 dx\ dy \tag{7.2E.16}$$

39.

$$\int_{-1/2}^{1/2}\int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y\ dx\ dy \tag{7.2E.17}$$

Answer

$$\int_{-1/2}^{1/2}\int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y\ dx\ dy=\int_1^2\int_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} y\ dy\ dx=0 \tag{7.2E.18}$$

7.2E.8 Exercises $\mathbf{7.2E.40-41}$

40. The region D is shown in the following figure. Evaluate the double integral

$$\iint_D (x^2+y)dA \tag{7.2E.19}$$

by using the easier order of integration.

A region is bounded by y = negative 4 + x squared and y = 4 minus x squared.

41. The region D is shown in the following figure. Evaluate the double integral

$$\iint_D (x^2-y^2)dA \tag{7.2E.20}$$

by using the easier order of integration.

A region is bounded by y to the fourth power = 1 minus x and y to the fourth power = 1 + x.

Answer

$$\iint_D (x^2-y^2)dA=\int_{-1}^1\int_{y^4-1}^{1-y^4}(x^2-y^2)dx\ dy=\frac{464}{4095} \tag{7.2E.21}$$

7.2E.9 Exercises $7.2E.42-45$

42. Find the volume of the solid under the surface $z=2x+y^2$ and above the region bounded by $y=x^5$ and $y=x$.

43. Find the volume of the solid under the plane $z=3x+y$ and above the region determined by $y=x^7$ and $y=x$.

Answer

$\frac{4}{5}$

44. Find the volume of the solid under the plane $z=3x+y$ and above the region bounded by $x=tan\,y,\ x=-tan\,y$, and $x=1$.

45. Find the volume of the solid under the surface $z=x^3$ and above the plane region bounded by $x=sin\,y,\ x=-sin\,y$, and $x=1$.

Answer

$\frac{5\pi}{32}$

7.2E.10 Exercises $7.2E.46-47$

46. Let g be a positive, increasing, and differentiable function on the interval $[a,b]$. Show that the volume of the solid under the surface $z=g'(x)$ and above the region bounded by $y=0,\ y=g(x),\ x=a$, and $x=b$ is given by $\frac{1}{2}(g^2(b)-g^2(a))$.

47. Let g be a positive, increasing, and differentiable function on the interval $[a,b]$ and let k be a positive real number. Show that the volume of the solid under the surface $z=g'(x)$ and above the region bounded by $y=g(x),\ y=g(x)+k,\ x=a$, and $x=b$ is given by $k(g(b)-g(a))$.

7.2E.11 Exercises $7.2E.48-51$

48. Find the volume of the solid situated in the first octant and determined by the planes $z=2$, $z=0,\ x+y=1,\ x=0$, and $y=0$.

49. Find the volume of the solid situated in the first octant and bounded by the planes $x+2y=1$, $x=0,\ z=4$, and $z=0$.

Answer

1

50. Find the volume of the solid bounded by the planes $x+y=1,\ x-y=1,\ x=0,\ z=0$, and $z=10$.

51. Find the volume of the solid bounded by the planes $x+y=1,\ x-y=1,\ x-y=-1,\ z=1$, and $z=0$

Answer

2

7.2E.12 Exercises $7.2E.52$

52. Let S_1 and S_2 be the solids situated in the first octant under the planes $x+y+z=1$ and $x+y+2z=1$ respectively, and let S be the solid situated between $S_1,\ S_2,\ x=0$, and $y=0$.

1. Find the volume of the solid S_1.
2. Find the volume of the solid S_2.
3. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

53. Let S_1 and S_2 be the solids situated in the first octant under the planes $2x+2y+z=2$ and $x+y+z=1$ respectively, and let S be the solid situated between $S_1,\ S_2,\ x=0$, and $y=0$.

1. Find the volume of the solid S_1.
2. Find the volume of the solid S_2.
3. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

Answer

a. $\frac{1}{3}$; b. $\frac{1}{6}$; c. $\frac{1}{6}$

54. Let S_1 and S_2 be the solids situated in the first octant under the plane $x+y+z=2$ and under the sphere $x^2+y^2+z^2=4$, respectively. If the volume of the solid S_2 is $\frac{4\pi}{3}$ determine the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of these solids.

55. Let S_1 and S_2 be the solids situated in the first octant under the plane $x+y+z=2$ and under the sphere $x^2+y^2=4$, respectively.

1. Find the volume of the solid S_1.
2. Find the volume of the solid S_2.
3. Find the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of the solids S_1 and S_2.

Answer

a. $\frac{4}{3}$; b. 2π; c. $\frac{6\pi-4}{3}$

7.2E.13 Exercises $7.2E.56-57$

56. [T] The following figure shows the region D bounded by the curves $y=sin\,x,\ x=0$, and $y=x^4$. Use a graphing calculator or CAS to find the -coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.

A region is bounded by y = sin x, y = x to the fourth power, and x = 0.

57. [T] The region D bounded by the curves $y=cos\,x,\ x=0$, and $y=x^3$ is shown in the following figure. Use a graphing calculator or CAS to find the *x*-coordinates of the intersection points of th

A region is bounded by y = cos x, y = x cubed, and x = 0.

Answer

0 and 0.865474; $A(D)=0.621135$

7.2E.14 Exercises $7.2E.58-59$

58. Suppose that (X,Y) is the outcome of an experiment that must occur in a particular region S in the xy-plane. In this context, the region S is called the sample space of the experiment and X and Y

$$p(x,y) = \frac{1}{9}(x,y) \in [0,3] \times [0,3], \tag{7.2E.22}$$

$$p(x,y) = 0 \text{ otherwise} \tag{7.2E.23}$$

Find the probability that the point (X, Y) is inside the unit square and interpret the result.

59. Consider X and Y two random variables of probability densities $p_1(x)$ and $p_2(x)$, respectively. The random variables X and Y are said to be independent if their joint density function is given by

$$p_1(x) = \frac{1}{3}e^{-x/3}\ x \geq 0, \tag{7.2E.24}$$

$$p_1(x) = 0 \text{ otherwise} \tag{7.2E.25}$$

$$p_2(y) = \frac{1}{5}e^{-y/5}\ y \geq 0 \tag{7.2E.26}$$

$$p_2(y) = 0 \text{ otherwise} \tag{7.2E.27}$$

respectively, the probability that a customer will spend less than 6 minutes in the drive-thru line is given by $P[X+Y \leq 6] = \iint_D p(x,y)dx\ dy$, where $D = (x,y)|x \geq 0,\ y \geq 0,\ x+y \leq 6$. Find P

Answer

$P[X+Y \leq 6] = 1 + \frac{3}{2e^2} - \frac{5}{e^{6/5}} \approx 0.45$; there is a 45%chance that a customer will spend minutes in the drive-thru line.

7.2E.15 Exercises $\mathbf{7.2E.60-61}$

60. [T] The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius *s* centered at the opposite vertex of the

An equilateral triangle with additional regions consisting of three arcs of a circle with radius equal to the length of the side of the triangle. These arcs connect two adjacent vertices, and the radius is taken from the opposite vertex.

61. [T] Show that the area of the lunes of Alhazen, the two blue lunes in the following figure, is the same as the area of the right triangle *ABC*. The outer boundaries of the lunes are semicircles of diam

A right triangle with points A, B, and C. Point B has the right angle. There are two lunes drawn from A to B and from B to C with outer diameters AB and AC, respectively, and with the inner boundaries formed by the circumcircle of the triangle ABC.

7.3: Double Integrals in Polar Coordinates

This page is a draft and is under active development.

Double integrals are sometimes much easier to evaluate if we change rectangular coordinates to polar coordinates. However, before we describe how to make this change, we need to establish the concept of a double integral in a polar rectangular region.

7.3.1 Polar Rectangular Regions of Integration

When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values. In polar coordinates, the shape we work with is a polar rectangle, whose sides have constant r-values and/or constant θ-values. This means we can describe a polar rectangle as in Figure $7.3.1a$, with $R = \{(r,\theta)\,|\,a \le r \le b,\ \alpha \le \theta \le \beta\}$.

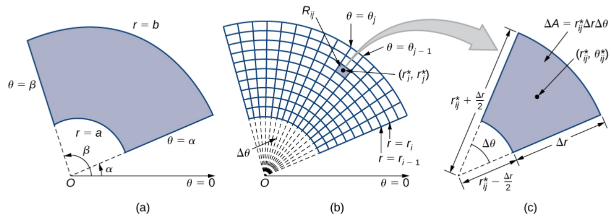


Figure 7.3.1: (a) A polar rectangle R (b) divided into subrectangles R_{ij} (c) Close-up of a subrectangle.

In this section, we are looking to integrate over polar rectangles. Consider a function $f(r,\theta)$ over a polar rectangle R. We divide the interval $[a,b]$ into m subintervals $[r_{i-1}, r_i]$ of length $\Delta r = (b-a)/m$ and divide the interval $[\alpha,\beta]$ into n subintervals $[\theta_{i-1},\theta_i]$ of width $\Delta\theta = (\beta-\alpha)/n$. This means that the circles $r = r_i$ and rays $\theta = \theta_i$ for $1 \le i \le m$ and $1 \le j \le n$ divide the polar rectangle R into smaller polar subrectangles R_ij (Figure $7.3.1b$).

As before, we need to find the area ΔA of the polar subrectangle R_{ij} and the "polar" volume of the thin box above R_{ij}. Recall that, in a circle of radius r the length s of an arc subtended by a central angle of θ radians is $s = r\theta$. Notice that the polar rectangle R_{ij} looks a lot like a trapezoid with parallel sides $r_{i-1}\Delta\theta$ and $r_i\Delta\theta$ and with a width Δr. Hence the area of the polar subrectangle R_{ij} is

$$\Delta A = \frac{1}{2}\Delta r(r_{i-1}\Delta\theta + r_1\delta\theta). \tag{7.3.1}$$

Simplifying and letting $r_{ij}^* = \frac{1}{2}(r_{i-1}+r_i)$, we have $\Delta A = r_{ij}^*\Delta r\Delta\theta$.

Therefore, the polar volume of the thin box above R_{ij} (Figure $7.3.2$) is

$$f(r_{ij}^*,\theta_{ij}^*)b\Delta A = f(r_{ij}^*,\theta_{ij}^*)r_{ij}^*\Delta r\Delta\theta. \tag{7.3.2}$$

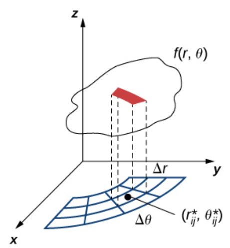


Figure 7.3.2: Finding the volume of the thin box above polar rectangle R_{ij}.

Using the same idea for all the subrectangles and summing the volumes of the rectangular boxes, we obtain a double Riemann sum as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)r_{ij}^*\Delta r\Delta\theta. \tag{7.3.3}$$

As we have seen before, we obtain a better approximation to the polar volume of the solid above the region R when we let m and n become larger. Hence, we define the polar volume as the limit of the double Riemann sum,

$$V = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)r_{ij}^*\Delta r\Delta\theta. \tag{7.3.4}$$

This becomes the expression for the double integral.

Definition: The double integral in polar coordinates

The double integral of the function $f(r,\theta)$ over the polar rectangular region R in the $r\theta$-plane is defined as

$$\iint_R f(r,\theta)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)\Delta A \tag{7.3.5}$$

$$= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)r_{ij}^*\Delta r\Delta\theta. \tag{7.3.6}$$

Again, just as in section on Double Integrals over Rectangular Regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence,

$$\iint_R f(r,\theta)\,dA = \iint_R f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} f(r,\theta)\,r\,dr\,d\theta. \tag{7.3.7}$$

Notice that the expression for dA is replaced by $r\,dr\,d\theta$ when working in polar coordinates. Another way to look at the polar double integral is to change the double integral in rectangular coordinates by substitution. When the function f is given in terms of x and y using $x = r\,\cos\,\theta,\ y = r\,\sin\,\theta$, and $dA = r\,dr\,d\theta$ changes it to

$$\iint_R f(x,y)\,dA = \iint_R f(r\,\cos\,\theta,\ r\,\sin\,\theta)\,r\,dr\,d\theta. \tag{7.3.8}$$

Note that all the properties listed in section on Double Integrals over Rectangular Regions for the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Example 7.3.1A: Sketching a Polar Rectangular Region

Sketch the polar rectangular region

$$R = \{(r, \theta) \mid 1 \le r \le 3, 0 \le \theta \le \pi\}.$$

Solution

As we can see from Figure 7.3.3, $r = 1$ and $r = 3$ are circles of radius 1 and 3 and $0 \le \theta \le \pi$ covers the entire top half of the plane. Hence the region R looks like a semicircular band.

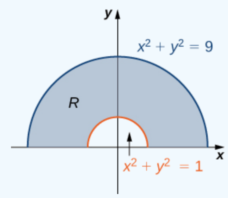


Figure 7.3.3: The polar region R lies between two semicircles.

Now that we have sketched a polar rectangular region, let us demonstrate how to evaluate a double integral over this region by using polar coordinates.

Example 7.3.1B: Evaluating a Double Integral over a Polar Rectangular Region

Evaluate the integral $\iint_R 3x\,dA$ over the region $R = \{(r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$.

Solution

First we sketch a figure similar to Figure 7.3.3, but with outer radius $r = 2$. From the figure we can see that we have

$$\begin{aligned} \iint_R 3x\,dA &= \int_{\theta=0}^{\theta=\pi} \int_{r=1}^{r=2} 3r\cos\theta\, r\,dr\,d\theta && \text{Use an integral with correct limits of integration.} \\ &= \int_{\theta+0}^{\theta=\pi} \cos\theta \left[r^3 \Big|_{r=1}^{r=2} \right] d\theta && \text{Integrate first with respect to } r. \\ &= \int_{\theta=0}^{\theta=\pi} 7\cos\theta\,d\theta = 7\sin\theta \Big|_{\theta=0}^{\theta=\pi} = 0. \end{aligned}$$

Exercise 7.3.1

Sketch the region $D = \{(r, \theta) | 1 \le r \le 2,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$, and evaluate $\iint_R x\,dA$.

Hint

Follow the steps in Example 7.3.1A

Answer

$\frac{14}{3}$

Example 7.3.2A: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (1 - x^2 - y^2)\,dA$$

where R is the unit circle on the xy-plane.

Solution

The region R is a unit circle, so we can describe it as $R = \{(r, \theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

Using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

$$\begin{aligned} \iint_R (1 - x^2 - y^2)\,dA &= \int_0^{2\pi} \int_0^1 (1 - r^2)\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 (r - r^3)\,dr\,d\theta \\ &= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{\pi}{2}. \end{aligned}$$

Example 7.3.2B: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (x + y)\,dA$$

where $R = \{(x, y) \mid 1 \le x^2 + y^2 \le 4,\ x \le 0\}$.

Solution

We can see that R is an annular region that can be converted to polar coordinates and described as $R = \{(r, \theta) \mid 1 \le r \le 2,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$ (see the following graph).

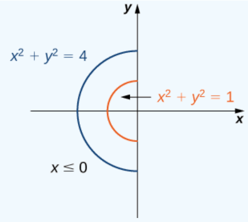


Figure 7.3.4: The annular region of integration R.

Hence, using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

$$\begin{aligned}\iint_R (x+y)\,dA &= \int_{\theta=\pi/2}^{\theta=3\pi/2}\int_{r=1}^{r=2}(r\cos\theta + r\sin\theta)r\,dr\,d\theta \\ &= \left(\int_{r=1}^{r=2} r^2\,dr\right)\left(\int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)\,d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_1^2 [\sin\theta - \cos\theta]\Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{14}{3}.\end{aligned}$$

Exercise 7.3.2

Evaluate the integral

$$\iint_R (4 - x^2 - y^2)\,dA$$

where R is the circle of radius 2 on the xy-plane.

Hint

Follow the steps in the previous example.

Answer

8π

7.3.2 General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular co

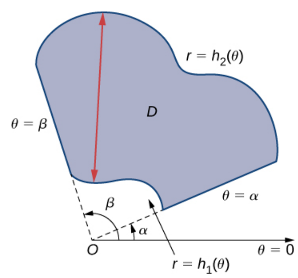


Figure 7.3.5: A general polar region between $\alpha < \theta < \beta$ and $h_1(\theta) < r < h_2(\theta)$.

Theorem: Double Integrals over General Polar Regions

If $f(r,\theta)$ is continuous on a general polar region D as described above, then

$$\iint_D f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)\,r\,dr\,d\theta. \tag{7.3.9}$$

Example 7.3.3: Evaluating a Double Integral over a General Polar Region

Evaluate the integral

$$\iint_D r^2\sin\theta\,r\,dr\,d\theta$$

where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1+\cos\theta$.

Solution

We can describe the region D as $\{(r,\theta)\,|\,0 \le \theta \le \pi,\ 0 \le r \le 1+\cos\theta\}$ as shown in Figure 7.3.6.

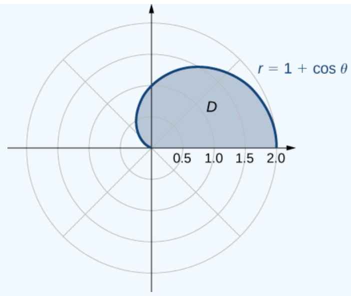


Figure 7.3.6: The region D is the top half of a cardioid.

Hence, we have

$$\begin{aligned}\iint_D r^2 \sin\theta\, r\, dr\, d\theta &= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1+\cos\theta} (r^2 \sin\theta)\, r\, dr\, d\theta \\ &= \frac{1}{4}\int_{\theta=0}^{\theta=\pi} [r^4]\Big|_{r=0}^{r=1+\cos\,\theta} \sin\theta\, d\theta \\ &= \frac{1}{4}\int_{\theta=0}^{\theta=\pi} (1+\cos\theta)^4 \sin\theta\, d\theta \\ &= -\frac{1}{4}\left[\frac{(1+\cos\theta)^5}{5}\right]_0^{\pi} = \frac{8}{5}.\end{aligned}$$

Exercise 7.3.3

Evaluate the integral

$$\iint_D r^2 \sin^2 2\theta\, r\, dr\, d\theta$$

where $D = \left\{(r,\theta) \mid 0 \le \theta \le \pi,\ 0 \le r \le 2\sqrt{\cos 2\theta}\right\}$.

Hint

Graph the region and follow the steps in the previous example.

Answer

$\frac{\pi}{8}$

7.3.3 Polar Areas and Volumes

As in rectangular coordinates, if a solid S is bounded by the surface $z = f(r,\theta)$, as well as by the surfaces $r = a,\ r = b,\ \theta = \alpha$, and $\theta = \beta$, we can find the volume V of S by double integration, as

$$V = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} f(r,\theta)\, r\, dr\, d\theta. \tag{7.3.10}$$

If the base of the solid can be described as $D = (r,\theta) | \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)$, then the double integral for the volume becomes

$$V = \iint_D f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta. \tag{7.3.11}$$

We illustrate this idea with some examples.

Example 7.3.4A: Finding a Volume Using a Double Integral

Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane (Figure 7.3.7).

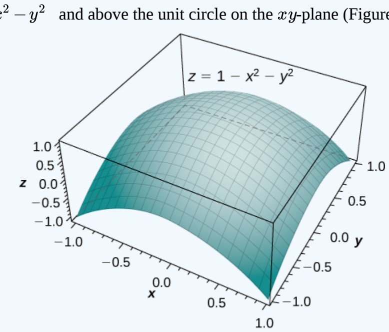


Figure 7.3.7: Finding the volume of a solid under a paraboloid and above the unit circle.

Solution

By the method of double integration, we can see that the volume is the iterated integral of the form $\iint_R (1 - x^2 - y^2)\, dA$ where $R = \left\{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\right\}$.

This integration was shown before in Example, so the volume is $\frac{\pi}{2}$ cubic units.

Example 7.3.4B: Finding a Volume Using Double Integration

Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x-1)^2 + y^2 = 1$ on the xy-plane. See the paraboloid in Figure 7.3.8 intersecting the cylinder $(x-1$

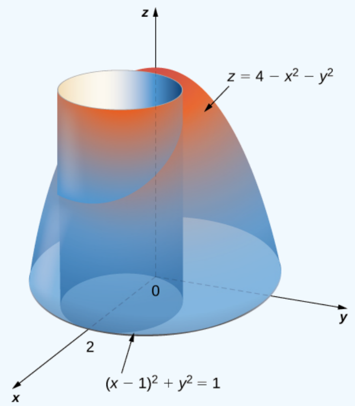


Figure 7.3.8: Finding the volume of a solid with a paraboloid cap and a circular base.

Solution

First change the disk $(x-1)^2 + y^2 = 1$ to polar coordinates. Expanding the square term, we have $x^2 - 2x + 1 + y^2 = 1$. Then simplify to get $x^2 + y^2 = 2x$, which in polar coordinates becomes r^2

$$D = \{(r,\theta) \mid 0 \le \theta \le \pi,\ 0 \le r \le 2\cos\theta\}.$$

Hence the volume of the solid bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by $r = 2\cos\theta$ is

$$\begin{aligned} V &= \iint_D f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\cos\theta} (4 - r^2)\, r\, dr\, d\theta \\ &= \int_{\theta=0}^{\theta=\pi} \left[4\frac{r^2}{2} - \frac{r^4}{4} \Big|_0^{2\cos\theta} \right] d\theta \\ &= \int_0^{\pi} [8\cos^2\theta - 4\cos^2\theta]\, d\theta = \left[\frac{5}{2}\theta + \frac{5}{2}\sin\theta\cos\theta - \sin\theta\cos^3\theta \right]_0^{\pi} = \frac{5}{2}\pi \text{ units}^3. \end{aligned}$$

Notice in the next example that integration is not always easy with polar coordinates. Complexity of integration depends on the function and also on the region over which we need to perform the integra

Example 7.3.5A: Finding a Volume Using a Double Integral

Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

Solution

First examine the region over which we need to set up the double integral and the accompanying paraboloid.

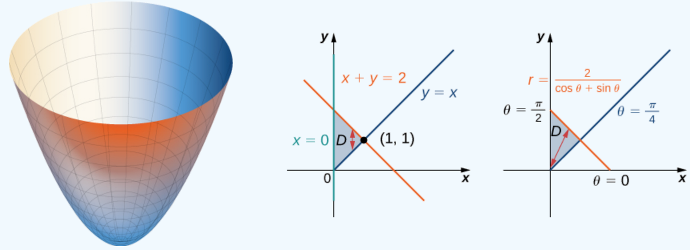


Figure 7.3.9: Finding the volume of a solid under a paraboloid and above a given triangle.

The region D is $\{(x,y) \mid 0 \le x \le 1,\ x \le y \le 2 - x\}$. Converting the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane to functions of r and θ we have $\theta = \pi/4$, $\theta = \pi/2$, and $r = 2/(\cos\theta + \sin$

$$\begin{aligned} V &= \iint_D f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/\cos\theta+\sin\theta)} r^2 r\, dr d\theta = \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4} \right]_0^{2/(\cos\theta+\sin\theta)} d\theta \\ &= \frac{1}{4} \int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\theta + \sin\theta} \right) d\theta = \frac{16}{4} \int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta + \sin\theta} \right) d\theta = 4 \int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta + \sin\theta} \right)^4 d\theta. \end{aligned}$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

$$V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx. \tag{7.3.12}$$

Evaluating gives

$$\begin{aligned} V &= \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right] \Big|_x^{2-x} dx \\ &= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3}\, dx \\ &= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3} \right] \Big|_0^1 = \frac{4}{3} \text{ units}^3. \end{aligned}$$

To answer the question of how the formulas for the volumes of different standard solids such as a sphere, a cone, or a cylinder are found, we want to demonstrate an example and find the volume of an a

Example $7.3.5B$: Finding a Volume Using a Double Integral

Use polar coordinates to find the volume inside the cone $z = 2 - \sqrt{x^2 + y^2}$ and above the xy-plane.

Solution

The region D for the integration is the base of the cone, which appears to be a circle on the xy-plane (Figure $7.3.10$).

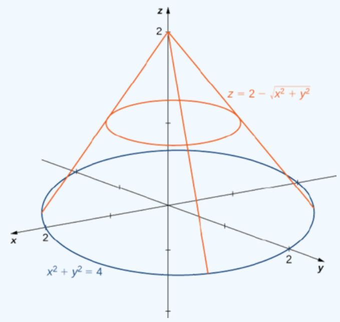


Figure 7.3.10: Finding the volume of a solid inside the cone and above the xy-plane.

We find the equation of the circle by setting $z = 0$:

$$\begin{aligned} 0 &= 2 - \sqrt{x^2 + y^2} \\ 2 &= \sqrt{x^2 + y^2} \\ x^2 + y^2 &= 4. \end{aligned}$$

This means the radius of the circle is 2 so for the integration we have $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ in the equation $z = 2 - \sqrt{x^2 + y^2}$ we have $z = 2 - r$. Th

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (2 - r)\, r\, dr\, d\theta = 2\pi \frac{4}{3} = \frac{8\pi}{3} \text{ cubic units.}$$

Analysis

Note that if we were to find the volume of an arbitrary cone with radius α units and height h units, then the equation of the cone would be $z = h - \frac{h}{a}\sqrt{x^2 + y^2}$.

We can still use Figure $7.3.10$ and set up the integral as

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \left(h - \frac{h}{a} r\right) r\, dr\, d\theta.$$

Evaluating the integral, we get $\frac{1}{3}\pi a^2 h$.

Exercise $7.3.5$

Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

Hint

Sketching the graphs can help.

Answer

$$V = \int_0^{2\pi} \int_0^{2\sqrt{2}} (16 - 2r^2)\, r\, dr\, d\theta = 64\pi \text{ cubic units.}$$

As with rectangular coordinates, we can also use polar coordinates to find areas of certain regions using a double integral. As before, we need to understand the region whose area we want to compute. S

$$\text{Area of } A = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} 1\, r\, dr\, d\theta. \tag{7.3.13}$$

Example $7.3.6A$: Finding an Area Using a Double Integral in Polar Coordinates

Evaluate the area bounded by the curve $r = \cos 4\theta$.

Solution

Sketching the graph of the function $r = \cos 4\theta$ reveals that it is a polar rose with eight petals (see the following figure).

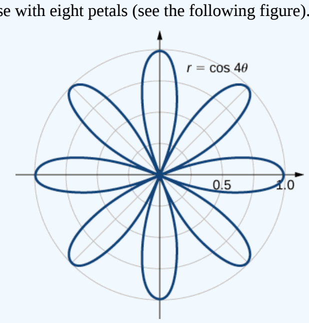


Figure 7.3.11: Finding the area of a polar rose with eight petals.

Using symmetry, we can see that we need to find the area of one petal and then multiply it by 8. Notice that the values of θ for which the graph passes through the origin are the zeros of the function c

$$A = 8 \int_{\theta=-\pi/8}^{\theta=\pi/8} \int_{r=0}^{r=\cos 4\theta} 1\, r\, dr\, d\theta \tag{7.3.14}$$

$$= 8 \int_{\theta=-\pi/8}^{\theta=\pi/8} \left[\frac{1}{2} r^2 \Big|_0^{\cos 4\theta} \right] d\theta \int_{-\pi/8}^{\pi/8} \frac{1}{2} \cos^2 4\theta\, d\theta = 8 \left[\frac{1}{4}\theta + \frac{1}{16} \sin 4\theta \cos 4\theta \Big|_{-\pi/8}^{\pi/8} \right] = 8 \left[\frac{\pi}{16} \right] = \frac{\pi}{2} \text{ units}^2. \tag{7.3.15}$$

Example 7.3.6B: Finding Area Between Two Polar Curves

Find the area enclosed by the circle $r = 3 \cos \theta$ and the cardioid $r = 1 + \cos \theta$.

Solution

First and foremost, sketch the graphs of the region (Figure 7.3.12).

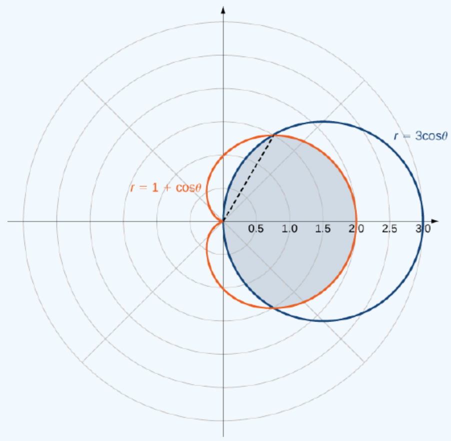


Figure 7.3.12: Finding the area enclosed by both a circle and a cardioid.

We can from see the symmetry of the graph that we need to find the points of intersection. Setting the two equations equal to each other gives

$$3 \cos \theta = 1 + \cos \theta.$$

One of the points of intersection is $\theta = \pi/3$. The area above the polar axis consists of two parts, with one part defined by the cardioid from $\theta = 0$ to $\theta = \pi/3$ and the other part defined by the circle fr

$$A = 2 \left[\int_{\theta=0}^{\theta=\pi/3} \int_{r=0}^{r=1+\cos \theta} 1\, r\, dr\, d\theta + \int_{\theta=\pi/3}^{\theta=\pi/2} \int_{r=0}^{r=3 \cos \theta} 1\, r\, dr\, d\theta \right].$$

Evaluating each piece separately, we find that the area is

$$A = 2 \left(\frac{1}{4}\pi + \frac{9}{16}\sqrt{3} + \frac{3}{8}\pi - \frac{9}{16}\sqrt{3} \right) = 2 \left(\frac{5}{8}\pi \right) = \frac{5}{4}\pi \text{ square units.}$$

Exercise 7.3.6

Find the area enclosed inside the cardioid $r = 3 - 3 \sin \theta$ and outside the cardioid $r = 1 + \sin \theta$.

Hint

Sketch the graph, and solve for the points of intersection.

Answer

$$A = 2 \int_{-\pi/2}^{\pi/6} \int_{1+\sin \theta}^{3-3\sin \theta} r\, dr\, d\theta = (8\pi + 9\sqrt{3}) \text{ units}^2$$

Example 7.3.7: Evaluating an Improper Double Integral in Polar Coordinates

Evaluate the integral

$$\iint_{R^2} e^{-10(x^2+y^2)}\, dx\, dy.$$

Solution

This is an improper integral because we are integrating over an unbounded region R^2. In polar coordinates, the entire plane R^2 can be seen as $0 \le \theta \le 2\pi$, $0 \le r \le \infty$.

Using the changes of variables from rectangular coordinates to polar coordinates, we have

$$\begin{aligned} \iint_{R^2} e^{-10(x^2+y^2)}\, dx\, dy &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\infty} e^{-10r^2}\, r\, dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} \left(\lim_{a\to\infty} \int_{r=0}^{r=a} e^{-10r^2}\, r\, dr \right) d\theta \\ &= \left(\int_{\theta=0}^{\theta=2\pi} \right) d\theta \left(\lim_{a\to\infty} \int_{r=0}^{r=a} e^{-10r^2}\, r\, dr \right) \\ &= 2\pi \left(\lim_{a\to\infty} \int_{r=0}^{r=a} e^{-10r^2}\, r\, dr \right) \\ &= 2\pi \lim_{a\to\infty} \left(-\frac{1}{20} \right) \left(e^{-10r^2} \Big|_0^a \right) \\ &= 2\pi \left(-\frac{1}{20} \right) \lim_{a\to\infty} \left(e^{-10a^2} - 1 \right) \\ &= \frac{\pi}{10}. \end{aligned}$$

Exercise 7.3.7

Evaluate the integral

$$\iint_{R^2} e^{-4(x^2+y^2)}dx\,dy.$$

Hint

Convert to the polar coordinate system.

Answer

$\frac{\pi}{4}$

7.3.4 Key Concepts

- To apply a double integral to a situation with circular symmetry, it is often convenient to use a double integral in polar coordinates. We can apply these double integrals over a polar rectangular regior
- The area dA in polar coordinates becomes $r\,dr\,d\theta$.
- Use $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$ to convert an integral in rectangular coordinates to an integral in polar coordinates.
- Use $r^2 = x^2 + y^2$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$ to convert an integral in polar coordinates to an integral in rectangular coordinates, if needed.
- To find the volume in polar coordinates bounded above by a surface $z = f(r,\theta)$ over a region on the xy-plane, use a double integral in polar coordinates.

7.3.5 Key Equations

- **Double integral over a polar rectangular region R**

$$\iint_R f(r,\theta)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r^*_{ij},\theta^*_{ij})\Delta A = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r^*_{ij},\theta^*_{ij})r^*_{ij}\Delta r\Delta\theta$$

- **Double integral over a general polar region**

$$\iint_D f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

7.3.5.0.1 Glossary

polar rectangle

the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r,\theta)\mid a \le r \le b,\ \alpha \le \theta \le \beta\}$

7.3.6 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://c

7.3E:

7.3E.1 Exercise $7.3E.1-6$

In the following exercises, express the region D in polar coordinates.

1. D is the region of the disk of radius 2 centered at the origin that lies in the first quadrant.

2. D is the region between the circles of radius 4 and radius 5 centered at the origin that lies in the second quadrant.

Answer

$$D = (r, \theta) | 4 \leq r \leq 5,\ \frac{\pi}{2} \leq \theta \leq \pi$$

3. D is the region bounded by the y-axis and $x = \sqrt{1 - y^2}$.

4. D is the region bounded by the x-axis and $y = \sqrt{2 - x^2}$.

Answer

$$D = (r, \theta) | 0 \leq r \leq \sqrt{2},\ 0 \leq \theta \leq \pi$$

5. $D = (x, y) | x^2 + y^2 \leq 4x$

6. $D = (x, y) | x^2 + y^2 \leq 4y$

Answer

$$D = (r, \theta) | 0 \leq r \leq 4\ sin\ \theta,\ 0 \leq \theta \leq \pi$$

7.3E.2 Exercise $7.3E.7-12$

In the following exercises, the graph of the polar rectangular region D is given. Express D in polar coordinates.

7.

Half an annulus D is drawn in the first and second quadrants with inner radius 3 and outer radius 5.

8.

A sector of an annulus D is drawn between theta = pi/4 and theta = pi/2 with inner radius 3 and outer radius 5.

Answer

$$D = (r, \theta) | 3 \leq r \leq 5,\ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

9.

Half of an annulus D is drawn between theta = pi/4 and theta = 5 pi/4 with inner radius 3 and outer radius 5.

10.

A sector of an annulus D is drawn between theta = 3 pi/4 and theta = 5 pi/4 with inner radius 3 and outer radius 5.

Answer

$$D = (r, \theta) | 3v \leq r \leq 5,\ \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$$

11. In the following graph, the region D is situated below $y = x$ and is bounded by $x = 1,\ x = 5$, and $y = 0$.

A region D is given that is bounded by y = 0, x = 1, x = 5, and y = x, that is, a right triangle with a corner cut off.

12. In the following graph, the region D is bounded by $y = x$ and y = x^2\).

A region D is drawn between y = x and y = x squared, which looks like a deformed lens, with the bulbous part below the straight part.

Answer

$D = (r, \theta) | 0 \leq r \leq tan\, \theta\, sec\, \theta,\ 0 \leq \theta \leq \frac{\pi}{4}$

7.3E.3 Exercise $\mathbf{7.3E.13-21}$

In the following exercises, evaluate the double integral

$$\iint_R f(x,y) dA \tag{7.3E.1}$$

over the polar rectangular region D.

13. $f(x,y) = x^2 + y^2$, $D = (r, \theta) | 3 \leq r \leq 5,\ 0 \leq \theta \leq 2\pi$

14. $f(x,y) = x + y$, $D = (r, \theta) | 3 \leq r \leq 5,\ 0 \leq \theta \leq 2\pi$

Answer

0

14. $f(x,y) = x^2 + xy,\ D = (r, \theta) | 1 \leq r \leq 2,\ \pi \leq \theta \leq 2\pi$

15. $f(x,y) = x^4 + y^4,\ D = (r, \theta) | 1 \leq r \leq 2,\ \frac{3\pi}{2} \leq \theta \leq 2\pi$

Answer

$\frac{63\pi}{16}$

16. $f(x,y) = \sqrt[3]{x^2 + y^2}$, where $D = (r, \theta) | 0 \leq r \leq 1,\ \frac{\pi}{2} \leq \theta \leq \pi$.

17. $f(x,y) = x^4 + 2x^2y^2 + y^4$, where $D = (r, \theta) | 3 \leq r \leq 4,\ \frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$.

Answer

$\frac{3367\pi}{18}$

18. $f(x,y) = sin(arctan \frac{y}{x})$, where $D = (r, \theta) | 1 \leq r \leq 2,\ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$

19. $f(x,y) = arctan\left(\frac{y}{x}\right)$, where $D = (r, \theta) | 2 \leq r \leq 3,\ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3}$

Answer

$\frac{35\pi^2}{576}$

20.

$$\iint_D e^{x^2+y^2} \left[1 + 2\, arctan\left(\frac{y}{x}\right)\right] dA,\ D = (r, \theta) | 1 \leq r \leq 2,\ \frac{\pi}{6} \leq \theta \frac{\pi}{3} \tag{7.3E.2}$$

21.

$$\iint_D \left(e^{x^2+y^2} + x^4 + 2x^2y^2 + y^4\right) arctan\left(\frac{y}{x}\right) dA,\ D = (r, \theta) | 1 \leq r \leq 2,\ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3} \tag{7.3E.3}$$

Answer

$\frac{7}{576}\pi^2(21 - e + e^4)$

7.3E.4 Exercise $7.3E.22-26$

In the following exercises, the integrals have been converted to polar coordinates. Verify that the identities are true and choose the easiest way to evaluate the integrals, in rectangular or polar coordinates.

22.

$$\int_1^2\int_0^x (x^2+y^2)dy\,dx = \int_0^{\frac{\pi}{4}}\int_{sec\,\theta}^{2\,sec\,\theta} r^3 dr\,d\theta \tag{7.3E.4}$$

23.

$$\int_2^3\int_0^x \frac{x}{\sqrt{x^2+y^2}}dy\,dx = \int_0^{\pi/4}\int_0^{tan\,\theta\,sec\,\theta} r\,cos\,\theta\,dr\,d\theta \tag{7.3E.5}$$

Answer

$\frac{5}{4}ln(3+2swrt2)$

25.

$$\int_0^1\int_{x^2}^x \frac{1}{\sqrt{x^2+y^2}}dy\,dx = \int_0^{\pi/4}\int_0^{tan\,\theta\,sec\,\theta} dr\,d\theta \tag{7.3E.6}$$

26.

$$\int_0^1\int_{x^2}^x \frac{y}{\sqrt{x^2+y^2}}dy\,dx = \int_0^{\pi/4}\int_0^{tan\,\theta\,sec\,\theta} r\,sin\,\theta\,dr\,d\theta \tag{7.3E.7}$$

Answer

$\frac{1}{6}(2-\sqrt{2})$

7.3E.5 Exercise $7.3E.27-31$

In the following exercises, convert the integrals to polar coordinates and evaluate them.

28.

$$\int_0^3\int_0^{\sqrt{9-y^2}} dx\,dy \tag{7.3E.8}$$

29.

$$\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\,dy \tag{7.3E.9}$$

Answer

$$\int_0^{\pi}\int_0^2 r^5 dr\,d\theta = \frac{32\pi}{3} \tag{7.3E.10}$$

30.

$$\int_0^1\int_0^{\sqrt{1-x^2}} (x+y)\,dy\,dx \tag{7.3E.11}$$

31.

$$\int_0^4\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} sin(x^2+y^2)\,dy\,dx \tag{7.3E.12}$$

Answer

$$\int_{-\pi/2}^{\pi/2}\int_0^4 r\ sin(r^2)\ dr\ d\theta = \pi\ sin^2 8 \tag{7.3E.13}$$

7.3E.6 Exercise $\mathbf{7.3E.32-36}$

32. Evaluate the integral

$$\iint_D rdA \tag{7.3E.14}$$

where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + cos\ \theta$.

33. Find the area of the region D bounded by the polar axis and the upper half of the cardioid $r = 1 + cos\ \theta$.

Answer

$\frac{3\pi}{4}$

34. Evaluate the integral

$$\iint_D rdA, \tag{7.3E.15}$$

where D is the region bounded by the part of the four-leaved rose $r = sin\ 2\theta$ situated in the first quadrant (see the following figure).

A region D is drawn in the first quadrant petal of the four petal rose given by r = sin (2 theta).

35. Find the total area of the region enclosed by the four-leaved rose $r = sin\ 2\theta$ (see the figure in the previous exercise).

Answer

\(\frac{\pi}{2}\

35. Find the area of the region D which is the region bounded by $y = \sqrt{4 - x^2}$, $x = \sqrt{3}$, $x = 2$, and $y = 0$.

36. Find the area of the region D, which is the region inside the disk $x^2 + y^2 \leq 4$ and to the right of the line $x = 1$.

Answer

$\frac{1}{3}(4\pi - 3\sqrt{3})$

7.3E.7 Exercise $\mathbf{7.3E.37-38}$

37. Determine the average value of the function $f(x,y) = x^2 + y^2$ over the region D bounded by the polar curve $r = cos\ 2\theta$, where $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ (see the following graph).

The first/fourth-quadrant petal of the four-petal rose given by r = cos (2 theta) is shown.

38. Determine the average value of the function $f(x,y) = \sqrt{x^2 + y^2}$ over the region D bounded by the polar curve $r = 3\ sin\ 2\theta$, where $0 \leq \theta \leq \frac{\pi}{2}$ (see the following graph).

The first-quadrant petal of the four-petal rose given by r = 3sin (2 theta) is shown.

Answer

$\frac{16}{3\pi}$

7.3E.8 Exercise $\mathbf{7.3E.39-46}$

39. Find the volume of the solid bounded by the paraboloid $z = 2 - 9x^2 - 9y^2$ and the plane $z = 1$.

Answer

$\frac{\pi}{18}$

40. Find the volume of the solid S_1 bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$.

41. Find the volume of the solid S_2 outside the double cone $z^2 = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 1$, and above the plane $z = 0$.

42. Find the volume of the solid inside the cone $z^2 = x^2 + y^2$ and below the plane $z = 1$ by subtracting the volumes of the solids S_1 and S_2.

43. Find the volume of the solid S_1 inside the unit sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.

44. Find the volume of the solid S_2 inside the double cone $(z-1)^2 = x^2 + y^2$ and above the plane $z = 0$.

45. Find the volume of the solid outside the double cone $(z-1)^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 1$.

Answer

$\frac{2\pi}{3}$; 2. $\frac{\pi}{2}$; 3. $\frac{\pi}{6}$

46. Find the volume of the solid situated in the first octant and bounded by the paraboloid $z = 1 - 4x^2 - 4y^2$ and the planes $x = 0,\ y = 0$, and $z = 0$.

7.3E.9 Exercise $\mathbf{7.3E.46-47}$

For the following two exercises, consider a spherical ring, which is a sphere with a cylindrical hole cut so that the axis of the cylinder passes through the center of the sphere (see the following figure).

A spherical ring is shown, that is, a sphere with a cylindrical hole going all the way through it.

46. If the sphere has radius 4 and the cylinder has radius 2 find the volume of the spherical ring.

47. A cylindrical hole of diameter 6 cm is bored through a sphere of radius 5 cm such that the axis of the cylinder passes through the center of the sphere. Find the volume of the resulting spherical ring.

Answer

$\frac{256\pi}{3}\ cm^3$

7.3E.10 Exercise $\mathbf{7.3E.48-51}$

48. Find the volume of the solid that lies under the double cone $z^2 = 4x^2 + 4y^2$, inside the cylinder $x^2 + y^2 = x$, and above the plane $z = 0$.

49. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 0$.

Answer

\(\frac{3\pi}{32}\

50. Find the volume of the solid that lies under the plane $x + y + z = 10$ and above the disk $x^2 + y^2 = 4x$.

51. Find the volume of the solid that lies under the plane $2x + y + 2z = 8$ and above the unit disk $x^2 + y^2 = 1$.

Answer

4π

7.3E.11 Exercise $\mathbf{7.3E.52-59}$

52. A radial function f is a function whose value at each point depends only on the distance between that point and the origin of the system of coordinates; that is, $f(x,y)=g(r)$, where $r=\sqrt{x^2+y^2}$. Show that if f is a continuous radial function, then

$$\iint_D f(x,y)dA=(\theta_2-\theta_1)[G(R_2)-G(R_1)],\ where\ G'(r)=rg(r) \tag{7.3E.16}$$

and $(x,y)\in D=(r,\theta)|R_1\leq r\leq R_2,\ 0\leq\theta\leq 2\pi$, with $0\leq R_1<R_2$ and $0\leq\theta_1<\theta_2\leq 2\pi$.

53. Use the information from the preceding exercise to calculate the integral

$$\iint_D (x^2+y^2)^3 dA, \tag{7.3E.17}$$

where D is the unit disk.

Answer

$\frac{\pi}{4}$

54. Let $f(x,y)=\frac{F'(r)}{r}$ be a continuous radial function defined on the annular region $D=(r,\theta)|R_1\leq r\leq R_2,\ 0\leq\theta 2\pi$, where $r=\sqrt{x^2+y^2}$, $0<R_1<R_2$, and F is a differentiable function. Show that

$$\iint_D f(x,y)dA=2\pi[F(R_2)-F(R_1)]. \tag{7.3E.18}$$

55. Apply the preceding exercise to calculate the integral

$$\iint_D \frac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}dx\ dy \tag{7.3E.19}$$

where D is the annular region between the circles of radii 1 and 2 situated in the third quadrant.

Answer

$\frac{1}{2}\pi e(e-1)$

56. Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x,y)=h(\theta)$, where $(x,y)\in D=(r,\theta)|R_1\leq r\leq R_2,\ \theta_1\leq\theta\leq\theta_2$, with $0\leq R_1<R_2$ and $0\leq\theta_1<\theta_2\leq 2\pi$. Show that

$$\iint_D f(x,y)dA=\frac{1}{2}(R_2^2-R_1^2)[H(\theta_2)-H(\theta_1)) \tag{7.3E.20}$$

, where H is an antiderivative of h.

57. Apply the preceding exercise to calculate the integral

$$\iint_D \frac{y^2}{x^2}dA, \tag{7.3E.21}$$

where $D=(r,\theta)|1\leq r\leq 2,\ \frac{\pi}{6}\leq\theta\leq\frac{\pi}{3}$.

Answer

\(\sqrt{3} - \frac{\pi}{4}\

58. Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is $f(x,y)=g(r)h(\theta)$, where (x,y) \in {(r, \theta)|R_1 \leq r \leq R_2, \space \theta_1 \leq \theta \leq \theta_2}\) with $0\leq R_1<R_2$ and $0\leq\theta_1<\theta_2\leq 2\pi$. Show that

$$\iint_D f(x,y)dA=[G(R_2)-G(R_1)]\ [H(\theta_2)-H(\theta_1)], \tag{7.3E.22}$$

where G and H are antiderivatives of g and h, respectively.

59. Evaluate

$$\iint_D arctan\left(\frac{y}{x}\right)\sqrt{x^2+y^2}\,dA, \tag{7.3E.23}$$

where $D = (r,\theta)|2 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}$.

Answer

$\frac{133\pi^3}{864}$

7.3E.12 Exercise $7.3E.60$

60. A spherical cap is the region of a sphere that lies above or below a given plane.

a. Show that the volume of the spherical cap in the figure below is $\frac{1}{6}\pi h(3a^2+h^2)$.

A sphere of radius R has a circle inside of it h units from the top of the sphere. This circle has radius a, which is less than R.

b. A spherical segment is the solid defined by intersecting a sphere with two parallel planes. If the distance between the planes is h show that the volume of the spherical segment in the figure below is $\frac{1}{6}\pi h(3a^2+3b^2+h^2)$.

A sphere has two parallel circles inside of it h units apart. The upper circle has radius b, and the lower circle has radius a. Note that a > b.

7.3E.13 Exercise $7.3E.61$

61. In statistics, the joint density for two independent, normally distributed events with a mean $\mu = 0$ and a standard distribution σ is defined by $p(x,y) = \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$. Consider (X,Y), the Cartesian coordinates of a ball in the resting position after it was released from a position on the *z*-axis toward the xy-plane. Assume that the coordinates of the ball are independently normally distributed with a mean $\mu = 0$ and a standard deviation of σ (in feet). The probability that the ball will stop no more than a feet from the origin is given by

$$P[X^2+Y^2 \le a^2] = \iint_D p(x,y)dy\,dx, \tag{7.3E.24}$$

where D is the disk of radius *a* centered at the origin. Show that

$$P[X^2+Y^2 \le a^2] = 1 - e^{-a^2/2\sigma^2}. \tag{7.3E.25}$$

The double improper integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}dy\,dx \tag{7.3E.26}$$

may be defined as the limit value of the double integrals

$$\iint_D e^{-x^2+y^2/2}dA \tag{7.3E.27}$$

over disks D_a of radii *a* centered at the origin, as *a* increases without bound; that is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}dy\,dx = \lim_{a\to\infty}\iint_{D_a} e^{-x^2+y^2/2}dA. \tag{7.3E.28}$$

Use polar coordinates to show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}dy\,dx = 2\pi. \tag{7.3E.29}$$

Show that

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi} \tag{7.3E.30}$$

by using the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2} dy\, dx = \left(\int_{-\infty}^{\infty} e^{-x^2/2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2} dy\right). \tag{7.3E.31}$$

7.4: Triple Integrals

This page is a draft and is under active development.

Previously, we discussed the double integral of a function $f(x, y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x, y, z)$ of three variables over a rectangular solid box in space, $\mathbb{R}^3$. Later in this section we extend the definition to more general regions in $\mathbb{R}^3$.

7.4.0.0.1 Integrable Functions of Three Variables

We can define a rectangular box B in $\mathbb{R}^3$ as

$$B = \{(x, y, z) \mid a \le x \le b,\ c \le y \le d,\ e \le z \le f\}. \tag{7.4.1}$$

We follow a similar procedure to what we did in previously. We divide the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length Δx with

$$\Delta x = \frac{x_i - x_{i-1}}{l}, \tag{7.4.2}$$

divide the interval $[c, d]$ into m subintervals $[y_{i-1}, y_i]$ of equal length Δy with

$$\Delta y = \frac{y_j - y_{j-1}}{m}, \tag{7.4.3}$$

and divide the interval $[e, f]$ into n subintervals $[z_{i-1}, z_i]$ of equal length Δz with

$$\Delta z = \frac{z_k - z_{k-1}}{n} \tag{7.4.4}$$

Then the rectangular box B is subdivided into lmn subboxes:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{i-1}, z_i], \tag{7.4.5}$$

as shown in Figure $7.4.1$.

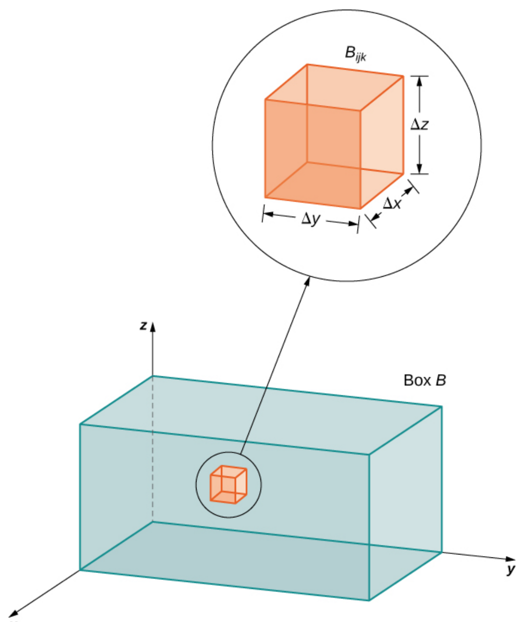


Figure $7.4.1$: A rectangular box in $\mathbb{R}^3$ divided into subboxes by planes parallel to the coordinate planes.

For each i, j, and k, consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is $\Delta V = \Delta x \Delta y \Delta z$. Form the triple Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta x \Delta y \Delta z. \tag{7.4.6}$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Definition: The triple integral

The triple integral of a function $f(x,y,z)$ over a rectangular box B is defined as

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta x\Delta y\Delta z = \iiint_B f(x,y,z)\,dV \tag{7.4.7}$$

if this limit exists.

When the triple integral exists on B the function $f(x,y,z)$ is said to be integrable on B. Also, the triple integral exists if $f(x,y,z)$ is continuous on B. Therefore, we will use continuous functions for our examples. However, continuity is sufficient but not necessary; in other words, f is bounded on B and continuous except possibly on the boundary of B. The sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ can be any point in the rectangular sub-box B_{ijk} and all the properties of a double integral apply to a triple integral. Just as the double integral has many practical applications, the triple integral also has many applications, which we discuss in later sections.

Now that we have developed the concept of the triple integral, we need to know how to compute it. Just as in the case of the double integral, we can have an iterated triple integral, and consequently, a version of **Fubini's theorem** for triple integrals exists.

Fubini's Theorem for Triple Integrals

If $f(x,y,z)$ is continuous on a rectangular box $B=[a,b]\times[c,d]\times[e,f]$, then

$$\iiint_B f(x,y,z)\,dV = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz. \tag{7.4.8}$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

For a,b,c,d,e and f real numbers, the iterated triple integral can be expressed in six different orderings:

$$\int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_e^f \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx\right) dy\right) dz \tag{7.4.9}$$

$$= \int_c^d \left(\int_e^f \left(\int_a^b f(x,y,z)\,dx\right) dz\right) dy \tag{7.4.10}$$

$$= \int_a^b \left(\int_e^f \left(\int_c^d f(x,y,z)\,dy\right) dz\right) dx \tag{7.4.11}$$

$$= \int_e^f \left(\int_a^b \left(\int_c^d f(x,y,z)\,dy\right) dx\right) dz \tag{7.4.12}$$

$$= \int_c^d \left(\int_a^b \left(\int_c^d f(x,y,z)\,dz\right) dx\right) dy \tag{7.4.13}$$

$$= \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)\,dz\right) dy\right) dx \tag{7.4.14}$$

For a rectangular box, the order of integration does not make any significant difference in the level of difficulty in computation. We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example 7.4.1: Evaluating a Triple Integral

Evaluate the triple integral

$$\int_{z=0}^{z=1}\int_{y=2}^{y=4}\int_{x=-1}^{x=5}(x+yz^2)\,dx\,dy\,dz.$$

Solution

The order of integration is specified in the problem, so integrate with respect to x first, then y, and then z.

$$\begin{aligned}
&\int_{z=0}^{z=1}\int_{y=2}^{y=4}\int_{x=-1}^{x=5}(x+yz^2)\,dx\,dy\,dz \\
&=\int_{z=0}^{z=1}\int_{y=2}^{y=4}\left[\frac{x^2}{2}+xyz^2\Big|_{x=-1}^{x=5}\right]dy\,dz && \text{Integrate with respect to } x. \\
&=\int_{z=0}^{z=1}\int_{y=2}^{y=4}\left[12+6yz^2\right]\,dy\,dz && \text{Evaluate.} \\
&=\int_{z=0}^{z=1}\left[12y+6\frac{y^2}{2}z^2\Big|_{y=2}^{y=4}\right]dz && \text{Integrate with respect to } y. \\
&=\int_{z=0}^{z=1}[24+36z^2]\,dz && \text{Evaluate.} \\
&=\left[24z+36\frac{z^3}{3}\right]_{z=0}^{z=1} && \text{Integrate with respect to } z. \\
&=36. && \text{Evaluate.}
\end{aligned}$$

Example 7.4.2: Evaluating a Triple Integral

Evaluate the triple integral

$$\iiint_B x^2yz\,dV \tag{7.4.15}$$

where $B=\{(x,y,z)\mid -2\le x\le 1,\ 0\le y\le 3,\ 1\le z\le 5\}$ as shown in Figure 7.4.2.

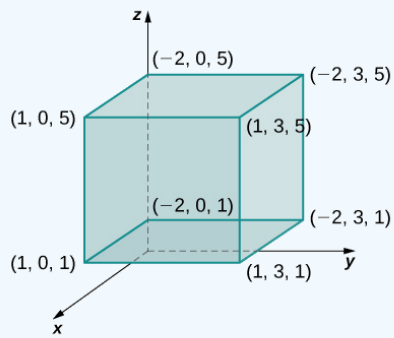


Figure 7.4.2: Evaluating a triple integral over a given rectangular box.

Solution

The order is not specified, but we can use the iterated integral in any order without changing the level of difficulty. Choose, say, to integrate y first, then x, and then z.

$$\begin{aligned}
\iiint_B x^2yz\,dV &= \int_1^5\int_{-2}^1\int_0^3[x^2yz]\,dy\,dx\,dz \\
&= \int_1^5\int_{-2}^1\left[x^2\frac{y^3}{3}z\Big|_0^3\right]dx\,dz \\
&= \int_1^5\int_{-2}^1\frac{y}{2}x^2z\,dx\,dz \\
&= \int_1^5\left[\frac{9}{2}\frac{x^3}{3}z\Big|_{-2}^1\right]dz=\int_1^5\frac{27}{2}z\,dz \\
&= \frac{27}{2}\frac{z^2}{2}\Big|_1^5=162.
\end{aligned}$$

Now try to integrate in a different order just to see that we get the same answer. Choose to integrate with respect to x first, then z, then y

$$\iiint_B x^2yz\,dV = \int_0^3\int_1^5\int_{-2}^1 [x^2yz]\,dx\,dz\,dy$$
$$= \int_0^3\int_1^5 \left[\frac{x^3}{3}yz\Big|_{-2}^{1}\right] dz\,dy$$
$$= \int_0^3\int_1^5 3yz\,dz\,dy$$
$$= \int_0^3 \left[3y\frac{z^2}{2}\Big|_1^5\right] dy$$
$$= \int_0^3 36y\,dy$$
$$= 36\frac{y^2}{2}\Big|_0^3 = 18(9-0) = 162.$$

Exercise 7.4.1

Evaluate the triple integral

$$\iint_B z\sin x\cos y\,dV$$

where $B = \left\{(x,y,z) \mid 0 \le x \le \pi,\ \frac{3\pi}{2} \le y \le 2\pi,\ 1 \le z \le 3\right\}$.

Hint

Follow the steps in the previous example.

Answer

$$\iint_B z\sin x\cos y\,dV = 8$$

7.4.0.0.1 Triple Integrals over a General Bounded Region

We now expand the definition of the triple integral to compute a triple integral over a more **general bounded region E** in $\mathbb{R}^3$. The general bounded regions we will consider are of three types. First, let D be the bounded region that is a projection of E onto the xy-plane. Suppose the region E in $\mathbb{R}^3$ has the form

$$E = \left\{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\right\}. \tag{7.4.16}$$

For two functions $z = u_1(x,y)$ and $u_2(x,y)$, such that $u_1(x,y) \le u_2(x,y)$ for all (x,y) in D as shown in the following figure.

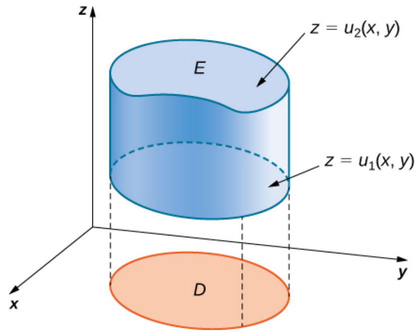


Figure 7.4.3: We can describe region E as the space between $u_1(x,y)$ and $u_2(x,y)$ above the projection D of E onto the xy-plane.

Triple Integral over a General Region

The triple integral of a continuous function $f(x,y,z)$ over a general three-dimensional region

$$E = \left\{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\right\} \tag{7.4.17}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA. \tag{7.4.18}$$

Similarly, we can consider a general bounded region D in the xy-plane and two functions $y = u_1(x,z)$ and $y = u_2(x,z)$ such that $u_1(x,z) \le u_2(x,z)$ for all $9x, z)$ in D. Then we can describe the solid region E in $\mathbb{R}^3$ as

$$E = \{(x,y,z) \mid (x,z) \in D,\ u_1(x,z) \le z \le u_2(x,z)\} \tag{7.4.19}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA. \tag{7.4.20}$$

Finally, if D is a general bounded region in the xy-plane and we have two functions $x = u_1(y,z)$ and $x = u_2(y,z)$ such that $u_1(y,z) \le u_2(y,z)$ for all (y,z) in D, then the solid region E in $\mathbb{R}^3$ can be described as

$$E = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le z \le u_2(y,z)\} \tag{7.4.21}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA. \tag{7.4.22}$$

Note that the region D in any of the planes may be of Type I or Type II as described in previously. If D in the xy-plane is of Type I (Figure 7.4.4), then

$$E = \{(x,y,z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y)\}. \tag{7.4.23}$$

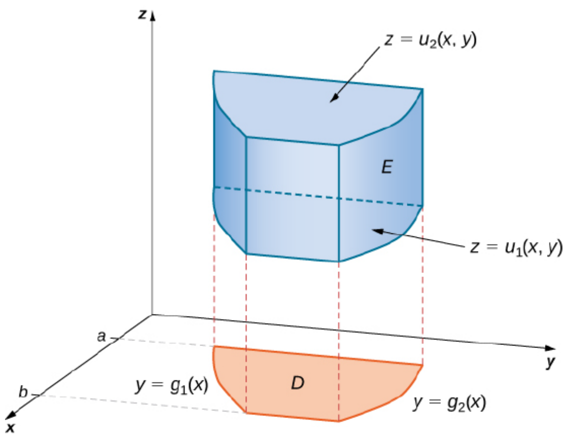


Figure 7.4.4: A box E where the projection D in the xy-plane is of Type I.

Then the triple integral becomes

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx. \tag{7.4.24}$$

If D in the xy-plane is of Type II (Figure 7.4.5), then

$$E = \{(x,y,z) \mid c \le x \le d, h_1(x) \le y \le h_2(x),\ u_1(x,y) \le z \le u_2(x,y)\}. \tag{7.4.25}$$

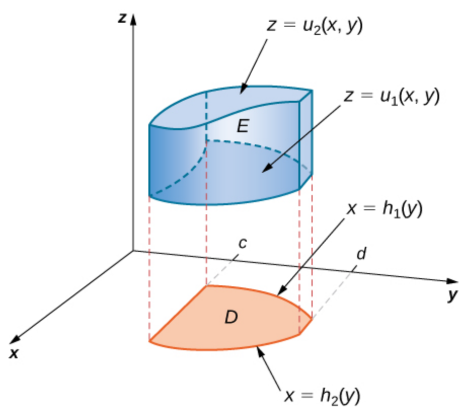


Figure 7.4.5: A box E where the projection D in the xy-plane is of Type II.

Then the triple integral becomes

$$\iiint_E f(x,y,z)\,dV = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\,dz\,dx\,dy. \tag{7.4.26}$$

Example 7.4.3A: Evaluating a Triple Integral over a General Bounded Region

Evaluate the triple integral of the function $f(x,y,z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution

Figure 7.4.6 shows the solid tetrahedron E and its projection D on the xy-plane.

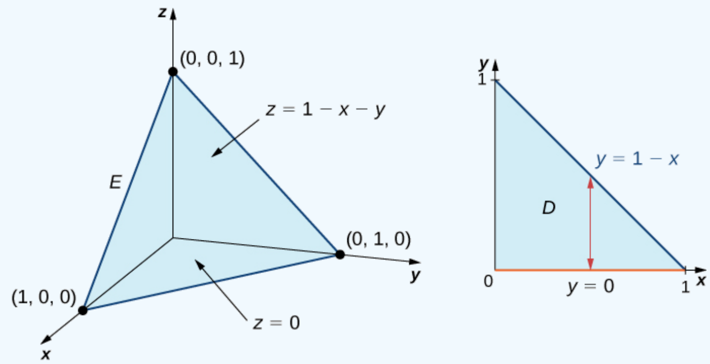


Figure 7.4.6: The solid E has a projection D on the xy-plane of Type I.

We can describe the solid region tetrahedron as

$$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Hence, the triple integral is

$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)\,dz\,dy\,dx.$$

To simplify the calculation, first evaluate the integral $\displaystyle\int_{z=0}^{z=1-x-y} (5x-3y)\,dz$. We have

$$\int_{z=0}^{z=1-x-y} (5x-3y)\,dz = (5x-3y)z\Big|_{z=0}^{z=1-x-y} = (5x-3y)(1-x-y).$$

Now evaluate the integral

$$\int_{y=0}^{y=1-x} (5x-3y)(1-x-y)\,dy,$$

obtaining

$$\int_{y=0}^{y=1-x} (5x-3y)(1-x-y)\,dy = \frac{1}{2}(x-1)^2(6x-1).$$

Finally evaluate

$$\int_{x=0}^{x=1} \frac{1}{2}(x-1)^2(6x-1)\,dx = \frac{1}{12}.$$

Putting it all together, we have

$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x-3y)\,dz\,dy\,dx = \frac{1}{12}.$$

Just as we used the double integral

$$\iint_D 1\,dA \tag{7.4.27}$$

to find the area of a general bounded region D we can use

$$\iiint_E 1\,dV \tag{7.4.28}$$

to find the volume of a general solid bounded region E. The next example illustrates the method.

Example 7.4.3B: Finding a Volume by Evaluating a Triple Integral

Find the volume of a right pyramid that has the square base in the xy-plane $[-1,1]\times[-1,1]$ and vertex at the point $(0,0,1)$ as shown in the following figure.

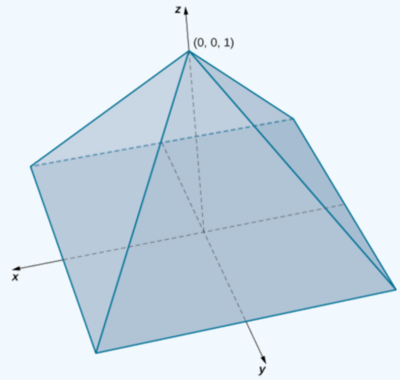


Figure 7.4.7: Finding the volume of a pyramid with a square base.

Solution

In this pyramid the value of z changes from 0 to 1 and at each height z the cross section of the pyramid for any value of z is the square

$$[-1+z,\,1-z]\times[-1+z,\,1-z].$$

Hence, the volume of the pyramid is

$$\iiint_E 1\,dV$$

where

$$E=\{(x,y,z)\,|\,0\le z\le 1,\ -1+z\le y\le 1-z,\ -1+z\le x\le 1-z\}.$$

Thus, we have

$$\begin{aligned}\iiint_E 1\,dV &= \int_{z=0}^{z=1}\int_{y=1+z}^{1-z}\int_{x=1+z}^{1-z} 1\,dx\,dy\,dz \\ &= \int_{z=0}^{z=1}\int_{y=1+z}^{1-z}(2-2z)\,dy\,dz \\ &= \int_{z=0}^{z=1}(2-2z)^2\,dz=\frac{4}{3}.\end{aligned}$$

Hence, the volume of the pyramid is $\frac{4}{3}$ cubic units.

Exercise 7.4.3

Consider the solid sphere $E=\{(x,y,z)\,|\,x^2+y^2+z^2=9\}$. Write the triple integral

$$\iiint_E f(x,y,z)\,dV$$

for an arbitrary function f as an iterated integral. Then evaluate this triple integral with $f(x,y,z)=1$. Notice that this gives the volume of a sphere using a triple integral.

Hint

Follow the steps in the previous example. Use symmetry.

Answer

$$\begin{aligned}\iiint_E 1\,dV &= 8\int_{x=-3}^{x=3}\int_{y=-\sqrt{9-z^2}}^{y=\sqrt{9-z^2}}\int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}} 1\,dz\,dy\,dx \\ &= 36\pi \text{ cubic units.}\end{aligned}$$

7.4.1 Changing the Order of Integration

As we have already seen in double integrals over general bounded regions, changing the order of the integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful. We demonstrate two examples here.

Example 7.4.4: Changing the Order of Integration

Consider the iterated integral

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y} f(x,y,z)\,dz\,dy\,dx. \tag{7.4.29}$$

The order of integration here is first with respect to z, then y, and then x. Express this integral by changing the order of integration to be first with respect to x, then z, and then y. Verify that the value of the integral is the same if we let $f(x,y,z) = xyz$.

Solution

The best way to do this is to sketch the region E and its projections onto each of the three coordinate planes. Thus, let

$$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}.$$

and

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=x^2} f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV.$$

We need to express this triple integral as

$$\int_{y=c}^{y=d}\int_{z=v_1(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\,dz\,dy.$$

Knowing the region E we can draw the following projections (Figure 7.4.8):

on the xy-plane is $D_1 = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x^2\} = \{(x,y) \mid 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$,

on the yz-plane is $D_2 = \{(y,z) \mid 0 \le y \le 1,\ 0 \le z \le y^2\}$, and

on the xz-plane is $D_3 = \{(x,z) \mid 0 \le x \le 1,\ 0 \le z \le x^2\}$.

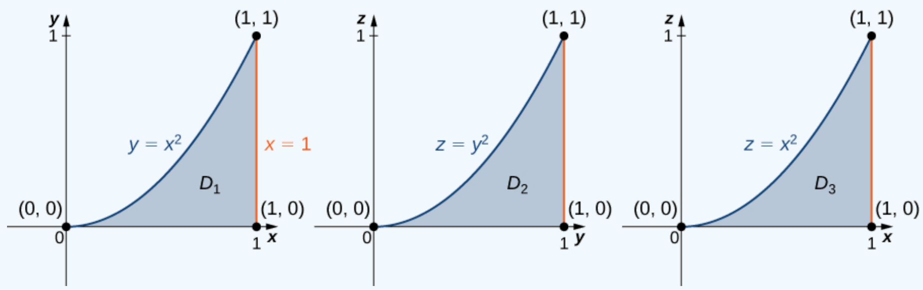


Figure 7.4.8. The three cross sections of E on the three coordinate planes.

Now we can describe the same region E as $\{(x,y,z) \mid 0 \le y \le 1,\ 0 \le z \le y^2,\ \sqrt{y} \le x \le 1\}$, and consequently, the triple integral becomes

$$\int_{y=c}^{y=d}\int_{z=v_1(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\,dz\,dy = \int_{y=0}^{y=1}\int_{z=0}^{z=x^2}\int_{x=\sqrt{y}}^{x=1} f(x,y,z)\,dx\,dz\,dy \tag{7.4.30}$$

Now assume that $f(x,y,z) = xyz$ in each of the integrals. Then we have

$$\begin{aligned}
&\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y^2} xyz\,dz\,dy\,dx \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left[xy\frac{z^2}{2}\Big|_{z=0}^{z=y^2}\right]dy\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left(x\frac{y^5}{2}\right)dy\,dx = \int_{x=0}^{x=1}\left[x\frac{y^6}{12}\Big|_{y=0}^{y=x^2}\right]dx = \int_{x=0}^{x=1}\frac{x^{13}}{12}dx \\
&= \frac{x^{14}}{168}\Big|_{x=0}^{x=1} = \frac{1}{168},
\end{aligned}$$

$$\begin{aligned}
&\int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\int_{x=\sqrt{y}}^{x=1} xyz\,dx\,dz\,dy \\
&= \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left[yz\frac{x^2}{2}\Big|_{\sqrt{y}}^{1}\right]dz\,dy = \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left(\frac{yz}{2} - \frac{y^2z}{2}\right)dz\,dy = \int_{y=0}^{y=1}\left[\frac{yz^2}{4} - \frac{y^2z^2}{4}\Big|_{z=0}^{z=y^2}\right]dy = \\
&\int_{y=0}^{y=1}\left(\frac{y^5}{4} - \frac{y^6}{4}\right)dy = \left(\frac{y^6}{24} - \frac{y^7}{28}\right)\Big|_{y=0}^{y=1} = \frac{1}{168}.
\end{aligned}$$

The answers match.

Exercise 7.4.4

Write five different iterated integrals equal to the given integral

$$\int_{z=0}^{z=4}\int_{y=0}^{y=4-z}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\,dx\,dy\,dz.$$

Hint

Follow the steps in the previous example, using the region E as $\{(x,y,z)\,|\,0\le z\le 4,\ 0\le y\le 4-z,\ 0\le x\le\sqrt{y}\}$, and describe and sketch the projections onto each of the three planes, five different times.

Answer

$$(i)\int_{z=0}^{z=4}\int_{x=0}^{x=\sqrt{4-z}}\int_{y=x^2}^{y=4-z} f(x,y,z)\,dy\,dx\,dz,\ (ii)\int_{y=0}^{y=4}\int_{z=0}^{z=4-y}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\,dx\,dz\,dy,\ (iii)\int_{y=0}^{y=4}\int_{x=0}^{x=\sqrt{y}}\int_{z=0}^{Z=4-y} f(x,y,z)\,dz\,dx\,dy,$$

$$(iv)\int_{x=0}^{x=2}\int_{y=x^2}^{y=4}\int_{z=0}^{z=4-y} f(x,y,z)\,dz\,dy\,dx,\ (v)\int_{x=0}^{x=2}\int_{z=0}^{z=4-x^2}\int_{y=x^2}^{y=4-z} f(x,y,z)\,dy\,dz\,dx$$

Example 7.4.5: Changing Integration Order and Coordinate Systems

Evaluate the triple integral

$$\iiint_E \sqrt{x^2+z^2}\,dV,$$

where E is the region bounded by the paraboloid $y=x^2+z^2$ (Figure 7.4.9) and the plane $y=4$.

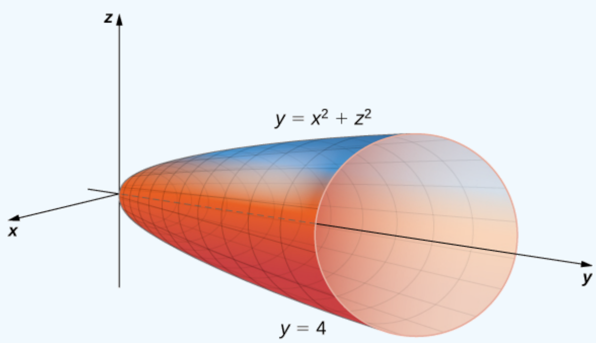


Figure 7.4.9. Integrating a triple integral over a paraboloid.

Solution

The projection of the solid region E onto the xy-plane is the region bounded above by $y=4$ and below by the parabola $y=x^2$ as shown.

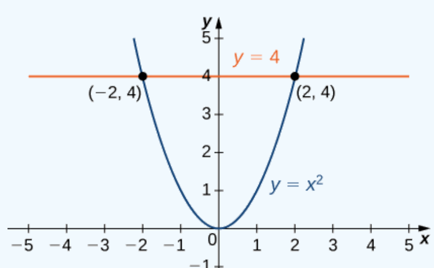


Figure 7.4.9.

Thus, we have

$$E=\{(x,y,z)\,|\,-2\le x\le 2,\ x^2\le y\le 4,\ -\sqrt{y-x^2}\le z\sqrt{y-x^2}\}.$$

The triple integral becomes

$$\iiint_E \sqrt{x^2+z^2}\,dV=\int_{x=-2}^{x=2}\int_{y=x^2}^{y=4}\int_{z=-\sqrt{y-x^2}}^{y=\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dz\,dy\,dx.$$

This expression is difficult to compute, so consider the projection of E onto the xz-plane. This is a circular disc $x^2+z^2\le 4$. So we obtain

$$\iiint_E \sqrt{x^2+z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dy\, dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2+z^2}\, dy\, dz\, dx.$$

Here the order of integration changes from being first with respect to z then y and then x to being first with respect to y then to z and then to x. It will soon be clear how this change can be beneficial for computation. We have

$$\int_{x=-2}^{x=2} \int_{z=\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2+z^2}\, dy\, dz\, dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4-x^2-z^2)\sqrt{x^2+z^2}\, dz\, dx.$$

Now use the polar substitution $x = r\cos\theta$, $z = r\sin\theta$, and $dz\,dx = r\,dr\,d\theta$ in the xz-plane. This is essentially the same thing as when we used polar coordinates in the xy-plane, except we are replacing y by z. Consequently the limits of integration change and we have, by using $r^2 = x^2+z^2$,

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4-x^2-z^2)\sqrt{x^2+z^2}\, dz\, dx = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4-r^2) r r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5}\Big|_0^2 \right] d\theta =$$

$$\int_0^{2\pi} \frac{64}{15}\, d\theta = \frac{128\pi}{15}$$

7.4.2 Average Value of a Function of Three Variables

Recall that we found the average value of a function of two variables by evaluating the double integral over a region on the plane and then dividing by the area of the region. Similarly, we can find the average value of a function in three variables by evaluating the triple integral over a solid region and then dividing by the volume of the solid.

Average Value of a Function of Three Variables

If $f(x,y,z)$ is integrable over a solid bounded region E with positive volume $V(E)$, then the average value of the function is

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x,y,z)\, dV. \tag{7.4.31}$$

Note that the volume is

$$V(E) = \iiint_E 1\, dV. \tag{7.4.32}$$

Example 7.4.6: Finding an Average Temperature

The temperature at a point (x,y,z) of a solid E bounded by the coordinate planes and the plane $x+y+z=1$ is $T(x,y,z) = (xy+8z+20)\,^\circ\mathrm{C}$. Find the average temperature over the solid.

Solution

Use the theorem given above and the triple integral to find the numerator and the denominator. Then do the division. Notice that the plane $x+y+z=1$ has intercepts $(1,0,0)$, $(0,1,0)$,and $(0,0,1)$. The region E looks like

$$E = \{(x,y,z) \,|\, 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}.$$

Hence the triple integral of the temperature is

$$\iiint_E f(x,y,z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (xy+8z+20)\, dz\, dy\, dx = \frac{147}{40}.$$

The volume evaluation is

$$V(E) = \iiint_E 1\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} 1\, dz\, dy\, dx = \frac{1}{6}.$$

Hence the average value is

$$T_{ave} = \frac{147/40}{1/6} = \frac{6(147)}{40} = \frac{441}{20}\,^\circ\mathrm{C}$$

.

Exercise 7.4.6

Find the average value of the function $f(x,y,z) = xyz$ over the cube with sides of length 4 units in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

Hint

Follow the steps in the previous example.

Answer

$f_{ave} = 8$

7.4.3 Key Concepts

- To compute a triple integral we use Fubini's theorem, which states that if $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then
$$\iiint_B f(x, y, z)\, dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$
and is also equal to any of the other five possible orderings for the iterated triple integral.
- To compute the volume of a general solid bounded region E we use the triple integral
$$V(E) = \iiint_E 1\, dV.$$
- Interchanging the order of the iterated integrals does not change the answer. As a matter of fact, interchanging the order of integration can help simplify the computation.
- To compute the average value of a function over a general three-dimensional region, we use
$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV.$$

7.4.3.0.1 Key Equations

- **Triple integral**

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta x \Delta y \Delta z = \iiint_B f(x, y, z)\, dV$$

7.4.3.1 Glossary

triple integral

the triple integral of a continuous function $f(x, y, z)$ over a rectangular solid box B is the limit of a Riemann sum for a function of three variables, if this limit exists

7.4.4 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

7.4E:

7.5: Triple Integrals in Cylindrical and Spherical Coordinates

This page is a draft and is under active development.

Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter prelude, which showed the opera house l'Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of 9000 twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

7.5.1 Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space $\mathbb{R}^2$ a point with rectangular coordinates (x, y) can be identified with (r, θ) in polar coordinates and vice versa, where $x = r\,\cos\theta$, $y = r\,\sin\theta$, $r^2 = x^2 + y^2$ and $\tan\,\theta = \left(\frac{y}{x}\right)$ are the relationships between the variables.

In three-dimensional space $\mathbb{R}^3$ a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa. We can use these same conversion relationships, adding z as the vertical distance to the point from the $(xy$-plane as shown in $7.5.1$.

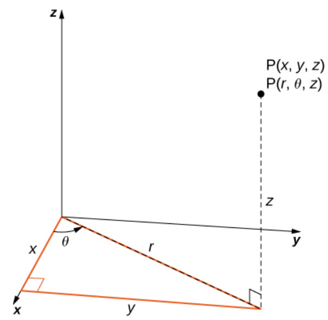


Figure $7.5.1$: Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion

- $x = r\,\cos\theta$
- $y = r\,\sin\theta$
- $z = z$

To convert from cylindrical to rectangular coordinates, we use

- $r^2 = x^2 + y^2$ and
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

Note that that z-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say $x = k$ (constant) we mean an unbounded vertical line parallel to the y-axis and when $y = l$ (constant) we mean an unbounded horizontal line parallel to the x-axis. With the polar coordinate system, when we say $r = c$ (constant), we mean a circle of radius c units and when $\theta = \alpha$ (constant) we mean an infinite ray making an angle α with the positive x-axis.

Similarly, in three-dimensional space with rectangular coordinates (x, y, z) the equations $x = k$, $y = l$ and $z = m$ where k, l and m are constants, represent unbounded planes parallel to the yz-plane, xz-plane and xy-plane, respectively. With cylindrical coordinates (r, θ, z), by $r = c$, $\theta = \alpha$, and $z = m$, where c, α, and m are constants, we mean an unbounded vertical cylinder with the z-axis as its radial axis; a plane making a constant angle α with the xy-plane; and an unbounded horizontal plane parallel to the xy-plane, respectively.

This means that the circular cylinder $x^2 + y^2 = c^2$ in rectangular coordinates can be represented simply as $r = c$ in cylindrical coordinates.

(Refer to Cylindrical and Spherical Coordinates for more review.)

7.5.2 Integration in Cylindrical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in

rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in Table $7.5.1$.

These equations will become handy as we proceed with solving problems using triple integrals.

Table $7.5.1$: Equations of Some Common Shapes

	Circular cylinder	Circular cone	Sphere	Paraboloid
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2(x^2 + y^2)$	$x^2 + y^2 + z^2 = c^2$	$z = c(x^2 + y^2)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

As before, we start with the simplest bounded region B in $\mathbb{R}^3$ to describe in cylindrical coordinates, in the form of a cylindrical box,

$B = \{(r, \theta, z) | a \leq r \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq z \leq d\}$ (Figure $7.5.2$).

Suppose we divide each interval into l, m, and n subdivisions such that $\Delta r = \frac{b\text{-}a}{l}$, $\Delta\theta = \frac{\beta\text{-}\alpha}{m}$, and

$\Delta z = \frac{d\text{-}c}{n}$. Then we can state the following definition for a triple integral in cylindrical coordinates.

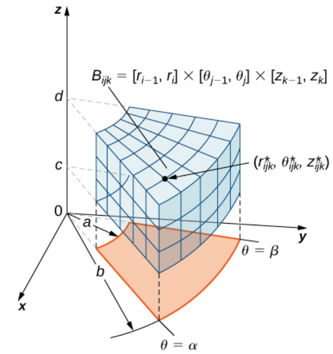


Figure 7.5.2: A cylindrical box B described by cylindrical coordinates.

DEFINITION: triple integral in cylindrical coordinates

Consider the cylindrical box (expressed in cylindrical coordinates)

$$B = \{(r,\theta,z)|a \leq r \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq z \leq d\}. \tag{7.5.1}$$

If the function $f(r,\theta,z)$ is continuous on B and if $(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})$ is any sample point in the cylindrical subbox

$B_{ijk} = |r_{i-1}, r_i| \times |\theta_{j-1}, \theta_j| \times |z_{k-1}, k_i|$ (Figure 7.5.2), then we can define

the *triple integral in cylindrical coordinates* as the limit of a triple Riemann sum, provided the following limit exists:

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk}) \Delta r \Delta\theta \Delta z. \tag{7.5.2}$$

Note that if $g(x,y,z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral

$$\iiint_B g(x,y,z)dV \tag{7.5.3}$$

is equal to the triple integral

$$\iiint_B g(r\ \cos\theta,\ r\ \sin\ \theta,\ z) r\, dr\, d\theta\, dz \tag{7.5.4}$$

and we have

$$\iiint_B g(x,y,z)dV = \iiint_B g(r\ \cos\theta,\ r\ \sin\ \theta,\ z) r\, dr\, d\theta\, dz = \iiint_B f(r,\theta\, z) r\, dr\, d\theta\, dz. \tag{7.5.5}$$

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini's theorem takes the following form:

Theorem: Fubini's Theorem in Cylindrical Coordinates

Suppose that $g(x,y,z)$ is continuous on a rectangular box B which when described in cylindrical coordinates looks like

$B = \{(r,\theta,z)|a \leq r \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq z \leq d\}$.

Then $g(x,y,z) = g(r\ \cos\theta, r\ \sin\ \theta, z) = f(r,\theta,z)$ and

$$\iiint_B g(x,y,z)dV = \int_c^d \int_\beta^\alpha \int_a^b f(r,\theta,z) r\, dr\, d\theta\, dz. \tag{7.5.6}$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example 7.5.1: Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral

$$\iiint_B (zr\ \sin\ \theta) r\, dr\, d\theta\, dz \tag{7.5.7}$$

where the cylindrical box B is $B = \{(r,\theta,z)|0 \leq r \leq 2,\ 0 \leq \theta \leq \pi/2,\ 0, \leq z \leq 4\}$.

Solution

As stated in Fubini's theorem, we can write the triple integral as the iterated integral

$$\iiint_B (zr\ \sin\ \theta) r\, dr\, d\theta\, dz = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr\ \sin\ \theta) r\, dz\, dr\, d\theta. \tag{7.5.8}$$

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(zr\sin\theta)r\,dz\,dr\,d\theta = \left(\int_0^{\pi/2}\sin\theta\,d\theta\right)\left(\int_0^2 r^2 dr\right)\left(\int_0^4 z\,dz\right) \\ = \left(-\cos\theta\big|_0^{\pi/2}\right)\left(\frac{r^3}{3}\Big|_0^2\right)\left(\frac{z^2}{2}\Big|_0^4\right) = \frac{64}{3}. \tag{7.5.9}$$

Exercise $7.5.1$:

Evaluate the triple integral

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=4} rz\sin\theta r\,dz\,dr\,d\theta. \tag{7.5.10}$$

Hint

Follow the same steps as in the previous example.

Answer

8

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous

function $f(r,\theta,z)$ over a general solid region $E=\{(r,\theta,z)|(r,\theta)\in D, u_1(r,\theta)\le z\le u_2(r,\theta)\}$ in $\mathbb{R}^3$ where D is the

projection of E onto the $r\theta$-plane, is

$$\iiint_E f(r,\theta,z)r\,dr\,d\theta\,dz = \iint_D\left[\int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)dz\right]r\,dr\,d\theta. \tag{7.5.11}$$

In particular, if $D=\{(r,\theta)|G_1(\theta)\le r\le g_2(\theta), \alpha\le\theta\le\beta\}$, then we have

$$\iiint_E f(r,\theta,z)r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)} f(r,\theta,z)r\,dz\,dr\,d\theta. \tag{7.5.12}$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

Example $7.5.2$: Setting up a Triple Integral in Cylindrical Coordinates over a General Region

Consider the region E inside the right circular cylinder with equation $r=2\sin\theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin (Figure 15.5.3). Set up a triple integral over this region with a function $f(r,\theta,z)$ in cylindrical coordinates.

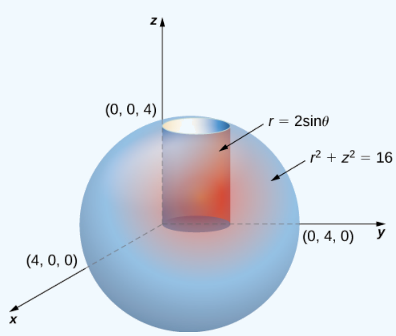


Figure $7.5.3$: Setting up a triple integral in cylindrical coordinates over a cylindrical region.

Solution

First, identify that the equation for the sphere is $r^2+z^2=16$. We can see that the limits for z are from 0 to $z=\sqrt{16-r^2}$. T
hen the limits for r are from 0 to $r=2\sin\theta$. Finally, the limits for θ are from 0 to π.

Hence the region is $E=\{(r,\theta,z)|0\le\theta\le\pi,\ 0\le r\le 2\sin\theta,\ 0\le z\le\sqrt{16-r^2}\}$.

Therefore, the triple integral is

$$\iiint_E f(r,\theta,z)r\,dz\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\,\sin\,\theta}\int_{z=0}^{z=\sqrt{16-r^2}} f(r,\theta,z)r\,dz\,dr\,d\theta. \tag{7.5.13}$$

Exercise 7.5.2:

Consider the region inside the right circular cylinder with equation $r = 2\,\sin\,\theta$ bounded below by the $r\theta$-plane and bounded above by $z = 4 - y$. Set up a triple integral with a function $f(r,\theta,z)$ in cylindrical coordinates.

Hint

Analyze the region, and draw a sketch.

Answer

$$\iiint_E f(r,\theta,z)r\,dz\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\,\sin\,\theta}\int_{z=0}^{z=4-r\,\sin\,\theta} f(r,\theta,z)r\,dz\,dr\,d\theta. \tag{7.5.14}$$

Example 7.5.3: Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. (Figure 15.5.4).

Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz\,dr\,d\theta$

b. $dr\,dz\,d\theta$

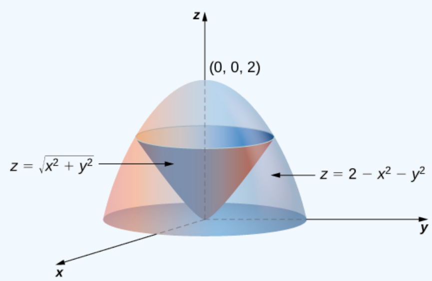


Figure 7.5.4: Setting up a triple integral in cylindrical coordinates over a conical region.

Solution

a. The cone is of radius 1 where it meets the paraboloid. Since $z = 2 - x^2 - y^2 = 2 - r^2$ and $z = \sqrt{x^2+y^2} = r^2$ (assuming r is nonnegative), we have $2 - r^2 = r$. Solving, we have $r^2 + r - 2 = (r+2)(r-1) = 0$. Since $r \geq 0$, we have $r = 1$. Therefore $z = 1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$.

The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.

Thus, we can describe the region as $E = \{(r,\theta,z)|0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ r \leq z \leq 2 - r^2\}$.

Hence the integral for the volume is

$$V = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=r}^{z=2-r^2} r\,dz\,dr\,d\theta. \tag{7.5.15}$$

b. We can also write the cone surface as $r = z$ and the paraboloid as $r^2 = 2 - z$. The lower bound for r is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane $z = 1$ divides the region into two regions. Then the region can be described as

$$E = \{(r,\theta,z)|0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq z\} \cup \{(r,\theta,z)|0 \leq \theta \leq 2\pi,\ 1 \leq z \leq 2,\ 0 \leq r \leq \sqrt{2-z}\}. \tag{7.5.16}$$

Now the integral for the volume becomes

$$V = \int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=1}\int_{r=0}^{r=z} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi}\int_{z=1}^{z=2}\int_{r=0}^{r=\sqrt{2-z}} r\,dr\,dz\,d\theta. \tag{7.5.17}$$

Exercise 7.5.3:

Redo the previous example with the order of integration $d\theta\,dz\,dr$.

Hint

Note that θ is independent of r and z.

Answer

$E = \{(r, \theta, z) | 0 \le \theta \le 2\pi,\ 0 \le z \le 1,\ 0 \le r \le 2 - z^2\}$ and

$$V = \int_{r=0}^{r=1} \int_{z=0}^{z=2-r^2} \int_{\theta=0}^{\theta=2\pi} r\, d\theta\, dz\, dr. \tag{7.5.18}$$

Example 7.5.4: Finding a Volume with Triple Integrals in Two Ways

Let E be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$ (Figure 15.5.5). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\, dr\, d\theta$
b. $dr\, dz\, d\theta$.

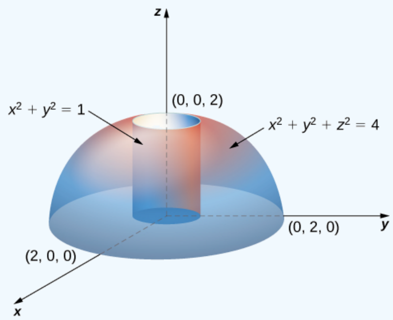


Figure 7.5.5: Finding a cylindrical volume with a triple integral in cylindrical coordinates.

Solution

a. Note that the equation for the sphere is

$$x^2 + y^2 + z^2 = 4 \text{ or } r^2 + z^2 = 4 \tag{7.5.19}$$

and the equation for the cylinder is

$$x^2 + y^2 = 1 \text{ or } r^2 = 1. \tag{7.5.20}$$

Thus, we have for the region E

$$E = \{(r, \theta, z) | 0 \le z \le \sqrt{4 - r^2},\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi\} \tag{7.5.21}$$

Hence the integral for the volume is

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} r\, dz\, dr\, d\theta \tag{7.5.22}$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left[\rangle rz|_{z=0}^{z=\sqrt{4-r^2}} \right] dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(r\sqrt{4-r^2} \right) dr\, d\theta \tag{7.5.23}$$

$$= \int_0^{2\pi} \left(\frac{8}{3} - \sqrt{3} \right) d\theta = 2\pi \left(\frac{8}{3} - \sqrt{3} \right) \text{ cubic units.} \tag{7.5.24}$$

b. Since the sphere is $x^2 + y^2 + z^2 = 4$, which is $r^2 + z^2 = 4$, and the cylinder is $x^2 + y^2 = 1$, which is $r^2 = 1$, we have $1 + z^2 = 4$, that is, $z^2 = 3$. Thus we have two regions, since the sphere and the cylinder intersect at $(1, \sqrt{3})$ in the rz-plane

$$E_1 = \{(r, \theta, z) | 0 \le r \le \sqrt{4 - r^2},\ \sqrt{3} \le z \le 2,\ 0 \le \theta \le 2\pi\} \tag{7.5.25}$$

and

$$E_2 = \{(r, \theta, z) | 0 \le r \le 1,\ 0 \le z \le \sqrt{3},\ 0 \le \theta \le 2\pi\}. \tag{7.5.26}$$

Hence the integral for the volume is

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{z=\sqrt{3}}^{z=2} \int_{r=0}^{r=\sqrt{4-r^2}} r\, dr\, dz\, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=\sqrt{3}} \int_{r=0}^{r=1} r\, dr\, dz\, d\theta \tag{7.5.27}$$

$$= \sqrt{3}\pi + \left(\frac{16}{3} - 3\sqrt{3} \right) \pi = 2\pi \left(\frac{8}{3} - \sqrt{3} \right) \text{ cubic units.} \tag{7.5.28}$$

Exercise 7.5.4

Redo the previous example with the order of integration $d\theta\, dz\, dr$.

Hint

A figure can be helpful. Note that θ is independent of r and z.

Answer

$E_2 = \{(r, \theta, z) | 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ r \le z \le \sqrt{4 - r^2}\}$ and

$$V = \int_{r=0}^{r=1} \int_{z=r}^{z=\sqrt{4-r^2}} \int_{\theta=0}^{\theta=2\pi} r\, d\theta\, dz\, dr. \tag{7.5.29}$$

7.5.3 Review of Spherical Coordinates

In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ

from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP

(Figure $7.5.6$). Note that $\rho > 0$ and $0 \le \varphi \le \pi$. (Refer to Cylindrical and Spherical Coordinates for a review.)

Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.

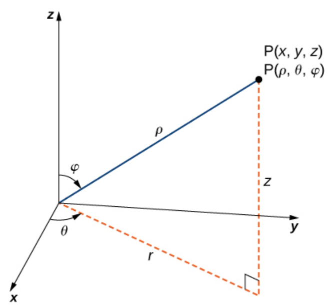


Figure $7.5.6$: The spherical coordinate system locates points with two angles and a distance from the origin.

Recall the relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates

$$x = \rho\, \sin\, \varphi\, \cos\theta,\ y = \rho\, \sin\, \varphi\, \sin\, \theta,\ and\, z = \rho\, \cos\, \varphi. \tag{7.5.30}$$

From rectangular coordinates to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan\, \theta = \frac{y}{x},\ \varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right). \tag{7.5.31}$$

Other relationships that are important to know for conversions are

- $r = \rho\, \sin\, \varphi$
- $\theta = \theta$ These equations are used to convert from spherical coordinates to cylindrical coordinates.
- $z = \rho\, \cos\, \varphi$

and

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$ These equations are used to convert from cylindrical coordinates to spherical coordinates.
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

$7.5.7$ shows a few solid regions that are convenient to express in spherical coordinates.

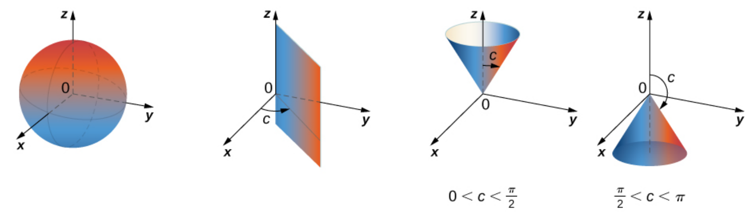


Figure 7.5.7: Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

7.5.4 Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system.

Let the function $f(\rho, \theta, \varphi)$ be continuous in a bounded spherical box,

$B = \{(\rho, \theta, \varphi) | a \leq \rho \leq b, \; \alpha \leq \theta \leq \beta, \; \gamma \leq \varphi \leq \psi\}$.

We then divide each interval into l, m, n and n subdivisions such that

\[\Delta \rho = \frac{b - a}{l}, \, \Delta \theta = \frac{\beta - \alpha}{m}, \, \Delta \varphi = \frac{\psi - \gamma}{n} .\]

Now we can illustrate the following theorem for triple integrals in spherical coordinates with $(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})$

being any sample point in the spherical subbox B_{ijk}. For the volume element of the subbox ΔV in spherical coordinates, we have

$\Delta V = (\Delta\rho)\,(\rho\Delta\varphi)\,(\rho\,\sin\,\varphi\,\Delta\theta)$, as shown in the following figure.

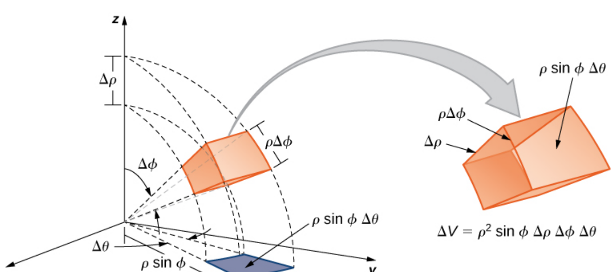


Figure 7.5.8: The volume element of a box in spherical coordinates.

Definition: triple integral in spherical coordinates

The **triple integral in spherical coordinates** is the limit of a triple Riemann sum,

\[\lim_{l,m,n \rightarrow \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f (\rho_{ijk}^*, \theta_{ijk}^*, \varphi_{ijk}^*) (\rho_{ijk}^*)^2 \sin \varphi \Delta \rho \Delta \theta \Delta \varphi \]

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals.

Fubini's theorem takes the following form.

Theorem: Fubini's Theorem for Spherical Coordinates

If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then

\[\iiint_B f(\rho,\theta,\varphi) \, \rho^2 \sin \, \varphi d\rho \, d\varphi \, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho,\theta,\varp

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before w

consider triple integrals in spherical coordinates on general spherical regions.

Example $7.5.5$: Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral

$$\int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/2} \int_{\rho=0}^{\rho=1} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta. \tag{7.5.32}$$

Solution

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:

\[\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \sin \, \varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin \, \varphi \, d\varphi

\int_0^1 \rho^2 d\rho = (2\pi) \, (1) \, \left(\frac{1}{3}\right) = \frac{2\pi}{3}\]

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes.

Note that dV and dA mean the increments in volume and area, respectively. The variables V and A are used as the variables for integration to

express the integrals.

The triple integral of a continuous function $f(\rho, \theta, \varphi)$ over a general solid region

$$E = \{(\rho, \theta, \varphi) | (\rho, \theta) \in D, u_1(\rho, \theta) \leq \varphi \leq u_2(\rho, \theta)\} \tag{7.5.33}$$

in $\mathbb{R}^3$, where D is the projection of E onto the $\rho\theta$-plane, is

$$\iiint_E f(\rho, \theta, \varphi) dV = \iint_D \left[\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho, \theta, \varphi) \, d\varphi \right] dA. \tag{7.5.34}$$

In particular, if $D = \{(\rho, \theta) | g_1(\theta) \leq \rho \leq g_2(\theta), \ \alpha \leq \theta \leq \beta\}$, the we have

\[\iiint_E f(\rho,\theta,\varphi) dV = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi \rho^2 \sin \, \varphi \, d\varphi \, d\rho \, d\th

Similar formulas occur for projections onto the other coordinate planes.

Example 7.5.6: Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone $z=\sqrt{3(x^2+y^2)}$ and the hemisphere $z=\sqrt{4-x^2-y^2}$ (see the figure below).

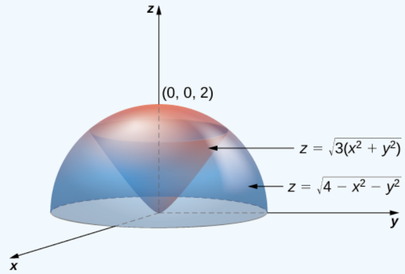


Figure 7.5.9: A region bounded below by a cone and above by a hemisphere.

Solution

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z=\sqrt{3(x^2+y^2)}$ or $\rho\cos\varphi=\sqrt{3}\rho\sin\varphi$ or $\tan\varphi=\frac{1}{\sqrt{3}}$ or $\varphi=\frac{\pi}{6}$.

For the sphere: $z=\sqrt{4-x^2-y^2}$ or $z^2+x^2+y^2=4$ or $\rho^2=4$ or $\rho=2$.

Thus, the triple integral for the volume is

$$V(E)=\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi+\pi/6}\int_{\rho=0}^{\rho=2}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta. \tag{7.5.35}$$

Exercise 7.5.5

Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho=2$ and bounded below by the cone $\varphi=\pi/3$.

Hint

Follow the steps of the previous example.

Answer

$$V(E)=\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/3}\int_{\rho=0}^{\rho=2}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \tag{7.5.36}$$

Example 7.5.7: Interchanging Order of Integration in Spherical Coordinates

Let E be the region bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the sphere $z=x^2+y^2+z^2$ (Figure 15.5.10). Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho\,d\phi\,d\theta$
b. $d\varphi\,d\rho\,d\theta$

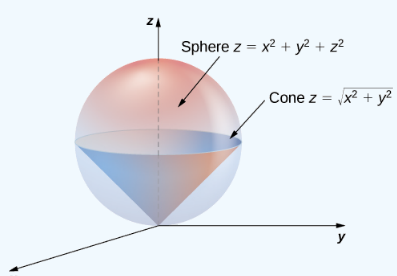


Figure 7.5.10:. A region bounded below by a cone and above by a sphere.

Solution

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.

For the sphere:

$$\begin{aligned} x^2+y^2+z^2 &= z && (7.5.37)\\ \rho^2 &= \rho \cos\varphi && (7.5.38)\\ \rho &= \cos\varphi. && (7.5.39)\end{aligned}$$

For the cone:

\[\begin{align} z &= \sqrt{x^2 + y^2}\\\rho \, \cos \, \varphi &= \sqrt{\rho^2 \sin^2 \, \varphi \, \cos^2 \phi } \\ \rho \, \cos \, \varphi&= \sqrt{\rho^2 \sin^2 \varphi \, (\cos^2\phi + \sin^2 \phi)}\\ \rho \, '

Hence the integral for the volume of the solid region E becomes

\[V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/4} \int_{\rho=0}^{\rho=\cos \, \varphi} \rho^2 \sin \, \varphi \, d\rho \, d\varphi \,
d\theta.\]

b. Consider the $\varphi\rho$-plane. Note that the ranges for φ and ρ (from part a.) are

$$\begin{aligned} &0 \leq \rho\sqrt{2}/2 && \text{and } \sqrt{2} \leq \rho 1 && (7.5.40)\\ &0 \leq \varphi \leq \pi/4 && 0 \leq \rho \leq \cos\varphi && (7.5.41)\end{aligned}$$

The curve $\rho = \cos\varphi$ meets the line $\varphi = \pi/4$ at the point $(\pi/4, \sqrt{2}/2)$ Thus, to change the order of integration, we need to use two pieces:

$$0 \leq \rho \leq \sqrt{2}/2,\ 0 \leq \varphi \leq \pi/4 \tag{7.5.42}$$

and

$$\sqrt{2}/2 \leq \rho \leq 1,\ 0 \leq \varphi \leq \cos^{-1}\rho. \tag{7.5.43}$$

Hence the integral for the volume of the solid region E becomes

\[V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=\sqrt{2}/2} \int_{\varphi=0}^{\varphi=\pi/4} \rho^2 \sin \, \varphi \, d\varphi \, d\rho \,
d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\sqrt{2}/2}^{\rho=1} \int_{\varphi=0}^{\varphi=\cos^{-1}\rho} \rho^2 \sin \, \varphi \, d\varphi \,
d\rho \, d\theta\]

In each case, the integration results in $V(E) = \frac{\pi}{8}$.

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

Example $7.5.8$: Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy. \tag{7.5.44}$$

Solution

The ranges of the variables are

$$\begin{aligned} -1 &\leq y \leq y && (7.5.45)\\ 0 &\leq x \leq \sqrt{1-y^2} && (7.5.46)\\ x^2+y^2 &\leq z \leq \sqrt{x^2+y^2}. && (7.5.47)\end{aligned}$$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the ranges for θ and r are

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ and } 0 \leq r \leq 1. \tag{7.5.48}$$

The limits of z are $r^2 \leq z \leq r$, hence

\[\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy = \int_{\theta=-\pi/2}^{\theta=\pi/2}
\int_{r=0}^{r=1} \int_{z=r^2}^{z=r} r(r \, \cos \theta) \, (r \, \sin \, \theta) \, z \, dz \, dr \, d\theta.\]

Example $7.5.9$: Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:

$$\int_{y=0}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)dz\,dx\,dy. \tag{7.5.49}$$

Solution

The ranges of the variables are

$$0 \leq y \leq 3 \tag{7.5.50}$$

$$0 \leq x \leq \sqrt{9-y^2} \tag{7.5.51}$$

$$\sqrt{x^2+y^2} \leq z \leq \sqrt{18-x^2-y^2}. \tag{7.5.52}$$

The first two ranges of variables describe a quarter disk in the first quadrant of the xy-plane. Hence the range for θ is $0 \leq \theta \leq \frac{\pi}{2}$.

The lower bound $z=\sqrt{x^2+y^2}$ is the upper half of a cone and the upper bound $z=\sqrt{18-x^2-y^2}$ is the upper half of a sphere.

Therefore, we have $0 \leq \rho \leq \sqrt{18}$, which is $0 \leq \rho \leq 3\sqrt{2}$.

