Erik 2024-06-21

Friday, June 21, 2024 4:22 PM

Two dimensions [edit]

The Laplace operator in two dimensions is given by:

In **Cartesian coordinates**,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

where x and y are the standard Cartesian coordinates of the xy-plane.

In **polar coordinates**,

$$\begin{aligned}\Delta f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2},\end{aligned}$$

where r represents the radial distance and θ the angle.

Three dimensions [edit]

See also: Del in cylindrical and spherical coordinates

In three dimensions, it is common to work with the Laplacian in a variety of different coordinate systems.

In **Cartesian coordinates**,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

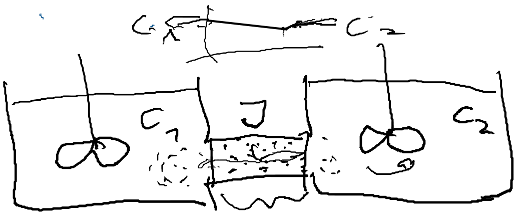
In **cylindrical coordinates**,

$$\Delta f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2},$$

where ρ represents the radial distance, φ the azimuth angle and z the height.

In **spherical coordinates**:

$$\Delta f = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2},$$



$$\cancel{\frac{\partial c}{\partial t}} = D\Delta c \qquad c\left(-\frac{L}{2}\right) = c_1$$
$$0 \qquad\qquad c\left(+\frac{L}{2}\right) = c_2$$

$$\Delta c = 0 \qquad c(x,y,z) = c(x)$$

$$c''(x) = 0 \qquad c = Ax + B$$

$$c\left(-\frac{L}{2}\right) = c_1 \qquad (1)\quad -A\frac{L}{2} + B = c_1$$

$$c\left(+\frac{L}{2}\right) = c_2 \qquad (2)\quad +A\frac{L}{2} + B = c_2$$

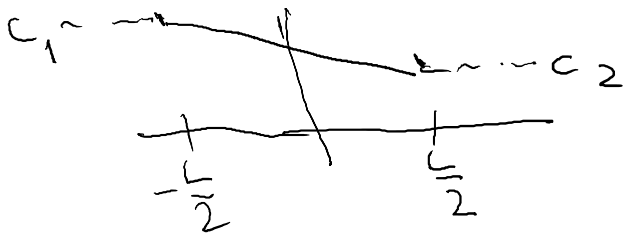
$$2B = c_1 + c_2$$

$$B = \frac{c_1 + c_2}{2}$$

$$(2) - (1) \qquad 2A\frac{L}{2} = c_2 - c_1$$

$$A = \frac{c_2 - c_1}{L}$$

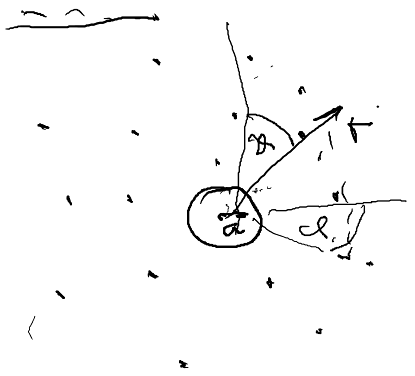
$$c(x) = \frac{c_2 - c_1}{L}x + \frac{c_1 + c_2}{2}$$



$$J = -D\,\mathrm{grad}\,c$$

$$J = -D\frac{c_2 - c_1}{L}$$

$$\mathcal{J} = AJ$$



Sférická symetria

Rel. súradnica r

$$\cancel{\frac{\partial c}{\partial t}} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right)$$

$$c(a) = 0$$
$$c(\infty) = c_0$$

$$\cancel{\frac{1}{r^2}}\frac{\partial}{\partial r} r^2\frac{\partial c}{\partial r} = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) = 0 \qquad r^2\frac{\partial c}{\partial r} = A \qquad \frac{\partial c}{\partial r} = \frac{A}{\dots}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right) = 0 \qquad r^2 \frac{\partial c}{\partial r} = A \qquad \frac{\partial c}{\partial r} = \frac{A}{r^2}$$

$$\partial c = \frac{A}{r^2}\,\partial r \qquad \int_{c(r)}^{c(\infty)} dc = A \int_{r}^{\infty} \frac{dr}{r^2}$$

$$c(\infty) - c(r) = A\left[\frac{r^{-1}}{-1}\right]_r^{\infty}$$

$$c(\infty) - c(r) = A\left[-1\,\frac{1}{\infty} - (-1)\,\frac{1}{r}\right]$$

$$c(\infty) - c(r) = -\frac{A}{r}$$

$$c(r) = c_0 + \frac{A}{r} \qquad c(a) = 0$$

$$c(r) = c_0 - \frac{c_0 a}{r} \qquad c_0 + \frac{A}{a} = 0$$

$$\boxed{c(r) = c_0\left(1 - \frac{a}{r}\right)} \qquad A = -c_0 a$$

$$J_a = -D \,\mathrm{grad}\, c = -D \frac{\partial c}{\partial r}\Big|_{r=a}$$

$$= -D\left(c_0(-a)(-1)r^{-2}\right) =$$

$$= -D c_0 a \frac{1}{r^2}\Big|_{r=a} = -D c_0 \frac{1}{a}$$

$$I = \oint \vec{J}\, d\vec{A} = 4\pi a^2 D c_0 \frac{1}{a} =$$

$$= \boxed{4\pi D a}\, c_0$$

$$\frac{I}{V} = \frac{4\pi D a c_0}{\frac{4}{3}\pi a^{\cancel{3}\,2}} = \frac{3Dc_0}{a^2}$$

