IB HL Mathematical Exploration

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1. Introduction

The concept of the center of mass (CM) is fundamental in the study of physics and engineering, serving as a crucial parameter for understanding the distribution of mass within a system. Traditionally, the center of mass has been computed using elementary methods, often involving simple geometric shapes. However, as we delve into more complex and irregularly shaped objects, the need for advanced mathematical techniques becomes apparent.

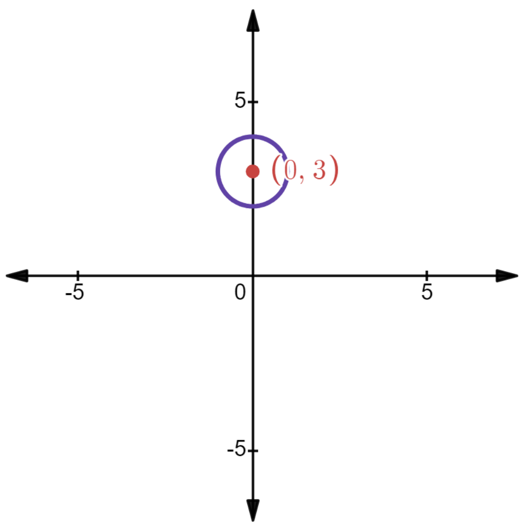
This internal assignment seeks to explore the method for determining the center of mass through the utilization of integration. By embracing integration in calculus, we aim to develop a more versatile approach to determining the center of mass, overcoming the time-consuming and complicated process posed by conventional methodologies.

Throughout this exploration, I will first use an example to show the "coincidence" and then develop the theorem. The theorem will then be rigorously proved and this method will be applied in both quick-calculating the volume and finding the center of mass. Our goal is to present an intuitive understanding of this approach, emphasizing its significance in practical scenarios and its potential to simplify the way we perceive and calculate the center of mass.

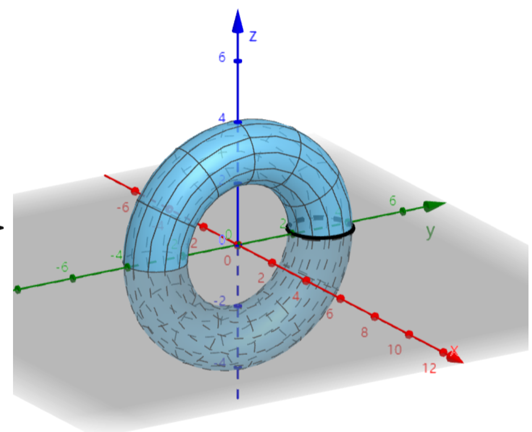
2. A Clear Example - Torus

Problem 2.1. *Consider the circle with the center* $(0, 3)$ *and radius 1 unit. When the circle revolves around the* x*-axis, we obtain a doughnut shape or torus. Calculate the volume of the torus.*

The normal approach is to divide the torus into two parts, the outer disk and inner disk with different shapes and then subtract them.



(a) The plane view



(b) The torus formed by rotating the circle

As the circle has the center $(0, 3)$ and radius 1, the expression of the circle is

$$(y-3)^2 + x^2 = 1^2,$$
$$(y-3)^2 = 1 - x^2,$$

so we can express it as a combination of 2 functions

$$y_U = 3 + \sqrt{1-x^2},$$
$$y_D = 3 - \sqrt{1-x^2},$$

where $y_U \geq y_D$ for all $x \in [-1, 1]$

Then, by applying

$$V = \pi \int_a^b ({y_U}^2 - {y_D}^2)dx,$$

the volume of the torus should be

$$\begin{aligned}
V_{\text{torus}} &= \pi\int_{-1}^{1}[(3+\sqrt{1-x^2})^2-(3-\sqrt{1-x^2})^2]dx \\
&= \pi\int_{-1}^{1}[(9+6\sqrt{1-x^2}+1-x^2)-(9-6\sqrt{1-x^2}+1-x^2)]dx \\
&= 12\pi\int_{-1}^{1}\sqrt{1-x^2}dx
\end{aligned}$$

Then, let $x=\sin\theta$, $\frac{dx}{d\theta}=\cos\theta \implies dx=\cos\theta d\theta$ and $\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\cos\theta$.