For the ranges of φ we need to find where the cone and the sphere intersect, so solve the equation

$$r^2+z^2 = 18 \tag{7.5.53}$$

$$(\sqrt{x^2+y^2})^2+z^2 = 18 \tag{7.5.54}$$

$$z^2+z^2 = 18 \tag{7.5.55}$$

$$2z^2 = 18 \tag{7.5.56}$$

$$z^2 = 9 \tag{7.5.57}$$

$$z = 3. \tag{7.5.58}$$

This gives

$$3\sqrt{2}\cos\varphi = 3 \tag{7.5.59}$$

$$\cos\varphi = \frac{1}{\sqrt{2}} \tag{7.5.60}$$

$$\varphi = \frac{\pi}{4}. \tag{7.5.61}$$

Putting this together, we obtain

\[\int_{y=0}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) dz \, dx \, dy = \int_{\varphi=0}^{\varphi=\pi/4} \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=3\sqrt{2}} \rho^4 \sin \, \varphi \, d\rho \, d\theta \, d\varphi.\]

Exercise $7.5.6$:

Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2+y^2+z^2=4$ but outside the cylinder $x^2+y^2=1$.

Answer: Rectangular

\[\int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz \, dy \, dx - \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz \, dy \, dx.\]

Answer: Cylindrical

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=1}^{r=2}\int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta. \tag{7.5.62}$$

Answer: Spherical

\[\int_{\varphi=\pi/6}^{\varphi=5\pi/6} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\csc \, \varphi}^{\rho=2} \rho^2 \sin \, \varphi \, d\rho \, d\theta \, d\varphi.\]

Now that we are familiar with the spherical coordinate system, let's find the volume of some known geometric figures, such as spheres and ellipsoids.

Example $7.5.10$: Chapter Opener: Finding the Volume of l'Hemisphèric

Find the volume of the spherical planetarium in l'Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately 50 ft, using the equation $x^2+y^2+z^2=r^2$.

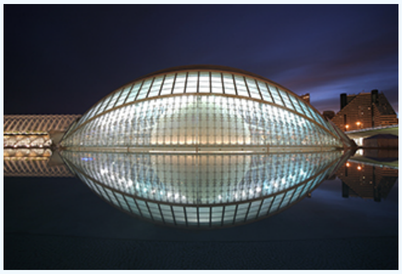


Figure 7.5.11: (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution

We calculate the volume of the ball in the first octant, where $x \leq 0,\ y \leq 0$, and $z \leq 0$, using spherical coordinates, and then multiply the result by 8 for symmetry. Since we consider the region D as the first octant in the integral, the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2},\ 0 \leq \rho \leq r,\ 0 \leq \theta \leq \frac{\pi}{2}. \tag{7.5.63}$$

Therefore,

\[\begin{align} V &= \iiint_D dx \, dy \, dz = 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=\pi} \int_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin \, \theta \, d\varphi \, d\rho \, d\varphi \\&=8 \int_{

This exactly matches with what we knew. So for a sphere with a radius of approximately 50 ft, the volume is \(\frac{4}{3} \pi (50)^3 \approx 523,600 \, ft^3\).

For the next example we find the volume of an ellipsoid.

Example $7.5.11$: Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant $x \leq 0,\ y \leq 0$, and $z \leq 0$ and then multiply the result by 8.

In this case the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2}\ 0 \leq \rho \leq 1, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}. \tag{7.5.64}$$

Also, we need to change the rectangular to spherical coordinates in this way:

$$x = a\rho\ \cos\ \varphi\ \sin\ \theta,\ y = b\rho\ \sin\ \varphi\ \sin\ \theta, \text{ and } z = cp\ \cos\theta. \tag{7.5.65}$$

Then the volume of the ellipsoid becomes

\[\begin{align} V &= \iiint_D dx \, dy \, dz \\&= 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\pi/2} abc \, \rho^2 \sin \, \theta \, d\varphi \, d\rho \, d\theta \\ \\&

Example $7.5.12$: Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid $\frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1$ and outside the sphere $x^2 + y^2 + z^2 = 50^2$.

Solution

This problem is directly related to the l'Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then substract.

First we find the volume of the ellipsoid using $a = 75$ ft, $b = 80$ ft, and $c = 90$ ft in the result from Example. Hence the volume of the ellipsoid is

$$V_{ellipsoid} = \frac{4}{3}\pi(75)(80)(90) \approx 2,262,000\ ft^3. \tag{7.5.66}$$

From Example, the volume of the sphere is

$$V_{sphere} \approx 523,600\ ft^3. \tag{7.5.67}$$

Therefore, the volume of the space inside the ellipsoid $\frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1$ and outside the sphere $x^2 + y^2 + z^2 = 50^2$ is approximately

$$V_{Hemispheric} = V_{ellipsoid} - V_{sphere} = 1,738,400\ ft^3. \tag{7.5.68}$$

Student Project: Hot air balloons

Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over 500 balloons participating each year.

Figure 7.5.12: Balloons lift off at the 2001 Albuquerque International Balloon Fiesta. (credit: David Herrera, Flickr)

As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using a vent near the top of the balloon to release heated air and descend.

The pilot has very little control over where the balloon goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28 feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off).

The radius of the large end of the frustum is 28 feet and the radius of the small end of the frustum is 28 feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the following figure.

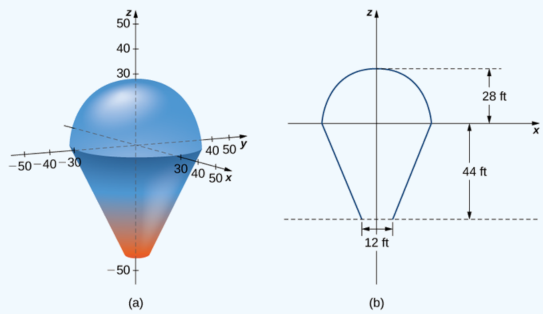


Figure 7.5.13: (a) Use a half sphere to model the top part of the balloon and a frustum of a cone to model the bottom part of the balloon.

(b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

1. Find the volume of the balloon in two ways.

a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)

b. Verify the answer using the formulas for the volume of a sphere, $V=\frac{4}{3}\pi r^3$, and for the volume of a cone, $V=\frac{1}{3}\pi r^2 h$.

In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function

$$T_0(r,\theta,z)=\frac{z-r}{10}+210. \tag{7.5.69}$$

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)

Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column in shown in the following figure

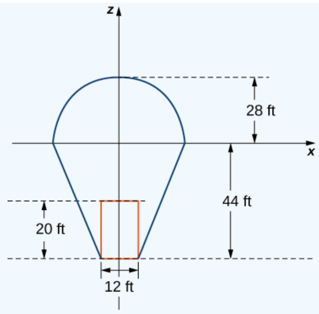


Figure 7.5.14: Activating the burner heats the air in a 20-foot-high, 12-foot-wide column directly above the burner.

Assume that after the pilot activates the burner for 10 seconds, the temperature of the air in the column described above *increases* according to the formula

$$H(r,\theta,z) = -2z - 48. \tag{7.5.70}$$

Then the temperature of the air in the column is given by

$$T_1(r,\theta,z) = \frac{z-r}{10} + 210 + (-2z-48), \tag{7.5.71}$$

while the temperature in the remainder of the balloon is still given by

$$T_0(r,\theta,z) = \frac{z-r}{10} + 210. \tag{7.5.72}$$

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for 10 seconds.

7.5.5 Key Concepts

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral

\[\int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)r \, dz \, dr \, d\theta. \nonumber\nonumber\]

- To evaluate a triple integral in spherical coordinates, use the iterated integral

\[\int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)} \int_{\varphi=u_1(r,\theta)}^{u_2(r,\theta)} f(\rho,\theta,\varphi) \, \rho^2 \sin \varphi \, d\varphi \, d\rho \, d\theta. \nonu

7.5.5.0.1 Key Equations

- **Triple integral in cylindrical coordinates**

$$\iiint_B g(s,y,z)dV = \iiint_B g(r\,\cos\theta,\; r\,\sin\,\theta,\; z)r\,dr\,d\theta\,dz = \iiint_B f(r,\theta,z)r\,dr\,d\theta\,dz$$

- **Triple integral in spherical coordinates**

\[\iiint_B f(\rho,\theta,\varphi) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi)

7.5.5.0.1 Glossary

triple integral in cylindrical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, s^*_{ijk}) r^*_{ijk} \Delta r \Delta\theta \Delta z$$

triple integral in spherical coordinates

the limit of a triple Riemann sum, provided the following limit exists:\[lim_{l,m,n\rightarrow\infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(\rho_{ijk}^*, \theta_{ijk}^*, \varphi_{ijk}^*) (\rh

7.5.6 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://c

7.5E.1 Exercise $7.5E.1$

In the following exercises, evaluate the triple integrals

$$\iiint_E f(x,y,z)dV \tag{7.5E.1}$$

over the solid E.

1. $f(x,y,z)=z,\ B=\{(x,y,z)|x^2+y^2\le 9,\ x\le 0,\ y\le 0,\ 0\le z\le 1\}$

A quarter section of a cylinder with height 1 and radius 3.

Answer

$\frac{9\pi}{8}$

2. $f(x,y,z)=xz^2,\ B=\{(x,y,z)|x^2+y^2\le 16,\ x\ge 0,\ y\le 0,\ -1\le z\le 1\}$
3. $f(x,y,z)=xy,\ B=\{(x,y,z)|x^2+y^2\le 1,\ x\ge 0,\ x\ge y,\ -1\le z\le 1\}$

A wedge with radius 1, height 1, and angle pi/4.

Answer

$\frac{1}{8}$st.

4. $f(x,y,z)=x^2+y^2,\ B=\{(x,y,z)|x^2+y^2\le 4,\ x\ge 0,\ x\le y,\ 0\le z\le 3\}$
5. $f(x,y,z)=e^{\sqrt{x^2+y^2}},\ B=\{(x,y,z)|1\le x^2+y^2\le 4,\ y\le 0,\ x\le y\sqrt{3},\ 2\le z\le 3\}$

Answer

$\frac{\pi e^2}{6}$

6. $f(x,y,z)=\sqrt{x^2+y^2},\ B=\{(x,y,z)|1\le x^2+y^2\le 9,\ y\le 0,\ 0\le z\le 1\}$

7.5E.2 Exercise $7.5E.2$

1.

a. Let B be a cylindrical shell with inner radius a outer radius b, and height c where $0<a<b$ and $c>0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x,y,z)=f(r)+h(z)$, where f and h are differentiable functions. If

$$\int_a^b \bar{f}(r)dr=0 \tag{7.5E.2}$$

and $\bar{h}(0)=0$, where $\bar{f}$ and $\bar{h}$ are antiderivatives of f and h, respectively, show that

$$\iiint_B F(x,y,z)dV=2\pi c(b\bar{f}(b)-a\bar{f}(a))+\pi(b^2-a^2)\bar{h}(c). \tag{7.5E.3}$$

b. Use the previous result to show that

$$\iiint_B \left(z+sin\sqrt{x^2+y^2}\right)dx\,dy\,dz=6\pi^2(\pi-2), \tag{7.5E.4}$$

where B is a cylindrical shell with inner radius π outer radius 2π, and height 2.

2.

a. Let B be a cylindrical shell with inner radius a outer radius b and height c where $0<a<b$ and $c>0$. Assume that a function F defined on B can be expressed in cylindrical coordinates as F(x,y,z) = f(r) g(\theta) f(z)\), where $f,\ g$, and h are differentiable functions. If

$$\int_a^b \tilde{f}(r)dr=0, \tag{7.5E.5}$$

where $\tilde{f}$ is an antiderivative of f, show that

$$\iiint_B F(x,y,z)dV=[b\tilde{f}(b)-a\tilde{f}(a)][\tilde{g}(2\pi)-\tilde{g}(0)][\tilde{h}(c)-\tilde{h}(0)], \tag{7.5E.6}$$

where $\tilde{g}$ and $\tilde{h}$ are antiderivatives of g and h, respectively.

b. Use the previous result to show that

$$\iiint_B z\,sin\sqrt{x^2+y^2}\,dx\,dy\,dz=-12\pi^2, \tag{7.5E.7}$$

where B is a cylindrical shell with inner radius π outer radius 2π, and height 2.

7.5E.3 Exercise $7.5E.3$

In the following exercises, the boundaries of the solid E are given in cylindrical coordinates.

a. Express the region E in cylindrical coordinates.

b. Convert the integral

$$\iiint_E f(x,y,z)dV \tag{7.5E.8}$$

to cylindrical coordinates.

1. E is bounded by the right circular cylinder $r=4\,sin\,\theta$, the $r\theta$-plane, and the sphere $r^2+z^2=16$.

Answer

a. $E = \{(r,\theta,z)|0 \leq \theta \leq \pi,\ 0 \leq r \leq 4\ sin\ \theta,\ 0 \leq z \leq \sqrt{16-r^2}\}$

b.

$$\int_0^{\pi}\int_0^{4\ sin\ \theta}\int_0^{\sqrt{16-r^2}} f(r,\theta,z)r\ dz\ dr\ d\theta \tag{7.5E.9}$$

2. E is bounded by the right circular cylinder $r = cos\ \theta$, the $r\theta$-plane, and the sphere $r^2 + z^2 = 9$.

3. E is located in the first octant and is bounded by the circular paraboloid $z = 9 - 3r^2$, the cylinder $r = \sqrt{r}$, and the plane $r(cos\ \theta + sin\ \theta) = 20 - z$.

Answer

a. \(E = \{(r,\theta,z) |0 \leq \theta \leq \frac{\pi}{2}, \space 0 \leq r \leq \sqrt{3}, \space 9 - r^2 \leq z \leq 10 - r(cos \space \theta + sin \space \theta)\};

b.

$$\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_{9-r^2}^{10-r(cos\ \theta+sin\ \theta)} f(r,\theta,z)r\ dz\ dr\ d\theta \tag{7.5E.10}$$

4. E is located in the first octant outside the circular paraboloid $z = 10 - 2r^2$ and inside the cylinder $r = \sqrt{5}$ and is bounded also by the planes $z = 20$ and $\theta = \frac{\pi}{4}$.

7.5E.4 Exercise $7.5E.4$

In the following exercises, the function f and region E are given.

a. Express the region E and the function f in cylindrical coordinates.

b. Convert the integral

$$\iiint_B f(x,y,z)dV \tag{7.5E.11}$$

into cylindrical coordinates and evaluate it.

1. $f(x,y,z) = x^2 + y^2$, $E = \{(x,y,z)|0 \leq x^2 + y^2 \leq 9,\ x \geq 0,\ y \geq 0,\ 0 \leq z \leq x + 3\}$

Answer

a. $E = \{(r,\theta,z)|0 \leq r \leq 3,\ 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq z \leq r\ cos\ \theta + 3\},$ f(r,\theta,z) = \frac{1}{r \space cos \space \theta + 3};

b.

$$\int_0^3\int_0^{\pi/2}\int_0^{r\ cos\ \theta+3} \frac{r}{r\ cos\ \theta + 3} dz\ d\theta\ dr = \frac{9\pi}{4} \tag{7.5E.12}$$

2. $f(x,y,z) = x^2 + y^2,\ E = \{(x,y,z)|0 \leq x^2 + y^2 \leq 4,\ y \geq 0,\ 0 \leq z \leq 3 - x$

$f(x,y,z) = x,\ E = \{(x,y,z)|1 \leq y^2 + z^2 \leq 9,\ 0 \leq x \leq 1 - y^2 - z^2\}$

Answer

a. $y = r\ cos\ \theta,\ z = r\ sin\ \theta,\ x = z,\ E = \{(r,\theta,z)|1 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1 - r^2\},\ f(r,\theta,z) = z$;

b.

$$\int_1^3\int_0^{2\pi}\int_0^{1-r^2} zr\ dz\ d\theta\ dr = \frac{356\pi}{3} \tag{7.5E.13}$$

3. $f(x,y,z) = y,\ E = \{(x,y,z)|1 \leq x^2 + z^2 \leq 9,\ 0 \leq y \leq 1 - x^2 - z^2\}$

7.5E.5 Exercise $7.5E.5$

In the following exercises, find the volume of the solid E whose boundaries are given in rectangular coordinates.

1. E is above the xy-plane, inside the cylinder $x^2 + y^2 = 1$, and below the plane $z = 1$.

Answer

π

2. E is below the plane $z = 1$ and inside the paraboloid $z = x^2 + y^2$.

3. E is bounded by the circular cone $z = \sqrt{x^2 + y^2}$ and $z = 1$.

Answer

$\frac{\pi}{3}$

4. E is located above the xy-plane, below $z = 1$, outside the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$, and inside the cylinder $x^2 + y^2 = 2$.

5. E is located inside the cylinder $x^2 + y^2 = 1$ and between the circular paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2$.

Answer

π

6. E is located inside the sphere $x^2 + y^2 + z^2 = 1$, above the xy-plane, and inside the circular cone $z = \sqrt{x^2 + y^2}$.

7. E is located outside the circular cone $x^2 + y^2 = (z - 1)^2$ and between the planes $z = 0$ and $z = 2$.

Answer

$\frac{4\pi}{3}$

8. E is located outside the circular cone $z = 1 - \sqrt{x^2 + y^2}$, above the xy-plane, below the circular paraboloid, and between the planes $z = 0$ and $z = 2$.

7.5E.6 Exercise $7.5E.6$

1. [T] Use a computer algebra system (CAS) to graph the solid whose volume is given by the iterated integral in cylindrical coordinates

$$\int_{-\pi/2}^{\pi/2}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta. \tag{7.5E.14}$$

Find the volume V of the solid. Round your answer to four decimal places.

Answer

$V=\frac{pi}{12}\approx 0.2618$

A quarter section of an ellipsoid with width 2, height 1, and depth 1.

2. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in cylindrical coordinates

$$\int_0^{\pi/2}\int_0^1\int_{r^4}^{r} r\,dz\,dr\,d\theta. \tag{7.5E.15}$$

Find the volume E of the solid Round your answer to four decimal places.

7.5E.7 Exercise $7.5E.7$

1. Convert the integral

$$\int_0^1\int_{-\sqrt{1-z^2}}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xz\,dz\,dx\,dy \tag{7.5E.16}$$

into an integral in cylindrical coordinates.

Answer

\[\int_0^1 \int_0^{\pi} \int_{r^2}^r zr^2 \space cos \space \theta \space dz \space d\theta \space dr\

2. Convert the integral

$$\int_0^2\int_0^x\int_0^1 (xy+z)dz\,dx\,dy \tag{7.5E.17}$$

into an integral in cylindrical coordinates.

7.5E.8 Exercise $7.5E.8$

In the following exercises, evaluate the triple integral

$$\iiint_B f(x,y,z)dV \tag{7.5E.18}$$

over the solid B.

1. $f(x,y,z)=1,\ B=\{(x,y,z)|x^2+y^2+z^2\le 90,\ z\ge 0\}$

A filled-in half-sphere with radius 3 times the square root of 10.

Answer

$180\pi\sqrt{10}$

2. $f(x,y,z)=1-\sqrt{x^2+y^2+z^2}\,,\ B=\{(x,y,z)|x^2+y^2+z^2\le 9,\ y\ge 0,\ z\ge 0\}$

A quarter section of an ovoid with height 8, width 8 and length 18.

3. $f(x,y,z)=\sqrt{x^2+y^2}\,,\ B$ is bounded above by the half-sphere $x^2+y^2+z^2=9$ with $z\ge 0$ and below by the cone $2z^2=x^2+y^2$.

Answer

$\frac{81\pi(\pi-2)}{16}$

4. $f(x,y,z)=\sqrt{x^2+y^2}\,,\ B$ is bounded above by the half-sphere $x^2+y^2+z^2=16$ with $z\ge 0$ and below by the cone $2z^2=x^2+y^2$.

7.5E.9 Exercise $7.5E.9$

Show that if $F(\rho,\theta,\varphi)=f(\rho)g(\theta)h(\varphi)$ is a continuous function on the spherical box $B=\{(\rho,\theta,\varphi)|a\le\rho\le b,\ \alpha\le\theta\le\beta,\ \gamma\le\varphi\le\psi\}$, then

$$\iiint_B F\,dV=\left(\int_a^b \rho^2 f(\rho)\,dr\right)\left(\int_\alpha^\beta g(\theta)\,d\theta\right)\left(\int_\gamma^\psi h(\varphi)\,sin\,\varphi\,d\varphi\right). \tag{7.5E.19}$$

7.5E.10 Exercise $7.5E.10$

1.

a. A function F is said to have spherical symmetry if it depends on the distance to the origin only, that is, it can be expressed in spherical coordinates as $F(x,y,z)=f(\rho),$ where $\rho=\sqrt{x^2+y^2+z^2}$. Show that

$$\iiint_B F(x,y,z)dV=2\pi\int_a^b \rho^2 f(\rho)d\rho, \tag{7.5E.20}$$

where B is the region between the upper concentric hemispheres of radii a and b centered at the origin, with $0<a<b$ and F a spherical function defined on B.

b. Use the previous result to show that

$$\iiint_B (x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}\,dV=21\pi, \tag{7.5E.21}$$

where $B=\{(x,y,z)|1\le x^2+y^2+z^2\le 2,\ z\ge 0\}$.

2.

a. Let B be the region between the upper concentric hemispheres of radii a and b centered at the origin and situated in the first octant, where $0 < a < b$. Consider F a function defined on B whose form in spherical coordinates (ρ, θ, φ) is $F(x, y, z) = f(\rho) cos\, \varphi$. Show that if $g(a) = g(b) = 0$ and

$$\int_a^b h(\rho) d\rho = 0, \tag{7.5E.22}$$

then

$$\iiint_B F(x, y, z) dV = \frac{\pi^2}{4}[ah(a) - bh(b)], \tag{7.5E.23}$$

where g is an antiderivative of f and h is an antiderivative of g.

b. Use the previous result to show that

$$\iiint_B = \frac{z\, cos \sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}} dV = \frac{3\pi^2}{2}, \tag{7.5E.24}$$

where B is the region between the upper concentric hemispheres of radii π and 2π centered at the origin and situated in the first octant.

7.5E.11 Exercise $\mathbf{7.5E.11}$

In the following exercises, the function f and region E are given.

a. Express the region E and function f in cylindrical coordinates.

b. Convert the integral

$$\iiint_B f(x, y, z) dV \tag{7.5E.25}$$

into cylindrical coordinates and evaluate it.

1. $f(x, y, z) = z;\ E = \{(x, y, z) | 0 \le x^2 + y^2 + z^2 \le 1,\ z \ge 0\}$
2. $f(x, y, z) = x + y;\ E = \{(x, y, z) | 1 \le x^2 + y^2 + z^2 \le 2,\ z \ge 0,\ y \ge 0\}$

Answer

a. $f(\rho, \theta, \varphi) = \rho\, sin\, \varphi\, (cos\, \theta + sin\, \theta),\ E = \{(\rho, \theta, \varphi) | 1 \le \rho \le 2,\ 0 \le \theta \le \pi,\ 0 \le \varphi \le \frac{\pi}{2}\}$;

b.

$$\int_0^\pi \int_0^{\pi/2} \int_1^2 \rho^3 cos\, \varphi\, sin\, \varphi\, d\rho\, d\varphi\, d\theta = \frac{15\pi}{8} \tag{7.5E.26}$$

3. $f(x, y, z) = 2xy;\ E = \{(x, y, z) | \sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2},\ x \ge 0,\ y \ge 0\}$
4. $f(x, y, z) = z;\ E = \{(x, y, z) | x^2 + y^2 + z^2 - 2x \le 0,\ \sqrt{x^2+y^2} \le z\}$

Answer

a. $f(\rho, \theta, \varphi) = \rho\, cos\, \varphi;\ E = \{(\rho, \theta, \varphi) | 0 \le \rho \le 2\, cos\, \varphi,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le \frac{\pi}{4}\}$;

b.

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{2\, cos\, \varphi} \rho^3 sin\, \varphi\, cos\, \varphi\, d\rho\, d\varphi\, d\theta = \frac{7\pi}{24} \tag{7.5E.27}$$

7.5E.12 Exercise $\mathbf{7.5E.12}$

In the following exercises, find the volume of the solid E whose boundaries are given in rectangular coordinates.

1. $E = \{(x, y, z) | \sqrt{x^2+y^2} \le z \le \sqrt{16-x^2-y^2},\ x \ge 0,\ y \ge 0\}$
2. $E = \{(x, y, z) | x^2 + y^2 + z^2 - 2z \le 0,\ \sqrt{x^2+y^2} \le z\}$

Answer

$\frac{\pi}{4}$

3. Use spherical coordinates to find the volume of the solid situated outside the sphere $\rho = 1$ and inside the sphere $\rho = cos\, \varphi$, with $\varphi \in [0, \frac{\pi}{2}]$.
4. Use spherical coordinates to find the volume of the ball $\rho \le 3$ that is situated between the cones $\varphi = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{3}$.

Answer

$9\pi(\sqrt{2} - 1)$

5. Convert the integral

$$\int_{-4} 64 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} (x^2+y^2+z^2)^2 dz\, dy\, dx \tag{7.5E.28}$$

into an integral in spherical coordinates.

Answer

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^4 \rho^6 sin\, \varphi\, d\theta \tag{7.5E.29}$$

6. Convert the integral

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dz\, dy\, dx \tag{7.5E.30}$$

into an integral in spherical coordinates and evaluate it.

7.5E.13 Exercise $7.5E.13$

1. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates

$$\int_{\pi/2}^{\pi}\int_{5\pi}^{\pi/6}\int_0^2 \rho^2 sin\,\varphi\, d\rho\, d\varphi\, d\theta. \tag{7.5E.31}$$

Find the volume V of the solid. Round your answer to three decimal places.

Answer

$V = \frac{4\pi\sqrt{3}}{3} \approx 7.255$

A sphere of radius 1 with a hole drilled into it of radius 0.5.

2. [T] Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates as

$$\int_0^{2\pi}\int_{3\pi/4}^{\pi/4}\int_0^1 \rho^2 sin\,\varphi\, d\rho\, d\varphi\, d\theta. \tag{7.5E.32}$$

Find the volume V of the solid. Round your answer to three decimal places.

3. [T] Use a CAS to evaluate the integral

$$\iiint_E (x^2+y^2)dV \tag{7.5E.33}$$

where E lies above the paraboloid $z = x^2+y^2$ and below the plane $z = 3y$.

Answer

$\frac{343\pi}{32}$

4. [T]

a. Evaluate the integral

$$\iiint_E e^{\sqrt{x^2+y^2+z^2}}dV, \tag{7.5E.34}$$

where E is bounded by spheres $4x^2+4y^2+4z^2 = 1$ and $x^2+y^2+z^2 = 1$.

b. Use a CAS to find an approximation of the previous integral. Round your answer to two decimal places.

7.5E.14 Exercise $7.5E.14$

Express the volume of the solid inside the sphere $x^2+y^2+z^2 = 16$ and outside the cylinder $x^2+y^2 = 4$ as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

7.5E.15 Exercise $7.5E.15$

1. The power emitted by an antenna has a power density per unit volume given in spherical coordinates by $p(\rho,\theta,\varphi) = \frac{P_0}{\rho^2}cos^2\theta\, sin^4\varphi$, where P_0 is a constant with units in watts. The total power wi

$$P = \iiint_B p(\rho,\theta,\varphi)\, dV. \tag{7.5E.35}$$

Find the total power P.

Answer

$P = \frac{32P_0\pi}{3}$ watts

2. Use the preceding exercise to find the total power within a sphere B of radius 5 meters when the power density per unit volume is given by $p(\rho,\theta,\varphi) = \frac{30}{\rho^2}cos^2\theta\, sin^4\varphi$.

3. A charge cloud contained in a sphere B of radius r centimeters centered at the origin has its charge density given by $q(x,y,z) = k\sqrt{x^2+y^2+z^2}\ \frac{\mu C}{cm^3}$, where $k > 0$.

The total charge contained in B is given by

$$Q = \iiint_B q(x,y,z)dV. \tag{7.5E.36}$$

Find the total charge Q.

Answer

$Q = kr^4\pi\mu C$

4. Use the preceding exercise to find the total charge cloud contained in the unit sphere if the charge density is $q(x,y,z) = 20\sqrt{x^2+y^2+z^2}\ \frac{\mu C}{cm^3}$.

7.6: Calculating Centers of Mass and Moments of Inertia

This page is a draft and is under active development.

We have already discussed a few applications of multiple integrals, such as finding areas, volumes, and the average value of a function over a bounded region. In this section we develop computational techniques for finding the center of mass and moments of inertia of several types of physical objects, using double integrals for a lamina (flat plate) and triple integrals for a three-dimensional object with variable density. The density is usually considered to be a constant number when the lamina or the object is homogeneous; that is, the object has uniform density.

7.6.1 Center of Mass in Two Dimensions

The center of mass is also known as the center of gravity if the object is in a uniform gravitational field. If the object has uniform density, the center of mass is the geometric center of the object, which is called the centroid. Figure $7.6.1$ shows a point P as the center of mass of a lamina. The lamina is perfectly balanced about its center of mass.

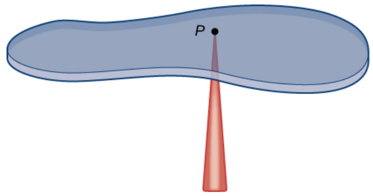


Figure $7.6.1$: A lamina is perfectly balanced on a spindle if the lamina's center of mass sits on the spindle.

To find the coordinates of the center of mass $P(\bar{x}, \bar{y})$ of a lamina, we need to find the moment M_x of the lamina about the x-axis and the moment M_y about the y-axis. We also need to find the mass m of the lamina. Then

$$\bar{x} = \frac{M_y}{m} \tag{7.6.1}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{7.6.2}$$

Refer to Moments and Centers of Mass for the definitions and the methods of single integration to find the center of mass of a one-dimensional object (for example, a thin rod). We are going to use a similar idea here except that the object is a two-dimensional lamina and we use a double integral.

If we allow a constant density function, then $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$ give the *centroid* of the lamina.

Suppose that the lamina occupies a region R in the xy-plane and let $\rho(x,y)$ be its density (in units of mass per unit area) at any point (x,y). Hence,

$$\rho(x,y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A} \tag{7.6.3}$$

where Δm and ΔA are the mass and area of a small rectangle containing the point (x,y) and the limit is taken as the dimensions of the rectangle go to 0 (see the following figure).

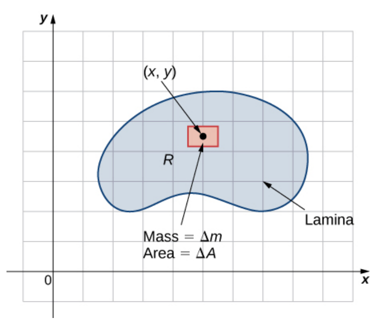


Figure $7.6.2$: The density of a lamina at a point is the limit of its mass per area in a small rectangle about the point as the area goes to zero.

Just as before, we divide the region R into tiny rectangles R_{ij} with area ΔA and choose (x^*_{ij}, y^*_{ij}) as sample points. Then the mass m_{ij} of each R_{ij} is equal to $\rho(x^*_{ij}, y^*_{ij})\Delta A$ (Figure $7.6.2$). Let k and l be the number of subintervals in x and y respectively. Also, note that the shape might not always be rectangular but the limit works anyway, as seen in previous sections.

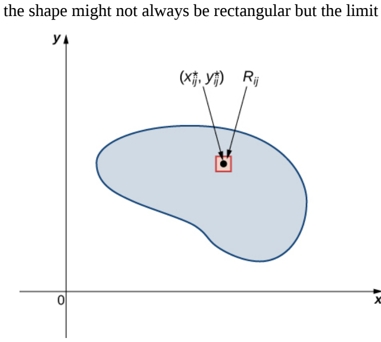


Figure $7.6.3$: Subdividing the lamina into tiny rectangles R_{ij} each containing a sample point (x^*_{ij}, y^*_{ij}).

Hence, the mass of the lamina is

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x^*_{ij}, y^*_{ij})\Delta A = \iint_R \rho(x,y)\,dA. \tag{7.6.4}$$

Let's see an example now of finding the total mass of a triangular lamina.

Example 7.6.1: Finding the Total Mass of a Lamina

Consider a triangular lamina R with vertices $(0,0),\ (0,3),\ (3,0)$and with density $\rho(x,y)=xy\ kg/m^2$. Find the total mass.

Solution

A sketch of the region R is always helpful, as shown in the following figure.

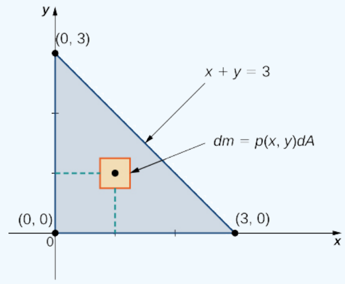


Figure 7.6.4: A lamina in the xy-plane with density $\rho(x,y)=xy$.

Using the expression developed for mass, we see that

$$m=\iint_R dm=\iint_R \rho(x,y)dA=\int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy\,dy\,dx=\int_{x=0}^{x=3}\left[x\frac{y^2}{2}\Big|_{y=0}^{y=3}\right]dx=\int_{x=0}^{x=3}\frac{1}{2}x(3-x)^2dx \tag{7.6.5}$$

$$=\left[\frac{9x^2}{4}-x^3+\frac{x^4}{8}\right]\Big|_{x=0}^{x=3}=\frac{27}{8}.$$

The computation is straightforward, giving the answer $m=\frac{27}{8}\,kg$.

Exercise 7.6.1

Consider the same region R as in the previous example, and use the density function $\rho(x,y)=\sqrt{xy}$. Find the total mass.

Answer

$\frac{9\pi}{8}\,kg$

Now that we have established the expression for mass, we have the tools we need for calculating moments and centers of mass. The moment M_z about the x-axis for R is the limit of the sums of moments of the regions R_{ij} about the x-axis. Hence

$$M_x=\lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)m_{ij}=\lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)\rho(x_{ij}^*,y_{ij}^*)\Delta A=\iint_R y\rho(x,y)dA \tag{7.6.6}$$

Similarly, the moment M_y about the y-axis for R is the limit of the sums of moments of the regions R_{ij} about the y-axis. Hence

$$M_x=\lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)m_{ij}=\lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)\rho(x_{ij}^*,y_{ij}^*)\Delta A=\iint_R x\rho(x,y)dA \tag{7.6.7}$$

Example 7.6.2: Finding Moments

Consider the same triangular lamina R with vertices $(0,0),\ (0,3),\ (3,0)$and with density $\rho(x,y)=xy$. Find the moments M_x and M_y.

Solution

Use double integrals for each moment and compute their values:

$$M_x=\iint_R y\rho(x,y)dA=\int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy^2\,dy\,dx=\frac{81}{20}, \tag{7.6.8}$$

$$M_y=\iint_R x\rho(x,y)dA=\int_{x=0}^{x=3}\int_{y=0}^{y=3-x} x^2y\,dy\,dx=\frac{81}{20}, \tag{7.6.9}$$

The computation is quite straightforward.

Exercise 7.6.2

Consider the same lamina R as above and use the density function $\rho(x,y)=\sqrt{xy}$. Find the moments M_x and M_y.

Answer

$M_x=\frac{81\pi}{64}$ and $M_y=\frac{81\pi}{64}$

Finally we are ready to restate the expressions for the center of mass in terms of integrals. We denote the x-coordinate of the center of mass by $\bar{x}$ and the y-coordinate by $\bar{y}$. Specifically,

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)dA}{\iint_R \rho(x,y)dA} \tag{7.6.10}$$

and

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)dA}{\iint_R \rho(x,y)dA} \tag{7.6.11}$$

Example 7.6.3: center of mass

Again consider the same triangular region R with vertices $(0,0)$, $(0,3)$, $(3,0)$and with density function $\rho(x,y)=xy$. Find the center of mass.

Solution

Using the formulas we developed, we have

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)dA}{\iint_R \rho(x,y)dA}=\frac{81/20}{27/8}=\frac{6}{5}, \tag{7.6.12}$$

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)dA}{\iint_R \rho(x,y)dA}=\frac{81/20}{27/8}=\frac{6}{5}. \tag{7.6.13}$$

Therefore, the center of mass is the point $\left(\frac{6}{5},\frac{6}{5}\right)$.

Analysis

If we choose the density $\rho(x,y)$ instead to be uniform throughout the region (i.e., constant), such as the value 1 (any constant will do), then we can compute the centroid,

$$x_c=\frac{M_y}{m}=\frac{\iint_R x\,dA}{\iint_R dA}=\frac{9/2}{9/2}=1, \tag{7.6.14}$$

$$y_c=\frac{M_x}{m}=\frac{\iint_R y\,dA}{\iint_R dA}=\frac{9/2}{9/2}=1. \tag{7.6.15}$$

Notice that the center of mass $\left(\frac{6}{5},\frac{6}{5}\right)$ is not exactly the same as the centroid $(1,1)$ of the triangular region. This is due to the variable density of R If the density is constant, then we just use $\rho(x,y)=c$ (constant). This value cancels out from the formulas, so for a constant density, the center of mass coincides with the centroid of the lamina.

Exercise 7.6.3

Again use the same region R as above and use the density function $\rho(x,y)=\sqrt{xy}$. Find the center of mass.

Answer

$\bar{x}=\frac{M_y}{m}=\frac{81\pi/64}{9\pi/8}=\frac{9}{8}$ and $\bar{y}=\frac{M_x}{m}=\frac{81\pi}{9\pi/8}=\frac{0}{8}$.

Once again, based on the comments at the end of Example $7.6.3$, we have expressions for the centroid of a region on the plane:

$$x_c=\frac{M_y}{m}=\frac{\iint_R x\,dA}{\iint_R dA} \text{ and } y_c=\frac{M_x}{m}=\frac{\iint_R y\,dA}{\iint_R dA}. \tag{7.6.16}$$

We should use these formulas and verify the centroid of the triangular region referred to in the last three examples.

Example 7.6.4: Finding Mass, Moments, and Center of Mass

Find the mass, moments, and the center of mass of the lamina of density $\rho(x,y)=x+y$ occupying the region R under the curve $y=x^2$ in the interval $0\leq x\leq 2$ (see the following figure).

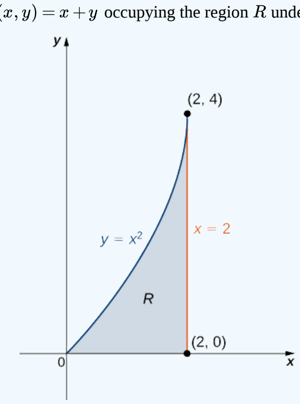


Figure 7.6.5: Locating the center of mass of a lamina R with density $\rho(x,y)=x+y$.

Solution

First we compute the mass m. We need to describe the region between the graph of $y=x^2$ and the vertical lines $x=0$ and $x=2$:

$$m=\iint_R dm=\iint_R \rho(x,y)dA=\int_{x=0} x=2\int_{y=0}^{y=x^2}(x+y)dy\,dx=\int_{x=0}^{x=2}\left[xy+\frac{y^2}{2}\Big|_{y=0}^{y=x^2}\right]dx \tag{7.6.17}$$

$$=\int_{x=0}^{x=2}\left[x^3+\frac{x^4}{2}\right]dx=\left[\frac{x^4}{4}+\frac{x^5}{10}\right]\Bigg|_{x=0}^{x=2}=\frac{36}{5}. \tag{7.6.18}$$

Now compute the moments M_x and M_y:

$$M_x=\iint_R y\rho(x,y)dA=\int_{x=0}^{x=2}\int_{y=0}^{y=x^2} y(x+y)dy\,dx=\frac{80}{7}, \tag{7.6.19}$$

$$M_y=\iint_R x\rho(x,y)dA=\int_{x=0}^{x=2}\int_{y=0}^{y=x^2} x(x+y)dy\,dx=\frac{176}{15}. \tag{7.6.20}$$

Finally, evaluate the center of mass,

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)dA}{\iint_R \rho(x,y)dA} = \frac{176/15}{36/5} = \frac{44}{27}, \tag{7.6.21}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)dA}{\iint_R \rho(x,y)dA} = \frac{80/7}{36/5} = \frac{100}{63}. \tag{7.6.22}$$

Hence the center of mass is $(\bar{x},\bar{y}) = \left(\frac{44}{27}, \frac{100}{63}\right)$.

Exercise 7.6.4

Calculate the mass, moments, and the center of mass of the region between the curves $y = x$ and $y = x^2$ with the density function $\rho(x,y) = x$ in the interval $0 \le x \le 1$.

Answer

$\bar{x} = \frac{M_y}{m} = \frac{1/20}{1/12} = \frac{3}{5}$ and $\bar{y} = \frac{M_x}{m} = \frac{1/24}{1/12} = \frac{1}{2}$

Example 7.6.5: Finding a Centroid

Find the centroid of the region under the curve $y = e^x$ over the interval $1 \le x \le 3$ (Figure 7.6.6).

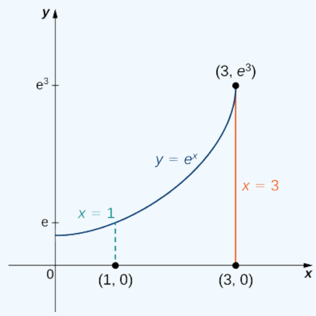


Figure 7.6.6: Finding a centroid of a region below the curve $y = e^x$.

Solution

To compute the centroid, we assume that the density function is constant and hence it cancels out:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \, and \, y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA}, \tag{7.6.23}$$

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} x\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} xe^x dx}{\int_{x=1}^{x=3} e^x dx} = \frac{2e^3}{e^3-e} = \frac{2e^2}{e^2-1}, \tag{7.6.24}$$

$$y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} y\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} \frac{e^{2x}}{2} dx}{\int_{x=1}^{x=3} e^x dx} = \frac{\frac{1}{4}e^2(e^4-1)}{e(e^2-1)} = \frac{1}{4}e(e^2+1). \tag{7.6.25}$$

Thus the centroid of the region is

$$(x_c, y_c) = \left(\frac{2e^2}{e^2-1}, \frac{1}{4}e(e^2+1)\right). \tag{7.6.26}$$

Exercise 7.6.5

Calculate the centroid of the region between the curves $y = x$ and $y = \sqrt{x}$ with uniform density in the interval $0 \le x \le 1$.

Answer

$x_c = \frac{M_y}{m} = \frac{1/15}{1/6} = \frac{2}{5}$ and $y_c = \frac{M_x}{m} = \frac{1/12}{1/6} = \frac{1}{2}$

7.6.2 Moments of Inertia

For a clear understanding of how to calculate moments of inertia using double integrals, we need to go back to the general definition in Section 6.6. The moment of inertia of a particle of mass m about a

The moment of inertia I_x about the x-axis for the region R is the limit of the sum of moments of inertia of the regions R_{ij} about the x-axis. Hence

$$I_x = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_R y^2\rho(x,y)dA. \tag{7.6.27}$$

Similarly, the moment of inertia I_y about the y-axis for R is the limit of the sum of moments of inertia of the regions R_{ij} about the y-axis. Hence

$$I_y = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_R x^2\rho(x,y)dA. \tag{7.6.28}$$

Sometimes, we need to find the moment of inertia of an object about the origin, which is known as the polar moment of inertia. We denote this by I_0 and obtain it by adding the moments of inertia I_x an

$$I_0 = I_x + I_y = \iint_R (x^2+y^2)\rho(x,y)dA. \tag{7.6.29}$$

All these expressions can be written in polar coordinates by substituting $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$. For example, $I_0 = \iint_R r^2\rho(r\cos\theta, r\sin\theta)dA$.

Example 7.6.6: Finding Moments of Inertia for a Triangular Lamina

Use the triangular region R with vertices $(0,0)$, $(2,2)$, and $(2,0)$ and with density $\rho(x,y)=xy$ as in previous examples. Find the moments of inertia.

Solution

Using the expressions established above for the moments of inertia, we have

$$I_x=\iint_R y^2\rho(x,y)dA=\int_{x=0}^{x=2}\int_{y=0}^{y=x} xy^3\,dy\,dx=\frac{8}{3}, \tag{7.6.30}$$

$$I_y=\iint_R x^2\rho(x,y)dA=\int_{x=0}^{x=2}\int_{y=0}^{y=x} x^3y dy\,dx=\frac{16}{3}, \tag{7.6.31}$$

$$I_0=\iint_R (x^2+y^2)\rho(x,y)dA=\int_0^2\int_0^x (x^2+y^2)xy\,dy\,dx=I_x+I_y=8 \tag{7.6.32}$$

Exercise 7.6.6

Again use the same region R as above and the density function $\rho(x,y)=\sqrt{xy}$. Find the moments of inertia.

Answer

$$I_x=\int_{x=0}^{x=2}\int_{y=0}^{y=x} y^2\sqrt{xy}\,dy\,dx=\frac{64}{35} \tag{7.6.33}$$

and

$$I_y=\int_{x=0}^{x=2}\int_{y=0}^{y=x} x^2\sqrt{xy}\,dy\,dx=\frac{64}{35}. \tag{7.6.34}$$

Also,

$$I_0=\int_{x=0}^{x=2}\int_{y=0}^{y=x} (x^2+y^2)\sqrt{xy}\,dy\,dx=\frac{128}{21} \tag{7.6.35}$$

As mentioned earlier, the moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis, also known as the **radius of gyration**.

Hence the radii of gyration with respect to the x-axis, the y-axis and the origin are

$$R_x=\sqrt{\frac{I_x}{m}},\ R_y=\sqrt{\frac{I_y}{m}},\ and\ R_0=\sqrt{\frac{I_0}{m}}, \tag{7.6.36}$$

respectively. In each case, the radius of gyration tells us how far (perpendicular distance) from the axis of rotation the entire mass of an object might be concentrated. The moments of an object are usefu

Example 7.6.7: Finding the Radius of Gyration for a Triangular Lamina

Consider the same triangular lamina R with vertices $(0,0)$, $(2,2)$, and $(2,0)$ and with density $\rho(x,y)=xy$ as in previous examples. Find the radii of gyration with respect to the x-axis the y-axis and

Solution

If we compute the mass of this region we find that $m=2$. We found the moments of inertia of this lamina in Example $7.6.4$. From these data, the radii of gyration with respect to the x-axis, y-axis an

$$R_x=\sqrt{\frac{I_x}{m}}=\sqrt{\frac{8/3}{2}}=\sqrt{\frac{8}{6}}=\frac{2\sqrt{3}}{3}, \tag{7.6.37}$$

$$R_y=\sqrt{\frac{I_y}{m}}=\sqrt{\frac{16/3}{2}}=\sqrt{\frac{8}{3}}=\frac{2\sqrt{6}}{3}, \tag{7.6.38}$$

$$R_0=\sqrt{\frac{I_0}{m}}=\sqrt{\frac{8}{2}}=\sqrt{4}=2. \tag{7.6.39}$$

Exercise 7.6.7

Use the same region R from Example $7.6.7$ and the density function $\rho(x,y)=\sqrt{xy}$. Find the radii of gyration with respect to the x-axis, the y-axis, and the origin.

Hint

Follow the steps shown in the previous example.

Answer

$R_x=\frac{6\sqrt{35}}{35}$, $R_y=\frac{6\sqrt{15}}{15}$, and $R_0=\frac{4\sqrt{42}}{7}$.

7.6.3 Center of Mass and Moments of Inertia in Three Dimensions

All the expressions of double integrals discussed so far can be modified to become triple integrals.

Definition

If we have a solid object Q with a density function $\rho(x,y,z)$ at any point (x,y,z) in space, then its mass is

$$m=\iiint_Q \rho(x,y,z)dV. \tag{7.6.40}$$

Its moments about the xy-plane the xz-plane and the yz-plane are

$$M_{xy}=\iiint_Q z\rho(x,y,z)dV,\ M_{xz}=\iiint_Q y\rho(x,y,z)dV,\ M_{yz}=\iiint_Q x\rho(x,y,z)dV. \tag{7.6.41}$$

If the center of mass of the object is the point $(\bar{x},\bar{y},\bar{z})$, then

$$\bar{x}=\frac{M_{yz}}{m},\ \bar{y}=\frac{M_{xz}}{m},\ \bar{z}=\frac{M_{xy}}{m}. \tag{7.6.42}$$

Also, if the solid object is homogeneous (with constant density), then the center of mass becomes the centroid of the solid. Finally, the moments of inertia about the yz-plane, xz-plane, and the xy-plai

$$I_x = \iiint_Q (y^2 + z^2)\,\rho(x,y,z)\,dV, \tag{7.6.43}$$

$$I_y = \iiint_Q (x^2 + z^2)\,\rho(x,y,z)\,dV, \tag{7.6.44}$$

$$I_z = \iiint_Q (x^2 + y^2)\,\rho(x,y,z)\,dV. \tag{7.6.45}$$

Example 7.6.8: Finding the Mass of a Solid

Suppose that Q is a solid region bounded by $x+2y+3z=6$ and the coordinate planes and has density $\rho(x,y,z)=x^2yz$. Find the total mass.

Solution

The region Q is a tetrahedron (Figure 7.6.7) meeting the axes at the points $(6,0,0)$, $(0,3,0)$,and $(0,0,2)$. To find the limits of integration, let $z=0$ in the slanted plane $z=\frac{1}{3}(6-x-2y)$. Then fc

$$m = \iiint_Q \rho(x,y,z)dV = \int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)} x^2yz\,dz\,dy\,dx = \frac{108}{35} \tag{7.6.46}$$

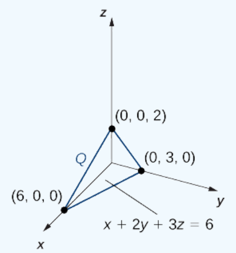


Figure 7.6.7: Finding the mass of a three-dimensional solid Q.

Exercise 7.6.8

Consider the same region Q (Figure 7.6.7), and use the density function $\rho(x,y,z)=xy^2z$. Find the mass.

Hint

Follow the steps in the previous example.

Answer

$\frac{54}{35} = 1.543$

Example 7.6.9: Finding the Center of Mass of a Solid

Suppose Q is a solid region bounded by the plane $x+2y+3z=6$ and the coordinate planes with density $\rho(x,y,z)=x^2yz$ (see Figure 7.6.7). Find the center of mass using decimal approximation.

Solution

We have used this tetrahedron before and know the limits of integration, so we can proceed to the computations right away. First, we need to find the moments about the xy-plane, the xz-plane, and th

$$M_{xy} = \iiint_Q z\rho(x,y,z)dV = \int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)} x^2yz^2\,dz\,dy\,dx = \frac{54}{35} \approx 1.543, \tag{7.6.47}$$

$$M_{xz} = \iiint_Q y\rho(x,y,z)dV = \int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)} x^2y^2z\,dz\,dy\,dx = \frac{81}{35} \approx 2.314, \tag{7.6.48}$$

$$M_{yz} = \iiint_Q x\rho(x,y,z)dV = \int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)} x^3yz\,dz\,dy\,dx = \frac{243}{35} \approx 6.943. \tag{7.6.49}$$

Hence the center of mass is

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m}, \tag{7.6.50}$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{243/35}{108/35} = \frac{243}{108} = 2.25, \tag{7.6.51}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{81/35}{108/35} = \frac{81}{108} = 0.75, \tag{7.6.52}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{54/35}{108/35} = \frac{54}{108} = 0.5 \tag{7.6.53}$$

The center of mass for the tetrahedron Q is the point $(2.25, 0.75, 0.5)$

Exercise 7.6.9

Consider the same region Q (Figure 7.6.7) and use the density function $\rho(x,y,z)=xy^2z$. Find the center of mass.

Hint

Check that $M_{xy} = \frac{27}{35}$, $M_{xz} = \frac{243}{140}$, and $M_{yz} = \frac{81}{35}$. Then use m from a previous checkpoint question.

Answer

$\left(\frac{3}{2}, \frac{9}{8}, \frac{1}{2}\right)$

We conclude this section with an example of finding moments of inertia I_x, I_y, and I_z.

Example 7.6.10: Finding the Moments of Inertia of a Solid

Suppose that Q is a solid region and is bounded by $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2yz$ (see Figure 7.6.7). Find the moments of inertia of the tetrahedron Q abo

Solution

Once again, we can almost immediately write the limits of integration and hence we can quickly proceed to evaluating the moments of inertia. Using the formula stated before, the moments of inertia o

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)dV, \tag{7.6.54}$$

$$I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z)dV, \tag{7.6.55}$$

and

$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)dV \; with \; \rho(x, y, z) = x^2yz. \tag{7.6.56}$$

Proceeding with the computations, we have

$$\begin{aligned} I_x &= \iiint_Q (y^2 + z^2)x^2\rho(x, y, z)dV \\ &= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (y^2 + z^2)x^2yz \, dz \, dy \, dx = \frac{117}{35} \approx 3.343, \\ I_y &= \iiint_Q (x^2 + z^2)x^2\rho(x, y, z)dV \\ &= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2 + z^2)x^2yz \, dz \, dy \, dx = \frac{684}{35} \approx 19.543, \\ I_z &= \iiint_Q (x^2 + y^2)x^2\rho(x, y, z)dV \\ &= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2 + y^2)x^2yz \, dz \, dy \, dx = \frac{729}{35} \approx 20.829. \end{aligned}$$

Thus, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $117/35$, $684/35$, and $729/35$, respectively.

Exercise 7.6.10

Consider the same region Q (Figure 7.6.7), and use the density function $\rho(x, y, z) = xy^2z$. Find the moments of inertia about the three coordinate planes.

Answer

The moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $99/35$, $36/7$, and $243/35$, respectively.

7.6.4 Key Concepts

Finding the mass, center of mass, moments, and moments of inertia in double integrals:

- For a lamina R with a density function $\rho(x, y)$ at any point (x, y) in the plane, the mass is

$$m = \iint_R \rho(x, y)dA. \tag{7.6.57}$$

- The moments about the x-axis and y-axis are

$$M_x = \iint_R y\rho(x, y)dA \; and \; M_y = \iint_R x\rho(x, y)dA. \tag{7.6.58}$$

- The center of mass is given by $\bar{x} = \frac{M_y}{m}$, $\bar{y} = \frac{M_x}{m}$.
- The center of mass becomes the centroid of the plane when the density is constant.
- The moments of inertia about the x-axis, y-axis, and the origin are

$$I_x = \iint_R y^2\rho(x, y)dA, \; I_y = \iint_R x^2\rho(x, y)dA, \; and \; I_0 = I_x + I_y = \iint_R (x^2 + y^2)\rho(x, y)dA. \tag{7.6.59}$$

Finding the mass, center of mass, moments, and moments of inertia in triple integrals:

- For a solid object Q with a density function $\rho(x, y, z)$ at any point (x, y, z) in space, the mass is

$$m = \iiint_Q \rho(x, y, z)dV. \tag{7.6.60}$$

- The moments about the xy-plane, the xz-plane, and the yz-plane are

$$M_{xy} = \iiint_Q z\rho(x, y, z)dV, \; M_{xz} = \iiint_Q y\rho(x, y, z)dV, \; M_{yz} = \iiint_Q x\rho(x, y, z)dV \tag{7.6.61}$$

- The center of mass is given by $\bar{x} = \frac{M_{yz}}{m}$, $\bar{y} = \frac{M_{xz}}{m}$, $\bar{z} = \frac{M_{xy}}{m}$.
- The center of mass becomes the centroid of the solid when the density is constant.
- The moments of inertia about the yz-plane, the xz-plane, and the xy-plane are

$$I_x = \iiint_Q (y^2 + z^2)\,\rho(x,y,z)\,dV,\ I_y = \iiint_Q (x^2 + z^2)\,\rho(x,y,z)\,dV,\ I_z = \iiint_Q (x^2 + y^2)\,\rho(x,y,z)\,dV. \tag{7.6.62}$$

7.6.5 Key Equations

- **Mass of a lamina**

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x^8_{ij}, y^*_{ij})\Delta A = \iint_R \rho(x,y)dA \tag{7.6.63}$$

- **Moment about the *x*-axis**

$$M_x = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y^*_{ij}) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y^*_{ij})\rho(x^8_{ij}, y^*_{ij})\Delta A = \iint_R y\rho(x,y)dA \tag{7.6.64}$$

- **Moment about the *y*-axis**

$$M_y = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x^*_{ij}) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x^*_{ij})\rho(x^8_{ij}, y^*_{ij})\Delta A = \iint_R x\rho(x,y)dA \tag{7.6.65}$$

- **Center of mass of a lamina**

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)dA}{\iint_R \rho(x,y)dA}\ and\ \bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)dA}{\iint_R \rho(x,y)dA} \tag{7.6.66}$$

7.6.5.1 Glossary

radius of gyration

the distance from an object’s center of mass to its axis of rotation

7.6.6 Contributors and Attributions

7.6E

1 Exercise 1: Mass

In the following exercises, the region R occupied by a lamina is shown in a graph. Find the mass of R with the density function ρ.

1. R is the triangular region with vertices $(0,0),\ (0,3)$, and $(6,0);\ \rho(x,y)=xy$.

A right triangle bounded by the x and y axes and the line y = negative x/2 + 3.

Answer

$\frac{27}{2}$

3. R is the triangular region with vertices $(0,0),\ (1,1)$, and $(0,5);\ \rho(x,y)=x+y$.

A triangle bounded by the y axis, the line x = y, and the line y = negative 4x + 5.

4. R is the rectangular region with vertices $(0,0),\ (0,3),\ (6,3)$ and $(6,0);\ \rho(x,y)=\sqrt{xy}$.

A rectangle bounded by the x and y axes and the lines x = 6 and y = 3.

Answer

$24\sqrt{2}$

5. R is the rectangular region with vertices $(0,1),\ (0,3),\ (3,3)$ and $(3,1);\ \rho(x,y)=x^2y$.

A rectangle bounded by the y axis, the lines y = 1 and 3, and the line x = 3.

6. R is the trapezoidal region determined by the lines $y=-\frac{1}{4}x+\frac{5}{2},\ y=0,\ y=2$, and $x=0;\ \rho(x,y)=3xy$.

A trapezoid bounded by the x and y axes, the line y = 2, and the line y = negative x/4 + 2.5.

Answer

76

7. R is the trapezoidal region determined by the lines $y=0,\ y=1,\ y=x$ and $y=-x+3;\ \rho(x,y)=2x+y$.

A trapezoid bounded by the x axis, the line y = 1, the line y = x, and the line y = negative x + 3.

8. R is the disk of radius 2 centered at $(1,2);\ \rho(x,y)=x^2+y^2-2x-4y+5$.

A circle with radius 2 centered at (1, 2), which is tangent to the x axis at (1, 0).

Answer

8π

10. R is the unit disk; $\rho(x,y)=3x^4+6x^2y^2+3y^4$.

A circle with radius 1 and center the origin.

11. R is the region enclosed by the ellipse $x^2+4y^2=1;\ \rho(x,y)=1$.

An ellipse with center the origin, major axis 2, and minor axis 0.5.

Answer

$\frac{\pi}{2}$

12. $R=\{(x,y)|9x^2+y^2\leq 1,\ x\geq 0,\ y\geq 0\};\ \rho(x,y)=\sqrt{9x^2+y^2}$.

The quarter section of an ellipse in the first quadrant with center the origin, major axis 2, and minor axis roughly 0.64.

13. R is the region bounded by $y=x,\ y=-x,\ y=x+2,\ y=-x+2;\ \rho(x,y)=1$.

A square with side length square root of 2 rotated 45 degrees, with corners at the origin, (2, 0), (1, 1), and (negative 1, 1).

Answer

2

14. R is the region bounded by $y=\frac{1}{x},\ y=\frac{2}{x},\ y=1$, and $y=2;\ \rho(x,y)=4(x+y)$.

A complex region between 2 and 1 that sweeps down and to the right with boundaries y = 1/x and y = 2/x.

2 Exercise 2 (CAS)

In the following exercises, consider a lamina occupying the region R and having the density function ρ given in the preceding group of exercises. Use a computer algebra system (CAS) to answer the

a. Find the moments M_x and M_y about the x-axis and y-axis, respectively.

b. Calculate and plot the center of mass of the lamina.

c. [T] Use a CAS to locate the center of mass on the graph of R.

1. [T] R is the triangular region with vertices $(0,0),\ (0,3)$, and $(6,0);\ \rho(x,y)=xy$.

Answer

a. $M_x=\frac{81}{5},\ M_y=\frac{162}{5}$; b. $\bar{x}=\frac{12}{5},\ \bar{y}=\frac{6}{5}$;

c.

A triangular region R bounded by the x and y axes and the line y = negative x/2 + 3, with a point marked at (12/5, 6/5).

2. [T] R is the triangular region with vertices $(0,0),\ (1,1)$, and $(0,5);\ \rho(x,y)=x+y$.

3. [T] R is the rectangular region with vertices $(0,0),\ (0,3),\ (6,3)$, and $(6,0);\ \rho(x,y)=\sqrt{xy}$.

Answer

a. $M_x = \frac{216\sqrt{2}}{5},\ M_y = \frac{432\sqrt{2}}{5}$; b. $\bar{x} = \frac{18}{5},\ \bar{y} = \frac{9}{5}$;

c.

A rectangle R bounded by the x and y axes and the lines x = 6 and y = 3 with point marked (18/5, 9/5).

4. [T] R is the rectangular region with vertices $(0,1),\ (0,3),\ (3,3)$, and $(3,1);\ \rho(x,y) = x^2y$.

[5. T] R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2},\ y = 0,\ y = 2$, and x = 0; \space \rho (x,y) = 3xy\).

Answer

a. $M_x = \frac{368}{5},\ M_y = \frac{1552}{5}$; b. $\bar{x} = \frac{92}{95},\ \bar{y} = \frac{388}{95}$;

c.

A trapezoid R bounded by the x and y axes, the line y = 2, and the line y = negative x/4 + 2.5 with the point marked (92/95, 388/95).

6. [T] R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$, and y = -x + 3; \space \rho (x,y) = 2x + y\).

7. [T] R is the disk of radius 2 centered at $(1,2);\ \rho(x,y) = x^2 + y^2 - 2x - 4y + 5$.

Answer

a. $M_x = 16\pi,\ M_y = 8\pi$; b. $\bar{x} = 1,\ \bar{y} = 2$;

c.

A circle with radius 2 centered at (1, 2), which is tangent to the x axis at (1, 0) and has pointed marked at the center (1, 2).

7. [T] R is the unit disk; $\rho(x,y) = 3x^4 + 6x^2y^2 + 3y^4$.

8. [T] R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x,y) = 1$.

Answer

a. $M_x = 0,\ M_y = 0)$; b. $\bar{x} = 0,\ \bar{y} = 0$;

c.

An ellipse R with center the origin, major axis 2, and minor axis 0.5, with point marked at the origin.

9. [T] $R = \{(x,y)|9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x,y) = \sqrt{9x^2 + y^2}$.

10. [T] R is the region bounded by $y = x,\ y = -x,\ y = x + 2$, and $y = -x + 2;\ \rho(x,y) = 1$.

Answer

a. $M_x = 2,\ M_y = 0)$; b. $\bar{x} = 0,\ \bar{y} = 1$;

c.

A square R with side length square root of 2 rotated 45 degrees, with corners at the origin, (2, 0), (1, 1), and (negative 1, 1). A point is marked at (0, 1).

11. [T] R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x,y) = 4(x+y)$.

3 Exercise 3

In the following exercises, consider a lamina occupying the region R and having the density function ρ given in the first two groups of Exercises.

a. Find the moments of inertia $I_x,\ I_y$ and I_0 about the x-axis, y-axis, and origin, respectively.

b. Find the radii of gyration with respect to the x-axis, y-axis, and origin, respectively.

1. R is the triangular region with vertices $(0,0),\ (0,3)$, and $(6,0);\ \rho(x,y) = xy$.

Answer

a. $I_x = \frac{243}{10},\ I_y = \frac{486}{5}$, and $I_0 = \frac{243}{2}$; b. $R_x = \frac{3\sqrt{5}}{5},\ R_y = \frac{6\sqrt{5}}{5}$, and $R_0 = 3$

2. R is the triangular region with vertices $(0,0),\ (1,1)$, and $(0,5);\ \rho(x,y) = x + y$.

3. R is the rectangular region with vertices $(0,0),\ (0,3),\ (6,3)$, and $(6,0);\ \rho(x,y) = \sqrt{xy}$.

Answer

a. $I_x = \frac{2592\sqrt{2}}{7},\ I_y = \frac{648\sqrt{2}}{7}$, and $I_0 = \frac{3240\sqrt{2}}{7}$; b. $R_x = \frac{6\sqrt{21}}{7},\ R_y = \frac{3\sqrt{21}}{7}$, and $R_0 = \frac{3\sqrt{106}}{7}$

4. R is the rectangular region with vertices $(0,1),\ (0,3),\ (3,3)$, and $(3,1);\ \rho(x,y) = x^2y$.

5. R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2},\ y = 0,\ y = 2$, and x = 0; \space \rho (x,y) = 3xy\).

Answer

a. $I_x = 88,\ I_y = 1560$, and $I_0 = 1648$; b. $R_x = \frac{\sqrt{418}}{19},\ R_y = \frac{\sqrt{7410}}{10}$, and $R_0 = \frac{2\sqrt{1957}}{19}$

6. R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x$, and y = -x + 3; \space \rho (x,y) = 2x + y\).

7. R is the disk of radius 2 centered at $(1,2);\ \rho(x,y) = x^2 + y^2 - 2x - 4y + 5$.

Answer

a. $I_x = \frac{128\pi}{3},\ I_y = \frac{56\pi}{3}$, and $I_0 = \frac{184\pi}{3}$; b. $R_x = \frac{4\sqrt{3}}{3},\ R_y = \frac{\sqrt{21}}{2}$, and $R_0 = \frac{\sqrt{69}}{3}$

8. R is the unit disk; $\rho(x,y) = 3x^4 + 6x^2y^2 + 3y^4$.

9. R is the region enclosed by the ellipse $x^2 + 4y^2 = 1;\ \rho(x,y) = 1$.

Answer

a. $I_x = \frac{\pi}{32},\ I_y = \frac{\pi}{8}$, and $I_0 = \frac{5\pi}{32}$; b. $R_x = \frac{1}{4},\ R_y = \frac{1}{2}$, and $R_0 = \frac{\sqrt{5}}{4}$

10. $R = \{(x, y) | 9x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\};\ \rho(x, y) = \sqrt{9x^2 + y^2}$.

11. R is the region bounded by $y = x,\ y = -x,\ y = x + 2$, and $y = -x + 2;\ \rho(x, y) = 1$.

Answer

a. $I_x = \frac{7}{3},\ I_y = \frac{1}{3}$, and $I_0 = \frac{8}{3}$; b. $R_x = \frac{\sqrt{42}}{6},\ R_y = \frac{\sqrt{6}}{6}$, and $R_0 = \frac{2\sqrt{3}}{3}$

12. R is the region bounded by $y = \frac{1}{x},\ y = \frac{2}{x},\ y = 1$, and $y = 2;\ \rho(x, y) = 4(x + y)$.

4 Exercise 4: Mass of a solid

1. Let Q be the solid unit cube. Find the mass of the solid if its density ρ is equal to the square of the distance of an arbitrary point of Q to the xy-plane.

Answer

$m = \frac{1}{3}$.

2. Let Q be the solid unit hemisphere. Find the mass of the solid if its density ρ is proportional to the distance of an arbitrary point of Q to the origin.

3. The solid Q of constant density 1 is situated inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the sphere $x^2 + y^2 + z^2 = 1$. Show that the center of mass of the solid is not located within the solid.

4. Find the mass of the solid $Q = \{(x, y, z) | 1 \le x^2 + z^2 \le 25,\ y \le 1 - x^2 - z^2\}$ whose density is $\rho(x, y, z) = k$, where $k > 0$.

5. [T] The solid $Q = \{(x, y, z) | x^2 + y^2 \le 9,\ 0 \le z \le 1,\ x \ge 0,\ y \ge 0\}$ has density equal to the distance to the xy-plane. Use a CAS to answer the following questions.

a. Find the mass of Q.

b. Find the moments $M_{xy},\ M_{xz}$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.

c. Find the center of mass of Q.

d. Graph Q and locate its center of mass.

Answer

a. $m = \frac{9\pi}{4}$; b. $M_{xy} = \frac{3\pi}{2},\ M_{xz} = \frac{81}{8},\ M_{yz} = \frac{81}{8}$; c. $\bar{x} = \frac{9}{2\pi},\ \bar{y} = \frac{9}{2\pi},\ \bar{z} = \frac{2}{3}$;

d.

A quarter cylinder in the first quadrant with height 1 and radius 3. A point is marked at (9/(2 pi), 9/(2 pi), 2/3).

6. Consider the solid $Q = \{(x, y, z) | 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$ with the density function $\rho(x, y, z) = x + y + 1$.

a. Find the mass of Q.

b. Find the moments $M_{xy},\ M_{xz}$ and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.

c. Find the center of mass of Q.

7. [T] The solid Q has the mass given by the triple integral

$$\int_{-1}^{1} \int_{0}^{\pi/4} \int_{0}^{1} r^2 dr\, d\theta\, dz. \tag{1}$$

8. Use a CAS to answer the following questions.

- Show that the center of mass of Q is located in the xy-plane.
- Graph Q and locate its center of mass.

$\bar{x} = \frac{3\sqrt{2}}{2\pi}$, $\bar{y} = \frac{3(2-\sqrt{2})}{2\pi},\ \bar{z} = 0$; 2. the solid Q and its center of mass are shown in the following figure.

A wedge from a cylinder in the first quadrant with height 2, radius 1, and angle roughly 45 degrees. A point is marked at (3 times the square root of 2/(2 pi), 3 times (2 minus the square root of 2)/(2 pi), 0).

9. The solid Q is bounded by the planes $x + 4y + z = 8,\ x = 0,\ y = 0$, and $z = 0$. Its density at any point is equal to the distance to the xz-plane. Find the moments of inertia of the solid about tl

10. The solid Q is bounded by the planes $x + y + z = 3,\ x = 0,\ y = 0$, and $z = 0$. Its density is $\rho(x, y, z) = x + ay$, where $a > 0$. Show that the center of mass of the solid is located in the plane z

11. Let Q be the solid situated outside the sphere $x^2 + y^2 + z^2 = z$ and inside the upper hemisphere $x^2 + y^2 + z^2 = R^2$, where $R > 1$. If the density of the solid is $\rho(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, find R s

12. The mass of a solid Q is given by

$$\int_{0}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)^n dz\, dy\, dx, \tag{2}$$

where n is an integer. Determine n such the mass of the solid is $(2 - \sqrt{2})\pi$.

Answer

$n = -2$

13. Let Q be the solid bounded above the cone $x^2 + y^2 = z^2$ and below the sphere $x^2 + y^2 + z^2 - 4z = 0$. Its density is a constant $k > 0$. Find k such that the center of mass of the solid is situated '

14. The solid $Q = \{(x, y, z) | 0 \le x^2 + y^2 \le 16,\ x \ge 0,\ y \ge 0,\ 0 \le z \le x\}$ has the density $\rho(x, y, z) = k$. Show that the moment M_{xy} about the xy-plane is half of the moment M_{yz} about the yz-p

15. The solid Q is bounded by the cylinder $x^2 + y^2 = a^2$, the paraboloid $b^2 - z = x^2 + y^2$, and the xy-plane, where $0 < a < b$. Find the mass of the solid if its density is given by $\rho(x, y, z) = \sqrt{x^2}$

16. Let Q be a solid of constant density k, where $k > 0$, that is located in the first octant, inside the circular cone $x^2 + y^2 = 9(z - 1)^2$, and above the plane $z = 0$. Show that the moment M_{xy} about t

17. The solid Q has the mass given by the triple integral

$$\int_{0}^{1} \int_{0}^{\pi/2} \int_{0}^{r^3} (r^4 + r)\, dz\, d\theta\, dr. \tag{3}$$

a. Find the density of the solid in rectangular coordinates.

b. Find the moment M_{xy} about the xy-plane.

18. The solid Q has the moment of inertia I_x about the yz-plane given by the triple integral

$$\int_{0}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\frac{1}{2}(x^2+y^2)}^{\sqrt{x^2+y^2}} (y^2 + z^2)(x^2 + y^2) dz\, dx\, dy. \tag{4}$$

a. Find the density of Q.

b. Find the moment of inertia I_z about the xy-plane.

Answer

a. $\rho(x, y, z) = x^2 + y^2$; b. $\frac{16\pi}{7}$

19. The solid Q has the mass given by the triple integral

$$\int_0^{\pi/4} \int_0^{2\,sec\,\theta} \int_0^1 (r^3 cos\,\theta\, sin\,\theta + 2r) dz\, dr\, d\theta. \tag{5}$$

a. Find the density of the solid in rectangular coordinates.

b. Find the moment M_{xz} about the xz-plane.

20. Let Q be the solid bounded by the xy-plane, the cylinder $x^2 + y^2 = a^2$, and the plane $z = 1$, where $a > 1$ is a real number. Find the moment M_{xy} of the solid about the xy-plane if its density give

Answer

$M_{xy} = \pi(f(0) - f(a) + af'(a))$

21. A solid Q has a volume given by \[\iint_D \int_a^b dA \space dz\), where D is the projection of the solid onto the xy-plane and $a < b$ are real numbers, and its density does not depend on the varia

22. Consider the solid enclosed by the cylinder $x^2 + z^2 = a^2$ and the planes $y = b$ and $y = c$, where $a > 0$ and $b < c$ are real numbers. The density of Q is given by $\rho(x, y, z) = f'(y)$, where f is a

23. [T] The average density of a solid Q is defined as

$$\rho_{ave} = \frac{1}{V(Q)} \iiint_Q \rho(x, y, z) dV = \frac{m}{V(Q)}, \tag{6}$$

where $V(Q)$ and m are the volume and the mass of Q, respectively. If the density of the unit ball centered at the origin is $\rho(x, y, z) = e^{-x^2-y^2-z^2}$, use a CAS to find its average density. Round your an

23. Show that the moments of inertia $I_x,\ I_y$, and I_z about the yz-plane, xz-plane, and xy-plane, respectively, of the unit ball centered at the origin whose density is $\rho(x, y, z) = e^{-x^2-y^2-z^2}$ are the sar

Answer

$I_x = I_y = I_z \approx 0.84$

7.7: Change of Variables in Multiple Integrals

This page is a draft and is under active development.

Recall from Substitution Rule the method of integration by substitution. When evaluating an integral such as

$$\int_2^3 x(x^2-4)^5 dx, \tag{7.7.1}$$

we substitute $u = g(x) = x^2 - 4$. Then $du = 2x\,dx$ or $x\,dx = \frac{1}{2}du$ and the limits change to $u = g(2) = 2^2 - 4 = 0$ and $u = g(3) = 9 - 4 = 5$. Thus the integral becomes

$$\int_0^5 \frac{1}{2}u^5 du \tag{7.7.2}$$

and this integral is much simpler to evaluate. In other words, when solving integration problems, we make appropriate substitutions to obtain an integral that becomes much simpler than the original integral.

We also used this idea when we transformed double integrals in rectangular coordinates to polar coordinates and transformed triple integrals in rectangular coordinates to cylindrical or spherical coordinates to make the computations simpler. More generally,

$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du, \tag{7.7.3}$$

Where $x = g(u)$, $dx = g'(u)du$, and $u = c$ and $u = d$ satisfy $c = g(a)$ and $d = g(b)$.

A similar result occurs in double integrals when we substitute

- $x = f(r, \theta) = r \cos\theta$
- $y = g(r, \theta) = r \sin\theta$, and
- $dA = dx\,dy = r\,dr\,d\theta$.

Then we get

$$\iint_R f(x,y)dA = \iint_S (r \cos\theta,\, r \sin\theta) r\,dr\,d\theta \tag{7.7.4}$$

where the domain R is replaced by the domain S in polar coordinates. Generally, the function that we use to change the variables to make the integration simpler is called a transformation or mapping.

7.7.1 Planar Transformations

A planar transformation T is a function that transforms a region G in one plane into a region R in another plane by a change of variables. Both G and R are subsets of R^2. For example, Figure $7.7.1$ shows a region G in the uv-plane transformed into a region R in the xy-plane by the change of variables $x = g(u, v)$ and $y = h(u, v)$, or sometimes we write $x = x(u, v)$ and $y = y(u, v)$. We shall typically assume that each of these functions has continuous first partial derivatives, which means g_u, g_v, h_u, and h_v exist and are also continuous. The need for this requirement will become clear soon.

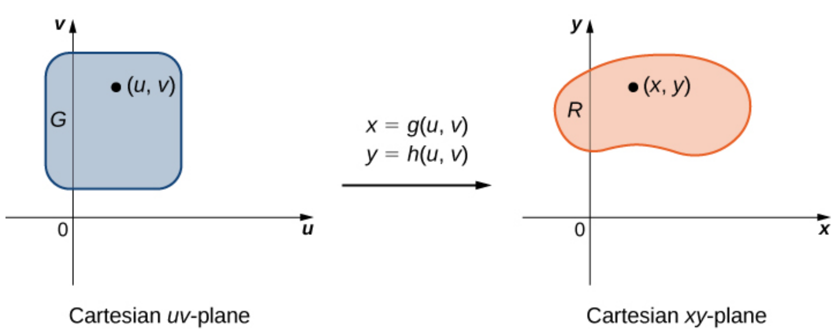


Figure $7.7.1$: The transformation of a region G in the uv-plane into a region R in the xy-plane.

Definition: one-to-one transformation

A transformation $T : G \to R$, defined as $T(u, v) = (x, y)$, is said to be a one-to-one transformation if no two points map to the same image point.

To show that T s a one-to-one transformation, we assume $T(u_1, v_1) = T(u_2, v_2)$ and show that as a consequence we obtain $(u_1, v_1) = (u_2, v_2)$. If the transformation T is one-to-one in the domain G, then the inverse T^{-1} exists with the domain R such that $T^{-1} \circ T$ and $T \circ T^{-1}$ are identity functions.

Figure $7.7.2$ shows the mapping $T(u, v) = (x, y)$ where x and y are related to u and v by the equations $x = g(u, v)$ and $y = h(u, v)$. The region G is the domain of T and the region R is the range of T, also known as the *image* of G under the transformation T.

Example 7.7.1A: Determining How the Transformation Works

Suppose a transformation T is defined as $T(r, \theta) = (x, y)$ where $x = r \cos \theta$, $y = r \sin \theta$. Find the image of the polar rectangle $G = \{(r, \theta) | 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2\}$ in the $r\theta$-plane to a region R in the xy-plane. Show that T is a one-to-one transformation in G and find $T^{-1}(x, y)$.

Solution

Since r varies from 0 to 1 in the $r\theta$-plane, we have a circular disc of radius 0 to 1 in the xy-plane. Because θ varies from 0 to $\pi/2$ in the $r\theta$-plane, we end up getting a quarter circle of radius 1 in the first quadrant of the xy-plane (Figure $7.7.2$). Hence R is a quarter circle bounded by $x^2 + y^2 = 1$ in the first quadrant.

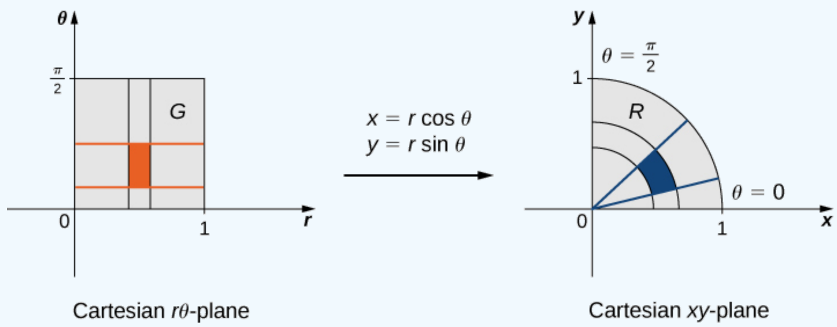


Figure $7.7.2$: A rectangle in the $r\theta$-plane is mapped into a quarter circle in the xy-plane.

In order to show that T is a one-to-one transformation, assume $T(r_1, \theta_1) = T(r_2, \theta_2)$ and show as a consequence that $(r_1, \theta_1) = (r_2, \theta_2)$. In this case, we have

$$T(r_1, \theta_1) = T(r_2, \theta_2), \tag{7.7.5}$$

$$(x_1, y_1) = (x_1, y_1), \tag{7.7.6}$$

$$(r_1 \cos \theta_1, r_1 \sin \theta_1) = (r_2 \cos \theta_2, r_2 \sin \theta_2), \tag{7.7.7}$$

$$r_1 \cos \theta_1 = r_2 \cos \theta_2, \; r_1 \sin \theta_1 = r_2 \sin \theta_2. \tag{7.7.8}$$

Dividing, we obtain

$$\frac{r_1 \cos \theta_1}{r_1 \sin \theta_1} = \frac{r_2 \cos \theta_2}{r_2 \sin \theta_2} \tag{7.7.9}$$

$$\frac{\cos \theta_1}{\sin \theta_1} = \frac{\cos \theta_2}{\sin \theta_2} \tag{7.7.10}$$

$$\tan \theta_1 = \tan \theta_2 \tag{7.7.11}$$

$$\theta_1 = \theta_2 \tag{7.7.12}$$

since the tangent function is one-one function in the interval $0 \leq \theta \leq \pi/2$. Also, since $0 \leq r \leq 1$, we have $r_1 = r_2$, $\theta_1 = \theta_2$. Therefore, $(r_1, \theta_1) = (r_2, \theta_2)$ and T is a one-to-one transformation from G to R.

To find $T^{-1}(x, y)$ solve for r, θ in terms of x, y. We already know that $r^2 = x^2 + y^2$ and $\tan \theta = \frac{y}{x}$. Thus $T^{-1}(x, y) = (r, \theta)$ is defined as $r = \sqrt{x^2 + y^2}$ and $\tan^{-1}\left(\frac{y}{x}\right)$.

Example 7.7.1B: Finding the Image under T

Let the transformation T be defined by $T(u,v)=(x,y)$ where $x=u^2-v^2$ and $y=uv$. Find the image of the triangle in the uv-plane with vertices $(0,0)$, $(0,1)$, and $(1,1)$.

Solution

The triangle and its image are shown in Figure 7.7.3. To understand how the sides of the triangle transform, call the side that joins $(0,0)$ and $(0,1)$ side A, the side that joins $(0,0)$ and $(1,1)$ side B, and the side that joins $(1,1)$ and $(0,1)$ side C.

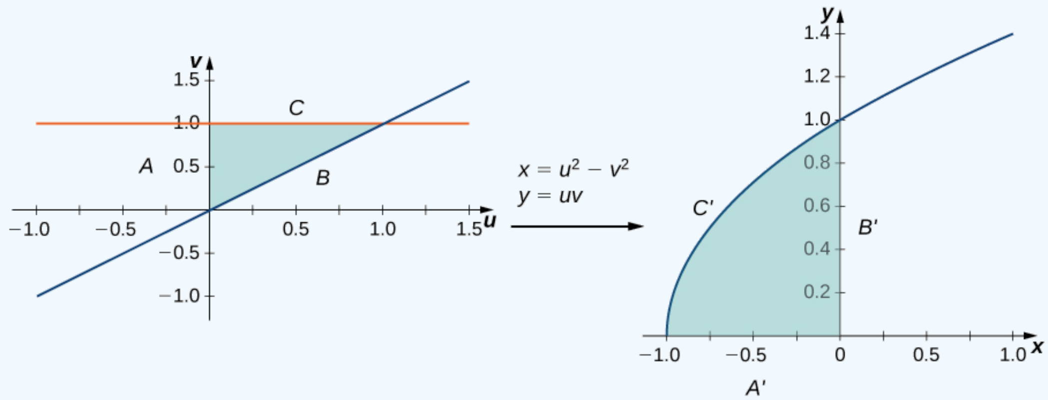


Figure 7.7.3: A triangular region in the uv-plane is transformed into an image in the xy-plane.

- For the side A: $u=0,\ 0\le v\le 1$ transforms to $x=-v^2,\ y=0$ so this is the side A' that joins $(-1,0)$ and $(0,0)$.
- For the side B: $u=v,\ 0\le u\le 1$ transforms to $x=0,\ y=u^2$ so this is the side B' that joins $(0,0)$ and $(0,1)$.
- For the side C: $0\le u\le 1,\ v=1$ transforms to $x=u^2-1,\ y=u$ (hence $x=y^2-1$ so this is the side C' that makes the upper half of the parabolic arc joining $(-1,0)$ and $(0,1)$.

All the points in the entire region of the triangle in the uv-plane are mapped inside the parabolic region in the xy-plane.

Exercise 7.7.1

Let a transformation T be defined as $T(u,v)=(x,y)$ where $x=u+v,\ y=3v$. Find the image of the rectangle $G=\{(u,v):\ 0\le u\le 1,\ 0\le v\le 2\}$ from the uv-plane after the transformation into a region R in the xy-plane. Show that T is a one-to-one transformation and find $T^{-1}(x,y)$.

Hint

Follow the steps of Example 7.7.1B

Answer

$T^{-1}(x,y)=(u,v)$ where $u=\frac{3x-y}{3}$ and $v=\frac{y}{3}$

7.7.2 Jacobians

Recall that we mentioned near the beginning of this section that each of the component functions must have continuous first partial derivatives, which means that g_u, g_v, h_u and h_v exist and are also continuous. A transformation that has this property is called a C^{-1} transformation (here C denotes continuous). Let $T(u,v)=(g(u,v),\ h(u,v))$, where $x=g(u,v)$ and $y=h(u,v)$ be a one-to-one C^1 transformation. We want to see how it transforms a small rectangular region S, Δu units by Δv units, in the uv-plane (Figure 7.7.4).

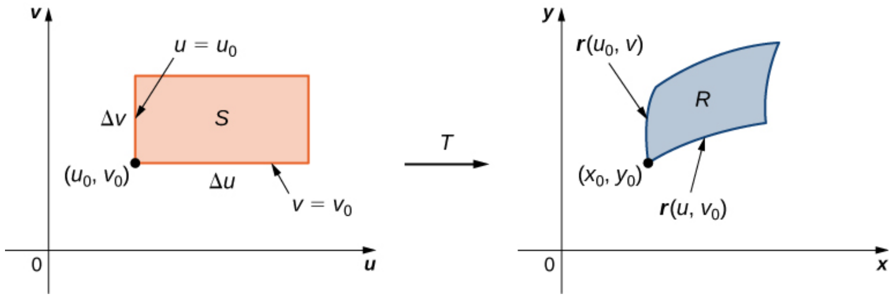


Figure 7.7.4: A small rectangle S in the uv-plane is transformed into a region R in the xy-plane.

Since $x = g(u,v)$ and $y = h(u,v)$, we have the position vector $r(u,v) = g(u,v)i + h(u,v)j$ of the image of the point (u,v). Suppose that $(u_0, v_0)$is the coordinate of the point at the lower left corner that mapped to $(x_0, y_0) = T(u_0, v_0)$ The line $v = v_0$ maps to the image curve with vector function $r(u, v_0)$, and the tangent vector at (x_0, y_0) to the image curve is

$$r_u = g_u(u_0, v_0)i + h_v(u_0, v_0)j = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j. \tag{7.7.13}$$

Similarly, the line $u = u_0$ maps to the image curve with vector function $r(u_0, v)$, and the tangent vector at (x_0, y_0) to the image curve is

$$r_v = g_v(u_0, v_0)i + h_u(u_0, v_0)j = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j. \tag{7.7.14}$$

Now, note that

$$r_u = \lim_{\Delta u \to 0} \frac{r(u_0 + \Delta u, v_0) - r(u_0, v_0)}{\Delta u} so r(u_0 + \Delta u, v_0) - r(u_0, v_0) \approx \Delta u r_u. \tag{7.7.15}$$

Similarly,

$$r_v = \lim_{\Delta v \to 0} \frac{r(u_0, v_0 + \Delta v) - r(u_0, v_0)}{\Delta v} so r(u_0, v_0 + \Delta v) - r(u_0, v_0) \approx \Delta v r_v. \tag{7.7.16}$$

This allows us to estimate the area ΔA of the image R by finding the area of the parallelogram formed by the sides $\Delta v r_v$ and $\Delta u r_u$. By using the cross product of these two vectors by adding the **k**th component as 0, the area ΔA of the image R (refer to The Cross Product) is approximately $|\Delta u r_u \times \Delta v r_v| = |r_u \times r_v| \Delta u \Delta v$. In determinant form, the cross product is

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} k = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right)$$

Since $|k| = 1$, we have

$\Delta A \approx |r_u \times r_v| \Delta u \Delta v = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \Delta u \Delta v.$

Definition: Jacobian

The *Jacobian* of the C^1 transformation $T(u,v) = (g(u,v),\ h(u,v))$ is denoted by $J(u,v)$ and is defined by the 2×2 determinant

$$J(u,v) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right).$$

Using the definition, we have

$$\Delta A \approx J(u,v) \Delta u \Delta v = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v. \tag{7.7.17}$$

Note that the Jacobian is frequently denoted simply by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}. \tag{7.7.18}$$

Note also that

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Hence the notation $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$ suggests that we can write the Jacobian determinant with partials of x in the first row and partials of y in the second row.

Example 7.7.2A: Finding the Jacobian

Find the Jacobian of the transformation given in Example 7.7.1A.

Solution

The transformation in the example is $T(r,\theta) = (r\,\cos\,\theta,\, r\,\sin\,\theta)$ where $x = r\,\cos\,\theta$ and $y = r\,\sin\,\theta$. Thus the Jacobian is

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\,\cos^2\theta + r\,\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r.$$

Example 7.7.2B: Finding the Jacobian

Find the Jacobian of the transformation given in Example 7.7.1B.

Solution

The transformation in the example is $T(u,v) = (u^2 - v^2, uv)$ where $x = u^2 - v^2$ and $y = uv$. Thus the Jacobian is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & v \\ -2v & u \end{vmatrix} = 2u^2 + 2v^2.$$

Exercise 7.7.2

Find the Jacobian of the transformation given in the previous checkpoint: $T(u,v) = (u+v, 2v)$.

Hint

Follow the steps in the previous two examples.

Answer

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

7.7.3 Change of Variables for Double Integrals

We have already seen that, under the change of variables $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$, a small region ΔA in the xy-plane is related to the area formed by the product $\Delta u \Delta v$ in the uv-plane by the approximation

$$\Delta A \approx J(u,v)\Delta u,\ \Delta v. \tag{7.7.19}$$

Now let's go back to the definition of double integral for a minute:

$$\iint_R f(x,y) f A = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ijh})\Delta A. \tag{7.7.20}$$

Referring to Figure $7.7.5$, observe that we divided the region S in the uv-plane into small subrectangles S_{ij} and we let the subrectangles R_{ij} in the xy-plane be the images of S_{ij} under the transformation $T(u,v) = (x,y)$.

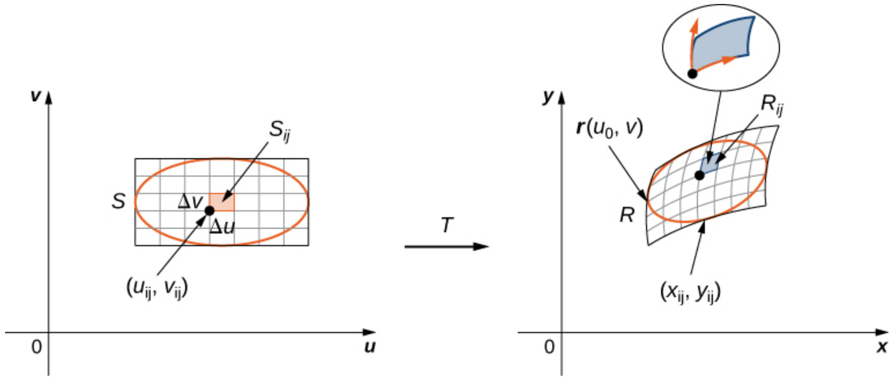


Figure $7.7.5$: The subrectangles S_{ij} in the uv-plane transform into subrectangles R_{ij} in the xy-plane.

Then the double integral becomes

$$\iint_R = f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_{ij}, v_{ij}),\ h(u_{ij}, v_{ij}))|J(u_{ij}, v_{ij})|\Delta u \Delta v. \tag{7.7.21}$$

Notice this is exactly the double Riemann sum for the integral

$$\iint_S f(g(u,v),\ h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv. \tag{7.7.22}$$

Change of Variables for Double Integrals

Let $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$ be a one-to-one C^1 transformation, with a nonzero Jacobian on the interior of the region S in the uv-plane it maps S into the region R in the xy-plane. If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\ h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv. \tag{7.7.23}$$

With this theorem for double integrals, we can change the variables from (x,y) to (u,v) in a double integral simply by replacing

$$dA = dx\, dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \tag{7.7.24}$$

when we use the substitutions $x = g(u,v)$ and $y = h(u,v)$ and then change the limits of integration accordingly. This change of variables often makes any computations much simpler.

Example 7.7.3: Changing Variables from Rectangular to Polar Coordinates

Consider the integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx. \tag{7.7.25}$$

Use the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, and find the resulting integral.

Solution

First we need to find the region of integration. This region is bounded below by $y = 0$ and above by $y = \sqrt{2x - x^2}$ (Figure $7.7.6$).

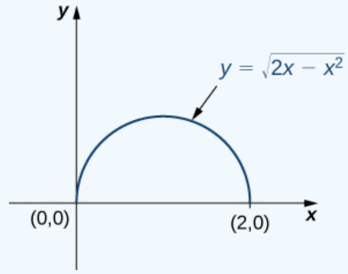


Figure 7.7.6: Changing a region from rectangular to polar coordinates.

Squaring and collecting terms, we find that the region is the upper half of the circle $x^2 + y^2 - 2x = 0$, that is $y^2 + (x-1)^2 = 1$. In polar coordinates, the circle is $r = 2\,cos\,\theta$ so the region of integration in polar coordinates is bounded by $0 \leq r \leq \cos\,\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$.

The Jacobian is $J(r,\theta) = r$, as shown in Example 7.7.2A. Since $r \geq 0$, we have $|J(r,\theta)| = r$.

The integrand $\sqrt{x^2+y^2}$ changes to r in polar coordinates, so the double iterated integral is

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2} \int_0^{2\,cos\,\theta} r|j(r,\theta)|\,dr\,d\theta = \int_0^{\pi/2} \int_0^{2\,cos\,\theta} r^2\,dr\,d\theta. \tag{7.7.26}$$

Exercise 7.7.3

Considering the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)dy\,dx,$ use the change of variables $x = r\,cos\,\theta$ and $y = r\,sin\,\theta$ and find the resulting integral.

Hint

Follow the steps in the previous example.

Answer

$$\int_0^{\pi/2} \int_0^1 r^3\,dr\,d\theta \tag{7.7.27}$$

Notice in the next example that the region over which we are to integrate may suggest a suitable transformation for the integration. This is a common and important situation.

Example 7.7.4: Changing Variables

Consider the integral

$$\iint_R (x-y)dy\,dx, \tag{7.7.28}$$

where R is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$ and $(6,5)$ (Figure 7.7.7). Make appropriate changes of variables, and write the resulting integral.

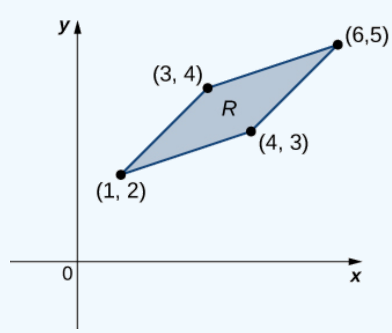


Figure 7.7.7: The region of integration for the given integral.

Solution

First, we need to understand the region over which we are to integrate. The sides of the parallelogram are $x-y+1,\ x-y-1=0,\ x-3y+5=0$ and $x-3y+9=0$ (Figure 7.7.8). Another way to look at them is $x-y=-1,\ x-y=1,\ x-3y=-5$, and $x-3y=9$.

Clearly the parallelogram is bounded by the lines $y=x+1,\ y=x-1,\ y=\frac{1}{3}(x+5)$, and $y=\frac{1}{3}(x+9)$.

Notice that if we were to make $u=x-y$ and $v=x-3y$, then the limits on the integral would be $-1\leq u\leq 1$ and $-9\leq v\leq -5$.

To solve for x and y, we multiply the first equation by 3 and subtract the second equation, $3u-v=(3x-3y)-(x-3y)=2x$. Then we have $x=\frac{3u-v}{2}$. Moreover, if we simply subtract the second equation from the first, we get $u-v=(x-y)-(x-3y)=2y$ and $y=\frac{u-v}{2}$.

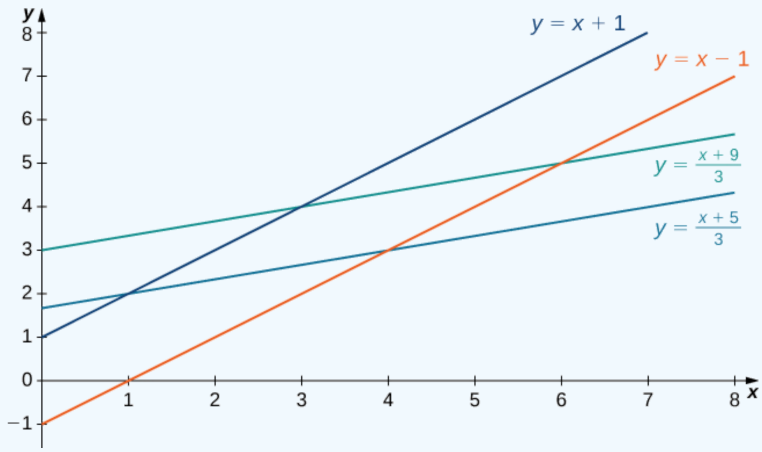


Figure 7.7.8: A parallelogram in the xy-plane that we want to transform by a change in variables.

Thus, we can choose the transformation

$$T(u,v)=\left(\frac{3u-v}{2},\frac{u-v}{2}\right) \tag{7.7.29}$$

and compute the Jacobian $J(u,v)$. We have

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}3/2 & -1/2\\ 1/2 & -1/2\end{vmatrix}=-\frac{3}{4}+\frac{1}{4}=-\frac{1}{2}$$

Therefore, $|J(u,v)|=\frac{1}{2}$. Also, the original integrand becomes

$$x-y=\frac{1}{2}[3u-v-u+v]=\frac{1}{2}[3u-u]=\frac{1}{2}[2u]=u. \tag{7.7.30}$$

Therefore, by the use of the transformation T, the integral changes to

$$\iint_R (x-y)dy\,dx=\int_{-9}^{-5}\int_{-1}^{1}J(u,v)u\,du\,dv=\int_{-9}^{-5}\int_{-1}^{1}\left(\frac{1}{2}\right)u\,du\,dv, \tag{7.7.31}$$

which is much simpler to compute.

Exercise 7.7.4

Make appropriate changes of variables in the integral

$$\iint_R \frac{4}{(x-y)^2}dy\,dx, \tag{7.7.32}$$

where R is the trapezoid bounded by the lines $x-y=2,\ x-y=4,\ x=0$, and $y=0$. Write the resulting integral.

Hint

Follow the steps in the previous example.

Answer

$x = \frac{1}{2}(v+u)$ and $y = \frac{1}{2}(v-u)$

and

$$\int_{-4}^{4}\int_{-2}^{2}\frac{4}{u^2}\left(\frac{1}{2}\right) du\, dv. \tag{7.7.33}$$

We are ready to give a problem-solving strategy for change of variables.

Problem-Solving Strategy: Change of Variables

1. Sketch the region given by the problem in the xy-plane and then write the equations of the curves that form the boundary.
2. Depending on the region or the integrand, choose the transformations $x = g(u, v)$ and $y = h(u, v)$.
3. Determine the new limits of integration in the uv-plane.
4. Find the Jacobian $J(u, v)$.
5. In the integrand, replace the variables to obtain the new integrand.
6. Replace $dy\, dx$ or $dx\, dy$, whichever occurs, by $J(u, v) du\, dv$.

In the next example, we find a substitution that makes the integrand much simpler to compute.

Example 7.7.5: Evaluating an Integral

Using the change of variables $u = x - y$ and $v = x + y$, evaluate the integral

$$\iint_R (x-y)e^{x^2-y^2}\, dA, \tag{7.7.34}$$

where R is the region bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$ (see the first region in Figure 7.7.9).

Solution

As before, first find the region R and picture the transformation so it becomes easier to obtain the limits of integration after the transformations are made (Figure 7.7.9).

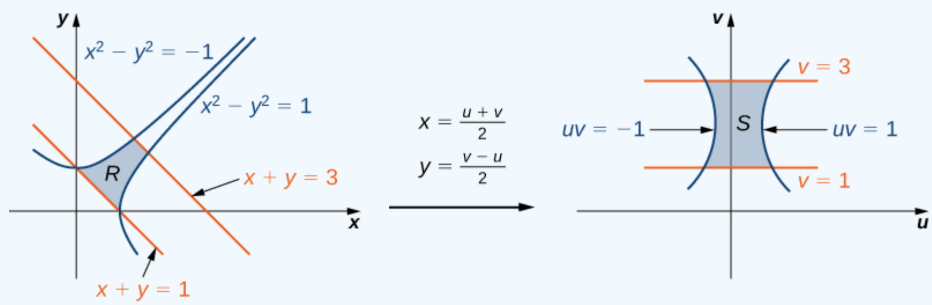


Figure 7.7.9: Transforming the region R into the region S to simplify the computation of an integral.

Given $u = x - y$ and $v = x + y$, we have $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$ and hence the transformation to use is $T(u, v) = \left(\frac{u+v}{2}, \frac{v-u}{2}\right)$. The lines $x + y = 1$ and $x + y = 3$ become $v = 1$ and $v = 3$, respectively. The curves $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ become $uv = 1$ and $uv = -1$, respectively.

Thus we can describe the region S (see the second region Figure 7.7.9) as

$$S = \left\{(u, v) | 1 \leq v \leq 3, \frac{-1}{v} \leq u \leq \frac{1}{v}\right\}. \tag{7.7.35}$$

The Jacobian for this transformation is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

Therefore, by using the transformation T, the integral changes to

$$\iint_R (x-y)e^{x^2-y^2}\,dA = \frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv. \tag{7.7.36}$$

Doing the evaluation, we have

$$\frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv = \frac{4}{3e} \approx 0.490. \tag{7.7.37}$$

Exercise 7.7.5

Using the substitutions $x=v$ and $y=\sqrt{u+v}$, evaluate the integral

$$\iint_R y\,\sin(y^2-x)dA, \tag{7.7.38}$$

where R is the region bounded by the lines $y=\sqrt{x},\ x=2$ and $y=0$.

Hint

Sketch a picture and find the limits of integration.

Answer

$\frac{1}{2}(\sin 2-2)$

7.7.4 Change of Variables for Triple Integrals

Changing variables in triple integrals works in exactly the same way. Cylindrical and spherical coordinate substitutions are special cases of this method, which we demonstrate here.

Suppose that G is a region in uvw-space and is mapped to D in xyz-space (Figure 7.7.10) by a one-to-one C^1 transformation $T(u,v,w)=(x,y,z)$ where $x=g(u,v,w),\ y=h(u,v,w)$, and $z=k(u,v,w)$.

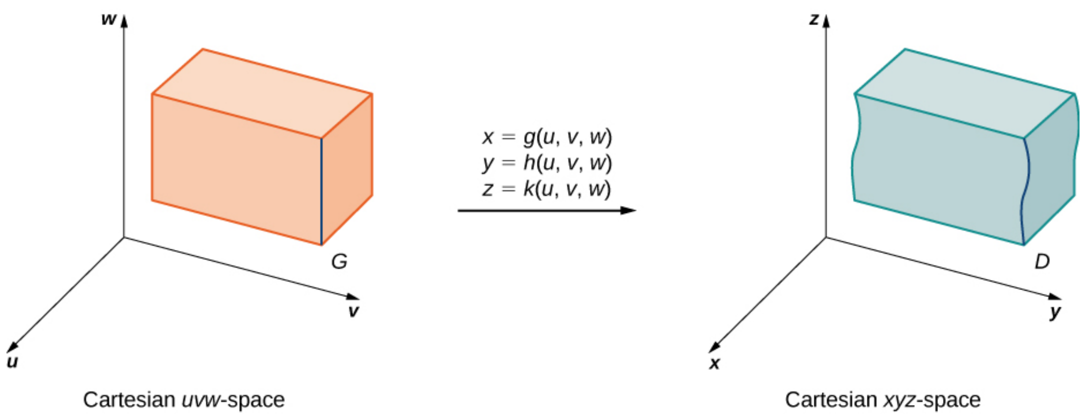


Figure 7.7.10: A region G in uvw-space mapped to a region D in xyz-space.

Then any function $F(x,y,z)$ defined on D can be thought of as another function $H(u,v,w)$ that is defined on G:

$$F(x,y,z) = F(g(u,v,w),\ h(u,v,w),\ k(u,v,w)) = H(u,v,w). \tag{7.7.39}$$

Now we need to define the Jacobian for three variables.

Definition: Jacobian determinant

The *Jacobian determinant* $J(u,v,w)$ in three variables is defined as follows:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

This is also the same as

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

The Jacobian can also be simply denoted as $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

With the transformations and the Jacobian for three variables, we are ready to establish the theorem that describes change of variables for triple integrals.

Change of Variables for Triple Integrals

Let $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$, be a one-to-one C^1 transformation, with a nonzero Jacobian, that maps the region G in the uvw-space into the region D in the xyz-space. As in the two-dimensional case, if F is continuous on D, then

$$\iiint_R F(x,y,z)dV = \iiint_G f(g(u,v,w)\,h(u,v,w),\ k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw \tag{7.7.40}$$

$$= \iiint_G H(u,v,w)|J(u,v,w)|du\,dv\,dq. \tag{7.7.41}$$

Let us now see how changes in triple integrals for cylindrical and spherical coordinates are affected by this theorem. We expect to obtain the same formulas as in Triple Integrals in Cylindrical and Spherical Coordinates.

Example $7.7.6A$: Obtaining Formulas in Triple Integrals for Cylindrical and Spherical Coordinates

Derive the formula in triple integrals for

a. cylindrical and
b. spherical coordinates.

Solution

A.

