Making Do with Less: An Introduction to Compressed Sensing 5
Classification

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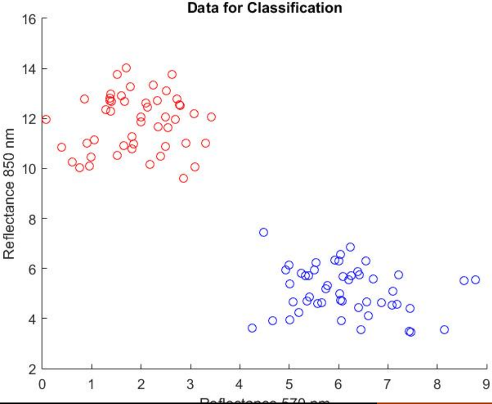
Goal: Develop an algorithm that uses $\mathbf{x}$ (the "feature vector") to determine the national origin and denomination of the note—*classify* the note.

Example

Two classes: US \$1 and \$20 notes, $x_1 =$ reflectance at 570 nm, $x_2 =$ reflectance at 850 nm. Fifty copies of each note are inserted into a bill validator and reflectance data collected:



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1. Collect *training data*:
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2. Find a function $\phi : \mathbb{R}^N \to \mathbb{R}$ such that
   - $\phi(\mathbf{x}_1^k) < 0$ for notes in class 1
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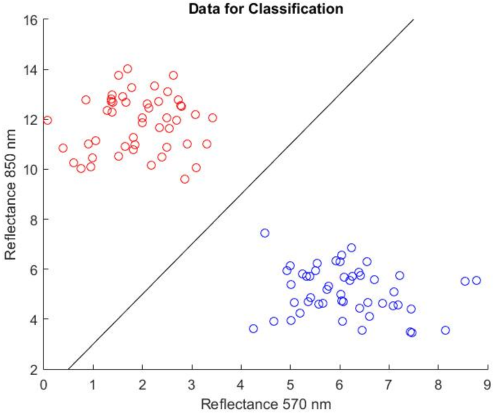
The function ϕ might be linear or nonlinear, implemented in a variety of ways.

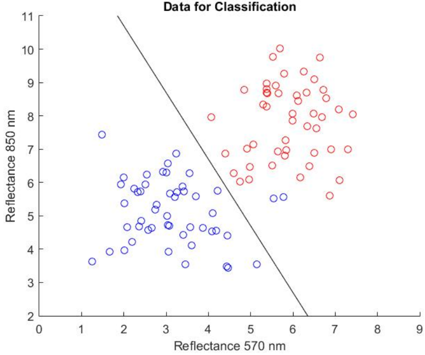
Example

This training data is *linearly separable* by the line $x_2 = 2x_1 + 1$, so we can use $\phi(\mathbf{x}) = x_2 - 2x_1 - 1$.



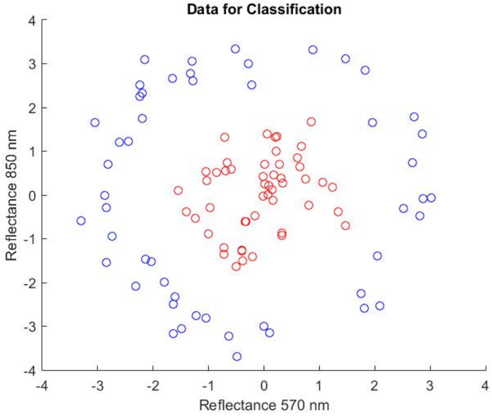
Noise

But the data might not be perfectly linearly separable due to noise, outliers, or the nature of the problem:



Nonlinear Classification

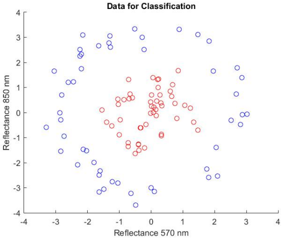
The training data may not be linearly separable at all:



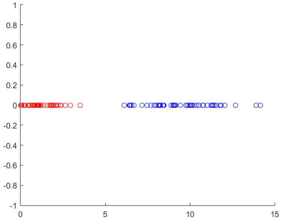
Nonlinear Classification

One could apply a nonlinear transformation $\mathbf{x} \rightarrow \psi(\mathbf{x})$ where $\psi : \mathbb{R}^N \rightarrow V$ to the data, where V is some other space. We can then try a classifier in V, where the data may be linearly separable. V may have dimension other than N.

Example: $\psi(\mathbf{x}) = x_1^2 + x_2^2$.



$\xrightarrow{\psi}$



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- Some of these techniques can be adapted for non-linearly separable data; other methods address nonlinearity directly.
- Binary classification techniques can be adapted to handle multiple classes (e.g., a “one versus all” approach.)