2D

$$\frac{\partial c}{\partial t} = 0 \qquad c(a) = 0$$

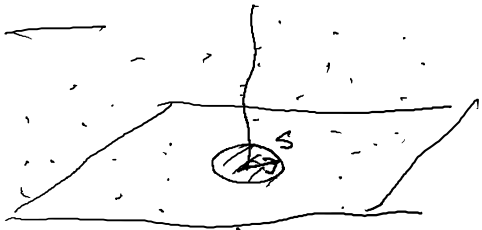
$$c(\infty) = c_0$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial c}{\partial r} = 0$$

$$r\frac{\partial}{\partial r} r \frac{\partial c}{\partial r} = 0$$

$$\frac{\partial}{\partial r} r \frac{\partial c}{\partial r} = 0 \qquad r\frac{\partial c}{\partial r} = A \qquad \frac{\partial c}{\partial r} = \frac{A}{r}$$

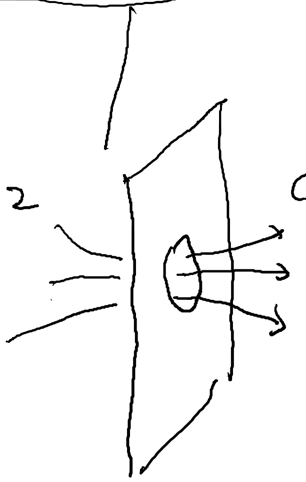
$$c_\infty - c(r) = A\left[\ln r\right]_a^\infty = \underbrace{A\ln\infty} - A\ln r$$



$I = \boxed{4Ds} c_0$

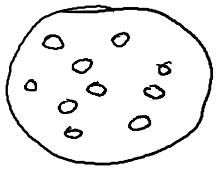
$I = 4\pi D \underline{S} c_0$

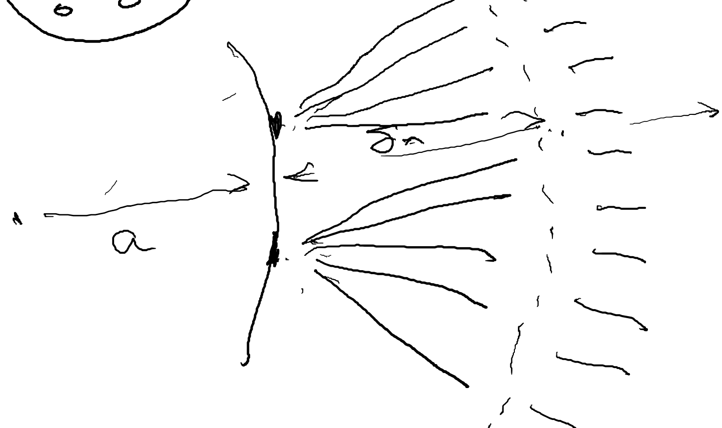
C_2

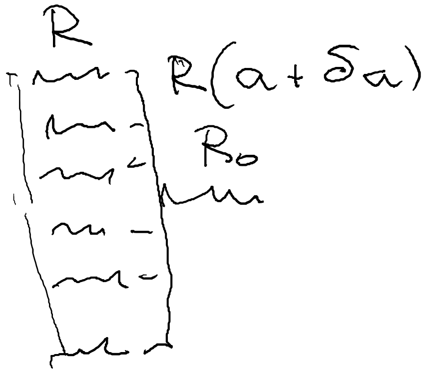


C_1 $\quad I = 2Ds\left(c_2 - c_1\right)$

$I = \frac{c_0}{R}$







$$R_a = (4\pi D a)^{-1}$$

$$R_{a+\delta a} = \frac{1}{4\pi D(a+\delta a)}$$

$$R_s = \frac{1}{4sD}$$

$$R = R_{a+\delta a} + \frac{R_s}{N} =$$

$$= \frac{1}{4\pi D\,(a+\delta a)} + \frac{1}{4sDN}$$

$$I = \frac{c_0}{R} = \frac{c_0}{\frac{1}{4\pi D(a+\delta a)} + \frac{1}{4sDN}} =$$

$$= \frac{4Dc_0}{\frac{1}{\pi(a+\delta a)} + \frac{1}{sN}} \approx \frac{4Dc_0}{\frac{1}{\pi a} + \frac{1}{sN}} =$$

$$= \frac{I_0}{1 + \frac{\pi a}{Ns}} = I_0 \frac{1}{1 + \frac{\pi a}{Ns}}$$

$$I_0 = 4\pi D a c_0$$

$N \sim 1 \quad a \gg s$

$\pi a \gg Ns$

$$I \simeq I_0 \frac{Ns}{\pi a} \sim N$$

$N \gg 1 \quad \pi a \ll Ns$

$$I \simeq I_0$$

$a = 5\,\mu m$

$a = 5\,\mu m$

$s = 10\,\mathring{A} \equiv 1\,nm$

$I = \frac{1}{2} I_0 \qquad N = \frac{\pi a}{s} = 15700$

$\frac{A}{A_0} \sim 10^{-4}$