For cylindrical coordinates, the transformation is $T(r,\theta,z) = (x,y,z)$ from the Cartesian $r\theta z$-space to the Cartesian xyz-space (Figure $7.7.11$). Here $x = r\,\cos\,\theta$, $y = r\,\sin\theta$ and $z = z$. The Jacobian for the transformation is

$$J(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$\begin{vmatrix} \cos\theta & -r\,\sin\theta & 0 \\ \sin\theta & r\,\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\,\cos^2\theta + r\,\sin^2\theta = r.$$

We know that $r \geq 0$, so $|J(r,\theta,z)| = r$. Then the triple integral is

$$\iiint_D f(x,y,z)dV = \iiint_G f(r\cos\theta,\ r\sin\theta,\ z) r\,dr\,d\theta\,dz. \tag{7.7.42}$$

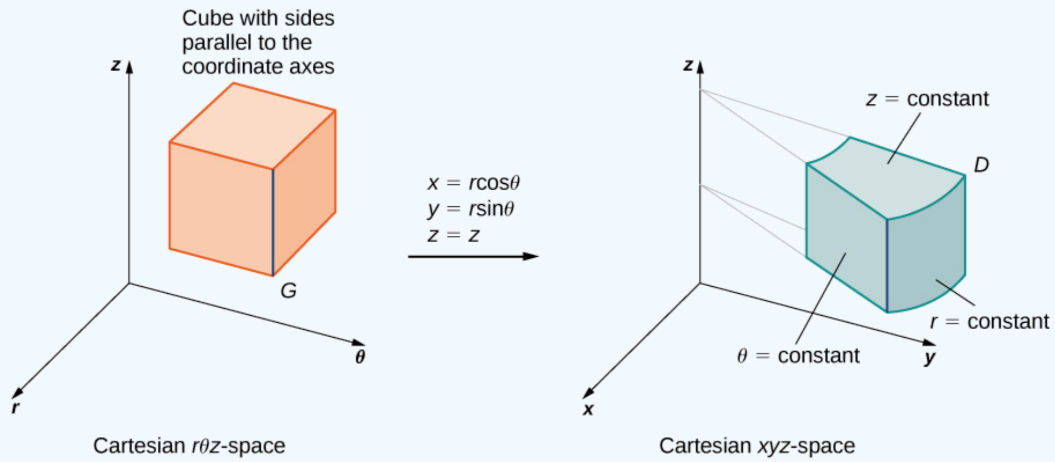


Figure 7.7.11: The transformation from rectangular coordinates to cylindrical coordinates can be treated as a change of variables from region G in $r\theta z$-space to region D in xyz-space.

B

For spherical coordinates, the transformation is $T(\rho,\theta,\varphi)$ from the Cartesian $\rho\theta\varphi$-space to the Cartesian xyz-space (Figure 7.7.12). Here $x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\varphi$, and $z=\rho\cos\varphi$. The Jacobian for the transformation is

$$J(\rho,\theta,\varphi)=\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}=\begin{vmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta\\ \sin\varphi\sin\theta & -\rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta\\ \cos\theta & 0 & -\rho\sin\varphi\end{vmatrix}.$$

Expanding the determinant with respect to the third row:

$$=\cos\varphi\begin{vmatrix}-\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta\\ \rho\, sin\,\varphi\sin\theta & \rho\cos\varphi\sin\theta\end{vmatrix}-\rho\sin\varphi\begin{vmatrix}\sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta\end{vmatrix}$$

$$=\cos\varphi(-\rho^2\sin\varphi\cos\varphi\sin^2\theta-\rho^2\sin\varphi\cos\varphi\cos^2\theta) \tag{7.7.43}$$

$$-\rho\sin\varphi(\rho\sin^2\varphi\cos^2\theta+\rho\sin^2\varphi\sin^2\theta) \tag{7.7.44}$$

$$=-\rho^2\sin\varphi\cos^2\varphi(\sin^2\theta+\cos^2\theta)-\rho^2\sin\varphi\sin^2\varphi(\sin^2\theta+\cos^2\theta) \tag{7.7.45}$$

$$=-\rho^2\sin\varphi\cos^2\varphi-\rho^2\sin\varphi\sin^2\varphi \tag{7.7.46}$$

$$=-\rho\sin\varphi(\cos^2\varphi+\sin^2\varphi)=-\rho^2\sin\varphi. \tag{7.7.47}$$

Since $0\le\varphi\le\pi$, we must have $\sin\varphi\ge 0$. Thus $|J(\rho,\theta\varphi)|=|-\rho^2\sin\varphi|=\rho^2\sin\varphi$.

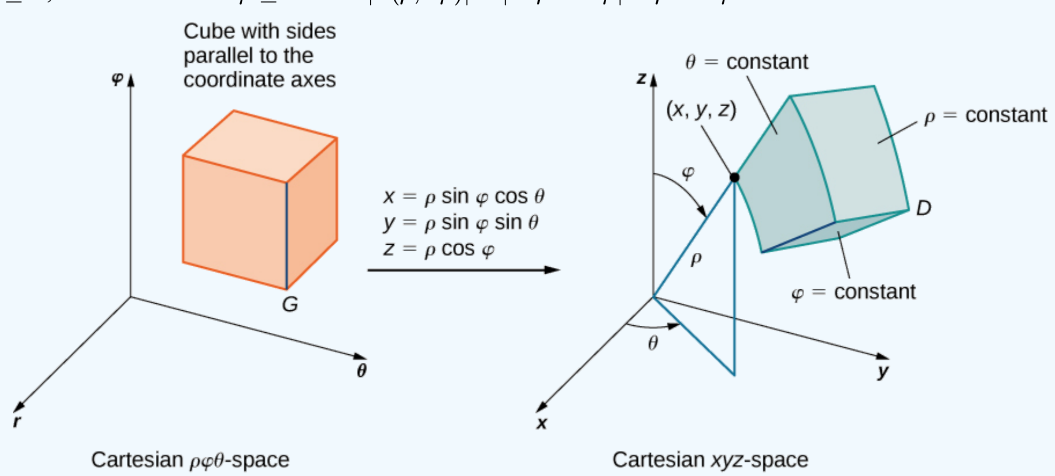


Figure 7.7.12: The transformation from rectangular coordinates to spherical coordinates can be treated as a change of variables from region G in $\rho\theta\varphi$-space to region D in xyz-space.

Then the triple integral becomes

$$\iiint_D f(x,y,z)dV = \iiint_G f(\rho \sin\varphi \cos\theta,\ \rho \sin\varphi \sin\theta, \rho \cos\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta. \tag{7.7.48}$$

Let's try another example with a different substitution.

Example 7.7.6B: Evaluating a Triple Integral with a Change of Variables

Evaluate the triple integral

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz \tag{7.7.49}$$

In xyz-space by using the transformation

$u = (2x - y)/2$, $v = y/2$, and $w = z/3$.

Then integrate over an appropriate region in uvw-space.

Solution

As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space (Figure 7.7.13). Clearly G in xyz-space is bounded by the planes $x = y/2$, $x = (y/2) + 1$, $y = 0$, $y = 4$, $z = 0$, and $z = 4$. We also know that we have to use $u = (2x - y)/2$, $v = y/2$, and $w = z/3$ for the transformations. We need to solve for x, y and z. Here we find that $x = u + v$, $y = 2v$, and $z = 3w$.

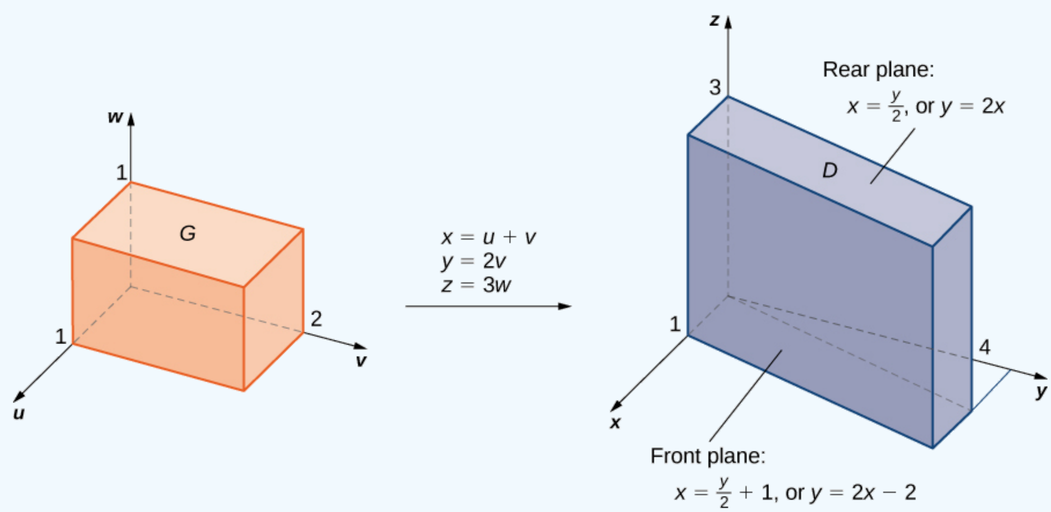


Figure 7.7.13: The region G in uvw-space is transformed to region D in xyz-space.

Using elementary algebra, we can find the corresponding surfaces for the region G and the limits of integration in uvw-space. It is convenient to list these equations in a table.

Equations in xyz for the region D	Corresponding equations in uvw for the region G	Limits for the integration in uvw
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = y/2$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

Now we can calculate the Jacobian for the transformation:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

The function to be integrated becomes

$$f(x,y,z) = x + \frac{z}{3} = u+v+\frac{3w}{3} = u+v+w. \tag{7.7.50}$$

We are now ready to put everything together and complete the problem.

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x+\frac{z}{3}\right) dx\, dy\, dz \tag{7.7.51}$$

$$= \int_0^1 \int_0^2 \int_0^1 (u+v+w)|J(u,v,w)| du\, dv\, dw = \int_0^1 \int_0^2 \int_0^1 (u+v+w)|6| du\, dv\, dw \tag{7.7.52}$$

$$= 6\int_0^1 \int_0^2 \int_0^1 (u+v+w)\, du\, dv\, dw = 6\int_0^1 \int_0^2 \left[\frac{u^2}{2}+vu+wu\right]_0^1 dv\, dw \tag{7.7.53}$$

$$= 6\int_0^1 \int_0^2 \left(\frac{1}{2}+v+u\right) dv\, dw = 6\int_0^1 \left[\frac{1}{2}v+\frac{v^2}{2}+wv\right]_0^2 dw \tag{7.7.54}$$

$$= 6\int_0^1 (3+2w)\, dw = 6[3w+w^2]_0^1 = 24. \tag{7.7.55}$$

Exercise 7.7.6

Let D be the region in xyz-space defined by $1 \le x \le 2,\ 0 \le xy \le 2$, and $0 \le z \le 1$.

Evaluate $\iiint_D (x^2y+3xyz)\, dx\, dy\, dz$ by using the transformation $u = x,\ v = xy$, and $w = 3z$.

Hint

Make a table for each surface of the regions and decide on the limits, as shown in the example.

Answer

$$\int_0^3 \int_0^2 \int_1^2 \left(\frac{v}{3}+\frac{vw}{3u}\right) du\, dv\, dw = 2+\ln 8 \tag{7.7.56}$$

7.7.5 Key Concepts

- A transformation T is a function that transforms a region G in one plane (space) into a region R. in another plane (space) by a change of variables.
- A transformation $T : G \to R$ defined as $T(u,v) = (x,y)$ (or $T(u,v,w) = (x,y,z)$)is said to be a one-to-one transformation if no two points map to the same image point.
- If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\ h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv. \tag{7.7.57}$$

- If F is continuous on R, then

$$\iiint_R F(x,y,z)dV = \iiint_G F(g(u,v,w),\ h(u,v,w),\ k(u,v,w) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw. \tag{7.7.58}$$

$$= \iint_G H(u,v,w)|J(u,v,w)|\, du\, dv\, dw. \tag{7.7.59}$$

[T] Lamé ovals (or superellipses) are plane curves of equations $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$, where *a*, *b*, and *n* are positive real numbers.

a. Use a CAS to graph the regions R bounded by Lamé ovals for $a=1,\ b=2,\ n=4$ and $n=6$ respectively.

b. Find the transformations that map the region R bounded by the Lamé oval $x^4+y^4=1$ also called a squircle and graphed in the following figure, into the unit disk.

A square of side length 2 with rounded corners.

c. Use a CAS to find an approximation of the area $A(R)$ of the region R bounded by $x^4+y^4=1$. Round your answer to two decimal places.

[T] Lamé ovals have been consistently used by designers and architects. For instance, Gerald Robinson, a Canadian architect, has designed a parking garage in a shopping center in Peterborough, Ontario, in the shape of a superellipse of the equation $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=1$ with $\frac{a}{b}=\frac{9}{7}$ and $n=e$. Use a CAS to find an approximation of the area of the parking garage in the case $a=900$ yards, $b=700$ yards, and $n=2.72$ yards.

[Hide Solution]

$A(R)\simeq 83,999.2$

7.7.6 Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

$$\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dy\,dx \tag{7.7.60}$$

Fubini’s theorem can be extended to three dimensions, as long as f is continuous in all variables.

[Hide solution]

True.

The integral

$$\int_0^{2\pi}\int_0^1\int_0^1 dz\,dr\,d\theta \tag{7.7.61}$$

represents the volume of a right cone.

The Jacobian of the transformation for $x=u^2-2v,\ y=3v-2uv$ is given by $-4u^2+6u+4v$.

[Hide Solution]

False.

Evaluate the following integrals.

$$\iint_R (5x^3y^2-y^2)\,dA,\ R=\{(x,y)|0\le x\le 2,\ 1\le y\le 4\} \tag{7.7.62}$$

$$\iint_D \frac{y}{3x^2+1}dA,\ D=\{(x,y)|0\le x\le 1,\ -x\le y\le x\} \tag{7.7.63}$$

[Hide Solution]

0

$$\iint_D \sin(x^2+y^2)dA \tag{7.7.64}$$

where D is a disk of radius 2 centered at the origin

$$\int_0^1\int_0^1 xye^{x^2}\,dx\,dy \tag{7.7.65}$$

[Hide Solution]

$\frac{1}{4}$

$$\int_{-1}^{1}\int_{0}^{z}\int_{0}^{x-z} 6dy\,dx\,dz \tag{7.7.66}$$

$$\iiint_R 3y\,dV, \tag{7.7.67}$$

where $R = \{(x,y,z)|0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le \sqrt{9-y^2}\}$

[Hide Solution]

1.475

$$\int_{0}^{2}\int_{0}^{2\pi}\int_{r}^{1} r\,dz\,d\theta\,dr \tag{7.7.68}$$

$$\int_{0}^{2\pi}\int_{0}^{\pi/2}\int_{1}^{3} \rho^2\,\sin(\varphi)d\rho\,d\varphi,\ d\theta \tag{7.7.69}$$

[Hide Solution]

$\frac{52}{3}\pi$

$$\int_{0}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,sx \tag{7.7.70}$$

For the following problems, find the specified area or volume.

The area of region enclosed by one petal of $r = \cos(4\theta)$.

[Hide Solution]

$\frac{\pi}{16}$

The volume of the solid that lies between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8$.

The volume of the solid bounded by the cylinder $x^2 + y^2 = 16$ and from $z = 1$ to $z + x = 2$.

[Hide Solution]

93.291

The volume of the intersection between two spheres of radius 1, the top whose center is $(0, 0, 0.25)$and the bottom, which is centered at $(0, 0, 0)$.

For the following problems, find the center of mass of the region.

$\rho(x, y) = xy$ on the circle with radius 1 in the first quadrant only.

[Hide Solution]

$\left(\frac{8}{15}, \frac{8}{15}\right)$

$\rho(x, y) = (y+1)\sqrt{x}$ in the region bounded by $y = e^x$, $y = 0$, and $x = 1$.

$\rho(x, y, z) = z$ on the inverted cone with radius 2 and height 2.

$\left(0, 0, \frac{8}{5}\right)$

The volume an ice cream cone that is given by the solid above $z = \sqrt{(x^2+y^2)}$ and below $z^2 + x^2 + y^2 = z$.

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

If the compacted trash used to build Mount Holly on average has a density $400\,lb/ft^3$, find the amount of work required to build the mountain.

[Hide Solution]

$1.452\pi \times 10^{15}$ ft-lb

In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density $400\,lb/ft^3$ and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

The following problems consider the temperature and density of Earth's layers.

[T] The temperature of Earth's layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Layer	Depth from center (km)	Temperature ^oC
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Source: http://www.enchantedlearning.com/sub...h/Inside.shtml

[Hide Solution]

$y=-1.238\times10^{-7}x^3+0.001196x^2-3.666x+7208$; average temperature approximately 2800^oC

[T] The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Layer	Depth from center (km)	Density (g/cm^3)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

Source: http://hyperphysics.phy-astr.gsu.edu...rthstruct.html

The following problems concern the Theorem of Pappus (see Moments and Centers of Mass for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by $V_x=2\pi A\bar{y}$, and when you revolve around the y-axis the volume is given by $V_y=2\pi A\bar{x}$, where A is the area of R. Consider the region bounded by $x^2+y^2=1$ and above $y=x+1$.

Find the volume when you revolve the region around the x-axis.

[Hide Solution]

$\frac{\pi}{3}$

Find the volume when you revolve the region around the y-axis.

7.7.6.1 Glossary

Jacobian

the Jacobian $J(u,v)$ in two variables is a 2×2 determinant:

$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix};$$

the Jacobian $J(u,v,w)$ in three variables is a 3×3 determinant:

$$J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

one-to-one transformation

a transformation $T:G\rightarrow R$ defined as $T(u,v)=(x,y)$ is said to be one-to-one if no two points map to the same image point

planar transformation

a function T that transforms a region G in one plane into a region R in another plane by a change of variables

transformation

a function that transforms a region GG in one plane into a region RR in another plane by a change of variables

7.7.7 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

7.7E:

7.7E.1 Exercise $\mathbf{7.7E.1}$: Transformation

In the following exercises, the function $T : S \rightarrow R,\ T(u, v) = (x, y)$ on the region $S = \{(u, v) | 0 \leq u \leq 1,\ 0 \leq v \leq 1\}$ bounded by the unit square is given, where $R \in R^2$ is the image of S under T.

a. Justify that the function T is a C^1 transformation.

b. Find the images of the vertices of the unit square S through the function T.

c. Determine the image R of the unit square S and graph it.

1. $x = 2u,\ y = 3v$

2. $x = \frac{u}{2},\ y = \frac{v}{3}$

3. $x = u - v,\ y = u + v$

4. $x = 2u - v,\ y = u + 2v$

5. $x = u^2,\ y = v^2$

6. $x = u^3,\ y = v^3$

Answer

2.

a. $T(u, v) = (g(u, v),\ h(u, v),\ x = g(u, v) = \frac{u}{2}$ and $y = h(u, v) = \frac{v}{3}$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u, v) = \frac{1}{2},\ g_v(u, v) = 0,\ h_u(u, v) = 0$ and $h_v(u, v) = \frac{1}{3}$ are continuous on S;

b. $T(0, 0) = (0, 0),\ T(1, 0) = \left(\frac{1}{2}, 0\right)$, $T(0, 1) = \left(0, \frac{1}{3}\right)$, and $T(1, 1) = \left(\frac{1}{2}, \frac{1}{3}\right)$;

c. R is the rectangle of vertices $(0, 0),\ \left(0, \frac{1}{3}\right),\ \left(\frac{1}{2}, \frac{1}{3}\right)$, and $\left(0, \frac{1}{3}\right)$ in the xy-plane; the following figure.

A rectangle with one corner at the origin, horizontal length 0.5, and vertical height 0.34.

4.

a. $T(u, v) = (g(u, v),\ h(u, v),\ x = g(u, v) = 2u - v$ and $y = h(u, v) = u + 2v$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u, v) = 2,\ g_v(u, v) = -1,\ h_u(u, v) = 1$ and $h_v(u, v) = 2$ are continuous on S;

b. $T(0, 0) = (0, 0),\ T(1, 0) = (2, 1),\ T(0, 1) = (-1, 2)$, and $T(1, 1) = (1, 3)$;

c. R is the parallelogram of vertices $(0, 0),\ (2, 1)\ (1, 3)$, and $(-1, 2)$ in the xy-plane; the following figure.

A square of side length square root of 5 with one corner at the origin and another at (2, 1).

6.

a. $T(u, v) = (g(u, v),\ h(u, v),\ x = g(u, v) = u^3$ and $y = h(u, v) = v^3$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u, v) = 3u^2,\ g_v(u, v) = 0,\ h_u(u, v) = 0$ and $h_v(u, v) = 3v^2$ are continuous on S;

b. $T(0, 0) = (0, 0),\ T(1, 0) = (1, 0),\ T(0, 1) = (0, 1)$, and $T(1, 1) = (1, 1)$;

c. R is the unit square in the xy-plane see the figure in the answer to the previous exercise.

7.7E.2 Exercise $\mathbf{7.7E.2}$: One to one

In the following exercises, determine whether the transformations $T : S \rightarrow R$ are one-to-one or not.

1. $x = u^2,\ y = v^2$, where S is the rectangle of vertices $(-1, 0),\ (1, 0),\ (1, 1)$, and $(-1, 1)$.

2. $x = u^4,\ y = u^2 + v$, where S is the triangle of vertices $(-2, 0),\ (2, 0)$, and $(0, 2)$.

3. $x=2u,\ y=3v$, where S is the square of vertices $(-1,1),\ (-1,1),\ (-1,-1)$, and $(1,-1)$.

4. $x=u+v+w,\ y=u+v,\ z=w$, where $S=R=R^3$.

5. $x=u^2+v+w,\ y=u^2+v,\ z=w$, where $S=R=R^3$.

Answer

2. T is not one-to-one: two points of S have the same image. Indeed, $T(-2,0)=T(2,0)=(16,4)$.

3. T is one-to-one: We argue by contradiction. $T(u_1,v_1)=T(u_2,v_2)$ implies $2u_1-v_1=2u_2-v_2$ and $u_1=u_2$. Thus, $u_1=u+2$ and $v_1=v_2$.

5. T is not one-to-one: $T(1,v,w)=(-1,v,w)$

7.7E.3 Exercise $7.7E.3$: Inverse Transformation

In the following exercises, the transformations $T:R\to S$ are one-to-one. Find their related inverse transformations $T^{-1}:R\to S$.

1. $x=4u,\ y=5v$, where $S=R=R^2$.
2. $x=u+2v,\ y=-u+v$, where $S=R=R^2$.
3. $x=e^{2u+v},\ y=e^{u-v}$, where $S=R^2$ and $R=\{(x,y)|x>0,\ y>0\}$
4. $x=\ln u,\ y=\ln(uv)$, where $S=\{(u,v)|u>0,\ v>0\}$ and $R=R^2$.
5. $x=u+v+w,\ y=3v,\ z=2w$, where $S=R=R^3$.
6. $x=u+v,\ y=v+w,\ z=u+w$, where $S=R=R^3$.

Answer

2. $u=\frac{x-2y}{3},\ v=\frac{x+y}{3}$.

4. $u=e^x,\ v=e^{-x+y}$.

6. $u=\frac{x-y+z}{2},\ v=\frac{x+y-z}{2},\ w=\frac{-x+y+z}{2}$.

7.7E.4 Exercise $7.7E.4$

In the following exercises, the transformation $T:S\to R,\ T(u,v)=(x,y)$ and the region $R\subset R^2$ are given. Find the region $S\subset R^2$.

1. $x=au,\ y=bv,\ R=\{(x,y)|x^2+y^2\leq a^2b^2\}$ where $a,b>0$
2. $x=au,\ y=bc,\ R=\{(x,y)|\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1\}$, where $a,b>0$
3. $S=\{(u,v)|u^2+v^2\leq 1\}$
4. $x=\frac{u}{a},\ y=\frac{v}{b},\ z=\frac{w}{c},\ R=\{(x,y)|x^2+y^2+z^2\leq 1\}$, where $a,b,c>0$
5. $x=au,\ y=bv,\ z=cw,\ R=\{(x,y)|\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\leq 1,\ z>0\}$, where $a,b,c>0$

Answer

2.$S=\{(u,v)|u^2+v^2\leq 1\}$.

5. $R=\{(u,v,w)|u^2-v^2-w^2\leq 1,\ w>0\}$

7.7E.5 Exercise $7.7E.5$: Jacobian

In the following exercises, find the Jacobian J of the transformation.

1. $x=u+2v,\ y=-u+v$
2. $x=\frac{u^3}{2},\ y=\frac{v}{u^2}$
3. $x=e^{2u-v},\ y=e^{u+v}$
4. $x=ue^v,\ y=e^{-v}$

5. $x = u\ \cos(e^v),\ y = u\ \sin(e^v)$
6. $x = v\ \sin(u^2),\ y = v\ \cos(u^2)$
7. $x = u\ \cosh v,\ y = u\ \sinh v,\ z = w$
8. $x = v\ \cosh\left(\frac{1}{u}\right),\ y = v\ \sinh\left(\frac{1}{u}\right),\ z = u + w^2$
9. $x = u + v,\ y = v + w,\ z = u$
10. $x = u - v,\ y = u + v,\ z = u + v + w$

Answer

2.$\frac{3}{2}$

4. -1

6. $2uv$

8. $\frac{v}{u^2}$

10. 2

7.7E.6 Exercise $\mathbf{7.7E.6}$

1. The triangular region R with the vertices $(0,0),\ (1,1)$, and $(1,2)$ is shown in the following figure.

A triangle with corners at the origin, (1, 1), and (1, 2).

a. Find a transformation $T : S \to R,\ T(u,v) = (x,y) = (au + bv + dv)$, where $a, b, c,$ and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0),\ T^{-1}(1,1) = (1,0)$, and $T^{-1}(1,2) = (0,1)$.

b. Use the transformation T to find the area $A(R)$ of the region R.

2.

The triangular region R with the vertices $(0,0),\ (2,0)$, and $(1,3)$ is shown in the following figure.

A triangle with corners at the origin, (2, 0), and (1, 3).

a. Find a transformation $T : S \to R,\ T(u,v) = (x,y) = (au + bv + dv)$, where $a, b, c,$ and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,0) = (0,0),\ T^{-1}(2,0) = (1,0)$, and $T^{-1}(1,3) = (0,1)$.

b. Use the transformation T to find the area $A(R)$ of the region R.

Answer

2.

a. $T(u,v) = (2u + v,\ 3v); b. The area of \backslash(R$ is

$$A(R) = \int_0^3 \int_{y/3}^{(6-y)/3} dx\ dy = \int_0^1 \int_0^{1-u} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\ du = \int_0^1 \int_0^{1-u} 6 dv\ du = 3. \tag{7.7E.1}$$

7.7E.7 Exercise $\mathbf{7.7E.7}$

In the following exercises, use the transformation $u = y - x,\ v = y$, to evaluate the integrals on the parallelogram R of vertices $(0,0),\ (1,0),\ (2,1)$, and $(1,1)$ shown in the following figure.

A rhombus with corners at the origin, (1, 0), (1, 1), and (2, 1).

1.

$$\iint_R (y - x) dA \tag{7.7E.2}$$

2.

$$\iint_R (y^2 - xy) dA \tag{7.7E.3}$$

Answer

$-\frac{1}{4}$

7.7E.8 Exercise $7.7E.8$

In the following exercises, use the transformation $y = x = u,\ x + y = v$ to evaluate the integrals on the square R determined by the lines $y = x,\ y = -x + 2,\ y = x + 2$, and $y = -x$ shown in the following figure.

A square with side lengths square root of 2 rotated 45 degrees with one corner at the origin and another at (1, 1).

$$\iint_R e^{x+y} dA \tag{7.7E.4}$$

$$\iint_R \sin(x - y) dA \tag{7.7E.5}$$

Answer

$-1 + cos2$

7.7E.9 Exercise $7.7E.9$

In the following exercises, use the transformation $x = u,\ 5y = v$ to evaluate the integrals on the region R bounded by the ellipse $x^2 + 25y^2 = 1$ shown inthe following figure.

An ellipse with center at the origin, major axis 2, and minor 0.4.

$$\iint_R \sqrt{x^2 + 25y^2} dA \tag{7.7E.6}$$

$$\iint_R (x^2 + 25y^2)^2 dA \tag{7.7E.7}$$

Answer

$\frac{\pi}{15}$

7.7E.10 Exercise $7.7E.10$

In the following exercises, use the transformation $u = x + y,\ v = x - y$ to evaluate the integrals on the trapezoidal region R determined by the points $(1, 0),\ (2, 0),\ (0, 2)$, and $(0, 1)$ shown in the following figure.

A trapezoid with corners at (1, 0), (0, 1), (0, 2), and (2, 0).

$$\iint_R (x^2 - 2xy + y^2)\, e^{x+y} dA \tag{7.7E.8}$$

$$\iint_R (x^3 + 3x^2y + 3xy^2 + y^3)\, dA \tag{7.7E.9}$$

Answer

$\frac{31}{5}$

7.7E.11 Exercise $7.7E.11$

1. The circular annulus sector R bounded by the circles $4x^2 + 4y^2 = 1$ and $9x^2 + 9y^2 = 64$, the line $x = y\sqrt{3}$, and the y-axis is shown in the following figure. Find a transformation T from a rectangular region S in the $r\theta$-plane to the region R in

the xy-plane. Graph S.

In the first quadrant, a section of an annulus described by an inner radius of 0.5, outer radius slightly more than 2.5, and center the origin. There is a line dividing this annulus that comes from approximately a 30 degree angle. The portion corresponding to 60 degrees is shaded.

2. The solid R bounded by the circular cylinder $x^2+y^2=9$ and the planes $z=0,\ z=1,\ x=0$, and $y=0$ is shown in the following figure. Find a transformation T from a cylindrical box S in $r\theta z$-space to the solid R in xyz-space.

A quarter of a cylinder with height 1 and radius 3. The center axis is the z axis.

3. Show that

$$\iint_R f\left(\sqrt{\frac{x^2}{3}+\frac{y^2}{3}}\right) dA = 2\pi\sqrt{15}\int_0^1 f(\rho)\rho\, d\rho, \tag{7.7E.10}$$

where f is a continuous function on $[0,1]$ and R is the region bounded by the ellipse $5x^2+3y^2=15$.

4. Show that

$$\iiint_R f\left(\sqrt{16x^2+4y^2+z^2}\right) dV = \frac{\pi}{2}\int_0^1 f(\rho)\rho^2 d\rho, \tag{7.7E.11}$$

where f is a continuous function on $[0,1]$ and R is the region bounded by the ellipsoid $16x^2+4y^2+z^2=1$.

Answer

2. $T(r,\theta,z)=(r\ \cos\theta,\ r\ \sin\theta,\ z);\ S=[0,3]\times[0,\frac{\pi}{2}]\times[0,1]$ in the $r\theta z$-space

7.7E.12 Exercise $7.7E.12$

1. [T] Find the area of the region bounded by the curves $xy=1,\ xy=3,\ y=2x$, and $y=3x$ by using the transformation $u=xy$ and $v=\frac{y}{x}$. Use a computer algebra system (CAS) to graph the boundary curves of the region R.

2. [T] Find the area of the region bounded by the curves $x^2y=2,\ x^2y=3,\ y=x$, and $y=2x$ by using the transformation $u=x^2y$ and $v=\frac{y}{x}$. Use a CAS to graph the boundary curves of the region R.

Answer

2. The area of R is $10-4\sqrt{6}$; the boundary curves of R are graphed in the following figure.

Four lines are drawn, namely, y = 3, y = 2, y = 3/(x squared), and y = 2/(x squared). The lines y = 3 and y = 2 are parallel to each other. The lines y = 3/(x squared) and y = 2/(x squared) are curves that run somewhat parallel to each other.

7.7E.13 Exercise $7.7E.13$

1. Evaluate the triple integral

$$\int_0^1\int_1^2\int_z^{z+1}(y+1)\,dx\,dy\,dz \tag{7.7E.12}$$

by using the transformation $u=x-z,\ v=3y$, and $w=\frac{z}{2}$.

2. Evaluate the triple integral

$$\int_0^2\int_4^6\int_{3z}^{3z+2}(5-4y)\,dx\,dy\,dz \tag{7.7E.13}$$

by using the transformation $u=x-3z,\ v=4y$, and $w=z$.

Answer

2. 8

7.7E.14 Exercise $\PageIndex{14}$

1. A transformation $T: R^2 \rightarrow R^2,\ T(u, v) = (x, y)$ of the form $x = au + bv,\ y = cu + dv$, where $a, b, c,$ and d are real numbers, is called linear. Show that a linear transformation for which $ad - bc \neq 0$ maps parallelograms to parallelograms.

2. A transformation $T_\theta: R^2 \rightarrow R^2,\ T_\theta(u, v) = (x, y)$ of the form $x = u\ \cos\theta - v\ \sin\theta,\ y = u\ \sin\theta + v\ \cos\theta$, is called a rotation angle θ. Show that the inverse transformation of T_θ satisfies $T_\theta^{-1} = T_{-\theta}$ where $T_{-\theta}$ is the rotation of angle $-\theta$.

7.7E.15 Exercise $\PageIndex{15}$

1. [T] Find the region S in the uv-plane whose image through a rotation of angle $\frac{\pi}{4}$ is the region R enclosed by the ellipse $x^2 + 4y^2 = 1$. Use a CAS to answer the following questions.

a. Graph the region S.

b. Evaluate the integral

$$\iint_S e^{-2uv}\, du\, dv. \tag{7.7E.14}$$

Round your answer to two decimal places.

2. [T] The transformations $T_i: \mathbb{R}^2 \rightarrow \mathbb{R}^2,\ i = 1, \ldots, 4,$ defined by $T_1(u, v) = (u, -v),\ T_2(u, v) = (-u, v),\ T_3(u, v) = (-u, -v)$, and $T_4(u, v) = (v, u)$ are called reflections about the x-axis, y-axis origin, and the line $y = x$, respectively.

a. Find the image of the region $S = \{(u, v) | u^2 + v^2 - 2u - 4v + 1 \leq 0\}$ in the xy-plane through the transformation $T_1 \circ T_2 \circ T_3 \circ T_4$.

b. Use a CAS to graph R.

c. Evaluate the integral

$$\iint_S \sin(u^2)\, du\, dv \tag{7.7E.15}$$

by using a CAS. Round your answer to two decimal places.

3. [T] The transformations $T_{k,1,1}: \mathbb{R}^3 \rightarrow \mathbb{R}^3,\ T_{k,1,1}(u, v, w) = (x, y, z)$ of the form $x = ku,\ y = v,\ z = w$, where $k \neq 1$ is a positive real number, is called a stretch if $k > 1$ and a compression if $0 < k < 1$ in the x-direction. Use a CAS to evaluate the integral

$$\iiint_S e^{-(4x^2 + 9y^2 + 25z^2)} dx\, dy\, dz \tag{7.7E.16}$$

on the solid $S = \{(x, y, z) | 4x^2 + 9y^2 + 25z^2 \leq 1\}$ by considering the compression $T_{2,3,5}(u, v, w) = (x, y, z)$ defined by $x = \frac{u}{2},\ y = \frac{v}{3}$, and $z = \frac{w}{5}$. Round your answer to four decimal places.

4. [T] The transformation $T_{a,0}: \mathbb{R}^2 \rightarrow \mathbb{R}^2,\ T_{a,0}(u, v) = (u + av, v)$, where $a \neq 0$ is a real number, is called a shear in the x-direction. The transformation, $T_{b,0}: R^2 \rightarrow R^2,\ T_{o,b}(u, v) = (u, bu + v)$, where $b \neq 0$ is a real number, is called a shear in the y-direction.

a. Find transformations $T_{0,2} \circ T_{3,0}$.

b. Find the image R of the trapezoidal region S bounded by $u = 0,\ v = 0,\ v = 1$, and $v = 2 - u$ through the transformation $T_{0,2} \circ T_{3,0}$.

c. Use a CAS to graph the image R in the xy-plane.

d. Find the area of the region R by using the area of region S.

Answer

2. a. $R = \{(x, y) | y^2 + x^2 - 2y - 4x + 1 \leq 0\}$; b. R is graphed in the following figure;

A circle with radius 2 and center (2, 1).

c. 3.16

4.

a. $T_{0,2} \circ T_{3,0}(u,v) = (u+3v, 2u+7v)$;

b. The image S is the quadrilateral of vertices $(0,0)$, $(3,7)$, $(2,4)$, and $(4,9)$;

c. S is graphed in the following figure;

A four-sided figure with points the origin, (2, 4), (4, 9), and (3, 7).

d. $\frac{3}{2}$

7.7E.16 Exercise $7.7E.16$

1. Use the transformation, $x = au,\ y = av,\ z = cw$ and spherical coordinates to show that the volume of a region bounded by the spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi a^2 c}{3}$.

2. Find the volume of a football whose shape is a spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$ whose length from tip to tip is 11 inches and circumference at the center is 22 inches. Round your answer to two decimal places.

Answer

2. $\frac{2662}{3\pi} \approx 282.45\ in^3$

7E: Chapter Review Excersies

Exercise $7E.1$

Sketch the regions and evaluate the integrals:

1) $\int_1^{\ln 8} \int_0^{\ln y} 2e^{x+y}\, dydx$

2) $\int_0^2 \int_x^2 3y^2 sin(xy)\, dydx$

3) $\int_0^1 \int_0^3 \frac{4x^2}{(y-1)^{2/3}}\, dydx$

4) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{3}{(x^2+1)(y^2+1)}\, dydx$

5) $\int\int_{x^2+y^2\leq 1} \ln(x^2+y^2)\, dA$

6) $\int_0^2 \int_{y/2}^1 ye^{x^3}\, dxdy$

7) $\int\int_Q \frac{dA}{(1+x^2)(1+y^2)}$, where Q is the first quadrant of the $xy-$plane.

8) $\int\int_R x\cos(y)\, dA$, where R is the region bounded by the coordinate axes and the curve $y=1-x^2$.

Answer

Add texts here. Do not delete this text first.

CHAPTER OVERVIEW

8: Partial Differential Equations

Previously, we studied differential equations in which the unknown function had one independent variable. A partial differential equation is an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives. Many important and interesting physical phenomena are modelled by the functions of several variables that satisfy certain partial differential equations.

In this chapter, we will learn a few particular partial differential equations that arise in physical sciences such as heat equation, Laplace equation etc. Furthermore, we will explore these partial differential equations in terms of polar, cylindrical and spherical coordinates.

Topic hierarchy

8.1: Laplace Equations
8.2: The heat equation
8.3: Wave Equations
8E: Excercise

8.1: Laplace Equations

8.1.1 Introduction

The following partial differential equation is called the two-dimensional Laplace equation:

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0 \tag{8.1.1}$$

where $w(x,\ y)$ is the unknown function with two variables x and y. The problem is to find a solution to this equation, namely, find a function $w(x,\ y)$ which satisfies the Equation 8.1.1. This equation is used to model various physical quantities.

Example 8.1.1

Let k be a real number. Show that the functions $w=e^{kx}cos(ky)$ and $w=e^{kx}sin(ky)$ satisfy the lapalce Equation 8.1.1 at every point in $\mathbb{R}^2$.

Solution

Let $w=e^{kx}cos(ky)$. Then we have,
$\frac{\partial w}{\partial x}=ke^{kx}\cos(ky),\ \frac{\partial w}{\partial y}=-ke^{kx}\sin(ky),$
which implies $\frac{\partial^2 w}{\partial x^2}=k^2e^{kx}\cos(ky),\ \frac{\partial^2 w}{\partial y^2}=-k^2e^{kx}\cos(ky)$.

Consider $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=k^2e^{kx}\cos(ky)-k^2e^{kx}\cos(ky)=0$.

Therefore, that the function $w=e^{kx}\cos(ky)$ satisfies the Equation 8.1.1. Similarly, the function $w=e^{kx}\sin(ky)$ satisfies the Equation 8.1.1.

Definition

A function $w(x,y)$ of two variables having continuous second partial derivatives in a region of the plane is said to be harmonic if it satisfies the Laplace Equation 8.1.1.

Exercise 8.1.1

Show that $\ln(y^2+x^2)$ is hamornic everywhere except at the origin.

8.1.2 Converting Laplace's equation to polar co-ordinates

Consider the transformation to polar coordinates, $x=r\cos(\theta), y=r\sin(\theta),$ imples that $r^2=x^2+y^2$ and $\tan(\theta)=y/x$. We can use these equations to express $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}$ in terms of partials of w with respect to r and θ.

8.2: The heat equation

$$\underset{\text{heat equation in two dimensions}}{u_t = c^2(u_{xx} + u_{yy})} \tag{8.2.1}$$

The unknown function u has three independent variables: t, x, and y with c is an arbitrary constant. The independent variables x and y are considered to be spatial variables, and the variable t represents time.

Example 8.2.1: A Solution to the Heat Equation

Verify that

$$u(x,y,t) = 2\sin\left(\frac{x}{3}\right)\sin\left(\frac{y}{4}\right)e^{-25t/16}$$

is a solution to the heat equation

$$u_t = 9(u_{xx} + u_{yy}).$$

Hint

Calculate the partial derivatives and substitute into the right-hand side.

Since the solution to the two-dimensional heat equation is a function of three variables, it is not easy to create a visual representation of the solution. We can graph the solution for fixed values of t, which amounts to snapshots of the heat distributions at fixed times. These snapshots show how the heat is distributed over a two-dimensional surface as time progresses. The graph of the preceding solution at time $t = 0$ appears in Figure $8.2.3$. As time progresses, the extremes level out, approaching zero as t approaches infinity.

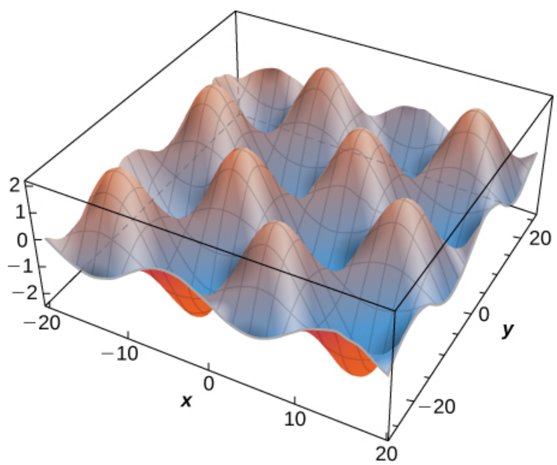


Figure 8.2.3

If we consider the heat equation in one dimension, then it is possible to graph the solution over time. The heat equation in one dimension becomes

$$u_t = c^2 u_{xx}, \tag{8.2.2}$$

where c^2 represents the thermal diffusivity of the material in question. A solution of this differential equation can be written in the form

$$u_m(x,t) = e^{-\pi^2 m^2 c^2 t}\sin(m\pi x) \tag{8.2.3}$$

where m is any positive integer. A graph of this solution using $m = 1$ appears in Figure $8.2.4$, where the initial temperature distribution over a wire of length 1 is given by $u(x,0) = \sin \pi x$. Notice that as time progresses, the wire cools off. This is seen because, from left to right, the highest temperature (which occurs in the middle of the wire) decreases and changes color from red to blue.

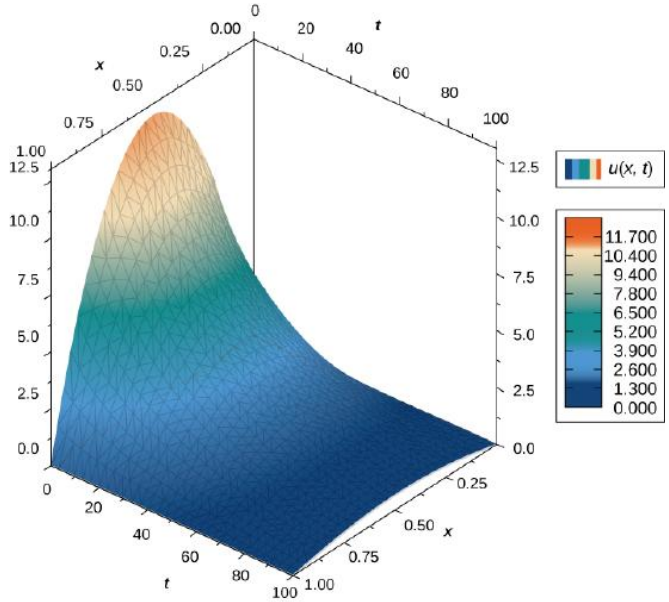


Figure 8.2.4: Graph of a solution of the heat equation in one dimension over time.

8.3: Wave Equations

$$\underset{\text{wave equation in two dimensions}}{u_{tt} = c^2(u_{xx} + u_{yy})} \tag{8.3.1}$$

The unknown function u has three independent variables: t, x, and y with c is an arbitrary constant. The independent variables x and y are considered to be spatial variables, and the variable t represents time.

Example 8.3.1: A Solution to the Wave Equation

Verify that

$$u(x,y,t) = 5\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \tag{8.3.2}$$

is a solution to the wave equation

$$u_{tt} = 4(u_{xx} + u_{yy}). \tag{8.3.3}$$

Solution

First, we calculate u_{tt}, u_{xx}, and u_{yy} :

$$\begin{aligned} u_{tt}(x,y,t) &= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}\right] \\ &= \frac{\partial}{\partial t}[5\sin(3\pi x)\sin(4\pi y)(-10\pi\sin(10\pi t))] \\ &= \frac{\partial}{\partial t}[-50\pi\sin(3\pi x)\sin(4\pi y)\sin(10\pi t)] \\ &= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \end{aligned} \tag{8.3.4}$$

$$\begin{aligned} u_{xx}(x,y,t) &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] \\ &= \frac{\partial}{\partial x}[15\pi\cos(3\pi x)\sin(4\pi y)\cos(10\pi t)] \\ &= -45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \end{aligned} \tag{8.3.5}$$

$$\begin{aligned} u_{yy}(x,y,t) &= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] \\ &= \frac{\partial}{\partial y}[5\sin(3\pi x)(4\pi\cos(4\pi y))\cos(10\pi t)] \\ &= \frac{\partial}{\partial y}[20\pi\sin(3\pi x)\cos(4\pi y)\cos(10\pi t)] \\ &= -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t). \end{aligned} \tag{8.3.6}$$

Next, we substitute each of these into the right-hand side of Equation 8.3.3 and simplify:

$$\begin{aligned} 4(u_{xx} + u_{yy}) &= 4(-45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) + -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\ &= 4(-125\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\ &= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\ &= u_{tt}. \end{aligned} \tag{8.3.7}$$

This verifies the solution.

8E: Excercise

8E.1 Exercise $8E.1$: Heat Equation

1. Let $u(x,t) = t^{-1/2} e^{-x^2/4t}$. Find $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}$.
2. Let $u(x,y,t) = t^{-1} e^{-(x^2+y^2)/4t}$. Find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t}$.

8E.2 Exercise $8E.2$: Laplace equation

Show that function $z(x,y) = 2e^{\pi x} sin(\pi y)$ satisfies the two dimensional Laplace equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0. \tag{8E.1}$$

CHAPTER OVERVIEW

9: Vector Calculus

Hurricanes are huge storms that can produce tremendous amounts of damage to life and property, especially when they reach land. Predicting where and when they will strike and how strong the winds will be is of great importance for preparing for protection or evacuation. Scientists rely on studies of rotational vector fields for their forecasts.

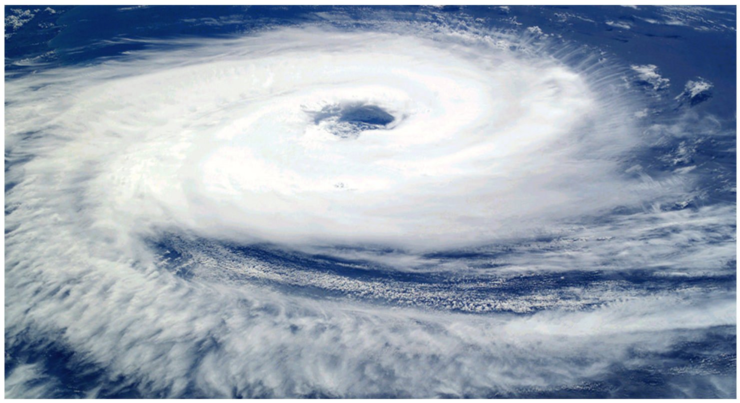


Figure 9.1: Hurricanes form from rotating winds driven by warm temperatures over the ocean. Meteorologists forecast the motion of hurricanes by studying the rotating vector fields of their wind velocity. Shown is Cyclone Catarina in the South Atlantic Ocean in 2004, as seen from the International Space Station. (credit: modification of work by NASA)

In this chapter, we learn to model new kinds of integrals over fields such as magnetic fields, gravitational fields, or velocity fields. We also learn how to calculate the work done on a charged particle traveling through a magnetic field, the work done on a particle with mass traveling through a gravitational field, and the volume per unit time of water flowing through a net dropped in a river.

All these applications are based on the concept of a vector field, which we explore in this chapter. Vector fields have many applications because they can be used to model real fields such as electromagnetic or gravitational fields. A deep understanding of physics or engineering is impossible without an understanding of vector fields. Furthermore, vector fields have mathematical properties that are worthy of study in their own right. In particular, vector fields can be used to develop several higher-dimensional versions of the Fundamental Theorem of Calculus.

9.1 Contributors

-

  Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors.

Topic hierarchy

9.1: Vector Fields
9.1E: Exercises
9.2: Line Integrals

9.2E: Exercises
9.3: Conservative vector Fields
9.3E: EXERCISES
9.4: Green's Theorem
9.4E: EXERCISES
9.5: Divergence and Curl
9.5E: EXERCISES
9.6: Surface Integrals
9.6E: Exercises
9.7: Stoke's Theorem
9.7E: EXERCISES
9.8: The Divergence Theorem
9.8E: Exercises
9E: Chapter Exercises

9.1: Vector Fields

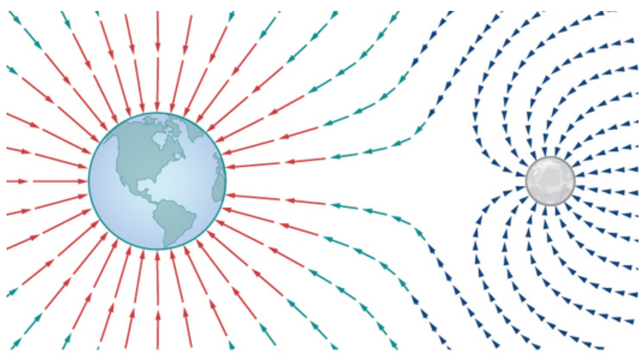
This page is a draft and is under active development.

Learning Objectives

- Recognize a vector field in a plane or in space.
- Sketch a vector field from a given equation.
- Identify a conservative field and its associated potential function.

Vector fields are an important tool for describing many physical concepts, such as gravitation and electromagnetism, which affect the behavior of objects over a large region of a plane or of space. They are also useful for dealing with large-scale behavior such as atmospheric storms or deep-sea ocean currents. In this section, we examine the basic definitions and graphs of vector fields so we can study them in more detail in the rest of this chapter.

How can we model the gravitational force exerted by multiple astronomical objects? How can we model the velocity of water particles on the surface of a river? Figure $9.1.1$ gives visual representations of such phenomena.



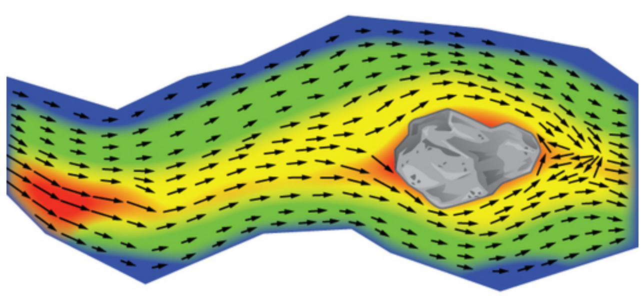


Figure 9.1.1 (a) The gravitational field exerted by two astronomical bodies on a small object. (b) The vector velocity field of water on the surface of a river shows the varied speeds of water. Red indicates that the magnitude of the vector is greater, so the water flows more quickly; blue indicates a lesser magnitude and a slower speed of water flow.

Figure $9.1.1a$ shows a gravitational field exerted by two astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of the largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

Figure 9.1.1b shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases and a whirlpool occurs in a part of the rapids.

Each figure illustrates an example of a vector field. Intuitively, a vector field is a map of vectors. In this section, we study vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

DEFINITION: vector field

- A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\vec{\mathbf{F}}(x,y)$ to each point (x,y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.
- A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\vec{\mathbf{F}}(x,y,z)$ to each point (x,y,z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field.

9.1.0.0.1 Vector Fields in $\mathbb{R}^2$

A vector field in $\mathbb{R}^2$ can be represented in either of two equivalent ways. The first way is to use a vector with components that are two-variable functions:

$$\vec{\mathbf{F}}(x,y)=\langle P(x,y),Q(x,y)\rangle \tag{9.1.1}$$

The second way is to use the standard unit vectors:

$$\vec{\mathbf{F}}(x,y)=P(x,y)\,\hat{\mathbf{i}}+Q(x,y)\,\hat{\mathbf{j}}. \tag{9.1.2}$$

A vector field is said to be *continuous* if its component functions are continuous.

Example 9.1.1: Finding a Vector Associated with a Given Point

Let $\vec{\mathbf{F}}(x,y)=(2y^2+x-4)\,\hat{\mathbf{i}}+\cos(x)\,\hat{\mathbf{j}}$ be a vector field in $\mathbb{R}^2$. Note that this is an example of a continuous vector field since both component functions are continuous. What vector is associated with point $(0,-1)$?

Solution:

Substitute the point values for x and y:

$$\begin{aligned}\vec{\mathbf{F}}(0,1)&=(2(-1)^2+0-4)\,\hat{\mathbf{i}}+\cos(0)\,\hat{\mathbf{j}}\\&=-2\,\hat{\mathbf{i}}+\hat{\mathbf{j}}.\end{aligned}$$

Exercise 9.1.1

Let $\vec{\mathbf{G}}(x,y)=x^2y\,\hat{\mathbf{i}}-(x+y)\,\hat{\mathbf{j}}$ be a vector field in $\mathbb{R}^2$. What vector is associated with the point $(-2,3)$?

Hint

Substitute the point values into the vector function.

Answer

$12\hat{\mathbf{i}} - \hat{\mathbf{j}}$

We can now represent a vector field in terms of its components of functions or unit vectors, but representing it visually by sketching it is more complex because the domain of a vector field is in $\mathbb{R}^2$, as is the range. Therefore the "graph" of a vector field in $\mathbb{R}^2$ lives in four-dimensional space. Since we cannot represent four-dimensional space visually, we instead draw vector fields in $\mathbb{R}^2$ in a plane itself. To do this, draw the vector associated with a given point at the point in a plane. For example, suppose the vector associated with point $(4,-1)$ is $\langle 3,1\rangle$. Then, we would draw vector $\langle 3,1\rangle$ at point $(4,-1)$.

We should plot enough vectors to see the general shape, but not so many that the sketch becomes a jumbled mess. If we were to plot the image vector at each point in the region, it would fill the region completely and is useless. Instead, we can choose points at the intersections of grid lines and plot a sample of several vectors from each quadrant of a rectangular coordinate system in $\mathbb{R}^2$.

There are two types of vector fields in $\mathbb{R}^2$ on which this chapter focuses: radial fields and rotational fields. Radial fields model certain gravitational fields and energy source fields, and rotational fields model the movement of a fluid in a vortex. In a radial field, all vectors either point directly toward or directly away from the origin. Furthermore, the magnitude of any vector depends only on its distance from the origin. In a radial field, the vector located at point (x,y) is perpendicular to the circle centered at the origin that contains point (x,y), and all other vectors on this circle have the same magnitude.

Example 9.1.2: Drawing a Radial Vector Field

Sketch the vector field $\vec{\mathbf{F}}(x,y)=\frac{x}{2}\hat{\mathbf{i}}+\frac{y}{2}\hat{\mathbf{j}}$.

Solution

To sketch this vector field, choose a sample of points from each quadrant and compute the corresponding vector. The following table gives a representative sample of points in a plane and the corresponding vectors.

Table 9.1.1

(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$
$(1,0)$	$\langle \frac{1}{2},0\rangle$	$(2,0)$	$\langle 1,0\rangle$	$(1,1)$	$\langle \frac{1}{2},\frac{1}{2}\rangle$
$(0,1)$	$\langle 0,\frac{1}{2}\rangle$	$(0,2)$	$\langle 0,1\rangle$	$(-1,1)$	$\langle -\frac{1}{2},\frac{1}{2}\rangle$
$(-1,0)$	$\langle -\frac{1}{2},0\rangle$	$(-2,0)$	$\langle -1,0\rangle$	$(-1,-1)$	$\langle -\frac{1}{2},-\frac{1}{2}\rangle$

(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$
$(0,-1)$	$\langle 0,-\frac{1}{2}\rangle$	$(0,-2)$	$\langle 0,-1\rangle$	$(1,-1)$	$\langle \frac{1}{2},-\frac{1}{2}\rangle$

Figure 9.1.2a shows the vector field. To see that each vector is perpendicular to the corresponding circle, Figure 9.1.2b shows circles overlain on the vector field.

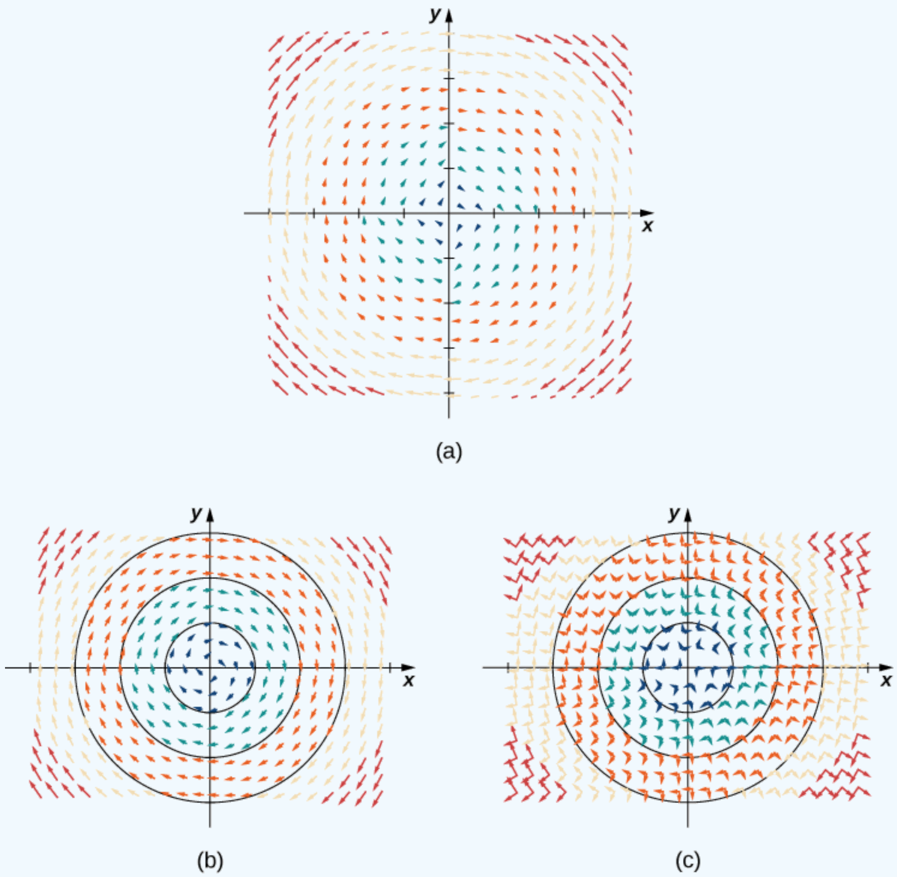


Figure 9.1.2: (a) A visual representation of the radial vector field $\vec{\mathbf{F}}(x,y)=\frac{x}{2}\hat{\mathbf{i}}+\frac{y}{2}\hat{\mathbf{j}}$. (b) The radial vector field $\vec{F}(x,y)=\frac{x}{2}\hat{\mathbf{i}}+\frac{y}{2}\hat{\mathbf{j}}$ with overlaid circles. Notice that each vector is perpendicular to the circle on which it is located.

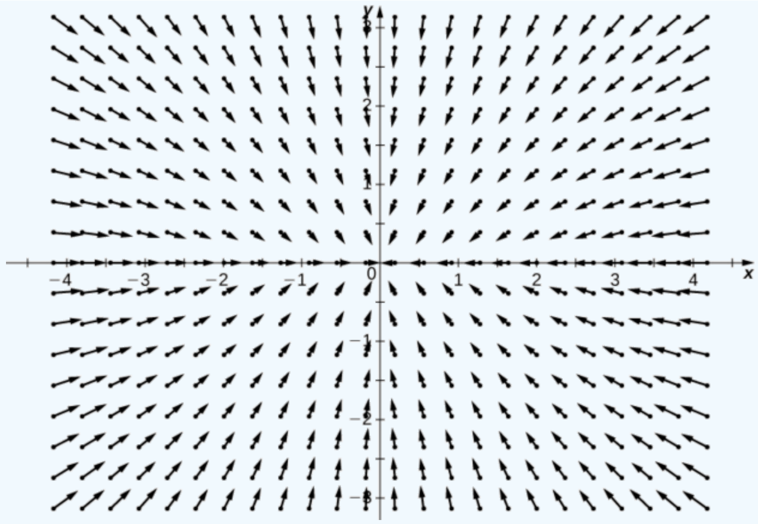
Exercise 9.1.2

Draw the radial field $\vec{\mathbf{F}}(x,y)=-\frac{x}{3}\hat{\mathbf{i}}-\frac{y}{3}\hat{\mathbf{j}}$.

Hint

Sketch enough vectors to get an idea of the shape.

Answer



In contrast to radial fields, in a *rotational field*, the vector at point (x, y) is tangent (not perpendicular) to a circle with radius $r = \sqrt{x^2 + y^2}$. In a standard rotational field, all vectors point either in a clockwise direction or in a counterclockwise direction, and the magnitude of a vector depends only on its distance from the origin. Both of the following examples are clockwise rotational fields, and we see from their visual representations that the vectors appear to rotate around the origin.

Example 9.1.3: Drawing a Rotational Vector Field

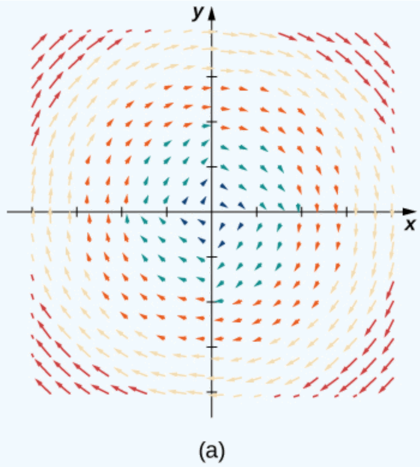
Sketch the vector field $\vec{\mathbf{F}}(x, y) = \langle y, -x \rangle$.

Solution

Create a table (see the one that follows) using a representative sample of points in a plane and their corresponding vectors. Figure 9.1.3 shows the resulting vector field.

(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$	(x, y)	$\vec{\mathbf{F}}(x, y)$
$(1, 0)$	$\langle 0, -1 \rangle$	$(2, 0)$	$\langle 0, -2 \rangle$	$(1, 1)$	$\langle 1, -1 \rangle$
$(0, 1)$	$\langle 1, 0 \rangle$	$(0, 2)$	$\langle 2, 0 \rangle$	$(-1, 1)$	$\langle 1, 1 \rangle$
$(-1, 0)$	$\langle 0, 1 \rangle$	$(-2, 0)$	$\langle 0, 2 \rangle$	$(-1, -1)$	$\langle -1, 1 \rangle$
$(0, -1)$	$\langle -1, 0 \rangle$	$(0, -2)$	$\langle -2, 0 \rangle$	$(1, -1)$	$\langle -1, -1 \rangle$

Table 9.1.2



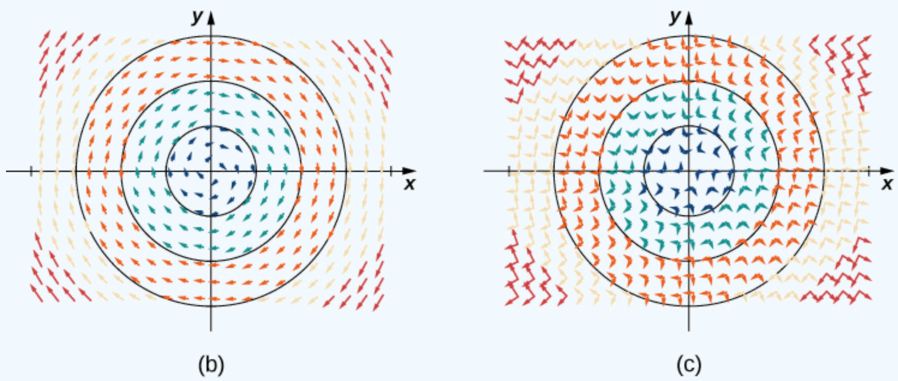


Figure 9.1.3: (a) A visual representation of vector field $\vec{\mathbf{F}}(x,y)=\langle y,-x\rangle$. (b) Vector field $\vec{\mathbf{F}}(x,y)=\langle y,-x\rangle$ with circles centered at the origin. (c) Vector $\vec{\mathbf{F}}(a,b)$ is perpendicular to radial vector $\langle a,b\rangle$ at point (a,b).

Analysis

Note that vector $\vec{\mathbf{F}}(a,b)=\langle b,-a\rangle$ points clockwise and is perpendicular to radial vector $\langle a,b\rangle$. (We can verify this assertion by computing the dot product of the two vectors: $\langle a,b\rangle\cdot\langle -b,a\rangle=-ab+ab=0$.) Furthermore, vector $\langle b,-a\rangle$ has length $r=\sqrt{a^2+b^2}$. Thus, we have a complete description of this rotational vector field: the vector associated with point (a,b) is the vector with length r tangent to the circle with radius r, and it points in the clockwise direction.

Sketches such as that in Figure 9.1.3 are often used to analyze major storm systems, including hurricanes and cyclones. In the northern hemisphere, storms rotate counterclockwise; in the southern hemisphere, storms rotate clockwise. (This is an effect caused by Earth's rotation about its axis and is called the Coriolis Effect.)

Figure 9.1.4: (credit: modification of work by NASA)

Example 9.1.4: Sketching a Vector Field

Sketch vector field $\vec{\mathbf{F}}(x,y)=\dfrac{y}{x^2+y^2}\,\hat{\mathbf{i}},-\dfrac{x}{x^2+y^2}\,\hat{\mathbf{j}}$.

Solution

To visualize this vector field, first note that the dot product $\vec{\mathbf{F}}(a,b)\cdot(a\,\hat{\mathbf{i}}+b\,\hat{\mathbf{j}})$ is zero for any point (a,b). Therefore, each vector is tangent to the circle on which it is located. Also, as $(a,b)\rightarrow(0,0)$, the magnitude of $\vec{\mathbf{F}}(a,b)$ goes to infinity. To see this, note that

$$||\vec{\mathbf{F}}(a,b)||=\sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}}=\sqrt{\frac{1}{a^2+b^2}}.$$

Since $\dfrac{1}{a^2+b^2}\rightarrow\infty$ as $(a,b)\rightarrow(0,0)$, then $||\vec{\mathbf{F}}(a,b)||\rightarrow\infty$ as $(a,b)\rightarrow(0,0)$. This vector field looks similar to the vector field in Example $9.1.3$, but in this case the magnitudes of the vectors close to the origin are large. Table $9.1.3$ shows a sample of points and the corresponding vectors, and Figure $9.1.5$ shows the vector field. Note that this vector field models the whirlpool motion of the river in Figure $9.1.5$(b). The domain of this vector field is all of $\mathbb{R}^2$ except for point $(0,0)$.

Table $9.1.3$

(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$	(x,y)	$\vec{\mathbf{F}}(x,y)$
$(1,0)$	$\langle 0,-1\rangle$	$(2,0)$	$\langle 0,-\frac{1}{2}\rangle$	$(1,1)$	$\langle\frac{1}{2},-\frac{1}{2}\rangle$
$(0,1)$	$\langle 1,0\rangle$	$(0,2)$	$\langle\frac{1}{2},0\rangle$	$(-1,1)$	$\langle\frac{1}{2},\frac{1}{2}\rangle$
$(-1,0)$	$\langle 0,1\rangle$	$(-2,0)$	$\langle 0,\frac{1}{2}\rangle$	$(-1,-1)$	$\langle-\frac{1}{2},\frac{1}{2}\rangle$
$(0,-1)$	$\langle -1,0\rangle$	$(0,-2)$	$\langle-\frac{1}{2},0\rangle$	$(1,-1)$	$\langle-\frac{1}{2},-\frac{1}{2}\rangle$

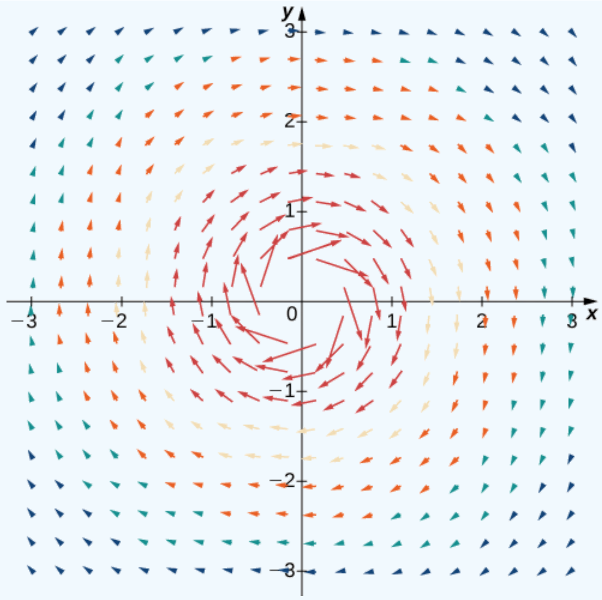


Figure 9.1.5: A visual representation of vector field $\vec{\mathbf{F}}(x, y) = \frac{y}{x^2+y^2}\,\hat{\mathbf{i}} - \frac{x}{x^2+y^2}\,\hat{\mathbf{j}}$. This vector field could be used to model whirlpool motion of a fluid.

Exercise 9.1.4

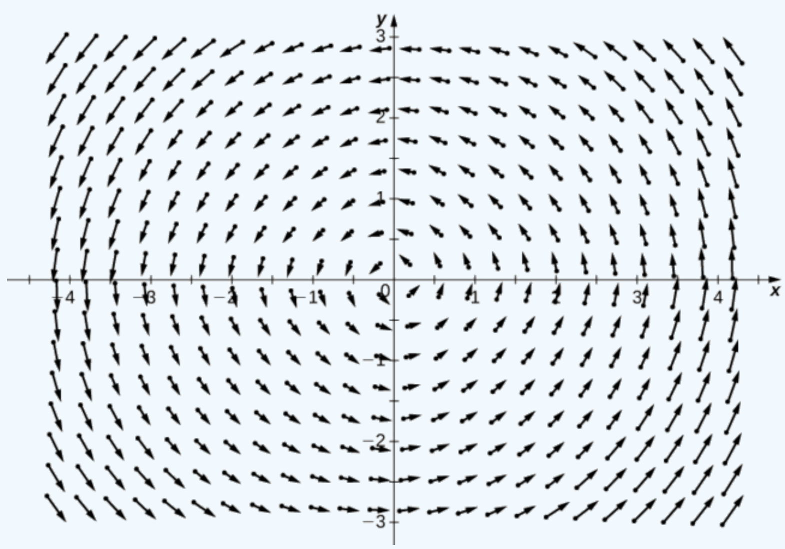
Sketch vector field $\vec{\mathbf{F}}(x, y) = \langle -2y,\, 2x \rangle$. Is the vector field radial, rotational, or neither?

Hint

Substitute enough points into $\vec{\mathbf{F}}$ to get an idea of the shape.

Answer

Rotational



Example 9.1.5: Velocity Field of a Fluid

Suppose that $\vec{\mathbf{v}}(x,y) = -\frac{2y}{x^2+y^2}\,\hat{\mathbf{i}} + \frac{2x}{x^2+y^2}\,\hat{\mathbf{j}}$ is the velocity field of a fluid. How fast is the fluid moving at point $(1,-1)$? (Assume the units of speed are meters per second.)

Solution

To find the velocity of the fluid at point $(1,-1)$, substitute the point into $\vec{\mathbf{v}}$:

$$\vec{\mathbf{v}}(1,-1) = \frac{-2(-1)}{1+1}\,\hat{\mathbf{i}} + \frac{2(1)}{1+1}\,\hat{\mathbf{j}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} \ .$$

The speed of the fluid at $(1,-1)$ is the magnitude of this vector. Therefore, the speed is $||\hat{\mathbf{i}} + \hat{\mathbf{j}}|| = \sqrt{2}$ m/sec.

Exercise 9.1.5

Vector field $\vec{\mathbf{v}}(x,y) = \langle 4|x|,\ 1\rangle$ models the velocity of water on the surface of a river. What is the speed of the water at point $(2,3)$? Use meters per second as the units.

Hint

Remember, speed is the magnitude of velocity.

Answer

$\sqrt{65}$ m/sec

We have examined vector fields that contain vectors of various magnitudes, but just as we have unit vectors, we can also have a unit vector field. A vector field $\vec{\mathbf{F}}$ is a **unit vector field** if the magnitude of each vector in the field is 1. In a unit vector field, the only relevant information is the direction of each vector.

Example 9.1.6: A Unit Vector Field

Show that vector field $\vec{\mathbf{F}}(x,y) = \langle \frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}} \rangle$ is a unit vector field.

Solution

To show that $\vec{\mathbf{F}}$ is a unit field, we must show that the magnitude of each vector is 1. Note that

$$\begin{aligned} \sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + \left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2} &= \sqrt{\frac{y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2}} \\ &= \sqrt{\frac{x^2+y^2}{x^2+y^2}} \\ &= 1 \end{aligned}$$

Therefore, $\vec{\mathbf{F}}$ is a unit vector field.

Exercise 9.1.6

Is vector field $\vec{\mathbf{F}}(x, y) = \langle -y,\, x\rangle$ a unit vector field?

Hint

Calculate the magnitude of $\vec{\mathbf{F}}$ at an arbitrary point (x, y).

Answer

No.

Why are unit vector fields important? Suppose we are studying the flow of a fluid, and we care only about the direction in which the fluid is flowing at a given point. In this case, the speed of the fluid (which is the magnitude of the corresponding velocity vector) is irrelevant, because all we care about is the direction of each vector. Therefore, the unit vector field associated with velocity is the field we would study.

If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is a vector field, then the corresponding unit vector field is $\langle \frac{P}{||\vec{\mathbf{F}}||}, \frac{Q}{||\vec{\mathbf{F}}||}, \frac{R}{||\vec{\mathbf{F}}||}\rangle$. Notice that if $\vec{\mathbf{F}}(x, y) = \langle y,\, -x\rangle$ is the vector field from Example 9.1.6, then the magnitude of $\vec{\mathbf{F}}$ is $\sqrt{x^2 + y^2}$, and therefore the corresponding unit vector field is the field $\vec{\mathbf{G}}$ from the previous example.

If $\vec{\mathbf{F}}$ is a vector field, then the process of dividing $\vec{\mathbf{F}}$ by its magnitude to form unit vector field $\vec{\mathbf{F}}/||\vec{\mathbf{F}}||$ is called normalizing the field $\vec{\mathbf{F}}$.

9.1.0.0.1 Vector Fields in $\mathbb{R}^3$

We have seen several examples of vector fields in $\mathbb{R}^2$; let's now turn our attention to vector fields in $\mathbb{R}^3$. These vector fields can be used to model gravitational or electromagnetic fields, and they can also be used to model fluid flow or heat flow in three dimensions. A two-dimensional vector field can really only model the movement of water on a two-dimensional slice of a river (such as the river's surface). Since a river flows through three spatial dimensions, to model the flow of the entire depth of the river, we need a vector field in three dimensions.

The extra dimension of a three-dimensional field can make vector fields in $\mathbb{R}^3$ more difficult to visualize, but the idea is the same. To visualize a vector field in $\mathbb{R}^3$, plot enough vectors to show the overall shape. We can use a similar method to visualizing a vector field in $\mathbb{R}^2$ by choosing points in each octant.

Just as with vector fields in $\mathbb{R}^2$, we can represent vector fields in $\mathbb{R}^3$ with component functions. We simply need an extra component function for the extra dimension. We write either

$$\vec{\mathbf{F}}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle \tag{9.1.3}$$

or

$$\vec{\mathbf{F}}(x, y, z) = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}. \tag{9.1.4}$$

Example 9.1.7: Sketching a Vector Field in Three Dimensions

Describe vector field $\vec{\mathbf{F}}(x,y,z) = \langle 1,\ 1,\ z\rangle$.

Solution

For this vector field, the x- and y-components are constant, so every point in $\mathbb{R}^3$ has an associated vector with x- and y-components equal to one. To visualize $\vec{\mathbf{F}}$, we first consider what the field looks like in the xy-plane. In the xy-plane, $z = 0$. Hence, each point of the form $(a, b, 0)$ has vector $\langle 1, 1, 0\rangle$ associated with it. For points not in the xy-plane but slightly above it, the associated vector has a small but positive z-component, and therefore the associated vector points slightly upward. For points that are far above the xy-plane, the z-component is large, so the vector is almost vertical. Figure 9.1.6 shows this vector field.

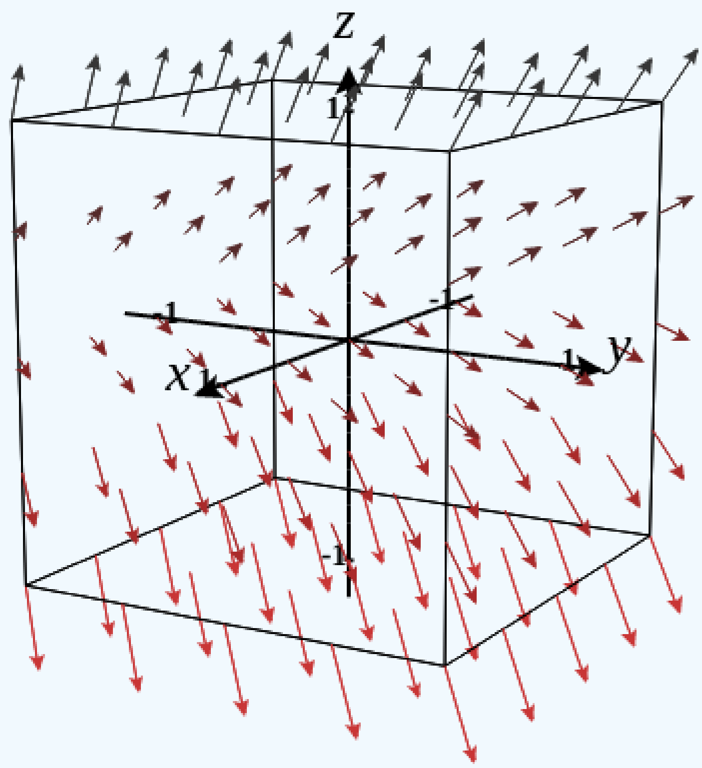


Figure* 9.1.6*: *A visual representation of vector field* $\vec{\mathbf{F}}(x,y,z) = \langle 1,1,z\rangle$.

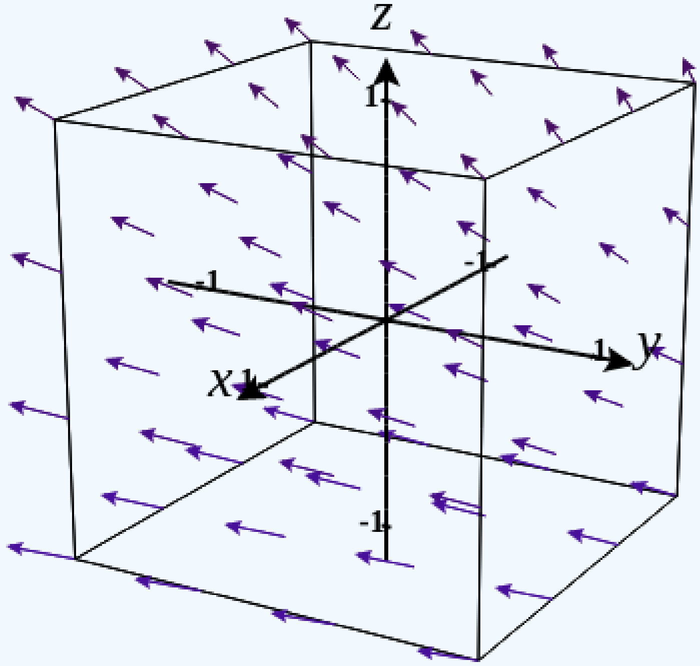
Exercise 9.1.7

Sketch vector field $\vec{\mathbf{G}}(x,y,z) = \langle 2,\ \frac{z}{2},\ 1\rangle$.

Hint

Substitute enough points into the vector field to get an idea of the general shape.

Answer



In the next example, we explore one of the classic cases of a three-dimensional vector field: a gravitational field.

Example 9.1.8: Describing a Gravitational Vector Field

Newton's law of gravitation states that $\vec{\mathbf{F}} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$, where G is the universal gravitational constant. It describes the gravitational field exerted by an object (object 1) of mass m_1 located at the origin on another object (object 2) of mass m_2 located at point (x, y, z). Field $\vec{\mathbf{F}}$ denotes the gravitational force that object 1 exerts on object 2, r is the distance between the two objects, and $\hat{\mathbf{r}}$ indicates the unit vector from the first object to the second. The minus sign shows that the gravitational force attracts toward the origin; that is, the force of object 1 is attractive. Sketch the vector field associated with this equation.

Solution

Since object 1 is located at the origin, the distance between the objects is given by $r=\sqrt{x^2+y^2+z^2}$. The unit vector from object 1 to object 2 is $\hat{\mathbf{r}}=\dfrac{\langle x,y,z\rangle}{||\langle x,y,z\rangle||}$, and hence $\hat{\mathbf{r}}=\langle \frac{x}{r},\frac{y}{r},\frac{z}{r}\rangle$. Therefore, gravitational vector field $\vec{\mathbf{F}}$ exerted by object 1 on object 2 is

$$\vec{\mathbf{F}}=-Gm_1m_2\langle \frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3}\rangle. \tag{9.1.5}$$

This is an example of a radial vector field in $\mathbb{R}^3$.

Figure 9.1.7 shows what this gravitational field looks like for a large mass at the origin. Note that the magnitudes of the vectors increase as the vectors get closer to the origin.

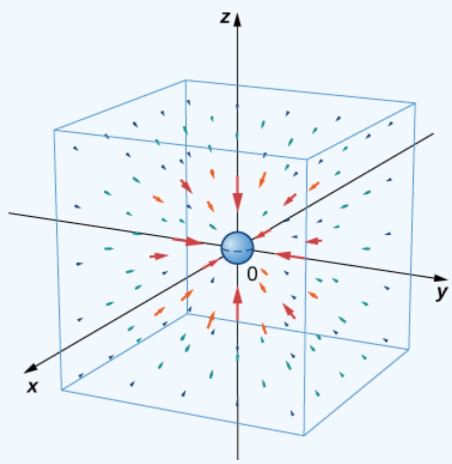


Figure 9.1.7: A visual representation of gravitational vector field $\vec{\mathbf{F}}=-Gm_1m_2\langle \frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3}\rangle$ for a large mass at the origin.

Exercise 9.1.8

The mass of asteroid 1 is 750,000 kg and the mass of asteroid 2 is 130,000 kg. Assume asteroid 1 is located at the origin, and asteroid 2 is located at $(15,-5,10)$, measured in units of 10 to the eighth power kilometers. Given that the universal gravitational constant is $G=6.67384\times 10^{-11} m^3 kg^{-1} s^{-2}$, find the gravitational force vector that asteroid 1 exerts on asteroid 2.

Hint

Follow Example 9.1.8 and first compute the distance between the asteroids.

Answer

1.49063×10^{-18}, 4.96876×10^{-19}, 9.93752×10^{-19} N

In this section, we study a special kind of vector field called a gradient field or a **conservative field**. These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved. Gravitational fields and electric fields associated with a static charge are examples of gradient fields.

Recall that if f is a (scalar) function of x and y, then the gradient of f is

$$\operatorname{grad} f = \vec{\nabla} f(x,y) = f_x(x,y)\hat{\mathbf{i}} + f_y(x,y)\hat{\mathbf{j}}. \tag{9.1.6}$$

We can see from the form in which the gradient is written that $\vec{\nabla} f$ is a vector field in $\mathbb{R}^2$. Similarly, if f is a function of x, y, and z, then the gradient of f is

$$\operatorname{grad} f = \vec{\nabla} f(x,y,z) = f_x(x,y,z)\hat{\mathbf{i}} + f_y(x,y,z)\hat{\mathbf{j}} + f_z(x,y,z)\hat{\mathbf{k}}. \tag{9.1.7}$$

The gradient of a three-variable function is a vector field in $\mathbb{R}^3$. A gradient field is a vector field that can be written as the gradient of a function, and we have the following definition.

DEFINITION: Gradient Field

A vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a gradient field if there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$.

Example 9.1.9: Sketching a Gradient Vector Field

Use technology to plot the gradient vector field of $f(x,y) = x^2y^2$.

Solution

The gradient of f is $\vec{\nabla} f(x,y) = \langle 2xy^2, 2x^2y \rangle$. To sketch the vector field, use a computer algebra system such as Mathematica. Figure 9.1.8 shows $\vec{\nabla} f$.

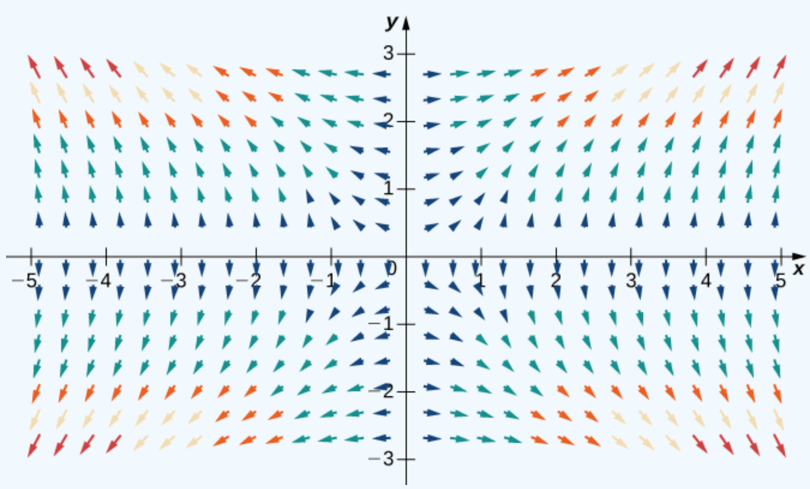


Figure 9.1.8: The gradient vector field is $\vec{\nabla} f$, where $f(x,y) = x^2y^2$.

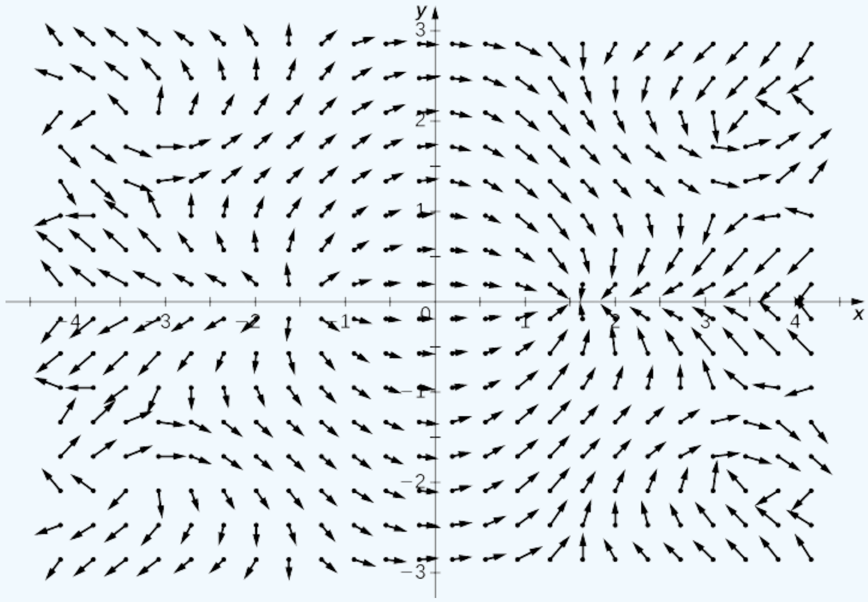
Exercise 9.1.9

Use technology to plot the gradient vector field of $f(x,y) = \sin x \cos y$.

Hint

Find the gradient of f.

Answer



Consider the function $f(x,y)=x^2y^2$ from Example 9.1.9. Figure 9.1.9 shows the level curves of this function overlaid on the function's gradient vector field. The gradient vectors are perpendicular to the level curves, and the magnitudes of the vectors get larger as the level curves get closer together, because closely grouped level curves indicate the graph is steep, and the magnitude of the gradient vector is the largest value of the directional derivative. Therefore, you can see the local steepness of a graph by investigating the corresponding function's gradient field.

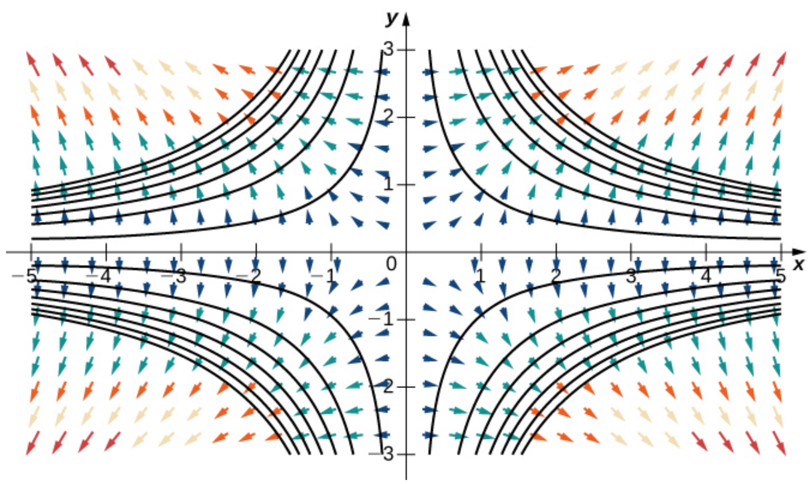


Figure 9.1.9: The gradient field of $f(x,y)=x^2y^2$ and several level curves of f. Notice that as the level curves get closer together, the magnitude of the gradient vectors increases.

As we learned earlier, a vector field $\vec{\mathbf{F}}$ is a conservative vector field, or a gradient field if there exists a scalar function f such that $\vec{\nabla} f=\vec{\mathbf{F}}$. In this situation, f is called a **potential function** for $\vec{\mathbf{F}}$. Conservative vector fields arise in many applications, particularly in physics. The reason such fields are called *conservative* is that they model forces of physical systems in which energy is conserved. We study conservative vector fields in more detail later in this chapter.

You might notice that, in some applications, a potential function f for $\vec{\mathbf{F}}$ is defined instead as a function such that $-\vec{\nabla} f = \vec{\mathbf{F}}$. This is the case for certain contexts in physics, for example.

Example 9.1.10: Verifying a Potential Function

Is $f(x, y, z) = x^2yz - \sin(xy)$ a potential function for vector field

$$\vec{\mathbf{F}}(x, y, z) = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y\rangle \ ?$$

Solution

We need to confirm whether $\vec{\nabla} f = \vec{\mathbf{F}}$. We have

$$\begin{aligned} f_x(x, y) &= 2xyz - y\cos(xy) \\ f_y(x, y) &= x^2z - x\cos(xy) \\ f_z(x, y) &= x^2y \end{aligned} \ .$$

Therefore, $\vec{\nabla} f = \vec{\mathbf{F}}$ and f is a potential function for $\vec{\mathbf{F}}$.

Exercise 9.1.10

Is $f(x, y, z) = x^2\cos(yz) + y^2z^2$ a potential function for $\vec{\mathbf{F}}(x, y, z) = \langle 2x\cos(yz), -x^2z\sin(yz) + 2yz^2, y^2\rangle$?

Hint

Compute the gradient of f.

Answer

No

Example 9.1.11: Verifying a Potential Function

The velocity of a fluid is modeled by field $v(x, y) = \langle xy, \frac{x^2}{2} - y\rangle$. Verify that $f(x, y) = \frac{x^2y}{2} - \frac{y^2}{2}$ is a potential function for $\vec{\mathbf{v}}$.

Solution

To show that f is a potential function, we must show that $\vec{\nabla} f = v$. Note that $f_x(x, y) = xy$ and $f_y(x, y) = \frac{x^2}{2} - y$. Therefore, $\vec{\nabla} f(x, y) = \langle xy, \frac{x^2}{2} - y\rangle$ and f is a potential function for $\vec{\mathbf{v}}$ (Figure 9.1.10).

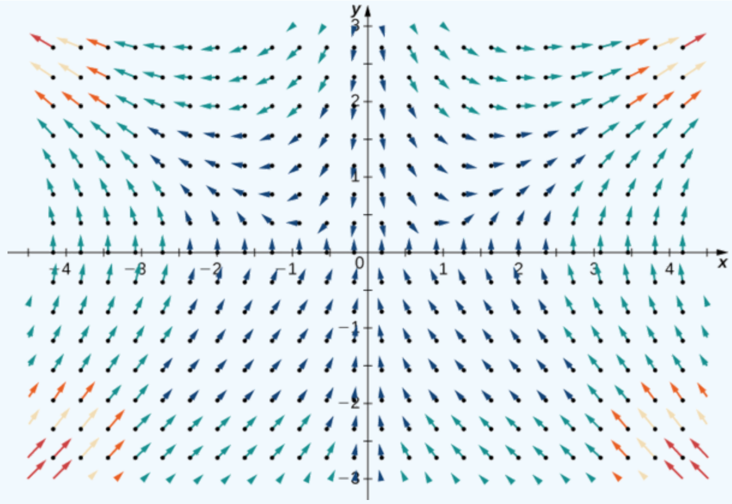


Figure 9.1.10: Velocity field $\vec{\mathbf{v}}(x, y)$ has a potential function and is a conservative field.

Exercise 9.1.11

Verify that $f(x, y) = x^2y^2 + x$ is a potential function for velocity field $\vec{\mathbf{v}}(x, y) = \langle 3x^2y^2 + 1, 2x^3y \rangle$.

Hint

Calculate the gradient.

Answer

$\vec{\nabla} f(x, y) = \vec{\mathbf{v}}(x, y)$

If $\vec{\mathbf{F}}$ is a conservative vector field, then there is at least one potential function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$. But, could there be more than one potential function? If so, is there any relationship between two potential functions for the same vector field? Before answering these questions, let's recall some facts from single-variable calculus to guide our intuition. Recall that if $k(x)$ is an integrable function, then k has infinitely many antiderivatives. Furthermore, if $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ are both antiderivatives of k, then $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ **differ only by a constant**. That is, there is some number C such that $\vec{\mathbf{F}}(x) = \vec{\mathbf{G}}(x) + C$.

Now let $\vec{\mathbf{F}}$ be a conservative vector field and let f and g be potential functions for $\vec{\mathbf{F}}$. Since the gradient is like a derivative, $\vec{\mathbf{F}}$ being conservative means that $\vec{\mathbf{F}}$ is "integrable" with "antiderivatives" f and g. Therefore, if the analogy with single-variable calculus is valid, we expect there is some constant C such that $f(x) = g(x) + C$. The next theorem says that this is indeed the case.

To state the next theorem with precision, we need to assume the domain of the vector field is connected and open. To be connected means if P_1 and P_2 are any two points in the domain, then you can walk from P_1 to P_2 along a path that stays entirely inside the domain.

UNIQUENESS OF POTENTIAL FUNCTIONS

Let $\vec{\mathbf{F}}$ be a conservative vector field on an open and connected domain and let f and g be functions such that $\vec{\nabla} f = \vec{\mathbf{F}}$ and $\vec{\nabla} g = \vec{\mathbf{G}}$. Then, there is a constant C such that $f = g + C$.

Proof

Since f and g are both potential functions for $\vec{\mathbf{F}}$, then $\vec{\nabla} f=(f-g)=\vec{\nabla} f-\vec{\nabla} g=\vec{\mathbf{F}}-\vec{\mathbf{F}}=\vec{\mathbf{0}}$. Let $h=f-g$, then we have $\vec{\nabla} h=\vec{\mathbf{0}}$.We would like to show that h is a constant function.

Assume h is a function of x and y (the logic of this proof extends to any number of independent variables). Since $\vec{\nabla} h=\vec{\mathbf{0}}$, we have $h_x(x,y)=0$ and $h_y(x,y)=0$. The expression $h_x(x,y)=0$ implies that h is a constant function with respect to x—that is, $h(x,y)=k_1(y)$ for some function k_1. Similarly, $h_y(x,y)=0$ implies $h(x,y)=k_2(x)$ for some function k_2. Therefore, function h depends only on y and also depends only on x. Thus, $h(x,y)=C$ for some constant C on the connected domain of $\vec{\mathbf{F}}$. Note that we really do need connectedness at this point; if the domain of $\vec{\mathbf{F}}$ came in two separate pieces, then k could be a constant C_1 on one piece but could be a different constant C_2 on the other piece. Since $f-g=h=C$, we have that $f-g+C$, as desired.

□

Conservative vector fields also have a special property called the ***cross-partial property***. This property helps test whether a given vector field is conservative.

THE CROSS-PARTIAL PROPERTY OF CONSERVATIVE VECTOR FIELDS

Let $\vec{\mathbf{F}}$ be a vector field in two or three dimensions such that the component functions of $\vec{\mathbf{F}}$ have continuous second-order mixed-partial derivatives on the domain of $\vec{\mathbf{F}}$.

If $\vec{\mathbf{F}}(x,y)=\langle P(x,y),Q(x,y)\rangle$ is a conservative vector field in $\mathbb{R}^2$, then

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}. \tag{9.1.8}$$

If $\vec{\mathbf{F}}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$ is a conservative vector field in $\mathbb{R}^3$, then

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

$$\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}$$

$$\frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}.$$

Proof

Since $\vec{\mathbf{F}}$ is conservative, there is a function $f(x,y)$ such that $\vec{\nabla} f=\vec{\mathbf{F}}$. Therefore, by the definition of the gradient, $f_x=P$ and $f_y=Q$. By Clairaut's theorem, $f_{xy}=f_{yx}$, But, $f_{xy}=P_y$ and $f_{yx}=Q_x$, and thus $P_y=Q_x$.

□

Clairaut's theorem gives a fast proof of the cross-partial property of conservative vector fields in $\mathbb{R}^3$, just as it did for vector fields in $\mathbb{R}^2$.

The Cross-Partial Property of Conservative Vector Fields shows that most vector fields are not conservative. The cross-partial property is difficult to satisfy in general, so most vector fields won't have equal cross-partials.

Example 9.1.12: Showing a Vector Field Is Not Conservative

Show that rotational vector field $\vec{\mathbf{F}}(x,y)=\langle y,\,-x\rangle$ is not conservative.

Solution

Let $P(x,y)=y$ and $Q(x,y)=-x$. If $\vec{\mathbf{F}}$ is conservative, then the cross-partials would be equal—that is, P_y would equal Q_x.Therefore, to show that $\vec{\mathbf{F}}$ is not conservative, check that $P_y\neq Q_x$. Since $P_y=1$ and $Q_x=-1$, the vector field is not conservative.

Exercise 9.1.12

Show that vector field $\vec{\mathbf{F}}(x,y)=xy\,\hat{\mathbf{i}}-x^2y\,\hat{\mathbf{j}}$ is not conservative.

Hint

Check the cross-partials.

Answer

$P_y(x,y)=x$ and $Q_x(x,y)=-2xy$. Since $P_y(x,y)\neq Q_x(x,y)$, $\vec{\mathbf{F}}$ is not conservative.

Example 9.1.13: Showing a Vector Field Is Not Conservative

Is vector field $\vec{\mathbf{F}}(x,y,z)=\langle 7,-2,x^3\rangle$ conservative?

Solution

Let $P(x,y,z)=7$, $Q(x,y,z)=-2$, and $R(x,y,z)=x^3$. If $\vec{\mathbf{F}}$ is conservative, then all three cross-partial equations will be satisfied—that is, if $\vec{\mathbf{F}}$ is conservative, then P_y would equal Q_x, Q_z would equal R_y, and R_x would equal P_z. Note that

$$P_y=Q_x=R_y=Q_z=0$$

so the first two necessary equalities hold. However, $R_x(x,y,z)=x^3$ and $P_z(x,y,z)=0$ so $R_x\neq P_z$. Therefore, $\vec{\mathbf{F}}$ is not conservative.

Exercise 9.1.13

Is vector field $\vec{\mathbf{G}}(x,y,z)=\langle y,\,x,\,xyz\rangle$ conservative?

Hint

Check the cross-partials.

Answer

No

We conclude this section with a word of warning: The Cross-Partial Property of Conservative Vector Fields says that if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ has the cross-partial property. The theorem does *not* say that, if $\vec{\mathbf{F}}$ has the cross-partial property, then $\vec{\mathbf{F}}$ is conservative (the converse of an implication is not logically equivalent to the original implication). In other words, The Cross-Partial Property of Conservative Vector Fields can only help determine that a field is not conservative; it does not let you conclude that a vector field is conservative.

For example, consider vector field $\vec{\mathbf{F}}(x,y) = \langle x^2y, \dfrac{x^3}{3}\rangle$. This field has the cross-partial property, so it is natural to try to use The Cross-Partial Property of Conservative Vector Fields to conclude this vector field is conservative. However, this is a misapplication of the theorem. We learn later how to conclude that $\vec{\mathbf{F}}$ is conservative.

9.1.0.1 Key Concepts

- A vector field assigns a vector $\vec{\mathbf{F}}(x,y)$ to each point (x,y) in a subset D of $\mathbb{R}^2$ or $\mathbb{R}^3$. $\vec{\mathbf{F}}(x,y,z)$ to each point (x,y,z) in a subset D of $\mathbb{R}^3$.
- Vector fields can describe the distribution of vector quantities such as forces or velocities over a region of the plane or of space. They are in common use in such areas as physics, engineering, meteorology, oceanography.
- We can sketch a vector field by examining its defining equation to determine relative magnitudes in various locations and then drawing enough vectors to determine a pattern.
- A vector field $\vec{\mathbf{F}}$ is called conservative if there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$.

9.1.0.1 Key Equations

- **Vector field in $\mathbb{R}^2$**
  $\vec{\mathbf{F}}(x,y) = \langle P(x,y),\, Q(x,y)\rangle$
  or
  $\vec{\mathbf{F}}(x,y) = P(x,y)\,\hat{\mathbf{i}} + Q(x,y)\,\hat{\mathbf{j}}$
- **Vector field in $\mathbb{R}^3$**
  $\vec{\mathbf{F}}(x,y,z) = \langle P(x,y,z),\, Q(x,y,z),\, R(x,y,z)\rangle$
  or
  $\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\hat{\mathbf{i}} + Q(x,y,z)\,\hat{\mathbf{j}} + R(x,y,z)\,\hat{\mathbf{k}}$

9.1.1 Glossary

conservative field

a vector field for which there exists a scalar function f such that $\vec{\boldsymbol{\nabla}} f = \vec{\mathbf{F}}$

gradient field

a vector field $\vec{\mathbf{F}}$ for which there exists a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$; in other words, a vector field that is the gradient of a function; such vector fields are also called *conservative*

potential function

a scalar function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$

radial field

a vector field in which all vectors either point directly toward or directly away from the origin; the magnitude of any vector depends only on its distance from the origin

rotational field

a vector field in which the vector at point (x, y) is tangent to a circle with radius $r = \sqrt{x^2 + y^2}$; in a rotational field, all vectors flow either clockwise or counterclockwise, and the magnitude of a vector depends only on its distance from the origin

unit vector field

a vector field in which the magnitude of every vector is 1

vector field

measured in $\mathbb{R}^2$, an assignment of a vector $\vec{\mathbf{F}}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$; in $\mathbb{R}^3$, an assignment of a vector $\vec{\mathbf{F}}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$

9.1.2 Contributors and Attributions

-

Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

9.1E: Exercises

9.1E.1 Exercise $9.1E.1$

For the following exercises, determine whether the statement is *true or false*.

1. The domain of vector field $\vec{\mathbf{F}} = \vec{\mathbf{F}}(x, y)$ is a set of points (x, y) in a plane, and the range of F is a set of *vectors* in the plane.

Answer

True.

2. Vector field $\vec{\mathbf{F}} = \langle 3x^2, 1\rangle$ is a gradient field for both $\phi_1(x, y) = x^3 + y$ and $\phi_2(x, y) = y + x^3 + 100$.

3. Vector field $\vec{\mathbf{F}} = \dfrac{\langle y, x\rangle}{\sqrt{x^2 + y^2}}$ is constant in direction and magnitude on a unit circle

Answer

False.

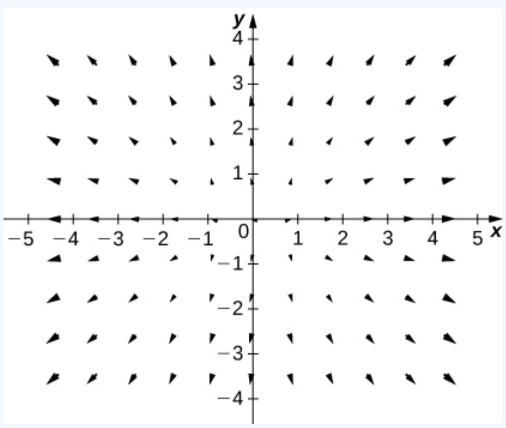
4. Vector field $\vec{\mathbf{F}} = \dfrac{\langle y, x\rangle}{\sqrt{x^2 + y^2}}$ is neither a radial field nor a rotation.

9.1E.2 Exercise $9.1E.2$

For the following exercises, describe each vector field by drawing some of its vectors.