Substitute these in the integral, we have

$$\begin{aligned}
\int\sqrt{1-x^2}dx &= \int\cos\theta\cdot\cos\theta d\theta \\
&= \int\cos^2\theta d\theta.
\end{aligned}$$

As $\cos 2\theta = 2\cos^2\theta-1$, $\cos^2\theta=\dfrac{1+\cos 2\theta}{2}$.

$$\begin{aligned}
\implies \int\cos^2\theta d\theta &= \frac{1}{2}\int(1+\cos 2\theta)d\theta \\
&= \frac{1}{2}(\theta+\frac{\sin 2\theta}{2}) \\
&= \frac{\arcsin x}{2}+\frac{x\sqrt{1-x^2}}{2}+C. \\
\implies \int_{-1}^{1}\sqrt{1-x^2}dx &= \left[\frac{\arcsin x}{2}+\frac{x\sqrt{1-x^2}}{2}\right]_{-1}^{1} \\
&= (\frac{\arcsin 1}{2}+\frac{1\cdot\sqrt{1-1^2}}{2})-(\frac{\arcsin(-1)}{2}+\frac{-1\cdot\sqrt{1-(-1)^2}}{2}) \\
&= \frac{\pi}{4}-(-\frac{\pi}{4}) \\
&= \frac{\pi}{2}
\end{aligned}$$

Therefore, the volume of the torus is $\frac{\pi}{2}\cdot 12\cdot\pi = 6\pi^2$.

This has driven my curiosity, as I discovered that circle has an area of $\pi r^2=\pi\cdot 1^2=\pi$, and the trace of the center of circle $(0,3)$ is $2\pi r=2\pi\cdot 3=6\pi$. Is it a coincidence that the volume of the torus is equal to the product of the area enclosed by the curve and the trace length of the centroid?

3. Pappus's Centroid Theorem

Actually, this is not a coincidence and can be generalized. The Pappus's Centroid Theorem suggests my above claim.

Theorem 3.1 (Pappus's Centroid Theorem)**.** *Suppose that a plane curve is rotated about an axis external to the curve. Then*

1. *the resulting surface area of revolution is equal to the product of the length of the curve and the displacement of its centroid;*
2. *in the case of a closed curve, the resulting volume of revolution is equal to the product of the plane area enclosed by the curve and the displacement of the centroid of this area.*

3.1. Deduce the Formula for Centroid

Start with deducing the formula of the centroid, we know that the centroid is the center of mass of a two-dimensional planar lamina or a three-dimensional solid [1]. And for computing, we can take the centroid as an equivalence of center of gravity (though they have different physical meaning: centroid is the center of **mass**, which does not depend on the gravity and the gravitational constant g; while the center of gravity is the equivalent point of sum of gravity all parts on this object experience).

If we have a curve $y(x)$ where its mass is the area A, the centroid $(\bar{x}, \bar{y})$ is the place where all torques produced by parts on the object are balanced. For any point, the torque produced equals the gravity multiplied by the distance to the centroid. The torque produced by the shaded area with respect to $x_0 = 0$ equals

$$\tau = mgy = (y' + \Delta y - y')\Delta x \cdot y'g = y'g\Delta y\Delta x,$$

and at the same time the area of shaded area is

$$\Delta y\Delta x = \Delta A.$$

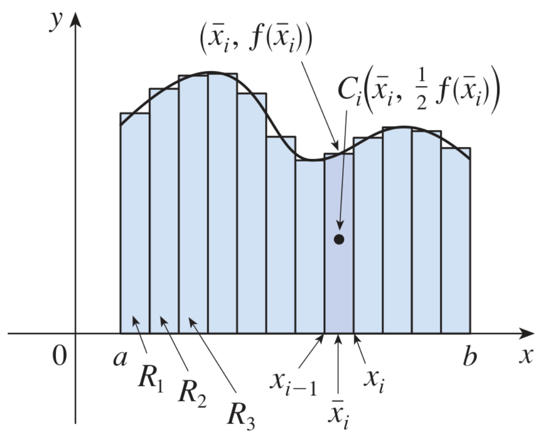
So the total torque produced by the bar is

$$\tau_{bar} = y \cdot dAg,$$

and integrate for whole $y(x)$, the entire torque is

$$\tau_y = \int y \cdot dAg.$$

[1] https://mathworld.wolfram.com/GeometricCentroid.html



This should be equivalent to the torque produced by the single equivalent point,

$$\tau_{centroid} = mg \cdot \bar{y}$$

As stated before the mass is equal to the entire area $m = A = \int dA$.