A Different Approach: Sparsity and Dictionary Learning

Consider data consisting of feature vectors $\mathbf{x} \in \mathbb{R}^2$ in one of two classes, of the forms

$$\mathbf{x} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ or } \mathbf{x} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for scalar k (each class lies on a line). Our measurement of $\mathbf{x}$ may contain noise or other errors.

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Suppose we measure feature vector

$$\mathbf{x}^* = \begin{bmatrix} 2.06 \\ -2.12 \end{bmatrix}.$$

To which class does $\mathbf{x}^*$ belong?

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$$\mathbf{D} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2.06 \\ -2.12 \end{bmatrix}.$$

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The solution is $\boldsymbol{\alpha}_1 = -0.03, \alpha_2 = 2.09$. We could classify $\mathbf{x}^*$ in class 2 based on $|\alpha_2|$ being much larger than $|\alpha_1|$, but this depends on the scaling of the dictionary columns.

An Alternate Approach: Example 1

Instead of using $|\alpha_2| > |\alpha_1|$ to classify, decompose

$$\boldsymbol{\alpha} = \langle -0.03, 2.09 \rangle = \underbrace{\langle -0.03, 0 \rangle}_{\boldsymbol{\alpha}^1} + \underbrace{\langle 0, 2.09 \rangle}_{\boldsymbol{\alpha}^2}$$

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Since $\mathbf{x}^*$ can be built from the class 2 atom much more accurately we assign $\mathbf{x}^*$ to class 2.

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$$\mathbf{D} = \begin{bmatrix} 1 & -1 & 2 & 1 & 2 & -1 & 1 & 3 & 1 \\ 1 & 0 & -1 & 1 & 1 & -2 & 1 & 0 & 2 \\ 3 & -1 & 0 & 0 & 1 & 1 & 2 & -3 & 1 \\ -5 & 2 & -1 & 3 & 3 & -6 & 0 & -3 & 0 \end{bmatrix}.$$

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We form a sparse solution $\boldsymbol{\alpha}$ to the system $\mathbf{D}\boldsymbol{\alpha} = \mathbf{x}^*$. The 3-sparse solution here is

$$\boldsymbol{\alpha} = \langle 0.267, 0, 1.1, 0, 0, 0, 0, -0.056, 0 \rangle.$$

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Define class 1, 2, 3 portions

$$\begin{aligned}
\boldsymbol{\alpha}^1 &= \langle 0.267, 0, 1.1, 0, 0, 0, 0, 0, 0 \rangle \\
\boldsymbol{\alpha}^2 &= \langle 0, 0, 0, 0, 0, 0, 0, 0, 0 \rangle \\
\boldsymbol{\alpha}^3 &= \langle 0, 0, 0, 0, 0, 0, 0, -0.056, 0 \rangle.
\end{aligned}$$

Note $\boldsymbol{\alpha} = \boldsymbol{\alpha}^1 + \boldsymbol{\alpha}^2 + \boldsymbol{\alpha}^3$.

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$\boldsymbol{\alpha}^1$ corresponds to atoms in class 1, $\boldsymbol{\alpha}^2$ to atoms in class 2,$\boldsymbol{\alpha}^3$ to atoms in class 3.

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The hope is that if $\mathbf{x}^*$ can be built sparsely and accurately as $\mathbf{D}\boldsymbol{\alpha}^1$ then $\mathbf{x}^*$ is in Class 1.

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Ditto for the other classes.

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Compute

$$\|\mathbf{D}\boldsymbol{\alpha}^1 - \mathbf{x}^*\| = 0.465, \quad \|\mathbf{D}\boldsymbol{\alpha}^2 - \mathbf{x}^*\| = 3.49, \quad \|\mathbf{D}\boldsymbol{\alpha}^3 - \mathbf{x}^*\| = 3.67.$$

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Since $\mathbf{x}^*$ can be built most accurately with a few vectors in class 1, we assign $\mathbf{x}^*$ to class 1.

The General Approach

We have K classes and m-dimensional feature vectors.

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where each $\mathbf{v}_k^j \in \mathbb{R}^m$ and $n = n_1 + \cdots + n_K$. Usually $m << n$.

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where each $\mathbf{v}_k^j \in \mathbb{R}^m$ and $n = n_1 + \cdots + n_K$. Usually $m << n$.

The vector (atom) $\mathbf{v}_k^j$ is in class k. The number of atoms n_k in any class may be large.

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we find a sparse solution $\boldsymbol{\alpha} \in \mathbb{R}^n$ to $\mathbf{D}\boldsymbol{\alpha} = \mathbf{x}^*$.

Let $\boldsymbol{\alpha}^k$ be the vector in $\mathbf{R}^n$ whose nonzero components are those of $\boldsymbol{\alpha}$ corresponding to atoms in class k.