5. [T] $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

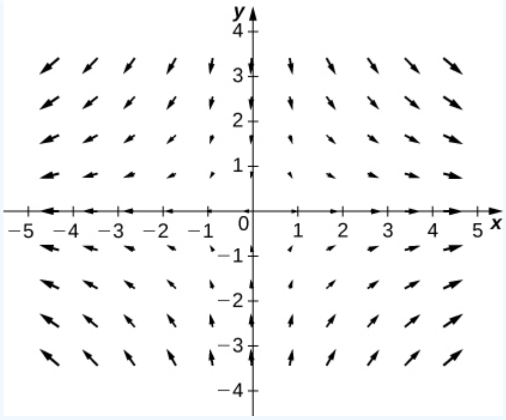
Answer



6. [T] $\vec{\mathbf{F}}(x, y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$

7. [T] $\vec{\mathbf{F}}(x, y) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$

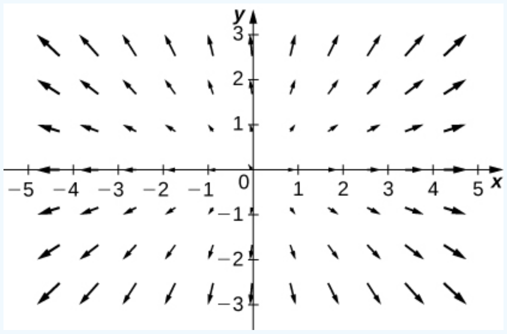
Answer



8. **[T]** $\vec{\mathbf{F}}(x,y)=\ \hat{\mathbf{i}}+\hat{\mathbf{j}}$

9. **[T]** $\vec{\mathbf{F}}(x,y)=2x\,\hat{\mathbf{i}}+3y\,\hat{\mathbf{j}}$

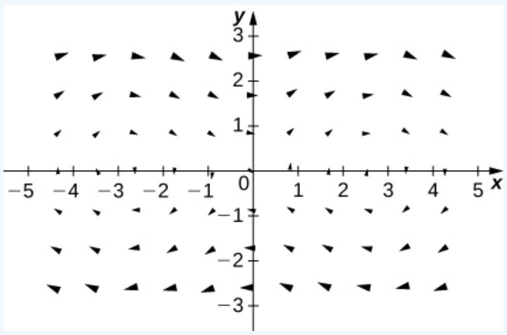
Answer



10. **[T]** $\vec{\mathbf{F}}(x,y)=3\,\hat{\mathbf{i}}+x\,\hat{\mathbf{j}}$

11. **[T]** $\vec{\mathbf{F}}(x,y)=y\,\hat{\mathbf{i}}+sinx\,\hat{\mathbf{j}}$

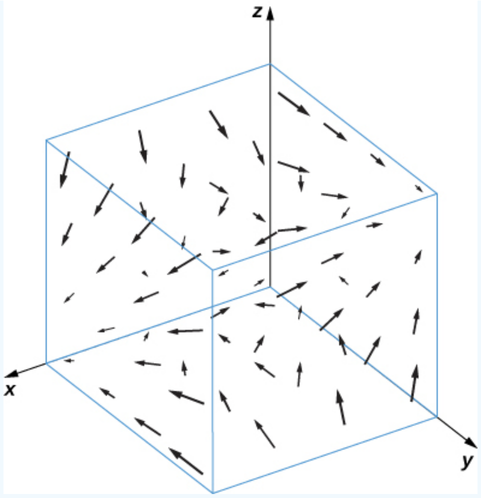
Answer



12. **[T]** $\vec{\mathbf{F}}(x,y,z)=x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}+z\,\hat{\mathbf{k}}$

13. **[T]** $\vec{\mathbf{F}}(x,y,z)=2x\,\hat{\mathbf{i}}-2y\,\hat{\mathbf{j}}-2z\,\hat{\mathbf{k}}$

Answer



14. [T] $\vec{\mathbf{F}}(x,y,z)=yz\,\hat{\mathbf{i}}-xz\,\hat{\mathbf{j}}$

9.1E.3 Exercise $9.1E.3$

For the following exercises, find the gradient vector field $\vec{\mathbf{F}}$ of each function f.

15. $f(x,y)=x\,sin(y)+cos(y)$

Answer

$\vec{\mathbf{F}}(x,y)=sin(y)\,\hat{\mathbf{i}}+(x\,cos(y)-sin(y))\,\hat{\mathbf{j}}$

16. $f(x,y,z)=ze^{-xy}$

17. $f(x,y,z)=x^2y+xy+y^2z$

Answer

$\vec{\mathbf{F}}(x,y,z)=(2xy+y)\,\hat{\mathbf{i}}+(x^2+x+2yz)\,\hat{\mathbf{j}}+y^2\,\hat{\mathbf{k}}$

18. $f(x,y)=x^2sin(5y)$

19. $f(x,y)=ln(1+x^2+2y^2)$

Answer

$$\vec{\mathbf{F}}(x,y)=(\frac{2x}{1+x^2+2y^2})\,\hat{\mathbf{i}}+(\frac{4y}{1+x^2+2y^2})\,\hat{\mathbf{j}}$$

20. $f(x,y,z)=x\,cos(\frac{y}{z})$

21. What is vector field $\vec{\mathbf{F}}(x,y)$ with a value at (x,y) that is of unit length and points toward $(1,0)$?

Answer

$$\vec{\mathbf{F}}(x,y)=\frac{(1-x)\,\hat{\mathbf{i}}-y\,\hat{\mathbf{j}}}{\sqrt{(1-x)^2+y^2}}$$

9.1E.4 Exercise $9.1E.4$

For the following exercises, write formulas for the vector fields with the given properties.

22. All vectors are parallel to the *x*-axis and all vectors on a vertical line have the same magnitude.

23. All vectors point toward the origin and have a constant length.

Solution: $\vec{\mathbf{F}}(x,y)=\dfrac{(y\,\hat{\mathbf{i}}-x\,\hat{\mathbf{j}})}{\sqrt{x^2+y^2}}$

24. All vectors are of unit length and are perpendicular to the position vector at that point.

25. Give a formula $\vec{\mathbf{F}}(x,y)=M(x,y)\,\hat{\mathbf{i}}+N(x,y)\,\hat{\mathbf{j}}$ for the vector field in a plane that has the properties that $\vec{\mathbf{F}}=0$ at $(0,0)$ and that at any other point (a,b), $\vec{\mathbf{F}}$ is tangent to circle $x^2+y^2=a^2+b^2$ and points in the clockwise direction with magnitude $|\vec{\mathbf{F}}|=\sqrt{a^2+b^2}$.

Solution: $\vec{\mathbf{F}}(x,y)=y\,\hat{\mathbf{i}}-x\,\hat{\mathbf{j}}$

26. Is vector field $\vec{\mathbf{F}}(x,y)=(P(x,y),Q(x,y))=(sinx+y)\,\hat{\mathbf{i}}+(cosy+x)\,\hat{\mathbf{j}}$ a gradient field?

27. Find a formula for vector field $\vec{\mathbf{F}}(x,y)=M(x,y)\,\hat{\mathbf{i}}+N(x,y)\,\hat{\mathbf{j}}$ given the fact that for all points (x,y), $\vec{\mathbf{F}}$ points toward the origin and $|\vec{\mathbf{F}}|=\dfrac{10}{x^2+y^2}$.

Answer

$$\vec{\mathbf{F}}(x,y)=\frac{-10}{(x^2+y^2)^{3/2}}(x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}})$$

9.1E.5 Exercise $\mathbf{9.1\mathit{E}.5}$

For the following exercises, assume that an electric field in the *xy*-plane caused by an infinite line of charge along the *x*-axis is a gradient field with potential function $V(x,y)=cln(\dfrac{r_0}{\sqrt{x^2+y^2}})$, where $c>0$ is a constant and r_0 is a reference distance at which the potential is assumed to be zero.

28. Find the components of the electric field in the *x*- and *y*-directions, where $\vec{\mathbf{E}}(x,y)=-\nabla V(x,y)$.

29. Show that the electric field at a point in the *xy*-plane is directed outward from the origin and has magnitude $|\vec{\mathbf{E}}|=\dfrac{c}{r}$, where $r=\sqrt{x^2=y^2}$.

Answer

$$\vec{\mathbf{E}}=\frac{c}{|r|^2 r}=\frac{c}{|r|}\frac{r}{|r|}$$

9.1E.6 Exercise $\mathbf{9.1\mathit{E}.6}$

A ***flow line*** (or *streamline*) of a vector field\vecs{F}is a curve **r**(t) such that \(d**r**/dt=\vecs{F}(**r**(t))\). If $\vec{\mathbf{F}}$ represents the velocity field of a moving particle, then the flow lines are paths taken by the particle. Therefore, flow lines are tangent to the vector field. For the following exercises, show that the given curve \(**c**(t)\) is a flow line of the given velocity vector field $\vec{\mathbf{F}}(x,y,z)$.

30. $c(t)=(e^{2t},ln|t|,\frac{1}{t}),t\neq 0;\vec{\mathbf{F}}(x,y,z)=\langle 2x,z,-z^2\rangle$

31. $c(t)=(sint,cost,e^t);\vec{\mathbf{F}}(x,y,z)=\langle y,-x,z\rangle$

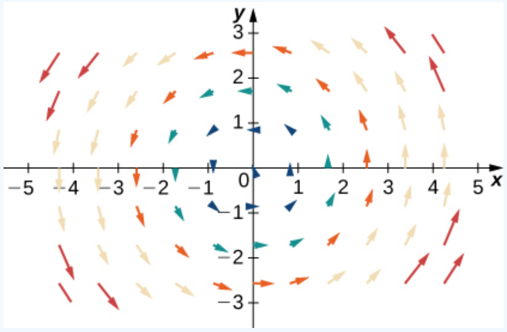
Answer

\(**c**′(t)=(cost,−sint,e^{−t})=\vecs{F}(**c**(t))\)

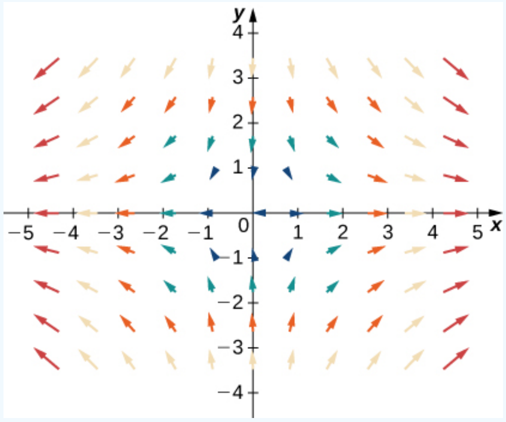
9.1E.7 Exercise $9.1E.7$

For the following exercises, let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}, \vec{\mathbf{G}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}},$ and $\vec{\mathbf{H}} = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}.$ Match $\vec{\mathbf{F}}$, $\vec{\mathbf{G}}$, and $\vec{\mathbf{H}}$ with their graphs.

32.



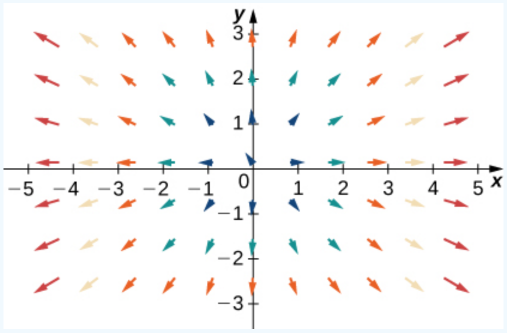
33.



Answer

$\vec{\mathbf{H}}$

34.



Answer

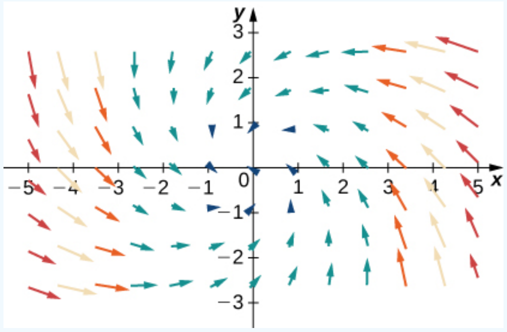
Add texts here. Do not delete this text first.

9.1E.8 Exercise $9.1E.8$

For the following exercises, let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}, \vec{\mathbf{G}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}, and \vec{\mathbf{H}} = -x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}.$ Match the vector fields with their graphs in (I)−(IV).

a. $\vec{\mathbf{F}} + \vec{\mathbf{G}}$
b. $\vec{\mathbf{F}} + \vec{\mathbf{H}}$
c. $\vec{\mathbf{H}} + \vec{\mathbf{G}}$
d. $-\vec{\mathbf{F}} + \vec{\mathbf{G}}$

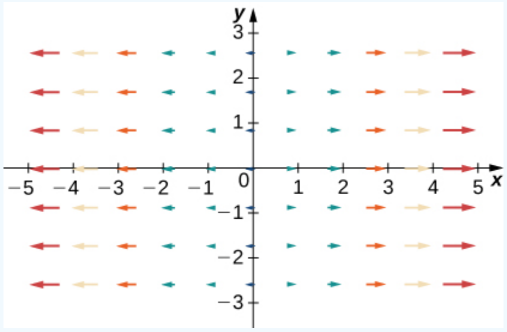
35.



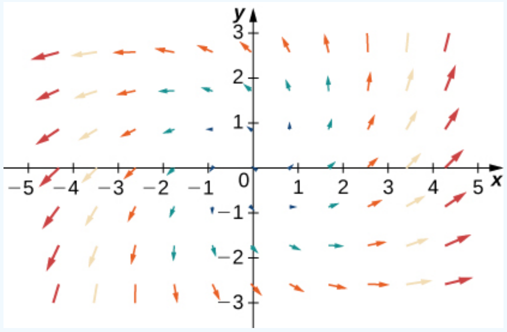
Answer

$-\vec{\mathbf{F}} + \vec{\mathbf{G}}$

36.



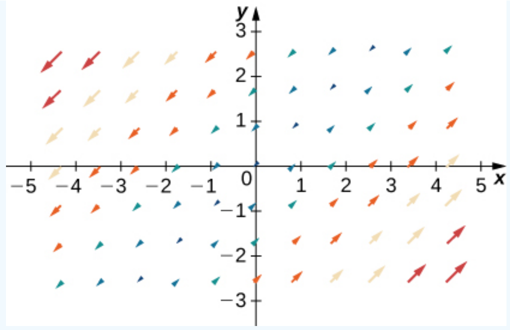
37.



Answer

$\vec{\mathbf{F}} + \vec{\mathbf{G}}$

38.



y
3
2
1
0
−1
−2
−3
−5
−4
−3
−2
−1
1
2
3
4
5
x

9.2: Line Integrals

Learning Objectives

- Calculate a scalar line integral along a curve.
- Calculate a vector line integral along an oriented curve in space.
- Use a line integral to compute the work done in moving an object along a curve in a vector field.
- Describe the flux and circulation of a vector field.

We are familiar with single-variable integrals of the form $\int_a^b f(x)\,dx$, where the domain of integration is an interval $[a,b]$. Such an interval can be thought of as a curve in the xy-plane, since the interval defines a line segment with endpoints $(a,0)$ and $(b,0)$—in other words, a line segment located on the x-axis. Suppose we want to integrate over *any* curve in the plane, not just over a line segment on the x-axis. Such a task requires a new kind of integral, called a *line integral.*

Line integrals have many applications to engineering and physics. They also allow us to make several useful generalizations of the Fundamental Theorem of Calculus. And, they are closely connected to the properties of vector fields, as we shall see.

9.2.1 Scalar Line Integrals

A line integral gives us the ability to integrate multivariable functions and vector fields over arbitrary curves in a plane or in space. There are two types of line integrals: scalar line integrals and vector line integrals. Scalar line integrals are integrals of a scalar function over a curve in a plane or in space. Vector line integrals are integrals of a vector field over a curve in a plane or in space. Let's look at scalar line integrals first.

A scalar line integral is defined just as a single-variable integral is defined, except that for a scalar line integral, the integrand is a function of more than one variable and the domain of integration is a curve in a plane or in space, as opposed to a curve on the x-axis.

For a scalar line integral, we let C be a smooth curve in a plane or in space and let ff be a function with a domain that includes C. We chop the curve into small pieces. For each piece, we choose point P in that piece and evaluate f at P. (We can do this because all the points in the curve are in the domain of f.) We multiply $f(P)$ by the arc length of the piece Δs, add the product $f(P)\Delta s$ over all the pieces, and then let the arc length of the pieces shrink to zero by taking a limit. The result is the scalar line integral of the function over the curve.

For a formal description of a scalar line integral, let C be a smooth curve in space given by the parameterization $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Let $f(x,y,z)$ be a function with a domain that includes curve C. To define the line integral of the function f over C, we begin as most definitions of an integral begin: we chop the curve into small pieces. Partition the parameter interval $[a,b]$ into n subintervals $[t_{i-l}, t_i]$ of equal width for $1 \le i \le n$, where $t_0 = a$ and $t_n = b$ (Figure $9.2.1$). Let t_i^* be a value in the i^{th} interval $[t_{i-l}, t_i]$. Denote the endpoints of $\vec{\mathbf{r}}(t_0)$, $\vec{\mathbf{r}}(t_1)$,..., $\vec{\mathbf{r}}(t_n)$ by P_0,..., P_n. Points P_i divide curve C into n pieces C_1, C_2,..., C_n,with lengths Δs_1, Δs_2,..., Δs_n, respectively. Let P_i^* denote the endpoint of $\vec{\mathbf{r}}(t_i^*)$ for $1 \le i \le n$. Now, we evaluate the function f at point P_i^* for $1 \le i \le n$. Note that P_i^* is in piece C_1, and therefore P_i^* is in the domain of f. Multiply $f(P_i^*)$ by the length Δs_1 of C_1, which gives the area of the "sheet" with base C_1, and height $f(P_i^*)$. This is analogous to using rectangles to approximate area in a single-variable integral. Now, we form the sum $\sum_{i=1}^{n} f(P_i^*)\,\Delta s_i$.

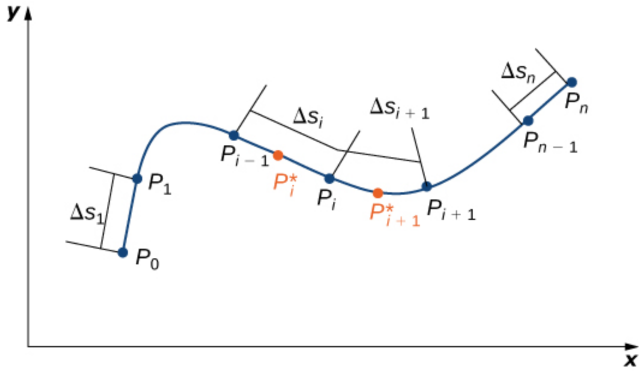


Figure 9.2.1: Curve C has been divided into n pieces, and a point inside each piece has been chosen.

Note the similarity of this sum versus a Riemann sum; in fact, this definition is a generalization of a Riemann sum to arbitrary curves in space. Just as with Riemann sums and integrals of form $\int_a^b g(x)\,dx$, we define an integral by letting the width of the pieces of the curve shrink to zero by taking a limit. The result is the scalar line integral of f along C.

You may have noticed a difference between this definition of a scalar line integral and a single-variable integral. In this definition, the arc lengths Δs_1, Δs_2,..., Δs_n aren't necessarily the same; in the definition of a single-variable integral, the curve in the x-axis is partitioned into pieces of equal length. This difference does not have any effect in the limit. As we shrink the arc lengths to zero, their values become close enough that any small difference becomes irrelevant.

DEFINITION: scalar line integral

Let f be a function with a domain that includes the smooth curve C that is parameterized by $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \leq t \leq b$. The *scalar line integral* of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\,\Delta s_i \tag{9.2.1}$$

if this limit exists t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t)$, $y = y(t)$, and $a \leq t \leq b$. If C is smooth and $f(x,y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\,\Delta s_i, \tag{9.2.2}$$

if this limit exists.

If f is a continuous function on a smooth curve C, then $\int_C f\,ds$ always exists. Since $\int_C f\,ds$ is defined as a limit of Riemann sums, the continuity of f is enough to guarantee the existence of the limit, just as the integral $\int_a^b g(x)\,dx$ exists if g is continuous over $[a,b]$.

Before looking at how to compute a line integral, we need to examine the geometry captured by these integrals. Suppose that $f(x,y) \geq 0$ for all points (x,y) on a smooth planar curve C. Imagine taking curve C and projecting it "up" to the surface defined by $f(x,y)$, thereby creating a new curve C' that lies in the graph of $f(x,y)$ (Figure 9.2.2). Now we drop a "sheet" from C' down to the xy-plane. The area of this sheet is $\int_C f(x,y)ds$. If $f(x,y) \leq 0$ for some points in C, then the value of $\int_C f(x,y)\,ds$ is the area above the xy-plane less the area below the xy-plane. (Note the similarity with integrals of the form $\int_a^b g(x)\,dx$.)

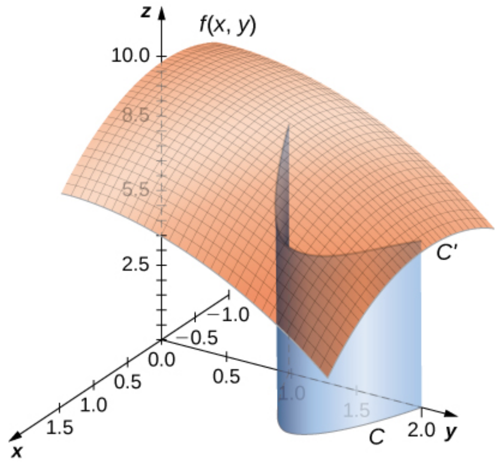


Figure 9.2.2: The area of the blue sheet is $\int_C f(x, y)\, ds$.

From this geometry, we can see that line integral $\int_C f(x, y)\, ds$ does not depend on the parameterization $\vec{\mathbf{r}}(t)$ of C. As long as the curve is traversed exactly once by the parameterization, the area of the sheet formed by the function and the curve is the same. This same kind of geometric argument can be extended to show that the line integral of a three-variable function over a curve in space does not depend on the parameterization of the curve.

Example 9.2.1: Finding the Value of a Line Integral

Find the value of integral $\int_C 2\, ds$, where C is the upper half of the unit circle.

Solution

The integrand is $f(x, y) = 2$. Figure 9.2.3 shows the graph of $f(x, y) = 2$, curve C, and the sheet formed by them. Notice that this sheet has the same area as a rectangle with width π and length 2. Therefore, $\int_C 2\, ds = 2\pi \text{ units}^2$.

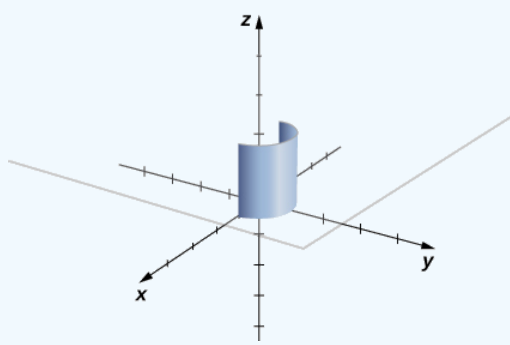


Figure 9.2.3: The sheet that is formed by the upper half of the unit circle in a plane and the graph of $f(x, y) = 2$.

To see that $\int_C 2\, ds = 2\pi$ using the definition of line integral, we let $\vec{\mathbf{r}}(t)$ be a parameterization of C. Then, $f(\vec{\mathbf{r}}(t_i)) = 2$ for any number t_i in the domain of $\vec{\mathbf{r}}$. Therefore,

$$\begin{aligned}
\int_C f\,ds &= \lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\,\Delta s_i \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} 2\,\Delta s_i \\
&= 2\lim_{n\to\infty}\sum_{i=1}^{n} \Delta s_i \\
&= 2(\text{length of } C) \\
&= 2\pi\,\text{units}^2.
\end{aligned}$$

Exercise 9.2.1

Find the value of $\displaystyle\int_C (x+y)\,ds$, where C is the curve parameterized by $x=t$, $y=t$, $0\le t\le 1$.

Hint

Find the shape formed by C and the graph of function $f(x,y)=x+y$.

Answer

$\sqrt{2}$

Note that in a scalar line integral, the integration is done with respect to arc length s, which can make a scalar line integral difficult to calculate. To make the calculations easier, we can translate $\displaystyle\int_C f\,ds$ to an integral with a variable of integration that is t.

Let $\vec{\mathbf{r}}(t)=\langle x(t),y(t),z(t)\rangle$ for $a\le t\le b$ be a parameterization of C. Since we are assuming that C is smooth, $\vec{\mathbf{r}}\,'(t)=\langle x'(t),y'(t),z'(t)\rangle$ is continuous for all t in $[a,b]$. In particular, $x'(t)$, $y'(t)$, and $z'(t)$ exist for all t in $[a,b]$. According to the arc length formula, we have

$$\text{length}(C_i)=\Delta s_i=\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt. \tag{9.2.3}$$

If width $\Delta t_i=t_i-t_{i-1}$ is small, then function $\displaystyle\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt\approx\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i$, $\|\vec{\mathbf{r}}\,'(t)\|$ is almost constant over the interval $[t_{i-1},t_i]$.Therefore,

$$\int_{t_{i-1}}^{t_i}\|\vec{\mathbf{r}}\,'(t)\|\,dt\approx\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i, \tag{9.2.4}$$

and we have

$$\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\,\Delta s_i\approx\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i. \tag{9.2.5}$$

See Figure 9.2.4.

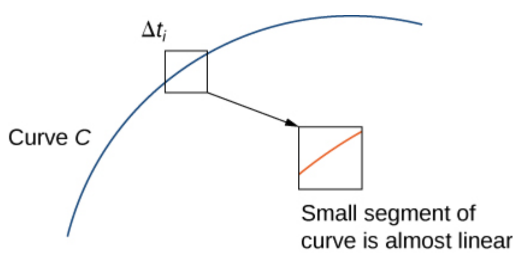


Figure 9.2.4: If we zoom in on the curve enough by making Δt_i very small, then the corresponding piece of the curve is approximately linear.

Note that

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i = \int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt. \tag{9.2.6}$$

In other words, as the widths of intervals $[t_{i-1}, t_i]$ shrink to zero, the sum $\sum_{i=1}^{n} f(\vec{\mathbf{r}}(t_i^*))\|\vec{\mathbf{r}}\,'(t_i^*)\|\,\Delta t_i$ converges to the integral $\int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt$. Therefore, we have the following theorem.

Theorem: EVALUATING A SCALAR LINE INTEGRAL

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$. Then

$$\int_C f\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\|\vec{\mathbf{r}}\,'(t)\|\,dt. \tag{9.2.7}$$

Although we have labeled Equation 9.2.4 as an equation, it is more accurately considered an approximation because we can show that the left-hand side of Equation 9.2.4 approaches the right-hand side as $n \to \infty$. In other words, letting the widths of the pieces shrink to zero makes the right-hand sum arbitrarily close to the left-hand sum. Since

$$\|\vec{\mathbf{r}}\,'(t)\| = \sqrt{(x\,'(t))^2 + (y\,'(t))^2 + (z\,'(t))^2}, \tag{9.2.8}$$

we obtain the following theorem, which we use to compute scalar line integrals.

Theorem: Scalar Line Integral Calculation

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Then

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x\,'(t))^2 + (y\,'(t))^2 + (z\,'(t))^2}\,dt. \tag{9.2.9}$$

Similarly,

$$\int_C f(x,y)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x\,'(t))^2 + (y\,'(t))^2}\,dt \tag{9.2.10}$$

if C is a planar curve and f is a function of two variables.

Note that a consequence of this theorem is the equation $ds = \|\vec{\mathbf{r}}\,'(t)\|\,dt$. In other words, the change in arc length can be viewed as a change in the t-domain, scaled by the magnitude of vector $\vec{\mathbf{r}}\,'(t)$.

Example 9.2.2: Evaluating a Line Integral

Find the value of integral $\int_C (x^2 + y^2 + z)\,ds$, where C is part of the helix parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 2\pi$.

Solution

To compute a scalar line integral, we start by converting the variable of integration from arc length s to t. Then, we can use Equation 9.2.1 to compute the integral with respect to t. Note that

$$f(\vec{\mathbf{r}}(t)) = \cos^2 t + \sin^2 t + t = 1 + t$$

and

$$\sqrt{(x\,'(t))^2 + (y\,'(t))^2 + (z\,'(t))^2} = \sqrt{(-\sin(t))^2 + \cos^2(t) + 1} = \sqrt{2}.$$

Therefore,

$$\int_C (x^2+y^2+z)\,ds = \int_0^{2\pi} (1+t)\sqrt{2}\,dt.$$

Notice that Equation 9.2.1 translated the original difficult line integral into a manageable single-variable integral. Since

$$\begin{aligned}\int_0^{2\pi} (1+t)\sqrt{2}\,dt &= \left[\sqrt{2}t + \frac{\sqrt{2}t^2}{2}\right]_0^{2\pi} \\ &= 2\sqrt{2}\pi + 2\sqrt{2}\pi^2,\end{aligned}$$

we have

$$\int_C (x^2+y^2+z)\,ds = 2\sqrt{2}\pi + 2\sqrt{2}\pi^2.$$

Exercise 9.2.2

Evaluate $\int_C (x^2+y^2+z)ds$, where C is the curve with parameterization $\vec{\mathbf{r}}(t) = \langle \sin(3t), \cos(3t)\rangle$, $0 \le t \le \frac{\pi}{4}$.

Hint

Use the two-variable version of scalar line integral definition (Equation 9.2.2).

Answer

$$\frac{1}{3} + \frac{\sqrt{2}}{6} + \frac{3\pi}{4} \tag{9.2.11}$$

Example 9.2.3: Independence of Parameterization

Find the value of integral $\int_C (x^2+y^2+z)\,ds$, where C is part of the helix parameterized by $\vec{\mathbf{r}}(t) = \langle \cos(2t), \sin(2t), 2t\rangle$, $0 \le t \le \pi$. Notice that this function and curve are the same as in the previous example; the only difference is that the curve has been reparameterized so that time runs twice as fast.

Solution

As with the previous example, we use Equation 9.2.1 to compute the integral with respect to t. Note that $f(\vec{\mathbf{r}}(t)) = \cos^2(2t) + \sin^2(2t) + 2t = 2t+1$ and

$$\begin{aligned}\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} &= \sqrt{(-\sin t + \cos t + 4)} \\ &= 22\end{aligned}$$

so we have

$$\begin{aligned}\int_C (x^2+y^2+z)ds &= 2\sqrt{2}\int_0^{\pi} (1+2t)dt \\ &= 2\sqrt{2}\left[t+t^2\right]_0^{\pi} \\ &= 2\sqrt{2}(\pi+\pi^2).\end{aligned}$$

Notice that this agrees with the answer in the previous example. Changing the parameterization did not change the value of the line integral. Scalar line integrals are independent of parameterization, as long as the curve is traversed exactly once by the parameterization.

Exercise 9.2.3

Evaluate line integral $\int_C (x^2 + yz)\, ds$, where C is the line with parameterization $\vec{\mathbf{r}}(t) = \langle 2t, 5t, -t\rangle$, $0 \le t \le 10$. Reparameterize C with parameterization $s(t) = \langle 4t, 10t, -2t\rangle$, $0 \le t \le 5$, recalculate line integral $\int_C (x^2 + yz)\, ds$, and notice that the change of parameterization had no effect on the value of the integral.

Hint

Use Equation 9.2.1.

Answer

Both line integrals equal $-\dfrac{1000\sqrt{30}}{3}$.

Now that we can evaluate line integrals, we can use them to calculate arc length. If $f(x,y,z) = 1$, then

$$\begin{aligned}\int_C f(x,y,z)\, ds &= \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\,\Delta s_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i \\ &= \lim_{n\to\infty} \text{length}(C) \\ &= \text{length}(C).\end{aligned}$$

Therefore, $\int_C 1\, ds$ is the arc length of C.

Example 9.2.4: Calculating Arc Length

A wire has a shape that can be modeled with the parameterization $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 4\pi$. Find the length of the wire.

Solution

The length of the wire is given by $\int_C 1\, ds$, where C is the curve with parameterization $\vec{\mathbf{r}}$. Therefore,

$$\begin{aligned}\text{The length of the wire} &= \int_C 1\, ds \\ &= \int_0^{4\pi} ||\vec{\mathbf{r}}\,'(t)||\, dt \\ &= \int_0^{4\pi} \sqrt{(-\sin t)^2 + \cos^2 t + t}\, dt \\ &= \int_0^{4\pi} \sqrt{1+t}\, dt \\ &= \left.\frac{2(1+t)^{\frac{3}{2}}}{3}\right|_0^{4\pi} \\ &= \frac{2}{3}\left((1+4\pi)^{3/2} - 1\right).\end{aligned}$$

Exercise 9.2.4

Find the length of a wire with parameterization $\vec{\mathbf{r}}(t)=\langle 3t+1, 4-2t, 5+2t\rangle$, $0\leq t\leq 4$.

Hint

Find the line integral of one over the corresponding curve.

Answer

$4\sqrt{17}$

9.2.2 Vector Line Integrals

The second type of line integrals are vector line integrals, in which we integrate along a curve through a vector field. For example, let

$$\vec{\mathbf{F}}(x,y,z)=P(x,y,z)\,\hat{\mathbf{i}}+Q(x,y,z)\,\hat{\mathbf{j}}+R(x,y,z)\,\hat{\mathbf{k}} \tag{9.2.12}$$

be a continuous vector field in $\mathbb{R}^3$ that represents a force on a particle, and let C be a smooth curve in $\mathbb{R}^3$ contained in the domain of $\vec{\mathbf{F}}$. How would we compute the work done by $\vec{\mathbf{F}}$ in moving a particle along C?

To answer this question, first note that a particle could travel in two directions along a curve: a forward direction and a backward direction. The work done by the vector field depends on the direction in which the particle is moving. Therefore, we must specify a direction along curve C; such a specified direction is called an orientation of a curve. The specified direction is the positive direction along C; the opposite direction is the negative direction along C. When C has been given an orientation, C is called an oriented curve (Figure $9.2.5$). The work done on the particle depends on the direction along the curve in which the particle is moving.

A closed curve is one for which there exists a parameterization $\vec{\mathbf{r}}(t)$, $a\leq t\leq b$, such that $\vec{\mathbf{r}}(a)=\vec{\mathbf{r}}(b)$, and the curve is traversed exactly once. In other words, the parameterization is one-to-one on the domain (a,b).

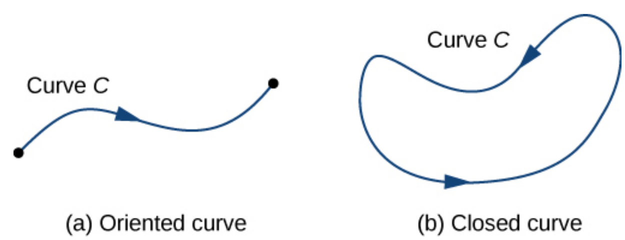


Figure $9.2.5$: (a) An oriented curve between two points. (b) A closed oriented curve.

Let $\vec{\mathbf{r}}(t)$ be a parameterization of C for $a\leq t\leq b$ such that the curve is traversed exactly once by the particle and the particle moves in the positive direction along C. Divide the parameter interval $[a,b]$ into n subintervals $[t_{i-1},t_i]$, $0\leq i\leq n$, of equal width. Denote the endpoints of $r(t_0)$, $r(t_1)$,…, $r(t_n)$ by P_0,…,P_n. Points P_i divide C into n pieces. Denote the length of the piece from P_{i-1} to P_i by Δs_i. For each i, choose a value t_i^* in the subinterval $[t_{i-1},t_i]$. Then, the endpoint of $\vec{\mathbf{r}}(t_i^*)$ is a point in the piece of C between P_{i-1} and P_i (Figure $9.2.6$). If Δs_i is small, then as the particle moves from P_{i-1} to P_i along C, it moves approximately in the direction of $\vec{\mathbf{T}}(P_i)$, the unit tangent vector at the endpoint of $\vec{\mathbf{r}}(t_i^*)$. Let P_i^* denote the endpoint of $\vec{\mathbf{r}}(t_i^*)$. Then, the work done by the force vector field in moving the particle from P_{i-1} to P_i is $\vec{\mathbf{F}}(P_i^*)\cdot(\Delta s_i\vec{\mathbf{T}}(P_i^*))$, so the total work done along C is

$$\sum_{i=1}^{n}\vec{\mathbf{F}}(P_i^*)\cdot(\Delta s_i\vec{\mathbf{T}}(P_i^*))=\sum_{i=1}^{n}\vec{\mathbf{F}}(P_i^*)\cdot\vec{\mathbf{T}}(P_i^*)\,\Delta s_i. \tag{9.2.13}$$

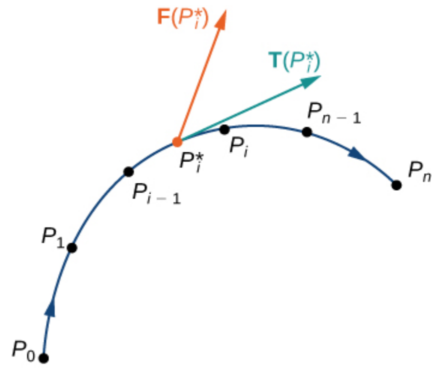


Figure 9.2.6: Curve C is divided into n pieces, and a point inside each piece is chosen. The dot product of any tangent vector in the ith piece with the corresponding vector $\vec{\mathbf{F}}$ is approximated by $\vec{\mathbf{F}}(P_i^*)\cdot\vec{\mathbf{T}}(P_i^*)$.

Letting the arc length of the pieces of C get arbitrarily small by taking a limit as $n\to\infty$ gives us the work done by the field in moving the particle along C. Therefore, the work done by $\vec{\mathbf{F}}$ in moving the particle in the positive direction along C is defined as

$$W=\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds, \tag{9.2.14}$$

which gives us the concept of a vector line integral.

DEFINITION: Line Integral of a vector field

The vector line integral of vector field $\vec{\mathbf{F}}$ along oriented smooth curve C is

$$\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds=\lim_{n\to\infty}\sum_{i=1}^{n}\vec{\mathbf{F}}(P_i^*)\cdot\vec{\mathbf{T}}(P_i^*)\Delta s_i \tag{9.2.15}$$

if that limit exists.

With scalar line integrals, neither the orientation nor the parameterization of the curve matters. As long as the curve is traversed exactly once by the parameterization, the value of the line integral is unchanged. With vector line integrals, the orientation of the curve does matter. If we think of the line integral as computing work, then this makes sense: if you hike up a mountain, then the gravitational force of Earth does negative work on you. If you walk down the mountain by the exact same path, then Earth's gravitational force does positive work on you. In other words, reversing the path changes the work value from negative to positive in this case. Note that if C is an oriented curve, then we let $-C$ represent the same curve but with opposite orientation.

As with scalar line integrals, it is easier to compute a vector line integral if we express it in terms of the parameterization function $\vec{\mathbf{r}}$ and the variable t. To translate the integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}ds$ in terms of t, note that unit tangent vector $\vec{\mathbf{T}}$ along C is given by $\vec{\mathbf{T}}=\dfrac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|}$ (assuming $\|\vec{\mathbf{r}}\,'(t)\|\neq 0$). Since $ds=\|\vec{\mathbf{r}}\,'(t)\|\,dt$, as we saw when discussing scalar line integrals, we have

$$\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds=\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|}\|\vec{\mathbf{r}}\,'(t)\|dt=\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{r}}\,'(t)\,dt. \tag{9.2.16}$$

Thus, we have the following formula for computing vector line integrals:

$$\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds=\int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{r}}\,'(t)\,dt. \tag{9.2.17}$$

Because of Equation 9.2.17, we often use the notation $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ for the line integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$.

If $\vec{\mathbf{r}}(t)=\langle x(t),y(t),z(t)\rangle$, then $\dfrac{d\vec{\mathbf{r}}}{dt}$ denotes vector $\langle x\,'(t),y\,'(t),z\,'(t)\rangle$, and $d\vec{\mathbf{r}}=\vec{\mathbf{r}}\,'(t)\,dt$.

Example 9.2.5: Evaluating a Vector Line Integral

Find the value of integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the semicircle parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq \pi$ and $\vec{\mathbf{F}} = \langle -y, x \rangle$.

Solution

We can use Equation 9.2.17 to convert the variable of integration from s to t. We then have

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -\sin t, \cos t \rangle \text{ and } \vec{\mathbf{r}}\,'(t) = \langle -\sin t, \cos t \rangle. \tag{9.2.18}$$

Therefore,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^\pi \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt \\ &= \int_0^\pi \sin^2 t + \cos^2 t \, dt \\ &= \int_0^\pi 1 \, dt = \pi. \end{aligned}$$

See Figure 9.2.7.

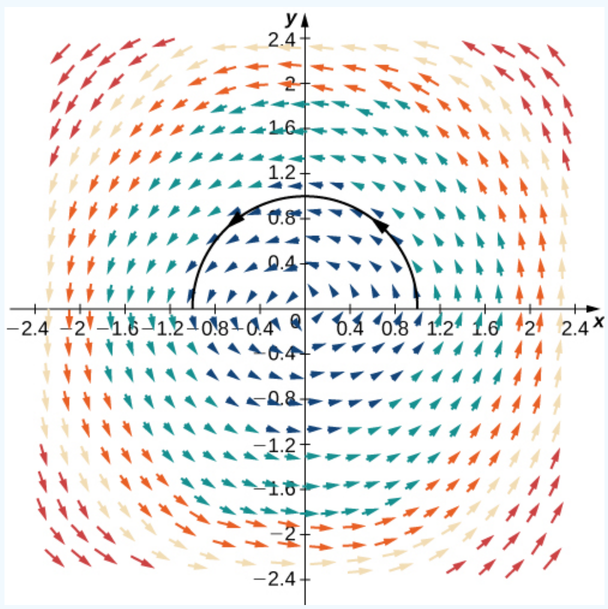


Figure 9.2.7: This figure shows curve $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq \pi$ in vector field $\vec{\mathbf{F}} = \langle -y, x \rangle$.

Example 9.2.6: Reversing Orientation

Find the value of integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the semicircle parameterized by $\vec{\mathbf{r}}(t) = \langle \cos(t+\pi), \sin t \rangle$, $0 \leq t \leq \pi$ and $\vec{\mathbf{F}} = \langle -y, x \rangle$.

Solution

Notice that this is the same problem as Example 9.2.5, except the orientation of the curve has been traversed. In this example, the parameterization starts at $\vec{\mathbf{r}}(0) = \langle -1, 0 \rangle$ and ends at $\vec{\mathbf{r}}(\pi) = \langle 1, 0 \rangle$. By Equation 9.2.17,

$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} &= \int_0^\pi \langle -\sin t, \cos(t+\pi)\rangle \cdot \langle -\sin(t+\pi), \cos t\rangle dt \\ &= \int_0^\pi \langle -\sin t, -\cos t\rangle \cdot \langle \sin t, \cos t\rangle dt \\ &= \int_0^\pi (-\sin^2 t - \cos^2 t) dt \\ &= \int_0^\pi -1 dt \\ &= -\pi.\end{aligned}$$

Notice that this is the negative of the answer in Example 9.2.5. It makes sense that this answer is negative because the orientation of the curve goes against the "flow" of the vector field.

Let C be an oriented curve and let $-C$ denote the same curve but with the orientation reversed. Then, the previous two examples illustrate the following fact:

$$\int_{-C} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = -\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}. \tag{9.2.19}$$

That is, reversing the orientation of a curve changes the sign of a line integral.

Exercise 9.2.6

Let $\vec{\mathbf{F}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ be a vector field and let C be the curve with parameterization $\langle t, t^2\rangle$ for $0 \le t \le 2$. Which is greater: $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$ or $\int_{-C} \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$?

Hint

Imagine moving along the path and computing the dot product $\vec{\mathbf{F}}\cdot\vec{\mathbf{T}}$ as you go.

Answer

$$\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds \tag{9.2.20}$$

Another standard notation for integral $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ is $\int_C P\,dx + Q\,dy + R\,dz$. In this notation, P, Q, and R are functions, and we think of $d\vec{\mathbf{r}}$ as vector $\langle dx, dy, dz\rangle$. To justify this convention, recall that $d\vec{\mathbf{r}} = \vec{\mathbf{T}}\,ds = \vec{\mathbf{r}}\,'(t)\,dt = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle dt$. Therefore,

$$\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle = P\,dx + Q\,dy + R\,dz. \tag{9.2.21}$$

If $d\vec{\mathbf{r}} = \langle dx, dy, dz\rangle$, then $\frac{dr}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle$, which implies that $d\vec{\mathbf{r}} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle dt$. Therefore

$$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_C P\,dx + Q\,dy + R\,dz \tag{9.2.22}$$

$$= \int_a^b \left(P(\vec{\mathbf{r}}(t))\frac{dx}{dt} + Q(\vec{\mathbf{r}}(t))\frac{dy}{dt} + R(\vec{\mathbf{r}}(t))\frac{dz}{dt}\right)\,dt. \tag{9.2.23}$$

Example 9.2.7: Finding the Value of an Integral of the Form $\int_C P\,dx + Q\,dy + R\,dz$

Find the value of integral $\int_C z\,dx + x\,dy + y\,dz$, where C is the curve parameterized by $\vec{\mathbf{r}}(t) = \langle t^2, \sqrt{t}, t\rangle$, $1 \le t \le 4$.

Solution

As with our previous examples, to compute this line integral we should perform a change of variables to write everything in terms of t. In this case, Equation 9.2.23 allows us to make this change:

$$\begin{aligned}\int_C z\,dx + x\,dy + y\,dz &= \int_1^4 \left(t(2t) + t^2\left(\frac{1}{2\sqrt{t}}\right) + \sqrt{t}\right)\,dt \\ &= \int_1^4 \left(2t^2 + \frac{t^{3/2}}{2} + \sqrt{t}\right)\,dt \\ &= \left[\frac{2t^3}{3} + \frac{t^{5/2}}{5} + \frac{2t^{3/2}}{3}\right]_{t=1}^{t=4} \\ &= \frac{793}{15}.\end{aligned}$$

Exercise 9.2.7

Find the value of $\int_C 4x\,dx + z\,dy + 4y^2\,dz$, where C is the curve parameterized by $\vec{\mathbf{r}}(t) = \langle 4\cos(2t), 2\sin(2t), 3\rangle$, $0 \le t \le \frac{\pi}{4}$.

Hint

Write the integral in terms of t using Equation 9.2.23

Answer

-26

We have learned how to integrate smooth oriented curves. Now, suppose that C is an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves. In this case, we say that C is a piecewise smooth curve. To be precise, curve C is piecewise smooth if C can be written as a union of n smooth curves C_1, C_2,…, C_n such that the endpoint of C_i is the starting point of C_{i+1} (Figure 9.2.8). When curves C_i satisfy the condition that the endpoint of C_i is the starting point of C_{i+1}, we write their union as $C_1 + C_2 + \cdots + C_n$.

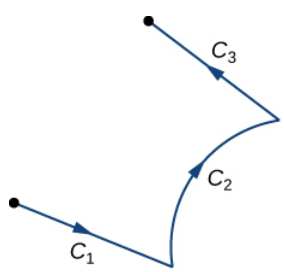


Figure 9.2.8: The union of C_1, C_2, C_3 is a piecewise smooth curve.

The next theorem summarizes several key properties of vector line integrals.

Theorem: PROPERTIES OF VECTOR LINE INTEGRALS

Let $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ be continuous vector fields with domains that include the oriented smooth curve C. Then

1. $\int_C (\vec{\mathbf{F}} + \vec{\mathbf{G}})\cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} + \int_C \vec{\mathbf{G}}\cdot d\vec{\mathbf{r}}$
2. $\int_C k\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = k\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, where k is a constant
3. $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_{-C} \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$
4. Suppose instead that C is a piecewise smooth curve in the domains of $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$, where $C = C_1 + C_2 + \cdots + C_n$ and $C_1, C_2, \ldots, C_n$ are smooth curves such that the endpoint of C_i is the starting point of C_{i+1}. Then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \cdots + \int_{C_n} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}. \tag{9.2.24}$$

Notice the similarities between these items and the properties of single-variable integrals. Properties i. and ii. say that line integrals are linear, which is true of single-variable integrals as well. Property iii. says that reversing the orientation of a curve changes the sign of the integral. If we think of the integral as computing the work done on a particle traveling along C, then this makes sense. If the particle moves backward rather than forward, then the value of the work done has the opposite sign. This is analogous to the equation $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. Finally, if $[a_1, a_2]$, $[a_2, a_3]$,..., $[a_{n-1}, a_n]$ are intervals, then

$$\int_{a_1}^{a_n} f(x)\,dx = \int_{a_1}^{a_2} f(x)\,dx + \int_{a_1}^{a_3} f(x)\,dx + \cdots + \int_{a_{n-1}}^{a_n} f(x)\,dx, \tag{9.2.25}$$

which is analogous to property iv.

Example 9.2.8: Using Properties to Compute a Vector Line Integral

Find the value of integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds$, where C is the rectangle (oriented counterclockwise) in a plane with vertices $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, and where $\vec{\mathbf{F}} = \langle x-2y, y-x\rangle$ (Figure 9.2.9).

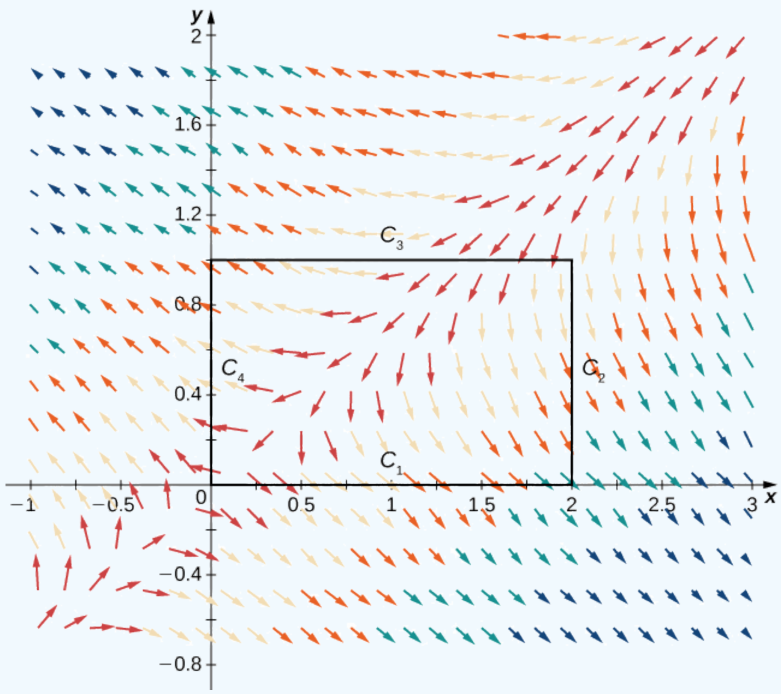


Figure 9.2.9: Rectangle and vector field for Example 9.2.8.

Solution

Note that curve C is the union of its four sides, and each side is smooth. Therefore C is piecewise smooth. Let C_1 represent the side from $(0,0)$ to $(2,0)$, let C_2 represent the side from $(2,0)$ to $(2,1)$, let C_3 represent the side from $(2,1)$ to $(0,1)$, and let C_4 represent the side from $(0,1)$ to $(0,0)$ (Figure 9.2.9). Then,

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,dr = \int_{C_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,dr + \int_{C_2} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,dr + \int_{C_3} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,dr + \int_{C_4} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,dr. \tag{9.2.26}$$

We want to compute each of the four integrals on the right-hand side using Equation 9.2.1. Before doing this, we need a parameterization of each side of the rectangle. Here are four parameterizations (note that they traverse C counterclockwise):

$$\begin{aligned} C_1 &: \langle t, 0\rangle, 0 \le t \le 2 \\ C_2 &: \langle 2, t\rangle, 0 \le t \le 1 \\ C_3 &: \langle 2-t, 1\rangle, 0 \le t \le 2 \\ C_4 &: \langle 0, 1-t\rangle, 0 \le t \le 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{C_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, dr &= \int_0^2 \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\, dt \\ &= \int_0^2 \langle t-2(0), 0-t\rangle \cdot \langle 1, 0\rangle\, dt = \int_0^2 t\, dt \\ &= \left[\tfrac{t^2}{2}\right]_0^2 = 2. \end{aligned}$$

Notice that the value of this integral is positive, which should not be surprising. As we move along curve C_1 from left to right, our movement flows in the general direction of the vector field itself. At any point along C_1, the tangent vector to the curve and the corresponding vector in the field form an angle that is less than 90°. Therefore, the tangent vector and the force vector have a positive dot product all along C_1, and the line integral will have positive value.

The calculations for the three other line integrals are done similarly:

$$\begin{aligned} \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \langle 2-2t, t-2\rangle \cdot \langle 0, 1\rangle\, dt \\ &= \int_0^1 (t-2)\, dt \\ &= \left[\tfrac{t^2}{2} - 2t\right]_0^1 = -\frac{3}{2}, \end{aligned}$$

$$\begin{aligned} \int_{C_3} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds &= \int_0^2 \langle (2-t)-2, 1-(2-t)\rangle \cdot \langle -1, 0\rangle\, dt \\ &= \int_0^2 t\, dt = 2, \end{aligned}$$

and

$$\begin{aligned} \int_{C_4} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \langle -2(1-t), 1-t\rangle \cdot \langle 0, -1\rangle\, dt \\ &= \int_0^1 (t-1)\, dt \\ &= \left[\tfrac{t^2}{2} - t\right]_0^1 = -\frac{1}{2}. \end{aligned}$$

Thus, we have $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2$.

Exercise 9.2.8

Calculate line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}$ is vector field $\langle y^2, 2xy+1\rangle$ and C is a triangle with vertices $(0,0)$, $(4,0)$, and $(0,5)$, oriented counterclockwise.

Hint

Write the triangle as a union of its three sides, then calculate three separate line integrals.

Answer

0

9.2.3 Applications of Line Integrals

Scalar line integrals have many applications. They can be used to calculate the length or mass of a wire, the surface area of a sheet of a given height, or the electric potential of a charged wire given a linear charge density. Vector line integrals are extremely useful in physics. They can be used to calculate the work done on a particle as it moves through a force field, or the flow rate of a fluid across a curve. Here, we calculate the mass of a wire using a scalar line integral and the work done by a force using a vector line integral.

Suppose that a piece of wire is modeled by curve C in space. The mass per unit length (the linear density) of the wire is a continuous function $\rho(x,y,z)$. We can calculate the total mass of the wire using the scalar line integral $\int_C \rho(x,y,z)\,ds$. The reason is that mass is density multiplied by length, and therefore the density of a small piece of the wire can be approximated by $\rho(x^*,y^*,z^*)\,\Delta s$ for some point (x^*,y^*,z^*) in the piece. Letting the length of the pieces shrink to zero with a limit yields the line integral $\int_C \rho(x,y,z)\,ds$.

Example 9.2.9: Calculating the Mass of a Wire

Calculate the mass of a spring in the shape of a curve parameterized by $\langle t, 2\cos t, 2\sin t\rangle$, $0 \le t \le \frac{\pi}{2}$, with a density function given by $\rho(x,y,z) = e^x + yz$ kg/m (Figure 9.2.10).

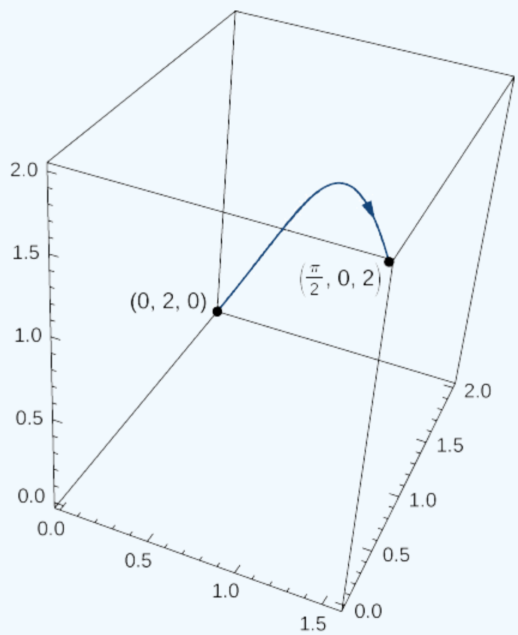


Figure 9.2.10: The wire from Example 9.2.9.

Solution

To calculate the mass of the spring, we must find the value of the scalar line integral $\int_C (e^x + yz)\,ds$, where C is the given helix. To calculate this integral, we write it in terms of t using Equation 9.2.1:

$$\begin{aligned}\int_C (e^x + yz)\, ds &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t \sin t)\sqrt{1 + (-2\cos t)^2 + (2\sin t)^2}\right)\, dt \\ &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t\sin t)\sqrt{5}\right)\, dt \\ &= \sqrt{5}\left[e^t + 2\sin^2 t\right]_{t=0}^{t=\pi/2} \\ &= \sqrt{5}(e^{\pi/2} + 1).\end{aligned}$$

Therefore, the mass is $\sqrt{5}(e^{\pi/2} + 1)$ kg.

Exercise 9.2.9

Calculate the mass of a spring in the shape of a helix parameterized by $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 6\pi$, with a density function given by $\rho(x, y, z) = x + y + z$ kg/m.

Hint

Calculate the line integral of ρ over the curve with parameterization $\vec{\mathbf{r}}$.

Answer

$18\sqrt{2}\pi^2$ kg

When we first defined vector line integrals, we used the concept of work to motivate the definition. Therefore, it is not surprising that calculating the **work done by a vector field** representing a force is a standard use of vector line integrals. Recall that if an object moves along curve C in force field $\vec{\mathbf{F}}$, then the work required to move the object is given by $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

Example 9.2.10: Calculating Work

How much work is required to move an object in vector force field $\vec{\mathbf{F}} = \langle yz, xy, xz\rangle$ along path $\vec{\mathbf{r}}(t) = \langle t^2, t, t^4\rangle$, $0 \le t \le 1$? See Figure 9.2.11.

Solution

Let C denote the given path. We need to find the value of $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. To do this, we use Equation 9.2.17:

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 (\langle t^5, t^3, t^6\rangle \cdot \langle 2t, 1, 4t^3\rangle)\, dt \\ &= \int_0^1 (2t^6 + t^3 + 4t^9)\, dt \\ &= \left[\frac{2t^7}{7} + \frac{t^4}{4} + \frac{2t^{10}}{5}\right]_{t=0}^{t=1} = \frac{131}{140}.\end{aligned}$$

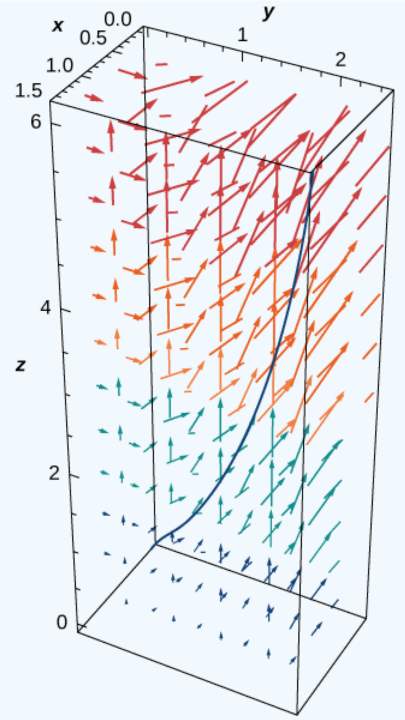


Figure 9.2.11: The curve and vector field for Example 9.2.10.

9.2.4 Flux and Circulation

We close this section by discussing two key concepts related to line integrals: flux across a plane curve and circulation along a plane curve. Flux is used in applications to calculate fluid flow across a curve, and the concept of circulation is important for characterizing conservative gradient fields in terms of line integrals. Both these concepts are used heavily throughout the rest of this chapter. The idea of flux is especially important for Green's theorem, and in higher dimensions for Stokes' theorem and the divergence theorem.

Let C be a plane curve and let $\vec{\mathbf{F}}$ be a vector field in the plane. Imagine C is a membrane across which fluid flows, but C does not impede the flow of the fluid. In other words, C is an idealized membrane invisible to the fluid. Suppose $\vec{\mathbf{F}}$ represents the velocity field of the fluid. How could we quantify the rate at which the fluid is crossing C?

Recall that the line integral of $\vec{\mathbf{F}}$ along C is $\displaystyle\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$—in other words, the line integral is the dot product of the vector field with the unit tangential vector with respect to arc length. If we replace the unit tangential vector with unit normal vector $\vec{\mathbf{N}}(t)$ and instead compute integral $int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,ds$, we determine the flux across C. To be precise, the definition of integral $\displaystyle\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,ds$ is the same as integral $\displaystyle\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$, except the $\vec{\mathbf{T}}$ in the Riemann sum is replaced with $\vec{\mathbf{N}}$. Therefore, the flux across C is defined as

$$\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} \vec{\mathbf{F}}(P_i^*)\cdot\vec{\mathbf{N}}(P_i^*)\,\Delta s_i, \tag{9.2.27}$$

where P_i^* and Δs_i are defined as they were for integral $\displaystyle\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$. Therefore, a flux integral is an integral that is *perpendicular* to a vector line integral, because $\vec{\mathbf{N}}$ and $\vec{\mathbf{T}}$ are perpendicular vectors.

If $\vec{\mathbf{F}}$ is a velocity field of a fluid and C is a curve that represents a membrane, then the flux of $\vec{\mathbf{F}}$ across C is the quantity of fluid flowing across C per unit time, or the rate of flow.

More formally, let C be a plane curve parameterized by $\vec{\mathbf{r}}(t) = \langle x(t),\, y(t)\rangle,\ a \le t \le b$. Let $\vec{\mathbf{n}}(t) = \langle y'(t),\, -x'(t)\rangle$ be the vector that is normal to C at the endpoint of $\vec{\mathbf{r}}(t)$ and points to the right as we traverse C in the positive direction (Figure $9.2.12$). Then,

$\vec{\mathbf{N}}(t) = \dfrac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}$ is the unit normal vector to C at the endpoint of $\vec{\mathbf{r}}(t)$ that points to the right as we traverse C.

DEFINITION: flux

The *flux* of $\vec{\mathbf{F}}$ across C is line integral

$$\int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds. \tag{9.2.28}$$

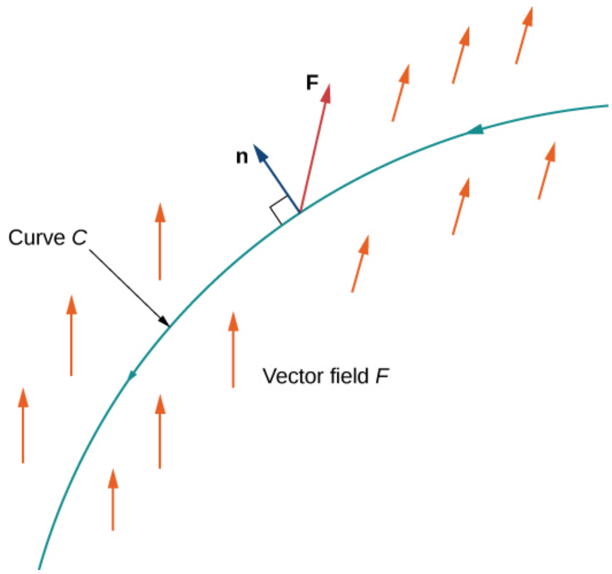


Figure 9.2.12: The flux of vector field $\vec{\mathbf{F}}$ across curve C is computed by an integral similar to a vector line integral.

We now give a formula for calculating the flux across a curve. This formula is analogous to the formula used to calculate a vector line integral (see Equation 9.2.17).

Theorem: CALCULATING FLUX ACROSS A CURVE

Let $\vec{\mathbf{F}}$ be a vector field and let C be a smooth curve with parameterization $r(t) = \langle x(t), y(t)\rangle,\ a \le t \le b$ .Let $\vec{\mathbf{n}}(t) = \langle y'(t), -x'(t)\rangle$. The flux of $\vec{\mathbf{F}}$ across C is

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{n}}(t)\, dt. \tag{9.2.29}$$

Proof

The proof of Equation 9.2.29 is similar to the proof of Equation 9.2.1. Before deriving the formula, note that

$$\|\vec{\mathbf{n}}(t)\| = \|\langle y'(t), -x'(t)\rangle\| = \sqrt{(y'(t))^2 + (x'(t))^2} = \|\vec{\mathbf{r}}\,'(t)\|. \tag{9.2.30}$$

Therefore,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds \\ &= \int_a^b \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|} \|\vec{\mathbf{r}}\,'(t)\|\, dt \\ &= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{n}}(t)\, dt. \end{aligned}$$

□

Example 9.2.11: Flux across a Curve

Calculate the flux of $\vec{\mathbf{F}} = \langle 2x, 2y\rangle$ across a unit circle oriented counterclockwise (Figure 9.2.13).

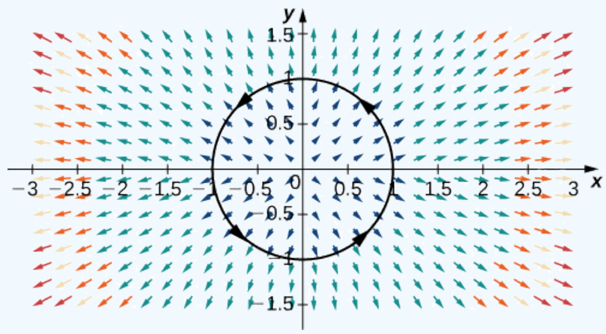


Figure 9.2.13: A unit circle in vector field $\vec{\mathbf{F}} = \langle 2x,\, 2y\rangle$.

Solution

To compute the flux, we first need a parameterization of the unit circle. We can use the standard parameterization $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. The normal vector to a unit circle is $\langle \cos t, \sin t\rangle$. Therefore, the flux is

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \int_0^{2\pi} \langle 2\cos t, 2\sin t\rangle \cdot \langle \cos t, \sin t\rangle\, dt \\ &= \int_0^{2\pi} (2\cos^2 t + 2\sin^2 t)\, dt \\ &= 2\int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt \\ &= 2\int_0^{2\pi} dt = 4\pi.\end{aligned}$$

Exercise 9.2.11

Calculate the flux of $\vec{\mathbf{F}} = \langle x+y, 2y\rangle$ across the line segment from $(0,0)$ to $(2,3)$, where the curve is oriented from left to right.

Hint

Use Equation 9.2.29.

Answer

$3/2$

Let $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a two-dimensional vector field. Recall that integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ is sometimes written as $\int_C P\, dx + Q\, dy$. Analogously, flux $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ is sometimes written in the notation $\int_C -Q\, dx + P\, dy$, because the unit normal vector $\vec{\mathbf{N}}$ is perpendicular to the unit tangent $\vec{\mathbf{T}}$. Rotating the vector $d\vec{\mathbf{r}} = \langle dx, dy\rangle$ by 90° results in vector $\langle dy, -dx\rangle$. Therefore, the line integral in Example 9.2.8 can be written as $\int_C -2y\, dx + 2x\, dy$.

Now that we have defined flux, we can turn our attention to circulation. The line integral of vector field $\vec{\mathbf{F}}$ along an oriented closed curve is called the circulation of $\vec{\mathbf{F}}$ along C. Circulation line integrals have their own notation: $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$. The circle on the integral symbol denotes that C is "circular" in that it has no endpoints. Example 9.2.5 shows a calculation of circulation.

To see where the term *circulation* comes from and what it measures, let $\vec{\mathbf{v}}$ represent the velocity field of a fluid and let C be an oriented closed curve. At a particular point P, the closer the direction of $\vec{\mathbf{v}}(P)$ is to the direction of $\vec{\mathbf{T}}(P)$, the larger the value of the dot product $\vec{\mathbf{v}}(P)\cdot\vec{\mathbf{T}}(P)$. The maximum value of $\vec{\mathbf{v}}(P)\cdot\vec{\mathbf{T}}(P)$ occurs when the two vectors are pointing in the exact same direction; the minimum value of $\vec{\mathbf{v}}(P)\cdot\vec{\mathbf{T}}(P)$ occurs when the two vectors are pointing in opposite directions. Thus, the value of the circulation $\oint_C \vec{\mathbf{v}}\cdot\vec{\mathbf{T}}\,ds$ measures the tendency of the fluid to move in the direction of C.

Example 9.2.12: Calculating Circulation

Let $\vec{\mathbf{F}} = \langle -y,\, x\rangle$ be the vector field from Example 9.2.3 and let C represent the unit circle oriented counterclockwise. Calculate the circulation of $\vec{\mathbf{F}}$ along C.

Solution

We use the standard parameterization of the unit circle: $\vec{\mathbf{r}}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. Then, $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -\sin t, \cos t\rangle$ and $\vec{\mathbf{r}}\,'(t) = \langle -\sin t, \cos t\rangle$. Therefore, the circulation of $\vec{\mathbf{F}}$ along C is

$$\begin{aligned}\oint_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds &= \int_0^{2\pi} \langle -\sin t, \cos t\rangle\cdot\langle -\sin t, \cos t\rangle\,dt \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

Notice that the circulation is positive. The reason for this is that the orientation of C “flows” with the direction of $\vec{\mathbf{F}}$. At any point along the circle, the tangent vector and the vector from $\vec{\mathbf{F}}$ form an angle of less than 90°, and therefore the corresponding dot product is positive.

In Example 9.2.12, what if we had oriented the unit circle clockwise? We denote the unit circle oriented clockwise by $-C$. Then

$$\oint_{-C} \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds = -\oint_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds = -2\pi. \tag{9.2.31}$$

Notice that the circulation is negative in this case. The reason for this is that the orientation of the curve flows against the direction of $\vec{\mathbf{F}}$.

Exercise 9.2.12

Calculate the circulation of $\vec{\mathbf{F}}(x,y) = \langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$ along a unit circle oriented counterclockwise.

Hint

Use Equation 9.2.29.

Answer

2π

Example 9.2.13: Calculating Work

Calculate the work done on a particle that traverses circle C of radius 2 centered at the origin, oriented counterclockwise, by field $\vec{\mathbf{F}}(x,y) = \langle -2,\, y\rangle$. Assume the particle starts its movement at $(1,\, 0)$.

Solution

The work done by $\vec{\mathbf{F}}$ on the particle is the circulation of $\vec{\mathbf{F}}$ along C: $\oint_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds$. We use the parameterization $\vec{\mathbf{r}}(t) = \langle 2\cos t,\, 2\sin t\rangle$, $0 \le t \le 2\pi$ for C. Then, $\vec{\mathbf{r}}\,'(t) = \langle -2\sin t,\, 2\cos t\rangle$ and $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle -2,\, 2\sin t\rangle$. Therefore, the circulation of $\vec{\mathbf{F}}$ along C is

$$\begin{aligned}\oint_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds &= \int_0^{2\pi}\langle -2, 2\sin t\rangle\cdot\langle -2\sin t, 2\cos t\rangle\,dt \\ &= \int_0^{2\pi}(4\sin t + 4\sin t\cos t)\,dt \\ &= \Big[-4\cos t + 4\sin^2 t\Big]_0^{2\pi} \\ &= \big(-4\cos(2\pi) + 2\sin^2(2\pi)\big) - \big(-4\cos(0) + 4\sin^2(0)\big) \\ &= -4 + 4 = 0.\end{aligned}$$

The force field does zero work on the particle.

Notice that the circulation of $\vec{\mathbf{F}}$ along C is zero. Furthermore, notice that since $\vec{\mathbf{F}}$ is the gradient of $f(x,y) = -2x + \frac{y^2}{2}$, $\vec{\mathbf{F}}$ is conservative. We prove in a later section that under certain broad conditions, the circulation of a conservative vector field along a closed curve is zero.

Exercise 9.2.14

Calculate the work done by field $\vec{\mathbf{F}}(x,y) = \langle 2x,\ 3y\rangle$ on a particle that traverses the unit circle. Assume the particle begins its movement at $(-1,\ 0)$.

Hint

Use Equation 9.2.29.

Answer

0

9.2.5 Key Concepts

- Line integrals generalize the notion of a single-variable integral to higher dimensions. The domain of integration in a single-variable integral is a line segment along the x-axis, but the domain of integration in a line integral is a curve in a plane or in space.
- If C is a curve, then the length of C is $\int_C ds$.
- There are two kinds of line integral: scalar line integrals and vector line integrals. Scalar line integrals can be used to calculate the mass of a wire; vector line integrals can be used to calculate the work done on a particle traveling through a field.
- Scalar line integrals can be calculated using Equation 9.2.1; vector line integrals can be calculated using Equation 9.2.17.
- Two key concepts expressed in terms of line integrals are flux and circulation. Flux measures the rate that a field crosses a given line; circulation measures the tendency of a field to move in the same direction as a given closed curve.

9.2.6 Key Equations

- **Calculating a scalar line integral**

  $$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt$$
- **Calculating a vector line integral**

  $$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{r}}\,'(t)\,dt$$

  or

  $$\int_C P\,dx + Q\,dy + R\,dz = \int_a^b \Big(P(\vec{\mathbf{r}}(t))\frac{dx}{dt} + Q(\vec{\mathbf{r}}(t))\frac{dy}{dt} + R(\vec{\mathbf{r}}(t))\frac{dz}{dt}\Big)\,dt$$

- **Calculating flux**
$$\int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{n}}(t)\, dt$$

9.2.7 Glossary

circulation

the tendency of a fluid to move in the direction of curve C. If C is a closed curve, then the circulation of $\vec{\mathbf{F}}$ along C is line integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$, which we also denote $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$.

closed curve

a curve for which there exists a parameterization $\vec{\mathbf{r}}(t), a \leq t \leq b$, such that $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$, and the curve is traversed exactly once

flux

the rate of a fluid flowing across a curve in a vector field; the flux of vector field $\vec{\mathbf{F}}$ across plane curve C is line integral $\int_C \vec{\mathbf{F}} \cdot \frac{\vec{\mathbf{n}}(t)}{\|\vec{\mathbf{n}}(t)\|}\, ds$

line integral

the integral of a function along a curve in a plane or in space

orientation of a curve

the orientation of a curve C is a specified direction of C

piecewise smooth curve

an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves

scalar line integral

the scalar line integral of a function f along a curve C with respect to arc length is the integral $\displaystyle\int_C f\, ds$, it is the integral of a scalar function f along a curve in a plane or in space; such an integral is defined in terms of a Riemann sum, as is a single-variable integral

vector line integral

the vector line integral of vector field $\vec{\mathbf{F}}$ along curve C is the integral of the dot product of $\vec{\mathbf{F}}$ with unit tangent vector $\vec{\mathbf{T}}$ of C with respect to arc length, $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$; such an integral is defined in terms of a Riemann sum, similar to a single-variable integral

9.2.8 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

9.2E: Exercises

9.2E.1 Exercise $\mathbf{9.2}\boldsymbol{E}\mathbf{.1}$

1. *True or False?* Line integral $\int_C f(x,y)ds$ is equal to a definite integral if C is a smooth curve defined on $[a,b]$ and if function f is continuous on some region that contains curve C.

Answer

True

2. *True or False?* Vector functions $\vec{\mathbf{r_1}}=t\,\hat{\mathbf{i}}+t^2\,\hat{\mathbf{j}}, 0\le t\le 1$ and $\vec{\mathbf{r_2}}=(1-t)\,\hat{\mathbf{i}}+(1-t)^2\,\hat{\mathbf{j}}, 0\le t\le 1$, define the same oriented curve.

3. *True or False?* $\int_{-C}(Pdx+Qdy)=\int_C(Pdx-Qdy)$

Answer

False

4. *True or False?* A piecewise smooth curve C consists of a finite number of smooth curves that are joined together end to end.

5. *True or False?* If is given by $x(t)=t, y(t)=t, 0\le t\le 1$, then $\int_C xyds=\int_0^1 t^2dt.$

Answer

False

9.2E.2 Exercise $\mathbf{9.2}\boldsymbol{E}\mathbf{.2}$

For the following exercises, use a computer algebra system (CAS) to evaluate the line integrals over the indicated path.

6. **[T]** $\int_C(x+y)ds$, where $C: x=t, y=(1-t), z=0$ from $(0,1,0)$ to $(1,0,0)$

7. [T] $\int_C(x-y)ds$, where $C:\vec{\mathbf{r}}(t)=4t\,\hat{\mathbf{i}}+3t\,\hat{\mathbf{j}}, 0\le t\le 2.$

Answer

10

8. [T] $\int_C(x^2+y^2+z^2)ds$ where $C:\vec{\mathbf{r}}(t)=\sin(t)\,\hat{\mathbf{i}}+\cos(t)\,\hat{\mathbf{j}}+8t\,\hat{\mathbf{k}}, 0\le t\le\frac{\pi}{2}.$

9. [T] Evaluate $\int_C xy^4ds$, where C is the right half of circle $x^2+y^2=16$ and is traversed in the clockwise direction.

Answer

$\frac{8192}{5}$

10. [T] Evaluate $\int_C 4x^3ds$, where C is the line segment from $(-2,-1)$ to $(1,2)$.

9.2E.3 Exercise $\mathbf{9.2}\boldsymbol{E}\mathbf{.3}$

For the following exercises, find the work done.

11. Find the work done by vector field $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + 3xy\,\hat{\mathbf{j}} - (x+z)\,\hat{\mathbf{k}}$ on a particle moving along a line segment that goes from $(1, 4, 2)$ to $(0, 5, 1)$.

Answer

8

12. Find the work done by a person weighing 150 lb walking exactly one revolution up a circular, spiral staircase of radius 3 ft if the person rises 10 ft.

13. Find the work done by force field $\vec{\mathbf{F}}(x, y, z) = -\frac{1}{2}x\,\hat{\mathbf{i}} - \frac{1}{2}y\,\hat{\mathbf{j}} + \frac{1}{4}\,\hat{\mathbf{k}}$ on a particle as it moves along the helix $\vec{\mathbf{r}}(t) = \cos(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ from point $(1, 0, 0)$ to point $(-1, 0, 3\pi)$.

Answer

$W = \frac{3\pi}{4}$

14. Find the work done by vector field $\vec{\mathbf{F}}(x, y) = y\,\hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}}$ in moving an object along path C, which joins points $(1, 0)$ and $(0, 1)$.

15. Find the work done by force $\vec{\mathbf{F}}(x, y) = 2y\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}} + (x+y)\,\hat{\mathbf{k}}$ in moving an object along curve $\vec{\mathbf{r}}(t) = \cos(t)\,\hat{\mathbf{i}} + \sin(t)\,\hat{\mathbf{j}} + 16\,\hat{\mathbf{k}}$, where $0 \leq t \leq 2\pi$.

Answer

$W = \pi$

16. Find the mass of wire in the shape of a circle of radius 2 centred at (3, 4) with linear mass density $\rho(x, y) = y^2$.

9.2E.4 Exercise $\mathbf{9.2E.4}$

For the following exercises, evaluate the line integrals.

17. Evaluate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}}$, where $\vec{\mathbf{F}}(x, y) = -\,\hat{\mathbf{j}}$ and C is the part of the graph of $y = 12x^3 - x$ from $(2, 2)$ to $(-2, -2)$.

Answer

4

18. Evaluate $\int_C (x^2+y^2+z^2)^{-1} ds$, where C is the helix $x = cost, y = sint, z = t, (0 \leq t \leq T)$.

19. Evaluate $\int_C yzdx + xzdy + xydz$ over the line segment from $(1, 1, 1)$ to $(3, 2, 0)$.

Answer

$\int_C yzdx + xzdy + xydz = -1$

20. Let C be the line segment from point (0, 1, 1) to point (2, 2, 3). Evaluate line integral $\int_C yds$.

21. [T] Use a computer algebra system to evaluate the line integral $\int_C y^2 dx + xdy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

Answer

$\int_C (y^2)dx + (x)dy = \frac{245}{6}$

22. [T] Use a computer algebra system to evaluate the line integral $\int_C (x+3y^2)dy$ over the path C given by $x=2t, y=10t$, where $0\le t\le 1$.

23. [T] Use a CAS to evaluate line integral $\int_C xydx+ydy$ over path C given by $x=2t, y=10t$,where $0\le t\le 1$.

Answer

$$\int_C xydx+ydy=\frac{190}{3}$$

24. Evaluate line integral $\int_C (2x-y)dx+(x+3y)dy$, where Clies along the x-axis from $x=0$ to $x=5$.

26. [T] Use a CAS to evaluate $\int_C \frac{y}{2x^2-y^2}ds$, where C is $x=t, y=t, 1\le t\le 5$.

Answer

$$\int_C \frac{y}{2x^2-y^2}ds=\sqrt{2}ln5$$

27. [T] Use a CAS to evaluate $\int_C xyds$, where C is $x=t^2, y=4t, 0\le t\le 1$.

9.2E.5 Exercise $9.2E.5$

In the following exercises, find the work done by force field $\vec{\mathbf{F}}$ on an object moving along the indicated path.

28. $\vec{\mathbf{F}}(x,y)=-x\,\hat{\mathbf{i}}-2y\,\hat{\mathbf{j}}$

$C: y=x^3$ from $(0,0)$ to $(2,8)$

Answer

$W=-66$

29. $\vec{\mathbf{F}}(x,y)=2x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}$

C: counterclockwise around the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$

30. $\vec{\mathbf{F}}(x,y,z)=x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}-5z\,\hat{\mathbf{k}}$

$C: \vec{\mathbf{r}}(t)=2cost\,\hat{\mathbf{i}}+2sint\,\hat{\mathbf{j}}+t\,\hat{\mathbf{k}}, 0\le t\le 2\pi$

Answer

$W=-10\pi^2$

9.2E.6 Exercise $9.2E.6$

31. Let $\vec{\mathbf{F}}$ be vector field $\vec{\mathbf{F}}(x,y)=(y^2+2xe^y+1)\,\hat{\mathbf{i}}+(2xy+x^2e^y+2y)\,\hat{\mathbf{j}}$. Compute the work of integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}$, where C is the path $\vec{\mathbf{r}}(t)=sint\,\hat{\mathbf{i}}+cost\,\hat{\mathbf{j}}, 0\le t\le\frac{\pi}{2}$.

32. Compute the work done by force $\vec{\mathbf{F}}(x,y,z)=2x\,\hat{\mathbf{i}}+3y\,\hat{\mathbf{j}}-zk$ along path $\vec{\mathbf{r}}(t)=t\,\hat{\mathbf{i}}+t^2\,\hat{\mathbf{j}}+t^3\,\hat{\mathbf{k}}$,where $0\le t\le 1$.

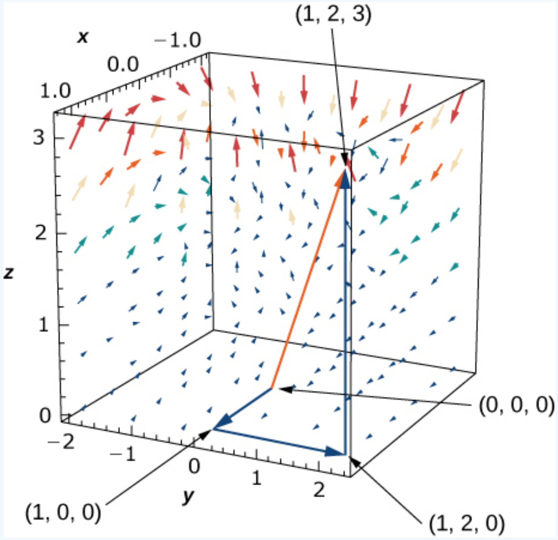
Answer

$W=2$

33. Evaluate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}}$, where $\vec{\mathbf{F}}(x,y) = \frac{1}{x+y}\,\hat{\mathbf{i}} + \frac{1}{x+y}\,\hat{\mathbf{j}}$ and C is the segment of the unit circle going counterclockwise from $(1,0)$ to $(0,1)$.

34. Force $\vec{\mathbf{F}}(x,y,z) = zy\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + z^2x\,\hat{\mathbf{k}}$ acts on a particle that travels from the origin to point $(1,2,3)$. Calculate the work done if the particle travels:

a. along the path $(0,0,0) \to (1,0,0) \to (1,2,0) \to (1,2,3)$ along straight-line segments joining each pair of endpoints;
b. along the straight line joining the initial and final points.
c. Is the work the same along the two paths?



Answer

a. $W = 11$; b. $W = 11$; c. Yes

35. Find the work done by vector field $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + 3xy\,\hat{\mathbf{j}} - (x+z)\,\hat{\mathbf{k}}$ on a particle moving along a line segment that goes from $(1,4,2)$ to $(0,5,1)$.

36. How much work is required to move an object in vector field $\vec{\mathbf{F}}(x,y) = yi + 3x\,\hat{\mathbf{j}}$ along the upper part of ellipse $\frac{x^2}{4} + y^2 = 1$ from $(2,0)$ to $(-2,0)$?

Answer

$W = 2\pi$

37. A vector field is given by $\vec{\mathbf{F}}(x,y) = (2x+3y)i + (3x+2y)\,\hat{\mathbf{j}}$. Evaluate the line integral of the field around a circle of unit radius traversed in a clockwise fashion.

38. Evaluate the line integral of scalar function xy along parabolic path $y = x^2$ connecting the origin to point $(1,1)$.

Answer

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = \frac{25\sqrt{5}+1}{120}$$

39. Find $\int_C y^2 dx + (xy - x^2)dy$ along $C : y = 3x$ from (0, 0) to (1, 3).

40. Find $\int_C y^2 dx + (xy - x^2)dy$ along $C : y^2 = 9x$ from (0, 0) to (1, 3).

Answer

$$\int_C y^2 dx + (xy - x^2)dy = 6.15$$

9.2E.7 Exercise $\boldsymbol{9.2E.7}$

For the following exercises, use a CAS to evaluate the given line integrals.

41. [T] Evaluate $\vec{\mathbf{F}}(x,y,z) = x^2 z\,\hat{\mathbf{i}} + 6y\,\hat{\mathbf{j}} + yz^2\,\hat{\mathbf{k}}$, where C is represented by $\vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + lnt\,\hat{\mathbf{k}}, 1 \le t \le 3$.

42. [T] Evaluate line integral $\int_C xe^y ds$ where, C is the arc of curve $x = e^y$ from $(1,0)$ to $(e,1)$.

Answer

$$\int_C xe^y ds \approx 7.157$$

43. [T] Evaluate the integral $\int_C xy^2 ds$, where C is a triangle with vertices $(0,1,2)$, $(1,0,3)$ and $(0,-1,0)$.

44. [T] Evaluate line integral $\int_C (y^2 - xy)dx$, where C is curve $y = lnx$ from $(1,0)$ toward $(e,1)$.

Answer

$$\int_C (y^2 - xy)dx \approx -1.379$$

45. [T] Evaluate line integral $\int_C xy^4 ds$, where C is the right half of circle $x^2 + y^2 = 16$.

46. [T] Evaluate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}}$, where $\vec{\mathbf{F}}(x,y,z) = x^2 y\,\hat{\mathbf{i}} + (x-z)\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$ and

$C : \vec{\mathbf{r}}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}, 0 \le t \le 1$.

Answer

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} \approx -1.133$$

47. Evaluate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}}$, where $\vec{\mathbf{F}}(x,y) = 2x sin(y) i + (x^2 cos(y) - 3y^2)\,\hat{\mathbf{j}}$ and

C is any path from $(-1,0)$ to $(5,1)$.

48. Find the line integral of $\vec{\mathbf{F}}(x,y,z) = 12x^2\,\hat{\mathbf{i}} - 5xy\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$ over path C defined by $y = x^2, z = x^3$ from point $(0,0,0)$ to point $(2,4,8)$.

Answer

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} \approx 22.857$$

49. Find the line integral of $\int_C (1 + x^2 y)ds$, where C is ellipse $\vec{\mathbf{r}}(t) = 2cost\,\hat{\mathbf{i}} + 3sint\,\hat{\mathbf{j}}$ from $0 \le t \le \pi$.

Answer

TBA

9.2E.8 Exercise $\mathbf{9.2E.8}$

For the following exercises, find the flux.

50. Compute the flux of $\vec{\mathbf{F}} = x^2\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$ across a line segment from $(0,0)$ to $(1,2)$.

Answer

$$Flux = -\frac{1}{3}$$

51. Let $\vec{\mathbf{F}} = 5\,\hat{\mathbf{i}}$ and let C be curve $y = 0, 0 \le x \le 4$. Find the flux across C.

52. Let $\vec{\mathbf{F}} = 5\,\hat{\mathbf{j}}$ and let C be curve $y = 0, 0 \le x \le 4$. Find the flux across C.

Answer

$$Flux = -20$$

53. Let $\vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$ and let $C : \vec{\mathbf{r}}(t) = cost\,\hat{\mathbf{i}} + sint\,\hat{\mathbf{j}}(0 \le t \le 2\pi)$. Calculate the flux across C.

54. Let $\vec{\mathbf{F}} = (x^2 + y^3)\,\hat{\mathbf{i}} + (2xy)\,\hat{\mathbf{j}}$. Calculate flux orientated counterclockwise across curve $C : x^2 + y^2 = 9$.

Answer

$$Flux = 0$$

9.2E.9 Exercise $\mathbf{9.2E.9}$

55. Find the line integral of $\int_C z^2dx + ydy + 2ydz,$ where C consists of two parts: C_1 and C_2. C_1 is the intersection of cylinder $x^2 + y^2 = 16$ and plane $z = 3$ from $(0,4,3)$ to $(-4,0,3)$. C_2 is a line segment from $(-4,0,3)$ to $(0,1,5)$.

56. A spring is made of a thin wire twisted into the shape of a circular helix $x = 2cost, y = 2sint, z = t.$ Find the mass of two turns of the spring if the wire has constant mass density.

Answer

$$m = 4\pi\rho\sqrt{5}$$

57. A thin wire is bent into the shape of a semicircle of radius *a*. If the linear mass density at point *P* is directly proportional to its distance from the line through the endpoints, find the mass of the wire.

58. An object moves in force field $\vec{\mathbf{F}}(x,y,z) = y^2\,\hat{\mathbf{i}} + 2(x+1)y\,\hat{\mathbf{j}}$ counterclockwise from point $(2,0)$ along elliptical path $x^2 + 4y^2 = 4$ to $(-2,0)$, and back to point $(2,0)$ along the *x*-axis. How much work is done by the force field on the object?

Answer

$$W = 0$$

59. Find the work done when an object moves in force field $\vec{\mathbf{F}}(x,y,z) = 2x\,\hat{\mathbf{i}} - (x+z)\,\hat{\mathbf{j}} + (y-x)\,\hat{\mathbf{k}}$ along the path given by $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (t^2 - t)\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}, 0 \le t \le 1$.

60. If an inverse force field $\vec{\mathbf{F}}$ is given by $\vec{\mathbf{F}}(x,y,z) = \frac{k}{\|r\|^3}r$, where k is a constant, find the work done by $\vec{\mathbf{F}}$ as its point of application moves along the *x*-axis from $A(1,0,0)$ to $B(2,0,0)$.

Answer

$$W = \frac{k}{2}$$

61. David and Sandra plan to evaluate line integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}}$ along a path in the *xy*-plane from (0, 0) to (1, 1). The force field is $\vec{\mathbf{F}}(x,y) = (x+2y)\,\hat{\mathbf{i}} + (-x+y^2)\,\hat{\mathbf{j}}$. David chooses the path that runs along the *x*-axis from (0, 0) to (1, 0) and then runs along the vertical line x=1 from (1, 0) to the final point (1, 1). Sandra chooses the direct path along the diagonal line $y = x$ from (0, 0) to (1, 1). Whose line integral is larger and by how much?

Answer

Add texts here. Do not delete this text first.

9.3: Conservative vector Fields

Learning Objectives

- Describe simple and closed curves; define connected and simply connected regions.
- Explain how to find a potential function for a conservative vector field.
- Use the Fundamental Theorem for Line Integrals to evaluate a line integral in a vector field.
- Explain how to test a vector field to determine whether it is conservative.

In this section, we continue the study of conservative vector fields. We examine the Fundamental Theorem for Line Integrals, which is a useful generalization of the Fundamental Theorem of Calculus to line integrals of conservative vector fields. We also discover show how to test whether a given vector field is conservative, and determine how to build a potential function for a vector field known to be conservative.

9.3.1 Curves and Regions

Before continuing our study of conservative vector fields, we need some geometric definitions. The theorems in the subsequent sections all rely on integrating over certain kinds of curves and regions, so we develop the definitions of those curves and regions here. We first define two special kinds of curves: closed curves and simple curves. As we have learned, a closed curve is one that begins and ends at the same point. A simple curve is one that does not cross itself. A curve that is both closed and simple is a simple closed curve (Figure $9.3.1$).

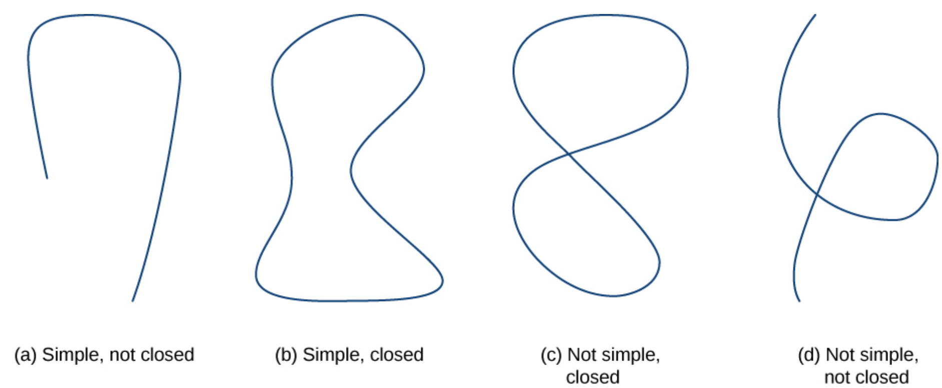


Figure $9.3.1$. Types of curves that are simple or not simple and closed or not closed.

DEFINITION: Closed Curves

Curve C is a *closed curve* if there is a parameterization $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$ of C such that the parameterization traverses the curve exactly once and $(\vec{\mathbf{r}}(a) = (\vec{\mathbf{r}}(b)$. Curve C is a simple curve if C does not cross itself. That is, C is simple if there exists a parameterization $(\vec{\mathbf{r}}(t)$, $a \leq t \leq b$ of C such that $(\vec{\mathbf{r}}$ is one-to-one over (a, b). It is possible for $(\vec{\mathbf{r}}(a) = (\vec{\mathbf{r}}(b)$, meaning that the simple curve is also closed.

Example $9.3.1$: Determining Whether a Curve Is Simple and Closed

Is the curve with parameterization $\vec{\mathbf{r}}(t) = \langle \cos t, \dfrac{\sin(2t)}{2} \rangle$, $0 \leq t \leq 2\pi$ a simple closed curve?

Solution

Note that $\vec{\mathbf{r}}(0) = \langle 1, 0 \rangle = r(2\pi)$; therefore, the curve is closed. The curve is not simple, however. To see this, note that $\vec{\mathbf{r}}(\dfrac{\pi}{2}) = \langle 0, 0 \rangle = \vec{\mathbf{r}}(\dfrac{3\pi}{2})$, and therefore the curve crosses itself at the origin (Figure $9.3.2$).

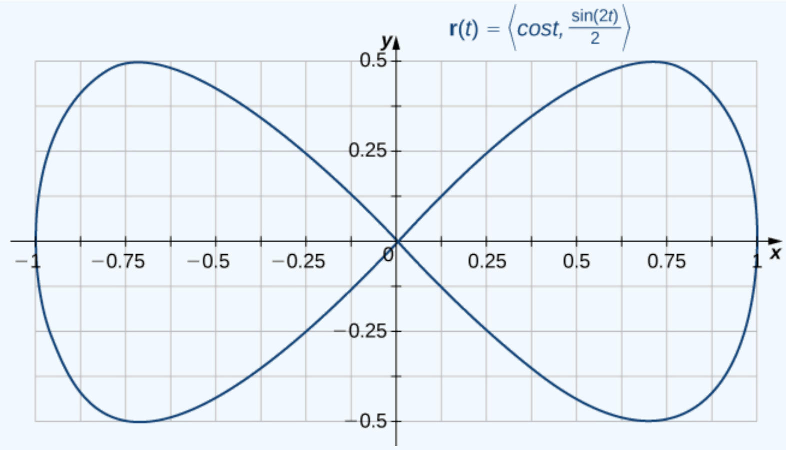


Figure $9.3.2$. A curve that is closed but not simple.

Exercise $9.3.1$

Is the curve given by parameterization $\vec{\mathbf{r}}(t) = \langle 2\cos t, 3\sin t\rangle$, $0 \le t \le 6\pi$, a simple closed curve?

Hint

Sketch the curve.

Answer

Yes

Many of the theorems in this chapter relate an integral over a region to an integral over the boundary of the region, where the region's boundary is a simple closed curve or a union of simple closed curves. To develop these theorems, we need two geometric definitions for regions: that of a connected region and that of a simply connected region. A connected region is one in which there is a path in the region that connects any two points that lie within that region. A simply connected region is a connected region that does not have any holes in it. These two notions, along with the notion of a simple closed curve, allow us to state several generalizations of the Fundamental Theorem of Calculus later in the chapter. These two definitions are valid for regions in any number of dimensions, but we are only concerned with regions in two or three dimensions.

DEFINITION: connected regions

A region D is a *connected region* if, for any two points P_1 and P_2, there is a path from P_1 to P_2 with a trace contained entirely inside D. A region D is a simply connected region if D is connected for any simple closed curve C that lies inside D, and curve C can be shrunk continuously to a point while staying entirely inside D. In two dimensions, a region is simply connected if it is connected and has no holes.

All simply connected regions are connected, but not all connected regions are simply connected (Figure $9.3.3$).

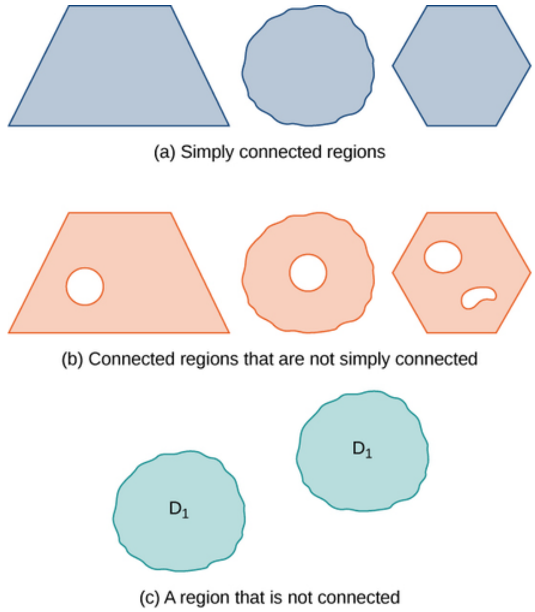
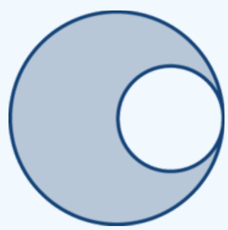


Figure 9.3.3: Not all connected regions are simply connected. (a) Simply connected regions have no holes. (b) Connected regions that are not simply connected may have holes but you can still find a path in the region between any two points. (c) A region that is not connected has some points that cannot be connected by a path in the region.

Exercise 9.3.2

Is the region in the below image connected? Is the region simply connected?



Hint

Consider the definitions.

Answer

The region in the figure is connected. The region in the figure is not simply connected.

9.3.2 Fundamental Theorem for Line Integrals

Now that we understand some basic curves and regions, let's generalize the Fundamental Theorem of Calculus to line integrals. Recall that the Fundamental Theorem of Calculus says that if a function f has an antiderivative F, then the integral of f from a to b depends only on the values of F at a and at b—that is,

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{9.3.1}$$

If we think of the gradient as a derivative, then the same theorem holds for vector line integrals. We show how this works using a motivational example.

Example 9.3.2: Evaluating a Line Integral and the Antiderivatives of the Endpoints

Let $\vec{\mathbf{F}}(x,y) = \langle 2x, 4y\rangle$. Calculate $\displaystyle\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, where C is the line segment from $(0,0)$ to $(2,2)$ (Figure 9.3.4).

Solution

We use the method from the previous section to calculate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. Curve C can be parameterized by $\vec{\mathbf{r}}(t) = \langle 2t, 2t \rangle$, $0 \leq t \leq 1$. Then, $\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \langle 4t, 8t \rangle$ and $\vec{\mathbf{r}}\,'(t) = \langle 2, 2 \rangle$, which implies that

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_0^1 \langle 4t, 8t \rangle \cdot \langle 2, 2 \rangle dt \\ &= \int_0^1 (8t + 16t) dt = \int_0^1 24t dt \\ &= [12t^2]_0^1 = 12. \end{aligned}$$

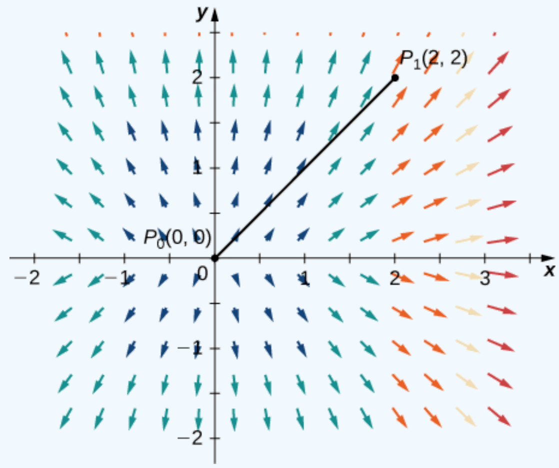


Figure 9.3.4: The value of line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ depends only on the value of the potential function of $\vec{\mathbf{F}}$ at the endpoints of the curve.

Notice that $\vec{\mathbf{F}} = \vec{\nabla} f$, where $f(x, y) = x^2 + 2y^2$. If we think of the gradient as a derivative, then f is an "antiderivative" of $\vec{\mathbf{F}}$. In the case of single-variable integrals, the integral of derivative $g'(x)$ is $g(b) - g(a)$, where a is the start point of the interval of integration and b is the endpoint. If vector line integrals work like single-variable integrals, then we would expect integral $\vec{\mathbf{F}}$ to be $f(P_1) - f(P_0)$, where P_1 is the endpoint of the curve of integration and P_0 is the start point. Notice that this is the case for this example:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = 12$$

and

$$f(2, 2) - f(0, 0) = 4 + 8 - 0 = 12.$$

In other words, the integral of a "derivative" can be calculated by evaluating an "antiderivative" at the endpoints of the curve and subtracting, just as for single-variable integrals.

The following theorem says that, under certain conditions, what happened in the previous example holds for any gradient field. The same theorem holds for vector line integrals, which we call the **Fundamental Theorem for Line Integrals**.

Theorem: THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

Let C be a piecewise smooth curve with parameterization $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$. Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on C. Then,

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)). \tag{9.3.2}$$

Proof

First,

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\,dt.$$

By the chain rule,

$$\frac{d}{dt}(f(\vec{\mathbf{r}}(t)) = \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)$$

Therefore, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} &= \int_a^b \vec{\nabla} f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)dt \\ &= \int_a^b \frac{d}{dt}(f(\vec{\mathbf{r}}(t))dt \\ &= [f(\vec{\mathbf{r}}(t))]_{t=a}^{t=b} \\ &= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).\end{aligned}$$

□

We know that if $\vec{\mathbf{F}}$ is a conservative vector field, there is a potential function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$. Therefore

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)). \tag{9.3.3}$$

In other words, just as with the Fundamental Theorem of Calculus, computing the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}$ is conservative, is a two-step process:

1. Find a potential function (“antiderivative”) f for $\vec{\mathbf{F}}$ and
2. Compute the value of f at the endpoints of C and calculate their difference $f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$.

Keep in mind, however, there is one major difference between the Fundamental Theorem of Calculus and the Fundamental Theorem for Line Integrals:
A function of one variable that is continuous must have an antiderivative. However, a vector field, even if it is continuous, does not need to have a potential function.

Example 9.3.3: Applying the Fundamental Theorem

Calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y,z) = \langle 2x \ln y, \frac{x^2}{y} + z^2, 2yz\rangle$ and C is a curve with parameterization $\vec{\mathbf{r}}(t) = \langle t^2, t, t\rangle$, $1 \leq t \leq e$

a. without using the Fundamental Theorem of Line Integrals and
b. using the Fundamental Theorem of Line Integrals.

Solution

1. First, let’s calculate the integral without the Fundamental Theorem for Line Integrals and instead use [link] or link:

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot dr &= \int_1^e \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\,dt \\ &= \int_1^e \langle 2t^2 \ln t, \frac{t^4}{t} + t^2, 2t^2 \rangle \cdot \langle 2t, 1, 1 \rangle \,dt \\ &= \int_1^e (4t^3 \ln t + t^3 + 3t^2)\,dt \\ &= \int_1^e 4t^3 \ln t\,dt + \int_1^e (t^3 + 3t^2)\,dt \\ &= \int_1^e 4t^3 \ln t\,dt + [\frac{t^4}{4} + t^3]_1^e.\end{aligned}$$

Integral $\int_1^e t^3 \ln t\,dt$ requires integration by parts. Let $u = \ln t$ and $dv = t^3$. Then $u = \ln t$, $dv = t^3$

and

$$du = \frac{1}{t}\,dt, \quad v = \frac{t^4}{4}.$$

Therefore,

$$\begin{aligned}\int_1^e t^3 \ln t\,dt &= [\frac{t^4}{4}\ln t]_1^e - \frac{1}{4}\int_1^e t^3\,dt \\ &= \frac{e^4}{4} - \frac{1}{r}(\frac{e^4}{4} - \frac{1}{4}).\end{aligned}$$

Thus,

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= 4\int_1^e t^3 \ln t\,dt \quad + \quad \frac{e^4}{4} + e^3 - \frac{5}{4} \\ &= 4(\frac{e^4}{4} - \frac{1}{4}(\frac{e^4}{4} - \frac{1}{4})) + \frac{e^4}{4} + e^3 - \frac{5}{4} \\ &= e^4 - \frac{e^4}{4} + \frac{1}{4} + \frac{e^4}{4} + e^3 - \frac{5}{4} \\ &= e^4 + e^3 - 1.\end{aligned}$$

2. Given that $f(x, y, z) = x^2 \ln y + yz^2$ is a potential function for $\vec{\mathbf{F}}$, let's use the Fundamental Theorem for Line Integrals to calculate the integral. Note that

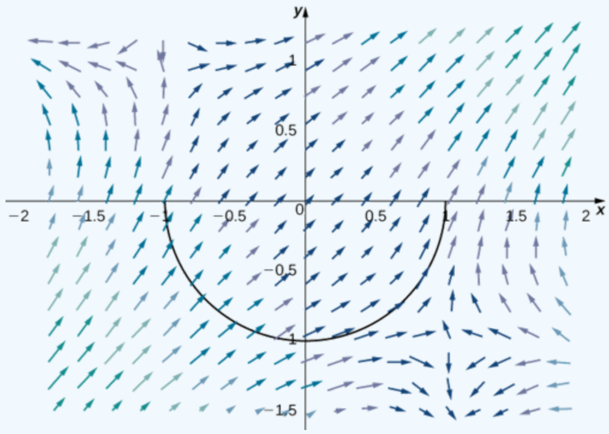
$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} \\ &= f(\vec{\mathbf{r}}(e)) - f(\vec{\mathbf{r}}(1)) \\ &= f(e^2, e, e) - f(1, 1, 1) \\ &= e^4 + e^3 - 1.\end{aligned}$$

This calculation is much more straightforward than the calculation we did in (a). As long as we have a potential function, calculating a line integral using the Fundamental Theorem for Line Integrals is much easier than calculating without the theorem.

