A Less-Used Approach to Calculating the Volume of Solids of Revolution and Its Practical Application

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1. Introduction

The calculation of the volume of solids of revolution is a fundamental problem in calculus, with practical applications through engineering, to daily lives. Traditional methods often rely on integration techniques, which, while accurate, can be cumbersome and complex, particularly for irregularly shaped objects. This paper explores an alternative approach using Pappus's Centroid Theorem, a powerful tool for calculating the volume of solids of revolution.

Pappus's Centroid Theorem provides a straightforward means to determine the volume of a solid of revolution by relating the area of a shape's cross-section to the distance its centroid revolved when rotating around the axis. Despite its simplicity, this theorem offers a highly effective method for calculating volumes that might otherwise require more sophisticated techniques.

This internal assignment seeks to explore the method for determining the volume of solids of revolution without using the complicated integration technique. By applying the Pappus's Centroid Theorem, we can overcome the time-consuming and complicated process posed by the common-used method.

Through this exploration, we aim to highlight the versatility of Pappus's Centroid Theorem as a practical tool for volume estimation, providing a novel approach that bridges theoretical geometry with real-world applications.

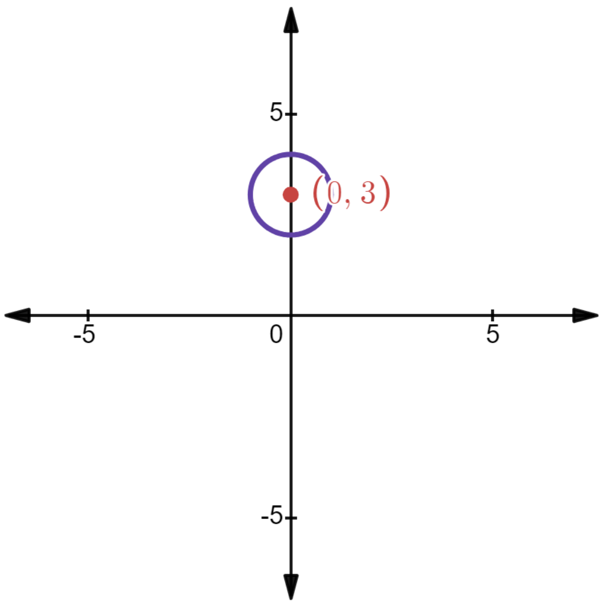
In this assignment, I will first use an example to show the "coincidence" and then develop the theorem. The theorem will then be rigorously proved and this method will be applied in quick-calculating the volume of a complicate shape that can be approximately taken as a solid of revolution, an apple, demonstrating how the theorem can be used in conjunction with experimental measurement to estimate volume.

Our goal is to present an intuitive understanding of this approach, emphasizing its significance in practical scenarios and its potential to simplify the way we perceive and calculate the volume.

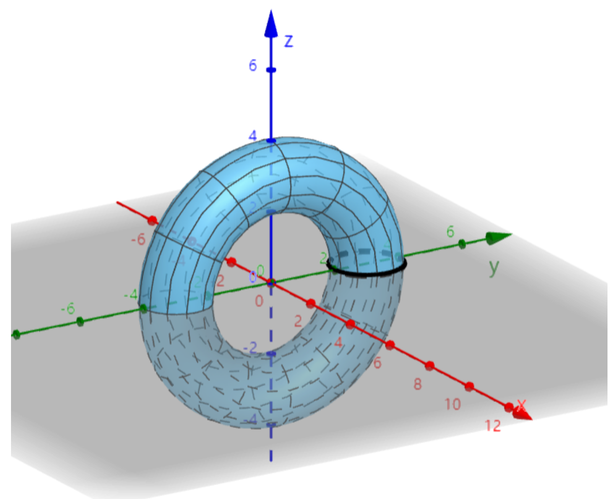
2. A Clear Example – Torus

Problem 2.1. *Consider the circle with the center* $(0, 3)$ *and radius 1 unit. When the circle revolves around the* x*-axis, we obtain a doughnut shape or torus. Calculate the volume of the torus.*

The normal approach is to divide the torus into two parts, the outer disk and inner disk with different shapes and then subtract them.