Assign $\mathbf{x}^*$ to the class with minimum residual $\|\mathbf{D}\boldsymbol{\alpha}^k - \mathbf{x}^*\|$, indicating that $\mathbf{x}^*$ can be built most accurately using only a few atoms from Class k.

Insights

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If the dictionary $\mathbf{D}$ is merely large and $\mathbf{x}^*$ is in class k then it should be possible to build $\mathbf{x}^*$ accurately using only a few atoms from class k.

USPS Classification Example

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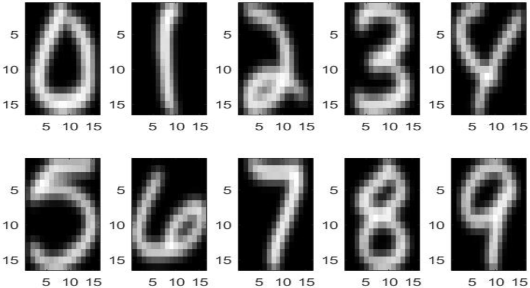
There are a total of 60000 digitized, handwritten digits in the MNIST database. Each is digitized on a 16×16 grid, normalized/registered:



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We select 500 samples to act as the (initial) dictionary, about 50 samples of each digit. The additional set of $>$ 59000 handwritten digits can be used to test the classification algorithm.

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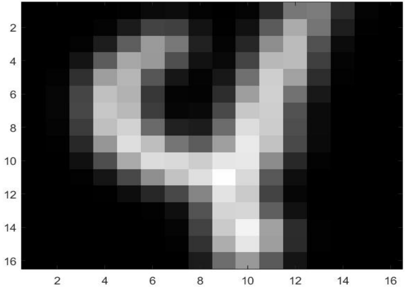
A typical digit from this set:



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The dictionary **D** consists of the 500 sample images, each arranged into a 256-dimensional vector, so **D** is a 256 by 500 matrix.

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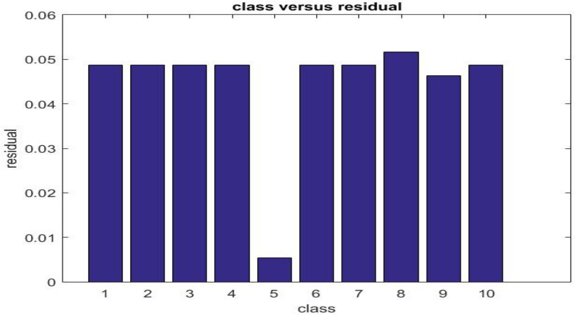
2. Form vectors $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{10}$ ($\boldsymbol{\alpha}^j$ composed of the components of $\boldsymbol{\alpha}$ with support in class j.)
3. Assign $\mathbf{x}^*$ to the class k with minimum residual $\|\mathbf{D}\boldsymbol{\alpha}^k - \mathbf{x}\|$.

Classification Example

Residual $\|\mathbf{D}\alpha^k - \mathbf{x}^*\|_2$ for each class $k = 1$ through $k = 10$, for some handwritten digit image $\mathbf{x}^*$:



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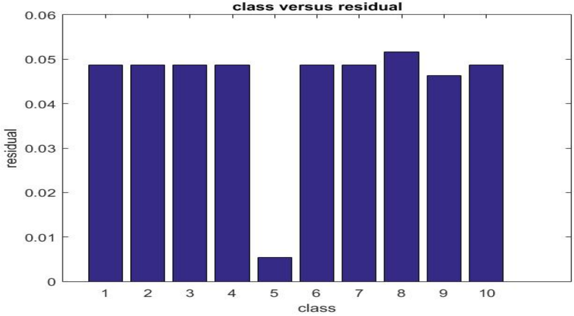

We conclude $\mathbf{x}^*$ is in class 5 (corresponding to a handwritten "4").

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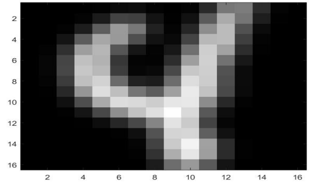

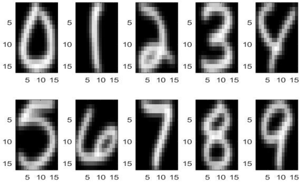

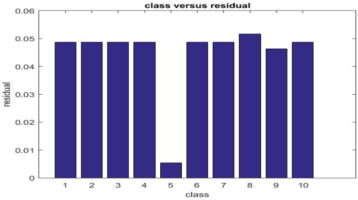

Class 5 (digit "4") has the smallest residual.

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The first step is to select additional samples $\mathbf{x}_1, \ldots, \mathbf{x}_N$; these will be used to "train" and improve the dictionary.