Example $9.3.3$ illustrates a nice feature of the Fundamental Theorem of Line Integrals: it allows us to calculate more easily many vector line integrals. As long as we have a potential function, calculating the line integral is only a matter of evaluating the potential function at the endpoints and subtracting.

Exercise 9.3.3

Given that $f(x,y) = (x-1)^2 y + (y+1)^2 x$ is a potential function for $\vec{\mathbf{F}}(x,y) = \langle 2xy - 2y + (y+1)^2, (x-1)^2 + 2yx + 2x \rangle$, calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where C is the lower half of the unit circle oriented counterclockwise.



Hint

The Fundamental Theorem for Line Intervals says this integral depends only on the value of f at the endpoints of C.

Answer

2

The Fundamental Theorem for Line Integrals has two important consequences. The first consequence is that if $\vec{\mathbf{F}}$ is conservative and C is a closed curve, then the circulation of $\vec{\mathbf{F}}$ along C is zero—that is, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$. To see why this is true, let f be a potential function for $\vec{\mathbf{F}}$. Since C is a closed curve, the terminal point $\vec{\mathbf{r}}(b)$ of C is the same as the initial $\vec{\mathbf{r}}(a)$ of C—that is, $\vec{\mathbf{r}}(a) = \vec{\mathbf{r}}(b)$. Therefore, by the Fundamental Theorem for Line Integrals,

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} \tag{9.3.4}$$

$$= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)) \tag{9.3.5}$$

$$= f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(b)) \tag{9.3.6}$$

$$= 0. \tag{9.3.7}$$

Recall that the reason a conservative vector field $\vec{\mathbf{F}}$ is called "conservative" is because such vector fields model forces in which energy is conserved. We have shown gravity to be an example of such a force. If we think of vector field $\vec{\mathbf{F}}$ in integral $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ as a gravitational field, then the equation $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$ follows. If a particle travels along a path that starts and ends at the same place, then the work done by gravity on the particle is zero.

The second important consequence of the Fundamental Theorem for Line Integrals (Equation 9.3.2) is that line integrals of conservative vector fields are independent of path—meaning, they depend only on the endpoints of the given curve, and do not depend on the path between the endpoints.

DEFINITION: Path Independence

Let $\vec{\mathbf{F}}$ be a vector field with domain D; it is independent of path (or path independent) if

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{9.3.8}$$

for any paths C_1 and C_2 in $D$with the same initial and terminal points.

The second consequence is stated formally in the following theorem.

Theorem: CONSERVATIVE FIELDS

If $\vec{\mathbf{F}}$ is a *conservative vector field*, then $\vec{\mathbf{F}}$ is independent of path.

Proof

Let D denote the domain of $\vec{\mathbf{F}}$ and let C_1 and C_2 be two paths in D with the same initial and terminal points (Figure $9.3.5$). Call the initial point P_1 and the terminal point P_2. Since $\vec{\mathbf{F}}$ is conservative, there is a potential function f for $\vec{\mathbf{F}}$. By the Fundamental Theorem for Line Integrals,

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(P_2) - f(P_1) = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Therefore, $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ is independent of path.

□

To visualize what independence of path means, imagine three hikers climbing from base camp to the top of a mountain. Hiker 1 takes a steep route directly from camp to the top. Hiker 2 takes a winding route that is not steep from camp to the top. Hiker 3 starts by taking the steep route but halfway to the top decides it is too difficult for him. Therefore he returns to camp and takes the non-steep path to the top. All three hikers are traveling along paths in a gravitational field. Since gravity is a force in which energy is conserved, the gravitational field is conservative. By independence of path, the total amount of work done by gravity on each of the hikers is the same because they all started in the same place and ended in the same place. The work done by the hikers includes other factors such as friction and muscle movement, so the total amount of energy each one expended is not the same, but the net energy expended against gravity is the same for all three hikers.

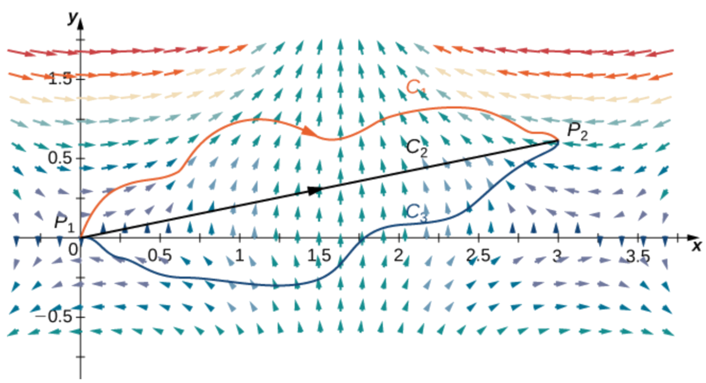


Figure $9.3.4$: The vector field is conservative, and therefore independent of path.

We have shown that if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ is independent of path. It turns out that if the domain of $\vec{\mathbf{F}}$ is open and connected, then the converse is also true. That is, if $\vec{\mathbf{F}}$ is independent of path and the domain of $\vec{\mathbf{F}}$ is open and connected, then $\vec{\mathbf{F}}$ is conservative. Therefore, the set of conservative vector fields on open and connected domains is precisely the set of vector fields independent of path.

THE PATH INDEPENDENCE TEST FOR CONSERVATIVE FIELDS

If $\vec{\mathbf{F}}$ is a continuous vector field that is independent of path and the domain D of $\vec{\mathbf{F}}$ is open and connected, then $\vec{\mathbf{F}}$ is conservative.

Proof

We prove the theorem for vector fields in $\mathbb{R}^2$. The proof for vector fields in $\mathbb{R}^3$ is similar. To show that $\vec{\mathbf{F}} = \langle P, Q \rangle$ is conservative, we must find a potential function f for $\vec{\mathbf{F}}$. To that end, let X be a fixed point in D. For any point (x, y) in D, let C be a path from X to (x, y). Define $f(x, y)$ by $f(x, y) = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. (Note that this definition of f makes sense only because

$\vec{\mathbf{F}}$ is independent of path. If $\vec{\mathbf{F}}$ was not independent of path, then it might be possible to find another path C' from X to (x, y) such that $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, and in such a case $f(x, y)$ would not be a function.) We want to show that f has the property $\vec{\nabla} f = \vec{\mathbf{F}}$.

Since domain D is open, it is possible to find a disk centered at (x, y) such that the disk is contained entirely inside D. Let (a, y) with $a < x$ be a point in that disk. Let C be a path from X to (x, y) that consists of two pieces: C_1 and C_2. The first piece, C_1, is any path from C to (a, y) that stays inside D; C_2 is the horizontal line segment from (a, y) to (x, y) (Figure 9.3.6). Then

$$f(x, y) = \int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

The first integral does not depend on x, so

$$f_x(x, y) = \frac{\partial}{\partial x} \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

If we parameterize C_2 by $\vec{\mathbf{r}}(t) = \langle t, y \rangle$, $a \leq t \leq x$, then

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \\ &= \frac{\partial}{\partial x} \int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\, dt \\ &= \frac{\partial}{\partial x} \int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{d}{dt}(\langle t, y \rangle)\, dt \\ &= \frac{\partial}{\partial x} \int_a^x \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \langle 1, 0 \rangle\, dt \\ &= \frac{\partial}{\partial x} \int_a^x P(t, y)\, dt. \end{aligned}$$

By the Fundamental Theorem of Calculus (part 1),

$$f_x(x, y) = \frac{\partial}{\partial x} \int_a^x P(t, y)\, dt = P(x, y).$$

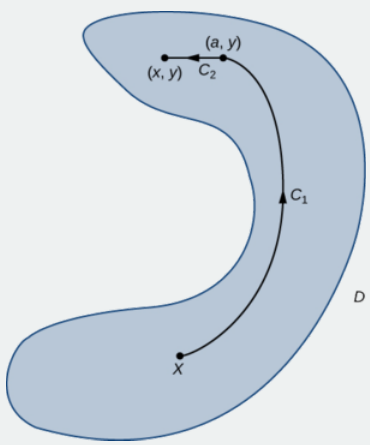


Figure 9.3.6. Here, C_1 is any path from C to (a, y) that stays inside D, and C_2 is the horizontal line segment from (a, y) to (x, y).

A similar argument using a vertical line segment rather than a horizontal line segment shows that $f_y(x, y) = Q(x, y)$.

Therefore $\vec{\nabla} f = \vec{\mathbf{F}}$ and $\vec{\mathbf{F}}$ is conservative.

□

We have spent a lot of time discussing and proving Note and Note, but we can summarize them simply: a vector field $\vec{\mathbf{F}}$ on an open and connected domain is conservative if and only if it is independent of path. This is important to know because conservative vector fields are extremely important in applications, and these theorems give us a different way of viewing what it means to be conservative using path independence.

Example 9.3.4: Showing That a Vector Field Is Not Conservative

Use path independence to show that vector field $\vec{\mathbf{F}}(x,y) = \langle x^2y, y+5\rangle$ is not conservative.

Solution

We can indicate that $\vec{\mathbf{F}}$ is not conservative by showing that $\vec{\mathbf{F}}$ is not path independent. We do so by giving two different paths, C_1 and C_2, that both start at $(0,0)$ and end at $(1,1)$, and yet $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

Let C_1 be the curve with parameterization $\vec{\mathbf{r}}_1(t) = \langle t,\, t\rangle,\ 0 \le t \le 1$ and let C_2 be the curve with parameterization $\vec{\mathbf{r}}_2(t) = \langle t,\, t^2\rangle,\ 0 \le t \le 1$ (Figure 9.3.7.). Then

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t)) \cdot \vec{\mathbf{r}}_1\,'(t)\,dt$$

$$= \int_0^1 \langle t^3, t+5\rangle \cdot \langle 1,1\rangle\,dt = \int_0^1 (t^3+t+5)\,dt$$

$$= [\frac{t^4}{4} + \frac{t^2}{2} + 5t]_0^1 = \frac{23}{4}$$

and

$$\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{r}}_2(t)) \cdot \vec{\mathbf{r}}_2\,'(t)\,dt$$

$$= \int_0^1 \langle t^4, t^2+5\rangle \cdot \langle 1, 2t\rangle\,dt = \int_0^1 (t^4+2t^3+10t)\,dt$$

$$= [\frac{t^5}{5} + \frac{t^4}{2} + 5t^2]_0^1 = \frac{57}{10}.$$

Since $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, the value of a line integral of $\vec{\mathbf{F}}$ depends on the path between two given points. Therefore, $\vec{\mathbf{F}}$ is not independent of path, and $\vec{\mathbf{F}}$ is not conservative.

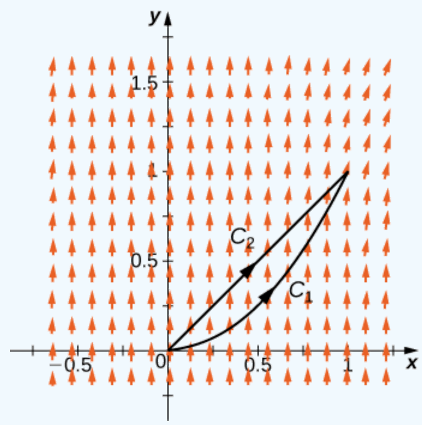


Figure 9.3.7: Curves C_1 and C_2 are both oriented from left to right.

Exercise 9.3.4

Show that $\vec{\mathbf{F}}(x,y) = \langle xy,\ x^2y^2\rangle$ is not path independent by considering the line segment from $(0,0)$ to $(0,2)$ and the piece of the graph of $y = \frac{x^2}{2}$ that goes from $(0,0)$ to $(0,2)$.

Hint

Calculate the corresponding line integrals.

Answer

If C_1 and C_2 represent the two curves, then

$$\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

9.3.3 Conservative Vector Fields and Potential Functions

As we have learned, the Fundamental Theorem for Line Integrals says that if $\vec{\mathbf{F}}$ is conservative, then calculating $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ has two steps: first, find a potential function f for $\vec{\mathbf{F}}$ and, second, calculate $f(P_1) - f(P_0)$, where P_1 is the endpoint of C and P_0 is the starting point. To use this theorem for a conservative field $\vec{\mathbf{F}}$, we must be able to find a potential function f for $\vec{\mathbf{F}}$. Therefore, we must answer the following question: Given a conservative vector field $\vec{\mathbf{F}}$, how do we find a function f such that $\vec{\nabla} f = \vec{\mathbf{F}}$? Before giving a general method for finding a potential function, let's motivate the method with an example.

Example 9.3.5: Finding a Potential Function

Find a potential function for $\vec{\mathbf{F}}(x,y) = \langle 2xy^3, 3x^2y^2 + \cos(y)\rangle$, thereby showing that $\vec{\mathbf{F}}$ is conservative.

Solution

Suppose that $f(x,y)$ is a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$, and therefore

$$f_x(x,y) = 2xy^3 \quad \text{and} \quad f_y(x,y) = 3x^2y^2 + \cos y.$$

Integrating the equation $f_x(x,y) = 2xy^3$ with respect to x yields the equation

$$f(x,y) = x^2y^3 + h(y).$$

Notice that since we are integrating a two-variable function with respect to x, we must add a constant of integration that is a constant with respect to x, but may still be a function of y. The equation $f(x,y) = x^2y^3 + h(y)$ can be confirmed by taking the partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^3) + \frac{\partial}{\partial x}(h(y)) = 2xy^3 + 0 = 2xy^3.$$

Since f is a potential function for $\vec{\mathbf{F}}$,

$$f_y(x,y) = 3x^2y^2 + \cos(y),$$

and therefore

$$3x^2y^2 + g'(y) = 3x^2y^2 + \cos(y).$$

This implies that $h'(y) = \cos y$, so $h(y) = \sin y + C$. Therefore, *any* function of the form $f(x,y) = x^2y^3 + \sin(y) + C$ is a potential function. Taking, in particular, $C = 0$ gives the potential function $f(x,y) = x^2y^3 + \sin(y)$.

To verify that f is a potential function, note that $\vec{\nabla} f(x,y) = \langle 2xy^3, 3x^2y^2 + \cos y\rangle = \vec{\mathbf{F}}$.

Exercise 9.3.5

Find a potential function for $\vec{\mathbf{F}}(x,y)=\langle e^x y^3+y, 3e^x y^2+x\rangle$.

Hint

Follow the steps in Example 9.3.5

Answer

$f(x,y)=e^x y^3+xy$

The logic of the previous example extends to finding the potential function for any conservative vector field in $\mathbb{R}^2$. Thus, we have the following problem-solving strategy for finding potential functions:

PROBLEM-SOLVING STRATEGY: FINDING A POTENTIAL FUNCTION FOR A CONSERVATIVE VECTOR FIELD $\vec{\mathbf{F}}(x,y)=\langle P(x,y),Q(x,y)\rangle$

1. Integrate P with respect to x. This results in a function of the form $g(x,y)+h(y)$, where $h(y)$ is unknown.
2. Take the partial derivative of $g(x,y)+h(y)$ with respect to y, which results in the function $gy(x,y)+h'(y)$.
3. Use the equation $gy(x,y)+h'(y)=Q(x,y)$ to find $h'(y)$.
4. Integrate $h'(y)$ to find $h(y)$.
5. Any function of the form $f(x,y)=g(x,y)+h(y)+C$, where C is a constant, is a potential function for $\vec{\mathbf{F}}$.

We can adapt this strategy to find potential functions for vector fields in $\mathbb{R}^3$, as shown in the next example.

Example 9.3.6: Finding a Potential Function in $\mathbb{R}^3$

Find a potential function for $F(x,y,z)=\langle 2xy, x^2+2yz^3, 3y^2z^2+2z\rangle$, thereby showing that $\vec{\mathbf{F}}$ is conservative.

Solution

Suppose that f is a potential function. Then, $\vec{\nabla} f=\vec{\mathbf{F}}$ and therefore $f_x(x,y,z)=2xy$. Integrating this equation with respect to x yields the equation $f(x,y,z)=x^2y+g(y,z)$ for some function g. Notice that, in this case, the constant of integration with respect to x is a function of y and z.

Since f is a potential function,

$$x^2+2yz^3=f_y(x,y,z)=x^2+g_y(y,z).$$

Therefore,

$$g_y(y,z)=2yz^3.$$

Integrating this function with respect to y yields

$$g(y,z)=y^2z^3+h(z)$$

for some function $h(z)$ of z alone. (Notice that, because we know that g is a function of only y and z, we do not need to write $g(y,z)=y^2z^3+h(x,z)$.) Therefore,

$$f(x,y,z)=x^2y+g(y,z)=x^2y+y^2z^3+h(z).$$

To find f, we now must only find h. Since f is a potential function,

$$3y^2z^2+2z=g_z(y,z)=3y^2z^2+h'(z).$$

This implies that $h'(z)=2z$, so $h(z)=z^2+C$. Letting $C=0$ gives the potential function

$$f(x,y,z)=x^2y+y^2z^3+z^2.$$

To verify that f is a potential function, note that $\vec{\nabla} f(x,y,z)=\langle 2xy, x^2+2yz^3, 3y^2z^2+2z\rangle=\vec{\mathbf{F}}(x,y,z)$.

Exercise $9.3.6$

Find a potential function for $\vec{\mathbf{F}}(x,y,z)=\langle 12x^2, \cos y\cos z, 1-\sin y\sin z\rangle$.

Hint

Following Example $9.3.6$, begin by integrating with respect to x.

Answer

$f(x,y,z)=4x^3+\sin y\cos z+z$

We can apply the process of finding a potential function to a gravitational force. Recall that, if an object has unit mass and is located at the origin, then the gravitational force in $\mathbb{R}^2$ that the object exerts on another object of unit mass at the point (x,y) is given by vector field

$$\vec{\mathbf{F}}(x,y)=-G\langle\frac{x}{(x^2+y^2)^{3/2}},\frac{y}{(x^2+y^2)^{3/2}}\rangle,$$

where G is the universal gravitational constant. In the next example, we build a potential function for $\vec{\mathbf{F}}$, thus confirming what we already know: that gravity is conservative.

Example $9.3.7$: Finding a Potential Function

Find a potential function f for $\vec{\mathbf{F}}(x,y)=-G\langle\frac{x}{(x^2+y^2)^{3/2}},\frac{y}{(x^2+y^2)^{3/2}}\rangle$.

Solution

Suppose that f is a potential function. Then, $\vec{\nabla}f=\vec{\mathbf{F}}$ and therefore

$$f_x(x,y)=\frac{-Gx}{(x^2+y^2)^{3/2}}.$$

To integrate this function with respect to x, we can use u-substitution. If $u=x^2+y^2$, then $\frac{du}{2}=x\,dx$, so

$$\begin{aligned}\int\frac{-Gx}{(x^2+y^2)^{3/2}}\,dx&=\int\frac{-G}{2u^{3/2}}\,du\\&=\frac{G}{\sqrt{u}}+h(y)\\&=\frac{G}{\sqrt{x^2+y^2}}+h(y)\end{aligned}$$

for some function $h(y)$. Therefore,

$$f(x,y)=\frac{G}{\sqrt{x^2+y^2}}+h(y).$$

Since f is a potential function for $\vec{\mathbf{F}}$,

$$f_y(x,y)=\frac{-Gy}{(x^2+y^2)^{3/2}}$$

.

Since $f(x,y)=\dfrac{G}{\sqrt{x^2+y^2}}+h(y)$, $f_y(x,y)$ also equals $\dfrac{-Gy}{(x^2+y^2)^{3/2}}+h'(y)$.

Therefore,

$$\frac{-Gy}{(x^2+y^2)^{3/2}} + h'(y) = \frac{-Gy}{(x^2+y^2)^{3/2}},$$

which implies that $h'(y) = 0$. Thus, we can take $h(y)$ to be any constant; in particular, we can let $h(y) = 0$. The function

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}}$$

is a potential function for the gravitational field $\vec{\mathbf{F}}$. To confirm that f is a potential function, note that

$$\begin{aligned} \vec{\nabla} f(x,y) &= \langle -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2x), -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2y)\rangle \\ &= \langle \frac{-Gx}{(x^2+y^2)^{3/2}}, \frac{-Gy}{(x^2+y^2)^{3/2}}\rangle \\ &= \vec{\mathbf{F}}(x,y). \end{aligned}$$

Exercise 9.3.7

Find a potential function f for the three-dimensional gravitational force $\vec{\mathbf{F}}(x,y,z) = \langle \frac{-Gx}{(x^2+y^2+z^2)^{3/2}}, \frac{-Gy}{(x^2+y^2+z^2)^{3/2}}, \frac{-Gz}{(x^2+y^2+z^2)^{3/2}}\rangle$.

Hint

Follow the Problem-Solving Strategy.

Answer

$$f(x,y,z) = \frac{G}{\sqrt{x^2+y^2+z^2}}$$

9.3.4 Testing a Vector Field

Until now, we have worked with vector fields that we know are conservative, but if we are not told that a vector field is conservative, we need to be able to test whether it is conservative. Recall that, if $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}}$ has the cross-partial property (see The Cross-Partial Property of Conservative Vector Fields). That is, if $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is conservative, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$. So, if $\vec{\mathbf{F}}$ has the cross-partial property, then is $\vec{\mathbf{F}}$ conservative? If the domain of $\vec{\mathbf{F}}$ is open and simply connected, then the answer is yes.

Theorem: THE CROSS-PARTIAL TEST FOR CONSERVATIVE FIELDS

If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is a vector field on an open, simply connected region D and $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D, then $\vec{\mathbf{F}}$ is conservative.

Although a proof of this theorem is beyond the scope of the text, we can discover its power with some examples. Later, we see why it is necessary for the region to be simply connected.

Combining this theorem with the cross-partial property, we can determine whether a given vector field is conservative:

Theorem: CROSS-PARTIAL PROPERTY OF CONSERVATIVE FIELDS

Let $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ be a vector field on an open, simply connected region D. Then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D if and only if $\vec{\mathbf{F}}$ is conservative.

The version of this theorem in $\mathbb{R}^2$ is also true. If $\vec{\mathbf{F}}(x,y) = \langle P, Q\rangle$ is a vector field on an open, simply connected domain in $\mathbb{R}^2$, then $\vec{\mathbf{F}}$ is conservative if and only if $P_y = Q_x$.

Example 9.3.8: Determining Whether a Vector Field Is Conservative

Determine whether vector field $\vec{\mathbf{F}}(x,y,z) = \langle xy^2z, x^2yz, z^2\rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^2$ and $\mathbb{R}^3$ is simply connected. Therefore, we can use The Cross-Partial Property of Conservative Vector Fields to determine whether $\vec{\mathbf{F}}$ is conservative. Let

$$P(x,y,z) = xy^2z$$

$$Q(x,y,z) = x^2yz$$

and

$$R(x,y,z) = z^2.$$

Since $Q_z(x,y,z) = x^2y$ and $R_y(x,y,z) = 0$, the vector field is not conservative.

Example 9.3.9: Determining Whether a Vector Field Is Conservative

Determine vector field $\vec{\mathbf{F}}(x,y) = \langle x\ln(y), \frac{x^2}{2y}\rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is the part of $\mathbb{R}^2$ in which $y > 0$. Thus, the domain of $\vec{\mathbf{F}}$ is part of a plane above the x-axis, and this domain is simply connected (there are no holes in this region and this region is connected). Therefore, we can use The Cross-Partial Property of Conservative Vector Fields to determine whether $\vec{\mathbf{F}}$ is conservative. Let

$$P(x,y) = x\ln(y) \text{ and } Q(x,y) = \frac{x^2}{2y}.$$

Then $P_y(x,y) = \frac{x}{y} = Q_x(x,y)$ and thus $\vec{\mathbf{F}}$ is conservative.

Exercise 9.3.8

Determine whether $\vec{\mathbf{F}}(x,y) = \langle \sin x\cos y, \cos x\sin y\rangle$ is conservative.

Hint

Use The Cross-Partial Property of Conservative Vector Fields.

Answer

It is conservative.

When using The Cross-Partial Property of Conservative Vector Fields, it is important to remember that a theorem is a tool, and like any tool, it can be applied only under the right conditions. In the case of The Cross-Partial Property of Conservative Vector Fields, the theorem can be applied only if the domain of the vector field is simply connected.

To see what can go wrong when misapplying the theorem, consider the vector field from Example 9.3.4:

$$\vec{\mathbf{F}}(x,y) = \frac{y}{x^2+y^2}\,\hat{\mathbf{i}} + \frac{-x}{x^2+y^2}\,\hat{\mathbf{j}}. \tag{9.3.9}$$

This vector field satisfies the cross-partial property, since

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \tag{9.3.10}$$

and

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)=\frac{-(x^2+y^2)+x(2x)}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}. \tag{9.3.11}$$

Since $\vec{\mathbf{F}}$ satisfies the cross-partial property, we might be tempted to conclude that $\vec{\mathbf{F}}$ is conservative. However, $\vec{\mathbf{F}}$ is not conservative. To see this, let

$$\vec{\mathbf{r}}(t)=\langle\cos t,\sin t\rangle,\quad 0\leq t\leq\pi \tag{9.3.12}$$

be a parameterization of the upper half of a unit circle oriented counterclockwise (denote this C_1) and let

$$\vec{\mathbf{s}}(t)=\langle\cos t,-\sin t\rangle,\quad 0\leq t\leq\pi \tag{9.3.13}$$

be a parameterization of the lower half of a unit circle oriented clockwise (denote this C_2). Notice that C_1 and C_2 have the same starting point and endpoint. Since $\sin^2 t+\cos^2 t=1$,

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{r}}\,'(t)=\langle\sin(t),-\cos(t)\rangle\cdot\langle-\sin(t),\cos(t)\rangle=-1 \tag{9.3.14}$$

and

$$\vec{\mathbf{F}}(\vec{\mathbf{s}}(t))\cdot\vec{\mathbf{s}}\,'(t)=\langle-\sin t,-\cos t\rangle\cdot\langle-\sin t,-\cos t\rangle=\sin^2 t+\cos^2 t=1. \tag{9.3.15}$$

Therefore,

$$\int_{C_1}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\int_0^{\pi}-1\,dt=-\pi \tag{9.3.16}$$

and

$$\int_{C_2}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\int_0^{\pi}1\,dt=\pi. \tag{9.3.17}$$

Thus, C_1 and C_2 have the same starting point and endpoint, but $\int_{C_1}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}\neq\int_{C_2}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$. Therefore, $\vec{\mathbf{F}}$ is not independent of path and $\vec{\mathbf{F}}$ is not conservative.

To summarize: $\vec{\mathbf{F}}$ satisfies the cross-partial property and yet $\vec{\mathbf{F}}$ is not conservative. What went wrong? Does this contradict The Cross-Partial Property of Conservative Vector Fields? The issue is that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^2$ except for the origin. In other words, the domain of $\vec{\mathbf{F}}$ has a hole at the origin, and therefore the domain is not simply connected. Since the domain is not simply connected, The Cross-Partial Property of Conservative Vector Fields does not apply to $\vec{\mathbf{F}}$.

We close this section by looking at an example of the usefulness of the Fundamental Theorem for Line Integrals. Now that we can test whether a vector field is conservative, we can always decide whether the Fundamental Theorem for Line Integrals can be used to calculate a vector line integral. If we are asked to calculate an integral of the form $\int_C\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, then our first question should be: Is $\vec{\mathbf{F}}$ conservative? If the answer is yes, then we should find a potential function and use the Fundamental Theorem for Line Integrals to calculate the integral. If the answer is no, then the Fundamental Theorem for Line Integrals cannot help us and we have to use other methods, such as using [link].

Example 9.3.10: Using the Fundamental Theorem for Line Integrals

Calculate line integral $\int_C\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y,z)=\langle 2xe^yz+e^xz,\ x^2e^yz,\ x^2e^y+e^x\rangle$ and C is any smooth curve that goes from the origin to $(1,1,1)$.

Solution

Before trying to compute the integral, we need to determine whether $\vec{\mathbf{F}}$ is conservative and whether the domain of $\vec{\mathbf{F}}$ is simply connected. The domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$, which is connected and has no holes. Therefore, the domain of $\vec{\mathbf{F}}$ is simply connected. Let

$$P(x,y,z)=2xe^yz+e^xz,\quad Q(x,y,z)=x^2e^yz,\quad\text{and}\quad R(x,y,z)=x^2e^y+e^x$$

so that $\vec{\mathbf{F}}(x,y,z) = \langle P,Q,R\rangle$. Since the domain of $\vec{\mathbf{F}}$ is simply connected, we can check the cross partials to determine whether $\vec{\mathbf{F}}$ is conservative. Note that

$$\begin{aligned} P_y(x,y,z) &= 2xe^yz = Q_x(x,y,z) \\ P_z(x,y,z) &= 2xe^y + e^x = R_x(x,y,z) \\ Q_z(x,y,z) &= x^2e^y = R_y(x,y,z). \end{aligned}$$

Therefore, $\vec{\mathbf{F}}$ is conservative.

To evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ using the Fundamental Theorem for Line Integrals, we need to find a potential function f for $\vec{\mathbf{F}}$. Let f be a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f = \vec{\mathbf{F}}$, and therefore $f_x(x,y,z) = 2xe^yz + e^xz$. Integrating this equation with respect to x gives $f(x,y,z) = x^2e^yz + e^xz + h(y,z)$ for some function h. Differentiating this equation with respect to y gives $x^2e^yz + h_y(y,z) = Q(x,y,z) = x^2e^yz$, which implies that $h_y(y,z) = 0$. Therefore, h is a function of z only, and $f(x,y,z) = x^2e^yz + e^xz + h(z)$. To find h, note that $f_z = x^2e^y + e^x + h'(z) = R = x^2e^y + e^x$. Therefore, $h'(z) = 0$ and we can take $h(z) = 0$. A potential function for $\vec{\mathbf{F}}$ is $f(x,y,z) = x^2e^yz + e^xz$.

Now that we have a potential function, we can use the Fundamental Theorem for Line Integrals to evaluate the integral. By the theorem,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} \\ &= f(1,1,1) - f(0,0,0) \\ &= 2e. \end{aligned}$$

Analysis

Notice that if we hadn't recognized that $\vec{\mathbf{F}}$ is conservative, we would have had to parameterize C and use [link]. Since curve C is unknown, using the Fundamental Theorem for Line Integrals is much simpler.

Exercise 9.3.9

Calculate integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = \langle \sin x \sin y, 5 - \cos x \cos y\rangle$ and C is a semicircle with starting point $(0,\pi)$ and endpoint $(0,-\pi)$.

Hint

Use the Fundamental Theorem for Line Integrals.

Answer

-10π

Example 9.3.11: Work Done on a Particle

Let $\vec{\mathbf{F}}(x,y) = \langle 2xy^2, 2x^2y\rangle$ be a force field. Suppose that a particle begins its motion at the origin and ends its movement at any point in a plane that is not on the x-axis or the y-axis. Furthermore, the particle's motion can be modeled with a smooth parameterization. Show that $\vec{\mathbf{F}}$ does positive work on the particle.

Solution

We show that $\vec{\mathbf{F}}$ does positive work on the particle by showing that $\vec{\mathbf{F}}$ is conservative and then by using the Fundamental Theorem for Line Integrals.

To show that $\vec{\mathbf{F}}$ is conservative, suppose $f(x,y)$ were a potential function for $\vec{\mathbf{F}}$. Then, $\vec{\nabla} f(x,y) = \vec{\mathbf{F}}(x,y) = \langle 2xy^2, 2x^2y\rangle$ and therefore $f_x(x,y) = 2xy^2$ and $f_y(x,y) = 2x^2y$. Equation $fx(x,y) = 2xy^2$ implies that $f(x,y) = x^2y^2 + h(y)$. Deriving both sides with respect to y yields $f_y(x,y) = 2x^2y + h'(y)$. Therefore, $h'(y) = 0$ and we can take $h(y) = 0$.

If $f(x,y) = x^2y^2$, then note that $\vec{\nabla} f(x,y) = \langle 2xy^2, 2x^2y\rangle = \vec{\mathbf{F}}$, and therefore f is a potential function for $\vec{\mathbf{F}}$.

Let (a, b) be the point at which the particle stops is motion, and let C denote the curve that models the particle's motion. The work done by $\vec{\mathbf{F}}$ on the particle is $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$. By the Fundamental Theorem for Line Integrals,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_C \nabla f \cdot d\vec{\mathbf{r}} \\ &= f(a,b) - f(0,0) \\ &= a^2b^2. \end{aligned}$$

Since $a \neq 0$ and $b \neq 0$, by assumption, $a^2b^2 > 0$. Therefore, $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} > 0$, and $\vec{\mathbf{F}}$ does positive work on the particle.

Analysis

Notice that this problem would be much more difficult without using the Fundamental Theorem for Line Integrals. To apply the tools we have learned, we would need to give a curve parameterization and use [link]. Since the path of motion C can be as exotic as we wish (as long as it is smooth), it can be very difficult to parameterize the motion of the particle.

Exercise 9.3.10

Let $\vec{\mathbf{F}}(x, y) = \langle 4x^3y^4, 4x^4y^3 \rangle$, and suppose that a particle moves from point $(4, 4)$ to $(1, 1)$ along any smooth curve. Is the work done by $\vec{\mathbf{F}}$ on the particle positive, negative, or zero?

Hint

Use the Fundamental Theorem for Line Integrals.

Answer

Negative

9.3.5 Key Concepts

- The theorems in this section require curves that are closed, simple, or both, and regions that are connected or simply connected.
- The line integral of a conservative vector field can be calculated using the Fundamental Theorem for Line Integrals. This theorem is a generalization of the Fundamental Theorem of Calculus in higher dimensions. Using this theorem usually makes the calculation of the line integral easier.
- Conservative fields are independent of path. The line integral of a conservative field depends only on the value of the potential function at the endpoints of the domain curve.
- Given vector field $\vec{\mathbf{F}}$, we can test whether $\vec{\mathbf{F}}$ is conservative by using the cross-partial property. If $\vec{\mathbf{F}}$ has the cross-partial property and the domain is simply connected, then $\vec{\mathbf{F}}$ is conservative (and thus has a potential function). If $\vec{\mathbf{F}}$ is conservative, we can find a potential function by using the Problem-Solving Strategy.
- The circulation of a conservative vector field on a simply connected domain over a closed curve is zero.

9.3.6 Key Equations

- **Fundamental Theorem for Line Integrals**
  $$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$$
- **Circulation of a conservative field over curve *C* that encloses a simply connected region**
  $$\oint_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = 0$$

9.3.7 Glossary

closed curve

a curve that begins and ends at the same point

connected region

a region in which any two points can be connected by a path with a trace contained entirely inside the region

Fundamental Theorem for Line Integrals

the value of line integral $\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$ depends only on the value of f at the endpoints of $C: \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$

independence of path

a vector field $\vec{\mathbf{F}}$ has path independence if $\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for any curves C_1 and C_2 in the domain of $\vec{\mathbf{F}}$ with the same initial points and terminal points

simple curve

a curve that does not cross itself

simply connected region

a region that is connected and has the property that any closed curve that lies entirely inside the region encompasses points that are entirely inside the region

9.3.8 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

9.3E: EXERCISES

Exercise $9.3E.1$: True or False?

1. If vector field $\vec{\mathbf{F}}$ is conservative on the open and connected region D, then line integrals of $\vec{\mathbf{F}}$ are path independent on D, regardless of the shape of D.
2. *The* Function $r(t)=a+t(b-a)$, where $0\le t\le 1$, parameterizes the straight-line segment from a to b.
3. *The* vector field $\vec{\mathbf{F}}(x,y,z)=(ysinz)\,\hat{\mathbf{i}}+(xsinz)\,\hat{\mathbf{j}}+(xycosz)\,\hat{\mathbf{k}}$ is conservative.
4. The Vector field $\vec{\mathbf{F}}(x,y,z)=y\,\hat{\mathbf{i}}+(x+z)\,\hat{\mathbf{j}}-y\,\hat{\mathbf{k}}$ is conservative.

Answer

1. T, 3. T

Exercise $9.3E.2$: Line integral over vector field

5. Justify the Fundamental Theorem of Line Integrals for $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}$ in the case when $\vec{\mathbf{F}}(x,y)=(2x+2y)\,\hat{\mathbf{i}}+(2x+2y)\,\hat{\mathbf{j}}$ and C is a portion of the positively oriented circle $x^2+y^2=25$ from $(5,0)$ to $(3,4)$.

Answer

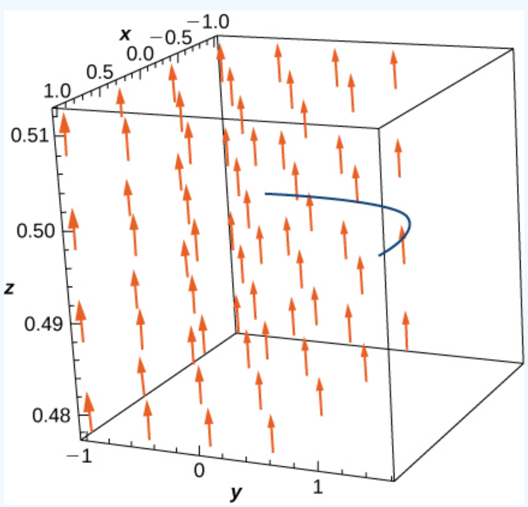
$\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}=24$

6. [T] Find $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}$, where \(\vecs{F}(x,y)=(ye^{xy}+ \cos(x)) \, \hat{ \mathbf i} + (xe^{xy}+\dfrac{1}{y^2+1}) \, \hat{ \mathbf j} \) and C is a portion of curve

$y=sinx$ from $x=0$ to $x=\dfrac{\pi}{2}$.

7. [T] Evaluate line integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}$, where $\vec{\mathbf{F}}(x,y)=(e^x siny-y)\,\hat{\mathbf{i}}+(e^x cosy-x-2)\,\hat{\mathbf{j}}$, and C is the path given by

$r(t)=(t^3 sin\dfrac{\pi t}{2})\,\hat{\mathbf{i}}-(\dfrac{\pi}{2}cos(\dfrac{\pi t}{2}+\dfrac{\pi}{2}))\,\hat{\mathbf{j}}$ for $0\le t\le 1$.



Answer

\(\int _C \vecs{F} · \vecs{dr}=e−\dfrac{3π}{2}\)

Exercise $9.3E.3$

For the following exercises, determine whether the vector field is conservative and, if it is, find the potential function.

8. $\vec{\mathbf{F}}(x,y)=2xy^3\,\hat{\mathbf{i}}+3y^2x^2\,\hat{\mathbf{j}}$
9. $\vec{\mathbf{F}}(x,y)=(-y+e^x siny)\,\hat{\mathbf{i}}+[(x+2)e^x cosy]\,\hat{\mathbf{j}}$

Answer

Not conservative

10. $\vec{\mathbf{F}}(x,y)=(e^{2x} siny)\,\hat{\mathbf{i}}+[e^{2x} cosy]\,\hat{\mathbf{j}}$
11. $\vec{\mathbf{F}}(x,y)=(6x+5y)\,\hat{\mathbf{i}}+(5x+4y)\,\hat{\mathbf{j}}$

Answer

Conservative, $\vec{\mathbf{F}}(x,y)=3x^2+5xy+2y^2$

12. $\vec{\mathbf{F}}(x,y)=[2xcos(y)-ycos(x)]\,\hat{\mathbf{i}}+[-x^2sin(y)-sin(x)]\,\hat{\mathbf{j}}$
13. $\vec{\mathbf{F}}(x,y)=[ye^x+sin(y)]\,\hat{\mathbf{i}}+[e^x+xcos(y)]\,\hat{\mathbf{j}}$

Answer

Conservative, $\vec{\mathbf{F}}(x,y)=ye^x+xsin(y)$

Exercise $9.3E.4$

For the following exercises, evaluate the line integrals using the Fundamental Theorem of Line Integrals.

14. $\oint_C (y\,\hat{\mathbf{i}}+x\,\hat{\mathbf{j}})\cdot dr,$ where C is any path from $(0,0)$ to $(2,4)$
15. $\oint_C (2ydx+2xdy)$, where C is the line segment from (0, 0) to (4, 4)

Answer

$\oint_C (2ydx+2xdy)=32$

16. **[T]** $\oint_C [arctan\frac{y}{x}-\frac{xy}{x^2+y^2}]dx+[\frac{x^2}{x^2+y^2}+e^{-y}(1-y)]dy$, where C is any smooth curve from $(1,1)$ to $(-1,2)$
17. Find the conservative vector field for the potential function

$$\vec{\mathbf{F}}(x,y)=5x^2+3xy+10y^2.$$

Answer

$\vec{\mathbf{F}}(x,y)=(10x+3y)i+(3x+10y)j$

Exercise $9.3E.5$

For the following exercises, determine whether the vector field is conservative and, if so, find a potential function.

18. $\vec{\mathbf{F}}(x,y)=(12xy)\,\hat{\mathbf{i}}+6(x^2+y^2)\,\hat{\mathbf{j}}$
19. $\vec{\mathbf{F}}(x,y)=(e^x cosy)\,\hat{\mathbf{i}}+6(e^x siny)\,\hat{\mathbf{j}}$

Answer

F is not conservative.

20. $\vec{\mathbf{F}}(x,y)=(2xye^{x^2y})\,\hat{\mathbf{i}}+6(x^2e^{x^2y})\,\hat{\mathbf{j}}$

21. $F(x,y,z)=(ye^z)\,\hat{\mathbf{i}}+(xe^z)\,\hat{\mathbf{j}}+(xye^z)\,\hat{\mathbf{k}}$

Answer

F is conservative and a potential function is $f(x,y,z)=xye^z$.

22. $F(x,y,z)=(siny)\,\hat{\mathbf{i}}-(xcosy)\,\hat{\mathbf{j}}+\hat{\mathbf{k}}$

23. $F(x,y,z)=(\frac{1}{y})\,\hat{\mathbf{i}}+(\frac{x}{y^2})\,\hat{\mathbf{j}}+(2z-1)\,\hat{\mathbf{k}}$

Answer

F is conservative and a potential function is $f(x,y,z)=z$.

24. $F(x,y,z)=3z^2\,\hat{\mathbf{i}}-cosy\,\hat{\mathbf{j}}+2xz\,\hat{\mathbf{k}}$

25. $F(x,y,z)=(2xy)\,\hat{\mathbf{i}}+(x^2+2yz)\,\hat{\mathbf{j}}+y^2\,\hat{\mathbf{k}}$

Answer

F is conservative and a potential function is $f(x,y,z)=x^2y+y^2z$.

Exercise $9.3E.6$

For the following exercises, determine whether the given vector field is conservative and find a potential function.

26. $\vec{\mathbf{F}}(x,y)=(e^xcosy)\,\hat{\mathbf{i}}+6(e^xsiny)\,\hat{\mathbf{j}}$

27. $\vec{\mathbf{F}}(x,y)=(2xye^{x^2y})\,\hat{\mathbf{i}}+6(x^2e^{x^2y})\,\hat{\mathbf{j}}$

Answer

$\vec{\mathbf{F}}$ is conservative and a potential function is $f(x,y)=e^{x^2y}$

Exercise $9.3E.7$

For the following exercises, evaluate the integral using the Fundamental Theorem of Line Integrals.

28. Evaluate $\int_C \vec{\nabla}\mathbf{f}\cdot\vec{\mathbf{dr}}$, where $f(x,y,z)=cos(\pi x)+sin(\pi y)-xyz$ and C is any path that starts at $(1,12,2)$ and ends at $(2,1,-1)$.

29. [T] Evaluate $\int_C \vec{\nabla}\mathbf{f}\cdot\vec{\mathbf{dr}}$, where $\vec{\mathbf{F}}(x,y)=xy+e^x$ and C is a straight line from $(0,0)$ to $(2,1)$.

Solution: $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}=e^2+1$

30. [T] Evaluate $\int_C \vec{\nabla}\mathbf{f}\cdot\vec{\mathbf{dr}}$, where $\vec{\mathbf{F}}(x,y)=x^2y-x$ and C is any path in a plane from (1, 2) to (3, 2).

31. Evaluate $\int_C \vec{\nabla}\mathbf{f}\cdot\vec{\mathbf{dr}}$, where $f(x,y,z)=xyz^2-yz$ and C has initial point (1, 2) and terminal point (3, 5).

Answer

$\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}=41$

Exercise $9.3E.8$

For the following exercises, evaluate the integral using the Fundamental Theorem of Line Integrals.

28. Evaluate $\int_C \vec{\nabla}\mathbf{f}\cdot\vec{\mathbf{dr}}$, where $f(x,y,z)=cos(\pi x)+sin(\pi y)-xyz$ and C is any path that starts at $(1,12,2)$ and ends at $(2,1,-1)$.

29. [T] Evaluate $\int_C \vec{\nabla} \mathbf{f} \cdot \vec{\mathbf{dr}}$, where $\vec{\mathbf{F}}(x,y) = xy + e^x$ and C is a straight line from $(0,0)$ to $(2,1)$.

Solution: $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = e^2 + 1$

30. [T] Evaluate $\int_C \vec{\nabla} \mathbf{f} \cdot \vec{\mathbf{dr}}$, where $\vec{\mathbf{F}}(x,y) = x^2 y - x$ and C is any path in a plane from (1, 2) to (3, 2).

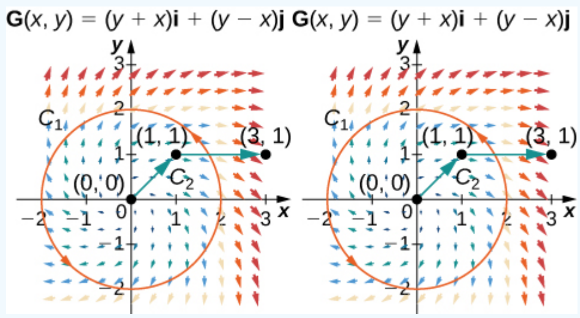
31. Evaluate $\int_C \vec{\nabla} \mathbf{f} \cdot \vec{\mathbf{dr}}$, where $f(x,y,z) = xyz^2 - yz$ and C has initial point (1, 2) and terminal point (3, 5).

Answer

$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = 41$

Exercise $9.3E.9$

For the following exercises, let $\vec{\mathbf{F}}(x,y) = 2xy^2\,\hat{\mathbf{i}} + (2yx^2 + 2y)\,\hat{\mathbf{j}}$ and $G(x,y) = (y+x)\,\hat{\mathbf{i}} + (y-x)\,\hat{\mathbf{j}}$, and let C_1 be the curve consisting of the circle of radius 2, centered at the origin and oriented counterclockwise, and C_2 be the curve consisting of a line segment from (0, 0) to (1, 1) followed by a line segment from (1, 1) to (3, 1).



32. Calculate the line integral of **F** over C_1.

33. Calculate the line integral of **G** over C_1.

Solution: $\oint_{C_1} \mathbf{G} \cdot \mathbf{dr} = -8\pi$

34. Calculate the line integral of **F** over C_2.

35. Calculate the line integral of **G** over C_2.

Solution: $\oint_{C_2} F \cdot dr = 7$

36. [T] Let $F(x,y,z) = x^2\,\hat{\mathbf{i}} + z\sin(yz)j + y\sin(yz)k$. Calculate $\oint_C F \cdot dr$, where C is a path from $A = (0,0,1)$ to $B = (3,1,2)$.

37. [T] Find line integral $\oint_C F \cdot dr$ of vector field $F(x,y,z) = 3x^2 z\,\hat{\mathbf{i}} + z^2\,\hat{\mathbf{j}} + (x^3 + 2yz)\,\hat{\mathbf{k}}$ along curve C parameterized by $r(t) = (\frac{\ln t}{\ln 2})\,\hat{\mathbf{i}} + t^{3/2}\,\hat{\mathbf{j}} + t\cos(\pi t)\,\hat{\mathbf{k}}, 1 \le t \le 4$.

Solution: $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = 150$

Answer

Add texts here. Do not delete this text first.

Exercise $9.3E.10$

For the following exercises, show that the following vector fields are conservative by using a computer. Calculate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}}$ for the given curve.

38. $\vec{\mathbf{F}} = (xy^2 + 3x^2 y)\,\hat{\mathbf{i}} + (x+y)x^2\,\hat{\mathbf{j}}$; C is the curve consisting of line segments from (1,1)to (0,2) to (3,0).

39. $\vec{\mathbf{F}} = \frac{2x}{y^2+1}\,\hat{\mathbf{i}} - \frac{2y(x^2+1)}{(y^2+1)^2}\,\hat{\mathbf{j}}$; C is parameterized by $x = t^3 - 1, y = t^6 - t, 0 \le t \le 1$.

Solution: $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = -1$

40. [T] $\vec{\mathbf{F}} = [\cos(xy^2) - xy^2 \sin(xy^2)]\,\hat{\mathbf{i}} - 2x^2 y \sin(xy^2)\,\hat{\mathbf{j}}$; C is curve $(e^t, e^{t+1}), -1 \le t \le 0$.

Answer

Add texts here. Do not delete this text first.

Exercise $9.3E.11$

41. The mass of Earth is approximately $6 \times 10^{27} g$ and that of the Sun is 330,000 times as much. The gravitational constant is $6.7 \times 10^{-8} cm^3/s^2 \cdot g$. The distance of Earth from the Sun is about $1.5 \times 10^{12} cm$. Compute, approximately, the work necessary to increase the distance of Earth from the Sun by 1cm.

Solution: $4 \times 10^{31} erg$

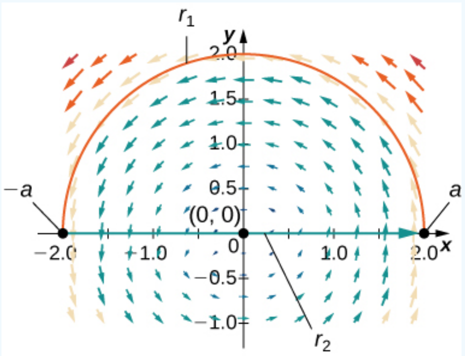
42. [T] Let $\vec{\mathbf{F}} = (x, y, z) = (e^x \sin y)\,\hat{\mathbf{i}} + (e^x \cos y)\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$. Evaluate the integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{ds}}$,where $C(t) = (\sqrt{t}, t^3, e^{\sqrt{t}}), 0 \le t \le 1$.

43. [T] Let $c : [1, 2] \to \mathbb{R}^2$ be given by $x = e^{t-1}, y = \sin(\frac{\pi}{t})$. Use a computer to compute the integral $\int_C F \cdot ds = \int_C 2x \cos y dx - x^2 \sin y dy$, where $\vec{\mathbf{F}} = (2x \cos y)i - (x^2 \sin y)j$.

Solution: $\int_C F \cdot ds = 0.4687$

44. [T] Use a computer algebra system to find the mass of a wire that lies along curve $r(t) = (t^2 - 1)\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}, 0 \le t \le 1$, if the density is $\frac{3}{2}t$.

45. Find the circulation and flux of field $\vec{\mathbf{F}} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}$ around and across the closed semicircular path that consists of semicircular arch $r_1(t) = (a\cos t)\,\hat{\mathbf{i}} + (a\sin t)\,\hat{\mathbf{j}}, 0 \le t \le \pi$, followed by line segment $r_2(t) = t\,\hat{\mathbf{i}}, -a \le t \le a$.



Solution: $circulation = \pi a^2$ and $flux = 0$

46. Compute $\int_C \cos x \cos y dx - \sin x \sin y dy$, where $C(t) = (t, t^2), 0 \le t \le 1$.

47. Complete the proof of Note by showing that $f_y = Q(x, y)$.

Answer

Add texts here. Do not delete this text first.

9.4: Green's Theorem

Learning Objectives

- Apply the circulation form of Green's theorem.
- Apply the flux form of Green's theorem.
- Calculate circulation and flux on more general regions.

In this section, we examine Green's theorem, which is an extension of the Fundamental Theorem of Calculus to two dimensions. Green's theorem has two forms: a circulation form and a flux form, both of which require region D in the double integral to be simply connected. However, we will extend Green's theorem to regions that are not simply connected.

Put simply, Green's theorem relates a line integral around a simply closed plane curve C and a double integral over the region enclosed by C. The theorem is useful because it allows us to translate difficult line integrals into more simple double integrals, or difficult double integrals into more simple line integrals.

9.4.1 Extending the Fundamental Theorem of Calculus

Recall that the Fundamental Theorem of Calculus says that

$$\int_a^b F'(x)\,dx = F(b) - F(a). \tag{9.4.1}$$

As a geometric statement, this equation says that the integral over the region below the graph of $F'(x)$ and above the line segment $[a, b]$ depends only on the value of F at the endpoints a and b of that segment. Since the numbers a and b are the boundary of the line segment $[a, b]$, the theorem says we can calculate integral $\int_a^b F'(x)\,dx$ based on information about the boundary of line segment $[a, b]$ (Figure 9.4.1). The same idea is true of the Fundamental Theorem for Line Integrals:

$$\int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)). \tag{9.4.2}$$

When we have a potential function (an "antiderivative"), we can calculate the line integral based solely on information about the boundary of curve C.

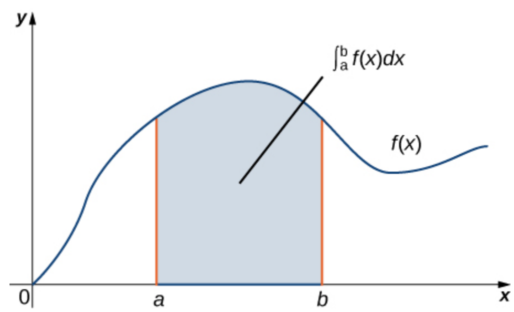


Figure 9.4.1: The Fundamental Theorem of Calculus says that the integral over line segment $[a, b]$ depends only on the values of the antiderivative at the endpoints of $[a, b]$.

Green's theorem takes this idea and extends it to calculating double integrals. Green's theorem says that we can calculate a double integral over region D based solely on information about the boundary of D. Green's theorem also says we can calculate a line integral over a simple closed curve C based solely on information about the region that C encloses. In particular, Green's theorem connects a double integral over region D to a line integral around the boundary of D.

9.4.2 Circulation Form of Green's Theorem

The first form of Green's theorem that we examine is the circulation form. This form of the theorem relates the vector line integral over a simple, closed plane curve C to a double integral over the region enclosed by C. Therefore, the circulation of a vector field along a simple closed curve can be transformed into a double integral and vice versa.

GREEN'S THEOREM (CIRCULATION FORM)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise (Figure 9.4.1). Let $\vec{\mathbf{F}} = \langle P, Q\rangle$ be a vector field with component functions that have continuous partial derivatives on D. Then,

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_C P\,dx + Q\,dy \tag{9.4.3}$$

$$= \iint_D (Q_x - P_y)\,dA. \tag{9.4.4}$$

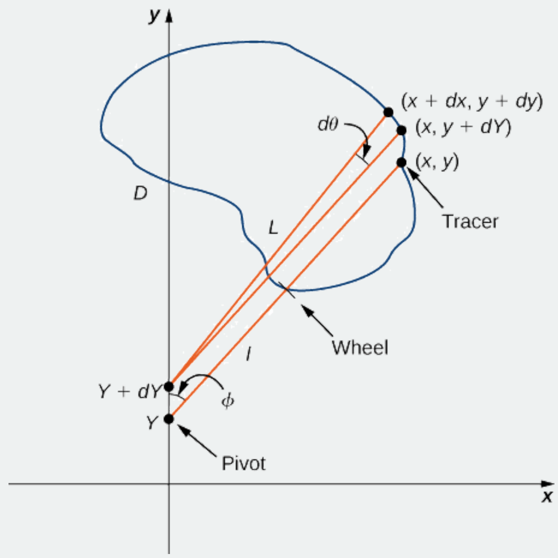


Figure 9.4.2: The circulation form of Green's theorem relates a line integral over curve C to a double integral over region D.

Notice that Green's theorem can be used only for a two-dimensional vector field $\vec{\mathbf{F}}$. If $\vec{\mathbf{F}}$ is a three-dimensional field, then Green's theorem does not apply. Since

$$\int_C P\,dx + Q\,dy = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds \tag{9.4.5}$$

this version of Green's theorem is sometimes referred to as the ***tangential form* of Green's theorem.**

The proof of Green's theorem is rather technical, and beyond the scope of this text. Here we examine a proof of the theorem in the special case that D is a rectangle. For now, notice that we can quickly confirm that the theorem is true for the special case in which $\vec{\mathbf{F}} = \langle P, Q\rangle$ is conservative. In this case,

$$\oint_C P\,dx + Q\,dy = 0 \tag{9.4.6}$$

because the circulation is zero in conservative vector fields. $\vec{\mathbf{F}}$ satisfies the cross-partial condition, so $P_y = Q_x$. Therefore,

$$\iint_D (Q_x - P_y)\,dA = \int_D 0\,dA = 0 = \oint_C P\,dx + Q\,dy \tag{9.4.7}$$

which confirms Green's theorem in the case of conservative vector fields.

Proof

Let's now prove that the circulation form of Green's theorem is true when the region D is a rectangle. Let D be the rectangle $[a, b] \times [c, d]$ oriented counterclockwise. Then, the boundary C of D consists of four piecewise smooth pieces C_1, C_2, C_3, and C_4 (Figure 9.4.3). We parameterize each side of D as follows:

$$C_1 : \vec{\mathbf{r}}_1(t) = \langle t, c\rangle\,,\ a \le t \le b$$

$$C_2 : \vec{\mathbf{r}}_2(t) = \langle b, t\rangle\,,\ c \le t \le d$$

$$-C_3 : \vec{\mathbf{r}}_3(t) = \langle t, d\rangle\,,\ a \le t \le b$$

$$-C_4 : \vec{\mathbf{r}}_4(t) = \langle a, t\rangle\,,\ c \le t \le d\,.$$

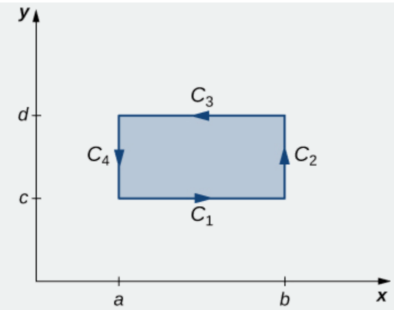


Figure 9.4.3: Rectangle D is oriented counterclockwise.

Then,

$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} &= \int_{C_1}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}+\int_{C_2}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}+\int_{C_3}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}+\int_{C_4}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}\\ &= \int_{C_1}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}+\int_{C_2}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}-\int_{-C_3}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}-\int_{-C_4}\vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}\\ &= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}_1(t))\cdot\vec{\mathbf{r}}_1'(t)\,dt+\int_c^d \vec{\mathbf{F}}(\vec{\mathbf{r}}_2(t))\cdot\vec{\mathbf{r}}_2'(t)\,dt-\int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}_3(t))\cdot\vec{\mathbf{r}}_3'(t)\,dt-\int_c^d \vec{\mathbf{F}}(\vec{\mathbf{r}}_4(t))\cdot\vec{\mathbf{r}}_4'(t)\,dt\\ &= \int_a^b P(t,c)\,dt+\int_c^d Q(b,t)\,dt-\int_a^b P(t,d)\,dt-\int_c^d Q(a,t)\,dt\\ &= \int_a^b (P(t,c)-P(t,d))\,dt+\int_c^d (Q(b,t)-Q(a,t))\,dt\\ &= -\int_a^b (P(t,d)-P(t,c))\,dt+\int_c^d (Q(b,t)-Q(a,t))\,dt.\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$P(t,d)-P(t,c)=\int_c^d \frac{\partial}{\partial y}P(t,y)dy$$

and

$$Q(b,t)-Q(a,t)=\int_a^b \frac{\partial}{\partial x}Q(x,t)\,dx.$$

Therefore,

$$-\int_a^b (P(t,d)-P(t,c))\,dt+\int_c^d (Q(b,t)-Q(a,t))\,dt=-\int_a^b\int_c^d \frac{\partial}{\partial y}P(t,y)\,dy\,dt+\int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,t)\,dx\,dt.$$

But,

$$\begin{aligned}-\int_a^b\int_c^d \frac{\partial}{\partial y}P(t,y)\,dy\,dt+\int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,t)\,dx\,dt &= -\int_a^b\int_c^d \frac{\partial}{\partial y}P(x,y)\,dy\,dx+\int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,y)\,dx\,dy\\ &= \int_a^b\int_c^d (Q_x-P_y)\,dy\,dx\\ &= \iint_D (Q_x-P_y)dA.\end{aligned}$$

Therefore, $\int_C \vec{\mathbf{F}}\bullet d\vec{\mathbf{r}}=\iint_D (Q_x-P_y)\,dA$ and we have proved Green’s theorem in the case of a rectangle.

□

To prove Green’s theorem over a general region D, we can decompose D into many tiny rectangles and use the proof that the theorem works over rectangles. The details are technical, however, and beyond the scope of this text.

Example 9.4.1: Applying Green's Theorem over a Rectangle

Calculate the line integral

$$\oint_C x^2 y dx + (y-3)dy,$$

where C is a rectangle with vertices $(1,1)$, $(4,1)$, $(4,5)$, and $(1,5)$ oriented counterclockwise.

Solution

Let $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x^2 y, y-3\rangle$. Then, $Q_x(x,y) = 0$ and $P_y(x,y) = x^2$. Therefore, $Q_x - P_y = -x^2$.

Let D be the rectangular region enclosed by C (Figure $9.4.4$). By Green's theorem,

$$\begin{aligned}\oint_C x^2 y dx + (y-3)\,dy &= \iint_D (Q_x - P_y)\,dA \\ &= \iint_D -x^2\,dA = \int_1^5 \int_1^4 -x^2\,dx\,dy \\ &= \int_1^5 -21\,dy = -84.\end{aligned}$$

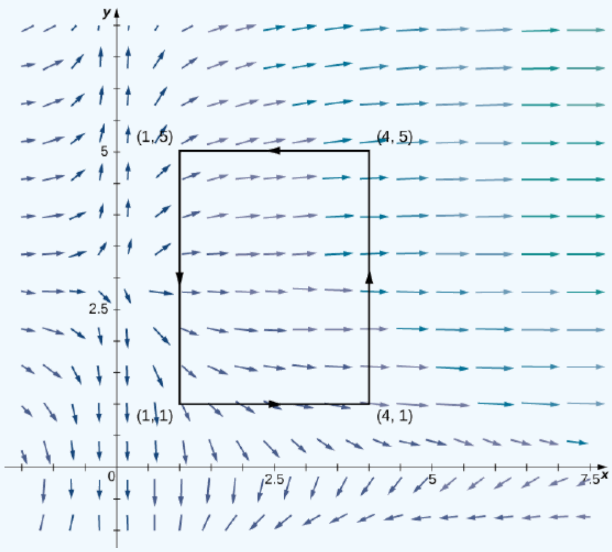


Figure $9.4.4$: The line integral over the boundary of the rectangle can be transformed into a double integral over the rectangle.

Analysis

If we were to evaluate this line integral without using Green's theorem, we would need to parameterize each side of the rectangle, break the line integral into four separate line integrals, and use the methods from Line Integrals to evaluate each integral. Furthermore, since the vector field here is not conservative, we cannot apply the Fundamental Theorem for Line Integrals. Green's theorem makes the calculation much simpler.

Example 9.4.2: Applying Green's Theorem to Calculate Work

Calculate the work done on a particle by force field

$$\vec{\mathbf{F}}(x,y) = \langle y + \sin x, e^y - x\rangle$$

as the particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2,0)$.

Solution

Let C denote the circle and let D be the disk enclosed by C. The work done on the particle is

$$W = \oint_C (y + \sin x)\,dx + (e^y - x)\,dy.$$

As with Example 9.4.1, this integral can be calculated using tools we have learned, but it is easier to use the double integral given by Green's theorem (Figure 9.4.5).

Let $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle y+\sin x, e^y - x\rangle$. Then, $Q_x = -1$ and $P_y = 1$. Therefore, $Q_x - P_y = -2$.

By Green's theorem,

\begin{align*} W &=\oint_C(y+\sin(x))dx+(e^y−x)\,dy \\[4pt] &=\iint_D (Q_x−P_y)\,dA \\[4pt] &=\iint_D−2\,dA \\[4pt] &=−2(area(D))=−2\pi (2^2)=−8\pi.

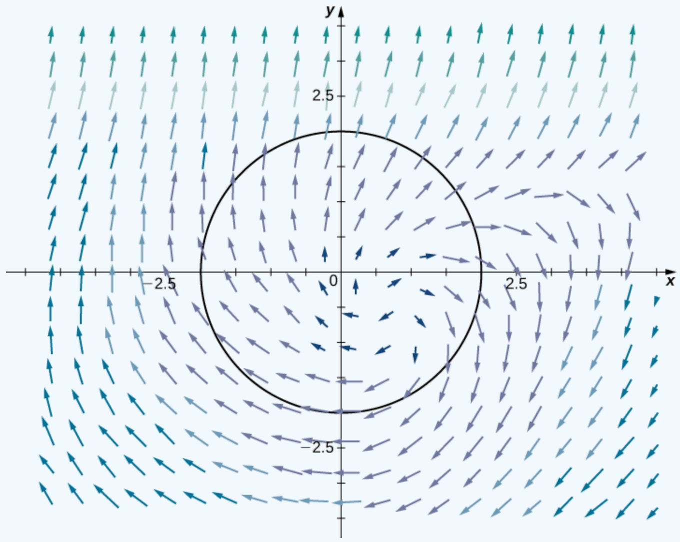


Figure 9.4.5: The line integral over the boundary circle can be transformed into a double integral over the disk enclosed by the circle.

Exercise 9.4.2

Use Green's theorem to calculate line integral

$$\oint_C \sin(x^2)\,dx + (3x-y)\,dy. \tag{9.4.8}$$

where C is a right triangle with vertices $(-1,2)$, $(4,2)$, and $(4,5)$ oriented counterclockwise.

Hint

Transform the line integral into a double integral.

Answer

$\frac{45}{2}$

In the preceding two examples, the double integral in Green's theorem was easier to calculate than the line integral, so we used the theorem to calculate the line integral. In the next example, the double integral is more difficult to calculate than the line integral, so we use Green's theorem to translate a double integral into a line integral.

Example 9.4.3: Applying Green's Theorem over an Ellipse

Calculate the area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Figure 9.4.6).

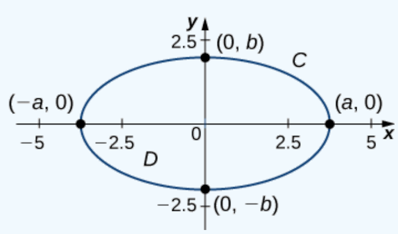


Figure 9.4.6: Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is denoted by C.

Solution

Let C denote the ellipse and let D be the region enclosed by C. Recall that ellipse C can be parameterized by

- $x = a\cos t$,
- $y = b\sin t$,
- $0 \leq t \leq 2\pi$.

Calculating the area of D is equivalent to computing double integral $\iint_D dA$. To calculate this integral without Green's theorem, we would need to divide D into two regions: the region above the *x*-axis and the region below. The area of the ellipse is

$$\int_{-a}^{a}\int_{0}^{\sqrt{b^2-(bx/a)^2}} dy\,dx + \int_{-a}^{a}\int_{-\sqrt{b^2-(bx/a)^2}}^{0} dy\,dx.$$

These two integrals are not straightforward to calculate (although when we know the value of the first integral, we know the value of the second by symmetry). Instead of trying to calculate them, we use Green's theorem to transform $\iint_D dA$ into a line integral around the boundary C.

Consider vector field

$$F(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle.$$

Then, $Q_x = \frac{1}{2}$ and $P_y = -\frac{1}{2}$, and therefore $Q_x - P_y = 1$. Notice that $\vec{\mathbf{F}}$ was chosen to have the property that $Q_x - P_y = 1$. Since this is the case, Green's theorem transforms the line integral of $\vec{\mathbf{F}}$ over C into the double integral of 1 over D.

By Green's theorem,

$$\begin{aligned}\iint_D dA &= \iint_D (Q_x - P_y)\,dA \\ &= \int_C \vec{\mathbf{F}}\bullet d\vec{\mathbf{r}} = \frac{1}{2}\int_C -y\,dx + x\,dy \\ &= \frac{1}{2}\int_0^{2\pi} -b\sin t(-a\sin t) + a(\cos t) b\cos t\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab.\end{aligned}$$

Therefore, the area of the ellipse is πab.

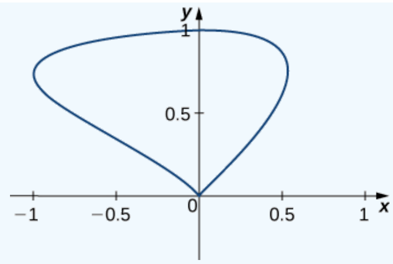
In Example $9.4.3$, we used vector field $\vec{\mathbf{F}}(x,y) = \langle P,Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$ to find the area of any ellipse. The logic of the previous example can be extended to derive a formula for the area of any region D. Let D be any region with a boundary that is a simple closed curve C oriented counterclockwise. If $F(x,y) = \langle P,Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$, then $Q_x - P_y = 1$. Therefore, by the same logic as in Example $9.4.3$,

$$\text{area of } D = \iint_D dA = \frac{1}{2}\oint_C -y dx + x dy. \tag{9.4.9}$$

It's worth noting that if $F = \langle P, Q\rangle$ is any vector field with $Q_x - P_y = 1$, then the logic of the previous paragraph works. So. Equation 9.4.9 is not the only equation that uses a vector field's mixed partials to get the area of a region.

Exercise 9.4.3

Find the area of the region enclosed by the curve with parameterization $r(t) = \langle \sin t\cos t, \sin t\rangle$, $0 \leq t \leq \pi$.



Hint

Use Equation 9.4.9.

Answer

$\frac{4}{3}$

9.4.3 Flux Form of Green's Theorem

The circulation form of Green's theorem relates a double integral over region D to line integral $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds$, where C is the boundary of D. The flux form of Green's theorem relates a double integral over region D to the flux across boundary C. The flux of a fluid across a curve can be difficult to calculate using the flux line integral. This form of Green's theorem allows us to translate a difficult flux integral into a double integral that is often easier to calculate.

GREEN'S THEOREM (FLUX FORM)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve that is oriented counterclockwise (Figure 9.4.7). Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on an open region containing D. Then,

$$\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}} \, ds = \iint_D P_x + Q_y \, dA. \tag{9.4.10}$$

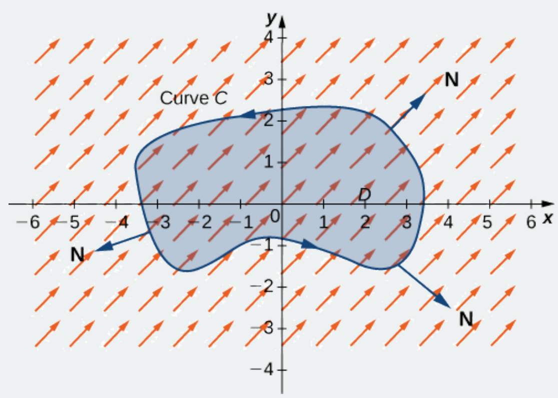


Figure 9.4.7: The flux form of Green's theorem relates a double integral over region D to the flux across curve C.

Because this form of Green's theorem contains unit normal vector **N**, it is sometimes referred to as the *normal form* of Green's theorem.

Proof

Recall that $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}} \, ds = \oint_C -Q \, dx + P \, dy$. Let $M = -Q$ and $N = P$. By the circulation form of Green's theorem,

$$\begin{aligned}\oint_C -Q\,dx + P\,dy &= \oint_C M\,dx + N\,dy \\ &= \iint_D N_x - M_y\,dA \\ &= \iint_D P_x - (-Q)_y\,dA \\ &= \iint_D P_x + Q_y\,dA.\end{aligned}$$

□

Example $9.4.4A$: Applying Green's Theorem for Flux across a Circle

Let C be a circle of radius r centered at the origin (Figure $9.4.8$) and let $\vec{\mathbf{F}}(x,y) = \langle x, y\rangle$. Calculate the flux across C.

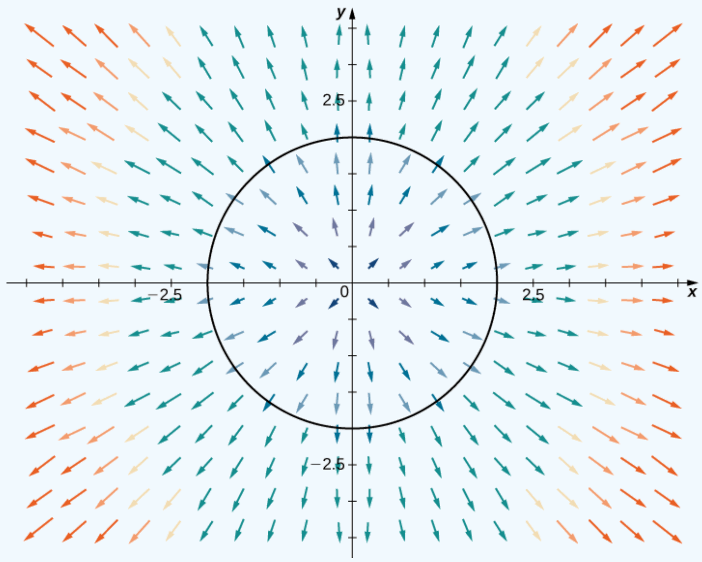


Figure $9.4.8$: Curve C is a circle of radius r centered at the origin.

Solution

Let D be the disk enclosed by C. The flux across C is $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,ds$. We could evaluate this integral using tools we have learned, but Green's theorem makes the calculation much more simple. Let $P(x,y) = x$ and $Q(x,y) = y$ so that $\vec{\mathbf{F}} = \langle P, Q\rangle$. Note that $P_x = 1 = Q_y$, and therefore $P_x + Q_y = 2$. By Green's theorem,

$$\int_C \vec{\mathbf{F}} \bullet \vec{\mathbf{N}}\,ds = \iint_D 2\,dA = 2\iint_D dA. \tag{9.4.11}$$

Since $\iint_D dA$ is the area of the circle, $\iint_D dA = \pi r^2$. Therefore, the flux across C is $2\pi r^2$.

Example $9.4.4B$: Applying Green's Theorem for Flux across a Triangle

Let S be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,3)$ oriented clockwise (Figure $9.4.9$). Calculate the flux of $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x^2 + e^y, x + y\rangle$ across S.

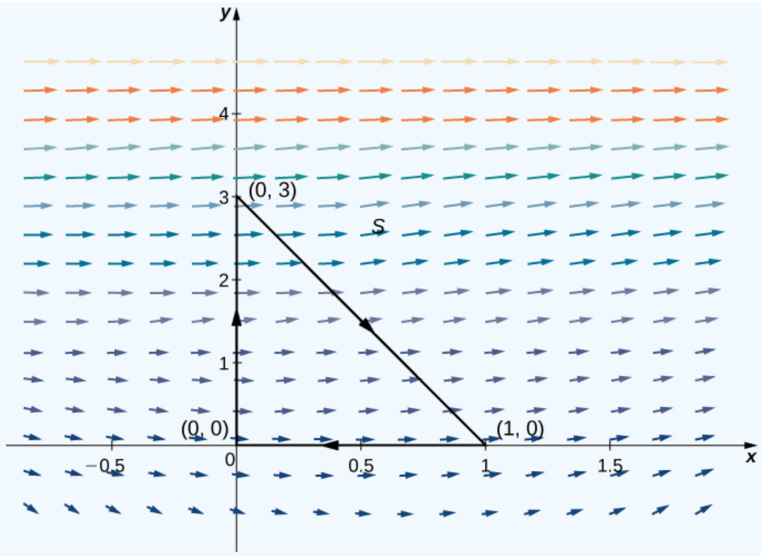


Figure 9.4.9: Curve S is a triangle with vertices $(0,0)$, $(1,0)$, and $(0,3)$ oriented clockwise.

Solution

To calculate the flux without Green's theorem, we would need to break the flux integral into three line integrals, one integral for each side of the triangle. Using Green's theorem to translate the flux line integral into a single double integral is much more simple.

Let D be the region enclosed by S. Note that $P_x = 2x$ and $Q_y = 1$; therefore, $P_x + Q_y = 2x + 1$. Green's theorem applies only to simple closed curves oriented counterclockwise, but we can still apply the theorem because $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = -\oint_{-S} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ and $-S$ is oriented counterclockwise. By Green's theorem, the flux is

$$\begin{aligned}\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds &= \oint_{-S} \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds \\ &= -\iint_D (P_x + Q_y)\, dA \\ &= -\iint_D (2x+1)\, dA.\end{aligned}$$

Notice that the top edge of the triangle is the line $y = -3x + 3$. Therefore, in the iterated double integral, the y-values run from $y = 0$ to $y = -3x + 3$, and we have

$$\begin{aligned}-\iint_D (2x+1)\, dA &= -\int_0^1 \int_0^{-3x+3} (2x+1)\, dy\, dx \\ = -\int_0^1 (2x+1)(-3x+3)\, dx &= -\int_0^1 (-6x^2+3x+3)\, dx \\ &= -\left[-2x^3 + \frac{3x^2}{2} + 3x\right]_0^1 = -\frac{5}{2}.\end{aligned}$$

Exercise 9.4.4

Calculate the flux of $\vec{\mathbf{F}}(x,y) = \langle x^3, y^3 \rangle$ across a unit circle oriented counterclockwise.

Hint

Apply Green's theorem and use polar coordinates.

Answer

$\frac{3\pi}{2}$

Example 9.4.5: Applying Green's Theorem for Water Flow across a Rectangle

Water flows from a spring located at the origin. The velocity of the water is modeled by vector field $\vec{\mathbf{v}}(x, y) = \langle 5x + y, x + 3y \rangle$ m/sec. Find the amount of water per second that flows across the rectangle with vertices $(-1, -2)$, $(1, -2)$, $(1, 3)$,and $(-1, 3)$, oriented counterclockwise (Figure $9.4.10$).

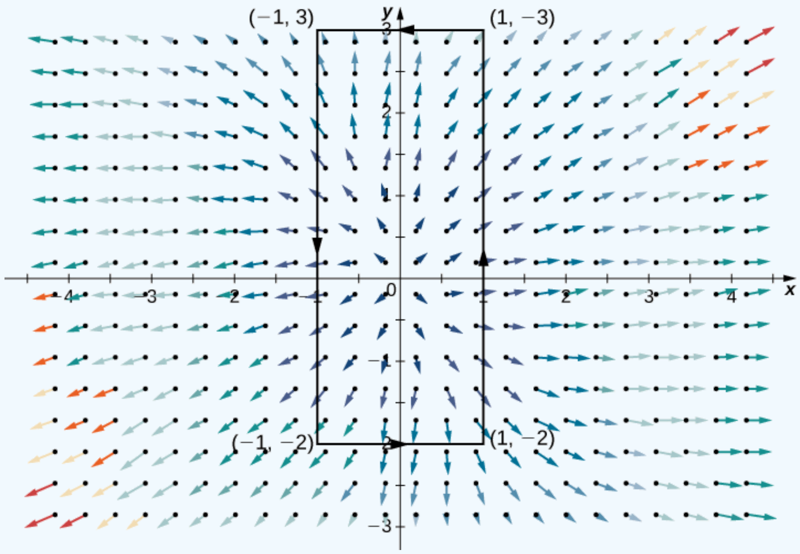


Figure 9.4.10: Water flows across the rectangle with vertices $(-1, -2)$, $(1, -2)$, $(1, 3)$,and $(-1, 3)$, oriented counterclockwise.

Solution

Let C represent the given rectangle and let D be the rectangular region enclosed by C. To find the amount of water flowing across C, we calculate flux $\int_C \vec{\mathbf{v}} \bullet d\vec{\mathbf{r}}$. Let $P(x, y) = 5x + y$ and $Q(x, y) = x + 3y$ so that $\vec{\mathbf{v}} = \langle P, Q \rangle$. Then, $P_x = 5$ and $Q_y = 3$. By Green's theorem,

$$\int_C \vec{\mathbf{v}} \bullet d\vec{\mathbf{r}} = \iint_D (P_x + Q_y)\, dA \tag{9.4.12}$$

$$= \iint_D 8\, dA \tag{9.4.13}$$

$$= 8(area\ of\ D) = 80. \tag{9.4.14}$$

Therefore, the water flux is 80 m^2/sec.

Recall that if vector field **F** is conservative, then **F** does no work around closed curves—that is, the circulation of **F** around a closed curve is zero. In fact, if the domain of **F** is simply connected, then **F** is conservative if and only if the circulation of **F** around any closed curve is zero. If we replace "circulation of **F**" with "flux of **F,**" then we get a definition of a source-free vector field. The following statements are all equivalent ways of defining a source-free field $F = \langle P, Q \rangle$ on a simply connected domain (note the similarities with properties of conservative vector fields):

1. The flux $\oint_C F \cdot N ds$ across any closed curve C is zero.
2. If C_1 and C_2 are curves in the domain of **F** with the same starting points and endpoints, then $\int_{C_1} F \cdot N ds = \int_{C_2} F \cdot N ds$. In other words, flux is independent of path.
3. There is a stream function $g(x, y)$ for **F**. A stream function for $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a function g such that $P = g_y$ and $Q = -g_x$.Geometrically, $F = (a, b)$ is tangential to the level curve of g at (a, b). Since the gradient of g is perpendicular to the level curve of g at (a, b), stream function g has the property $F(a, b) \bullet \nabla g(a, b) = 0$ for any point (a, b) in the domain of g. (Stream functions play the same role for source-free fields that potential functions play for conservative fields.)
4. $P_x + Q_y = 0$

Example 9.4.6: Finding a Stream Function

Verify that rotation vector field $F(x, y) = \langle y, -x \rangle$ is source free, and find a stream function for **F**.

Solution

Note that the domain of **F** is all of $\mathbb{R}^2$, which is simply connected. Therefore, to show that **F** is source free, we can show any of items 1 through 4 from the previous list to be true. In this example, we show that item 4 is true. Let $P(x, y) = y$ and $Q(x, y) = -x$. Then

$P_x + 0 = Q_y$, and therefore $P_x + Q_y = 0$. Thus, **F** is source free.

To find a stream function for **F**, proceed in the same manner as finding a potential function for a conservative field. Let g be a stream function for **F**. Then $g_y = y$, which implies that

$$g(x,y) = \frac{y^2}{2} + h(x) .$$

Since $-g_x = Q = -x$, we have $h\,'(x) = x$. Therefore,

$$h(x) = \frac{x^2}{2} + C .$$

Letting $C = 0$ gives stream function

$$g(x,y) = \frac{x^2}{2} + \frac{y^2}{2} .$$

To confirm that g is a stream function for **F**, note that $g_y = y = P$ and $-g_x = -x = Q$.

Notice that source-free rotation vector field $F(x,y) = \langle y, -x\rangle$ is perpendicular to conservative radial vector field $\nabla g = \langle x, y\rangle$ (Figure 9.4.11).

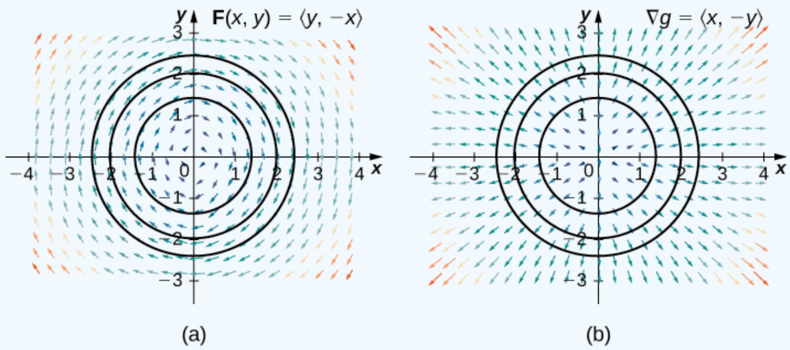


Figure 9.4.11: (a) In this image, we see the three-level curves of g and vector field **F**. Note that the **F** vectors on a given level curve are tangent to the level curve. (b) In this image, we see the three-level curves of g and vector field ∇g. The gradient vectors are perpendicular to the corresponding level curve. Therefore, $F(a,b) \bullet \nabla g(a,b) = 0$ for any point in the domain of g.

Exercise 9.4.6

Find a stream function for vector field $F(x,y) = \langle x \sin y, \cos y\rangle$.

Hint

Follow the outline provided in the previous example.

Answer

$g(x,y) = -x \cos y$

Vector fields that are both conservative and source free are important vector fields. One important feature of conservative and source-free vector fields on a simply connected domain is that any potential function ff of such a field satisfies Laplace's equation $f_{xx} + f_{yy} = 0$. Laplace's equation is foundational in the field of partial differential equations because it models such phenomena as gravitational and magnetic potentials in space, and the velocity potential of an ideal fluid. A function that satisfies Laplace's equation is called a *harmonic* function. Therefore any potential function of a conservative and source-free vector field is harmonic.

To see that any potential function of a conservative and source-free vector field on a simply connected domain is harmonic, let ff be such a potential function of vector field $F = \langle P, Q\rangle$. Then, $f_x = P$ and $f_x = Q$ because $\nabla f = F$. Therefore, $f_{xx} = P_x$ and $f_{yy} = Q_y$. Since **F** is source free, $f_{xx} + f_{yy} = P_x + Q_y = 0$, and we have that f is harmonic.

Example 9.4.7: Satisfying Laplace's Equation

For vector field F(x,y)=⟨exsiny,excosy⟩,F(x,y)=⟨exsiny,excosy⟩, verify that the field is both conservative and source free, find a potential function for **F**, and verify that the potential function is harmonic.

Solution

Let $P(x, y) = e^x \sin y$ and $Q(x, y) = e^x \cos y$. Notice that the domain of **F** is all of two-space, which is simply connected. Therefore, we can check the cross-partials of **F** to determine whether **F** is conservative. Note that $P_y = e^x \cos y = Q_x$, so **F** is conservative. Since $P_x = e^x \sin y$ and $Q_y = e^x \sin y$, $P_x + Q_y = 0$ and the field is source free.

To find a potential function for **F**, let ff be a potential function. Then, $\nabla f = F$, so $f_x = e^x \sin y$. Integrating this equation with respect to *x* gives $f(x, y) = e^x \sin y + h(y)$. Since $f_y = e^x \cos y$, differentiating f with respect to *y* gives $e^x \cos y = e^x \cos y + h'(y)$. Therefore, we can take $h(y) = 0$, and $f(x, y) = e^x \sin y$ is a potential function for f.

To verify that f is a harmonic function, note that $f_{xx} = \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$ and

$f_{yy} = \frac{\partial}{\partial x}(e^x \cos y) = -e^x \sin y$. Therefore, $f_{xx} + f_{yy} = 0$, and ff satisfies Laplace's equation.

Exercise 9.4.7

Is the function $f(x, y) = e^{x+5y}$ harmonic?

Hint

Determine whether the function satisfies Laplace's equation.

Answer

No

9.4.4 Green's Theorem on General Regions

Green's theorem, as stated, applies only to regions that are simply connected—that is, Green's theorem as stated so far cannot handle regions with holes. Here, we extend Green's theorem so that it does work on regions with finitely many holes (Figure $9.4.12$).

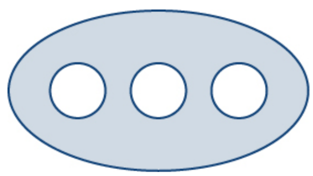


Figure $9.4.12$: Green's theorem, as stated, does not apply to a nonsimply connected region with three holes like this one.

Before discussing extensions of Green's theorem, we need to go over some terminology regarding the boundary of a region. Let *D* be a region and let *C* be a component of the boundary of *D*. We say that *C* is *positively oriented* if, as we walk along *C* in the direction of orientation, region *D* is always on our left. Therefore, the counterclockwise orientation of the boundary of a disk is a positive orientation, for example. Curve *C* is *negatively oriented* if, as we walk along *C* in the direction of orientation, region *D* is always on our right. The clockwise orientation of the boundary of a disk is a negative orientation, for example.

Let *D* be a region with finitely many holes (so that *D* has finitely many boundary curves), and denote the boundary of *D* by ∂D (Figure $9.4.13$). To extend Green's theorem so it can handle *D*, we divide region *D* into two regions, D_1 and D_2 (with respective boundaries ∂D_1 and ∂D_2), in such a way that $D = D_1 \cup D_2$ and neither D_1 nor D_2 has any holes (Figure $9.4.13$).

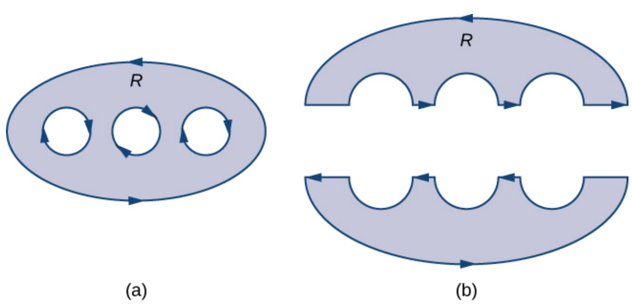


Figure $9.4.13$: (a) Region *D* with an oriented boundary has three holes. (b) Region *D* split into two simply connected regions has no holes.

Assume the boundary of *D* is oriented as in the figure, with the inner holes given a negative orientation and the outer boundary given a positive orientation. The boundary of each simply connected region D_1 and D_2 is positively oriented. If **F** is a vector field defined on *D*, then Green's theorem says that

$$\oint_{\partial D} F \cdot dr = \oint_{\partial D_1} F \cdot dr + \oint_{\partial D_2} F \cdot dr \tag{9.4.15}$$

$$= \iint_{D_1} Q_x - P_y dA + \iint_{D_2} Q_x - P_y dA \tag{9.4.16}$$

$$= \iint_D (Q_x - P_y) dA. \tag{9.4.17}$$

Therefore, Green's theorem still works on a region with holes.

To see how this works in practice, consider annulus D in Figure $9.4.14$ and suppose that $F = \langle P, Q\rangle$ is a vector field defined on this annulus. Region D has a hole, so it is not simply connected. Orient the outer circle of the annulus counterclockwise and the inner circle clockwise (Figure $9.4.14$) so that, when we divide the region into D_1 and D_2, we are able to keep the region on our left as we walk along a path that traverses the boundary. Let D_1 be the upper half of the annulus and D_2 be the lower half. Neither of these regions has holes, so we have divided D into two simply connected regions.

We label each piece of these new boundaries as P_i for some i, as in Figure $9.4.14$. If we begin at P and travel along the oriented boundary, the first segment is P_1, then P_2, P_3, and P_4. Now we have traversed D_1 and returned to P. Next, we start at P again and traverse D_2. Since the first piece of the boundary is the same as P_4 in D_1, but oriented in the opposite direction, the first piece of D_2 is $-P_4$. Next, we have P_5, then $-P_2$, and finally P_6.

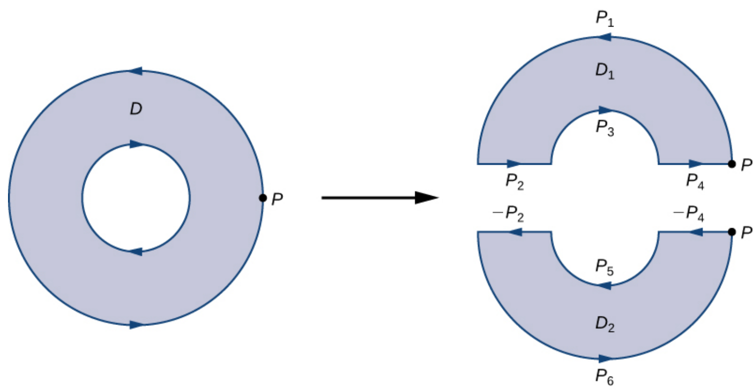


Figure 9.4.14: Breaking the annulus into two separate regions gives us two simply connected regions. The line integrals over the common boundaries cancel out.

Figure $9.4.14$ shows a path that traverses the boundary of D. Notice that this path traverses the boundary of region D_1, returns to the starting point, and then traverses the boundary of region D_2. Furthermore, as we walk along the path, the region is always on our left. Notice that this traversal of the PiPi paths covers the entire boundary of region D. If we had only traversed one portion of the boundary of D, then we cannot apply Green's theorem to D.

The boundary of the upper half of the annulus, therefore, is $P_1 \cup P_2 \cup P_3 \cup P_4$ and the boundary of the lower half of the annulus is $-P_4 \cup P_5 \cup -P_2 \cup P_6$. Then, Green's theorem implies

$$\oint_{\partial D} F \cdot dr = \int_{P_1} F \cdot dr + \int_{P_2} F \cdot dr + \int_{P_3} F \cdot dr + \int_{P_4} F \cdot dr + \int_{-P_4} F \cdot dr + \int_{P_5} F \cdot dr + \int_{-P_2} F \cdot dr + \int_{P_6} F \cdot dr \tag{9.4.18}$$

$$= \int_{P_1} F \cdot dr + \int_{P_2} F \cdot dr + \int_{P_3} F \cdot dr + \int_{P_4} F \cdot dr + \int_{P_4} F \cdot dr + \int_{P_5} F \cdot dr + \int_{-P_2} F \cdot dr + \int_{P_6} F \cdot dr \tag{9.4.19}$$

$$= \int_{P_1} F \cdot dr + \int_{P_3} F \cdot dr + \int_{P_5} F \cdot dr + \int_{P_6} F \cdot dr \tag{9.4.20}$$

$$= \oint_{\partial D_1} F \cdot dr + \oint_{\partial D_2} F \cdot dr \tag{9.4.21}$$

$$= \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (Q_x - P_y)\, dA \tag{9.4.22}$$

$$= \iint_D (Q_x - P_y)\, dA. \tag{9.4.23}$$

Therefore, we arrive at the equation found in Green's theorem—namely,

$$\oint_{\partial D} F \cdot dr = \iint_D (Q_x - P_y)\, dA. \tag{9.4.24}$$

The same logic implies that the flux form of Green's theorem can also be extended to a region with finitely many holes:

$$\oint_C F \cdot N\, ds = \iint_D (P_x + Q_y)\, dA. \tag{9.4.25}$$

Example $9.4.8A$: Using Green's Theorem on a Region with Holes

Calculate the integral

$$\oint_{\partial D} (\sin x - \frac{y^3}{3})dx + (\frac{y^3}{3} + \sin y)dy, \tag{9.4.26}$$

where D is the annulus given by the polar inequalities $1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.

Solution

Although D is not simply connected, we can use the extended form of Green's theorem to calculate the integral. Since the integration occurs over an annulus, we convert to polar coordinates:

$$\begin{aligned} \oint_{\partial D} (\sin x - \frac{y^3}{3})dx + (\frac{x^3}{3} + \sin y)dy &= \iint_D (Q_x - P_y)dA \\ &= \iint_D (x^2 + y^2)dA \\ = \int_0^{2\pi} \int_1^2 r^3 dr d\theta &= \int_0^{2\pi} \frac{15}{4} d\theta \\ &= \frac{15\pi}{2}. \end{aligned}$$

Example $9.4.8B$: Using the Extended Form of Green's Theorem

Let $F = \langle P, Q \rangle = \langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \rangle$ and let C be any simple closed curve in a plane oriented counterclockwise. What are the possible values of $\oint_C F \cdot dr$?

Solution

We use the extended form of Green's theorem to show that $\oint_C F \cdot dr$ is either 0 or -2π—that is, no matter how crazy curve C is, the line integral of **F** along C can have only one of two possible values. We consider two cases: the case when C encompasses the origin and the case when C does not encompass the origin.

9.4.4.1 Case 1: C Does Not Encompass the Origin

In this case, the region enclosed by C is simply connected because the only hole in the domain of **F** is at the origin. We showed in our discussion of cross-partials that **F** satisfies the cross-partial condition. If we restrict the domain of **F** just to C and the region it encloses, then **F** with this restricted domain is now defined on a simply connected domain. Since **F** satisfies the cross-partial property on its restricted domain, the field **F** is conservative on this simply connected region and hence the circulation $\oint_C F \cdot dr$ is zero.

9.4.4.2 Case 2: C Does Encompass the Origin

In this case, the region enclosed by C is not simply connected because this region contains a hole at the origin. Let C_1 be a circle of radius a centered at the origin so that C_1 is entirely inside the region enclosed by C (Figure $9.4.15$). Give C_1 a clockwise orientation.

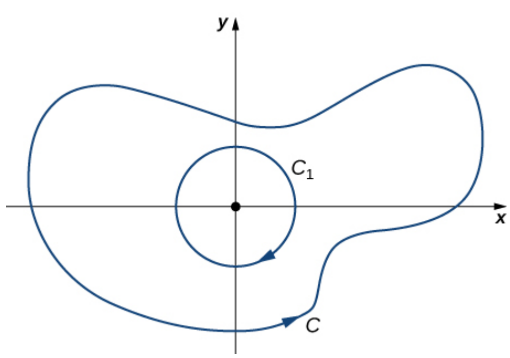


Figure $9.4.15$: Choose circle C_1 centered at the origin that is contained entirely inside C.

Let D be the region between C_1 and C, and C is orientated counterclockwise. By the extended version of Green's theorem,

$$\int_C F\cdot dr+\int_{C_1} F\cdot dr=\iint_D Qx_{-}P_y\,dA \tag{9.4.27}$$

$$=\iint_D -\frac{y^2-x^2}{\left(x^2+y^2\right)^2}+\frac{y^2-x^2}{\left(x^2+y^2\right)^2}dA \tag{9.4.28}$$

$$=0, \tag{9.4.29}$$

and therefore

$$\int_C F\cdot dr=-\int_{C_1} F\cdot dr. \tag{9.4.30}$$

Since C_1 is a specific curve, we can evaluate $\int_{C_1} F\cdot dr$. Let

$$x=a\cos t,\quad y=a\sin t,\quad 0\le t\le 2\pi \tag{9.4.31}$$

be a parameterization of C_1. Then,

$$\int_{C_1} F\cdot dr=\int_0^{2\pi} F(r(t))\cdot r'(t)dt \tag{9.4.32}$$

$$=\int_0^{2\pi}\langle-\frac{\sin(t)}{a},-\frac{\cos(t)}{a}\rangle\cdot\langle-a\sin(t),-a\cos(t)\rangle dt \tag{9.4.33}$$

$$=\int_0^{2\pi}\sin^2(t)+\cos^2(t)dt=\int_0^{2\pi}dt=2\pi. \tag{9.4.34}$$

Therefore, $\int_C F\cdot ds=-2\pi$.

Exercise 9.4.8

Calculate integral $\oint_{\partial D} F\cdot dr$, where D is the annulus given by the polar inequalities $2\le r\le 5$, $0\le\theta\le 2\pi$, and $F(x,y)=\langle x^3,5x+e^y\sin y\rangle$.

Hint

Use the extended version of Green’s theorem.

Answer

105π

MEASURING AREA FROM A BOUNDARY: THE PLANIMETER

Imagine you are a doctor who has just received a magnetic resonance image of your patient’s brain. The brain has a tumor (Figure 9.4.16). How large is the tumor? To be precise, what is the area of the red region? The red cross-section of the tumor has an irregular shape, and therefore it is unlikely that you would be able to find a set of equations or inequalities for the region and then be able to calculate its area by conventional means. You could approximate the area by chopping the region into tiny squares (a Riemann sum approach), but this method always gives an answer with some error.

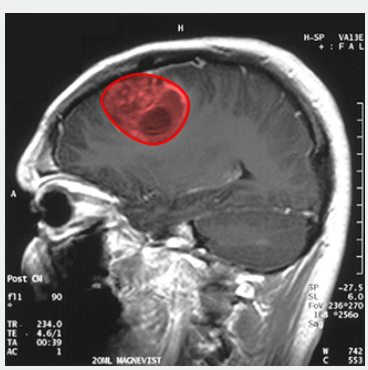


Figure 9.4.16: This magnetic resonance image of a patient’s brain shows a tumor, which is highlighted in red. (credit: modification of work by Christaras A, Wikimedia Commons)

Instead of trying to measure the area of the region directly, we can use a device called a *rolling planimeter* to calculate the area of the region exactly, simply by measuring its boundary. In this project you investigate how a planimeter works, and you use Green’s theorem to show the device calculates area correctly.

A rolling planimeter is a device that measures the area of a planar region by tracing out the boundary of that region (Figure 9.4.17). To measure the area of a region, we simply run the tracer of the planimeter around the boundary of the region. The planimeter measures the number of turns through which the wheel rotates as we trace the boundary; the area of the shape is proportional to this number of wheel turns. We can derive the precise proportionality equation using Green's theorem. As the tracer moves around the boundary of the region, the tracer arm rotates and the roller moves back and forth (but does not rotate).

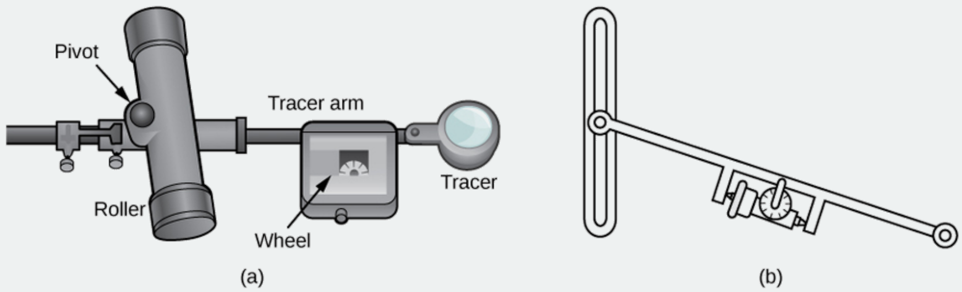


Figure 9.4.17: (a) A rolling planimeter. The pivot allows the tracer arm to rotate. The roller itself does not rotate; it only moves back and forth. (b) An interior view of a rolling planimeter. Notice that the wheel cannot turn if the planimeter is moving back and forth with the tracer arm perpendicular to the roller.

Let *C* denote the boundary of region *D*, the area to be calculated. As the tracer traverses curve *C*, assume the roller moves along the *y*-axis (since the roller does not rotate, one can assume it moves along a straight line). Use the coordinates (x, y) to represent points on boundary *C*, and coordinates $(0, Y)$ to represent the position of the pivot. As the planimeter traces *C*, the pivot moves along the *y*-axis while the tracer arm rotates on the pivot.

Watch a short animation of a planimeter in action.

Begin the analysis by considering the motion of the tracer as it moves from point (x, y) counterclockwise to point $(x+dx, y+dy)$ that is close to (x, y) (Figure 9.4.18). The pivot also moves, from point $(0, Y)$ to nearby point $(0, Y+dY)$. How much does the wheel turn as a result of this motion? To answer this question, break the motion into two parts. First, roll the pivot along the *y*-axis from $(0, Y)$ to $(0, Y+dY)$ without rotating the tracer arm. The tracer arm then ends up at point $(x, y+dY)$ while maintaining a constant angle ϕ with the *x*-axis. Second, rotate the tracer arm by an angle $d\theta$ without moving the roller. Now the tracer is at point $(x+dx, y+dy)$. Let ll be the distance from the pivot to the wheel and let *L* be the distance from the pivot to the tracer (the length of the tracer arm).

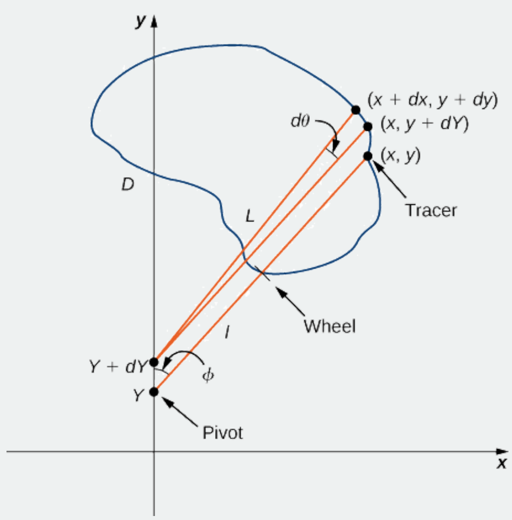


Figure 9.4.18: Mathematical analysis of the motion of the planimeter.

1. Explain why the total distance through which the wheel rolls the small motion just described is $\sin\phi dY + l d\theta = \frac{x}{L} dY + l d\theta$.
2. Show that $\oint_C d\theta = 0$.
3. Use step 2 to show that the total rolling distance of the wheel as the tracer traverses curve *C* is

   Total wheel roll $= \frac{1}{L} \oint_C x dY$.

   Now that you have an equation for the total rolling distance of the wheel, connect this equation to Green's theorem to calculate area *D* enclosed by *C*.
4. Show that $x^2 + (y-Y)^2 = L^2$.
5. Assume the orientation of the planimeter is as shown in Figure 9.4.18. Explain why $Y \le y$, and use this inequality to show there is a unique value of *Y* for each point (x, y): $Y = y = \sqrt{L^2 - x^2}$.

6. Use step 5 to show that $dY = dy + \frac{x}{L^2 - x^2} dx$.
7. Use Green's theorem to show that $\oint_C \frac{x}{L^2 - x^2} dx = 0$.
8. Use step 7 to show that the total wheel roll is

$$\text{Total wheel roll} \quad = \quad 1L \oint_C x\, dy. \tag{9.4.35}$$

   It took a bit of work, but this equation says that the variable of integration Y in step 3 can be replaced with y.

9. Use Green's theorem to show that the area of D is $\oint_C x dy$. The logic is similar to the logic used to show that the area of $D = 12 \oint_C -y\, dx + x\, dy$.
10. Conclude that the area of D equals the length of the tracer arm multiplied by the total rolling distance of the wheel.

You now know how a planimeter works and you have used Green's theorem to justify that it works. To calculate the area of a planar region D, use a planimeter to trace the boundary of the region. The area of the region is the length of the tracer arm multiplied by the distance the wheel rolled.

9.4.5 Key Concepts

- Green's theorem relates the integral over a connected region to an integral over the boundary of the region. Green's theorem is a version of the Fundamental Theorem of Calculus in one higher dimension.
- Green's Theorem comes in two forms: a circulation form and a flux form. In the circulation form, the integrand is $F \cdot T$. In the flux form, the integrand is $F \cdot N$.
- Green's theorem can be used to transform a difficult line integral into an easier double integral, or to transform a difficult double integral into an easier line integral.
- A vector field is source free if it has a stream function. The flux of a source-free vector field across a closed curve is zero, just as the circulation of a conservative vector field across a closed curve is zero.