Thus,

$$\bar{y} = \frac{\int y \cdot dAg}{\int dAg} = \frac{\int ydA}{\int dA}.$$

Similarly, for curve's centroid, it is trivial that the centroid is

$$\bar{y} = \frac{\int yds}{\int ds},$$

where s is the length of the curve.

$$M_y = \rho \int_a^b xf(x)dx$$
$$M_x = \rho \int_a^b \frac{1}{2}[f(x)]^2 dx$$

and the total mass is

$$m = \rho A = \rho \int_a^b f(x)dx,$$

When density is constant, the ρ term can be canceled out and

$$\bar{x} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2 dx}{\rho \int_a^b f(x) dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2 dx}{\int_a^b f(x) dx}$$
$$\bar{y} = \frac{M_y}{m} = \frac{\rho \int_a^b x \cdot f(x) dx}{\rho \int_a^b f(x) dx} = \frac{\int_a^b x \cdot f(x) dx}{\int_a^b f(x) dx}$$

so with an area A, the centroid coordinates $(\bar{x}, \bar{y})$ is

$$\bar{x} = \frac{1}{A} \int_a^b x f(x) dx, \quad \bar{y} = \frac{1}{A} \int_a^b \frac{1}{2}[f(x)]^2 dx$$

3.2. Proving the First Theorem

Theorem 3.2 (Pappus's Centroid Theorem, Part 1)**.** *The surface area of a solid of revolution is the arc length of the generating curve multiplied by the distance traveled by the centroid of the curve.*

Let x-axis be the rotation axis and $y = f(x), x \in [a, b]$ be the generating curve. The y-axis of the centroid of a line segment $f(x), x \in [a, b]$ can be computed by the formula

$$\bar{y} = \frac{\int y ds}{\int ds},$$

where ds is the length.

The length ds can be computed as

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

Plug in ds, we can get

$$\bar{y} = \frac{\int y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy}{s},$$

so the distance traveled by the centroid of the curve is

$$\text{distance} = 2\pi \frac{\int y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy}{s}.$$

At the same time, the surface area of a solid of revolution is

$$S = \int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

So we have

$$\begin{aligned} LHS &= \text{surface area} \\ &= \int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= s \int \frac{2\pi y \sqrt{1 + (\frac{dy}{dx})^2} dx}{s} \\ &= s \cdot 2\pi \cdot \frac{\int y \sqrt{1 + (\frac{dx}{dy})^2} dy}{s} \\ &= \text{arc length} \cdot \text{distance traveled by centroid} \\ &= RHS. \end{aligned}$$

3.3. Proving the Second Theorem

Theorem 3.3 (Pappus's Centroid Theorem, Part 2)**.** *The volume of a solid of revolution generated by rotating a plane figure about an external axis is equal to the product of the area of the plane and the distance traveled by the geometric centroid of the plane.*

The area bounded by the two functions:

$$\begin{aligned} y &= f(x), y \geq 0 \\ y &= g(x), g(x) \geq f(x) \end{aligned}$$

and bounded by the interval $x \in [a, b]$ is given by:

$$A = \int_a^b dA = \int_a^b [f(x) - g(x)] dx$$

With reference to §3.1, x component of the centroid of this area is given by

$$\bar{x} = \frac{1}{A} \int_a^b x[f(x) - g(x)] dx.$$

If this area is rotated about the y-axis, the volume generated can be calculated using the shell method. It is given by:

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx$$

Using the last two equations to eliminate the integral we have:

$$V = 2\pi\bar{x}A$$

4. Application: Estimate the Volume of an Apple

4.1. Estimation

4.1.1. Assumption Used to Simplify the Problem

- The apple has density that is same everyw

4.1.2. Locate the Centroid for an Apple Slice

4.1.3. Calculate the Theoretical Volume of Apple

4.2. Validation