Optimizing the Dictionary: Example

Consider the previous example dictionary

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & 2 & 1 & 2 & -1 & 1 & 3 & 1 \\ 1 & 0 & -1 & 1 & 1 & -2 & 1 & 0 & 2 \\ 3 & -1 & 0 & 0 & 1 & 1 & 2 & -3 & 1 \\ -5 & 2 & -1 & 3 & 3 & -6 & 0 & -3 & 0 \end{bmatrix}.$$

The first three atoms are class 1, the second three class 2, the last three class 3.

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Suppose we have four additional samples, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, classes 1,2,3 and 3, respectively.

Optimizing the Dictionary: Example

Suppose $\mathbf{x}_1 = \langle 0.0.068, 1.31, 3.12, -5.11 \rangle$. Let's classify $\mathbf{x}_1$ as previously: Find a 3-sparse solution to $\mathbf{D}\boldsymbol{\alpha} = \mathbf{x}_1$:

$$\boldsymbol{\alpha} = \langle 1.000, 0, 0, 0, -0.174, 0, 0, -0.144, 0 \rangle.$$

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Define class 1/2/3 subsections

$$\begin{aligned} \boldsymbol{\alpha}^1 &= \langle 1.000, 0, 0, 0, 0, 0, 0, 0, 0 \rangle \\ \boldsymbol{\alpha}^2 &= \langle 0, 0, 0, 0, -0.174, 0, 0, 0, 0 \rangle \\ \boldsymbol{\alpha}^3 &= \langle 0, 0, 0, 0, -0, 0, 0, -0.144, 0 \rangle. \end{aligned}$$

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We find $\|\mathbf{D}\boldsymbol{\alpha}^1 - \mathbf{x}_1\|_2 = 0.996$, $\|\mathbf{D}\boldsymbol{\alpha}^2 - \mathbf{x}_1\|_2 = 5.85$, $\|\mathbf{D}\boldsymbol{\alpha}^3 - \mathbf{x}_1\|_2 = 6.31$, so $\mathbf{x}_1$ is assigned to Class 1 (correct).

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Optimizing the Dictionary: Example

A similar computation with class 2 training vector $\mathbf{x}_2 = \langle -0.380, 0.015, -0.510, 0.668 \rangle$ yields

$$\|\mathbf{D}\boldsymbol{\alpha}^1 - \mathbf{x}_2\|_2 = 0.217, \|\mathbf{D}\boldsymbol{\alpha}^2 - \mathbf{x}_2\|_2 = 0.922, \|\mathbf{D}\boldsymbol{\alpha}^3 - \mathbf{x}_2\|_2 = 1.103.$$

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This training sample gets assigned to the wrong class (class 1 instead of class 2).

We'd like $\|\mathbf{D}\boldsymbol{\alpha}^2 - \mathbf{x}_2\|_2 = 0$.

Optimizing the Dictionary: Example

Training vectors $\mathbf{x}_3$ and $\mathbf{x}_4$ (not shown, but both class 3) yield residuals

$$\|\mathbf{D}\alpha^1 - \mathbf{x}_3\|_2 = 1.17, \quad \|\mathbf{D}\alpha^2 - \mathbf{x}_3\|_2 = 0.934, \quad \|\mathbf{D}\alpha^3 - \mathbf{x}_3\|_2 = 1.35$$
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A perfect dictionary would give $\|\mathbf{D}\alpha^3 - \mathbf{x}_3\|_2 = 0$ and $\|\mathbf{D}\alpha^3 - \mathbf{x}_4\|_2 = 0$.

Optimizing the Dictionary: Example

The training vectors $\mathbf{x}_1, \ldots, \mathbf{x}_4$ are given, and for each $\mathbf{x}_i$ the sparse vector α is computed—think of them as now fixed.

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Consider the quantity

$$E(\mathbf{D}) = \|\mathbf{D}\boldsymbol{\alpha}^1 - \mathbf{x}_1\|_2^2 + \|\mathbf{D}\boldsymbol{\alpha}^2 - \mathbf{x}_2\|_2^2 + \|\mathbf{D}\boldsymbol{\alpha}^3 - \mathbf{x}_3\|_2^2 + \|\mathbf{D}\boldsymbol{\alpha}^3 - \mathbf{x}_4\|_2^2$$

as a (quadratic) function of $\mathbf{D}$. A perfect dictionary would give $E(\mathbf{D}) = 0$.

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We can improve the dictionary by minimizing $E(\mathbf{D})$ above, using any optimization algorithm.

We then go back and recompute each α for $\mathbf{x}_1, \ldots, \mathbf{x}_4$, and repeat the whole process.

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These techniques have been used in a wide variety of image classification problems, medical imaging, facial recognition, . . .