(a) The plane view



(b) The torus formed by rotating the circle

As the circle has the center $(0, 3)$ and radius 1, the expression of the circle is

$$(y-3)^2 + x^2 = 1^2,$$
$$(y-3)^2 = 1 - x^2,$$

so we can express it as a combination of 2 functions

$$y_U = 3 + \sqrt{1-x^2},$$
$$y_D = 3 - \sqrt{1-x^2},$$

where $y_U \geq y_D$ for all $x \in [-1, 1]$.

Then, by applying

$$V = \pi \int_a^b ({y_U}^2 - {y_D}^2) dx,$$

the volume of the torus should be

$$\begin{aligned} V_{\text{torus}} &= \pi \int_{-1}^{1} [(3 + \sqrt{1-x^2})^2 - (3 - \sqrt{1-x^2})^2] dx \\ &= \pi \int_{-1}^{1} [(9 + 6\sqrt{1-x^2} + 1 - x^2) - (9 - 6\sqrt{1-x^2} + 1 - x^2)] dx \\ &= 12\pi \int_{-1}^{1} \sqrt{1-x^2} dx \end{aligned}$$

Then, let $x = \sin\theta$, $\frac{dx}{d\theta} = \cos\theta \implies dx = \cos\theta d\theta$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$.

Substitute these in the integral, we have

$$\begin{aligned} \int \sqrt{1-x^2} dx &= \int \cos\theta \cdot \cos\theta d\theta \\ &= \int \cos^2\theta d\theta. \end{aligned}$$

As $\cos 2\theta = 2\cos^2\theta - 1$, $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$.

$$\begin{aligned} \implies \int \cos^2\theta d\theta &= \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2}(\theta + \frac{\sin 2\theta}{2}) \\ &= \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C. \\ \implies \int_{-1}^{1} \sqrt{1-x^2} dx &= \left[\frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} \right]_{-1}^{1} \\ &= \frac{\arcsin 1}{2} - \frac{\arcsin(-1)}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

Therefore, the volume of the torus is $\frac{\pi}{2} \cdot 12 \cdot \pi = 6\pi^2$.

This has driven my curiosity, as I discovered that circle has an area of $\pi r^2 = \pi \cdot 1^2 = \pi$, and the trace of the center of circle $(0, 3)$ is $2\pi r = 2\pi \cdot 3 = 6\pi$. Is it a coincidence that the volume of the torus is equal to the product of the area enclosed by the curve and the trace length of the centroid?

3. Pappus's Centroid Theorem

Actually, this is not a coincidence and can be generalized. The Pappus's Centroid Theorem suggests my above claim.

Theorem 3.1 (Pappus's Centroid Theorem)**.** *Suppose that a plane curve is rotated about an axis external to the curve. Then*

1. *the resulting surface area of revolution is equal to the product of the length of the curve and the displacement of its centroid;*

2. *in the case of a closed curve, the resulting volume of revolution is equal to the product of the plane area enclosed by the curve and the displacement of the centroid of this area.*

3.1. Deduce the Formula for Centroid

3.1.1. For Two Dimensional Plate

To begin deriving the formula for the centroid, we recognize that the centroid represents the center of mass for a two-dimensional planar lamina or a three-dimensional solid when the density is uniform [1]. It is also the geometric center of the object. In the context of deriving the formula, we can consider the centroid to be analogous to the center of gravity, despite their different physical interpretations: the centroid is the geometric center and center of mass under uniform density, independent of gravity and the gravitational constant g; while the center of gravity is the point where the total gravitational force on the object can be considered to act.

If we consider a region R bounded by the x-axis and a function $f(x)$, the centroid $(\bar{x}, \bar{y})$ can be determined by treating it as the center of gravity.

To calculate the centroid, we first analyze the torques about the x-axis and y-axis. The total torque experienced by all small elements of the plate must be equal to the torque produced by the entire region acting at the centroid.

We begin by calculating the total torque about the x-axis and y-axis.

Let the area density of the plate be σ, which implies that for a plate with area A, its mass is σA.