9.4.6 Key Equations

- **Green's theorem, circulation form**
  $\oint CPdx + Qdy = \iint DQx - PydA$, $\oint CPdx + Qdy = \iint DQx - PydA$, where C is the boundary of D
- **Green's theorem, flux form**
  $\oint CF \cdot dr = \iint DQx - PydA$, $\oint CF \cdot dr = \iint DQx - PydA$, where C is the boundary of D
- **Green's theorem, extended version**
  $\oint \partial DF \cdot dr = \iint DQx - PydA \oint \partial DF \cdot dr = \iint DQx - PydA$

9.4.7 Glossary

Green's theorem

relates the integral over a connected region to an integral over the boundary of the region

stream function

if $F = \langle P, Q \rangle$ is a source-free vector field, then stream function g is a function such that $P = g_y$ and $Q = -g_x$

9.4.8 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

9.4E: EXERCISES

Exercise $9.4E.1$

For the following exercises, evaluate the line integrals by applying Green's theorem.

1. $\int_C 2xydx + (x+y)dy$, where C is the path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along the graph of $y = x$ oriented in the counterclockwise direction.

2. $\int_C 2xydx + (x+y)dy$, where C is the boundary of the region lying between the graphs of $y = 0$ and $y = 4 - x^2$ oriented in the counterclockwise direction.

Answer

$\int_C 2xydx + (x+y)dy = \frac{32}{3}$

3. $\int_C 2arctan(\frac{y}{x})dx + ln(x^2+y^2)dy$, where C is defined by $x = 4 + 2cos\theta, y = 4sin\theta$ oriented in the counterclockwise direction

4. $\int_C sinxcosydx + (xy + cosxsiny)dy$, where C is the boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$ oriented in the counterclockwise direction

Answer

$\int_C sinxcosydx + (xy + cosxsiny)dy = \frac{1}{12}$

5. $\int_C xydx + (x+y)dy$, where C is the boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ oriented in the counterclockwise direction

6. $\oint_C (-ydx + xdy)$, where C consists of line segment C_1 from $(-1,0)$ to $(1,0)$, followed by the semicircular arc C_2 from $(1,0)$ back to $(1,0)$

Answer

$\oint_C (-ydx + xdy) = \pi$

Exercise $9.4E.2$

For the following exercises, use Green's theorem.

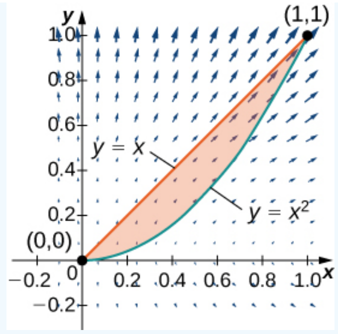
7. Let C be the curve consisting of line segments from $(0,0)$ to $(1,1)$ to $(0,1)$ and back to $(0,0)$. Find the value of $\int_C xydx + \sqrt{y^2+1}\,dy$.

8. Evaluate line integral $\int_C xe^{-2x}dx + (x^4 + 2x^2y^2)dy$, where C is the boundary of the region between circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and is a positively oriented curve.

Answer

$\int_C xe^{-2x}dx + (x^4 + 2x^2y^2)dy = 0$

9. Find the counterclockwise circulation of field $F(x,y) = xy\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}}$ around and over the boundary of the region enclosed by curves $y = x^2$ and $y = x$ in the first quadrant and oriented in the counterclockwise direction.

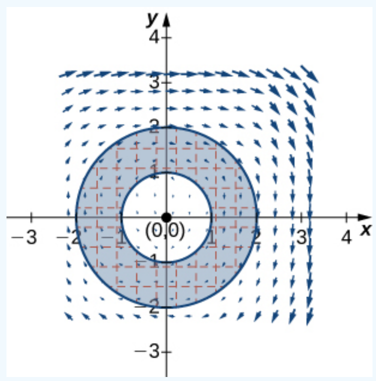


10. Evaluate $\oint_C y^3 dx - x^3 y^2 dy$, where C is the positively oriented circle of radius 2 centered at the origin.

Answer

$\oint_C y^3 dx - x^3 dy = -24\pi$

11. Evaluate $\oint_C y^3 dx - x^3 dy$, where C includes the two circles of radius 2 and radius 1 centered at the origin, both with positive orientation.



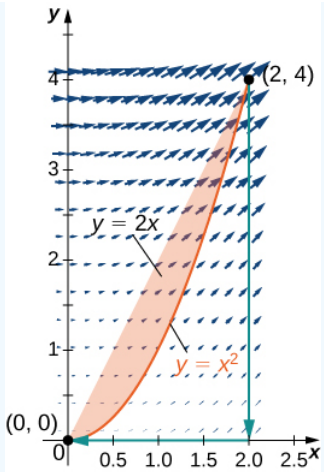
12. Calculate $\oint_C -x^2 y dx + x y^2 dy$, where C is a circle of radius 2 centered at the origin and oriented in the counterclockwise direction.

Answer

$\oint_C -x^2 y dx + x y^2 dy = 8\pi$

13. Calculate integral $\oint_C 2[y + x sin(y)]dx + [x^2 cos(y) - 3y^2]dy$ along triangle C with vertices $(0, 0), (1, 0)$ and $(1, 1)$, oriented counterclockwise, using Green's theorem.

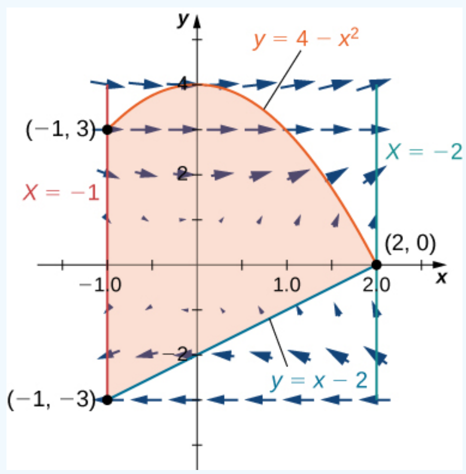
14. Evaluate integral $\oint_C (x^2 + y^2)dx + 2xydy$, where C is the curve that follows parabola $y = x^2$ from $(0, 0), (2, 4)$, then the line from $(2, 4)$ to $(2, 0)$, and finally the line from $(2, 0)$ to $(0, 0)$.



Answer

$\oint_C (x^2+y^2)dx+2xydy=0$

15. Evaluate line integral $\oint_C (y-sin(y)cos(y))dx+2xsin^2(y)dy$, where C is oriented in a counterclockwise path around the region bounded by $x=-1, x=2, y=4-x^2$, and $y=x-2$.



Answer

TBA

Exercise $9.4E.3$

For the following exercises, use Green's theorem to find the area.

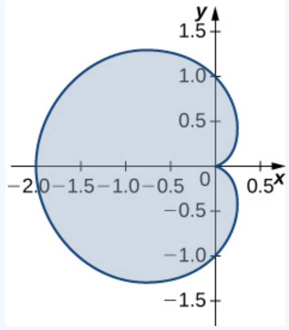
16. Find the area between ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and circle $x^2+y^2=25$.

Answer

$A=19\pi$

17. Find the area of the region enclosed by parametric equation

$p(\theta)=(\cos(\theta)-\cos^2(\theta))\,\hat{\mathbf{i}}-(\sin(\theta)-\cos(\theta)\sin(\theta))\,\hat{\mathbf{j}}$ for $0\le\theta\le 2\pi$.



18. Find the area of the region bounded by hypocycloid $\vec{\mathbf{r}}(t) = \cos^3(t)\,\hat{\mathbf{i}} + \sin^3(t)\,\hat{\mathbf{j}}$. The curve is parameterized by $t \in [0, 2\pi]$.

Answer

$A = \frac{3}{8\pi}$

19. Find the area of a pentagon with vertices $(0, 4)$, $(4, 1)$, $(3, 0)$, $(-1, -1)$,and $(-2, 2)$.

Answer

TBA

Exercise $9.4E.4$

20. Use Green's theorem to evaluate $\int_{C+} (y^2 + x^3)dx + x^4 dy$, where C^+ is the perimeter of square $[0, 1] \times [0, 1]$ oriented counterclockwise.

Answer

$\int_C +(y^2 + x^3)dx + x^4 dy = 0$

21. Use Green's theorem to prove the area of a disk with radius a is $A = \pi a^2$.

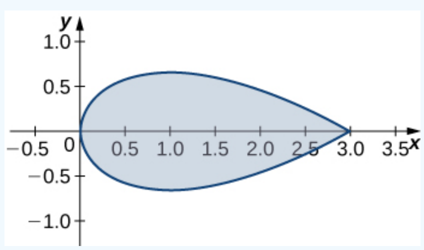
22. Use Green's theorem to find the area of one loop of a four-leaf rose $r = 3sin2\theta$. (*Hint*: $xdy - ydx = r^2 d\theta$).

Answer

$A = \frac{9\pi}{8}$

23. Use Green's theorem to find the area under one arch of the cycloid given by parametric plane $x = t - sint, y = 1 - cost, t \geq 0$.

24. Use Green's theorem to find the area of the region enclosed by curve $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + \left(\frac{t^3}{3} - t\right)\hat{\mathbf{j}}, -\sqrt{3} \leq t \leq \sqrt{3}$.



Answer

$A = \frac{8\sqrt{3}}{5}$

25. [T] Evaluate Green’s theorem using a computer algebra system to evaluate the integral $\int_C xe^y dx + e^x dy$, where C is the circle given by $x^2+y^2=4$ and is oriented in the counterclockwise direction.

26. Evaluate $\int_C (x^2y-2xy+y^2)ds$, where C is the boundary of the unit square $0\le x\le 1, 0\le y\le 1$, traversed counterclockwise.

Answer

$\int_C (x^2y-2xy+y^2)ds=3$

27. Evaluate $\int_C \frac{-(y+2)dx+(x-1)dy}{(x-1)^2+(y+2)^2}$, where C is any simple closed curve with an interior that does not contain point $(1,-2)$ traversed counterclockwise.

28. Evaluate $\int_C \frac{xdx+ydy}{x^2+y^2}$, where C is any piecewise, smooth simple closed curve enclosing the origin, traversed counterclockwise.

Answer

$\int_C \frac{xdx+ydy}{x^2+y^2}=2\pi$

Exercise $9.4E.5$

For the following exercises, use Green’s theorem to calculate the work done by force $\vec{\mathbf{F}}$ on a particle that is moving counterclockwise around closed path C.

29. $\vec{\mathbf{F}}(x,y)=xy\,\hat{\mathbf{i}}+(x+y)\,\hat{\mathbf{j}}, C: x^2+y^2=4$

30. $\vec{\mathbf{F}}(x,y)=\left(x^{\frac{3}{2}}-3y\right)\,\hat{\mathbf{i}}+(6x+5\sqrt{y})\,\hat{\mathbf{j}}$, C : boundary of a triangle with vertices $(0,0)$, $(5,0)$, and $(0,5)$

Answer

$W=\frac{225}{2}$

Exercise $9.4E.6$

31. Evaluate $\int_C (2x^3-y^3)dx+(x^3+y^3)dy$, where C is a unit circle oriented in the counterclockwise direction.

32. A particle starts at point $(-2,0)$, moves along the *x*-axis to $(2,0)$, and then travels along semicircle $y=\sqrt{4-x^2}$ to the starting point. Use Green’s theorem to find the work done on this particle by force field $\vec{\mathbf{F}}(x,y)=x\,\hat{\mathbf{i}}+(x^3+3xy^2)\,\hat{\mathbf{j}}$.

Answer

$W=12\pi$

33. David and Sandra are skating on a frictionless pond in the wind. David skates on the inside, going along a circle of radius 2 in a counterclockwise direction. Sandra skates once around a circle of radius 3, also in the counterclockwise direction. Suppose the force of the wind at point $(x,y)(x,y)(x,y)$ is $\vec{\mathbf{F}}(x,y)=(x^2y+10y)\,\hat{\mathbf{i}}+(x^3+2xy^2)\,\hat{\mathbf{j}}$. Use Green’s theorem to determine who does more work.

34. Use Green’s theorem to find the work done by force field $\vec{\mathbf{F}}(x,y)=(3y-4x)\,\hat{\mathbf{i}}+(4x-y)\,\hat{\mathbf{j}}$ when an object moves once counterclockwise around ellipse $4x^2+y^2=4$.

Answer

$W=12\pi$

35. Use Green's theorem to evaluate line integral $\oint_C e^{2x} sin2ydx + e^{2x} cos2ydy$, where C is ellipse $9(x-1)^2 + 4(y-3)^2 = 36$ oriented counterclockwise.

36. Evaluate line integral $\oint_C y^2 dx + x^2 dy$, where C is the boundary of a triangle with vertices $(0,0)$, $(1,1)$, and $(1,0)$, with the counterclockwise orientation.

Answer

$\oint_C y^2 dx + x^2 dy = \frac{1}{3}$

37. Use Green's theorem to evaluate line integral $\int_C \vec{\mathbf{H}} \cdot \vec{\mathbf{dr}}$ if $\vec{\mathbf{H}}(x,y) = e^y \,\hat{\mathbf{i}} - sin(\pi x)\,\hat{\mathbf{j}}$, where C is a triangle with vertices $(1,0)$, $(0,1)$, and $(-1,0)$, $(-1,0)$ traversed counterclockwise.

38. Use Green's theorem to evaluate line integral $\int_C \sqrt{1+x^3}\,dx + 2xydy$ where C is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$ oriented clockwise.

Answer

$\int_C \sqrt{1+x^3}\,dx + 2xydy = 3$

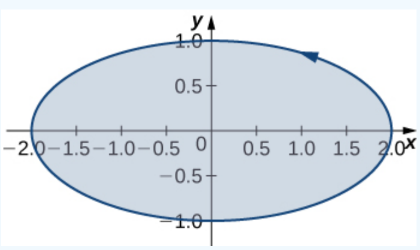
39. Use Green's theorem to evaluate line integral $\int_C x^2 ydx - xy^2 dy$ where C is a circle $x^2 + y^2 = 4$ oriented counterclockwise.

40. Use Green's theorem to evaluate line integral $\int_C (3y - e^{sin x})dx + (7x + \sqrt{y^4+1})dy$ where C is circle $x^2 + y^2 = 9$ oriented in the counterclockwise direction.

Answer

$\int_C (3y - e^{sin x})dx + (7x + \sqrt{y^4+1})dy = 36\pi$

41. Use Green's theorem to evaluate line integral $\int_C (3x - 5y)dx + (x - 6y)dy$, where C is ellipse $\frac{x^2}{4} + y^2 = 1$ and is oriented in the counterclockwise direction.



Answer

TBA

Exercise $9.4E.7$: Use Green's theorem

42. Let C be a triangular closed curve from (0, 0) to (1, 0) to (1, 1) and finally back to (0, 0). Let \(\vecs{F}(x,y)=4y \, \hat{ \mathbf i}+6x^2 \, \hat{ \mathbf j}.\) Use Green's theorem to evaluate $\oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{ds}}$.

Answer

\(∮_C**F**·d**r**=2\)

43. Use Green's theorem to evaluate line integral $\oint_C ydx - xdy$, where C is circle $x^2 + y^2 = a^2$ oriented in the clockwise direction.

44. Use Green's theorem to evaluate line integral $\oint_C (y+x)dx + (x + siny)dy$, where C is any smooth simple closed curve joining the origin to itself oriented in the counterclockwise direction.

Answer

$\oint_C (y+x)dx+(x+siny)dy=0$

45. Use Green's theorem to evaluate line integral $\oint C(y-ln(x^2+y^2))dx+(2arctan(y)x)dy,$ where C is the positively oriented circle $(x-2)^2+(y-3)^2=1.$

46. Use Green's theorem to evaluate $\oint_C xydx+x^3y^3dy,$ where C is a triangle with vertices $(0,0),(1,0),$ and $(1,2)$ with positive orientation.

Answer

$\oint_C xydx+x^3y^3dy=2221$

47. Use Green's theorem to evaluate line integral $\int_C sinydx+xcosydy,$ where C is ellipse $x^2+xy+y^2=1$ oriented in the counterclockwise direction.

48. Let $\vec{\mathbf{F}}(x,y)=(\cos(x^5))-13y^3)\,\hat{\mathbf{i}}+13x^3\,\hat{\mathbf{j}}$. Find the counterclockwise circulation $\oint_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}},$ where C is a curve consisting of the line segment joining $(-2,0)$ and $(-1,0),$ half circle $y=\sqrt{1-x^2}$, the line segment joining $(1,0)$ and $(2,0),$ and half circle $y=\sqrt{4-x^2}$

Answer

$\oint_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}=15\pi^4$

49. Use Green's theorem to evaluate line integral $\int_C sin(x^3)dx+2ye^{x^2}dy,$ where C is a triangular closed curve that connects the points $(0,0),(2,2),$ and $(0,2)$ counterclockwise.

50. Let C be the boundary of square $0\le x\le\pi, 0\le y\le\pi,$ traversed counterclockwise. Use Green's theorem to find $\int_C sin(x+y)dx+cos(x+y)dy.$

Answer

$\int_C sin(x+y)dx+cos(x+y)dy=4$

51. Use Green's theorem to evaluate line integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}$, where $\vec{\mathbf{F}}(x,y)=(y^2-x^2)\,\hat{\mathbf{i}}+(x^2+y^2)\,\hat{\mathbf{j}},$ and C is a triangle bounded by $y=0, x=3,$ and $y=x,$ oriented counterclockwise.

52. Use Green's Theorem to evaluate integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}$, where$\vec{\mathbf{F}}(x,y)=(xy^2)\,\hat{\mathbf{i}}+x\,\hat{\mathbf{j}}$, and C is a unit circle oriented in the counterclockwise direction.

Answer

$\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{dr}}=\pi$

53. Use Green's theorem in a plane to evaluate line integral $\oint_C (xy+y^2)dx+x^2dy,$ where C is a closed curve of a region bounded by $y=x$ and $y=x^2$ oriented in the counterclockwise direction.

54. Calculate the outward flux of $\vec{\mathbf{F}}=-x\,\hat{\mathbf{i}}+2y\,\hat{\mathbf{j}}$ over a square with corners $(\pm1,\pm1),$ where the unit normal is outward pointing and oriented in the counterclockwise direction.

Answer

\(∮_C \vecs{F}·\vecs{n} ds=4\)

55. **[T]** Let C be circle $x^2+y^2=4$ oriented in the counterclockwise direction. Evaluate $\oint_C (3y-e^{tan^{-1}(x)})dx+(7x+y^4+1)dy$ using a computer algebra system.

56. Find the flux of field $\vec{\mathbf{F}}=-x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}$ across $x^2+y^2=16$ oriented in the counterclockwise direction.

Answer

\(∮_C \vecs{F}·\vecs{n} ds=32π\)

57. Let $\vec{\mathbf{F}} = (y^2 - x^2)\,\hat{\mathbf{i}} + (x^2 + y^2)\,\hat{\mathbf{j}},$ and let C be a triangle bounded by $y = 0, x = 3,$ and $y = x$ oriented in the counterclockwise direction. Find the outward flux of $\vec{\mathbf{F}}$ through C.

58. **[T]** Let C be unit circle $x^2 + y^2 = 1$ traversed once counterclockwise. Evaluate $\int_C [-y^3 + sin(xy) + xycos(xy)]dx + [x^3 + x^2cos(xy)]dy$ by using a computer algebra system.

Answer

$\int_C [-y^3 + sin(xy) + xycos(xy)]dx + [x^3 + x^2cos(xy)]dy = 4.7124$

59. **[T]** Find the outward flux of vector field $\vec{\mathbf{F}} = xy^2\,\hat{\mathbf{i}} + x^2y\,\hat{\mathbf{j}}$ across the boundary of annulus $R = (x, y) : 1 \leq x^2 + y^2 \leq 4 = (r, \theta) : 1 \leq r \leq 2, 0 \leq \theta \leq 2\pi$ using a computer algebra system.

60. Consider region *R* bounded by parabolas $y = x^2$ and $x = y^2$. Let be the boundary of *R* oriented counterclockwise. Use Green’s theorem to evaluate

$\oint_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy$

Answer

$\oint_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy = 13$

9.5: Divergence and Curl

In this section, we examine two important operations on a vector field: divergence and curl. They are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higher-dimensional versions of the Fundamental Theorem of Calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism, and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is conservative if and only if its curl is zero.

In addition to defining curl and divergence, we look at some physical interpretations of them, and show their relationship to conservative and source-free vector fields.

9.5.1 Divergence

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. Locally, the divergence of a vector field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ at a particular point P is a measure of the "outflowing-ness" of the vector field at P. If $\vec{\mathbf{F}}$ represents the velocity of a fluid, then the divergence of $\vec{\mathbf{F}}$ at P measures the net rate of change with respect to time of the amount of fluid flowing away from P (the tendency of the fluid to flow "out of" P). In particular, if the amount of fluid flowing into P is the same as the amount flowing out, then the divergence at P is zero.

Definition: divergence in $\mathbb{R}^3$

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z all exist, then the divergence of $\vec{\mathbf{F}}$ is defined by

$$div\, F = P_x + Q_y + R_z \tag{9.5.1}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{9.5.2}$$

Note the divergence of a vector field is not a vector field, but a scalar function. In terms of the gradient operator

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \tag{9.5.3}$$

divergence can be written symbolically as the dot product

$$div\, F = \nabla \cdot \vec{\mathbf{F}}. \tag{9.5.4}$$

Note this is merely helpful notation, because the dot product of a vector of operators and a vector of functions is not meaningfully defined given our current definition of dot product.

If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, and P_x and Q_y both exist, then the divergence of $\vec{\mathbf{F}}$ is defined similarly as

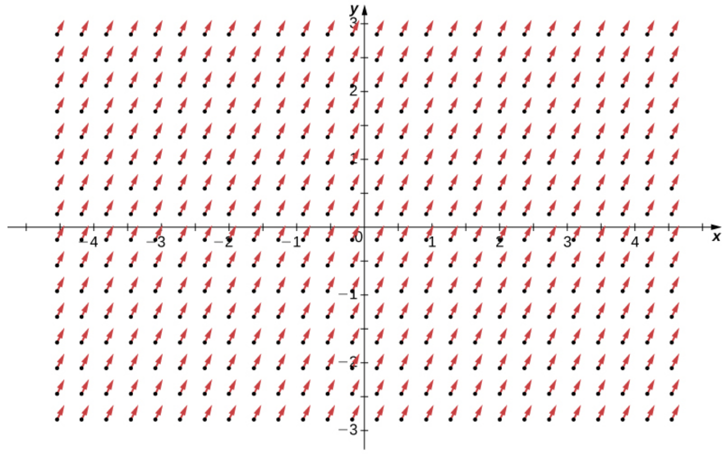
$$\begin{aligned} div\, \vec{\mathbf{F}} &= P_x + Q_y \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \\ &= \nabla \cdot \vec{\mathbf{F}}. \end{aligned}$$

To illustrate this point, consider the two vector fields in Figure $9.5.1$. At any particular point, the amount flowing in is the same as the amount flowing out, so at every point the "outflowing-ness" of the field is zero. Therefore, we expect the divergence of both fields to be zero, and this is indeed the case, as

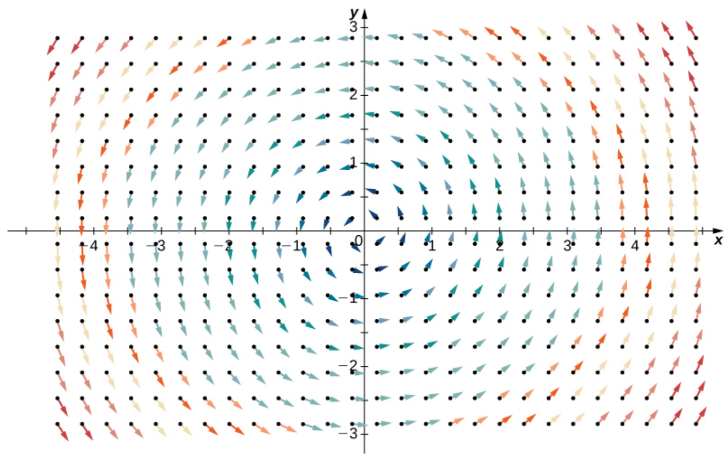
$$div(\langle 1, 2 \rangle) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) = 0 \tag{9.5.5}$$

and

$$div(\langle -y, x \rangle) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0. \tag{9.5.6}$$



(a)



(b)

Figure 9.5.1: (a) Vector field $\langle 1, 2\rangle$ has zero divergence. (b) Vector field $\langle -y, x\rangle$ also has zero divergence.

By contrast, consider radial vector field $\vec{\mathbf{R}}(x, y) = \langle -x, -y\rangle$ in Figure 9.5.2. At any given point, more fluid is flowing in than is flowing out, and therefore the “outgoingness” of the field is negative. We expect the divergence of this field to be negative, and this is indeed the case, as

$$div(\vec{\mathbf{R}}) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2. \tag{9.5.7}$$

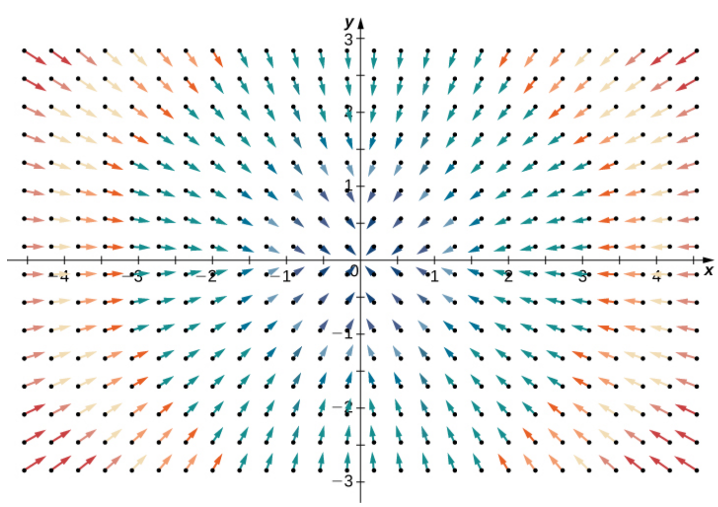


Figure 9.5.2: This vector field has negative divergence.

To get a global sense of what divergence is telling us, suppose that a vector field in $\mathbb{R}^2$ represents the velocity of a fluid. Imagine taking an elastic circle (a circle with a shape that can be changed by the vector field) and dropping it into a fluid. If the circle maintains its exact area as it flows through the fluid, then the divergence is zero. This would occur for both vector fields in Figure $9.5.1$. On the other hand, if the circle's shape is distorted so that its area shrinks or expands, then the divergence is not zero. Imagine dropping such an elastic circle into the radial vector field in Figure $9.5.2$ so that the center of the circle lands at point $(3,3)$. The circle would flow toward the origin, and as it did so the front of the circle would travel more slowly than the back, causing the circle to "scrunch" and lose area. This is how you can see a negative divergence.

Example $9.5.1$: Calculating Divergence at a Point

If $\vec{\mathbf{F}}(x,y,z)=e^x\,\hat{i}+yz\hat{j}-y^2\hat{k}$, then find the divergence of $\vec{\mathbf{F}}$ at $(0,2,-1)$.

Solution

The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(e^x)+\frac{\partial}{\partial y}(yz)-\frac{\partial}{\partial z}(yz^2)=e^x+z-2yz.$$

Therefore, the divergence at $(0,2,-1)$ is $e^0-1+4=4$. If $\vec{\mathbf{F}}$ represents the velocity of a fluid, then more fluid is flowing out than flowing in at point $(0,2,-1)$.

Exercise $9.5.1$

Find $div\,\vec{\mathbf{F}}$ for

$$\vec{\mathbf{F}}(x,y,z)=\langle xy,\,5-z^2,\,x^2+y^2\rangle.$$

Hint

Follow Example $9.5.1$

Answer

$y-z^2$

One application for divergence occurs in physics, when working with magnetic fields. A magnetic field is a vector field that models the influence of electric currents and magnetic materials. Physicists use divergence in Gauss's law for magnetism, which states that if $\vec{\mathbf{B}}$ is a magnetic field, then $\nabla\cdot\vec{\mathbf{B}}=0$; in other words, the divergence of a magnetic field is zero.

Example $9.5.2$: Determining Whether a Field Is Magnetic

Is it possible for $\vec{\mathbf{F}}(x,y)=\langle x^2y,\,y-xy^2\rangle$ to be a magnetic field?

Solution

If $\vec{\mathbf{F}}$ were magnetic, then its divergence would be zero. The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(x^2y)+\frac{\partial}{\partial y}(y-xy^2)=2xy+1-2xy=1$$

and therefore $\vec{\mathbf{F}}$ cannot model a magnetic field (Figure $9.5.3$).

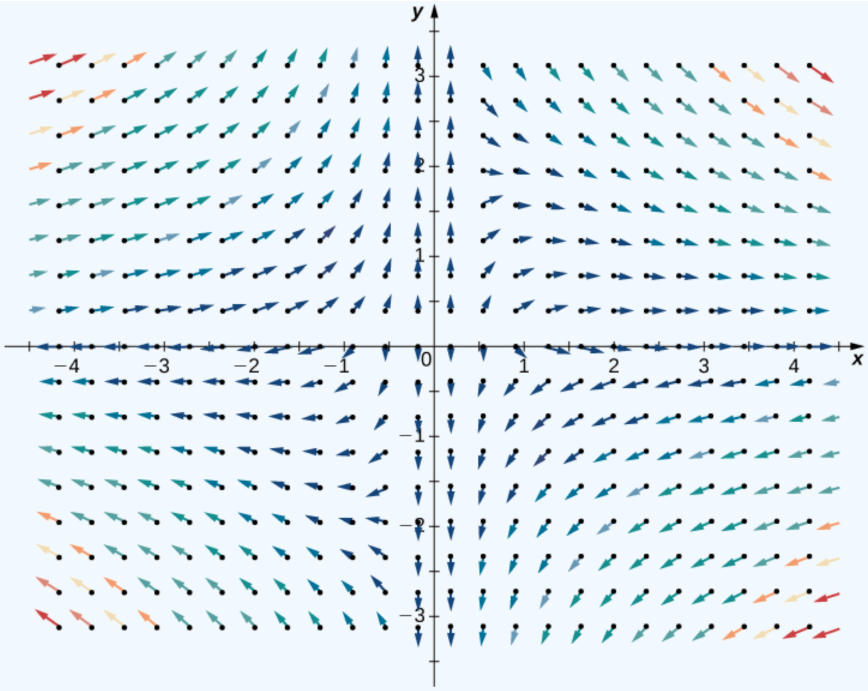


Figure 9.5.3: The divergence of vector field $\vec{\mathbf{F}}(x, y) = \langle x^2 y,\, y - xy^2 \rangle$ is one, so it cannot model a magnetic field.

Another application for divergence is detecting whether a field is source free. Recall that a source-free field is a vector field that has a stream function; equivalently, a source-free field is a field with a flux that is zero along any closed curve. The next two theorems say that, under certain conditions, source-free vector fields are precisely the vector fields with zero divergence.

Theorem: Divergence of a Source-Free Vector Field

If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a source-free continuous vector field with differentiable component functions, then $div\,\vec{\mathbf{F}} = 0$.

Proof

Since $\vec{\mathbf{F}}$ is source free, there is a function $g(x, y)$ with $g_y = P$ and $-g_x = Q$. Therefore, $\vec{\mathbf{F}} = \langle g_y, -g_x \rangle$ and $div\,\vec{\mathbf{F}} = g_{yx} - g_{xy} = 0$ by Clairaut's theorem.

□

The converse of Divergence of a Source-Free Vector Field is true on simply connected regions, but the proof is too technical to include here. Thus, we have the following theorem, which can test whether a vector field in $\mathbb{R}^2$ is source free.

Theorem: Divergence Test for Source-Free Vector Fields

Let $\vec{\mathbf{F}} = \langle P, Q \rangle$ be a continuous vector field with differentiable component functions with a domain that is simply connected. Then, $div\,\vec{\mathbf{F}} = 0$ if and only if $\vec{\mathbf{F}}$ is source free.

Example 9.5.3: Determining Whether a Field Is Source Free

Is field $\vec{\mathbf{F}}(x, y) = \langle x^2 y,\, 5 - xy^2 \rangle$ source free?

Solution

Note the domain of $\vec{\mathbf{F}}$ is $\mathbb{R}^2$ which is simply connected. Furthermore, $\vec{\mathbf{F}}$ is continuous with differentiable component functions. Therefore, we can use the Divergence Test for Source-Free Vector Fields to analyze $\vec{\mathbf{F}}$. The divergence of $\vec{\mathbf{F}}$ is

$$\frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(5 - xy^2) = 2xy - 2xy = 0.$$

Therefore, $\vec{\mathbf{F}}$ is source free by the Divergence Test for Source-Free Vector Fields.

Exercise 9.5.2

Let $\vec{\mathbf{F}}(x,y)=\langle -ay, bx\rangle$ be a rotational field where a and b are positive constants. Is $\vec{\mathbf{F}}$ source free?

Hint

Calculate the divergence.

Answer

Yes

Recall that the flux form of Green's theorem says that

$$\oint_C F\cdot N ds=\iint_D P_x+Q_y dA, \tag{9.5.8}$$

where C is a simple closed curve and D is the region enclosed by C. Since $P_x+Q_y=div\, F$, Green's theorem is sometimes written as

$$\oint_C F\cdot N ds=\iint_D div\, F dA. \tag{9.5.9}$$

Therefore, Green's theorem can be written in terms of divergence. If we think of divergence as a derivative of sorts, then Green's theorem says the "derivative" of $\vec{\mathbf{F}}$ on a region can be translated into a line integral of $\vec{\mathbf{F}}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on a line segment $[a,b]$ can be translated into a statement about f on the boundary of $[a,b]$. Using divergence, we can see that Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can use all of what we have learned in the application of divergence. Let $\vec{\mathbf{v}}$ be a vector field modeling the velocity of a fluid. Since the divergence of $\vec{\mathbf{v}}$ at point P measures the "outflowing-ness" of the fluid at P, $div\, v(P)>0$ implies that more fluid is flowing out of P than flowing in. Similarly, $div\, v(P)<0$ implies the more fluid is flowing in to P than is flowing out, and $div\,\vec{\mathbf{v}}(P)=0$ implies the same amount of fluid is flowing in as flowing out.

Example 9.5.4: Determining Flow of a Fluid

Suppose $\vec{\mathbf{v}}(x,y)=\langle -xy, y\rangle$, $y>0$ models the flow of a fluid. Is more fluid flowing into point $(1,4)$ than flowing out?

Solution

To determine whether more fluid is flowing into $(1,4)$ than is flowing out, we calculate the divergence of **v** at $(1,4)$:

$$div(\vec{\mathbf{v}})=\frac{\partial}{\partial x}(-xy)+\frac{\partial}{\partial y}(y)=-y+1.$$

To find the divergence at $(1,4)$ substitute the point into the divergence: $-4+1=-3$. Since the divergence of **v** at $(1,4)$ is negative, more fluid is flowing in than flowing out (Figure 9.5.4).

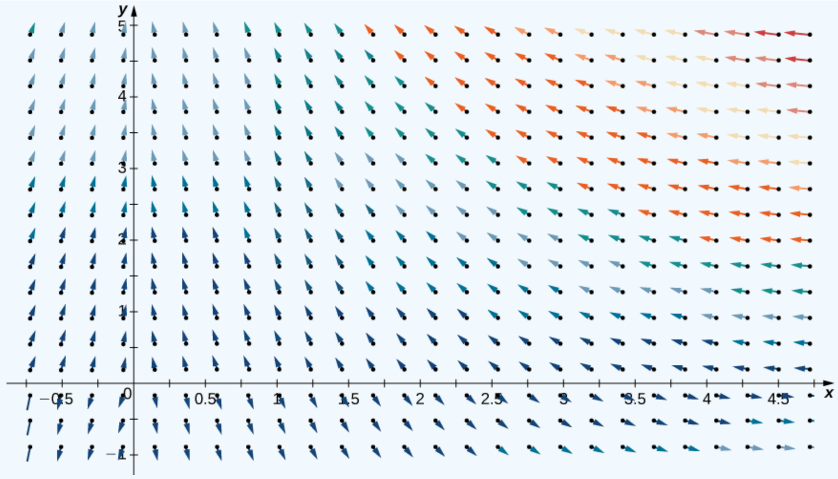


Figure 9.5.4: Vector field $\vec{\mathbf{v}}(x,y)=\langle -xy, y\rangle$ has negative divergence at $(1,4)$

Exercise 9.5.3

For vector field $\vec{\mathbf{v}}(x,y)=\langle -xy, y\rangle$, $y>0$, find all points P such that the amount of fluid flowing in to P equals the amount of fluid flowing out of P.

Hint

Find where the divergence is zero.

Answer

All points on line $y=1$.

9.5.2 Curl

The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. Suppose that $\vec{\mathbf{F}}$ represents the velocity field of a fluid. Then, the curl of $\vec{\mathbf{F}}$ at point P is a vector that measures the tendency of particles near P to rotate about the axis that points in the direction of this vector. The magnitude of the curl vector at P measures how quickly the particles rotate around this axis. In other words, the curl at a point is a measure of the vector field's "spin" at that point. Visually, imagine placing a paddlewheel into a fluid at P, with the axis of the paddlewheel aligned with the curl vector (Figure 9.5.5). The curl measures the tendency of the paddlewheel to rotate.

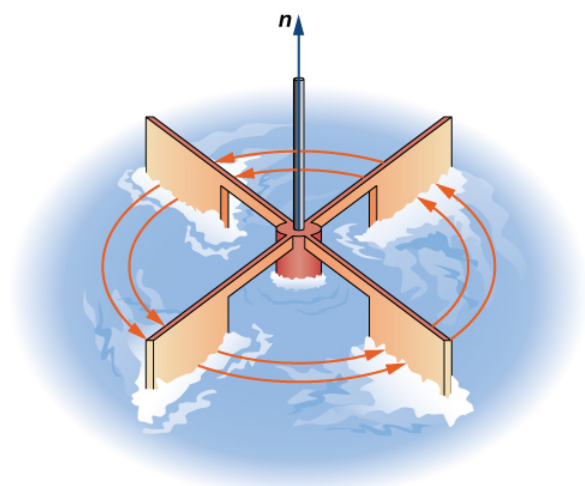


Figure 9.5.5: To visualize curl at a point, imagine placing a small paddlewheel into the vector field at a point.

Consider the vector fields in Figure 9.5.1. In part (a), the vector field is constant and there is no spin at any point. Therefore, we expect the curl of the field to be zero, and this is indeed the case. Part (b) shows a rotational field, so the field has spin. In particular, if you place a paddlewheel into a field at any point so that the axis of the wheel is perpendicular to a plane, the wheel rotates counterclockwise. Therefore, we expect the curl of the field to be nonzero, and this is indeed the case (the curl is $2k$).

To see what curl is measuring globally, imagine dropping a leaf into the fluid. As the leaf moves along with the fluid flow, the curl measures the tendency of the leaf to rotate. If the curl is zero, then the leaf doesn't rotate as it moves through the fluid.

Definition: Curl

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$, and P_x, Q_y, and R_z all exist, then the curl of $\vec{\mathbf{F}}$ is defined by

$$curl\, \vec{\mathbf{F}} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k} \tag{9.5.10}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}. \tag{9.5.11}$$

Note that the curl of a vector field is a vector field, in contrast to divergence.

The definition of curl can be difficult to remember. To help with remembering, we use the notation $\nabla \times \vec{\mathbf{F}}$ to stand for a "determinant" that gives the curl formula:

$$\begin{vmatrix} \hat{i}\,\hat{j}\,\hat{k} \\ \dfrac{\partial}{\partial x}\,\dfrac{\partial}{\partial y}\,\dfrac{\partial}{\partial z} \\ P\,Q\,R \end{vmatrix}. \tag{9.5.12}$$

The determinant of this matrix is

$$(R_y - Q_z)\hat{i} - (R_x - P_z)\hat{j} + (Q_x - P_y)\hat{k} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k} = curl\, \vec{\mathbf{F}}. \tag{9.5.13}$$

Thus, this matrix is a way to help remember the formula for curl. Keep in mind, though, that the word *determinant* is used very loosely. A determinant is not really defined on a matrix with entries that are three vectors, three operators, and three functions.

If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, then the curl of $\vec{\mathbf{F}}$, by definition, is

$$curl\, \vec{\mathbf{F}} = (Q_x - P_y)k = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k. \tag{9.5.14}$$

Example 9.5.5: Finding the Curl of a Three-Dimensional Vector Field

Find the curl of $\vec{\mathbf{F}}(P, Q, R) = \langle x^2 z, e^y + xz, xyz \rangle$.

Solution

The curl is

$$\begin{aligned} curl\, f &= \nabla \times \vec{\mathbf{F}} \\ &= \begin{vmatrix} \hat{i}\,\hat{j}\,\hat{k} \\ \partial/\partial x \partial/\partial y \partial/\partial z \\ P\,Q\,R \end{vmatrix} \\ &= (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k} \\ &= (xz - x)\hat{i} + (x^2 - yz)\hat{j} + z\hat{k}. \end{aligned}$$

Exercise 9.5.4

Find the curl of $\vec{\mathbf{F}} = \langle \sin x\ \cos z,\ \sin y\ \sin z,\ \cos x\ \cos y \rangle$ at point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

Hint

Find the determinant of matrix $\nabla \times \vec{\mathbf{F}}$.

Answer

$-\hat{i}$

Example 9.5.6: Finding the Curl of a Two-Dimensional Vector Field

Find the curl of $\vec{\mathbf{F}} = \langle P, Q \rangle = \langle y, 0 \rangle$.

Solution

Notice that this vector field consists of vectors that are all parallel. In fact, each vector in the field is parallel to the *x*-axis. This fact might lead us to the conclusion that the field has no spin and that the curl is zero. To test this theory, note that

$$curl\,\vec{\mathbf{F}} = (Q_x - P_y)k = -k \neq 0. \tag{9.5.15}$$

Therefore, this vector field does have spin. To see why, imagine placing a paddlewheel at any point in the first quadrant (Figure 9.5.6). The larger magnitudes of the vectors at the top of the wheel cause the wheel to rotate. The wheel rotates in the clockwise (negative) direction, causing the coefficient of the curl to be negative.

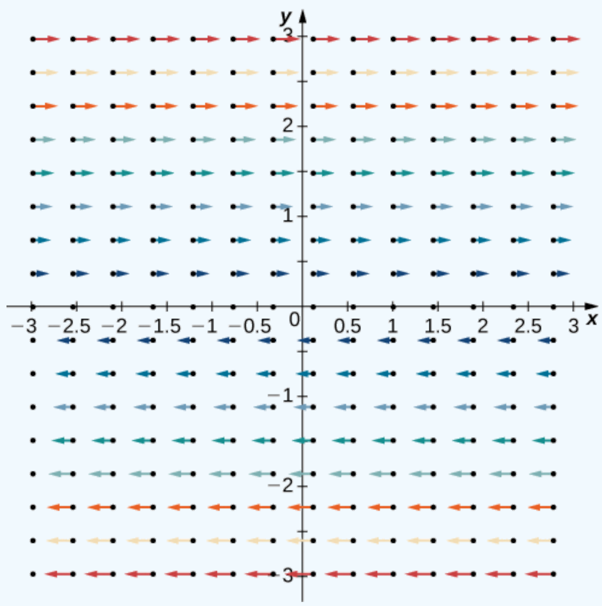


Figure 9.5.6: Vector field $\vec{\mathbf{F}}(x,y) = \langle y, 0 \rangle$ consists of vectors that are all parallel.

Note that if $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a vector field in a plane, then $curl\,\vec{\mathbf{F}} \cdot k = (Q_x - P_y)k \cdot k = Q_x - P_y$. Therefore, the circulation form of Green's theorem is sometimes written as

$$\oint_C \vec{\mathbf{F}} \cdot dr = \iint_D curl\,F \cdot k dA, \tag{9.5.16}$$

where *C* is a simple closed curve and *D* is the region enclosed by *C*. Therefore, the circulation form of Green's theorem can be written in terms of the curl. If we think of curl as a derivative of sorts, then Green's theorem says that the "derivative" of $\vec{\mathbf{F}}$ on a region can be translated into a line integral of $\vec{\mathbf{F}}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on line segment $[a, b]$ can be translated into a statement about f on the boundary of $[a, b]$. Using curl, we can see the circulation form of Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can now use what we have learned about curl to show that gravitational fields have no "spin." Suppose there is an object at the origin with mass m_1 at the origin and an object with mass m_2. Recall that the gravitational force that object 1 exerts on object 2 is given by field

$$\vec{\mathbf{F}}(x,y,z) = -Gm_1m_2\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle. \tag{9.5.17}$$

Example 9.5.7: Determining the Spin of a Gravitational Field

Show that a gravitational field has no spin.

Solution

To show that $\vec{\mathbf{F}}$ has no spin, we calculate its curl. Let

- $P(x,y,z) = \dfrac{x}{(x^2+y^2+z^2)^{3/2}}$,
- $Q(x,y,z) = \dfrac{y}{(x^2+y^2+z^2)^{3/2}}$, and
- $R(x,y,z) = \dfrac{z}{(x^2+y^2+z^2)^{3/2}}$.

Then,

$$\begin{aligned} curl\,\vec{\mathbf{F}} &= -Gm_1m_2[(R_y - Q_z)i + (P_z - R_x)j + (Q_x - P_y)k] \\ &= -Gm_1m_2 \begin{pmatrix} \left(\dfrac{-3yz}{(x^2+y^2+z^2)^{5/2}} - \left(\dfrac{-3yz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{i} \\ +\left(\dfrac{-3xz}{(x^2+y^2+z^2)^{5/2}} - \left(\dfrac{-3xz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{j} \\ +\left(\dfrac{-3xy}{(x^2+y^2+z^2)^{5/2}} - \left(\dfrac{-3xy}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{k} \end{pmatrix} \\ &= 0. \end{aligned}$$

Since the curl of the gravitational field is zero, the field has no spin.

Exercise 9.5.7

Field $\vec{\mathbf{v}}(x,y) = \langle -\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \rangle$ models the flow of a fluid. Show that if you drop a leaf into this fluid, as the leaf moves over time, the leaf does not rotate.

Hint

Calculate the curl.

Answer

$curl\,\vec{\mathbf{v}} = 0$

9.5.3 Using Divergence and Curl

Now that we understand the basic concepts of divergence and curl, we can discuss their properties and establish relationships between them and conservative vector fields.

If $\vec{\mathbf{F}}$ is a vector field in $\mathbb{R}^3$ then the curl of $\vec{\mathbf{F}}$ is also a vector field in $\mathbb{R}^3$. Therefore, we can take the divergence of a curl. The next theorem says that the result is always zero. This result is useful because it gives us a way to show that some vector fields are not the curl of any other field. To give this result a physical interpretation, recall that divergence of a velocity field $\vec{\mathbf{v}}$ at point P measures the tendency of the corresponding fluid to flow out of P. Since $div\,curl\,(v) = 0$, the net rate of flow in vector field curl(**v**) at any point is zero. Taking the curl of vector field $\vec{\mathbf{F}}$ eliminates whatever divergence was present in $\vec{\mathbf{F}}$.

Theorem: Divergence of the Curl

Let $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ be a vector field in $\mathbb{R}^3$ such that the component functions all have continuous second-order partial derivatives. Then,

$$div\, curl\,(\vec{\mathbf{F}}) = \nabla\cdot(\nabla\times F) = 0. \tag{9.5.18}$$

Proof

By the definitions of divergence and curl, and by Clairaut's theorem,

$$\begin{aligned} div\, curl\, \vec{\mathbf{F}} &= div[(R_y - Q_z)i + (P_z - R_x)j + (Q_x - P_y)k] \\ &= R_{yx} - Q_{xz} + P_{yz} - R_{yx} + Q_{zx} - P_{zy} \\ &= 0. \end{aligned}$$

□

Example 9.5.8: Showing That a Vector Field Is Not the Curl of Another

Show that $\vec{\mathbf{F}}(x,y,z) = e^x i + yzj + xz^2 k$ is not the curl of another vector field. That is, show that there is no other vector $\vec{\mathbf{G}}$ with $curl\, G = F$.

Solution

Notice that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$ and the second-order partials of $\vec{\mathbf{F}}$ are all continuous. Therefore, we can apply the previous theorem to $\vec{\mathbf{F}}$.

The divergence of $\vec{\mathbf{F}}$ is $e^x + z + 2xz$. If $\vec{\mathbf{F}}$ were the curl of vector field $\vec{\mathbf{G}}$, then $div\, F = div\, curl G = 0$. But, the divergence of $\vec{\mathbf{F}}$ is not zero, and therefore $\vec{\mathbf{F}}$ is not the curl of any other vector field.

Exercise 9.5.8

Is it possible for $G(x,y,z) = \langle \sin x,\ \cos y,\ \sin(xyz)\rangle$ to be the curl of a vector field?

Hint

Find the divergence of $\vec{\mathbf{G}}$.

Answer

No.

With the next two theorems, we show that if $\vec{\mathbf{F}}$ is a conservative vector field then its curl is zero, and if the domain of $\vec{\mathbf{F}}$ is simply connected then the converse is also true. This gives us another way to test whether a vector field is conservative.

Theorem: Curl of a Conservative Vector Field

If $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ is conservative, then $curl\, \vec{\mathbf{F}} = 0$.

Proof

Since conservative vector fields satisfy the cross-partials property, all the cross-partials of **F** are equal. Therefore,

$$\begin{aligned} curl\, \vec{\mathbf{F}} &= (R_y - Q_z)i + (P_z - R_x)j + (Q_x - P_y)k \\ &= 0. \end{aligned}$$

□

The same theorem is true for vector fields in a plane.

Since a conservative vector field is the gradient of a scalar function, the previous theorem says that $curl\,(\nabla f) = 0$ for any scalar function f. In terms of our curl notation, $\nabla \times \nabla(f) = 0$. This equation makes sense because the cross product of a vector with itself is always the zero vector. Sometimes equation $\nabla \times \nabla(f) = 0$ is simplified as $\nabla \times \nabla = 0$.

Theorem: Curl Test for a Conservative Field

Let $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ be a vector field in space on a simply connected domain. If $curl\,F = 0$, then $\vec{\mathbf{F}}$ is conservative.

Proof

Since $curl\,F = 0$, we have that $R_y = Q_z$, $P_z = R_x$, and $Q_x = P_y$. Therefore, $\vec{\mathbf{F}}$ satisfies the cross-partials property on a simply connected domain, and the Cross-Partial Property of Conservative Fields implies that $\vec{\mathbf{F}}$ is conservative.

□

The same theorem is also true in a plane. Therefore, if $\vec{\mathbf{F}}$ is a vector field in a plane or in space and the domain is simply connected, then $\vec{\mathbf{F}}$ is conservative if and only if $curl\,F = 0$.

Example 9.5.9: Testing Whether a Vector Field Is Conservative

Use the curl to determine whether $\vec{\mathbf{F}}(x, y, z) = \langle yz, xz, xy \rangle$ is conservative.

Solution

Note that the domain of $\vec{\mathbf{F}}$ is all of $\mathbb{R}^3$ which is simply connected (Figure 9.5.7). Therefore, we can test whether $\vec{\mathbf{F}}$ is conservative by calculating its curl.

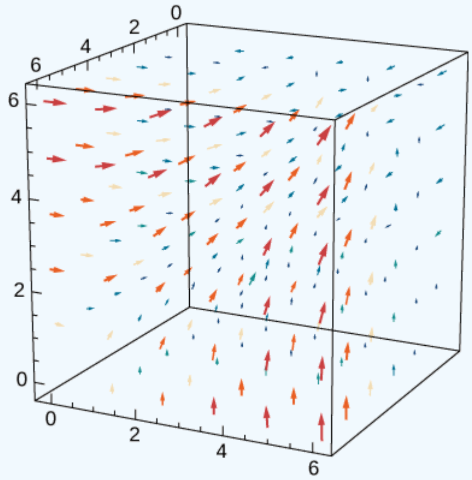


Figure 9.5.7: The curl of vector field $\vec{\mathbf{F}}(x, y, z) = \langle yz, xz, xy \rangle$ is zero.

The curl of $\vec{\mathbf{F}}$ is

$$\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right)\hat{i} + \left(\frac{\partial}{\partial y}yz - \frac{\partial}{\partial z}xy\right)\hat{j} + \left(\frac{\partial}{\partial y}xz - \frac{\partial}{\partial z}yz\right)\hat{k} = (x - x)\hat{i} + (y - y)\hat{j} + (z - z)\hat{k} = 0.$$

Thus, $\vec{\mathbf{F}}$ is conservative.

We have seen that the curl of a gradient is zero. What is the divergence of a gradient? If f is a function of two variables, then $div(\nabla f) = \nabla \cdot (\nabla f) = f_{xx} + f_{yy}$. We abbreviate this “double dot product” as ∇^2. This operator is called the *Laplace operator*, and in this notation Laplace’s equation becomes $\nabla^2 f = 0$. Therefore, a harmonic function is a function that becomes zero after taking the divergence of a gradient.

Similarly, if f is a function of three variables then

$$div(\nabla f) = \nabla \cdot (\nabla f) = f_{xx} + f_{yy} + f_{zz}. \tag{9.5.19}$$

Using this notation we get Laplace's equation for harmonic functions of three variables:

$$\nabla^2 f = 0. \tag{9.5.20}$$

Harmonic functions arise in many applications. For example, the potential function of an electrostatic field in a region of space that has no static charge is harmonic.

Example $9.5.10$: Finding a Potential Function

Is it possible for $f(x,y) = x^2 + x - y$ to be the potential function of an electrostatic field that is located in a region of $\mathbb{R}^2$ free of static charge?

Solution

If f were such a potential function, then f would be harmonic. Note that $f_{xx} = 2$ and $f_{yy} = 0$, and so $f_{xx} + f_{yy} \neq 0$. Therefore, f is not harmonic and f cannot represent an electrostatic potential.

Exercise $9.5.10$

Is it possible for function $f(x,y) = x^2 - y^2 + x$ to be the potential function of an electrostatic field located in a region of $\mathbb{R}^2$ free of static charge?

Hint

Determine whether the function is harmonic.

Answer

Yes.

9.5.4 Key Concepts

- The divergence of a vector field is a scalar function. Divergence measures the "outflowing-ness" of a vector field. If $\vec{\mathbf{v}}$ is the velocity field of a fluid, then the divergence of $\vec{\mathbf{v}}$ at a point is the outflow of the fluid less the inflow at the point.
- The curl of a vector field is a vector field. The curl of a vector field at point P measures the tendency of particles at P to rotate about the axis that points in the direction of the curl at P.
- A vector field with a simply connected domain is conservative if and only if its curl is zero.

9.5.5 Key Equations

- **Curl**

$$\nabla \times \vec{\mathbf{F}} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}$$

- **Divergence**

$$\nabla \cdot \vec{\mathbf{F}} = P_x + Q_y + R_z$$

- **Divergence of curl is zero**

$$\nabla \cdot (\nabla \times F) = 0$$

- **Curl of a gradient is the zero vector**

$$\nabla \times (\nabla f) = 0$$

9.5.6 Glossary

curl

the curl of vector field $\vec{\mathbf{F}} = \langle P, Q, R\rangle$, denoted $\nabla \times \vec{\mathbf{F}}$ is the "determinant" of the matrix

$$\begin{vmatrix} \hat{i}\hat{j}\hat{k} \\ \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial y}\dfrac{\partial}{\partial z} \\ PQR \end{vmatrix}.$$

and is given by the expression $(R_y - Q_z)i + (P_z - R_x)j + (Q_x - P_y)k$; it measures the tendency of particles at a point to rotate about the axis that points in the direction of the curl at the point

divergence

the divergence of a vector field $\vec{\mathbf{F}} = \langle P, Q, R\rangle$, denoted $\nabla \times \vec{\mathbf{F}}$, is $P_x + Q_y + R_z$; it measures the "outflowing-ness" of a vector field

9.5.7 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

9.5E: EXERCISES

Exercise $9.5E.1$: True or False

For the following exercises, determine whether the statement is *true or false*.

1. If the coordinate functions of $\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3$ have continuous second partial derivatives, then $curl\,(div(F))$ equals zero.

2. $\nabla \cdot (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}) = 1$.

Answer

False

3. All vector fields of the form $\vec{\mathbf{F}}(x,y,z) = f(x)\,\hat{\mathbf{i}} + g(y)\,\hat{\mathbf{j}} + h(z)\,\hat{\mathbf{k}}$ are conservative.

4. If $curl\,\vec{\mathbf{F}} = 0$, then$\vec{\mathbf{F}}$ is conservative.

Answer

True

5. If $\vec{\mathbf{F}}$ is a constant vector field then $div\,(\vec{\mathbf{F}}) = 0$.

6. If $\vec{\mathbf{F}}$ is a constant vector field then $curl\,(\vec{\mathbf{F}}) = 0$.

Answer

True

Exercise $9.5E.2$: Curl

For the following exercises, find the curl of $\vec{\mathbf{F}}$.

1. $\vec{\mathbf{F}}(x,y,z) = xy^2z^4\,\hat{\mathbf{i}} + (2x^2y+z)\,\hat{\mathbf{j}} + y^3z^2\,\hat{\mathbf{k}}$
2. $\vec{\mathbf{F}}(x,y,z) = x^2z\,\hat{\mathbf{i}} + y^2x\,\hat{\mathbf{j}} + (y+2z)\,\hat{\mathbf{k}}$

Answer

$curl$ \\($\vec{\mathbf{F}}$= \, \hat{ \mathbf i} + x^2 \, \hat{ \mathbf j} + y^2 \, \hat{ \mathbf k}\)

3. $\vec{\mathbf{F}}(x,y,z) = 3xyz^2\,\hat{\mathbf{i}} + y^2\sin z\,\hat{\mathbf{j}} + xe^{2z}\,\hat{\mathbf{k}}$

4. $\vec{\mathbf{F}}(x,y,z) = x^2yz\,\hat{\mathbf{i}} + xy^2z\,\hat{\mathbf{j}} + xyz^2\,\hat{\mathbf{k}}$

Answer

$curl$ \\($\vec{\mathbf{F}}$= (xz^2 - xy^2) \, \hat{ \mathbf i}+ (x^2 y - yz^2)\, \hat{ \mathbf j} + (y^2z - x^2z) \, \hat{ \mathbf k}\)

5. $\vec{\mathbf{F}}(x,y,z) = (x\,\cos y)\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}}$

6. $\vec{\mathbf{F}}(x,y,z) = (x-y)\,\hat{\mathbf{i}} + (y-z)\,\hat{\mathbf{j}} + (z-x)\,\hat{\mathbf{k}}$

Answer

$curl$ \\($\vec{\mathbf{F}}$= \, \hat{ \mathbf i} + \, \hat{ \mathbf j} + \, \hat{ \mathbf k}\)

7. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + x^2y^2z^2\,\hat{\mathbf{j}} + y^2z^3\,\hat{\mathbf{k}}$

8. $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$

Answer

$curl$ \($\vec{\mathbf{F}}$= - y \, \hat{ \mathbf i} - z\, \hat{ \mathbf j} - x\, \hat{ \mathbf k}\)

9. $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$

10. $\vec{\mathbf{F}}(x,y,z) = ax\,\hat{\mathbf{i}} + by\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$ for constants a, b, c

Answer

$curl$ \($\vec{\mathbf{F}}$= 0\)

Exercise $9.5E.3$: Divergence

For the following exercises, find the divergence of $\vec{\mathbf{F}}$.

1. $\vec{\mathbf{F}}(x,y,z) = x^2z\,\hat{\mathbf{i}} + y^2x\,\hat{\mathbf{j}} + (y+2z)\,\hat{\mathbf{k}}$

2. $\vec{\mathbf{F}}(x,y,z) = 3xyz^2\,\hat{\mathbf{i}} + y^2\sin z\,\hat{\mathbf{j}} + xe^2\,\hat{\mathbf{k}}$

Answer

$div(\vec{\mathbf{F}}) = 3yz^2 + 2y\,\sin z + 2xe^{2z}$

3. $\vec{\mathbf{F}}(x,y) = (\sin x)\,\hat{\mathbf{i}} + (\cos y)\,\hat{\mathbf{j}}$

4. $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$

Answer

div \($\vec{\mathbf{F}}$= 2(x + y + z)\)

5. $\vec{\mathbf{F}}(x,y,z) = (x-y)\,\hat{\mathbf{i}} + (y-z)\,\hat{\mathbf{j}} + (z-x)\,\hat{\mathbf{k}}$

6. $\vec{\mathbf{F}}(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\,\hat{\mathbf{i}} + \dfrac{y}{\sqrt{x^2+y^2}}\,\hat{\mathbf{j}}$

Answer

div \($\vec{\mathbf{F}}$= \dfrac{1}{\sqrt{x^2+y^2}}\)

8. $\vec{\mathbf{F}}(x,y) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$

9. $\vec{\mathbf{F}}(x,y,z) = ax\,\hat{\mathbf{i}} + by\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$ for constants a, b, c

Answer

div \($\vec{\mathbf{F}}$= a + b\)

10. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + x^2y^2z^2\,\hat{\mathbf{j}} + y^2z^3\,\hat{\mathbf{k}}$

11. $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$

Answer

div \($\vec{\mathbf{F}}$= x + y + z\)

Exercise $9.5E.4$: Harmonic

For the following exercises, determine whether each of the given scalar functions is harmonic.

1. $u(x,y,z) = e^{-x}(\cos y - \sin y)$

2. $w(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

Answer

Harmonic

Exercise $9.5E.5$: CURL

1. If $\vec{\mathbf{F}}(x,y,z) = 2\,\hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}} + 3y\,\hat{\mathbf{k}}$ and $\vec{\mathbf{G}}(x,y,z) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, find $curl\,(F \times G)$.

2. If $\vec{\mathbf{F}}(x,y,z) = 2\,\hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}} + 3y\,\hat{\mathbf{k}}$ and $\vec{\mathbf{G}}(x,y,z) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, find $div\,(F \times G)$.

Answer

$div\,(F \times G) = 2z + 3x$

3. Find $div\,F$, given that $F = \nabla f$, where $f(x,y,z) = xy^3z^2$.

4. Find the divergence of $\vec{\mathbf{F}}$ for vector field $\vec{\mathbf{F}}(x,y,z) = (y^2+z^2)(x+y)i + (z^2+x^2)(y+z)\,\hat{\mathbf{j}} + (x^2+y^2)(z+x)\,\hat{\mathbf{k}}$.

Answer

div \($\vec{\mathbf{F}}$= 2r^2\)

5. Find the divergence of $\vec{\mathbf{F}}$ for vector field $\vec{\mathbf{F}}(x,y,z) = f_1(y,z)\,\hat{\mathbf{i}} + f_2(x,z)\,\hat{\mathbf{j}} + f_3(x,y)\,\hat{\mathbf{k}}$.

6. For the following exercises, use $r = |r|$ and $r = (x,y,z)$.

a) Find the $curl\,r$

Answer

$curl\,r = 0$

b) Find the $curl\,\frac{r}{r}$.

c) Find the $curl\,\frac{r}{r^3}$.

Answer

$curl\,\frac{r}{r^3} = 0$

7. Let $\vec{\mathbf{F}}(x,y) = \dfrac{-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}}{x^2+y^2}$, where$\vec{\mathbf{F}}$is defined on $\{(x,y) \in \mathbb{R} | (x,y) \neq (0,0)\}$. Find $curl\,F$.

Exercise $9.5E.6$: Curl

For the following exercises, use a computer algebra system to find the curl of the given vector fields.

1. [T] $\vec{\mathbf{F}}x,y,z) = arctan\left(\frac{x}{y}\right)\,\hat{\mathbf{i}} + \ln\sqrt{x^2+y^2}\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Answer

$curl$ \($\vec{\mathbf{F}}$= \dfrac{2x}{x^2+y^2} \, \hat{ \mathbf k}\)

2. [T] $\vec{\mathbf{F}}(x,y,z) = \sin(x-y)\,\hat{\mathbf{i}} + \sin(y-z)\,\hat{\mathbf{j}} + \sin(z-x)\,\hat{\mathbf{k}}$

Exercise $9.5E.7$: Divergence

For the following exercises, find the divergence of $\vec{\mathbf{F}}$ at the given point.

1. $\vec{\mathbf{F}}(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ at $(2,-1,3)$

Answer

$div\,(\vec{\mathbf{F}}) = 0$

2. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,3)$
3. $\vec{\mathbf{F}}(x,y,z) = e^{-xy}\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + e^{yz}\,\hat{\mathbf{k}}$ at $(3,2,0)$

Answer

$div\,\vec{\mathbf{F}} = 2 - 2e^{-6}$

4. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,1)$
5. $\vec{\mathbf{F}}(x,y,z) = e^x \sin y\,\hat{\mathbf{i}} - e^x \cos y\,\hat{\mathbf{j}}$ at $(0,0,3)$

Answer

div \($\vec{\mathbf{F}}$= 0\)

Exercise $9.5E.8$: CURL

For the following exercises, find the curl of $\vec{\mathbf{F}}$ at the given point.

1. $\vec{\mathbf{F}}(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ at $(2,-1,3)$
2. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,3)$

Answer

$curl$ \($\vec{\mathbf{F}}$= \, \hat{ \mathbf j} - 3\, \hat{ \mathbf k}\)

3. $\vec{\mathbf{F}}(x,y,z) = e^{-xy}\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + e^{yz}\,\hat{\mathbf{k}}$ at $(3,2,0)$
4. $\vec{\mathbf{F}}(x,y,z) = xyz\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ at $(1,2,1)$

Answer

$curl$ \($\vec{\mathbf{F}}$= 2\, \hat{ \mathbf j} - \, \hat{ \mathbf k}\

5. $\vec{\mathbf{F}}(x,y,z) = e^x \sin y\,\hat{\mathbf{i}} - e^x \cos y\,\hat{\mathbf{j}}$ at $(0,0,3)$

Exercise $9.5E.9$

Let $\vec{\mathbf{F}}(x,y,z) = (3x^2y + az)\,\hat{\mathbf{i}} + x^3\,\hat{\mathbf{j}} + (3x + 3z^2)\,\hat{\mathbf{k}}$. For what value of a is $\vec{\mathbf{F}}$ conservative?

Answer

$a = 3$

Exercise $9.5E.10$

1. Given vector field $\vec{\mathbf{F}}(x,y)=\frac{1}{x^2+y^2}(-y,x)$ on domain $D=\frac{\mathbb{R}^2}{\{(0,0)\}}=\{(x,y)\in\mathbb{R}^2|(x,y)\neq(0,0)\}$, is $\vec{\mathbf{F}}$conservative?

2. Given vector field $\vec{\mathbf{F}}(x,y)=\frac{1}{x^2+y^2}(x,y)$ on domain $D=\frac{\mathbb{R}^2}{\{(0,0)\}}$, is$\vec{\mathbf{F}}$conservative?

Answer

$\vec{\mathbf{F}}$ is conservative.

3. Find the work done by force field $\vec{\mathbf{F}}(x,y)=e^{-y}\,\hat{\mathbf{i}}-xe^{-y}\,\hat{\mathbf{j}}$ in moving an object from *P*(0, 1) to *Q*(2, 0). Is the force field conservative?

Exercise $9.5E.11$

1. Compute divergence $\vec{\mathbf{F}}=(\sinh x)\,\hat{\mathbf{i}}+(\cosh y)\,\hat{\mathbf{j}}-xyz\,\hat{\mathbf{k}}$.

Answer

$div\,(\vec{\mathbf{F}})=\cosh x+\sinh y-xy$

2. Compute $curl\,(\vec{\mathbf{F}})=(\sinh x)\,\hat{\mathbf{i}}+(\cosh y)\,\hat{\mathbf{j}}-xyz\,\hat{\mathbf{k}}$.

Answer

TBA

Exercise $9.5E.12$

For the following exercises, consider a rigid body that is rotating about the *x*-axis counterclockwise with constant angular velocity $\omega=\langle a,b,c\rangle$. If *P* is a point in the body located at $\vec{\mathbf{r}}=x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}+z\,\hat{\mathbf{k}}$, the velocity at *P* is given by vector field $\vec{\mathbf{F}}=\omega\times\vec{\mathbf{r}}$.

A three dimensional diagram of an object rotating about the x axis in a counterclockwise manner with constant angular velocity w = <a,b,c>. The object is roughly a sphere with pointed ends on the x axis, which cuts it in half. An arrow r is drawn from (0,0,0) to P(x,y,z) and down from P(x,y,z) to the x axis.

a) Express$\vec{\mathbf{F}}$ in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ vectors.

Answer

$(bz-cy)\,\hat{\mathbf{i}}+(cx-az)\,\hat{\mathbf{j}}+(ay-bx)\,\hat{\mathbf{k}}$

b) Find $div\,\vec{\mathbf{F}}$.

c) Find $curl\,\vec{\mathbf{F}}$

Answer

$curl\,(\vec{\mathbf{F}})=2\omega$

Exercise $9.5E.13$

In the following exercises, suppose that $\nabla\cdot\vec{\mathbf{F}}=0$ and $\nabla\cdot\vec{\mathbf{G}}=0$.

a) Does $\vec{\mathbf{F}}+\vec{\mathbf{G}}$ necessarily have zero divergence?

b) Does $\vec{\mathbf{F}}\times\vec{\mathbf{G}}$ necessarily have zero divergence?

Answer

$\vec{\mathbf{F}} \times \vec{\mathbf{G}}$ does not have zero divergence.

Exercise $9.5E.14$

In the following exercises, suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x, y, z)$. The heat flow vector field in the object is $\vec{\mathbf{F}} = -k\nabla T$, where $k > 0$ is a property of the material. The heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\nabla \cdot \vec{\mathbf{F}} = -k\nabla \cdot \nabla T = -k\nabla^2 T$.

a) Compute the heat flow vector field.

b) Compute the divergence.

Answer

$\nabla \cdot \vec{\mathbf{F}} = -200k[1 + 2(x^2 + y^2 + z^2)]e^{-x^2+y^2+z^2}$

Exercise $9.5E.15$

[T] Consider rotational velocity field $\vec{\mathbf{v}} = \langle 0, 10z, -10y\rangle$. If a paddlewheel is placed in plane $x + y + z = 1$ with its axis normal to this plane, using a computer algebra system, calculate how fast the paddlewheel spins in revolutions per unit time.

A three dimensional diagram of a rotational velocity field. The arrows are showing a rotation in a clockwise manner. A paddlewheel is shown in plan x + y + z = 1 with n extended out perpendicular to the plane.

9.5E.1 Glossary

curl

the curl of vector field $F = \langle P, Q, R\rangle$, denoted $\nabla \times F$, is the "determinant" of the matrix $\begin{vmatrix} i\,\hat{\mathbf{j}}\,\hat{\mathbf{k}} \\ \frac{\partial}{\partial x}\,\frac{\partial}{\partial y}\,\frac{\partial}{\partial z} \\ PQR \end{vmatrix}$ and is given by the expression $(R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}}$; it measures the tendency of particles at a point to rotate about the axis that points in the direction of the curl at the point

divergence

the divergence of a vector field $F = \langle P, Q, R\rangle$, denoted $\nabla \times F$, is $P_x + Q_y + R_z$; it measures the "outflowing-ness" of a vector field

9.6: Surface Integrals

We have seen that a line integral is an integral over a path in a plane or in space. However, if we wish to integrate over a surface (a two-dimensional object) rather than a path (a one-dimensional object) in space, then we need a new kind of integral that can handle integration over objects in higher dimensions. We can extend the concept of a line integral to a surface integral to allow us to perform this integration.

Surface integrals are important for the same reasons that line integrals are important. They have many applications to physics and engineering, and they allow us to develop higher dimensional versions of the Fundamental Theorem of Calculus. In particular, surface integrals allow us to generalize Green's theorem to higher dimensions, and they appear in some important theorems we discuss in later sections.

9.6.1 Parametric Surfaces

A surface integral is similar to a line integral, except the integration is done over a surface rather than a path. In this sense, surface integrals expand on our study of line integrals. Just as with line integrals, there are two kinds of surface integrals: a surface integral of a scalar-valued function and a surface integral of a vector field.

However, before we can integrate over a surface, we need to consider the surface itself. Recall that to calculate a scalar or vector line integral over curve C, we first need to parameterize C. In a similar way, to calculate a surface integral over surface S, we need to parameterize S. That is, we need a working concept of a *parameterized surface* (or a *parametric surface*), in the same way that we already have a concept of a parameterized curve.

A parameterized surface is given by a description of the form

$$\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle. \tag{9.6.1}$$

Notice that this parameterization involves two parameters, u and v, because a surface is two-dimensional, and therefore two variables are needed to trace out the surface. The parameters u and v vary over a region called the parameter domain, or parameter space—the set of points in the uv-plane that can be substituted into $\vec{\mathbf{r}}$. Each choice of u and v in the parameter domain gives a point on the surface, just as each choice of a parameter t gives a point on a parameterized curve. The entire surface is created by making all possible choices of u and v over the parameter domain.

Definition: Parameter Domain

Given a parameterization of surface

$$\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle. \tag{9.6.2}$$

the *parameter domain* of the parameterization is the set of points in the uv-plane that can be substituted into $\vec{\mathbf{r}}$.

Example 9.6.1: Parameterizing a Cylinder

Describe surface S parameterized by

$$\vec{\mathbf{r}}(u,v) = \langle \cos u,\, \sin u,\, v\rangle,\ -\infty < u < \infty,\ -\infty < v < \infty.$$

Solution

To get an idea of the shape of the surface, we first plot some points. Since the parameter domain is all of $\mathbb{R}^2$, we can choose any value for u and v and plot the corresponding point. If $u = v = 0$, then $\vec{\mathbf{r}}(0,0) = \langle 1,0,0\rangle$, so point (1, 0, 0) is on S. Similarly, points $\vec{\mathbf{r}}(\pi,2) = (-1,0,2)$ and $\vec{\mathbf{r}}\left(\frac{\pi}{2},4\right) = (0,1,4)$ are on S.

Although plotting points may give us an idea of the shape of the surface, we usually need quite a few points to see the shape. Since it is time-consuming to plot dozens or hundreds of points, we use another strategy. To visualize S, we visualize two families of curves that lie on S. In the first family of curves we hold u constant; in the second family of curves we hold v constant. This allows us to build a "skeleton" of the surface, thereby getting an idea of its shape.

- Suppose that u is a constant K. Then the curve traced out by the parameterization is $\langle \cos K,\, \sin K,\, v\rangle$, which gives a vertical line that goes through point $(\cos K, \sin K, v\rangle$ in the xy-plane.

- Suppose that v is a constant K. Then the curve traced out by the parameterization is $\langle \cos u,\ \sin u,\ K\rangle$, which gives a circle in plane $z=K$ with radius 1 and center $(0,0,K)$.

If u is held constant, then we get vertical lines; if v is held constant, then we get circles of radius 1 centered around the vertical line that goes through the origin. Therefore the surface traced out by the parameterization is cylinder $x^2+y^2=1$ (Figure $9.6.1$).

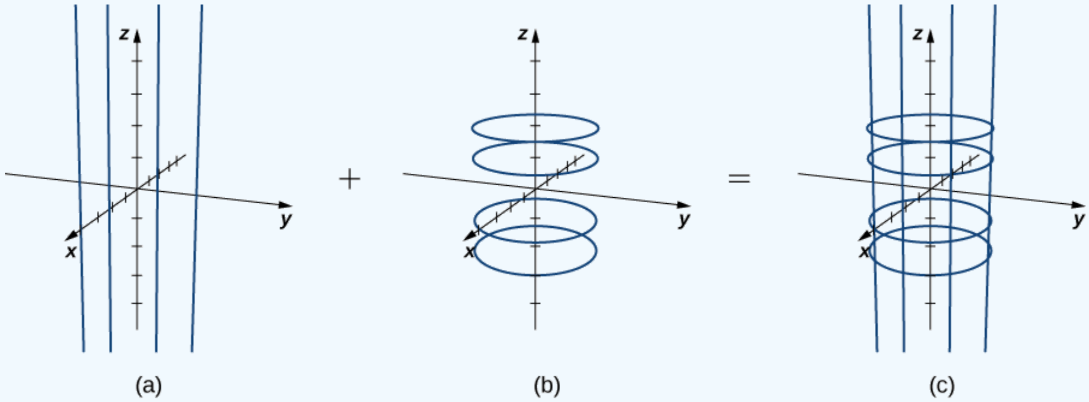


Figure $9.6.1$: (a) Lines $\langle \cos K,\ \sin K, v\rangle$ for $K=0,\ \dfrac{\pi}{2},\ \pi$, and $\dfrac{3\pi}{2}$. (b) Circles $\langle \cos u,\ \sin u,\ K\rangle$ for $K=-2,-1,1,$ and 2 (c) The lines and circles together. As u and v vary, they describe a cylinder.

Notice that if $x=\cos u$ and $y=\sin u$, then $x^2+y^2=1$, so points from S do indeed lie on the cylinder. Conversely, each point on the cylinder is contained in some circle $\langle \cos u,\ \sin u,\ k\rangle$ for some k, and therefore each point on the cylinder is contained in the parameterized surface (Figure $9.6.2$).

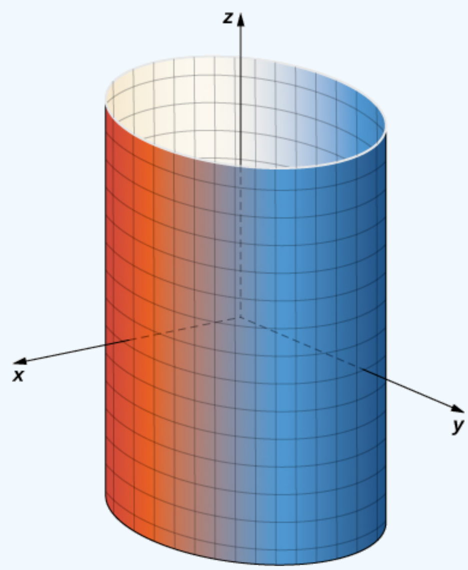


Figure $9.6.2$: Cylinder $x^2+y^2=r^2$ has parameterization $\vec{\mathbf{r}}(u,v)=\langle r\ \cos u,\ r\ \sin u,\ v\rangle,\ 0\le u\le 2\pi,\ -\infty<v<\infty$.

Analysis

Notice that if we change the parameter domain, we could get a different surface. For example, if we restricted the domain to $0\le u\le \pi,\ -\infty<v<6$, then the surface would be a half-cylinder of height 6.

Exercise $9.6.1$

Describe the surface with parameterization

$$\vec{\mathbf{r}}(u,v)=\langle 2\ \cos u,\ 2\ \sin u,\ v\rangle,\ 0\le u\le 2\pi,\ -\infty<v<\infty$$

Hint

Hold u and v constant, and see what kind of curves result.

Answer

Cylinder $x^2 + y^2 = 4$

It follows from Example 9.6.1 that we can parameterize all cylinders of the form $x^2 + y^2 = R^2$. If S is a cylinder given by equation $x^2 + y^2 = R^2$, then a parameterization of S is $\vec{\mathbf{r}}(u,v) = \langle R\cos u,\ R\sin u,\ v\rangle,\ 0 \le u \le 2\pi,\ -\infty < v < \infty.$

We can also find different types of surfaces given their parameterization, or we can find a parameterization when we are given a surface.

Example 9.6.2: Describing a Surface

Describe surface S parameterized by $\vec{\mathbf{r}}(u,v) = \langle u\cos v,\ u\sin v,\ u^2\rangle,\ 0 \le u < \infty,\ 0 \le v < 2\pi$.

Solution

Notice that if u is held constant, then the resulting curve is a circle of radius u in plane $z = u$. Therefore, as u increases, the radius of the resulting circle increases. If v is held constant, then the resulting curve is a vertical parabola. Therefore, we expect the surface to be an elliptic paraboloid. To confirm this, notice that

$$\begin{aligned} x^2 + y^2 &= (u\cos v)^2 + (u\sin v)^2 \\ &= u^2\cos^2 v + u^2 sin^2 v \\ &= u^2 \\ &= z \end{aligned}$$

Therefore, the surface is the elliptic paraboloid $x^2 + y^2 = z$ (Figure 9.6.3).

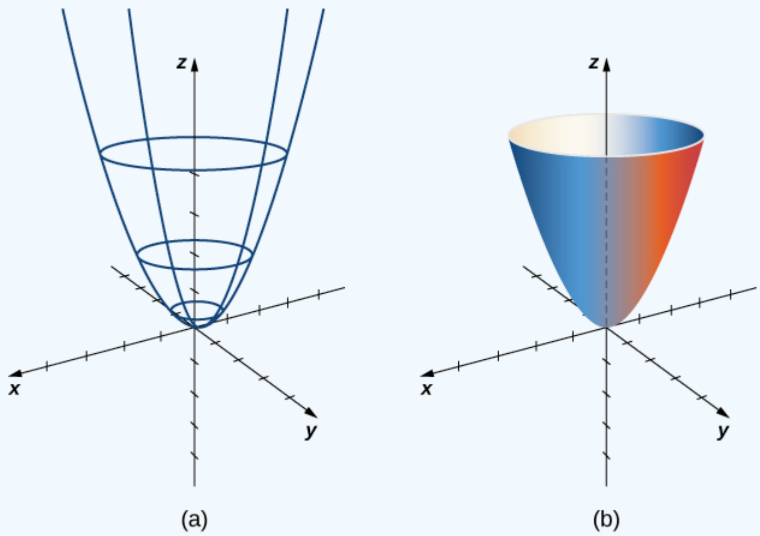


Figure 9.6.3: (a) Circles arise from holding u constant; the vertical parabolas arise from holding v constant. (b) An elliptic paraboloid results from all choices of u and v in the parameter domain.

Exercise 9.6.2

Describe the surface parameterized by $\vec{\mathbf{r}}(u,v) = \langle u\cos v,\ u\sin v,\ u\rangle,\ -\infty < u < \infty,\ 0 \le v < 2\pi$.

Hint

Hold u constant and see what kind of curves result. Imagine what happens as u increases or decreases.

Answer

Cone $x^2 + y^2 = z^2$

Example 9.6.3: Finding a Parameterization

Give a parameterization of the cone $x^2+y^2=z^2$ lying on or above the plane $z=-2$.

Solution

The horizontal cross-section of the cone at height $z=u$ is circle $x^2+y^2=u^2$. Therefore, a point on the cone at height u has coordinates $(u\cos v, u\sin v, u)$ for angle v. Hence, a parameterization of the cone is $\vec{\mathbf{r}}(u,v)=\langle u\cos v, u\sin v, u\rangle$. Since we are not interested in the entire cone, only the portion on or above plane $z=-2$, the parameter domain is given by $-2<u<\infty, 0\le v<2\pi$ (Figure 9.6.4).

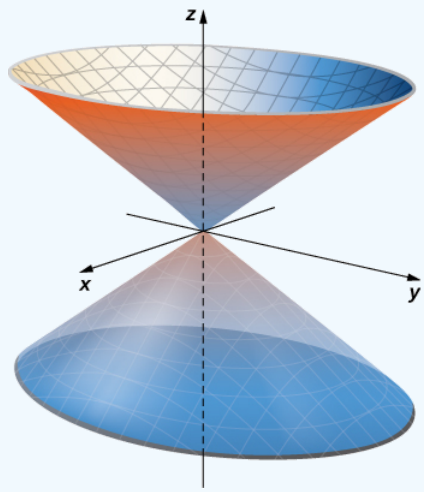


Figure 9.6.4: Cone $x^2+y^2=z^2$ has parametrization $\vec{\mathbf{r}}(u,v)=\langle u\cos v, u\sin v\rangle, -\infty<u<\infty, 0\le v\le 2\pi$.

Exercise 9.6.3

Give a parameterization for the portion of cone $x^2+y^2=z^2$ lying in the first octant.

Hint

Consider the parameter domain for this surface.

Answer

$\vec{\mathbf{r}}(u,v)=\langle u\cos v, u\sin v, u\rangle, 0<u<\infty, 0\le v<\frac{\pi}{2}$

We have discussed parameterizations of various surfaces, but two important types of surfaces need a separate discussion: spheres and graphs of two-variable functions. To parameterize a sphere, it is easiest to use spherical coordinates. The sphere of radius ρ centered at the origin is given by the parameterization

$$\vec{\mathbf{r}}(\phi,\theta)=\langle \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi\rangle, 0\le\theta\le 2\pi, 0\le\phi\le\pi.$$

The idea of this parameterization is that as ϕ sweeps downward from the positive z-axis, a circle of radius $\rho\sin\phi$ is traced out by letting θ run from 0 to 2π. To see this, let ϕ be fixed. Then

$$\begin{aligned} x^2+y^2 &=(\rho\cos\theta\sin\phi)^2+(\rho\sin\theta\sin\phi)^2 \\ &=\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta) \\ &=\rho^2\sin^2\phi \\ &=(\rho\sin\phi)^2. \end{aligned}$$

This results in the desired circle (Figure 9.6.5).

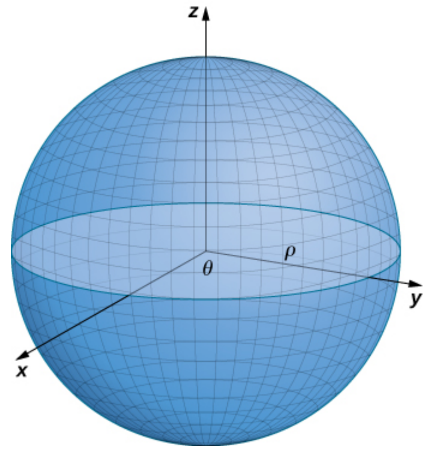


Figure 9.6.5: The sphere of radius ρ has parameterization $\vec{\mathbf{r}}(\phi, \theta) = \langle \rho \cos\theta \sin\phi,\, \rho \sin\theta \sin\phi,\, \rho \cos\phi \rangle,\, 0 \leq \theta \leq 2\pi,\, 0 \leq \phi \leq \pi.$

Finally, to parameterize the graph of a two-variable function, we first let $z = f(x, y)$ be a function of two variables. The simplest parameterization of the graph of f is $\vec{\mathbf{r}}(x, y) = \langle x, y, f(x, y) \rangle$, where x and y vary over the domain of f (Figure 9.6.6). For example, the graph of $f(x, y) = x^2 y$ can be parameterized by $\vec{\mathbf{r}}(x, y) = \langle x, y, x^2 y \rangle$, where the parameters x and y vary over the domain of f. If we only care about a piece of the graph of f - say, the piece of the graph over rectangle $[1, 3] \times [2, 5]$ - then we can restrict the parameter domain to give this piece of the surface:

$$\vec{\mathbf{r}}(x, y) = \langle x, y, x^2 y \rangle,\; 1 \leq x \leq 3,\; 2 \leq y \leq 5. \tag{9.6.3}$$

Similarly, if S is a surface given by equation $x = g(y, z)$ or equation $y = h(x, z)$, then a parameterization of S is $\vec{\mathbf{r}}(y, z) = \langle g(y, z),\, y,\, z \rangle$ or $\vec{\mathbf{r}}(x, z) = \langle x, h(x, z), z \rangle$, respectively. For example, the graph of paraboloid $2y = x^2 + z^2$ can be parameterized by $\vec{\mathbf{r}}(x, y) = \left\langle x, \dfrac{x^2 + z^2}{2}, z \right\rangle,\, 0 \leq x < \infty,\, 0 \leq z < \infty$. Notice that we do not need to vary over the entire domain of y because x and z are squared.

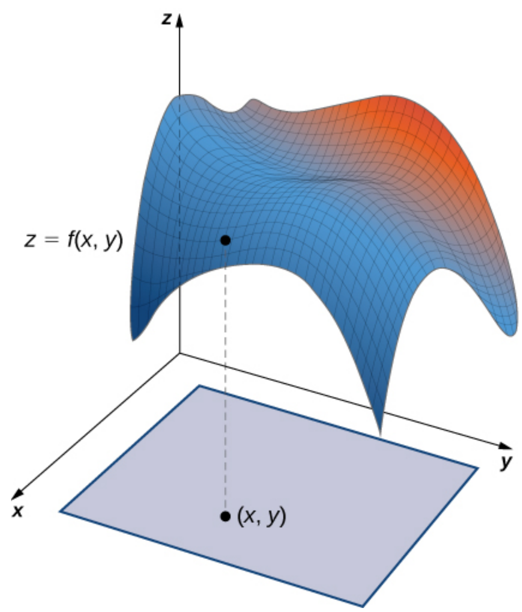


Figure 9.6.6: The simplest parameterization of the graph of a function is $\vec{\mathbf{r}}(x, y) = \langle x, y, f(x, y) \rangle$.

Let's now generalize the notions of smoothness and regularity to a parametric surface. Recall that curve parameterization $\vec{\mathbf{r}}(t),\, a \leq t \leq b$ is regular (or **smooth**) if $\vec{\mathbf{r}}'(t) \neq \vec{\mathbf{0}}$ for all t in $[a, b]$. For a curve, this condition ensures that the image of $\vec{\mathbf{r}}$ really is a curve, and not just a point. For example, consider curve parameterization $\vec{\mathbf{r}}(t) = \langle 1, 2 \rangle,\, 0 \leq t \leq 5$. The image of this parameterization is simply point $(1, 2)$, which is not a curve. Notice also that $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$. The fact that the derivative is the zero vector indicates we are not actually looking at a curve.

Analogously, we would like a notion of regularity (or smoothness) for surfaces so that a surface parameterization really does trace out a surface. To motivate the definition of regularity of a surface parameterization, consider the parameterization

$$\vec{\mathbf{r}}(u,v) = \langle 0,\, \cos v,\, 1\rangle,\; 0 \le u \le 1,\; 0 \le v \le \pi. \tag{9.6.4}$$

Although this parameterization appears to be the parameterization of a surface, notice that the image is actually a line (Figure $9.6.7$). How could we avoid parameterizations such as this? Parameterizations that do not give an actual surface? Notice that $\vec{\mathbf{r}}_u = \langle 0,0,0\rangle$ and $\vec{\mathbf{r}}_v = \langle 0, -\sin v, 0\rangle$, and the corresponding cross product is zero. The analog of the condition $\vec{\mathbf{r}}'(t) = \vec{\mathbf{0}}$ is that $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for point (u,v) in the parameter domain, which is a regular parameterization.

A three-dimensional diagram of a line on the x,z plane where the z component is 1, the x component is 1, and the y component exists between -1 and 1.

Figure $9.6.7$: The image of parameterization $\vec{\mathbf{r}}(u,v) = \langle 0, \cos v, 1\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$ is a line.

DEFINITION: regular parameterization

Parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ is a *regular parameterization* if $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for point (u,v) in the parameter domain.

If parameterization $\vec{r}$ is regular, then the image of $\vec{r}$ is a two-dimensional object, as a surface should be. Throughout this chapter, parameterizations $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ are assumed to be regular.

Recall that curve parameterization $\vec{\mathbf{r}}(t)$, $a \le t \le b$ is smooth if $\vec{\mathbf{r}}'(t)$ is continuous and $\vec{\mathbf{r}}'(t) \ne \vec{\mathbf{0}}$ for all t in $[a,b]$. Informally, a curve parameterization is smooth if the resulting curve has no sharp corners. The definition of a smooth surface parameterization is similar. Informally, a surface parameterization is *smooth* if the resulting surface has no sharp corners.

DEFINITION: Smooth Surfaces

A surface parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ is *smooth* if vector $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for any choice of u and v in the parameter domain.

A surface may also be *piecewise smooth* if it has smooth faces but also has locations where the directional derivatives do not exist.

Example $9.6.4$: Identifying Smooth and Nonsmooth Surfaces

Which of the figures in Figure $9.6.8$ is smooth?

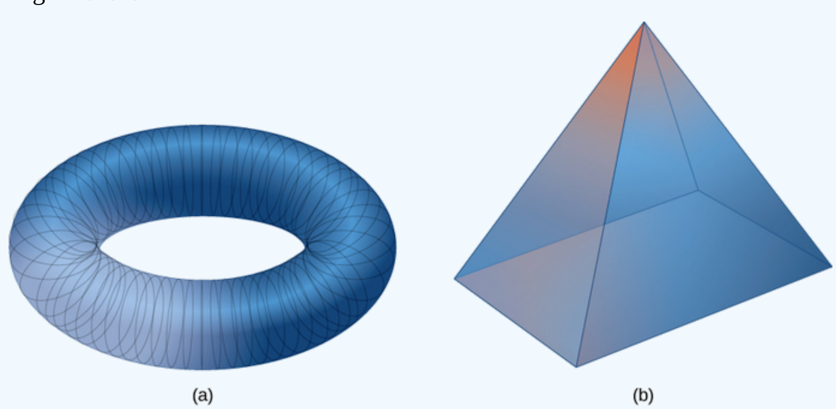


Figure $9.6.8$: (a) This surface is smooth. (b) This surface is piecewise smooth.

Solution

The surface in Figure $9.6.8a$ can be parameterized by

$$\vec{\mathbf{r}}(u,v) = \langle (2+\cos v)\cos u,\; (2+\cos v)\sin u,\; \sin v\rangle,\; 0 \le u < 2\pi,\; 0 \le v < 2\pi$$

(we can use technology to verify). Notice that vectors

$$\vec{\mathbf{r}}_u = \langle -(2+\cos v)\sin u,\; (2+\cos v)\cos u, 0\rangle$$

and

$$\vec{\mathbf{r}}_v = \langle -\sin v\,\cos u,\ -\sin v\,\sin u,\ \cos v\rangle$$

exist for any choice of u and v in the parameter domain, and

$$\begin{aligned}\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -(2+\cos v)\sin u & (2+\cos v)\cos u & 0 \\ -\sin v\,\cos u & -\sin v\,\sin u & \cos v \end{vmatrix} \\ &= [(2+\cos v)\cos u\ \cos v]\,\hat{\mathbf{i}} + [2+\cos v)\sin u\ \cos v]\,\hat{\mathbf{j}} + [(2+\cos v)\sin v\,\sin^2 u + (2+\cos v)\sin v\,\cos^2 u]\hat{\mathbf{k}} \\ &= [(2+\cos v)\cos u\ \cos v]\,\hat{\mathbf{i}} + [(2+\cos v)\sin u\ \cos v]\,\hat{\mathbf{j}} + [(2+\cos v)\sin v]\hat{\mathbf{k}}.\end{aligned}$$

The $\hat{\mathbf{k}}$ component of this vector is zero only if $v = 0$ or $v = \pi$. If $v = 0$ or $v = \pi$, then the only choices for u that make the $\hat{\mathbf{j}}$ component zero are $u = 0$ or $u = \pi$. But, these choices of u do not make the $\hat{\mathbf{i}}$ component zero. Therefore, $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is not zero for any choice of u and v in the parameter domain, and the parameterization is smooth. Notice that the corresponding surface has no sharp corners.

In the pyramid in Figure $9.6.8b$, the sharpness of the corners ensures that directional derivatives do not exist at those locations. Therefore, the pyramid has no smooth parameterization. However, the pyramid consists of four smooth faces, and thus this surface is piecewise smooth.

Exercise $9.6.4$

Is the surface parameterization $\vec{\mathbf{r}}(u,v) = \langle u^{2v}, v+1,\ \sin u\rangle,\ 0 \le u \le 2,\ 0 \le v \le 3$ smooth?

Hint

Investigate the cross product $\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$.

Answer

Yes

9.6.2 Surface Area of a Parametric Surface

Our goal is to define a surface integral, and as a first step we have examined how to parameterize a surface. The second step is to define the surface area of a parametric surface. The notation needed to develop this definition is used throughout the rest of this chapter.

Let S be a surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ over some parameter domain D. We assume here and throughout that the surface parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ is continuously differentiable—meaning, each component function has continuous partial derivatives. Assume for the sake of simplicity that D is a rectangle (although the following material can be extended to handle nonrectangular parameter domains). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that *i* ranges from 1 to *m* and *j* ranges from 1 to *n* so that D is subdivided into *mn* rectangles. This division of D into subrectangles gives a corresponding division of surface S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij}. Point P_{ij} corresponds to point (u_i, v_j) in the parameter domain.

Note that we can form a grid with lines that are parallel to the u-axis and the v-axis in the uv-plane. These grid lines correspond to a set of grid curves on surface S that is parameterized by $\vec{\mathbf{r}}(u,v)$. Without loss of generality, we assume that P_{ij} is located at the corner of two grid curves, as in Figure $9.6.9$. If we think of $\vec{\mathbf{r}}$ as a mapping from the uv-plane to $\mathbb{R}^3$, the grid curves are the image of the grid lines under $\vec{\mathbf{r}}$. To be precise, consider the grid lines that go through point (u_i, v_j). One line is given by $x = u_i,\ y = v$; the other is given by $x = u,\ y = v_j$. In the first grid line, the horizontal component is held constant, yielding a vertical line through (u_i, v_j). In the second grid line, the vertical component is held constant, yielding a horizontal line through (u_i, v_j). The corresponding grid curves are $\vec{\mathbf{r}}(u_i, v)$ and (u, v_j) and these curves intersect at point P_{ij}.

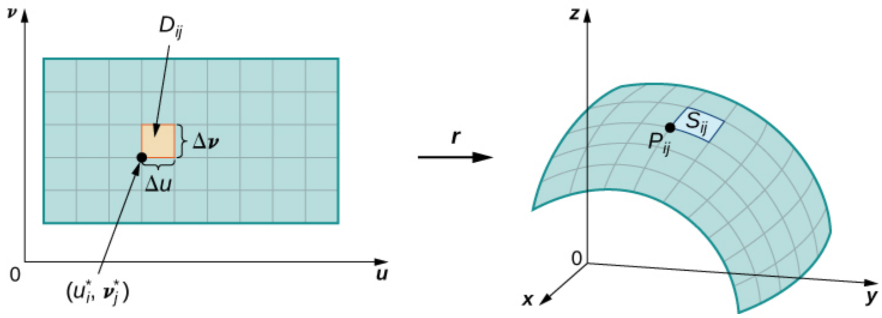


Figure 9.6.9: Grid lines on a parameter domain correspond to grid curves on a surface.

Now consider the vectors that are tangent to these grid curves. For grid curve $\vec{\mathbf{r}}(u_i, v)$, the tangent vector at P_{ij} is

$$\vec{\mathbf{t}}_v(P_{ij}) = \vec{\mathbf{r}}_v(u_i, v_j) = \langle x_v(u_i, v_j),\, y_v(u_i, v_j),\, z_v(u_i, v_j)\rangle. \tag{9.6.5}$$

For grid curve $\vec{\mathbf{r}}(u, v_j)$, the tangent vector at P_{ij} is

$$\vec{\mathbf{t}}_u(P_{ij}) = \vec{\mathbf{r}}_u(u_i, v_j) = \langle x_u(u_i, v_j),\, y_u(u_i, v_j),\, z_u(u_i, v_j)\rangle. \tag{9.6.6}$$

If vector $\vec{\mathbf{N}} = \vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})$ exists and is not zero, then the tangent plane at P_{ij} exists (Figure $9.6.10$). If piece S_{ij} is small enough, then the tangent plane at point P_{ij} is a good approximation of piece S_{ij}.

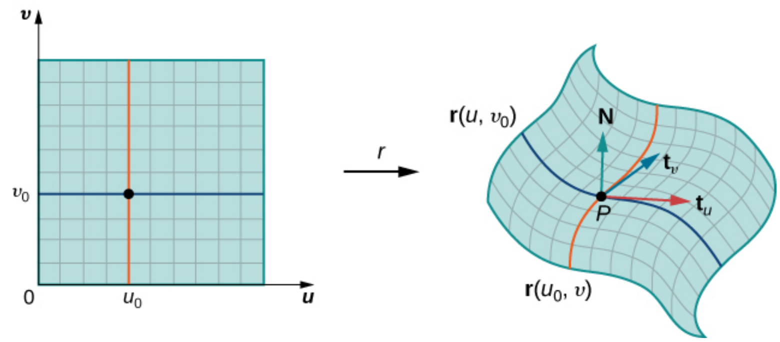


Figure 9.6.10: If the cross product of vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$ exists, then there is a tangent plane.

The tangent plane at P_{ij} contains vectors $\vec{\mathbf{t}}_u(P_{ij})$ and $\vec{\mathbf{t}}_v(P_{ij})$ and therefore the parallelogram spanned by $\vec{\mathbf{t}}_u(P_{ij})$ and $\vec{\mathbf{t}}_v(P_{ij})$ is in the tangent plane. Since the original rectangle in the uv-plane corresponding to S_{ij} has width Δu and length Δv, the parallelogram that we use to approximate S_{ij} is the parallelogram spanned by $\Delta u \vec{\mathbf{t}}_u(P_{ij})$ and $\Delta v \vec{\mathbf{t}}_v(P_{ij})$. In other words, we scale the tangent vectors by the constants Δu and Δv to match the scale of the original division of rectangles in the parameter domain. Therefore, the area of the parallelogram used to approximate the area of S_{ij} is

$$\Delta S_{ij} \approx ||(\Delta u \vec{\mathbf{t}}_u(P_{ij})) \times (\Delta v \vec{\mathbf{t}}_v(P_{ij}))|| = ||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})||\Delta u\, \Delta v. \tag{9.6.7}$$

Varying point P_{ij} over all pieces S_{ij} and the previous approximation leads to the following definition of surface area of a parametric surface (Figure $9.6.11$).

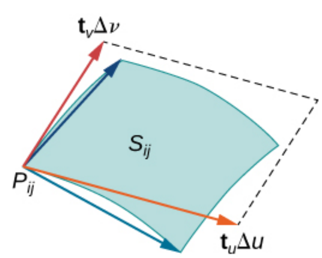


Figure 9.6.11: The parallelogram spanned by $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$ approximates the piece of surface S_{ij}.

definition: smooth parameterization of surface

Let $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$ with parameter domain D be a smooth parameterization of surface S. Furthermore, assume that S is traced out only once as (u,v) varies over D. The surface area of S is

$$\iint_D ||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||\, dA, \tag{9.6.8}$$

where $\vec{\mathbf{t}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$

and

$$\vec{\mathbf{t}}_v = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle. \tag{9.6.9}$$

Example $9.6.5$: Calculating Surface Area

Calculate the lateral surface area (the area of the "side," not including the base) of the right circular cone with height *h* and radius *r*.

Solution

Before calculating the surface area of this cone using Equation 9.6.8, we need a parameterization. We assume this cone is in $\mathbb{R}^3$ with its vertex at the origin (Figure $9.6.12$). To obtain a parameterization, let α be the angle that is swept out by starting at the positive *z*-axis and ending at the cone, and let $k = \tan\alpha$. For a height value v with $0 \le v \le h$, the radius of the circle formed by intersecting the cone with plane $z = v$ is kv. Therefore, a parameterization of this cone is

$$\vec{\mathbf{s}}(u,v) = \langle kv\cos u,\, kv\sin u,\, v\rangle,\, 0 \le u < 2\pi,\, 0 \le v \le h.$$

The idea behind this parameterization is that for a fixed v-value, the circle swept out by letting u vary is the circle at height v and radius kv. As v increases, the parameterization sweeps out a "stack" of circles, resulting in the desired cone.

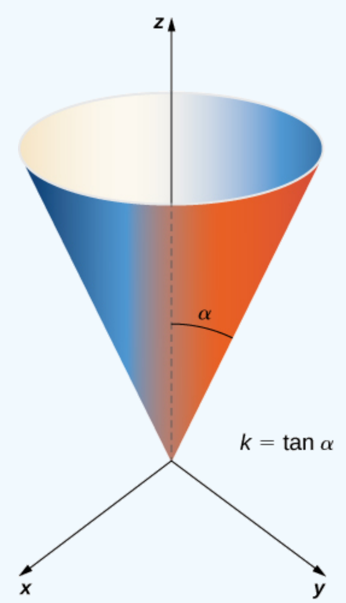


Figure 9.6.12: The right circular cone with radius $r = kh$ and height h has parameterization

$$\vec{\mathbf{s}}(u,v) = \langle kv\cos u,\, kv\sin u,\, v\rangle,\, 0 \le u < 2\pi,\, 0 \le v \le h. \tag{9.6.10}$$

With a parameterization in hand, we can calculate the surface area of the cone using Equation 9.6.8. The tangent vectors are $\vec{\mathbf{t}}_u = \langle -kv\sin u,\, kv\cos u,\, 0\rangle$ and $\vec{\mathbf{t}}_v = \langle k\cos u,\, k\sin u,\, 1\rangle$. Therefore,

$$\begin{aligned}\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -kv\sin u & kv\cos u & 0 \\ k\cos u & k\sin u & 1 \end{vmatrix} \\ &= \langle kv\,\cos u,\ kv\,\sin u,\ -k^2v\,\sin^2 u - k^2v\,\cos^2 u\rangle \\ &= \langle kv\,\cos u,\ kv\,\sin u,\ -k^2v\rangle.\end{aligned}$$

The magnitude of this vector is

$$\begin{aligned}||\langle kv\,\cos u,\ kv\,\sin u,\ -k^2v\rangle|| &= \sqrt{k^2v^2\cos^2 u + k^2v^2\sin^2 u + k^4v^2} \\ &= \sqrt{k^2v^2 + k^4v^2} \\ &= kv\sqrt{1+k^2}.\end{aligned}$$

By Equation 9.6.8, the surface area of the cone is

$$\begin{aligned}\iint_D ||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||\,dA &= \int_0^h \int_0^{2\pi} kv\sqrt{1+k^2}\,du\,dv \\ &= 2\pi k\sqrt{1+k^2}\int_0^h v\,dv \\ &= 2\pi k\sqrt{1+k^2}\left[\frac{v^2}{2}\right]_0^h \\ &= \pi kh^2\sqrt{1+k^2}.\end{aligned}$$

Since $k = \tan\alpha = r/h$,

$$\begin{aligned}\pi kh^2\sqrt{1+k^2} &= \pi\frac{r}{h}h^2\sqrt{1+\frac{r^2}{h^2}} \\ &= \pi rh\sqrt{1+\frac{r^2}{h^2}} \\ &= \pi r\sqrt{h^2+h^2\left(\frac{r^2}{h^2}\right)} \\ &= \pi r\sqrt{h^2+r^2}.\end{aligned}$$

Therefore, the lateral surface area of the cone is $\pi r\sqrt{h^2+r^2}$.

Analysis

The surface area of a right circular cone with radius r and height h is usually given as $\pi r^2 + \pi r\sqrt{h^2+r^2}$. The reason for this is that the circular base is included as part of the cone, and therefore the area of the base πr^2 is added to the lateral surface area $\pi r\sqrt{h^2+r^2}$ that we found.

Exercise 9.6.5

Find the surface area of the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle u+v,\ u^2,\ 2v\rangle,\ 0 \le u \le 3,\ 0 \le v \le 2$.

Hint

Use Equation 9.6.8.

Answer

≈ 43.02

Example 9.6.6: Calculating Surface Area

Show that the surface area of the sphere $x^2 + y^2 + z^2 = r^2$ is $4\pi r^2$.

Solution

The sphere has parameterization

$$r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi\rangle,\ 0 \le \theta < 2\pi,\ 0 \le \phi \le \pi.$$

The tangent vectors are

$$\vec{\mathbf{t}}_\theta = \langle -r\sin\theta\sin\phi,\ r\cos\theta\sin\phi,\ 0\rangle$$

and

$$\vec{\mathbf{t}}_\phi = \langle r\cos\theta\cos\phi,\ r\sin\theta\cos\phi,\ -r\sin\phi\rangle.$$

Therefore,

$$\begin{aligned}\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta &= \langle r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin^2\theta\sin\phi\cos\phi + r^2\cos^2\theta\sin\phi\cos\phi\rangle \\ &= \langle r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin\phi\cos\phi\rangle.\end{aligned}$$

Now,

$$\begin{aligned}||\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta|| &= \sqrt{r^4\sin^4\phi\cos^2\theta + r^4\sin^4\phi\sin^2\theta + r^4\sin^2\phi\cos^2\phi} \\ &= \sqrt{r^4\sin^4\phi + r^4\sin^2\phi\cos^2\phi} \\ &= r^2\sqrt{\sin^2\phi} \\ &= r\sin\phi.\end{aligned}$$

Notice that $\sin\phi \ge 0$ on the parameter domain because $0 \le \phi < \pi$, and this justifies equation $\sqrt{\sin^2\phi} = \sin\phi$. The surface area of the sphere is

$$\int_0^{2\pi}\int_0^{\pi} r^2\sin\phi\,d\phi\,d\theta = r^2\int_0^{2\pi} 2\,d\theta = 4\pi r^2.$$

We have derived the familiar formula for the surface area of a sphere using surface integrals.

Exercise 9.6.6

Show that the surface area of cylinder $x^2 + y^2 = r^2,\ 0 \le z \le h$ is $2\pi rh$. Notice that this cylinder does not include the top and bottom circles.

Hint

Use the standard parameterization of a cylinder and follow the previous example.

Answer

With the standard parameterization of a cylinder, Equation 9.6.8shows that the surface area is $2\pi rh$.

In addition to parameterizing surfaces given by equations or standard geometric shapes such as cones and spheres, we can also parameterize surfaces of revolution. Therefore, we can calculate the surface area of a surface of revolution by using the same techniques. Let $y = f(x) \ge 0$ be a positive single-variable function on the domain $a \le x \le b$ and let S be the surface obtained by rotating f about the x-axis (Figure 9.6.13). Let θ be the angle of rotation. Then, S can be parameterized with parameters x and θ by

$$\vec{\mathbf{r}}(x,\theta) = \langle x, f(x)\,\cos\theta,\ f(x)\sin\theta\rangle,\ a \le x \le b,\ 0 \le x \le 2\pi. \tag{9.6.11}$$

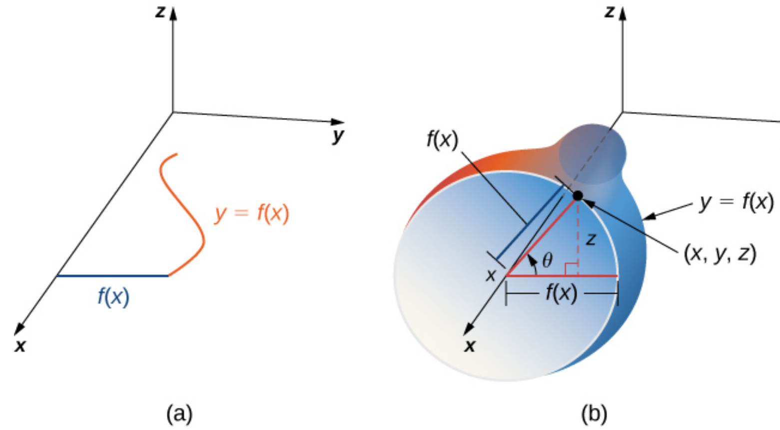


Figure 9.6.13: We can parameterize a surface of revolution by $\vec{\mathbf{r}}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta\rangle$, $a \le x \le b$, $0 \le x < 2\pi$.

Example 9.6.7: Calculating Surface Area

Find the area of the surface of revolution obtained by rotating $y = x^2,\ 0 \le x \le b$ about the *x*-axis (Figure 9.6.14).

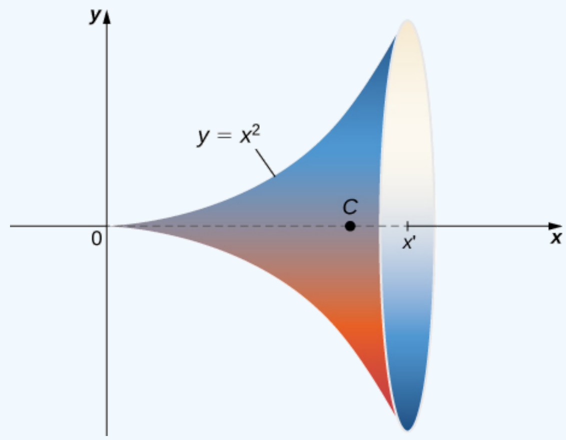


Figure 9.6.14: A surface integral can be used to calculate the surface area of this solid of revolution.

Solution

This surface has parameterization $\vec{\mathbf{r}}(x,\theta) = \langle x,\ x^2\cos\theta,\ x^2\sin\theta\rangle,\ 0 \le x \le b,\ 0 \le x < 2\pi.$

The tangent vectors are $\vec{\mathbf{t}}_x = \langle 1,\ 2x\ \cos\theta,\ 2x\ \sin\theta\rangle$ and $\vec{\mathbf{t}}_\theta = \langle 0,\ -x^2\sin\theta,\ -x^2\cos\theta\rangle$.

Therefore,

$$\begin{aligned}\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_\theta &= \langle 2x^3\cos^2\theta + 2x^3\sin^2\theta,\ -x^2\cos\theta,\ -x^2\sin\theta\rangle \\ &= \langle 2x^3,\ -x^2\cos\theta,\ -x^2\sin\theta\rangle\end{aligned}$$

and

$$\begin{aligned}\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_\theta &= \sqrt{4x^6 + x^4\cos^2\theta + x^4\sin^2\theta} \\ &= \sqrt{4x^6 + x^4} \\ &= x^2\sqrt{4x^2+1}\end{aligned}$$

The area of the surface of revolution is

$$\int_0^b \int_0^\pi x^2 \sqrt{4x^2+1}\, d\theta\, dx = 2\pi \int_0^b x^2 \sqrt{4x^2+1}\, dx$$

$$= 2\pi \left[\frac{1}{64} \left(2\sqrt{4x^2+1}(8x^3+x) \sinh^{-1}(2x) \right) \right]_0^b$$

$$= 2\pi \left[\frac{1}{64} \left(2\sqrt{4b^2+1}(8b^3+b) \sinh^{-1}(2b) \right) \right].$$

Exercise 9.6.7

Use Equation 9.6.8 to find the area of the surface of revolution obtained by rotating curve $y = \sin x$, $0 \le x \le \pi$ about the x-axis.

Hint

Use the parameterization of surfaces of revolution given before Example 9.6.7.

Answer

$2\pi(\sqrt{2} + \sinh^{-1}(1))$

9.6.3 Surface Integral of a Scalar-Valued Function

Now that we can parameterize surfaces and we can calculate their surface areas, we are able to define surface integrals. First, let's look at the surface integral of a scalar-valued function. Informally, the surface integral of a scalar-valued function is an analog of a scalar line integral in one higher dimension. The domain of integration of a scalar line integral is a parameterized curve (a one-dimensional object); the domain of integration of a scalar surface integral is a parameterized surface (a two-dimensional object). Therefore, the definition of a surface integral follows the definition of a line integral quite closely. For scalar line integrals, we chopped the domain curve into tiny pieces, chose a point in each piece, computed the function at that point, and took a limit of the corresponding Riemann sum. For scalar surface integrals, we chop the domain *region* (no longer a curve) into tiny pieces and proceed in the same fashion.

Let S be a piecewise smooth surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$ with parameter domain D and let $f(x,y,z)$ be a function with a domain that contains S. For now, assume the parameter domain D is a rectangle, but we can extend the basic logic of how we proceed to any parameter domain (the choice of a rectangle is simply to make the notation more manageable). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges from 1 to m and j ranges from 1 to n so that D is subdivided into mn rectangles. This division of D into subrectangles gives a corresponding division of S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij} evaluate P_{ij} at f, and multiply by area S_{ij} to form the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij})\, \Delta S_{ij}. \tag{9.6.12}$$

To define a surface integral of a scalar-valued function, we let the areas of the pieces of S shrink to zero by taking a limit.

DEFINITION: surface integral of a scalar-valued function

The *surface integral of a scalar-valued function* of f over a piecewise smooth surface S is

$$\iint_S f(x,y,z)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}. \tag{9.6.13}$$

Again, notice the similarities between this definition and the definition of a scalar line integral. In the definition of a line integral we chop a curve into pieces, evaluate a function at a point in each piece, and let the length of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. In the definition of a surface integral, we chop a surface into pieces, evaluate a function at a point in each piece, and let the area of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. Thus, a surface integral is similar to a line integral but in one higher dimension.

The definition of a scalar line integral can be extended to parameter domains that are not rectangles by using the same logic used earlier. The basic idea is to chop the parameter domain into small pieces, choose a sample point in each piece, and so on. The exact shape of each piece in the sample domain becomes irrelevant as the areas of the pieces shrink to zero.

Scalar surface integrals are difficult to compute from the definition, just as scalar line integrals are. To develop a method that makes surface integrals easier to compute, we approximate surface areas ΔS_{ij} with small pieces of a tangent plane, just as we did in the previous subsection. Recall the definition of vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$:

$$\vec{\mathbf{t}}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \vec{\mathbf{t}}_v = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle. \tag{9.6.14}$$

From the material we have already studied, we know that

$$\Delta S_{ij} \approx ||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})||\, \Delta u \, \Delta v. \tag{9.6.15}$$

Therefore,

$$\iint_S f(x,y,z)\, dS \approx \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})||\vec{\mathbf{t}}_u(P_{ij}) \times \vec{\mathbf{t}}_v(P_{ij})||\, \Delta u\, \Delta v. \tag{9.6.16}$$

This approximation becomes arbitrarily close to $\lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}$ as we increase the number of pieces S_{ij} by letting m and n go to infinity. Therefore, we have the following equation to calculate **scalar surface integrals**:

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{\mathbf{r}}(u,v))||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||\, dA. \tag{9.6.17}$$

Equation 9.6.17 allows us to calculate a surface integral by transforming it into a double integral. This equation for surface integrals is analogous to Equation ??? for line integrals:

$$\iint_C f(x,y,z)\, ds = \int_a^b f(\vec{\mathbf{r}}(t))||\vec{\mathbf{r}}'(t)||\, dt. \tag{9.6.18}$$

In this case, vector $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v$ is perpendicular to the surface, whereas vector $\vec{\mathbf{r}}'(t)$ is tangent to the curve.

Example 9.6.8: Calculating a Surface Integral

Calculate surface integral

$$\iint_S 5\, dS, \tag{9.6.19}$$

where S is the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle u,\, u^2,\, v\rangle$ for $0 \le u \le 2$ and $0 \le v \le u$.

Solution

Notice that this parameter domain$D$ is a triangle, and therefore the parameter domain is not rectangular. This is not an issue though, because Equation does not place any restrictions on the shape of the parameter domain.

To use Equation to calculate the surface integral, we first find vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$. Note that $\vec{\mathbf{t}}_u = \langle 1, 2u, 0\rangle$ and $\vec{\mathbf{t}}_v = \langle 0, 0, 1\rangle$. Therefore,

$$\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2u,\, -1,\, 0\rangle$$

and

$$||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v|| = \sqrt{1+4u^2}. \tag{9.6.20}$$

By Equation,

$$\begin{aligned}\iint_S 5\,dS &= 5\iint_D u\sqrt{1+4u^2}\,dA \\ &= 5\int_0^2\int_0^u \sqrt{1+4u^2}\,dv\,du = 5\int_0^2 u\sqrt{1+4u^2}\,du \\ &= 5\left[\frac{(1+4u^2)^{3/2}}{3}\right]_0^2 \\ &= \frac{5(17^{3/2}-1)}{3} \approx 115.15.\end{aligned}$$

Example 9.6.9: Calculating the Surface Integral of a Cylinder

Calculate surface integral

$$\iint_S (x+y^2)\,dS, \tag{9.6.21}$$

where S is cylinder $x^2+y^2=4,\ 0 \le z \le 3$ (Figure 9.6.15).

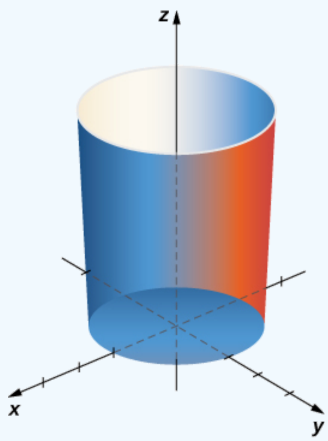


Figure 9.6.15: Integrating function $f(x,y,z)=x+y^2$ over a cylinder.

Solution

To calculate the surface integral, we first need a parameterization of the cylinder. Following Example, a parameterization is $\vec{\mathbf{r}}(u,v)=\langle\cos u,\ \sin u,\ v\rangle, 0\le u\le 2\pi,\ 0\le v\le 3.$

The tangent vectors are $\vec{\mathbf{t}}_u=\langle\sin u,\ \cos u,\ 0\rangle$ and $\vec{\mathbf{t}}_v=\langle 0,0,1\rangle$. Then,

$$\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -\sin u & \cos u & 0\\ 0 & 0 & 1\end{vmatrix}=\langle\cos u,\ \sin u,\ 0\rangle \tag{9.6.22}$$

and $||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||=\sqrt{\cos^2 u+\sin^2 u}=1$. By Equation,

$$\begin{aligned}\iint_S f(x,y,z)dS &= \iint_D f(\vec{\mathbf{r}}(u,v))||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||\,dA \\ &= \int_0^3\int_0^{2\pi}(\cos u+\sin^2 u)\,du\,dv \\ &= \int_0^3\left[\sin u+\frac{u}{2}-\frac{\sin(2u)}{4}\right]_0^{2\pi}dv \\ &= \int_0^3 \pi\,dv = 3\pi.\end{aligned}$$

Exercise 9.6.8

Calculate

$$\iint_S (x^2 - z)\, dS, \tag{9.6.23}$$

where S is the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle v,\, u^2 + v^2,\, 1\rangle,\ 0 \le u \le 2,\ 0 \le v \le 3.$

Hint

Use Equation.

Answer

24

Example 9.6.10: Calculating the Surface Integral of a Piece of a Sphere

Calculate surface integral

$$\iint_S f(x,y,z)\, dS, \tag{9.6.24}$$

where $f(x,y,z) = z^2$ and S is the surface that consists of the piece of sphere $x^2 + y^2 + z^2 = 4$ that lies on or above plane $z = 1$ and the disk that is enclosed by intersection plane $z = 1$ and the given sphere (Figure $9.6.16$).

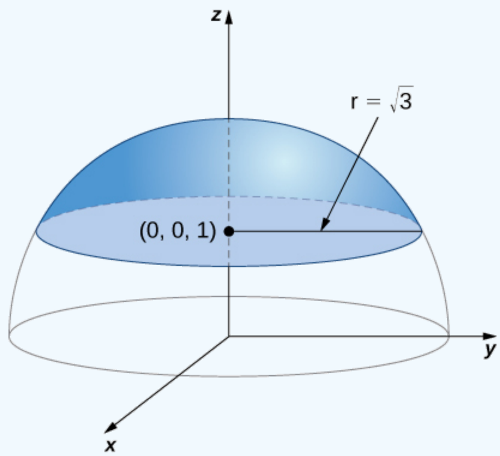


Figure 9.6.16: Calculating a surface integral over surface S.

Solution

Notice that S is not smooth but is piecewise smooth; S can be written as the union of its base S_1 and its spherical top S_2, and both S_1 and S_2 are smooth. Therefore, to calculate

$$\iint_S z^2 dS, \tag{9.6.25}$$

we write this integral as

$$\iint_{S_1} z^2\, dS + \iint_{S_2} z^2\, dS \tag{9.6.26}$$

and we calculate integrals

$$\iint_{S_1} z^2\, dS \tag{9.6.27}$$

and

$$\iint_{S_2} Z^2\,dS. \tag{9.6.28}$$

First, we calculate $\iint_{S_1} z^2\,dS$. To calculate this integral we need a parameterization of S_1. This surface is a disk in plane $z=1$ centered at $(0,0,1)$. To parameterize this disk, we need to know its radius. Since the disk is formed where plane $z=1$ intersects sphere $x^2+y^2+z^2=4$, we can substitute $z=1$ into equation $x^2+y^2+z^2=4$:

$$x^2+y^2+1=4 \Rightarrow x^2+y^2=3. \tag{9.6.29}$$

Therefore, the radius of the disk is $\sqrt{3}$ and a parameterization of S_1 is $\vec{\mathbf{r}}(u,v)=\langle u\,\cos v,\,u\,\sin v,\,1\rangle,\,0\le u\le\sqrt{3},\,0\le v\le 2\pi$. The tangent vectors are $\vec{\mathbf{t}}_u=\langle\cos v,\,\sin v,\,0\rangle$ and $\vec{\mathbf{t}}_v=\langle -u\,\sin v,\,u\,\cos v,\,0\rangle$, and thus

$$\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \cos v & \sin v & 0\\ -u\sin v & u\cos v & 0\end{vmatrix}=\langle 0,\,0,u\,\cos^2 v+u\,\sin^2 v\rangle=\langle 0,0,u\rangle. \tag{9.6.30}$$

The magnitude of this vector is u. Therefore,

$$\begin{aligned}\iint_{S_1} z^2\,dS &= \int_0^{\sqrt{3}}\int_0^{2\pi} f(r(u,v))||t_u\times t_v||\,dv\,du\\ &=\int_0^{\sqrt{3}}\int_0^{2\pi} u\,dv\,du\\ &=2\pi\int_0^{\sqrt{3}} u\,du\\ &=2\pi\sqrt{3}.\end{aligned}$$

Now we calculate

$$\iint_{S_2} dS. \tag{9.6.31}$$

To calculate this integral, we need a parameterization of S_2. The parameterization of full sphere $x^2+y^2+z^2=4$ is

$$\vec{\mathbf{r}}(\phi,\theta)=\langle 2\,\cos\theta\,\sin\phi,\,2\,\sin\theta\,\sin\phi,\,2\,\cos\phi\rangle,\,0\le\theta\le 2\pi, 0\le\phi\le\pi. \tag{9.6.32}$$

Since we are only taking the piece of the sphere on or above plane $z=1$, we have to restrict the domain of ϕ. To see how far this angle sweeps, notice that the angle can be located in a right triangle, as shown in Figure $9.6.17$(the $\sqrt{3}$ comes from the fact that the base of S is a disk with radius $\sqrt{3}$). Therefore, the tangent of ϕ is $\sqrt{3}$, which implies that ϕ is $\pi/6$. We now have a parameterization of S_2:

$\vec{\mathbf{r}}(\phi,\theta)=\langle 2\,\cos\theta\,\sin\phi,\,2\,\sin\theta\,\sin\phi,\,2\,\cos\phi\rangle,\,0\le\theta\le 2\pi,\,0\le\phi\le\pi/3.$

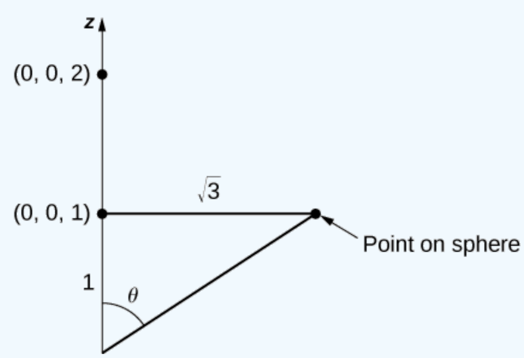


Figure 9.6.17: The maximum value of ϕ has a tangent value of $\sqrt{3}$.

The tangent vectors are $\vec{\mathbf{t}}_\phi=\langle 2\,\cos\theta\,\cos\phi,\,2\,\sin\theta\,\cos\phi,\,-2\,\sin\phi\rangle$ and $\vec{\mathbf{t}}_\theta=\langle -2\sin\theta\sin\phi,\,u\cos\theta\sin\phi,\,0\rangle$, and thus

$$\begin{aligned} \vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \end{vmatrix} \\ &= \langle 4\,\cos\theta\,\sin^2\phi,\ 4\,\sin\theta\,\sin^2\phi,\ 4\,\cos^2\theta\,\cos\phi\,\sin\phi + 4\,\sin^2\theta\,\cos\phi\,\sin\phi\rangle \\ &= \langle 4\,\cos\theta\,\sin^2\phi,\ 4\,\sin\theta\,\sin^2\phi,\ 4\,\cos\phi\,\sin\phi\rangle. \end{aligned}$$

The magnitude of this vector is

$$\begin{aligned} \vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta &= \sqrt{16\,\cos^2\theta\,\sin^4\phi + 16\,\sin^2\theta\,\sin^4\phi + 16\,\cos^2\phi\,\sin^2\phi} \\ &= 4\sqrt{sin^4\phi + \cos^2\phi\,\sin^2\phi}. \end{aligned}$$

Therefore,

$$\begin{aligned} \iint_{S_2} z\,dS &= \int_0^{\pi/6}\int_0^{2\pi} f(\vec{\mathbf{r}}(\phi,\theta))||\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta||\,d\theta\,d\phi \\ &= \int_0^{\pi/6}\int_0^{2\pi} 16\,\cos^2\phi\sqrt{\sin^4\phi + \cos^2\phi\,\sin^2\phi}\,d\theta\,d\phi \\ &= 32\pi\int_0^{\pi/6} \cos^2\phi\sqrt{\sin^4\phi + \cos^2\phi\,\sin^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6} \cos^2\phi\,sin\phi\sqrt{\sin^2\phi + \cos^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6} \cos^2\phi\,sin\phi\,d\phi \\ &= 32\pi\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/6} \\ &= 32\pi\left[\frac{1}{3} - \frac{\sqrt{3}}{8}\right] = \frac{32\pi}{3} - 4\sqrt{3}. \end{aligned}$$

Since

$$\iint_S z^2\,dS = \iint_{S_1} z^2\,dS + \iint_{S_2} z^2\,dS, \tag{9.6.33}$$

we have

$$\iint_S z^2\,dS = (2\pi - 4)\sqrt{3} + \frac{32\pi}{3}. \tag{9.6.34}$$

Analysis

In this example we broke a surface integral over a piecewise surface into the addition of surface integrals over smooth subsurfaces. There were only two smooth subsurfaces in this example, but this technique extends to finitely many smooth subsurfaces.

Exercise 9.6.9

Calculate line integral $\iint_S (x-y)\,dS,$ where S is cylinder $x^2+y^2=1,\ 0 \le z \le 2$, including the circular top and bottom.

Hint

Break the integral into three separate surface integrals.

Answer

0

Scalar surface integrals have several real-world applications. Recall that scalar line integrals can be used to compute the mass of a wire given its density function. In a similar fashion, we can use scalar surface integrals to compute the mass of a sheet given its density function. If a thin sheet of metal has the shape of surface S and the density of the sheet at point (x, y, z) is $\rho(x, y, z)$ then mass m of the sheet is $m = \iint_S \rho(x, y, z)\,dS$.

Example 9.6.11: Calculating the Mass of a Sheet

A flat sheet of metal has the shape of surface $z = 1 + x + 2y$ that lies above rectangle $0 \leq x \leq 4$ and $0 \leq y \leq 2$. If the density of the sheet is given by $\rho(x, y, z) = x^2yz$, what is the mass of the sheet?

Solution

Let S be the surface that describes the sheet. Then, the mass of the sheet is given by $m = \iint_S x^2yx\,dS$. To compute this surface integral, we first need a parameterization of S. Since S is given by the function $f(x, y) = 1 + x + 2y$, a parameterization of S is $\vec{\mathbf{r}}(x, y) = \langle x,\, y,\, 1 + x + 2y \rangle,\ 0 \leq x \leq 4,\ 0 \leq y \leq 2$.

The tangent vectors are $\vec{\mathbf{t}}_x = \langle 1, 0, 1 \rangle$ and $\vec{\mathbf{t}}_y = \langle 1, 0, 2 \rangle$. Therefore, $\vec{\mathbf{t}}_x + \vec{\mathbf{t}}_y = \langle -1, -2, 1 \rangle$ and $||\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y|| = \sqrt{6}$.

By the definition of the line integral (Section 16.2),

$$\begin{aligned} m &= \iint_S x^2yz\,dS \\ &= \sqrt{6}\int_0^4 \int_0^2 x^2y(1 + x + 2y)\,dy\,dx \\ &= \sqrt{6}\int_0^4 \frac{22x^2}{3} + 2x^3\,dx \\ &= \frac{2560\sqrt{6}}{9} \approx 696.74. \end{aligned}$$

Exercise 9.6.10

A piece of metal has a shape that is modeled by paraboloid $z = x^2 + y^2,\ 0 \leq z \leq 4,$ and the density of the metal is given by $\rho(x, y, z) = z + 1$. Find the mass of the piece of metal.

Hint

The mass of a sheet is given by

$$m = \iint_S \rho(x, y, z)\,dS. \tag{9.6.35}$$

A useful parameterization of a paraboloid was given in a previous example.

Answer

$38.401\pi \approx 120.640$

9.6.4 Orientation of a Surface

Recall that when we defined a scalar line integral, we did not need to worry about an orientation of the curve of integration. The same was true for scalar surface integrals: we did not need to worry about an "orientation" of the surface of integration.

On the other hand, when we defined vector line integrals, the curve of integration needed an orientation. That is, we needed the notion of an oriented curve to define a vector line integral without ambiguity. Similarly, when we define a surface integral of a vector field, we need the notion of an oriented surface. An oriented surface is given an "upward" or "downward" orientation or, in the case of surfaces such as a sphere or cylinder, an "outward" or "inward" orientation.

Let S be a smooth surface. For any point (x, y, z) on S, we can identify two unit normal vectors $\vec{\mathbf{N}}$ and $-\vec{\mathbf{N}}$. If it is possible to choose a unit normal vector $\vec{\mathbf{N}}$ at every point (x, y, z) on S so that $\vec{\mathbf{N}}$ varies continuously over S, then S is "*orientable*." Such a choice of unit normal vector at each point gives the **orientation of a surface** S. If you think of the normal field as describing water flow, then the side of the surface that water flows toward is the "negative" side and the side of the surface at which the water flows away is the "positive" side. Informally, a choice of orientation gives S an "outer" side and an "inner" side (or an "upward" side and a "downward" side), just as a choice of orientation of a curve gives the curve "forward" and "backward" directions.

Closed surfaces such as spheres are orientable: if we choose the outward normal vector at each point on the surface of the sphere, then the unit normal vectors vary continuously. This is called the *positive orientation of the closed surface* (Figure). We also could choose the inward normal vector at each point to give an "inward" orientation, which is the negative orientation of the surface.

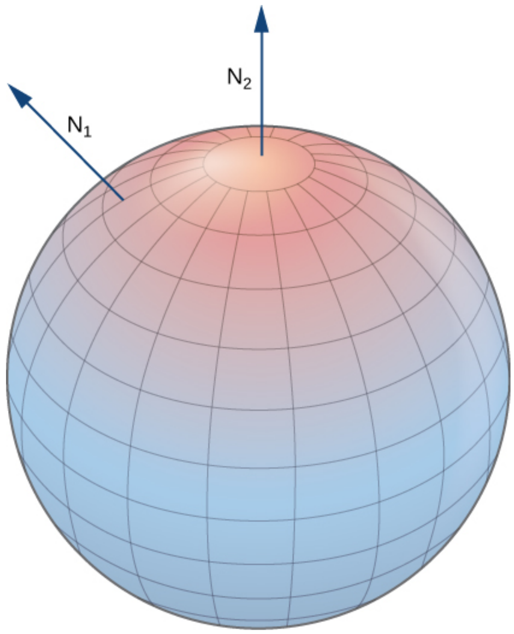


Figure 9.6.18: An oriented sphere with positive orientation.

A portion of the graph of any smooth function $z = f(x, y)$ is also orientable. If we choose the unit normal vector that points "above" the surface at each point, then the unit normal vectors vary continuously over the surface. We could also choose the unit normal vector that points "below" the surface at each point. To get such an orientation, we parameterize the graph of f in the standard way: $\vec{\mathbf{r}}(x, y) = \langle x,\ y,\ f(x, y)\rangle$, where x and y vary over the domain of f. Then, $\vec{\mathbf{t}}_x = \langle 1, 0, f_x\rangle$ and $\vec{\mathbf{t}}_y = \langle 0, 1, f_y\rangle$, and therefore the cross product $\vec{\mathbf{t}}_x \times \vec{\mathbf{t}}_y$ (which is normal to the surface at any point on the surface) is $\langle -f_x,\ -f_y,\ 1\rangle$Since the z-component of this vector is one, the corresponding unit normal vector points "upward," and the upward side of the surface is chosen to be the "positive" side.

Let S be a smooth orientable surface with parameterization $\vec{\mathbf{r}}(u, v)$. For each point $\vec{\mathbf{r}}(a, b)$ on the surface, vectors $\vec{\mathbf{t}}_u$ and $\vec{\mathbf{t}}_v$ lie in the tangent plane at that point. Vector $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v$ is normal to the tangent plane at $\vec{\mathbf{r}}(a, b)$ and is therefore normal to S at that point. Therefore, the choice of unit normal vector

$$\vec{\mathbf{N}} = \frac{\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||} \tag{9.6.36}$$

gives an orientation of surface S.

Example 9.6.12: Choosing an Orientation

Give an orientation of cylinder $x^2 + y^2 = r^2$, $0 \leq z \leq h$.

Solution

This surface has parameterization $\vec{\mathbf{r}}(u, v) = \langle r\ \cos u,\ r\ \sin u,\ v\rangle$, $0 \leq u < 2\pi$, $0 \leq v \leq h$.

The tangent vectors are $\vec{t}_u = \langle -r\,\sin u,\ r\,\cos u,\ 0\rangle$ and $\vec{t}_v = \langle 0,0,1\rangle$. To get an orientation of the surface, we compute the unit normal vector

$$\vec{N} = \frac{\vec{t}_u \times \vec{t}_v}{||\vec{t}_u \times \vec{t}_v||} \tag{9.6.37}$$

In this case, $\vec{t}_u \times \vec{t}_v = \langle r\,\cos u,\ r\,\sin u,\ 0\rangle$ and therefore

$$||\vec{t}_u \times \vec{t}_v|| = \sqrt{r^2\cos^2 u + r^2\sin^2 u} = r. \tag{9.6.38}$$

An orientation of the cylinder is

$$\vec{N}(u,v) = \frac{\langle r\,\cos u,\ r\,\sin u,\ 0\rangle}{r} = \langle \cos u,\ \sin u,\ 0\rangle. \tag{9.6.39}$$

Notice that all vectors are parallel to the xy-plane, which should be the case with vectors that are normal to the cylinder. Furthermore, all the vectors point outward, and therefore this is an outward orientation of the cylinder (Figure $9.6.19$).

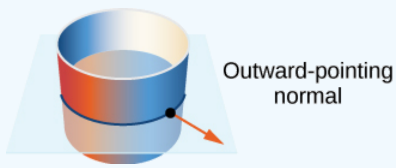


Figure 9.6.19:. If all the vectors normal to a cylinder point outward, then this is an outward orientation of the cylinder.

Exercise $9.6.11$

Give the "upward" orientation of the graph of $f(x,y) = xy$.

Hint

Parameterize the surface and use the fact that the surface is the graph of a function.

Answer

$$\vec{N}(x,y) = \left\langle \frac{-y}{\sqrt{1+x^2+y^2}}, \frac{-x}{\sqrt{1+x^2+y^2}}, \frac{1}{\sqrt{1+x^2+y^2}} \right\rangle \tag{9.6.40}$$

Since every curve has a "forward" and "backward" direction (or, in the case of a closed curve, a clockwise and counterclockwise direction), it is possible to give an orientation to any curve. Hence, it is possible to think of every curve as an oriented curve. This is not the case with surfaces, however. Some surfaces cannot be oriented; such surfaces are called *nonorientable*. Essentially, a surface can be oriented if the surface has an "inner" side and an "outer" side, or an "upward" side and a "downward" side. Some surfaces are twisted in such a fashion that there is no well-defined notion of an "inner" or "outer" side.

The classic example of a nonorientable surface is the Möbius strip. To create a Möbius strip, take a rectangular strip of paper, give the piece of paper a half-twist, and the glue the ends together (Figure $9.6.20$). Because of the half-twist in the strip, the surface has no "outer" side or "inner" side. If you imagine placing a normal vector at a point on the strip and having the vector travel all the way around the band, then (because of the half-twist) the vector points in the opposite direction when it gets back to its original position. Therefore, the strip really only has one side.

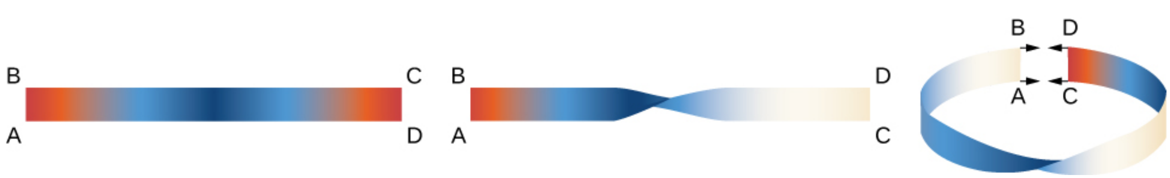


Figure 9.6.20: The construction of a Möbius strip.

Since some surfaces are *nonorientable*, it is not possible to define a vector surface integral on all piecewise smooth surfaces. This is in contrast to vector line integrals, which can be defined on any piecewise smooth curve.

9.6.5 Surface Integral of a Vector Field

With the idea of orientable surfaces in place, we are now ready to define a surface integral of a vector field. The definition is analogous to the definition of the flux of a vector field along a plane curve. Recall that if $\vec{\mathbf{F}}$ is a two-dimensional vector field and C is a plane curve, then the definition of the flux of $\vec{\mathbf{F}}$ along C involved chopping C into small pieces, choosing a point inside each piece, and calculating $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ at the point (where $\vec{\mathbf{N}}$ is the unit normal vector at the point). The definition of a surface integral of a vector field proceeds in the same fashion, except now we chop surface S into small pieces, choose a point in the small (two-dimensional) piece, and calculate $\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}$ at the point.

To place this definition in a real-world setting, let S be an oriented surface with unit normal vector $\vec{\mathbf{N}}$. Let $\vec{\mathbf{v}}$ be a velocity field of a fluid flowing through S, and suppose the fluid has density $\rho(x, y, z)$ Imagine the fluid flows through S, but S is completely permeable so that it does not impede the fluid flow (Figure $9.6.21$). The mass flux of the fluid is the rate of mass flow per unit area. The mass flux is measured in mass per unit time per unit area. How could we calculate the mass flux of the fluid across S?

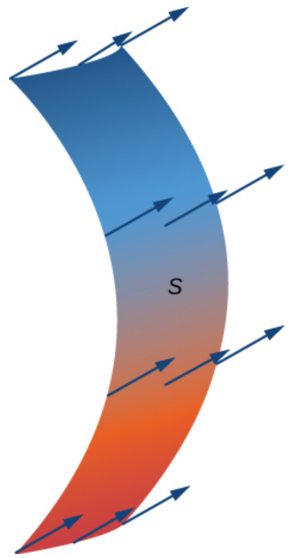


Figure $9.6.21$: Fluid flows across a completely permeable surface S.

The rate of flow, measured in mass per unit time per unit area, is $\rho\vec{\mathbf{N}}$. To calculate the mass flux across S, chop S into small pieces S_{ij}. If S_{ij} is small enough, then it can be approximated by a tangent plane at some point P in S_{ij}. Therefore, the unit normal vector at P can be used to approximate $\vec{\mathbf{N}}(x, y, z)$ across the entire piece S_{ij} because the normal vector to a plane does not change as we move across the plane. The component of the vector ρv at P in the direction of $\vec{\mathbf{N}}$ is $\rho\vec{\mathbf{v}} \cdot \vec{\mathbf{N}}$ at P. Since S_{ij} is small, the dot product $\rho v \cdot N$ changes very little as we vary across S_{ij} and therefore $\rho\vec{\mathbf{v}} \cdot \vec{\mathbf{N}}$ can be taken as approximately constant across S_{ij}. To approximate the mass of fluid per unit time flowing across S_{ij} (and not just locally at point P), we need to multiply $(\rho\vec{\mathbf{v}} \cdot \vec{\mathbf{N}})(P)$ by the area of S_{ij}. Therefore, the mass of fluid per unit time flowing across S_{ij} in the direction of $\vec{\mathbf{N}}$ can be approximated by $(\rho\vec{\mathbf{v}} \cdot \vec{\mathbf{N}})\Delta S_{ij}$ where $\vec{\mathbf{N}}$, ρ and $\vec{\mathbf{v}}$ are all evaluated at P (Figure $9.6.22$). This is analogous to the flux of two-dimensional vector field $\vec{\mathbf{F}}$ across plane curve C, in which we approximated flux across a small piece of C with the expression $(\vec{\mathbf{F}} \cdot \vec{\mathbf{N}})\,\Delta s$. To approximate the mass flux across S, form the sum

$$\sum_{i=1} m \sum_{j=1}^{n} (\rho\vec{\mathbf{v}} \cdot \vec{\mathbf{N}})\Delta S_{ij}.$$

As pieces S_{ij} get smaller, the sum

$$\sum_{i=1} m \sum_{j=1}^{n} (\rho\vec{\mathbf{v}} \cdot \vec{\mathbf{N}})\Delta S_{ij}$$

gets arbitrarily close to the mass flux. Therefore, the mass flux is

$$\iint_s \rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}}\,dS = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}. \tag{9.6.41}$$

This is a surface integral of a vector field. Letting the vector field $\rho\vec{\mathbf{v}}$ be an arbitrary vector field $\vec{\mathbf{F}}$ leads to the following definition.

A diagram in three dimensions of a surface S. A small section S_ij is labeled. Coming out of this section are two vectors, labeled N and F = v. The latter points in the same direction as several other arrows with positive z and y components but negative x components.

Figure 9.6.22: The mass of fluid per unit time flowing across S_{ij} in the direction of $\vec{\mathbf{N}}$ can be approximated by $(\rho\vec{\mathbf{v}}\cdot\vec{\mathbf{N}})\Delta S_{ij}$.

Definition: Surface Integrals

Let $\vec{\mathbf{F}}$ be a continuous vector field with a domain that contains oriented surface S with unit normal vector $\vec{\mathbf{N}}$. The surface integral of $\vec{\mathbf{F}}$ over S is

$$\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS. \tag{9.6.42}$$

Notice the parallel between this definition and the definition of vector line integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$. A surface integral of a vector field is defined in a similar way to a flux line integral across a curve, except the domain of integration is a surface (a two-dimensional object) rather than a curve (a one-dimensional object). Integral $\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$ is called the *flux of* $\vec{\mathbf{F}}$ *across* S, just as integral $\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$ is the flux of $\vec{\mathbf{F}}$ across curve C. A surface integral over a vector field is also called a **flux integral**.

Just as with vector line integrals, surface integral $\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS$ is easier to compute after surface S has been parameterized. Let $\vec{\mathbf{r}}(u,v)$ be a parameterization of S with parameter domain D. Then, the unit normal vector is given by $\vec{\mathbf{N}} = \dfrac{\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||}$ and, from Equation 9.6.42, we have

$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS &= \iint_S \vec{\mathbf{F}}\cdot\frac{\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||}\,dS \\ &= \iint_D \left(\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot\frac{\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v}{||\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||}\right)\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v||\,dA \\ &= \iint_D (\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v))\,dA.\end{aligned}$$

Therefore, to compute a surface integral over a vector field we can use the equation

$$\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS = \iint_D (\vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v))\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v))\,dA. \tag{9.6.43}$$

Example 9.6.13: Calculating a Surface Integral

Calculate the surface integral

$$\iint_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,dS, \tag{9.6.44}$$

where $\vec{\mathbf{F}} = \langle -y, x, 0\rangle$ and S is the surface with parameterization $\vec{\mathbf{r}}(u,v) = \langle u, v^2-u,\, u+v\rangle,\ 0\le u\le 3,\ 0\le v\le 4$.

Solution

The tangent vectors are $\vec{\mathbf{t}}_u = \langle 1,-1,1\rangle$ and $\vec{\mathbf{t}}_v = \langle 0,2v,1\rangle$. Therefore,

$$\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v = \langle -1-2v, -1, 2v\rangle. \tag{9.6.45}$$

By Equation 9.6.43,

$$\begin{aligned}\iint_S \vec{\mathbf{F}} \cdot dS &= \int_0^4 \int_0^3 F(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\,du\,dv \\ &= \int_0^4 \int_0^3 \langle u - v^2,\, u,\, 0\rangle \cdot \langle -1-2v,\, -1,\, 2v\rangle\,du\,dv \\ &= \int_0^4 \int_0^3 [(u-v^2)(-1-2v) - u]\,du\,dv \\ &= \int_0^4 \int_0^3 (2v^3 + v^2 - 2uv - 2u)\,du\,dv \\ &= \int_0^4 \left[2v^3u + v^2u - vu^2 - u^2\right]\Big|_0^3\,dv \\ &= \int_0^4 (6v^3 + 3v^2 - 9v - 9)\,dv \\ &= \left[\frac{3v^4}{2} + v^3 - \frac{9v^2}{2} - 9v\right]_0^4 \\ &= 340.\end{aligned}$$

Therefore, the flux of $\vec{\mathbf{F}}$ across S is 340.

Exercise 9.6.12

Calculate surface integral

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS,$$

where $\vec{\mathbf{F}} = \langle 0, -z, y\rangle$ and S is the portion of the unit sphere in the first octant with outward orientation.

Hint

Use Equation 9.6.43

Answer

0

Example 9.6.14:Calculating Mass Flow Rate

Let $\vec{\mathbf{v}}(x,y,z) = \langle 2x,\, 2y,\, z\rangle$ represent a velocity field (with units of meters per second) of a fluid with constant density 80 kg/m^3. Let S be hemisphere $x^2 + y^2 + z^2 = 9$ with $z \le 0$ such that S is oriented outward. Find the mass flow rate of the fluid across S.

Solution

A parameterization of the surface is

$$\vec{\mathbf{r}}(\phi,\theta) = \langle 3\,\cos\theta\,\sin\phi,\, 3\,\sin\theta\,\sin\phi,\, 3\,\cos\phi\rangle,\, 0 \le \theta \le 2\pi,\, 0 \le \phi \le \pi/2. \tag{9.6.46}$$

As in Example, the tangent vectors are $\vec{\mathbf{t}}_\theta = \langle -3\,\sin\theta\,\sin\phi,\, 3\,\cos\theta\,\sin\phi,\, 0\rangle$ and $\vec{\mathbf{t}}_\phi = \langle 3\,\cos\theta\,\cos\phi,\, 3\,\sin\theta\,\cos\phi,\, -3\,\sin\phi\rangle$, and their cross product is

$$\vec{\mathbf{t}}_\phi \times \vec{\mathbf{t}}_\theta = \langle 9\,\cos\theta\,\sin^2\phi,\, 9\,\sin\theta\,\sin^2\phi,\, 9\,\sin\phi\,\cos\phi\rangle. \tag{9.6.47}$$

Notice that each component of the cross product is positive, and therefore this vector gives the outward orientation. Therefore we use the orientation

$$\vec{\mathbf{N}} = \langle 9\,\cos\theta\,\sin^2\phi,\, 9\,\sin\theta\,\sin^2\phi,\, 9\,\sin\phi\,\cos\phi\rangle$$

for the sphere.

By Equation,

$$\begin{aligned} \iint_S \rho v \cdot\, dS &= 80\int_0^{2\pi}\int_0^{\pi/2} v(r(\phi,\theta))\cdot(t_\phi\times t_\theta)\,d\phi\,d\theta \\ &= 80\int_0^{2\pi}\int_0^{\pi/2} \langle 6\,\cos\theta\,\sin\phi,\, 6\,\sin\theta\,\sin\phi,\, 3\,\cos\phi\rangle\cdot\langle 9\,\cos\theta\,\sin^2\phi,\, 9\,\sin\theta\,\sin^2\phi,\, 9\,\sin\phi\,\cos\phi\rangle\,d\phi\,d\theta \\ &= 80\int_0^{2\pi}\int_0^{\pi/2} 54\,\sin^3\phi + 27\,\cos^2\phi\,\sin\phi\,d\phi\,d\theta \\ &= 80\int_0^{2\pi}\int_0^{\pi/2} 54(1-\cos^2\phi)\,\sin\phi + 27\cos^2\phi\,\sin\phi\,d\phi\,d\theta \\ &= 80\int_0^{2\pi}\int_0^{\pi/2} 54\,\sin\phi - 27\,\cos^2\phi\,\sin\phi\,d\phi\,d\theta \\ &= 80\int_0^{2\pi}\Big[-54\,\cos\phi + 9\,\cos^3\phi\Big]_{\phi=0}^{\phi=2\pi}\,d\theta \\ &= 80\int_0^{2\pi} 45\,d\theta \\ &= 7200\pi. \end{aligned}$$

Therefore, the mass flow rate is $7200\pi\ \mathrm{kg/sec/m^2}$.

Exercise 9.6.13

Let $\vec{\mathbf{v}}(x,y,z) = \langle x^2+y^2,\, z,\, 4y\rangle$ m/sec represent a velocity field of a fluid with constant density 100 kg/m^3. Let S be the half-cylinder $\vec{\mathbf{r}}(u,v) = \langle\cos u,\, \sin u,\, v\rangle$, $0\le u\le\pi$, $0\le v\le 2$ oriented outward. Calculate the mass flux of the fluid across S.

Hint

Use Equation.

Answer

400 kg/sec/m

In Example 9.6.14, we computed the mass flux, which is the rate of mass flow per unit area. If we want to find the flow rate (measured in volume per time) instead, we can use flux integral

$$\iint_S \vec{\mathbf{v}}\cdot\vec{\mathbf{N}}\,dS, \tag{9.6.48}$$

which leaves out the density. Since the flow rate of a fluid is measured in volume per unit time, flow rate does not take mass into account. Therefore, we have the following characterization of the flow rate of a fluid with velocity $\vec{\mathbf{v}}$ across a surface S:

Flow rate of fluid across

$$S = \iint_S \vec{\mathbf{v}}\cdot dS. \tag{9.6.49}$$

To compute the flow rate of the fluid in Example, we simply remove the density constant, which gives a flow rate of $90\pi\ m^3/sec$.

Both mass flux and flow rate are important in physics and engineering. Mass flux measures how much mass is flowing across a surface; flow rate measures how much volume of fluid is flowing across a surface.

In addition to modeling fluid flow, surface integrals can be used to model heat flow. Suppose that the temperature at point (x,y,z) in an object is $T(x,y,z)$. Then the heat flow is a vector field proportional to the negative temperature gradient in the object. To be precise, the heat flow is defined as vector field $F = -k\nabla T$, where the constant k is the *thermal conductivity* of the substance from which the object is made (this constant is determined experimentally). The rate of heat flow across surface S in the object is given by the flux integral

$$\iint_S \vec{\mathbf{F}} \cdot dS = \iint_S -k\vec{\nabla}T \cdot dS. \tag{9.6.50}$$

Example 9.6.15: Calculating Heat Flow

A cast-iron solid cylinder is given by inequalities $x^2+y^2 \le 1,\ 1 \le z \le 4$. The temperature at point (x,y,z) in a region containing the cylinder is $T(x,y,z) = (x^2+y^2)z$. Given that the thermal conductivity of cast iron is 55, find the heat flow across the boundary of the solid if this boundary is oriented outward.

Solution

Let S denote the boundary of the object. To find the heat flow, we need to calculate flux integral

$$\iint_S -k\vec{\nabla}T \cdot dS. \tag{9.6.51}$$

Notice that S is not a smooth surface but is piecewise smooth, since S is the union of three smooth surfaces (the circular top and bottom, and the cylindrical side). Therefore, we calculate three separate integrals, one for each smooth piece of S. Before calculating any integrals, note that the gradient of the temperature is $\vec{\nabla}T = \langle 2xz,\ 2yz,\ x^2+y^2\rangle$.

First we consider the circular bottom of the object, which we denote S_1. We can see that S_1 is a circle of radius 1 centered at point $(0,0,1)$ sitting in plane $z=1$. This surface has parameterization $\vec{\mathbf{r}}(u,v) = \langle v\cos u,\ v\sin u,\ 1\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le 1$.

Therefore,

$\vec{\mathbf{t}}_u = \langle -v\sin u,\ v\cos u,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle \cos u,\ v\sin u,\ 0\rangle$, and $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle 0,\ 0,\ -v\sin^2 u - v\cos^2 u\rangle = \langle 0,\ 0,\ -v\rangle$.

Since the surface is oriented outward and S_1 is the bottom of the object, it makes sense that this vector points downward. By Equation, the heat flow across S_1 is

$$\begin{aligned}
\iint_{S_1} -k\vec{\nabla}T \cdot dS &= -55\int_0^{2\pi}\int_0^1 \vec{\nabla}T(u,v)\cdot(\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\,dv\,du \\
&= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos u,\ 2v\sin u,\ v^2\cos^2 u + v^2\sin^2 u\rangle \cdot \langle 0,0,-v\rangle\,dv\,du \\
&= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos u,\ 2v\sin u,\ v^2\rangle \cdot \langle 0,0,-v\rangle\,dv\,du \\
&= -55\int_0^{2\pi}\int_0^1 -v^3\,dv\,du \\
&= -55\int_0^{2\pi} -\frac{1}{4}du \\
&= \frac{55\pi}{2}.
\end{aligned}$$

Now let's consider the circular top of the object, which we denote S_2. We see that S_2 is a circle of radius 1 centered at point $(0,0,4)$, sitting in plane $z=4$. This surface has parameterization $\vec{\mathbf{r}}(u,v) = \langle v\cos u,\ v\sin u,\ 4\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le 1$.

Therefore, $\vec{\mathbf{t}}_u = \langle -v\sin u,\ v\cos u,\ 0\rangle$ and $\vec{\mathbf{t}}_v = \langle \cos u,\ v\sin u,\ 0\rangle$, and $\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v = \langle 0,\ 0,\ -v\sin^2 u - v\cos^2 u\rangle = \langle 0,0,-v\rangle$.

Since the surface is oriented outward and S_1 is the top of the object, we instead take vector $\vec{\mathbf{t}}_v \times \vec{\mathbf{t}}_u = \langle 0,0,v\rangle$. By Equation, the heat flow across S_1 is

$$\begin{aligned}\iint_{S_2} -k\vec{\nabla} T\cdot dS &= -55\int_0^{2\pi}\int_0^1 \vec{\nabla} T(u,v)\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v)\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 8v\cos u,\, 8v\sin u,\, v^2\cos^2 u+v^2\sin^2 u\rangle\cdot\langle 0,0,-v\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 8v\cos u,\, 8v\sin u,\, v^2\rangle\cdot\langle 0,0,-v\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 -v^3\,dv\,du = -55\int_0^{2\pi} -\frac{1}{4}\,du = -\frac{55\pi}{2}.\end{aligned}$$

Last, let's consider the cylindrical side of the object. This surface has parameterization $\vec{\mathbf{r}}(u,v)=\langle\cos u,\,\sin u,\,v\rangle,\ 0\le u<2\pi,\ 1\le v\le 4$. By Example, we know that $\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v=\langle\cos u,\,\sin u,\,0\rangle$. By Equation,

$$\begin{aligned}\iint_{S_3} -k\vec{\nabla} T\cdot dS &= -55\int_0^{2\pi}\int_1^4 \vec{\nabla} T(u,v)\cdot(\vec{\mathbf{t}}_u\times\vec{\mathbf{t}}_v)\,dv\,du\\ &= -55\int_0^{2\pi}\int_1^4 \langle 2v\cos u,\, 2v\sin u,\, \cos^2 u+\sin^2 u\rangle\cdot\langle\cos u,\,\sin u,\,0\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos^2 u,\, 2v\sin u,\, 1\rangle\cdot\langle\cos u,\,\sin u,\,0\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 (2v\cos^2 u+2v\sin^2 u)\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 2v\,dv\,du\\ &= -55\int_0^{2\pi} du\\ &= -110\pi.\end{aligned}$$

Therefore, the rate of heat flow across S is

$$\frac{55\pi}{2}-\frac{55\pi}{2}-110\pi=-110\pi.$$

Exercise 9.6.14

A cast-iron solid ball is given by inequality $x^2+y^2+z^2\le 1$. The temperature at a point in a region containing the ball is $T(x,y,z)=\frac{1}{3}(x^2+y^2+z^2)$. Find the heat flow across the boundary of the solid if this boundary is oriented outward.

Hint

Follow the steps of Example 9.6.15

Answer

$-\frac{440\pi}{3}$

9.6.6 Key Concepts

- Surfaces can be parameterized, just as curves can be parameterized. In general, surfaces must be parameterized with two parameters.
- Surfaces can sometimes be oriented, just as curves can be oriented. Some surfaces, such as a Möbius strip, cannot be oriented.
- A surface integral is like a line integral in one higher dimension. The domain of integration of a surface integral is a surface in a plane or space, rather than a curve in a plane or space.

- The integrand of a surface integral can be a scalar function or a vector field. To calculate a surface integral with an integrand that is a function, use Equation. To calculate a surface integral with an integrand that is a vector field, use Equation.
- If S is a surface, then the area of S is

$$\iint_S dS. \tag{9.6.52}$$

9.6.7 Key Equations

- **Scalar surface integral**

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{\mathbf{r}}(u,v))||\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v||\,dA$$

- **Flux integral**

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS = \iint_S \vec{\mathbf{F}} \cdot dS = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{t}}_u \times \vec{\mathbf{t}}_v)\,dA$$

9.6.7.0.1 Glossary

flux integral
another name for a surface integral of a vector field; the preferred term in physics and engineering

grid curves
curves on a surface that are parallel to grid lines in a coordinate plane

heat flow
a vector field proportional to the negative temperature gradient in an object

mass flux
the rate of mass flow of a fluid per unit area, measured in mass per unit time per unit area

orientation of a surface
if a surface has an “inner” side and an “outer” side, then an orientation is a choice of the inner or the outer side; the surface could also have “upward” and “downward” orientations

parameter domain (parameter space)
the region of the uv-plane over which the parameters u and v vary for parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$

parameterized surface (parametric surface)
a surface given by a description of the form $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$, where the parameters u and v vary over a parameter domain in the uv-plane

regular parameterization
parameterization $\vec{\mathbf{r}}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$ such that $r_u \times r_v$ is not zero for point (u,v) in the parameter domain

surface area
the area of surface S given by the surface integral

$$\iint_S dS \tag{9.6.53}$$

surface integral
an integral of a function over a surface

surface integral of a scalar-valued function
a surface integral in which the integrand is a scalar function

surface integral of a vector field

a surface integral in which the integrand is a vector field

9.6.8 Contributors and Attributions

9.6E: Exercises

9.6E.1 Exercise $9.6E.1$: True or false.

For the following exercises, determine whether the statements are *true or false*.

1. If surface S is given by $\{(x,y,z):\ 0\leq x\leq 1,\ 0\leq y\leq 1,\ z=10\}$, then

$$\iint_S f(x,y,z)\,dS=\int_0^1\int_0^1 f(x,y,10)\,dxdy. \tag{9.6E.1}$$

Answer

True

2. If surface S is given by $\{(x,y,z):\ 0\leq x\leq 1,\ 0\leq y\leq 1,\ z=x\}$, then

$$\iint_S f(x,y,z)\,dS=\int_0^1\int_0^1 f(x,y,x)\,dxdy. \tag{9.6E.2}$$

3. Surface $r=\langle v\cos u,\ v\sin u,\ v^2\rangle,\ for\ 0\leq u\leq \pi,\ 0\leq v\leq 2$ is the same surface $r=\langle \sqrt{v}\cos 2u,\ \sqrt{v}\sin 2u,\ v\rangle,\ for\ 0\leq u\leq \frac{\pi}{2},\ 0\leq v\leq 4$.

Answer

True

Exercise $9.6E.2$

Given the standard parameterization of a sphere, normal vectors $t_u\times t_v$ are outward normal vectors. For the following exercises, find parametric descriptions for the following surfaces.

1. Plane $3x-2y+z=2$

Answer

$r(u,v)=\langle u,\ v,\ 2-3u+2v\rangle$ for $-\infty\leq u<\infty$ and $-\infty\leq v<\infty$.

2. Paraboloid $z=x^2+y^2$, for $0\leq z\leq 9$.

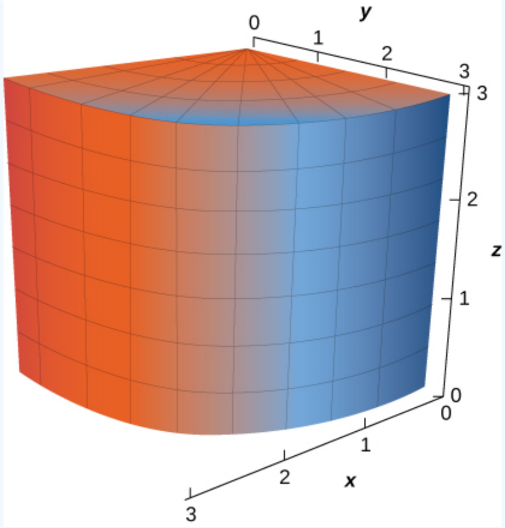
3. Plane $2x-4y+3z=16$

Answer

$r(u,v)=\langle u,\ v,\ \frac{1}{3}(16-2u+4v)\rangle$ for $|u|<\infty$ and $|v|<\infty$.

4. The frustum of cone $z^2=x^2+y^2$, for $2\leq z\leq 8$

5. The portion of cylinder $x^2+y^2=9$ in the first octant, for $0\leq z\leq 3$



Answer

$r(u,v) = \langle 3\cos u,\ 3\sin u,\ v\rangle$ for $0 \le u \le \frac{\pi}{2},\ 0 \le v \le 3$

6. A cone with base radius r and height h, where r and h are positive constants

Answer

TBA

9.6E.2 Exercise $\mathbf{9.6E.3}$: Surface Area

For the following exercises, use a computer algebra system to approximate the area of the following surfaces using a parametric description of the surface.

1. [T] Half cylinder $\{(r,\theta,z):\ r=4,\ 0\le\theta\le\pi,\ 0\le z\le 7\}$

Answer

$A = 87.9646$

2. [T] Plane $z = 10 - z - y$ above square $|x| \le 2,\ |y| \le 2$

9.6E.3 Exercise $\mathbf{9.6E.4}$

For the following exercises, let S be the hemisphere $x^2+y^2+z^2=4$, with $z\ge 0$, and evaluate each surface integral, in the counterclockwise direction.

1.

$$\iint_S z\,dS \tag{9.6E.3}$$

Answer

$$\iint_S z\,dS = 8\pi \tag{9.6E.4}$$

2.

$$\iint_S (x-2y)\,dS \tag{9.6E.5}$$

3.

$$\iint_S (x^2+y^2)\,dS \tag{9.6E.6}$$

Answer

$$\iint_S z\,dS = 8\pi \tag{9.6E.7}$$

9.6E.4 Exercise $\mathbf{9.6E.5}$

For the following exercises, evaluate

$$\int\int_S \vec{\mathbf{F}}\cdot\vec{\mathbf{N}}\,ds \tag{9.6E.8}$$

for vector field $\vec{\mathbf{F}}$, where $\vec{\mathbf{N}}$ is an outward normal vector to surface *S*.

1. $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} + 3z\,\hat{\mathbf{k}}$, and S is that part of plane $15x-12y+3z=6$ that lies above unit square $0\le x\le 1,\ 0\le y\le 1$.
2. $\vec{\mathbf{F}}(x,y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, and S is hemisphere $z=\sqrt{1-x^2-y^2}$.

Answer

$$\iint_S F\cdot N\,dS = \frac{4\pi}{3} \tag{9.6E.9}$$

3. $F(x,y,z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$, and S is the portion of plane $z=y+1$ that lies inside cylinder $x^2+y^2=1$.

A cylinder and an intersecting plane shown in three-dimensions. S is the portion of the plane z = y + 1 inside the cylinder x^2 + y ^2 = 1.

9.6E.5 Exercise $\mathbf{9.6E.6}$

For the following exercises, approximate the mass of the homogeneous lamina that has the shape of given surface *S*. Round to four decimal places.

1. [T] S is surface $z=4-x-2y$, with $z\ge 0,\ x\ge 0,\ y\ge 0; \xi = x$.

Answer

$m \approx 13.0639$

2. [T] S is surface $z=x^2+y^2$, with $z\le 1;\ \xi = z$.
3. [T] S is surface $x^2+y^2+x^2=5$, with $z\ge 1;\ \xi=\theta^2$.

Answer

$m \approx 228.5313$

Exercise $9.6E.6$

1. Evaluate

$$\iint_S (y^2z\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}})\cdot dS, \tag{9.6E.10}$$

where S is the surface of cube $-1\le x\le 1,\ -1\le y\le 1$, and $0\le z\le 2$ in a counterclockwise direction.

2. Evaluate surface integral

$$\iint_S g dS, \tag{9.6E.11}$$

where $g(x, y, z) = xz + 2x^2 - 3xy$ and S is the portion of plane $2x - 3y + z = 6$ that lies over unit square R: $0 \leq x \leq 1,\ 0 \leq y \leq 1$.

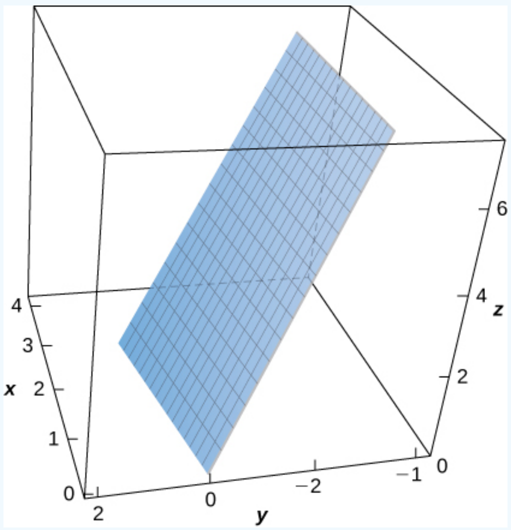
Answer

$$\iint_S g dS = 3\sqrt{4} \tag{9.6E.12}$$

3. Evaluate

$$\iint_S (x + y + z) dS, \tag{9.6E.13}$$

where S is the surface defined parametrically by $R(u, v) = (2u + v)\,\hat{\mathbf{i}} + (u - 2v)\,\hat{\mathbf{j}} + (u + 3v)\,\hat{\mathbf{k}}$ for $0 \leq u \leq 1,$ and $0 \leq v \leq 2$.



Answer

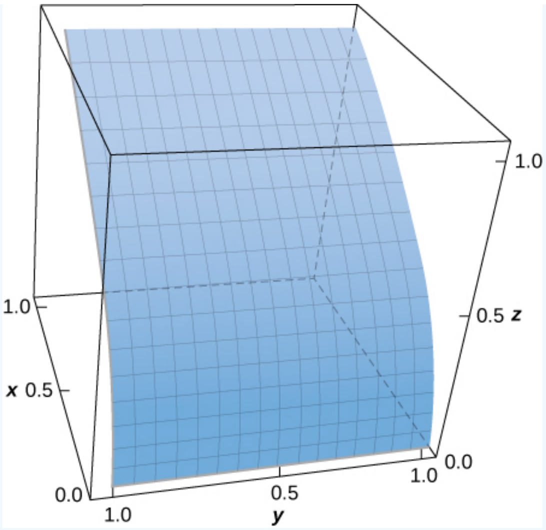
TBA

Exercise $9.6E.7$

1. [T] Evaluate

$$\iint_S (x - y^2 + z) dS, \tag{9.6E.14}$$

where S is the surface defined parametrically by $R(u, v) = u^2\,\hat{\mathbf{i}} + v\,\hat{\mathbf{j}} + u\,\hat{\mathbf{k}}$ for $0 \leq u \leq 1,\ 0 \leq v \leq 1$.



Answer

$$\iint_S (x^2 + y - z)\, dS \approx 0.9617 \tag{9.6E.15}$$

2. [T] Evaluate where S is the surface defined by $R(u,v) = u\,\hat{\mathbf{i}} - u^2\,\hat{\mathbf{j}} + v\,\hat{\mathbf{k}},\ 0 \le u \le 2,\ 0 \le v \le 1$ for $0 \le u \le 1,\ 0 \le v \le 2$.

3. Evaluate

$$\iint_S (x^2 + y^2)\, dS, \tag{9.6E.16}$$

where S is the surface bounded above hemisphere $z = \sqrt{1 - x^2 - y^2}$, and below by plane $z = 0$.

Answer

$$\iint_S (x^2 + y^2)\, dS = \frac{4\pi}{3} \tag{9.6E.17}$$

4. Evaluate

$$\iint_S (x^2 + y^2 + z^2)\, dS, \tag{9.6E.18}$$

where S is the portion of plane that lies inside cylinder $x^2 + y^2 = 1$.

5. [T] Evaluate

$$\iint_S x^2 z dS, \tag{9.6E.19}$$

where S is the portion of cone $z^2 = x^2 + y^2$ that lies between planes $z = 1$ and $z = 4$.

A diagram of the given upward opening cone in three dimensions. The cone is cut by planes z=1 and z=4.

Answer

$$\iint_S x^2 z dS = \frac{1023\sqrt{2}\pi}{5} \tag{9.6E.20}$$

6. [T] Evaluate

$$\iint_S (xz/y)\, dS, \tag{9.6E.21}$$

where S is the portion of cylinder $x = y^2$ that lies in the first octant between planes $z = 0,\ z = 5$, and $y = 4$.

A diagram of the given cylinder in three-dimensions. It is cut by the planes z=0, z=5, y=1, and y=4.

7. [T] Evaluate

$$\iint_S (z + y)\, dS, \tag{9.6E.22}$$

where S is the part of the graph of $z = \sqrt{1 - x^2}$ in the first octant between the *xz*-plane and plane $y = 3$.

A diagram of the given surface in three dimensions in the first octant between the xz-plane and plane y=3. The given graph of z= the square root of (1-x^2) stretches down in a concave down curve from along (0,y,1) to along (1,y,0). It looks like a portion of a horizontal cylinder with base along the xz-plane and height along the y axis.

Answer

$$\iint_S (z + y)\, dS \approx 10.1 \tag{9.6E.23}$$

Exercise $9.6E.8$

1. Evaluate

$$\iint_S xyz\, dS \tag{9.6E.24}$$

if S is the part of plane $z = x + y$ that lies over the triangular region in the *xy*-plane with vertices (0, 0, 0), (1, 0, 0), and (0, 2, 0).

2. Find the mass of a lamina of density $\xi(x, y, z) = z$ in the shape of hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$.

Answer

$m = \pi a^3$

3. Compute

$$\int\int_S F \cdot N\, dS, \tag{9.6E.25}$$

where \(F(x,y,z) = x \, \hat{ \mathbf i} - 5y \hat{j}+ 4z\, \hat{ \mathbf k}\) and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the union of two squares $S_1 : x = 0,\ 0 \le y \le 1,\ 0 \le z \le 1$ and $S_2\ :\ x = 0,\ 0 \le x \le 1,\ 0 \le y \le 1$.

A diagram in three dimensions. It shows the square formed by the components x=0, 0 <= y <= 1, and 0 <= z <= 1. It also shows the square formed by the components z=1, 0 <= x <= 1, and 0 <= y <= 1.

4. Compute

$$\int\int_S F \cdot N\, dS, \tag{9.6E.26}$$

where $F(x, y, z) = xy\,\hat{\mathbf{i}} + z\,\hat{\mathbf{j}} + (x + y)\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the triangular region cut off from plane $x + y + z = 1$ by the positive coordinate axes.

Answer

$$\iint_S F \cdot N\, dS = \frac{13}{24} \tag{9.6E.27}$$

Exercise $9.6E.9$

Compute

$$\int\int_S F \cdot N\, dS, \tag{9.6E.28}$$

where $F(x,y,z) = 2yz\,\hat{\mathbf{i}} + (tan^{-1}xz)\,\hat{\mathbf{j}} + e^{xy}\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the surface of sphere $x^2 + y^2 + z^2 = 1$.

Compute

$$\int\int_S F \cdot N\,dS, \tag{9.6E.29}$$

where $F(x,y,z) = xyz\,\hat{\mathbf{i}} + xyz\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$ and $\vec{\mathbf{N}}$ is an outward normal vector S, where S is the surface of the five faces of the unit cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$ missing $z = 0$.

[Hide Solution]

$$\iint_S F \cdot N\,dS = \frac{3}{4} \tag{9.6E.30}$$

For the following exercises, express the surface integral as an iterated double integral by using a projection on S on the yz-plane.

$$\iint_S xy^2z^3\,dS; \tag{9.6E.31}$$

S is the first-octant portion of plane $2x + 3y + 4z = 12$.

$$\iint_S (x^2 - 2y + z\,dS; \tag{9.6E.32}$$

S is the portion of the graph of $4x + y = 8$ bounded by the coordinate planes and plane $z = 6$.

[Hide Solution]

$$\int_0^8 \int_0^6 \left(4 - 3y + \frac{1}{16}y^2 + z\right)\left(\frac{1}{4}\sqrt{17}\right)\,dzdy \tag{9.6E.33}$$

Exercise $9.6E.10$

For the following exercises, express the surface integral as an iterated double integral by using a projection on S on the xz-plane

$$\iint_S xy^2z^3\,dS; \tag{9.6E.34}$$

S is the first-octant portion of plane $2x + 3y + 4z = 12$.

$$\iint_S (x^2 - 2y + z)\,dS; \tag{9.6E.35}$$

is the portion of the graph of $4x + y = 8$ bounded by the coordinate planes and plane $z = 6$.

[Hide Solution]

$$\int_0^2 \int_0^6 [x^2 - 2(8 - 4x) + z]\sqrt{17}\,dzdx \tag{9.6E.36}$$

Answer

Add texts here. Do not delete this text first.

Exercise $9.6E.11$

Evaluate surface integral

$$\iint_S yz\,dS, \tag{9.6E.37}$$

where S is the first-octant part of plane $x+y+z=\lambda$, where λ is a positive constant.

Evaluate surface integral

$$\iint_S (x^2z+y^2z)\,dS, \tag{9.6E.38}$$

where S is hemisphere $x^2+y^2+z^2=a^2,\ z\geq 0$.

[Hide Solution]

$$\iint_S (x^2z+y^2z)\,dS = \frac{\pi a^5}{2} \tag{9.6E.39}$$

Evaluate surface integral

$$\iint_S z dA, \tag{9.6E.40}$$

where S is surface $z=\sqrt{x^2+y^2}$, $0\leq z\leq 2$.

Evaluate surface integral

$$\iint_S x^2yz\,dS, \tag{9.6E.41}$$

where S is the part of plane $z=1+2x+3y$ that lies above rectangle $0\leq x\leq 3$ and $0\leq y\leq 2$.

[Hide Solution]

$$\iint_S x^2yz\,dS = 171\sqrt{14} \tag{9.6E.42}$$

Evaluate surface integral

$$\iint_S yz\,dS, \tag{9.6E.43}$$

where S is plane $x+y+z=1$ that lies in the first octant.

Evaluate surface integral

$$\iint_S yz\,dS, \tag{9.6E.44}$$

where S is the part of plane $z=y+3$ that lies inside cylinder $x^2+y^2=1$.

[Hide Solution]

$$\iint_S yz\,dS = \frac{\sqrt{2}\pi}{4} \tag{9.6E.45}$$

Exercise $9.6E.12$

For the following exercises, use geometric reasoning to evaluate the given surface integrals.

$$\iint_S \sqrt{x^2+y^2+z^2}\ dS, \tag{9.6E.46}$$

where S is surface $x^2+y^2+z^2=4,\ z\geq 0$

$$\iint_S (xi+yj)\cdot dS, \tag{9.6E.47}$$

where S is surface $x^2+y^2=4,\ 1\leq z\leq 3$, oriented with unit normal vectors pointing outward

[Hide Solution]

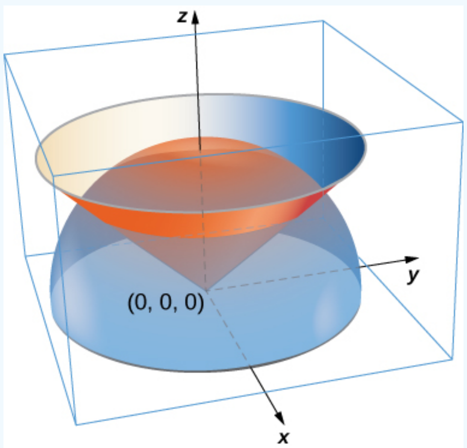
$$\iint_S (xi+yj)\cdot dS=16\pi \tag{9.6E.48}$$

$$\iint_S (zk)\cdot dS, \tag{9.6E.49}$$

where S is disc $x^2+y^2\leq 9$ on plane $z=4$ oriented with unit normal vectors pointing upward

Exercise $9.6E.13$

A lamina has the shape of a portion of sphere $x^2+y^2+z^2=a^2$ that lies within cone $z=\sqrt{x^2+y^2}$. Let S be the spherical shell centered at the origin with radius a, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the z-axis. Determine the mass of the lamina if $\delta(x,y,z)=x^2y^2z$.



[Hide Solution]

$m=\dfrac{\pi a^7}{192}$

A lamina has the shape of a portion of sphere $x^2+y^2+z^2=a^2$ that lies within cone $z=\sqrt{x^2+y^2}$. Let S be the spherical shell centered at the origin with radius a, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the z-axis. Suppose the vertex angle of the cone is ϕ_0, with $0\leq\phi_0<\dfrac{\pi}{2}$. Determine the mass of that portion of the shape enclosed in the intersection of S and C. Assume $\delta(x,y,z)=x^2y^2z$.

A diagram in three dimensions. A cone opens upward with point at the origin and an asic of symmetry that coincides with the z-axis. The upper half of a hemisphere with center at the origin opens downward and is cut off by the xy-plane.

A paper cup has the shape of an inverted right circular cone of height 6 in. and radius of top 3 in. If the cup is full of water weighing $62.5\ lb/ft^3$, find the total force exerted by the water on the inside surface of the cup.

[Hide Solution]

$F\approx 4.57\ lb$

Exercise $9.6E.14$

For the following exercises, the heat flow vector field for conducting objects $\vec{\mathbf{F}}=-k\nabla T$, where $T(x,y,z)$ is the temperature in the object and $k>0$ is a constant that depends on the material. Find the outward flux of $\vec{\mathbf{F}}$ across the following surfaces S for the given temperature distributions and assume $k=1$.

$T(x,y,z)=100e^{-x-y}$; S consists of the faces of cube $|x|\leq 1,\ |y|\leq 1,\ |z|\leq 1$.

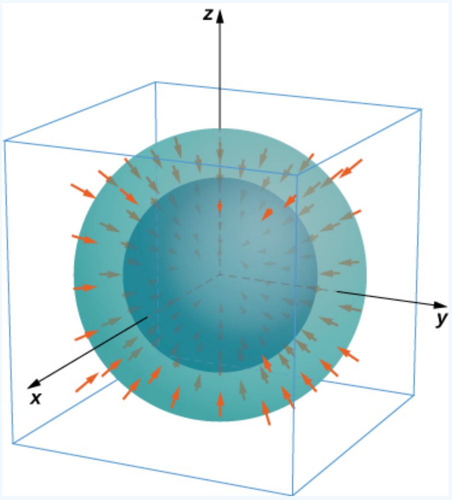
$T(x,y,z)=-\ln(x^2+y^2+z^2)$; S is sphere $x^2+y^2+z^2=a^2$.

[Hide Solution]

$8\pi a$

Exercise $9.6E.15$

For the following exercises, consider the radial fields $F = \dfrac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{\frac{p}{2}}} = \dfrac{r}{|r|^p}$, where p is a real number. Let S consist of spheres A and B centered at the origin with radii $0 < a < b$. The total outward flux across S consists of the outward flux across the outer sphere B less the flux into S across inner sphere A.



Find the total flux across S with $p = 0$.

Show that for $p = 3$ the flux across S is independent of a and b.

Answer

The net flux is zero.

9.7: Stoke's Theorem

In this section, we study Stokes' theorem, a higher-dimensional generalization of Green's theorem. This theorem, like the Fundamental Theorem for Line Integrals and Green's theorem, is a generalization of the Fundamental Theorem of Calculus to higher dimensions. Stokes' theorem relates a vector surface integral over surface S in space to a line integral around the boundary of S. Therefore, just as the theorems before it, Stokes' theorem can be used to reduce an integral over a geometric object S to an integral over the boundary of S. In addition to allowing us to translate between line integrals and surface integrals, Stokes' theorem connects the concepts of curl and circulation. Furthermore, the theorem has applications in fluid mechanics and electromagnetism. We use Stokes' theorem to derive Faraday's law, an important result involving electric fields.

9.7.1 Stokes' Theorem

Stokes' theorem says we can calculate the flux of $curl\,\vec{\mathbf{F}}$ across surface S by knowing information only about the values of $\vec{\mathbf{F}}$ along the boundary of S. Conversely, we can calculate the line integral of vector field $\vec{\mathbf{F}}$ along the boundary of surface S by translating to a double integral of the curl of $\vec{\mathbf{F}}$ over S.

Let S be an oriented smooth surface with unit normal vector $\vec{\mathbf{N}}$. Furthermore, suppose the boundary of S is a simple closed curve C. The orientation of S induces the positive orientation of C if, as you walk in the positive direction around C with your head pointing in the direction of $\vec{\mathbf{N}}$, the surface is always on your left. With this definition in place, we can state *Stokes' theorem*.

Theorem 9.7.1: Stokes' Theorem

Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation (Figure 9.7.1). If $\vec{\mathbf{F}}$ is a vector field with component functions that have continuous partial derivatives on an open region containing S, then

$$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \iint_S curl\,\vec{\mathbf{F}}\cdot dS. \tag{9.7.1}$$

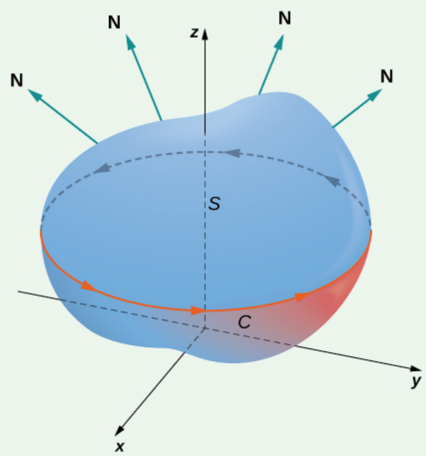


Figure 9.7.1: Stokes' theorem relates the flux integral over the surface to a line integral around the boundary of the surface. Note that the orientation of the curve is positive.

Suppose surface S is a flat region in the xy-plane with upward orientation. Then the unit normal vector is $\vec{\mathbf{k}}$ and surface integral

$$\iint_S curl\,\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} \tag{9.7.2}$$

is actually the double integral

$$\iint_S curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{k}}\,dA. \tag{9.7.3}$$

In this special case, Stokes' theorem gives

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{k}}\, dA. \tag{9.7.4}$$

However, this is the flux form of Green's theorem, which shows us that Green's theorem is a special case of Stokes' theorem. Green's theorem can only handle surfaces in a plane, but Stokes' theorem can handle surfaces in a plane or in space.

The complete proof of Stokes' theorem is beyond the scope of this text. We look at an intuitive explanation for the truth of the theorem and then see proof of the theorem in the special case that surface S is a portion of a graph of a function, and S, the boundary of S, and $\vec{\mathbf{F}}$ are all fairly tame.

Proof

First, we look at an informal proof of the theorem. This proof is not rigorous, but it is meant to give a general feeling for why the theorem is true. Let S be a surface and let D be a small piece of the surface so that D does not share any points with the boundary of S. We choose D to be small enough so that it can be approximated by an oriented square E. Let D inherit its orientation from S, and give E the same orientation. This square has four sides; denote them E_l, E_r, E_u, and E_d for the left, right, up, and down sides, respectively. On the square, we can use the flux form of Green's theorem:

$$\int_{E_l+E_d+E_r+E_u} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_E curl\, \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, d\vec{\mathbf{S}} = \iint_E curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}. \tag{9.7.5}$$

To approximate the flux over the entire surface, we add the values of the flux on the small squares approximating small pieces of the surface (Figure $9.7.2$).

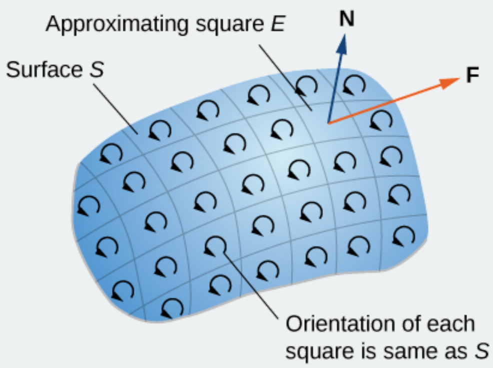


Figure $9.7.2$: Chop the surface into small pieces. The pieces should be small enough that they can be approximated by a square.

By Green's theorem, the flux across each approximating square is a line integral over its boundary. Let F be an approximating square with an orientation inherited from S and with a right side E_l (so F is to the left of E). Let F_r denote the right side of F; then, $E_l = -f_r$. In other words, the right side of F is the same curve as the left side of E, just oriented in the opposite direction. Therefore,

$$\int_{E_l} F \cdot dr = -\int_{F_r} F \cdot dr.$$

As we add up all the fluxes over all the squares approximating surface S, line integrals

$$\int_{E_l} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{9.7.6}$$

and

$$\int_{F_r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{9.7.7}$$

cancel each other out. The same goes for the line integrals over the other three sides of E. These three line integrals cancel out with the line integral of the lower side of the square above E, the line integral over the left side of the square to the right of E, and the line integral over the upper side of the square below E (Figure $9.7.3$). After all this cancelation occurs over all the approximating squares, the only line integrals that survive are the line integrals over sides approximating the boundary of S.

Therefore, the sum of all the fluxes (which, by Green's theorem, is the sum of all the line integrals around the boundaries of approximating squares) can be approximated by a line integral over the boundary of S. In the limit, as the areas of the approximating squares go to zero, this approximation gets arbitrarily close to the flux.

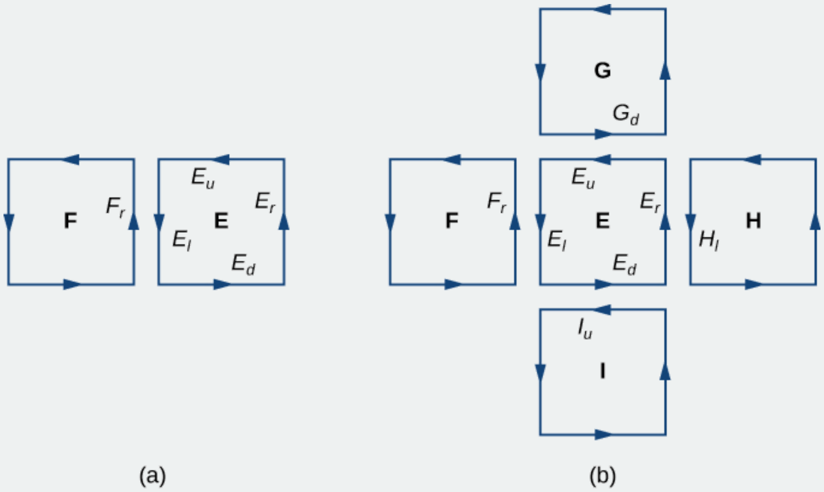


Figure 9.7.3: (a) The line integral along E_l cancels out the line integral along F_r because $E_l = -f_r$. (b) The line integral along any of the sides of E cancels out with the line integral along a side of an adjacent approximating square.

Let's now look at a rigorous proof of the theorem in the special case that S is the graph of function $z = f(x, y)$, where x and y vary over a bounded, simply connected region D of finite area (Figure 9.7.4). Furthermore, assume that f has continuous second-order partial derivatives. Let C denote the boundary of S and let C' denote the boundary of D. Then, D is the "shadow" of S in the plane and C' is the "shadow" of C. Suppose that S is oriented upward. The counterclockwise orientation of C is positive, as is the counterclockwise orientation of C'. Let $F(x, y, z) = \langle P, Q, R\rangle$ be a vector field with component functions that have continuous partial derivatives.

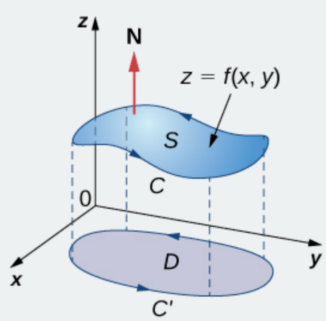


Figure 9.7.4: D is the "shadow," or projection, of S in the plane and C' is the projection of C.

We take the standard parameterization of $S: x = x,\ y = y,\ z = g(x, y)$. The tangent vectors are $t_x = \langle 1, 0, g_x\rangle$ and $t_y = \langle 0, 1, g_y\rangle$, and therefore $t_x \cdot t_y = \langle -g_x,\ -g_y,\ 1\rangle$.

$$\iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D [-(R_y - Q_z)z_x - (P_z - R_x)z_y + (Q_x - P_y)]\, dA,$$

where the partial derivatives are all evaluated at $(x, y, g(x, y))$, making the integrand depend on x and y only. Suppose $\langle x(t),\ y(t)\rangle,\ a \le t \le b$ is a parameterization of C'. Then, a parameterization of C is $\langle x(t),\ y(t),\ g(x(t),\ y(t))\rangle,\ a \le t \le b$. Armed with these parameterizations, the Chain rule, and Green's theorem, and keeping in mind that P, Q, and R are all functions of x and y, we can evaluate line integral

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_a^b (Px'(t) + Qy'(t) + Rz'(t))\, dt \\ &= \int_a^b \left[Px'(t) + Qy'(t) + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right] dt \\ &= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right) x'(t) + \left(Q + R\frac{\partial z}{\partial y}\right) y'(t)\right] dt \\ &= \int_{C'} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy \\ &= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dA \\ &= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x \partial y}\right) - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y \partial x}\right)\end{aligned}$$

By Clairaut’s theorem,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Therefore, four of the terms disappear from this double integral, and we are left with

$$\iint_D [-(R_y - Q_z)Z_x - (P_z - R_x)z_y + (Q_x - P_y)]\, dA,$$

which equals

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}.$$

□

We have shown that Stokes’ theorem is true in the case of a function with a domain that is a simply connected region of finite area. We can quickly confirm this theorem for another important case: when vector field $\vec{\mathbf{F}}$ is a conservative field. If $\vec{\mathbf{F}}$ is conservative, the curl of $\vec{\mathbf{F}}$ is zero, so

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 0. \tag{9.7.8}$$

Since the boundary of S is a closed curve, the integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}. \tag{9.7.9}$$

is also zero.

Example 9.7.1: Verifying Stokes’ Theorem for a Specific Case

Verify that Stokes’ theorem is true for vector field $\vec{\mathbf{F}}(x, y) = \langle -z, x, 0\rangle$ and surface S, where S is the hemisphere, oriented outward, with parameterization $r(\phi, \theta) = \langle \sin\phi\, \cos\theta,\ \sin\phi\, \sin\theta,\ \cos\phi\rangle,\ 0 \le \theta \le \pi,\ 0 \le \phi \le \pi$ as shown in Figure 9.7.5.

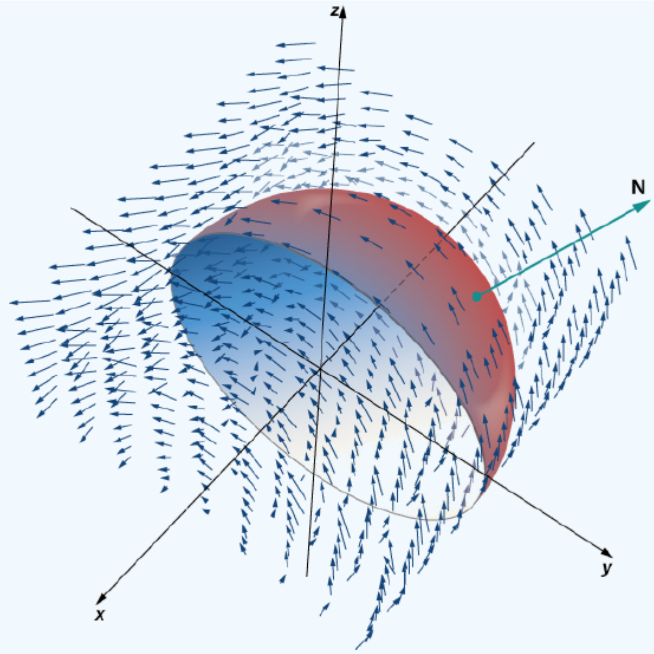


Figure 9.7.5: Verifying Stokes' theorem for a hemisphere in a vector field.

Solution

Let C be the boundary of S. Note that C is a circle of radius 1, centered at the origin, sitting in plane $y=0$. This circle has parameterization $\langle \cos t,\ 0,\ \sin t\rangle,\ 0\le t\le 2\pi$. the equation for scalar surface integrals

$$\begin{aligned}\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} &= \int_0^{2\pi} \langle -\sin t,\ \cos t,\ 0\rangle\cdot\langle -\sin t,\ 0,\ \cos t\rangle\, dt \\ &= \int_0^{2\pi} \sin^2 t\, dt \\ &= \pi.\end{aligned}$$

By the equation for vector line integrals,

$$\begin{aligned}\iint_S curl\,\vec{\mathbf{F}}\cdot dS &= \iint_D curl\,\vec{\mathbf{F}}(r(\phi,\theta))\cdot(t_\phi\times t_\theta)\, dA \\ &= \iint_D \langle 0,-1,1\rangle\cdot\langle \cos\theta\,\sin^2\phi,\ \sin\theta\,\sin^2\phi,\ \sin\phi\,\cos\phi\rangle\, dA \\ &= \int_0^\pi\int_0^\pi (\sin\phi\,\cos\phi - \sin\theta\,\sin^2\phi)\, d\phi d\theta \\ &= \frac{\pi}{2}\int_0^\pi \sin\theta\, d\theta \\ &= \pi.\end{aligned}$$

Therefore, we have verified Stokes' theorem for this example.

Exercise 9.7.1

Verify that Stokes' theorem is true for vector field $\vec{\mathbf{F}}(x,y,z)=\langle y,x,-z\rangle$ and surface S, where S is the upwardly oriented portion of the graph of $f(x,y)=x^2y$ over a triangle in the xy-plane with vertices $(0,0)$, $(2,0)$, and $(0,2)$.

Hint

Calculate the double integral and line integral separately.

Answer

Both integrals give $-\frac{136}{45}$:

9.7.2 Applying Stokes' Theorem

Stokes' theorem translates between the flux integral of surface S to a line integral around the boundary of S. Therefore, the theorem allows us to compute surface integrals or line integrals that would ordinarily be quite difficult by translating the line integral into a surface integral or vice versa. We now study some examples of each kind of translation.

Example 9.7.2: Calculating a Surface Integral

Calculate surface integral

$$\iint_S curl\,\vec{\mathbf{F}} \cdot dS,$$

where S is the surface, oriented outward, in Figure 9.7.6 and $\vec{\mathbf{F}} = \langle z,\, 2xy,\, x+y\rangle$.

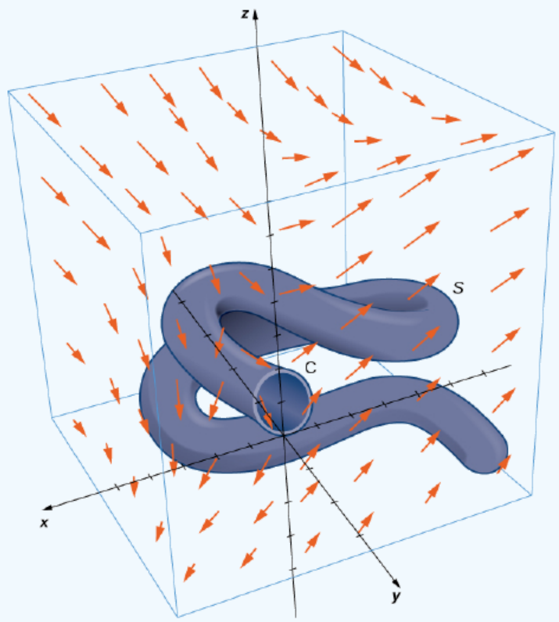


Figure 9.7.6: A complicated surface in a vector field.

Solution

Note that to calculate

$$\iint_S curl\,F \cdot dS$$

without using Stokes' theorem, we would need the equation for scalar surface integrals. Use of this equation requires a parameterization of S. Surface S is complicated enough that it would be extremely difficult to find a parameterization. Therefore, the methods we have learned in previous sections are not useful for this problem. Instead, we use Stokes' theorem, noting that the boundary C of the surface is merely a single circle with radius 1.

The curl of $\vec{\mathbf{F}}$ is $\langle 1, 1, 2y\rangle$. By Stokes' theorem,

$$\iint_S curl\,F \cdot dS = \int_C F \cdot dr,$$

where C has parameterization $\langle \cos t,\, \sin t,\, 1\rangle, 0 \le t \le 2\pi$. By the equation for vector line integrals,

$$\begin{aligned}\iint_S curl\, F\cdot dS &= \int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}\\ &= \int_0^2 \langle 1,\, \sin t\, \cos t,\, \cos t+\sin t\rangle\cdot\langle -\sin t,\, \cos t,\, 0\rangle\, dt\\ &= \int_0^{2\pi}(-\sin t+2\,\sin t\,\cos^2 t)\, dt\\ &= \left[\cos t-\frac{2\,\cos^3 t}{3}\right]_0^{2\pi}\\ &= \cos(2\pi)-\frac{2\,\cos^3(2\pi)}{3}-\left(\cos(0)-\frac{2\,\cos^3(0)}{3}\right)\\ &= 0.\end{aligned}$$

An amazing consequence of Stokes' theorem is that if *S′* is any other smooth surface with boundary *C* and the same orientation as *S*, then

$$\iint_S curl\, F\cdot dS=\int_C F\cdot dr=0 \tag{9.7.10}$$

because Stokes' theorem says the surface integral depends on the line integral around the boundary only.

In Example $9.7.2$, we calculated a surface integral simply by using information about the boundary of the surface. In general, let S_1 and S_2 be smooth surfaces with the same boundary *C* and the same orientation. By Stokes' theorem,

$$\iint_{S_1} curl\,\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}=\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}=\iint_{S_2} curl\,\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}. \tag{9.7.11}$$

Therefore, if

$$\iint_{S_1} curl\,\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} \tag{9.7.12}$$

is difficult to calculate but

$$\iint_{S_2} curl\,\vec{\mathbf{F}}\cdot dS \tag{9.7.13}$$

is easy to calculate, Stokes' theorem allows us to calculate the easier surface integral. In Example $9.7.2$, we could have calculated

$$\iint_{S} curl\,\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}} \tag{9.7.14}$$

by calculating

$$\iint_{S'} curl\,\vec{\mathbf{F}}\cdot d\vec{\mathbf{S}}, \tag{9.7.15}$$

where $\vec{\mathbf{S}}'$ is the disk enclosed by boundary curve *C* (a much more simple surface with which to work).

Equation 9.7.11 shows that flux integrals of curl vector fields are surface independent in the same way that line integrals of gradient fields are path independent. Recall that if $\vec{\mathbf{F}}$ is a two-dimensional conservative vector field defined on a simply connected domain, f is a *potential function* for $\vec{\mathbf{F}}$, and *C* is a curve in the domain of $\vec{\mathbf{F}}$, then

$$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} \tag{9.7.16}$$

depends only on the endpoints of *C*. Therefore if *C′* is any other curve with the same starting point and endpoint as *C* (that is, *C′* has the same orientation as *C*), then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \tag{9.7.17}$$

In other words, the value of the integral depends on the boundary of the path only; it does not really depend on the path itself.

Analogously, suppose that S and S' are surfaces with the same boundary and same orientation, and suppose that $\vec{\mathbf{G}}$ is a three-dimensional vector field that can be written as the curl of another vector field $\vec{\mathbf{F}}$ (so that $\vec{\mathbf{F}}$ is like a "potential field" of $\vec{\mathbf{G}}$). By Equation 9.7.11,

$$\iint_S G \cdot dS = \iint_S curl\, F \cdot dS = \int_C F \cdot dr = \iint_{S'} curl\, F \cdot dS = \iint_{S'} G \cdot dS.$$

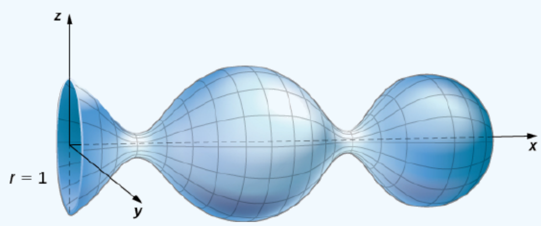
Therefore, the flux integral of $\vec{\mathbf{G}}$ does not depend on the surface, only on the boundary of the surface. Flux integrals of vector fields that can be written as the curl of a vector field are surface independent in the same way that line integrals of vector fields that can be written as the gradient of a scalar function are path independent.

Exercise 9.7.1

Use Stokes' theorem to calculate surface integral

$$\iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}, \tag{9.7.18}$$

where $\vec{\mathbf{F}} = \langle x, y, z\rangle$ and S is the surface as shown in the following figure.



Hint

Parameterize the boundary of S and translate to a line integral.

Answer

$-\pi$

Example 9.7.3: Calculating a Line Integral

Calculate the line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where $\vec{\mathbf{F}} = \langle xy,\, x^2+y^2+z^2,\, yz\rangle$ and C is the boundary of the parallelogram with vertices $(0,0,1)$, $(0,1,0)$, $(2,0,-1)$ and $(2,1,-2)$.

Solution

To calculate the line integral directly, we need to parameterize each side of the parallelogram separately, calculate four separate line integrals, and add the result. This is not overly complicated, but it is time-consuming.

By contrast, let's calculate the line integral using Stokes' theorem. Let S denote the surface of the parallelogram. Note that S is the portion of the graph of $z = 1 - x - y$ for (x, y) varying over the rectangular region with vertices $(0,0)$, $(0,1)$, $(2,0)$ and $(2,1)$ in the *xy*-plane. Therefore, a parameterization of S is $\langle x, y, 1-x-y\rangle$, $0 \le x \le 2$, $0 \le y \le 1$. The curl of $\vec{\mathbf{F}}$ is $-\langle x,\, 0,\, s\rangle$,and Stokes' theorem and the equation for scalar surface integrals

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\,\vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

$$= \int)9^2 \int_0^1 curl\,\vec{\mathbf{F}}(x,y) \cdot (t_x \cdot t_y)\,dy\,dx$$

$$= \int)9^2 \int_0^1 \langle -(1-x-y),\, 0,\, x\rangle \cdot (\langle 1,\, 0,\, -1\rangle \cdot \langle 0,\, 1,\, -1\rangle)\,dy\,dx$$

$$= \int)9^2 \int_0^1 \langle x+y-1,\, 0,\, x\rangle \cdot \langle 1,1,1\rangle\,dy\,dx$$

$$= \int)9^2 \int_0^1 2x+y-1\,dy\,dx$$

$$= 3.$$

Exercise 9.7.3

Use Stokes' theorem to calculate line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where $\vec{\mathbf{F}} = \langle z, x, y\rangle$ and C is the boundary of a triangle with vertices $(0,0,1)$, $(3,0,-2)$ and $(0,1,2)$.

Hint

This triangle lies in plane $z = 1-x+y$.

Answer

$\frac{3}{2}$

9.7.3 Interpretation of Curl

In addition to translating between line integrals and flux integrals, Stokes' theorem can be used to justify the physical interpretation of curl that we have learned. Here we investigate the relationship between curl and circulation, and we use Stokes' theorem to state Faraday's law—an important law in electricity and magnetism that relates the curl of an electric field to the rate of change of a magnetic field.

Recall that if C is a closed curve and $\vec{\mathbf{F}}$ is a vector field defined on C, then the circulation of $\vec{\mathbf{F}}$ around C is line integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}. \tag{9.7.19}$$

If $\vec{\mathbf{F}}$ represents the velocity field of a fluid in space, then the circulation measures the tendency of the fluid to move in the direction of C.

Let $\vec{\mathbf{F}}$ be a continuous vector field and let D_τ be a small disk of radius r with center \PD_0\) (Figure $9.7.7$). If D_τ is small enough, then $(curl\,\vec{\mathbf{F}})(P) \approx (curl\,F)(P_0)$ for all points P in D_τ because the curl is continuous. Let C_τ be the boundary circle of D_τ: By Stokes' theorem,

$$\int_{C_\tau} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{D_\tau} curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\,dS \approx \iint_{D_\tau} (curl\,\vec{\mathbf{F}})(P_0) \cdot \vec{\mathbf{N}}(P_0)\,dS. \tag{9.7.20}$$

A Disk D_r is a small disk in a continuous vector field in three dimensions. The radius of the disk is labeled r, and the center is labeled P_0. The arrows appear to have negative x components, slightly positive y components, and positive z components that become larger as z becomes larger.

Figure $9.7.7$: Disk D_τ is a small disk in a continuous vector field.

The quantity $(curl\,F)(P_0) \cdot N(P_0)$ is constant, and therefore

$$\iint_{D_\tau} (curl\, F)(P_0) \cdot N(P_0)\, dS = \pi r^2 [(curl\, F)(P_0) \cdot N(P_0)].$$

Thus

$$\int_{C_\tau} F \cdot dr \approx \pi r^2 [(curl\, F)(P_0) \cdot N(P_0)],$$

and the approximation gets arbitrarily close as the radius shrinks to zero. Therefore Stokes' theorem implies that

$$(curl\, F)(P_0) \cdot N(P_0) = \lim_{r \to 0^+} \frac{1}{\pi r^2} \int_{C_\tau} F \cdot dr.$$

This equation relates the curl of a vector field to the circulation. Since the area of the disk is πr^2, this equation says we can view the curl (in the limit) as the circulation per unit area. Recall that if **F** is the velocity field of a fluid, then circulation

$$\oint_{C_\tau} F \cdot dr = \oint_{C_\tau} F \cdot T\, ds \tag{9.7.21}$$

is a measure of the tendency of the fluid to move around C_τ: The reason for this is that $F \cdot T$ is a component of **F** in the direction of **T**, and the closer the direction of **F** is to **T**, the larger the value of $F \cdot T$ (remember that if **a** and **b** are vectors and **b** is fixed, then the dot product a·b is maximal when **a** points in the same direction as **b**). Therefore, if **F** is the velocity field of a fluid, then $curl\, F \cdot N$ is a measure of how the fluid rotates about axis **N**. The effect of the curl is largest about the axis that points in the direction of **N**, because in this case $curl\, F \cdot N$ is as large as possible.

To see this effect in a more concrete fashion, imagine placing a tiny paddlewheel at point P_0 (Figure $9.7.8$). The paddlewheel achieves its maximum speed when the axis of the wheel points in the direction of curl **F**. This justifies the interpretation of the curl we have learned: curl is a measure of the rotation in the vector field about the axis that points in the direction of the normal vector **N**, and Stokes' theorem justifies this interpretation.

A diagram of a tiny paddlewheel in water. A segment is stretching up out of its center, and that has an arrow labeled curl v. Red arrows are drawn to show the rotation of the wheel in a counterclockwise direction.

Figure $9.7.8$: To visualize curl at a point, imagine placing a tiny paddlewheel at that point in the vector field.

Now that we have learned about Stokes' theorem, we can discuss applications in the area of electromagnetism. In particular, we examine how we can use Stokes' theorem to translate between two equivalent forms of Faraday's law. Before stating the two forms of Faraday's law, we need some background terminology.

Let *C* be a closed curve that models a thin wire. In the context of electric fields, the wire may be moving over time, so we write $C(t)$ to represent the wire. At a given time *t*, curve $C(t)$ may be different from original curve *C* because of the movement of the wire, but we assume that $C(t)$ is a closed curve for all times *t*. Let $D(t)$ be a surface with $C(t)$ as its boundary, and orient $C(t)$ so that $D(t)$ has positive orientation. Suppose that $C(t)$is in a magnetic field $B(t)$ that can also change over time. In other words, $\vec{\mathbf{B}}$ has the form

$$B(x, y, z) = \langle P(x, y, z),\, Q(x, y, z),\, R(x, y, z) \rangle, \tag{9.7.22}$$

where *P*, *Q*, and *R* can all vary continuously over time. We can produce current along the wire by changing field $B(t)$ (this is a consequence of Ampere's law). Flux $\phi(t) = \iint_{D(t)} B(t) \cdot dS$ creates electric field $E(t)$ that does work. The integral form of Faraday's law states that

$$Work = \int_{C(t)} E(t) \cdot dr = -\frac{\partial \phi}{\partial t}. \tag{9.7.23}$$

In other words, the work done by $\vec{\mathbf{E}}$ is the line integral around the boundary, which is also equal to the rate of change of the flux with respect to time. The differential form of Faraday's law states that

$$curl\, \vec{\mathbf{E}} = -\frac{\partial B}{\partial t}. \tag{9.7.24}$$

Using Stokes' theorem, we can show that the differential form of Faraday's law is a consequence of the integral form. By Stokes' theorem, we can convert the line integral in the integral form into surface integral

$$-\frac{\partial \phi}{\partial t}=\int_{C(t)} E(t) \cdot d r=\iint_{D(t)} \operatorname{curl} E(t) \cdot d S. \tag{9.7.25}$$

Since

$$\phi(t)=\iint_{D(t)} B(t) \cdot d S, \tag{9.7.26}$$

then as long as the integration of the surface does not vary with time we also have

$$-\frac{\partial \phi}{\partial t}=\iint_{D(t)}-\frac{\partial B}{\partial t} \cdot d S. \tag{9.7.27}$$

Therefore,

$$\iint_{D(t)}-\frac{\partial B}{\partial t} \cdot d S=\iint_{D(t)} \operatorname{curl} E \cdot d S. \tag{9.7.28}$$

To derive the differential form of Faraday's law, we would like to conclude that $\operatorname{curl} E=-\frac{\partial B}{\partial t}$: In general, the equation

$$\iint_{D(t)}-\frac{\partial B}{\partial t} \cdot d S=\iint_{D(t)} \operatorname{curl} E \cdot d S \tag{9.7.29}$$

is not enough to conclude that $\operatorname{curl} E=-\frac{\partial B}{\partial t}$: The integral symbols do not simply "cancel out," leaving equality of the integrands. To see why the integral symbol does not just cancel out in general, consider the two single-variable integrals $\int_0^1 x\,dx$ and $\int_0^1 f(x)\,dx$, where

$$f(x)=\begin{cases}1,0 \leq \mathrm{x} \leq 2 \\ 01/2 \leq \mathrm{x} \leq 1.\end{cases} \tag{9.7.30}$$

Both of these integrals equal $\frac{1}{2}$, so $\int_0^1 x\,dx=\int_0^1 f(x)\,dx$.

However, $x \neq f(x)$. Analogously, with our equation

$$\iint_{D(t)}-\frac{\partial B}{\partial t} \cdot d S=\iint_{D(t)} \operatorname{curl} E \cdot d S, \tag{9.7.31}$$

we cannot simply conclude that$\operatorname{curl} E=-\frac{\partial B}{\partial t}$ just because their integrals are equal. However, in our context, equation

$$\iint_{D(t)}-\frac{\partial B}{\partial t} \cdot d S=\iint_{D(t)} \operatorname{curl} E \cdot d S \tag{9.7.32}$$

is true for *any* region, however small (this is in contrast to the single-variable integrals just discussed). If **F** and **G** are three-dimensional vector fields such that

$$\iint_S F \cdot d S=\iint_S G \cdot d S \tag{9.7.33}$$

for any surface *S*, then it is possible to show that $F=G$ by shrinking the area of *S* to zero by taking a limit (the smaller the area of *S*, the closer the value of $\iint_S F \cdot dS$ to the value of **F** at a point inside *S*). Therefore, we can let area $D(t)$ shrink to zero by taking a limit and obtain the differential form of Faraday's law:

$$\operatorname{curl} E=-\frac{\partial B}{\partial t}. \tag{9.7.34}$$

In the context of electric fields, the curl of the electric field can be interpreted as the negative of the rate of change of the corresponding magnetic field with respect to time.

Example 9.7.4: Using Faraday's Law

Calculate the curl of electric field $\vec{\mathbf{E}}$ if the corresponding magnetic field is constant field $B(t) = \langle 1, -4, 2\rangle$.

Solution

Since the magnetic field does not change with respect to time, $-\dfrac{\partial B}{\partial t} = 0$. By Faraday's law, the curl of the electric field is therefore also zero.

Analysis

A consequence of Faraday's law is that the curl of the electric field corresponding to a constant magnetic field is always zero.

Exercise 9.7.4

Calculate the curl of electric field $\vec{\mathbf{E}}$ if the corresponding magnetic field is $B(t) = \langle tx,\, ty,\, -2tz\rangle$, $0 \le t < \infty$.

Hint

- Use the differential form of Faraday's law.
- Notice that the curl of the electric field does not change over time, although the magnetic field does change over time.

Answer

$curl\, \vec{\mathbf{E}} = \langle x,\, y,\, -2z\rangle$

9.7.4 Key Concepts

- Stokes' theorem relates a flux integral over a surface to a line integral around the boundary of the surface. Stokes' theorem is a higher dimensional version of Green's theorem, and therefore is another version of the Fundamental Theorem of Calculus in higher dimensions.
- Stokes' theorem can be used to transform a difficult surface integral into an easier line integral, or a difficult line integral into an easier surface integral.
- Through Stokes' theorem, line integrals can be evaluated using the simplest surface with boundary C.
- Faraday's law relates the curl of an electric field to the rate of change of the corresponding magnetic field. Stokes' theorem can be used to derive Faraday's law.

9.7.5 Key Equations

- **Stokes' theorem**

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S curl\, \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

9.7.5.1 Glossary

Stokes' theorem

relates the flux integral over a surface S to a line integral around the boundary C of the surface S

surface independent

flux integrals of curl vector fields are surface independent if their evaluation does not depend on the surface but only on the boundary of the surface

9.7.6 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

9.7E: EXERCISES

9.7E.1 Exercise $\mathbf{9.7}\boldsymbol{E}\mathbf{.1}$

For the following exercises, without using Stokes' theorem, calculate directly both the flux of $curl\,\vec{\mathbf{F}}\cdot N$ over the given surface and the circulation integral around its boundary, assuming all are oriented clockwise.

1. $\vec{\mathbf{F}}(x,y,z)=y^2\,\hat{\mathbf{i}}+z^2\,\hat{\mathbf{j}}+x^2\,\hat{\mathbf{k}}$; S is the first-octant portion of plane $x+y+z=1$.
2. $\vec{\mathbf{F}}(x,y,z)=z\,\hat{\mathbf{i}}+x\,\hat{\mathbf{j}}+y\,\hat{\mathbf{k}}$; S is hemisphere $z=(a^2-x^2-y^2)^{1/2}$.

Answer

$$\iint_S (curl\,\vec{\mathbf{F}}\cdot\vec{\mathbf{N}})\,dS=\pi a^2 \tag{9.7E.1}$$

3. $\vec{\mathbf{F}}(x,y,z)=y^2\,\hat{\mathbf{i}}+2x\,\hat{\mathbf{j}}+5\,\hat{\mathbf{k}}$; S is hemisphere $z=(4-x^2-y^2)^{1/2}$.
4. $\vec{\mathbf{F}}(x,y,z)=z\,\hat{\mathbf{i}}+2x\,\hat{\mathbf{j}}+3y\,\hat{\mathbf{k}}$; S is upper hemisphere $z=\sqrt{9-x^2-y^2}$.

Answer

$$\iint_S (curl\,(\vec{\mathbf{F}})\cdot\vec{\mathbf{N}})\,dS=18\pi \tag{9.7E.2}$$

5. $\vec{\mathbf{F}}(x,y,z)=(x+2z)\,\hat{\mathbf{i}}+(y-x)\,\hat{\mathbf{j}}+(z-y)\,\hat{\mathbf{k}}$; S is a triangular region with vertices $(3,0,0)$, $(0,3/2,0)$,and $(0,0,3)$.
6. $\vec{\mathbf{F}}(x,y,z)=2y\,\hat{\mathbf{i}}+6z\,\hat{\mathbf{i}}+3x\,\hat{\mathbf{k}}$; S is a portion of paraboloid $z=4-x^2-y^2$ and is above the $xy-$plane.

Answer

$$\iint_S (curl\,(\vec{\mathbf{F}})\cdot\vec{\mathbf{N}})\,dS=-8\pi \tag{9.7E.3}$$

9.7E.2 Exercise $\mathbf{9.7}\boldsymbol{E}\mathbf{.2}$

For the following exercises, use Stokes' theorem to evaluate

$$\iint_S (curl\,(\vec{\mathbf{F}})\cdot\vec{\mathbf{N}})\,dS \tag{9.7E.4}$$

for the vector fields and surface.

1. $\vec{\mathbf{F}}(x,y,z)=xy\,\hat{\mathbf{i}}-z\,\hat{\mathbf{j}}$ and S is the surface of the cube $0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1$, except for the face where $z=0$ and using the outward unit normal vector.
2. $\vec{\mathbf{F}}(x,y,z)=xy\,\hat{\mathbf{i}}+x^2\,\hat{\mathbf{j}}+z^2\,\hat{\mathbf{k}}$; and C is the intersection of paraboloid $z=x^2+y^2$ and plane $z=y$, and using the outward normal vector.

Answer

$$\iint_S (curl\,(\vec{\mathbf{F}})\cdot\vec{\mathbf{N}})\,dS=0 \tag{9.7E.5}$$

3. $\vec{\mathbf{F}}(x,y,z)=4y\,\hat{\mathbf{i}}+z\,\hat{\mathbf{j}}+2y\,\hat{\mathbf{k}}$; and C is the intersection of sphere $x^2+y^2+z^2=4$ with plane $z=0$, and using the outward normal vector.

9.7E.3 Exercise $\mathbf{9.7}\boldsymbol{E}\mathbf{.3}$

1. Use Stokes' theorem to evaluate

$$\int_C [2xy^2z\,dx + 2x^2yz\,dy + (x^2y^2 - 2z)\,dz], \tag{9.7E.6}$$

where C is the curve given by $x = \cos t,\ y = \sin t,\ 0 \leq t \leq 2\pi$, traversed in the direction of increasing \(t\).

A vector field in three dimensional space. The arrows are larger the further they are from the x, y plane. The arrows curve up from below the x, y plane and slightly above it. The rest tend to curve down and horizontally. An oval-shaped curve is drawn through the middle of the space.

Answer

$$\int_C F \cdot dS = 0 \tag{9.7E.7}$$

2. [T] Use a computer algebraic system (CAS) and Stokes' theorem to approximate line integral

$$\int_C (y\,dx + z\,dy + x\,dz), \tag{9.7E.8}$$

where C is the intersection of plane $x + y = 2$ and surface $x^2 + y^2 + z^2 = 2(x + y)$, traversed counterclockwise viewed from the origin.

3. [T] Use a CAS and Stokes' theorem to approximate line integral

$$\int_C (3y\,dx + 2z\,dy - 5x\,dz), \tag{9.7E.9}$$

where C is the intersection of the $xy-$plane and hemisphere $z = \sqrt{1 - x^2 - y^2}$, traversed counterclockwise viewed from the top—that is, from the positive *z*-axis toward the $xy-$plane.

Answer

$$\int_C F \cdot dS = -9.4248 \tag{9.7E.10}$$

4. [T] Use a CAS and Stokes' theorem to approximate line integral

$$\int_C [(1 + y)\,zdx + (1 + z)xdy + (1 + x)ydz], \tag{9.7E.11}$$

where C is a triangle with vertices $(1, 0, 0),\ (0, 1, 0)$ and $(0, 0, 1)$ oriented counterclockwise.

5. Use Stokes' theorem to evaluate line integral

$$\int_C (z\,dx + x\,dy + y\,dz), \tag{9.7E.12}$$

where C is a triangle with vertices $(3, 0, 0)$, $(0, 0, 2)$,and $(0, 6, 0)$ traversed in the given order.

6. Use Stokes' theorem to evaluate

$$\int_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right), \tag{9.7E.13}$$

where C is the curve of intersection of plane $x + z = 1$ and ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented clockwise from the origin.

A diagram of an intersecting plane and ellipsoid in three dimensional space. There is an orange curve drawn to show the intersection.

Answer

$$\int_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right) = -\frac{\pi}{4} \tag{9.7E.14}$$

7. Use Stokes' theorem to evaluate

$$\iint_S (curl\,F \cdot N)dS, \tag{9.7E.15}$$

where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + ze^{xy}k$ and S is the part of surface $z = 1 - x^2 - 2y^2$ with $z \geq 0$, oriented counterclockwise.

8. Use Stokes' theorem for vector field $\vec{\mathbf{F}}(x,y,z) = z\,\hat{\mathbf{i}} + 3x\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}}$ where S is surface $z = 1 - x^2 - 2y^2$, $z \geq 0$, C is boundary circle $x^2 + y^2 = 1$, and S is oriented in the positive z-direction.

Answer

\[\iint_S (curl \, F \cdot N)dS = -3\pi]

9. Use Stokes' theorem for vector field $\vec{\mathbf{F}}(x,y,z) = -\frac{3}{2}y^2\,\hat{\mathbf{i}} - 2xy\,\hat{\mathbf{j}} + yz\,\hat{\mathbf{k}}$, where S is that part of the surface of plane $x + y + z = 1$ contained within triangle C with vertices $(1,0,0), (0,1,0)$, and $(0,0,1)$, traversed counterclockwise as viewed from above.

10. A certain closed path C in plane $2x + 2y + z = 1$ is known to project onto unit circle $x^2 + y^2 = 1$ in the \(xy\)-plane. Let \(c \) be a constant and let $R(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$. Use Stokes' theorem to evaluate

$$\int_C (ck \times R) \cdot dS. \tag{9.7E.16}$$

Answer

$$\int_C (ck \times R) \cdot dS = 2\pi c \tag{9.7E.17}$$

11. Use Stokes' theorem and let C be the boundary of surface $z = x^2 + y^2$ with $0 \leq x \leq 2$ and $0 \leq y \leq 1$ oriented with upward facing normal. Define

$\vec{\mathbf{F}}(x,y,z) = [\sin(x^3) + xz]\,\hat{\mathbf{i}} + (x - yz)\,\hat{\mathbf{j}} + \cos(z^4)\,\hat{\mathbf{k}}$ and evaluate $\int_C F \cdot dS$.

12. Let S be hemisphere $x^2 + y^2 + z^2 = 4$ with $z \geq 0$, oriented upward. Let $\vec{\mathbf{F}}(x,y,z) = x^2 e^{yz}\,\hat{\mathbf{i}} + y^2 e^{xz}\,\hat{\mathbf{j}} + z^2 e^{xy}\,\hat{\mathbf{k}}$ be a vector field. Use Stokes' theorem to evaluate

$$\iint_S curl\, F \cdot dS. \tag{9.7E.18}$$

Answer

$$\iint_S curl\, F \cdot dS = 0 \tag{9.7E.19}$$

13. Let $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} + (e^{z^2} + y)\,\hat{\mathbf{j}} + (x + y)\,\hat{\mathbf{k}}$ and let S be the graph of function $y = \frac{x^2}{9} + \frac{z^2}{9} - 1$ with $z \leq 0$ oriented so that the normal vector S has a positive y component. Use Stokes' theorem to compute integral

$$\iint_S curl\, F \cdot dS. \tag{9.7E.20}$$

14. Use Stokes' theorem to evaluate

$$\oint F \cdot dS, \tag{9.7E.21}$$

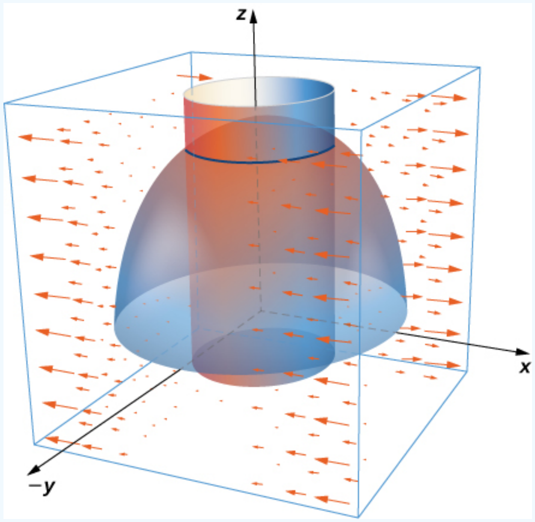
where $\vec{\mathbf{F}}(x,y,z) = y\,\hat{\mathbf{i}} + z\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$ and C is a triangle with vertices $(0,0,0), (2,0,0)$ and $0,-2,2)$ oriented counterclockwise when viewed from above.

Answer

$$\oint F \cdot dS = -4 \tag{9.7E.22}$$

15. Use the surface integral in Stokes' theorem to calculate the circulation of field **F**, $\vec{\mathbf{F}}(x,y,z) = x^2y^3\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ around C, which is the intersection of cylinder $x^2 + y^2 = 4$ and hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, oriented counterclockwise

when viewed from above.



16. Use Stokes' theorem to compute

$$\iint_S curl\, F \cdot dS. \tag{9.7E.23}$$

where $\vec{\mathbf{F}}(x, y, z) = \hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}} + xy^2\,\hat{\mathbf{k}}$ and S is a part of plane $y+z=2$ inside cylinder $x^2+y^2=1$ and oriented counterclockwise.

A diagram of a vector field in three dimensional space showing the intersection of a plane and a cylinder. The curve where the plane and cylinder intersect is drawn in blue.

Answer

$$\iint_S curl\, F \cdot dS = 0 \tag{9.7E.24}$$

17. Use Stokes' theorem to evaluate

$$\iint_S curl\, F \cdot dS, \tag{9.7E.25}$$

where $\vec{\mathbf{F}}(x, y, z) = -y^2\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$ and S is the part of plane $x+y+z=1$ in the positive octant and oriented counterclockwise $x \geq 0,\ y \geq 0,\ z \geq 0$.

18. Let $\vec{\mathbf{F}}(x, y, z) = xy\,\hat{\mathbf{i}} + 2z\,\hat{\mathbf{j}} - 2y\,\hat{\mathbf{k}}$ and let C be the intersection of plane $x+z=5$ and cylinder $x^2+y^2=9$, which is oriented counterclockwise when viewed from the top. Compute the line integral of **F** over C using Stokes' theorem.

Answer

$$\iint_S curl\, F \cdot dS = -36\pi \tag{9.7E.26}$$

19. [T] Use a CAS and let $\vec{\mathbf{F}}(x, y, z) = xy^2\,\hat{\mathbf{i}} + (yz - x)\,\hat{\mathbf{j}} + e^{yxz}\,\hat{\mathbf{k}}$. Use Stokes' theorem to compute the surface integral of curl **F** over surface S with inward orientation consisting of cube $[0,1] \times [0,1] \times [0,1]$ with the right side missing.

20. Let S be ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$ oriented counterclockwise and let **F** be a vector field with component functions that have continuous partial derivatives.

Answer

$$\iint_S curl\, F \cdot N = 0 \tag{9.7E.27}$$

21. Let S be the part of paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$. Verify Stokes' theorem for vector field $\vec{\mathbf{F}}(x,y,z) = 3z\,\hat{\mathbf{i}} + 4x\,\hat{\mathbf{j}} + 2y\,\hat{\mathbf{k}}$.

22. Use Stokes' theorem to evaluate

$$\iint_S curl\, F \cdot dS, \tag{9.7E.28}$$

where $\vec{\mathbf{F}}(x,y,z) = e^{xy} cos\, z\,\hat{\mathbf{i}} + x^2 z\,\hat{\mathbf{j}} + xy\,\hat{\mathbf{k}}$, and S is half of sphere $x = \sqrt{1 - y^2 - z^2}$, oriented out toward the positive *x*-axis.

Answer

$$\iint_S F \cdot dS = 0 \tag{9.7E.29}$$

23. [T] Use a CAS and Stokes' theorem to evaluate

$$\iint_S (curl\, F \cdot N)\, dS, \tag{9.7E.30}$$

where $\vec{\mathbf{F}}(x,y,z) = x^2 y\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}} + z^3\,\hat{\mathbf{k}}$ and C is the curve of the intersection of plane $3x + 2y + z = 6$ and cylinder $x^2 + y^2 = 4$, oriented clockwise when viewed from above.

24. [T] Use a CAS and Stokes' theorem to evaluate

$$\iint_S curl\, F \cdot dS, \tag{9.7E.31}$$

where $\vec{\mathbf{F}}(x,y,z) = \left(\sin(y+z) - yx^2 - \dfrac{y^3}{3}\right)\,\hat{\mathbf{i}} + x\,\cos(y+z)\,\hat{\mathbf{j}} + \cos(2y)\,\hat{\mathbf{k}}$ and S consists of the top and the four sides but not the bottom of the cube with vertices $(\pm 1,\ \pm 1,\ \pm 1)$, oriented outward.

Answer

$$\iint_S curl\, F \cdot dS = 2.6667 \tag{9.7E.32}$$

25. [T] Use a CAS and Stokes' theorem to evaluate

$$\iint_S curl\, F \cdot dS, \tag{9.7E.33}$$

where $\vec{\mathbf{F}}(x,y,z) = z^2\,\hat{\mathbf{i}} + 3xy\,\hat{\mathbf{j}} + x^3 y^3\,\hat{\mathbf{k}}$ and S is the top part of $z = 5 - x^2 - y^2$ above plane $z = 1$ and S is oriented upward.

26. Use Stokes' theorem to evaluate

$$\iint_S (curl\, F \cdot N) dS, \tag{9.7E.34}$$

where $\vec{\mathbf{F}}(x,y,z) = z^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$ and S is a triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counterclockwise orientation.

Answer

\[\iint_S (curl \, F \cdot N)dS = -\dfrac{1}{6}\

9.7E.4 Exercise $9.7E.4$

1. Let S be paraboloid $z=a(1-x^2-y^2)$, for $z \geq 0$, where $a>0$ is a real number. Let $\vec{\mathbf{F}}(x,y,z)=\langle x-y,\ y+z,\ z-x\rangle$. For what value(s) of a (if any) does

$$\iint_S (\nabla \times F)\cdot n\,dS \tag{9.7E.35}$$

have its maximum value?

2. [T] Use a CAS and Stokes' theorem to evaluate

$$\oint F\cdot dS, \tag{9.7E.36}$$

if $\vec{\mathbf{F}}(x,y,z)=(3z-\sin x)\,\hat{\mathbf{i}}+(x^2+e^y)\,\hat{\mathbf{j}}+(y^3-\cos z)\,\hat{\mathbf{k}}$, where C is the curve given by $x=\cos t,\ y=\sin t,\ z=1;\ 0\leq t\leq 2\pi$.

Answer

$$\oint_C F\cdot dr=0 \tag{9.7E.37}$$

3. [T] Use a CAS and Stokes' theorem to evaluate $\vec{\mathbf{F}}(x,y,z)=2y\,\hat{\mathbf{i}}+e^z\,\hat{\mathbf{j}}-arctan\,x\,\hat{\mathbf{k}}$ with S as a portion of paraboloid $z=4-x^2-y^2$ cut off by the $xy-$plane oriented counterclockwise.

4. [T] Use a CAS to evaluate

$$\iint_S curl(F)\cdot dS, \tag{9.7E.38}$$

where $\vec{\mathbf{F}}(x,y,z)=2z\,\hat{\mathbf{i}}+3x\,\hat{\mathbf{j}}+5y\,\hat{\mathbf{k}}$ and S is the surface parametrically by $r(r,\theta)=r\,\cos\theta\,\hat{\mathbf{i}}+r\,\sin\theta\,\hat{\mathbf{j}}+(4-r^2)\,\hat{\mathbf{k}}\,(0\leq\theta\leq 2\pi,\ 0\leq r\leq 3)$.

Answer

$$\iint_S curl(F)\cdot dS=84.8230 \tag{9.7E.39}$$

9.7E.5 Exercise $\boldsymbol{9.7E.5}$

1. For the following application exercises, the goal is to evaluate

$$A=\iint_S (\nabla\times F)\cdot n\,dS, \tag{9.7E.40}$$

where $\vec{\mathbf{F}}=\langle xz,\ -xz,\ xy\rangle$ and S is the upper half of ellipsoid $x^2+y^2+8z^2=1$, where $z\geq 0$.

a) Evaluate a surface integral over a more convenient surface to find the value of \(A.\)

Answer

$$A=\iint_S (\nabla\times F)\cdot n\,dS=0 \tag{9.7E.41}$$

Evaluate \(A\) using a line integral.

2. Take paraboloid $z=x^2+y^2$, for $0\leq z\leq 4$, and slice it with plane $y=0$. Let S be the surface that remains for $y\geq 0$, including the planar surface in the *xz*-plane. Let C be the semicircle and line segment that bounded the cap of S in plane $z=4$ with counterclockwise orientation. Let $\vec{\mathbf{F}}=\langle 2z+y,\ 2x+z,\ 2y+x\rangle$. Evaluate

$$\iint_S (\nabla\times F)\cdot n\,dS. \tag{9.7E.42}$$

A diagram of a vector field in three dimensional space where a paraboloid with vertex at the origin, plane at y=0, and plane at z=4 intersect. The remaining surface is the half of a paraboloid under z=4 and above y=0.

Answer

$$\iint_S (\nabla \times F)\cdot n\,dS = 2\pi \tag{9.7E.43}$$

Exercise $9.7E.7$

1. For the following exercises, let S be the disk enclosed by curve $C: r(t) = \langle \cos\varphi \, \cos t, \, \sin t, \, \sin\varphi \, \cos t \rangle$, for $0 \le t \le 2\pi$, where $0 \le \varphi \le \frac{\pi}{2}$ is a fixed angle.

a) What is the length of C in terms of φ?

b) What is the circulation of $C$of vector field $\vec{\mathbf{F}} = \langle -y, \, -z, \, x \rangle$ as a function of φ?

Answer

$C = \pi(\cos\varphi - \sin\varphi)$

c) For what value of φ is the circulation a maximum?

2. Circle C in plane $x+y+z=8$ has radius 4 and center (2, 3, 3). Evaluate

$$\oint_C F\cdot dr \tag{9.7E.44}$$

for $\vec{\mathbf{F}} = \langle 0, \, -z, \, 2y \rangle$, where C has a counterclockwise orientation when viewed from above.

Answer

$$\oint_C F\cdot dr = 48\pi \tag{9.7E.45}$$

3. Velocity field $v = \langle 0, \, 1-x^2, \, 0 \rangle$, for $|x| \le 1$ and $|z| \le 1$, represents a horizontal flow in the *y*-direction. Compute the curl of **v** in a clockwise rotation.

4. Evaluate integral

$$\iint_S (\nabla \times F)\cdot n\,dS, \tag{9.7E.46}$$

where $\vec{\mathbf{F}} = -xz\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xye^z\,\hat{\mathbf{k}}$ and S is the cap of paraboloid $z = 5-x^2-y^2$ above plane $z=3$, and **n** points in the positive *z*-direction on *S*.

Answer

$$\iint_S (\nabla \times F)\cdot n = 0 \tag{9.7E.47}$$

5. For the following exercises, use Stokes' theorem to find the circulation of the following vector fields around any smooth, simple closed curve C.

a) $\vec{\mathbf{F}} = \nabla(x \, \sin ye^z)$

b) $\vec{\mathbf{F}} = \langle y^2z^3, \, z2xyz^3, 3xy^2z^2 \rangle$

Answer

0

9.8: The Divergence Theorem

We have examined several versions of the Fundamental Theorem of Calculus in higher dimensions that relate the integral around an oriented boundary of a domain to a “derivative” of that entity on the oriented domain. In this section, we state the divergence theorem, which is the final theorem of this type that we will study. The divergence theorem has many uses in physics; in particular, the divergence theorem is used in the field of partial differential equations to derive equations modeling heat flow and conservation of mass. We use the theorem to calculate flux integrals and apply it to electrostatic fields.

Overview of Theorems

Before examining the divergence theorem, it is helpful to begin with an overview of the versions of the Fundamental Theorem of Calculus we have discussed:

1. The Fundamental Theorem of Calculus:
$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{9.8.1}$$
This theorem relates the integral of derivative f' over line segment $[a,b]$ along the *x*-axis to a difference of f evaluated on the boundary.
2. The Fundamental Theorem for Line Integrals:
$$\int_C \nabla f \cdot dr = f(P_1) - f(P_0), \tag{9.8.2}$$
where P_0 is the initial point of *C* and P_1 is the terminal point of *C*. The Fundamental Theorem for Line Integrals allows path *C* to be a path in a plane or in space, not just a line segment on the *x*-axis. If we think of the gradient as a derivative, then this theorem relates an integral of derivative ∇f over path *C* to a difference of f evaluated on the boundary of *C*.
3. Green’s theorem, circulation form:
$$\iint_D (Q_x - P_y)dA = \int_C F \cdot dr. \tag{9.8.3}$$
Since $Q_x - P_y = curl\, F \cdot k$ and curl is a derivative of sorts, Green’s theorem relates the integral of derivative curl **F** over planar region *D* to an integral of **F** over the boundary of *D*.
4. Green’s theorem, flux form:
$$\iint_D (P_x + Q_y)dA = \int_C F \cdot N\, dS. \tag{9.8.4}$$
Since $P_x + Q_y = div\, F$ and divergence is a derivative of sorts, the flux form of Green’s theorem relates the integral of derivative div **F** over planar region *D* to an integral of **F** over the boundary of *D*.
5. Stokes’ theorem:
$$\iint_S curl\, F \cdot dS = \int_C F \cdot dr. \tag{9.8.5}$$
If we think of the curl as a derivative of sorts, then Stokes’ theorem relates the integral of derivative curl**F** over surface *S* (not necessarily planar) to an integral of **F** over the boundary of *S*.

9.8.1 Stating the Divergence Theorem

The divergence theorem follows the general pattern of these other theorems. If we think of divergence as a derivative of sorts, then the divergence theorem relates a triple integral of derivative div **F** over a solid to a flux integral of **F** over the boundary of the solid. More specifically, the divergence theorem relates a flux integral of vector field **F** over a closed surface *S* to a triple integral of the divergence of **F** over the solid enclosed by *S*.

The Divergence Theorem

Let S be a piecewise, smooth closed surface that encloses solid E in space. Assume that S is oriented outward, and let $\mathbf{F}$ be a vector field with continuous partial derivatives on an open region containing E (Figure 9.8.1). Then

$$\iiint_E div\, F\, dV = \iint_S F \cdot dS. \tag{9.8.6}$$

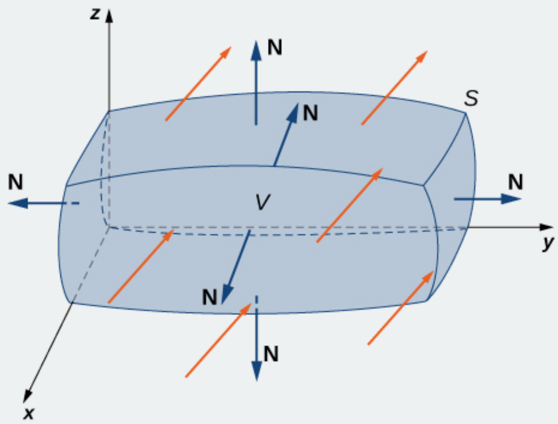


Figure 9.8.1: The divergence theorem relates a flux integral across a closed surface S to a triple integral over solid E enclosed by the surface.

Recall that the flux form of Green's theorem states that

$$\iint_D div\, dA = \int_C F \cdot N\, dS. \tag{9.8.7}$$

Therefore, the divergence theorem is a version of Green's theorem in one higher dimension.

The proof of the divergence theorem is beyond the scope of this text. However, we look at an informal proof that gives a general feel for why the theorem is true, but does not prove the theorem with full rigor. This explanation follows the informal explanation given for why Stokes' theorem is true.

Proof

Let B be a small box with sides parallel to the coordinate planes inside E (Figure $9.8.2a$). Let the center of B have coordinates (x, y, z) and suppose the edge lengths are Δx, Δy, and Δz. (Figure $9.8.1b$). The normal vector out of the top of the box is $\hat{\mathbf{k}}$ and the normal vector out of the bottom of the box is $-k$. The dot product of $\vec{\mathbf{F}} = \langle P, Q, R\rangle$ with $\hat{\mathbf{k}}$ is R and the dot product with $-k$ is $-R$. The area of the top of the box (and the bottom of the box)

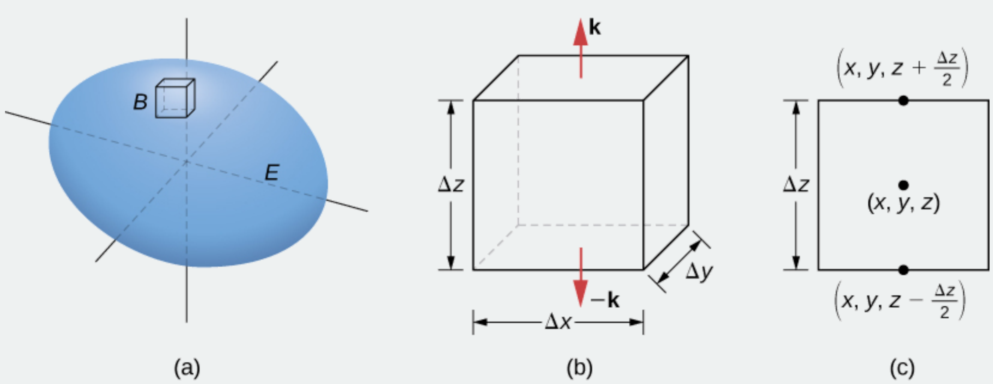


Figure 9.8.2: (a) A small box B inside surface E has sides parallel to the coordinate planes. (b) Box B has side lengthsΔx, Δy, and Δz (c) If we look at the side view of B, we see that, since (x, y, z) is the center of the box, to get to the top of the box we must travel a vertical distance of Δ/z up from (x, y, z). Similarly, to get to the bottom of the box we must travel a distance Δ/z down from (x, y, z).

The flux out of the top of the box can be approximated by $R\left(x, y, z+\frac{\Delta z}{2}\right)\Delta x\Delta y$ (Figure 9.8.2c) and the flux out of the bottom of the box is $-R\left(x, y, z-\frac{\Delta z}{2}\right)\Delta x\Delta y$. If we denote the difference between these values as ΔR, then the net flux in the vertical direction can be approximated by $\Delta R\Delta x\Delta y$. However,

$$\Delta R\Delta x\Delta y=\left(\frac{\Delta R}{\Delta z}\right)\Delta x\Delta y\Delta z\approx\left(\frac{\partial R}{\partial z}\right)\Delta V.$$

Therefore, the net flux in the vertical direction can be approximated by $\left(\frac{\partial R}{\partial z}\right)\Delta V$. Similarly, the net flux in the *x*-direction can be approximated by $\left(\frac{\partial P}{\partial x}\right)\Delta V$ and the net flux in the *y*-direction can be approximated by $\left(\frac{\partial Q}{\partial y}\right)\Delta V$. Adding the fluxes in all three directions gives an approximation of the total flux out of the box:

$$Total\ flux\approx\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\Delta V=div\,F\Delta V.$$

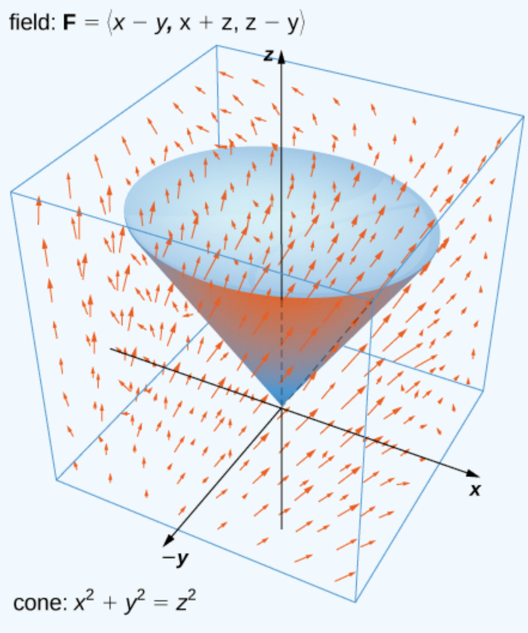
This approximation becomes arbitrarily close to the value of the total flux as the volume of the box shrinks to zero.

The sum of $divF\Delta V$ over all the small boxes approximating *E* is approximately $\iiint_E div\,F dV$. On the other hand, the sum of $div\,F\Delta V$ over all the small boxes approximating *E* is the sum of the fluxes over all these boxes. Just as in the informal proof of Stokes' theorem, adding these fluxes over all the boxes results in the cancelation of a lot of the terms. If an approximating box shares a face with another approximating box, then the flux over one face is the negative of the flux over the shared face of the adjacent box. These two integrals cancel out. When adding up all the fluxes, the only flux integrals that survive are the integrals over the faces approximating the boundary of *E*. As the volumes of the approximating boxes shrink to zero, this approximation becomes arbitrarily close to the flux over *S*.

□

Example 9.8.1: Verifying the Divergence Theorem

Verify the divergence theorem for vector field $F=\langle x-y, x+z, z-y\rangle$ and surface *S* that consists of cone $x^2+y^2=z^2$, $0\leq z\leq 1$, and the circular top of the cone (see the following figure). Assume this surface is positively oriented.



Solution

Let *E* be the solid cone enclosed by *S*. To verify the theorem for this example, we show that

$$\iiint_E div\, F dV = \iint_S F \cdot dS$$

by calculating each integral separately.

To compute the triple integral, note that $div\, F = P_x + Q_y + R_z = 2$, and therefore the triple integral is

$$\begin{aligned}\iiint_E div\, dV &= 2\iiint_E dV \\ &= 2\,(volume\, of\; E).\end{aligned}$$

The volume of a right circular cone is given by $\pi r^2 \frac{h}{3}$. In this case, $h = r = 1$. Therefore,

$$\iiint_E div\, F dV = 2\,(volume\, of\; E) = \frac{2\pi}{3}.$$

To compute the flux integral, first note that S is piecewise smooth; S can be written as a union of smooth surfaces. Therefore, we break the flux integral into two pieces: one flux integral across the circular top of the cone and one flux integral across the remaining portion of the cone. Call the circular top S_1 and the portion under the top S_2. We start by calculating the flux across the circular top of the cone. Notice that S_1 has parameterization

$$r(u,v) = \langle u\, \cos v,\; u\, \sin v,\; 1\rangle,\; 0 \le u \le 1,\; 0 \le v \le 2\pi.$$

Then, the tangent vectors are $t_u = \langle \cos v,\, \sin v,\, 0\rangle$ and $t_v = \langle -u\, \cos v,\, u\, \sin v, 0\rangle$. Therefore, the flux across S_1 is

$$\begin{aligned}\iint_{S_1} F \cdot dS &= \int_0^1 \int_0^{2\pi} F(r(u,v)) \cdot (t_u \times t_v)\, dA \\ &= \int_0^1 \int_0^{2\pi} \langle u\, \cos v - u\, \sin v,\; u\, \cos v + 1,\; 1 - u\, \sin v\rangle \cdot \langle 0, 0, v\rangle\, dv\, du \\ &= \int_0^1 \int_0^{2\pi} u - u^2 \sin v\, dv du \\ &= \pi.\end{aligned}$$

We now calculate the flux over S_2. A parameterization of this surface is

$$r(u,v) = \langle u\, \cos v,\; u\, \sin v,\; u\rangle,\; 0 \le u \le 1,\; 0 \le v \le 2\pi.$$

The tangent vectors are $t_u = \langle \cos v,\, \sin v,\, 1\rangle$ and $t_v = \langle -u\, \sin v,\, u\, \cos v, 0\rangle$, so the cross product is

$$t_u \times t_v = \langle -u\, \cos v,\; -u\, \sin v,\; u\rangle.$$

Notice that the negative signs on the x and y components induce the negative (or inward) orientation of the cone. Since the surface is positively oriented, we use vector $t_v \times t_u = \langle u\, \cos v,\; u\, \sin v,\; -u\rangle$ in the flux integral. The flux across S_2 is then

$$\begin{aligned}\iint_{S_1} F \cdot dS &= \int_0^1 \int_0^{2\pi} F(r(u,v)) \cdot (t_u \times t_v)\, dA \\ &= \int_0^1 \int_0^{2\pi} \langle u\, \cos v - u\, \sin v,\; u\, \cos v + u,\; u\, - \sin v\rangle \cdot \langle u\, \cos v,\; u\, \sin v,\; -u\rangle \\ &= \int_0^1 \int_0^{2\pi} u^2 \cos^2 v + 2u^2 \sin v - u^2 dv\, du \qquad = \frac{\pi}{3}\end{aligned}$$

The total flux across S is

$$\iint_{S_2} F \cdot dS = \iint_{S_1} F \cdot dS + \iint_{S_2} F \cdot dS = \frac{2\pi}{3} = \iiint_E div\, F dV,$$

and we have verified the divergence theorem for this example.

Exercise 9.8.1

Verify the divergence theorem for vector field $F(x,y,z)=\langle x+y+z,\ y,\ 2x-y\rangle$ and surface S given by the cylinder $x^2+y^2=1,\ 0\le z\le 3$ plus the circular top and bottom of the cylinder. Assume that *S* is positively oriented.

Hint

Calculate both the flux integral and the triple integral with the divergence theorem and verify they are equal.

Answer

Both integrals equal 6π.

Recall that the divergence of continuous field **F** at point *P* is a measure of the "outflowing-ness" of the field at *P*. If **F** represents the velocity field of a fluid, then the divergence can be thought of as the rate per unit volume of the fluid flowing out less the rate per unit volume flowing in. The divergence theorem confirms this interpretation. To see this, let *P* be a point and let B_τ be a ball of small radius *r* centered at *P* (Figure $9.8.3$). Let S_τ be the boundary sphere of B_τ. Since the radius is small and **F** is continuous, $div\, F(Q)\approx div\, F(P)$ for all other points *Q* in the ball. Therefore, the flux across S_τ can be approximated using the divergence theorem:

$$\iint_{S_\tau} F\cdot dS=\iiint_{B_\tau} div\, F dV\approx\iiint_{B_\tau} divF(P)dV.$$

Since $divF(P)$ is a constant,

$$\iiint_{B_\tau} divF(P)dV=div\, F(P)\, V(B_\tau).$$

Therefore, flux

$$\iint_{S_\tau} F\cdot dS \tag{9.8.8}$$

can be approximated by $F(P)\, V(B_\tau)$. This approximation gets better as the radius shrinks to zero, and therefore

$$div\, F(P)=\lim_{\tau\to 0}\frac{1}{V(B_\tau)}\iint_{S_\tau} F\cdot dS.$$

This equation says that the divergence at *P* is the net rate of outward flux of the fluid per unit volume.

This figure is a diagram of ball B_r, with small radius r centered at P. Arrows are drawn pointing up and to the right across the ball.

Figure $9.8.3$: Ball B_τ of small radius *r* centered at *P*.

9.8.2 Using the Divergence Theorem

The divergence theorem translates between the flux integral of closed surface *S* and a triple integral over the solid enclosed by *S*. Therefore, the theorem allows us to compute flux integrals or triple integrals that would ordinarily be difficult to compute by translating the flux integral into a triple integral and vice versa.

Example 9.8.2: Applying the Divergence Theorem

Calculate the surface integral

$$\iint_S F\cdot dS,$$

where S is cylinder $x^2+y^2=1,\ 0\le z\le 2$, including the circular top and bottom, and $F=\left\langle \frac{x^3}{3}+yz,\ \frac{y^3}{3}-\sin(xz),\ z-x-y\right\rangle$.

Solution

We could calculate this integral without the divergence theorem, but the calculation is not straightforward because we would have to break the flux integral into three separate integrals: one for the top of the cylinder, one for the bottom, and one for the side. Furthermore, each integral would require parameterizing the corresponding surface, calculating tangent vectors and their cross product..

By contrast, the divergence theorem allows us to calculate the single triple integral

$$\iiint_E div\, F\, dV,$$

where E is the solid enclosed by the cylinder. Using the divergence theorem (Equation 9.8.6) and converting to cylindrical coordinates, we have

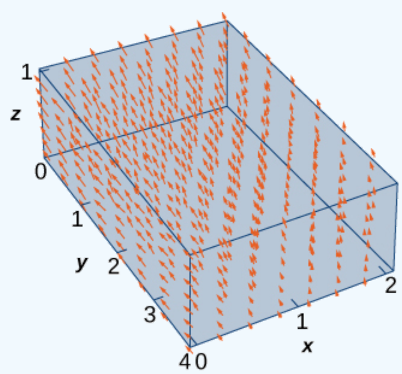
$$\begin{aligned}\iint_S F \cdot dS &= \iiint_E div\, F\, dV, \\ &= \iiint_E (x^2+y^2+1)\, dV \\ &= \int_0^{2\pi}\int_0^1\int_0^2 (r^2+1)\, r\, dz\, dr\, d\theta \\ &= \frac{3}{2}\int_0^{2\pi} d\theta \\ &= 3\pi.\end{aligned}$$

Exercise 9.8.2

Use the divergence theorem to calculate flux integral

$$\iint_S F \cdot dS,$$

where S is the boundary of the box given by $0 \le x \le 2,\ 1 \le y \le 4,\ 0 \le z \le 1$ and $F = \langle x^2+yz,\ y-z,\ 2x+2y+2z\rangle$ (see the following figure).



Hint

Calculate the corresponding triple integral.

Answer

30

Example 9.8.3: Applying the Divergence Theorem

Let $v = \left\langle -\frac{y}{z},\ \frac{x}{z},\ 0\right\rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \le x \le 4,\ 2 \le y \le 5,\ 1 \le z \le 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.

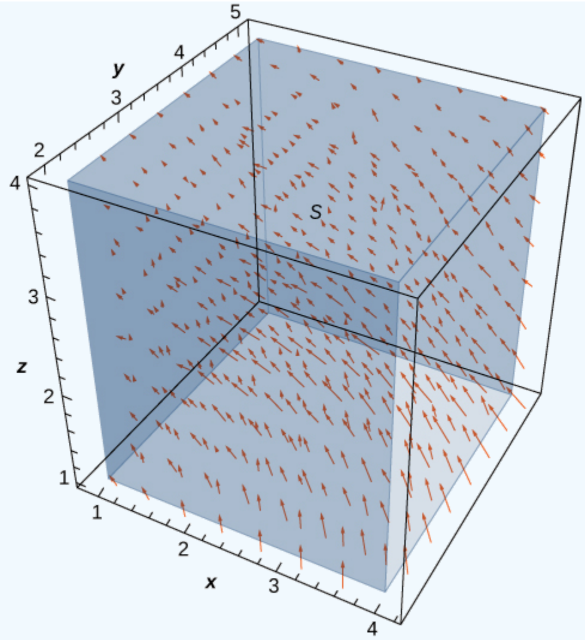


Figure 9.8.4: Vector field $v = \left\langle -\frac{y}{z}, \frac{x}{z}, 0 \right\rangle$.

Solution

The flow rate of the fluid across *S* is $\iint_S v \cdot dS$. Before calculating this flux integral, let's discuss what the value of the integral should be. Based on Figure 9.8.4, we see that if we place this cube in the fluid (as long as the cube doesn't encompass the origin), then the rate of fluid entering the cube is the same as the rate of fluid exiting the cube. The field is rotational in nature and, for a given circle parallel to the *xy*-plane that has a center on the *z*-axis, the vectors along that circle are all the same magnitude. That is how we can see that the flow rate is the same entering and exiting the cube. The flow into the cube cancels with the flow out of the cube, and therefore the flow rate of the fluid across the cube should be zero.

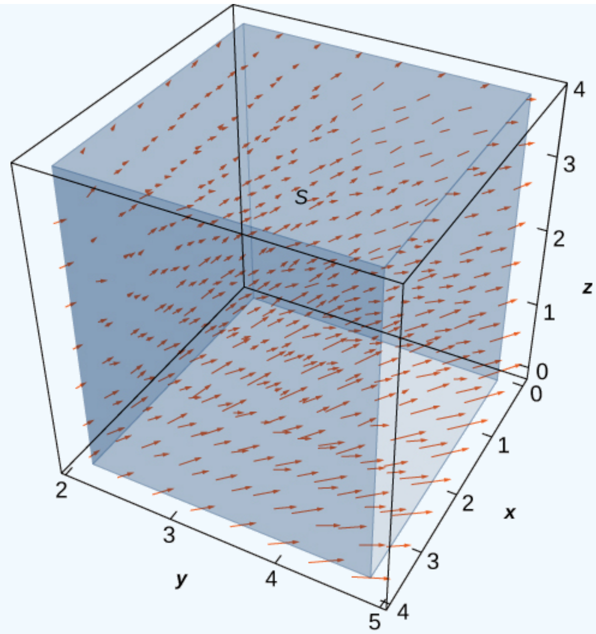
To verify this intuition, we need to calculate the flux integral. Calculating the flux integral directly requires breaking the flux integral into six separate flux integrals, one for each face of the cube. We also need to find tangent vectors, compute their cross product. However, using the divergence theorem makes this calculation go much more quickly:

$$\begin{aligned}\iint_S v \cdot dS &= \iiint_C div(v)\, dV \\ &= \iiint_C 0\, dV = 0.\end{aligned}$$

Therefore the flux is zero, as expected.

Exercise 9.8.3

Let $v = \left\langle \frac{x}{z}, \frac{y}{z}, 0 \right\rangle$ be the velocity field of a fluid. Let *C* be the solid cube given by $1 \le x \le 4,\ 2 \le y \le 5,\ 1 \le z \le 4$, and let *S* be the boundary of this cube (see the following figure). Find the flow rate of the fluid across *S*.



Hint

Use the divergence theorem and calculate a triple integral

Answer

$9\ \ln(16)$

Example illustrates a remarkable consequence of the divergence theorem. Let *S* be a piecewise, smooth closed surface and let **F** be a vector field defined on an open region containing the surface enclosed by *S*. If **F** has the form $F = \langle f(y,z),\, g(x,z),\, h(x,y) \rangle$, then the divergence of **F** is zero. By the divergence theorem, the flux of **F** across *S* is also zero. This makes certain flux integrals incredibly easy to calculate. For example, suppose we wanted to calculate the flux integral $\iint_S F \cdot dS$ where *S* is a cube and

$$F = \langle \sin(y)\, e^{yz},\, x^2 z^2,\, \cos(xy)\, e^{\sin x} \rangle. \tag{9.8.9}$$

Calculating the flux integral directly would be difficult, if not impossible, using techniques we studied previously. At the very least, we would have to break the flux integral into six integrals, one for each face of the cube. But, because the divergence of this field is zero, the divergence theorem immediately shows that the flux integral is zero.

We can now use the divergence theorem to justify the physical interpretation of divergence that we discussed earlier. Recall that if **F** is a continuous three-dimensional vector field and *P* is a point in the domain of **F**, then the divergence of **F** at *P* is a measure of the "outflowing-ness" of **F** at *P*. If **F** represents the velocity field of a fluid, then the divergence of **F** at *P* is a measure of the net flow rate out of point *P* (the flow of fluid out of *P* less the flow of fluid in to *P*). To see how the divergence theorem justifies this interpretation, let B_τ be a ball of very small radius *r* with center *P*, and assume that B_τ is in the domain of **F**. Furthermore, assume that B_τ has a positive, outward orientation. Since the radius of B_τ is small and **F** is continuous, the divergence of **F** is approximately constant on B_τ. That is, ifv P' is any point in B_τ, then $div\, F(P) \approx div\, F(P')$. Let S_τ denote the boundary sphere of B_τ. We can approximate the flux across S_τ using the divergence theorem as follows:

$$\iint_{S_\tau} F \cdot dS = \iiint_{B_\tau} div\, F\, dV \tag{9.8.10}$$

$$\approx \iiint_{B_\tau} div\, F(P)\, dV \tag{9.8.11}$$

$$= div\, F(P)\, V(B_\tau). \tag{9.8.12}$$

As we shrink the radius *r* to zero via a limit, the quantity $div\, F(P)\, V(B_\tau)$ gets arbitrarily close to the flux. Therefore,

$$div\,F(P) = \lim_{\tau \to 0} \frac{1}{V(B_\tau)} \iint_{S_\tau} F \cdot dS \tag{9.8.13}$$

and we can consider the divergence at *P* as measuring the net rate of outward flux per unit volume at *P*. Since "outflowing-ness" is an informal term for the net rate of outward flux per unit volume, we have justified the physical interpretation of divergence we discussed earlier, and we have used the divergence theorem to give this justification.

9.8.3 Application to Electrostatic Fields

The divergence theorem has many applications in physics and engineering. It allows us to write many physical laws in both an integral form and a differential form (in much the same way that Stokes' theorem allowed us to translate between an integral and differential form of Faraday's law). Areas of study such as fluid dynamics, electromagnetism, and quantum mechanics have equations that describe the conservation of mass, momentum, or energy, and the divergence theorem allows us to give these equations in both integral and differential forms.

One of the most common applications of the divergence theorem is to **electrostatic fields**. An important result in this subject is **Gauss' law**. This law states that if *S* is a closed surface in electrostatic field **E**, then the flux of **E** across *S* is the total charge enclosed by *S* (divided by an electric constant). We now use the divergence theorem to justify the special case of this law in which the electrostatic field is generated by a stationary point charge at the origin.

If (x, y, z) is a point in space, then the distance from the point to the origin is $r = \sqrt{x^2 + y^2 + z^2}$. Let F_τ denote radial vector field $F_\tau = \frac{1}{\tau^2} \left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau} \right\rangle$.The vector at a given position in space points in the direction of unit radial vector $\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau} \right\rangle$ and is scaled by the quantity $1/\tau^2$. Therefore, the magnitude of a vector at a given point is inversely proportional to the square of the vector's distance from the origin. Suppose we have a stationary charge of *q* Coulombs at the origin, existing in a vacuum. The charge generates electrostatic field **E** given by

$$E = \frac{q}{4\pi\epsilon_0} F_\tau, \tag{9.8.14}$$

where the approximation $\epsilon_0 = 8.854 \times 10^{-12}$ farad (F)/m is an electric constant. (The constant ϵ_0 is a measure of the resistance encountered when forming an electric field in a vacuum.) Notice that **E** is a radial vector field similar to the gravitational field described in [link]. The difference is that this field points outward whereas the gravitational field points inward. Because

$$E = \frac{q}{4\pi\epsilon_0} F_\tau = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{\tau^2} \left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau} \right\rangle \right), \tag{9.8.15}$$

we say that electrostatic fields obey an inverse-square law. That is, the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge (which in this case is at the origin). Given this vector field, we show that the flux across closed surface *S* is zero if the charge is outside of *S*, and that the flux is $q/epsilon_0$ if the charge is inside of *S*. In other words, the flux across *S* is the charge inside the surface divided by constant ϵ_0. This is a special case of Gauss' law, and here we use the divergence theorem to justify this special case.

To show that the flux across *S* is the charge inside the surface divided by constant ϵ_0, we need two intermediate steps. First we show that the divergence of F_τ is zero and then we show that the flux of F_τ across any smooth surface *S* is either zero or 4π. We can then justify this special case of Gauss' law.

Example 9.8.4: The Divergence of F_τ is Zero

Verify that the divergence of F_τ is zero where F_τ is defined (away from the origin).

Solution

Since $\tau = \sqrt{x^2 + y^2 + z^2}$, the quotient rule gives us

$$\frac{\partial}{\partial x}\left(\frac{x}{\tau^3}\right) = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{(x^2+y^2+z^2)^{3/2} - x\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}2x\right]}{(x^2+y^2+z^2)^3}$$

$$= \frac{\tau^3 - 3x^2\tau}{\tau^6} = \frac{\tau^2-3x^2}{\tau^5}.$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{y}{\tau^3}\right) = \frac{\tau^2-3y^2}{\tau^5} \, and \, \frac{\partial}{\partial z}\left(\frac{z}{\tau^3}\right) = \frac{\tau^2-3z^2}{\tau^5}.$$

Therefore,

$$div\, F_\tau = \frac{\tau^2-3x^2}{\tau^5} + \frac{\tau^2-3y^2}{\tau^5} + \frac{\tau^2-3z^2}{\tau^5}$$

$$= \frac{3\tau^2 - 3(x^2+y^2+z^2)}{\tau^5}$$

$$= \frac{3\tau^2-3\tau^2}{\tau^5} = 0.$$

Notice that since the divergence of F_τ is zero and **E** is F_τ scaled by a constant, the divergence of electrostatic field **E** is also zero (except at the origin).

Flux across a Smooth Surface

Let S be a connected, piecewise smooth closed surface and let $F_\tau = \frac{1}{\tau^2}\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau}\right\rangle$. Then,

$$\iint_S F_\tau \cdot dS = \begin{cases} 0 \text{if S does not encompass the origin} \\ 4\pi \text{if S encompasses the origin.} \end{cases} \tag{9.8.16}$$

In other words, this theorem says that the flux of F_τ across any piecewise smooth closed surface S depends only on whether the origin is inside of S.

Proof

The logic of this proof follows the logic of [link], only we use the divergence theorem rather than Green's theorem.

First, suppose that S does not encompass the origin. In this case, the solid enclosed by S is in the domain of F_τ, and since the divergence of F_τ is zero, we can immediately apply the divergence theorem and find that

$$\iint_S F \cdot dS \tag{9.8.17}$$

is zero.

Now suppose that S does encompass the origin. We cannot just use the divergence theorem to calculate the flux, because the field is not defined at the origin. Let S_a be a sphere of radius a inside of S centered at the origin. The outward normal vector field on the sphere, in spherical coordinates, is

$$t_\phi \times t_\theta = \langle a^2 \cos\theta \sin^2\phi,\, a^2 \sin\theta \sin^2\phi,\, a^2 \sin\phi \cos\phi\rangle \tag{9.8.18}$$

(see [link]). Therefore, on the surface of the sphere, the dot product $F_\tau \cdot N$ (in spherical coordinates) is

$$\begin{aligned} F_\tau \cdot N &= \left\langle \frac{\sin\phi\,\cos\theta}{a^2}, \frac{\sin\phi\,\sin\theta}{a^2}, \frac{\cos\phi}{a^2}\right\rangle \cdot \langle a^2\cos\theta\,\sin^2\phi, a^2\sin\theta\,\sin^2\phi,\, a^2\sin\phi\,\cos\phi\rangle \\ &= \sin\phi(\langle\sin\phi\,\cos\theta,\,\sin\phi\,\sin\theta,\,\cos\phi\rangle\cdot\langle\sin\phi\,\cos\theta,\sin\phi\,\sin\theta,\,\cos\phi\rangle) \\ &= \sin\phi. \end{aligned}$$

The flux of F_τ across S_a is

$$\iint_{S_a} F_\tau \cdot N dS = \int_0^{2\pi}\int_0^\pi \sin\phi\, d\phi\, d\theta = 4\pi. \tag{9.8.19}$$

Now, remember that we are interested in the flux across S, not necessarily the flux across S_a. To calculate the flux across S, let E be the solid between surfaces S_a and S. Then, the boundary of E consists of S_a and S. Denote this boundary by $S - S_a$ to indicate that S is oriented outward but now S_a is oriented inward. We would like to apply the divergence theorem to solid E. Notice that the divergence theorem, as stated, can't handle a solid such as E because E has a hole. However, the divergence theorem can be extended to handle solids with holes, just as Green's theorem can be extended to handle regions with holes. This allows us to use the divergence theorem in the following way. By the divergence theorem,

$$\begin{aligned} \iint_{S-S_a} F_\tau \cdot dS = \iint_S F_\tau \cdot dS - \iint_{S_a} F_\tau \cdot dS \\ &= \iiint_E div\, F_\tau\, dV \\ &= \iiint_E 0\, dV = 0. \end{aligned}$$

Therefore,

$$\iint_S F_\tau \cdot dS = \iint_{S_a} F_\tau \cdot dS = 4\pi,$$

and we have our desired result.

□

Now we return to calculating the flux across a smooth surface in the context of electrostatic field $E = \frac{q}{4\pi\epsilon_0}F_\tau$ of a point charge at the origin. Let S be a piecewise smooth closed surface that encompasses the origin. Then

$$\begin{aligned} \iint_S E\cdot dS &= \iint_S \frac{q}{4\pi\epsilon_0}F_\tau \cdot dS \\ &= \frac{q}{4\pi\epsilon_0}\iint_S F_\tau\cdot dS \\ &= \frac{q}{\epsilon_0}. \end{aligned}$$

If S does not encompass the origin, then

$$\iint_S E\cdot dS = \frac{q}{4\pi\epsilon_0}\iint_S F_\tau \cdot dS = 0.$$

Therefore, we have justified the claim that we set out to justify: the flux across closed surface S is zero if the charge is outside of S, and the flux is q/ϵ_0 if the charge is inside of S.

This analysis works only if there is a single point charge at the origin. In this case, Gauss' law says that the flux of **E** across S is the total charge enclosed by S. Gauss' law can be extended to handle multiple charged solids in space, not just a single point charge at the origin. The logic is similar to the previous analysis, but beyond the scope of this text. In full generality, Gauss' law states that if S is a piecewise smooth closed surface and Q is the total amount of charge inside of S, then the flux of **E** across S is Q/ϵ_0.

Example 9.8.5: Using Gauss' law

Suppose we have four stationary point charges in space, all with a charge of 0.002 Coulombs (C). The charges are located at $(0, 0, 1)$, $(1, 1, 4)$, $(-1, 0, 0)$ and $(-2, -2, 2)$. Let **E** denote the electrostatic field generated by these point charges. If S is the sphere of radius 2 oriented outward and centered at the origin, then find

$$\iint_S E \cdot dS.$$

Solution

According to Gauss' law, the flux of **E** across S is the total charge inside of S divided by the electric constant. Since S has radius 2, notice that only two of the charges are inside of S: the charge at $0, 1, 1)$ and the charge at $(-1, 0, 0)$. Therefore, the total charge encompassed by S is 0.004 and, by Gauss' law,

$$\iint_S E \cdot dS = \frac{0.004}{8.854 \times 10^{-12}} \approx 4.418 \times 10^9\, V - m.$$

Exercise 9.8.4

Work the previous example for surface S that is a sphere of radius 4 centered at the origin, oriented outward.

Hint

Use Gauss' law.

Answer

$\approx 6.777 \times 10^9$

9.8.4 Key Concepts

- The divergence theorem relates a surface integral across closed surface S to a triple integral over the solid enclosed by S. The divergence theorem is a higher dimensional version of the flux form of Green's theorem, and is therefore a higher dimensional version of the Fundamental Theorem of Calculus.
- The divergence theorem can be used to transform a difficult flux integral into an easier triple integral and vice versa.
- The divergence theorem can be used to derive Gauss' law, a fundamental law in electrostatics.

9.8.5 Key Equations

- **Divergence theorem**

$$\iiint_E div\, F\, dV = \iint_S F \cdot dS$$

9.8.6 Glossary

divergence theorem

a theorem used to transform a difficult flux integral into an easier triple integral and vice versa

Gauss' law

if S is a piecewise, smooth closed surface in a vacuum and Q is the total stationary charge inside of S, then the flux of electrostatic field **E** across S is Q/ϵ_0

inverse-square law

the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge

9.8.7 Contributors and Attributions

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

9.8E: Exercises

9.8E.1 Exercise $\mathbf{9.8E.1}$

For the following exercises, use a computer algebraic system (CAS) and the divergence theorem to evaluate surface integral $\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds$ for the given choice of $\vec{\mathbf{F}}$ and the boundary surface S. For each closed surface, assume $\vec{\mathbf{N}}$ is the outward unit normal vector.

1. [T] $\vec{\mathbf{F}} x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$; S is the surface of cube $0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 < z \leq 1$.

2. [T] $\vec{\mathbf{F}}(x, y, z) = (\cos yz)\,\hat{\mathbf{i}} + e^{xz}\,\hat{\mathbf{j}} + 3z^2\,\hat{\mathbf{k}}$; S is the surface of hemisphere $z = \sqrt{4 - x^2 - y^2}$ together with disk $x^2 + y^2 \leq 4$ in the *xy*-plane.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = 75.3982 \tag{9.8E.1}$$

3. [T] $\vec{\mathbf{F}}(x, y, z) = (x^2 + y^2 - x^2)\,\hat{\mathbf{i}} + x^2 y\,\hat{\mathbf{j}} + 3z\,\hat{\mathbf{k}}$; S is the surface of the five faces of unit cube $0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 < z \leq 1$.

4. [T] $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$; S is the surface of paraboloid $z = x^2 + y^2$ for $0 \leq z \leq 9$.

Answer

\[\int_S \vecs{F} \cdot \vecs{N} \, ds = 127.2345\

5. [T] $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$; $S$is the surface of sphere $x^2 + y^2 + z^2 = 4$.

6. [T] $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + (z^2 - 1)\,\hat{\mathbf{k}}$; $S$is the surface of the solid bounded by cylinder $x^2 + y^2 = 4$ and planes $z = 0$ and $z = 1$.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = 37.699 \tag{9.8E.2}$$

7. [T] $\vec{\mathbf{F}}(x, y, z) = x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$; S is the surface bounded above by sphere $\rho = 2$ and below by cone $\varphi = \frac{\pi}{4}$ in spherical coordinates. (Think of S as the surface of an "ice cream cone.")

8. [T] $\vec{\mathbf{F}}(x, y, z) = x^3\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + 3a^2 z\,\hat{\mathbf{k}}\,(constant\ a > 0)$; S is the surface bounded by cylinder $x^2 + y^2 = a^2$ and planes $z = 0$ and $z = 1$.

Answer

$$\int_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, ds = \frac{9\pi a^4}{2} \tag{9.8E.3}$$

9. [T] Surface integral $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$, where S is the solid bounded by paraboloid $z = x^2 + y^2$ and plane $z = 4$, and $\vec{\mathbf{F}}(x, y, z) = (x + y^2 z^2)\,\hat{\mathbf{i}} + (y + z^2 x^2)\,\hat{\mathbf{j}} + (z + x^2 y^2)\,\hat{\mathbf{k}}$

10. [T] Use a CAS and the divergence theorem to calculate flux $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$, where $\vec{\mathbf{F}}(x, y, z) = (x^3 + y^3)\,\hat{\mathbf{i}} + (y^3 + z^3) j\,\hat{\mathbf{j}} + (z^3 + x^3)\,\hat{\mathbf{k}}$ and S is a sphere with center (0, 0) and radius 2.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = 241.2743 \tag{9.8E.4}$$

9.8E.2 Exercise $\mathbf{9.8E.2}$

1. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$, where $\vec{\mathbf{F}}(x,y,z) = (e^{y^2}\,\hat{\mathbf{i}} + (y+\sin(z^2))\,\hat{\mathbf{j}} + (z-1)\,\hat{\mathbf{k}}$ and S is upper hemisphere $x^2+y^2+z^2=1,\ z\geq 0$, oriented upward.

Answer

\[\iint_S \vecs{F} \cdot \vecs{dS}= \dfrac{\pi}{3}\

2. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$, where $\vec{\mathbf{F}}(x,y,z) = x^4\,\hat{\mathbf{i}} - x^3z^2\,\hat{\mathbf{j}} + 4xy^2z\,\hat{\mathbf{k}}$ and S is the surface bounded by cylinder $x^2+y^2=1$ and planes $z=x+2$ and $z=0$.

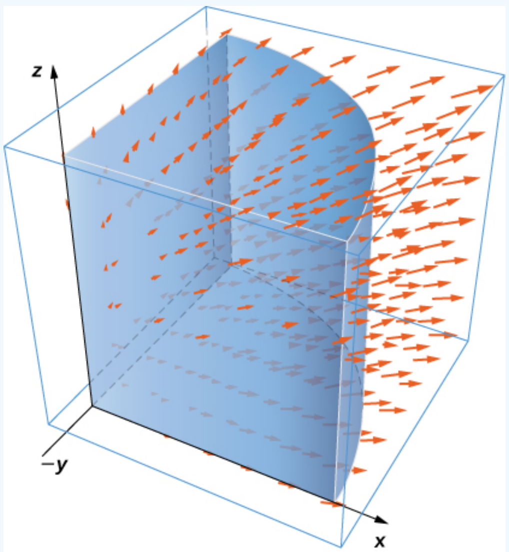
3. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$, when $\vec{\mathbf{F}}(x,y,z) = x^2z^3\,\hat{\mathbf{i}} + 2xyz^3\,\hat{\mathbf{j}} + xz^4\,\hat{\mathbf{k}}$ and S is the surface of the box with vertices $(\pm 1,\ \pm 2,\ \pm 3)$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = 0 \tag{9.8E.5}$$

4. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$, when $\vec{\mathbf{F}}(x,y,z) = z\,tan^{-1}(y^2)\,\hat{\mathbf{i}} + z^3\ln(x^2+1)\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is a part of paraboloid $x^2+y^2+z=2$ that lies above plane $z=1$ and is oriented upward.

5. Use the divergence theorem to compute the value of flux integral $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$, where $\vec{\mathbf{F}}(x,y,z) = (y^3+3x)\,\hat{\mathbf{i}} + (xz+y)\,\hat{\mathbf{j}} + [z+x^4\cos(x^2y)]\,\hat{\mathbf{k}}$ and S is the area of the region bounded by $x^2+y^2=1,\ x\geq 0,\ y\geq 0$, and $0\leq z\leq 1$.



6. Use the divergence theorem to compute flux integral $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$, where $\vec{\mathbf{F}}(x,y,z) = y\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}}$ and S consists of the union of paraboloid $y=x^2+z^2,\ 0\leq y\leq 1$, and disk $x^2+z^2\leq 1,\ y=1$, oriented outward. What is the flux through just the paraboloid?

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = -\pi \tag{9.8E.6}$$

7. Use the divergence theorem to compute flux integral $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$, where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z^4\,\hat{\mathbf{k}}$ and S is a part of cone $z=\sqrt{x^2+y^2}$ beneath top plane $z=1$ oriented downward.

8. Use the divergence theorem to calculate surface integral $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$ for $\vec{\mathbf{F}}(x,y,z) = x^4\,\hat{\mathbf{i}} - x^3z^2\,\hat{\mathbf{j}} + 4xy^2z\,\hat{\mathbf{k}}$, where S is the surface bounded by cylinder $x^2+y^2=1$ and planes $z=x+2$ and $z=0$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = \frac{2\pi}{3} \tag{9.8E.7}$$

9. Consider $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} + (z+1)\,\hat{\mathbf{k}}$. Let E be the solid enclosed by paraboloid $z=4-x^2-y^2$ and plane $z=0$ with normal vectors pointing outside E. Compute flux F across the boundary of E using the divergence theorem.

9.8E.3 Exercise $\mathbf{9.8}\boldsymbol{E}\mathbf{.3}$

For the following exercises, use a CAS along with the divergence theorem to compute the net outward flux for the fields across the given surfaces S.

1. [T] $\vec{\mathbf{F}} = \langle x,\ -2y,\ 3z\rangle$; S is sphere $\{(x,y,z) : x^2+y^2+z^2=6\}$.

Answer

\(15\sqrt{6}\pi\

2. [T] $\vec{\mathbf{F}} = \langle x,\ 2y,\ z\rangle$; S is the boundary of the tetrahedron in the first octant formed by plane $x+y+z=1$.

3. [T[$\vec{\mathbf{F}} = \langle y-2x,\ x^3-y,\ y^2-z\rangle$; S is sphere $\{(x,y,z) : x^2+y^2+z^2=4\}$.

Answer

$-\frac{128}{3}\pi$

4. [T] $\vec{\mathbf{F}} = \langle x,y,z\rangle$; S is the surface of paraboloid $z=4-x^2-y^2$, for $z \geq 0$, plus its base in the *xy*-plane.

9.8E.4 Exercise $\mathbf{9.8}\boldsymbol{E}\mathbf{.4}$

For the following exercises, use a CAS and the divergence theorem to compute the net outward flux for the vector fields across the boundary of the given regions D.

1. [T] $\vec{\mathbf{F}} = \langle z-x,\ x-y,\ 2y-z\rangle$; D is the region between spheres of radius 2 and 4 centered at the origin.

Answer

\(-703.7168\

2. [T] $\vec{\mathbf{F}} = \frac{r}{|r|} = \frac{\langle x,y,z\rangle}{\sqrt{x^2+y^2+z^2}}$; D is the region between spheres of radius 1 and 2 centered at the origin.

3.[T] $\vec{\mathbf{F}} = \langle x^2,\ -y^2,\ z^2\rangle$; D is the region in the first octant between planes $z=4-x-y$ and $z=2-x-y$.

Answer

20

9.8E.5 Exercise $\mathbf{9.8}\boldsymbol{E}\mathbf{.5}$

1. Let $\vec{\mathbf{F}}(x,y,z) = 2x\,\hat{\mathbf{i}} - 3xy\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}$. Use the divergence theorem to calculate $\iint_S F \cdot dS$, where S is the surface of the cube with corners at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,0,1)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$, oriented outward.

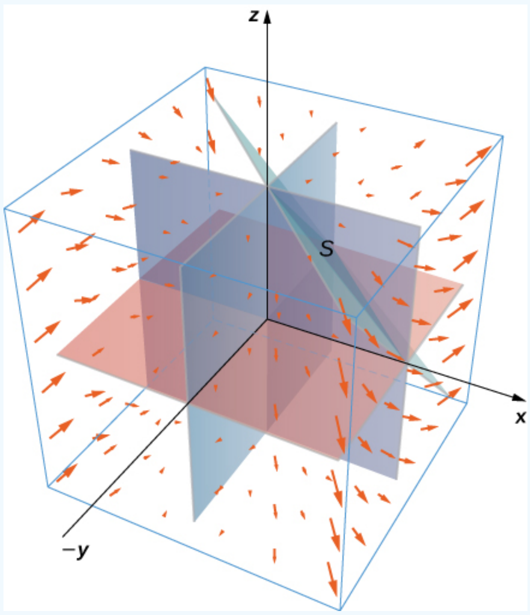
2. Use the divergence theorem to find the outward flux of field $\vec{\mathbf{F}}(x,y,z) = (x^3-3y)\,\hat{\mathbf{i}} + (2yz+1)\,\hat{\mathbf{j}} + xyz\,\hat{\mathbf{k}}$ through the cube bounded by planes $x=\pm 1$, $y=\pm 1$, and $z=\pm 1$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = 8 \tag{9.8E.8}$$

3. Let $\vec{\mathbf{F}}(x,y,z) = 2x\,\hat{\mathbf{i}} - 3y\,\hat{\mathbf{j}} + 5z\,\hat{\mathbf{k}}$ and let S be hemisphere $z = \sqrt{9 - x^2 - y^2}$ together with disk $x^2 + y^2 \leq 9$ in the *xy*-plane. Use the divergence theorem.

4. Evaluate $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$, where $\vec{\mathbf{F}}(x,y,z) = x^2\,\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} + x^3y^3\,\hat{\mathbf{k}}$ and S is the surface consisting of all faces except the tetrahedron bounded by plane $x + y + z = 1$ and the coordinate planes, with outward unit normal vector $\vec{\mathbf{N}}$.



Answer

\[\iint_S \vecs{F} \cdot \vecs{dS} = \dfrac{1}{8}\]

5. Find the net outward flux of field $F = \langle bz - cy,\ cx - az,\ ay - bx \rangle$ across any smooth closed surface in R^3 where *a*, *b*, and *c* are constants.

6. Use the divergence theorem to evaluate

$$\iint_S ||R||R \cdot n\, ds, \tag{9.8E.9}$$

where $R(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is sphere $x^2 + y^2 + z^2 = a^2$, with constant $a > 0$.

Answer

$$\iint_S ||R||R \cdot n\, ds = 4\pi a^4 \tag{9.8E.10}$$

7. Use the divergence theorem to evaluate

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}, \tag{9.8E.11}$$

where $\vec{\mathbf{F}}(x,y,z) = y^2z\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$ and S is the boundary of the cube defined by $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, and $0 \leq z \leq 2$.

8. Let R be the region defined by $x^2 + y^2 + z^2 \leq 1$. Use the divergence theorem to find

$$\iiint_R z^2 dV. \tag{9.8E.12}$$

Answer

$$\iiint_R z^2 dV = \frac{4\pi}{15} \tag{9.8E.13}$$

9. Let E be the solid bounded by the *xy*-plane and paraboloid $z = 4 - x^2 - y^2$ so that S is the surface of the paraboloid piece together with the disk in the *xy*-plane that forms its bottom. If $\vec{\mathbf{F}}(x, y, z) = (xz\,\sin(yz) + x^3)\,\hat{\mathbf{i}} + \cos(yz)\,\hat{\mathbf{j}} + (3zy^2 - e^{x^2+y^2})\,\hat{\mathbf{k}}$, find

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} \tag{9.8E.14}$$

using the divergence theorem.

A vector field in three dimensions with all of the arrows pointing down. They seem to follow the path of the paraboloid drawn opening down with vertex at the origin. S is the surface of this paraboloid and the disk in the (x, y) plane that forms its bottom.

10. Let *E* be the solid unit cube with diagonally opposite corners at the origin and (1, 1, 1), and faces parallel to the coordinate planes. Let *S* be the surface of *E*, oriented with the outward-pointing normal. Use a CAS to find

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} \tag{9.8E.15}$$

using the divergence theorem if $\vec{\mathbf{F}}(x, y, z) = 2xy\,\hat{\mathbf{i}} + 3ye^z\,\hat{\mathbf{j}} + x\,\sin z\,\hat{\mathbf{k}}$.

Answer

\[\iint_S \vecs{F} \cdot \vecs{dS}= 6.5759\

11. Use the divergence theorem to calculate the flux of $\vec{\mathbf{F}}(x, y, z) = x^3\,\hat{\mathbf{i}} + y^3\,\hat{\mathbf{j}} + z^3\,\hat{\mathbf{k}}$ through sphere $x^2 + y^2 + z^2 = 1$.

12. Find

$$\iint_S F \cdot dS, \tag{9.8E.16}$$

where $\vec{\mathbf{F}}(x, y, z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the outwardly oriented surface obtained by removing cube $[1, 2] \times [1, 2] \times [1, 2]$ from cube $[0, 2] \times [0, 2] \times [0, 2]$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = 21 \tag{9.8E.17}$$

13. Consider radial vector field $F = \frac{r}{|r|} = \frac{\langle x, y, z\rangle}{(x^2 + y^2 + z^2)^{1/2}}$. Compute the surface integral, where S is the surface of a sphere of radius *a* centered at the origin.

9.8E.6 Exercise $9.8E.6$

1. Compute the flux of water through parabolic cylinder $S : y = x^2$, from $0 \le x \le 2$, $0 \le z \le 3$, if the velocity vector is $\vec{\mathbf{F}}(x, y, z) = 3z^2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}} + 6xz\,\hat{\mathbf{k}}$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = 72 \tag{9.8E.18}$$

2. [T} Use a CAS to find the flux of vector field $\vec{\mathbf{F}}(x,y,z) = z\,\hat{\mathbf{i}} + z\,\hat{\mathbf{j}} + \sqrt{x^2+y^2}\,\hat{\mathbf{k}}$ across the portion of hyperboloid $x^2+y^2 = z^2+1$ between planes $z=0$ and $z=\frac{\sqrt{3}}{3}$, oriented so the unit normal vector points away from the z-axis.

3. Use a CAS to find the flux of vector field $\vec{\mathbf{F}}(x,y,z) = (e^y+x)\,\hat{\mathbf{i}} + (3\,\cos(xz)-y)\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ through surface S, where S is given by $z^2 = 4x^2+4y^2$ from $0 \le z \le 4$, oriented so the unit normal vector points downward.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = -33.5103 \tag{9.8E.19}$$

4. [T] Use a CAS to compute

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}, \tag{9.8E.20}$$

where $\vec{\mathbf{F}}(x,y,z) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}}$ and S is a part of sphere $x^2+y^2+z^2=2$ with $0 \le z \le 1$.

5. Evaluate

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}, \tag{9.8E.21}$$

where $\vec{\mathbf{F}}(x,y,z) = bxy^2\,\hat{\mathbf{i}} + bx^2y\,\hat{\mathbf{j}} + (x^2+y^2)z^2\,\hat{\mathbf{k}}$ and S is a closed surface bounding the region and consisting of solid cylinder $x^2+y^2 \le a^2$ and $0 \le z \le b$.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = \pi a^4 b^2 \tag{9.8E.22}$$

6. [T] Use a CAS to calculate the flux of $\vec{\mathbf{F}}(x,y,z) = (x^3+y\,\sin z)\,\hat{\mathbf{i}} + (y^3+z\,\sin x)\,\hat{\mathbf{j}} + 3z\,\hat{\mathbf{k}}$ across surface S, where S is the boundary of the solid bounded by hemispheres $z=\sqrt{4-x^2-y^2}$ and $z=\sqrt{1-x^2-y^2}$, and plane $z=0$.

7. Use the divergence theorem to evaluate

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}, \tag{9.8E.23}$$

where $\vec{\mathbf{F}}(x,y,z) = xy\,\hat{\mathbf{i}} - \frac{1}{2}y^2\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the surface consisting of three pieces: $z=4-3x^2-3y^2,\ 1 \le z \le 4$ on the top; $x^2+y^2=1,\ 0 \le z \le 1$ on the sides; and $z=0$ on the bottom.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = \frac{5}{2}\pi \tag{9.8E.24}$$

8. [T] Use a CAS and the divergence theorem to evaluate

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}, \tag{9.8E.25}$$

where $\vec{\mathbf{F}}(x,y,z) = (2x+y\,\cos z)\,\hat{\mathbf{i}} + (x^2-y)\,\hat{\mathbf{j}} + y^2z\,\hat{\mathbf{k}}$ and S is sphere $x^2+y^2+z^2=4$ orientated outward.

9. Use the divergence theorem to evaluate

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}, \tag{9.8E.26}$$

where $\vec{\mathbf{F}}(x,y,z) = xi+yj+zk$ and S is the boundary of the solid enclosed by paraboloid $y=x^2+z^2-2$, cylinder $x^2+z^2=1$, and plane $x+y=2$, and S is oriented outward.

Answer

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}} = \frac{21\pi}{2} \tag{9.8E.27}$$

9.8E.7 Exercise $9.8E.7$

For the following exercises, **Fourier's law of heat transfer** states that the heat flow vector $\vec{\mathbf{F}}$ at a point is proportional to the negative gradient of the temperature; that is, $\vec{\mathbf{F}} = -k\nabla T$, which means that heat energy flows hot regions to cold regions. The constant $k > 0$ is called the *conductivity*, which has metric units of joules per meter per second-kelvin or watts per meter-kelvin. A temperature function for region D is given. Use the divergence theorem to find net outward heat flux

$$\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}}\, dS = -k \iint_S \nabla T \cdot \vec{\mathbf{N}}\, dS \tag{9.8E.28}$$

across the boundary S of D, where $k = 1$.

1. $T(x,y,z) = 100 + x + 2y + z$; $D = \{(x,y,z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$
2. $T(x,y,z) = 100 + e^{-z}$; $D = \{(x,y,z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$

Answer

$-(1 - e^{-1})$

3. $T(x,y,z) = 100e^{-x^2-y^2-z^2}$; D is the sphere of radius a centered at the origin.

9E: Chapter Exercises

9E.1 Exercise $\PageIndex{1}$: *True or False?*

Justify your answer with a proof or a counterexample.

1. Vector field $\vecs{F}(x,y)=x^2y\,\hat{\mathbf{i}}+y^2x\,\hat{\mathbf{j}}$ is conservative.

Answer

False

2. For vector field $\vecs{F}(x,y)=P(x,y)\,\hat{\mathbf{i}}+Q(x,y)\,\hat{\mathbf{j}}$, if $P_y(x,y)=Q_z(x,y)$ in open region D, then

$$\int_{\partial D} P\,dx+Q\,dy=0. \tag{9E.1}$$

3. The divergence of a vector field is a vector field.

Answer

False

4. If $curl\,\vecs{F}=0$, then $\vecs{F}$ is a conservative vector field.

9E.2 Exercise $\PageIndex{2}$

Draw the following vector fields.

1. $\vecs{F}(x,y)=\frac{1}{2}\,\hat{\mathbf{i}}+2x\,\hat{\mathbf{j}}$

Answer

A vector field in two dimensions. All quadrants are shown. The arrows are larger the further from the y axis they become. They point up and to the right for positive x values and down and to the right for negative x values. The further from the y axis they are, the steeper the slope they have.

2. $\vecs{F}(x,y)=\dfrac{y\,\hat{\mathbf{i}}+3x\,\hat{\mathbf{j}}}{x^2+y^2}$

9E.3 Exercise $\PageIndex{3}$

Are the following the vector fields conservative? If so, find the potential function f such that $\vecs{F}=\nabla f$.

1. $\vecs{F}(x,y)=y\,\hat{\mathbf{i}}+(x-2e^y)\,\hat{\mathbf{j}}$

Answer

Conservative, $f(x,y)=xy-2e^y$

2. $\vecs{F}(x,y)=(6xy)\,\hat{\mathbf{i}}+(3x^2-ye^y)\,\hat{\mathbf{j}}$
3. $\vecs{F}(x,y)=(2xy+z^2)\,\hat{\mathbf{i}}+(x^2+2yz)\,\hat{\mathbf{j}}+(2xz+y^2)\,\hat{\mathbf{k}}$

Answer

Conservative, $f(x,y,z)=x^2y+y^2z+z^2x$

4. $\vecs{F}(x,y,z)=(e^xy)\,\hat{\mathbf{i}}+(e^x+z)\,\hat{\mathbf{j}}+(e^x+y^2)\,\hat{\mathbf{k}}$

9E.4 Exercise $9E.4$

Evaluate the following integrals.

1. $\int_C x^2 dy + (2x - 3xy)\,dx$, along $C : y = \dfrac{1}{2}x$ from (0, 0) to (4, 2)

Answer

$-\dfrac{16}{3}$

2. $\int_C y dx + xy^2 dy$, where $C : x = \sqrt{t},\ y = t - 1,\ 0 \le t \le 1$
3. $\iint_S xy^2 dS$, where S is surface $z = x^2 - y,\ 0 \le x \le 1,\ 0 \le y \le 4$

Answer

A$\dfrac{32\sqrt{2}}{9}(3\sqrt{3} - 1)$

9E.5 Exercise $9E.5$

Find the divergence and curl for the following vector fields.

1. $\vec{\mathbf{F}}(x, y, z) = 3xyz\,\hat{\mathbf{i}} + xye^x\,\hat{\mathbf{j}} - 3xy\,\hat{\mathbf{k}}$
2. $\vec{\mathbf{F}}(x, y, z) = e^x\ \hat{\mathbf{i}} + e^{xy}\,\hat{\mathbf{j}} - e^{xyz}\,\hat{\mathbf{k}}$

Answer

Divergence: $e^x + x\,e^{xy} + xy\,e^{xyz}$, curl: $xze^{xyz}\ \hat{\mathbf{i}} - yze^{xyz}\hat{j} + ye^{xy}\hat{k}$

9E.6 Exercise $9E.6$

Use Green’s theorem to evaluate the following integrals.

1. $\int_C 3xydx + 2xy^2dy$, where C is a square with vertices $(0, 0)$, $(0, 2)$, $(2, 2)$and $(2, 0)$.
2. $\oint_C 3ydx + (x + e^y)dy$, where C is a circle centered at the origin with radius 3.

Answer

-2π

9E.7 Exercise $9E.7$

Use Stokes’ theorem to evaluate $\iint_S curl\,\vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$.

1. $\vec{\mathbf{F}}(x, y, z) = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, where S is the upper half of the unit sphere
2. $\vec{\mathbf{F}}(x, y, z) = y\,\hat{\mathbf{i}} + xyz\,\hat{\mathbf{j}} - 2zx\,\hat{\mathbf{k}}$, where S is the upward-facing paraboloid $z = x^2 + y^2$ lying in cylinder $x^2 + y^2 = 1$

Answer

$-\pi$

9E.8 Exercise $9E.8$

Use the divergence theorem to evaluate $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{dS}}$.

1. $\vec{\mathbf{F}}(x, y, z) = (x^3y)\,\hat{\mathbf{i}} + (3y - e^x)\,\hat{\mathbf{j}} + (z + x)\,\hat{\mathbf{k}}$, over cube S defined by $-1 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 2$
2. $\vec{\mathbf{F}}(x, y, z) = (2xy)\,\hat{\mathbf{i}} + (-y^2)\,\hat{\mathbf{j}} + (2z^3)\,\hat{\mathbf{k}}$, where S is bounded by paraboloid $z = x^2 + y^2$ and plane $z = 2$

Answer

$31\pi/2$

9E.9 Exercise $9E.9$

1. Find the amount of work performed by a 50-kg woman ascending a helical staircase with radius 2 m and height 100 m. The woman completes five revolutions during the climb.

2. Find the total mass of a thin wire in the shape of a semicircle with radius $\sqrt{2}$, and a density function of $\rho(x,y)=y+x^2$.

Answer

$\sqrt{2}(2+\pi)$

3. Find the total mass of a thin sheet in the shape of a hemisphere with radius 2 for $z\geq 0$ with a density function $\rho(x,y,z)=x+y+z$.

4. Use the divergence theorem to compute the value of the flux integral over the unit sphere with $\vec{\mathbf{F}}(x,y,z)=3z\,\hat{\mathbf{i}}+2y\,\hat{\mathbf{j}}+2x\,\hat{\mathbf{k}}$.

Answer

$2\pi/3$

CHAPTER OVERVIEW

Summary Tables

This page is a draft and is under active development.

Topic hierarchy

Summary Table Of Integrals
Summary of Convergence Tests
Summary of Theorems
Summary table of derivatives

Summary Table Of Integrals

This page is a draft and is under active development.

1 Indefinite Integral

$$\int u^{\alpha}\,du = \frac{u^{\alpha+1}}{\alpha+1}+c, \quad \alpha \neq -1$$

$$\int \frac{du}{u} = \ln|u|+c$$

$$\int \cos u\,du = \sin u+c$$

$$\int \sin u\,du = -\cos u+c$$

$$\int \tan u\,du = -\ln|\cos u|+c$$

$$\int \cot u\,du = \ln|\sin u|+c$$

$$\int \sec^2 u\,du = \tan u+c$$

$$\int \csc^2 u\,du = -\cot u+c$$

$$\int \sec u\,du = \ln|\sec u+\tan u|+c$$

$$\int \csc(u)du = \ln|\csc(u)-\cot(u)|+c$$

$$\int \cos^2 u\,du = \frac{u}{2}+\frac{1}{4}\sin 2u+c$$

$$\int \sin^2 u\,du = \frac{u}{2}-\frac{1}{4}\sin 2u+c$$

$$\int \frac{du}{1+u^2}\,du = \tan^{-1}u+c$$

$$\int \frac{du}{\sqrt{1-u^2}}\,du = \sin^{-1}u+c$$

$$\int \frac{1}{u^2-1}\,du = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+c$$

2

3 Integration Rules

$$\int (Af(x)+Bg(x)dx = A\int f(x)dx + B\int g(x)dx$$

$$\int f'(g(x))g'(x)dx = f(g(x))+C$$

4 Definite Integral

$$\int_a^a f(x)\,dx = 0$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)dx$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)$$

for constant c. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

Let f be continuous on $[a,b]$ and let F be any anti-derivative of f. Then $\int_a^b f(x)dx = F(b)-F(a)$.

4.1 Reduction formulas

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx. \tag{1}$$

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx. \tag{2}$$

$$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx, n \neq 1. \tag{3}$$

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx, n \geq 2.$$

5 Integration by parts

$$\int u\,dv = uv - \int v\,du$$

$$\int u\cos u\,du = u\sin u + \cos u + c$$

$$\int u\sin u\,du = -u\cos u + \sin u + c$$

$$\int ue^u\,du = ue^u - e^u + c$$

$$\int e^{\lambda u}\cos\omega u\,du = \frac{e^{\lambda u}(\lambda\cos\omega u + \omega\sin\omega u)}{\lambda^2+\omega^2} + c$$

$$\int e^{\lambda u}\sin\omega u\,du = \frac{e^{\lambda u}(\lambda\sin\omega u - \omega\cos\omega u)}{\lambda^2+\omega^2} + c$$

$$\int \ln|u|\,du = u\ln|u| - u + c$$

$$\int u\ln|u|\,du = \frac{u^2\ln|u|}{2} - \frac{u^2}{4} + c$$

$$\int \cos\omega_1 u\cos\omega_2 u\,du = \frac{\sin(\omega_1+\omega_2)u}{2(\omega_1+\omega_2)} + \frac{\sin(\omega_1-\omega_2)u}{2(\omega_1-\omega_2)} + c \quad (\omega_1 \neq \pm\omega_2)$$

$$\int \sin\omega_1 u\sin\omega_2 u\,du = -\frac{\sin(\omega_1+\omega_2)u}{2(\omega_1+\omega_2)} + \frac{\sin(\omega_1-\omega_2)u}{2(\omega_1-\omega_2)} + c \quad (\omega_1 \neq \pm\omega_2)$$

$$\int \sin\omega_1 u\cos\omega_2 u\,du = -\frac{\cos(\omega_1+\omega_2)u}{2(\omega_1+\omega_2)} - \frac{\cos(\omega_1-\omega_2)u}{2(\omega_1-\omega_2)} + c \quad (\omega_1 \neq \pm\omega_2)$$

6 Improper Integrals

Let $f(x)$ be continuous over an interval of the form $[a,+\infty)$. Then

$$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty}\int_a^t f(x)dx, \tag{5}$$

provided this limit exists.

Let $f(x)$ be continuous over an interval of the form $(-\infty,b]$. Then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx, \tag{6}$$

provided this limit exists.

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

Let $f(x)$ be continuous over $(-\infty,+\infty)$. Then

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^{+\infty} f(x)dx, \tag{7}$$

provided that $\int_{-\infty}^0 f(x)dx$ and $\int_0^{+\infty} f(x)dx$ both converge. If either of these two integrals diverge, then $\int_{-\infty}^{+\infty} f(x)dx$ diverges. (It can be shown that, in fact, $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^{+\infty} f(x)dx$ for any value of a.).

Let $f(x)$ be continuous over $[a,b)$. Then,

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx. \tag{8}$$

Let $f(x)$ be continuous over $(a,b]$. Then,

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx. \tag{9}$$

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

If $f(x)$ is continuous over $[a,b]$ except at a point c in (a,b) , then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \tag{10}$$

provided both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge. If either of these integrals diverges, then $\int_a^b f(x)dx$ diverges.

Summary of Convergence Tests

This page is a draft and is under active development.

1 Summary of Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n\to\infty} a_n$.	If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive. If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.	This test cannot prove convergence of a series.
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\lvert r\rvert < 1$, the series converges to $a/(1-r)$. If $\lvert r\rvert \geq 1$, the series diverges.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
p-Series $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges. If $p \leq 1$, the series diverges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx$.	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} \lvert \frac{a_{n+1}}{a_n} \rvert$	If $0 \leq \rho < 1$, the series converges absolutely. If $\rho > 1$ or $\rho = \infty$, the series diverges. If $\rho = 1$, the test is inconclusive.	Often used for series involving factorials or exponentials.
Root Test For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert}$.	If $0 \leq \rho < 1$, the series converges absolutely. If $\rho > 1$ or $\rho = \infty$, the series diverges. If $\rho = 1$, the test is inconclusive.	Often used for series where $\lvert a_n\rvert = b_n^n$.

Summary of Theorems

1 Fundamental Theorem of Line Integrals

Suppose a curve C is given by the vector function $\mathbf{r}(t)$, with $\mathbf{a}=\mathbf{r}(a)$ and $\mathbf{b}=\mathbf{r}(b)$. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}), \tag{1}$$

provided that $\mathbf{r}$ is sufficiently nice.

2 Green's Theorem

If the vector field $\mathbf{F}=\langle P, Q\rangle$ and the region D are sufficiently nice, and if C is the boundary of D (C is a closed curve), then

$$\iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_C P\, dx + Q\, dy, \tag{2}$$

provided the integration on the right is done counter-clockwise around C.

3 Stoke's Theorem

Provided that the quantities involved are sufficiently nice, and in particular if D is orientable,

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint\limits_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS, \tag{3}$$

if ∂D is oriented counter-clockwise relative to $\mathbf{N}$.

4 Green's Theorem(3D)

If the vector field $\mathbf{F}=\langle P, Q\rangle$ and the region D are sufficiently nice, and if C is the boundary of D (C is a closed curve), then

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N}\, ds = \iint\limits_D \nabla \cdot \mathbf{F}\, dA.$$

5 Divergence Theorem

Under suitable conditions, if E is a region of three dimensional space and D is its boundary surface, oriented outward, then

$$\iint\limits_D \mathbf{F} \cdot \mathbf{N}\, dS = \iiint\limits_E \nabla \cdot \mathbf{F}\, dV.$$

Summary table of derivatives

This page is a draft and is under active development.

1 Differentiation Rules

Sum Rule	$\frac{d}{dx}(f(x)+g(x))=(f(x)+g(x))'=f'(x)+g'(x)$
Constant Multiple Rule	$\frac{d}{dx}(cf(x))=(cf(x))'=cf'(x)$
Product Rule	$\frac{d}{dx}(f(x)g(x))=(f(x)g(x))'=f'(x)g(x)+f(x)g'(x)$
	$\frac{d}{dx}(\frac{1}{f(x)})=-\frac{f'(x)}{(f(x))^2}$
Quotient Rule	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\left(\frac{f(x)}{g(x)}\right)'=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$
Chain Rule	$\frac{d}{dx}f(g(x))=(f(g(x)))'=f'(g(x))g'(x)$

2 Derivatives for Elementary Trancendental Functions

$\frac{d}{dx}x^n=nx^{n-1}$
$\frac{d}{dx}e^x=e^x$
$\frac{d}{dx}b^x=b^x ln(b)$, where $b>0$
$\frac{d}{dx}\ln(\lvert x\rvert)=\frac{1}{x}$,$x\neq 0$
$\frac{d}{dx}\log_b(\lvert x\rvert)=\frac{1}{x\ \ln(b)}$, $x\neq 0$
$\frac{d}{dx}\sin(x)=\cos(x)$
$\frac{d}{dx}\cos(x)=-\sin(x)$
$\frac{d}{dx}\tan(x)=\sec^2(x)$
$\frac{d}{dx}\sec(x)=\sec(x)tan(x)$
$\frac{d}{dx}\csc(x)=-\csc(x)cot(x)$
$\frac{d}{dx}\cot(x)=-\csc^2(x)$
$\frac{d}{dx}\sin^{-1}(x)=\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\tan^{-1}(x)=\frac{1}{1+x^2}$
$\frac{d}{dx}\sec^{-1}(x)=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\cos^{-1}(x)=-\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\cot^{-1}(x)=-\frac{1}{1+x^2}$

$\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\lvert x\rvert = sgn(x) = \frac{x}{\lvert x\rvert}, x \neq 0$

Index

M

method of Frobenius

3.6: The Method of Frobenius II

Glossary

Sample Word 1 | Sample Definition 1

Detailed Licensing

Overview

Title: MATH 3200: Mathematical Methods

Webpages: 132

Applicable Restrictions: Noncommercial

All licenses found:

- CC BY-NC-SA 4.0: 60.6% (80 pages)
- Undeclared: 39.4% (52 pages)

By Page

- MATH 3200: Mathematical Methods - *CC BY-NC-SA 4.0*
  - Front Matter - *Undeclared*
    - TitlePage - *Undeclared*
    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - 1: Power Series - *CC BY-NC-SA 4.0*
    - 1 E: Chapter Exercises - *CC BY-NC-SA 4.0*
    - 1.1: Power Series - *CC BY-NC-SA 4.0*
      - 1.1E: Exercises - *CC BY-NC-SA 4.0*
    - 1.2: Properties of Power Series - *CC BY-NC-SA 4.0*
      - 1.2E: Exercises - *CC BY-NC-SA 4.0*
    - 1.3: Taylor and Maclaurin Series - *CC BY-NC-SA 4.0*
      - 1.3E: Exercises - *CC BY-NC-SA 4.0*
    - 1.4: Working with Taylor Series - *CC BY-NC-SA 4.0*
      - 1.4E: Exercises - *CC BY-NC-SA 4.0*
  - 2: Ordinary differential equations - *CC BY-NC-SA 4.0*
    - 2.1: Linear Second Order Homogeneous Equations - *CC BY-NC-SA 4.0*
      - 2.1E: Exercises - *CC BY-NC-SA 4.0*
    - 2.2: Linear Second Order Constant Coefficient Homogeneous Equations - *CC BY-NC-SA 4.0*
      - 2.2E: Exercises - *CC BY-NC-SA 4.0*
    - 2.3: Linear Second Order Nonhomogeneous Linear Equations - *CC BY-NC-SA 4.0*
      - 2.3E: Exercises - *CC BY-NC-SA 4.0*
    - 2.4: The Method of Undetermined Coefficient - *CC BY-NC-SA 4.0*
      - 2.4E: Exercises - *CC BY-NC-SA 4.0*
    - 2E: Exercises - *CC BY-NC-SA 4.0*
  - 3: Series Solutions of Linear Second order Equations - *CC BY-NC-SA 4.0*
    - 3.1: Review of Power Series - *CC BY-NC-SA 4.0*
      - 3.1E: Exercises - *CC BY-NC-SA 4.0*
    - 3.2: Series Solutions Near an Ordinary Point I - *CC BY-NC-SA 4.0*
      - 3.2E: Exercises - *CC BY-NC-SA 4.0*
    - 3.3: Series Solutions Near an Ordinary Point II - *CC BY-NC-SA 4.0*
      - 3.3E: Exercises - *CC BY-NC-SA 4.0*
    - 3.4: Regular Singular Points: Euler Equations - *CC BY-NC-SA 4.0*
      - 3.4E: Exercises - *CC BY-NC-SA 4.0*
    - 3.5: The Method of Frobenius I - *CC BY-NC-SA 4.0*
      - 3.5E: Exercises - *CC BY-NC-SA 4.0*
    - 3.6: The Method of Frobenius II - *CC BY-NC-SA 4.0*
      - 3.6E: Exercises - *CC BY-NC-SA 4.0*
    - 3.7: The Method of Frobenius III - *CC BY-NC-SA 4.0*
      - 3.7E: Exercises - *CC BY-NC-SA 4.0*
  - 4: Linear Systems of Ordinary Differential Equations (LSODE) - *CC BY-NC-SA 4.0*
    - 4.1: Introduction to Systems of Differential Equations - *CC BY-NC-SA 4.0*
      - 4.1E: Exercises - *CC BY-NC-SA 4.0*
    - 4.2: Linear Systems of Differential Equations - *CC BY-NC-SA 4.0*
      - 4.2E: Exercises - *CC BY-NC-SA 4.0*
    - 4.3: Basic Theory of Homogeneous Linear System - *CC BY-NC-SA 4.0*
      - 4.3E: Exercises - *CC BY-NC-SA 4.0*
    - 4.4: Constant Coefficient Homogeneous Systems I - *CC BY-NC-SA 4.0*
      - 4.4E: Exercises - *CC BY-NC-SA 4.0*
    - 4.5: Constant Coefficient Homogeneous Systems II - *CC BY-NC-SA 4.0*
      - 4.5E: Exercises - *CC BY-NC-SA 4.0*
    - 4.6: Constant Coefficient Homogeneous Systems III - *CC BY-NC-SA 4.0*

- 4.6E: Exercises - *CC BY-NC-SA 4.0*
- 4.7: Variation of Parameters for Nonhomogeneous Linear Systems - *CC BY-NC-SA 4.0*
  - 4.7E: Exercises - *CC BY-NC-SA 4.0*
- 5: Vector-Valued Functions - *CC BY-NC-SA 4.0*
  - 5.1: Vector-Valued Functions and Space Curves - *CC BY-NC-SA 4.0*
    - 5.1E: - *Undeclared*
  - 5.2: Calculus of Vector-Valued Functions - *CC BY-NC-SA 4.0*
    - 5.2E: - *Undeclared*
  - 5.3: Arc Length and Curvature - *CC BY-NC-SA 4.0*
    - 5.3E: Exercises - *Undeclared*
  - 5.4: Motion in Space - *CC BY-NC-SA 4.0*
    - 5.4E:Exercises - *Undeclared*
  - 5E: Excercises - *Undeclared*
- 6: Differentiation of Functions of Several Variables - *CC BY-NC-SA 4.0*
  - 6.1: Functions of Several Variables - *CC BY-NC-SA 4.0*
    - 6.1E: - *Undeclared*
  - 6.2: Limits and Continuity - *CC BY-NC-SA 4.0*
    - 6.2E: - *Undeclared*
  - 6.3: Partial Derivatives - *CC BY-NC-SA 4.0*
    - 6.3E: - *Undeclared*
  - 6.4: Tangent Planes and Linear Approximations - *CC BY-NC-SA 4.0*
    - 6.4E:Exercises - *Undeclared*
  - 6.5: The Chain Rule for Multivariable Functions - *CC BY-NC-SA 4.0*
    - 6.5E: Exercises - *Undeclared*
  - 6.6: Directional Derivatives and the Gradient - *CC BY-NC-SA 4.0*
    - 6.6E: Exercises - *Undeclared*
  - 6.7: Maxima/Minima Problems - *CC BY-NC-SA 4.0*
    - 6.7E: - *Undeclared*
  - 6.8: Lagrange Multipliers - *CC BY-NC-SA 4.0*
    - 6.8E: - *Undeclared*
      - 6E: Chapter Review Excersies - *Undeclared*
- 7: Multiple Integration - *CC BY-NC-SA 4.0*
  - 7.1 :Double Integrals over Rectangular Regions - *CC BY-NC-SA 4.0*
    - 7.1E: Exercises - *Undeclared*
  - 7.2: Double Integrals over General Regions - *CC BY-NC-SA 4.0*
    - 7.2E: Exercises - *Undeclared*
  - 7.3: Double Integrals in Polar Coordinates - *CC BY-NC-SA 4.0*
    - 7.3E: - *Undeclared*
  - 7.4: Triple Integrals - *CC BY-NC-SA 4.0*
    - 7.4E: - *Undeclared*
  - 7.5: Triple Integrals in Cylindrical and Spherical Coordinates - *CC BY-NC-SA 4.0*
    - 7.5E: - *Undeclared*
  - 7.6: Calculating Centers of Mass and Moments of Inertia - *CC BY-NC-SA 4.0*
    - 7.6E - *Undeclared*
  - 7.7: Change of Variables in Multiple Integrals - *CC BY-NC-SA 4.0*
    - 7.7E: - *Undeclared*
      - 7E: Chapter Review Excersies - *Undeclared*
- 8: Partial Differential Equations - *CC BY-NC-SA 4.0*
  - 8.1: Laplace Equations - *Undeclared*
  - 8.2: The heat equation - *Undeclared*
  - 8.3: Wave Equations - *Undeclared*
  - 8E: Excercise - *Undeclared*
- 9: Vector Calculus - *CC BY-NC-SA 4.0*
  - 9.1: Vector Fields - *CC BY-NC-SA 4.0*
    - 9.1E: Exercises - *Undeclared*
  - 9.2: Line Integrals - *Undeclared*
    - 9.2E: Exercises - *Undeclared*
  - 9.3: Conservative vector Fields - *Undeclared*
    - 9.3E: EXERCISES - *Undeclared*
  - 9.4: Green's Theorem - *Undeclared*
    - 9.4E: EXERCISES - *Undeclared*
  - 9.5: Divergence and Curl - *Undeclared*
    - 9.5E: EXERCISES - *Undeclared*
  - 9.6: Surface Integrals - *Undeclared*
    - 9.6E: Exercises - *Undeclared*
  - 9.7: Stoke's Theorem - *Undeclared*
    - 9.7E: EXERCISES - *Undeclared*
  - 9.8: The Divergence Theorem - *Undeclared*
    - 9.8E: Exercises - *Undeclared*
  - 9E: Chapter Exercises - *Undeclared*
- Summary Tables - *CC BY-NC-SA 4.0*
  - Summary of Convergence Tests - *CC BY-NC-SA 4.0*
  - Summary of Theorems - *Undeclared*
  - Summary table of derivatives - *CC BY-NC-SA 4.0*
  - Summary Table Of Integrals - *CC BY-NC-SA 4.0*
- Back Matter - *Undeclared*
  - Index - *Undeclared*
  - Glossary - *Undeclared*

- Detailed Licensing - *Undeclared*