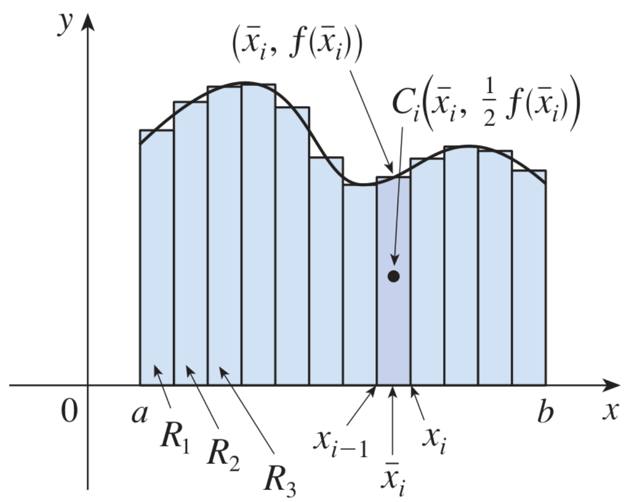


Figure 1: The region R, between $x = a$ and $x = b$, bounded by the x-axis and $f(x)$, where $f(x)$ is a continuous function [2].

Assume that the region R, as shown in Figure 1, can be divided into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. The sample point x_i^* is chosen to be the midpoint $\bar{x}_i$ of the i-th subinterval, such that $\bar{x}_i = \frac{x_{i-1}+x_i}{2}$. The centroid of the i-th approximating rectangle R_i is approximated by its center $C_i(\bar{x}_i, \frac{1}{2}f(x_i))$, with area $f(\bar{x}_i)\Delta x$, giving a mass of

$$\sigma f(\bar{x}_i)\Delta x.$$

The gravitational force on this subinterval is

$$\sigma f(\bar{x}_i)\Delta x g.$$

Thus, the torque about the y-axis is the product of this gravitational force and the distance from C_i to the y-axis, which is $\bar{x}_i$, yielding

$$\tau_y(R_i) = [\sigma f(\bar{x}_i)\Delta x g]\bar{x}_i.$$

Taking the limit as $n \to \infty$, the total torque of R about the y-axis is

$$\tau_y = \lim_{n\to\infty} \sum_{i=1}^{n} [\sigma f(\bar{x}_i)\Delta x g]\bar{x}_i = \sigma g \int_a^b x f(x) dx.$$

Similarly, the torque about the x-axis, which involves the gravitational force and the distance from C_i to the x-axis (half the height of R_i, $\frac{f(\bar{x}_i)}{2}$), is given by

$$\tau_x = \lim_{n\to\infty} \sum_{i=1}^{n} [\sigma f(\bar{x}_i)\Delta x g]\frac{f(\bar{x}_i)}{2} = \sigma g \int_a^b \frac{1}{2}[f(x)]^2 dx.$$

Next, we calculate the equivalent torque produced by the entire region acting at the centroid, $(\bar{x}, \bar{y})$:

$$\tau_x = A\sigma g\bar{y},$$

$$\tau_y = A\sigma g\bar{x}.$$

Therefore, the formulas for the centroid of the plate R are:

$$\bar{x} = \frac{\tau_y}{A\sigma g} = \frac{1}{A}\int_a^b xf(x)dx,$$
$$\bar{y} = \frac{\tau_x}{A\sigma g} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2 dx.$$

3.1.2. For One-Dimensional Curve

By the same approach, we can derive the coordinates $(\bar{x}, \bar{y})$ for the one-dimensional curve. Suppose the curve is expressed by $f(x)$, with the domain $x \in [a, b]$, and set the linear density as λ, which means that for the curve with length l, the mass of the curve is λl.

Consider the torque experienced by the x-axis, for every differential small segment of the curve, ds, we have

$$\tau_x = \bar{y} \cdot s\lambda g = \int_a^b f(x) \cdot ds\lambda g,$$

so the y coordinate of the centroid

$$\bar{y} = \frac{\tau_x}{s\lambda g} = \frac{\int_a^b f(x) \cdot ds\lambda g}{s\lambda g} = \frac{\int_a^b f(x)ds}{s}.$$

Similarly, the x coordinate of the centroid can be calculated by the equality between $\bar{x} \cdot s\lambda g$ and $\int_a^b x \cdot ds\lambda g$,

$$\bar{x} = \frac{\tau_y}{s\lambda g} = \frac{\int_a^b x \cdot ds\lambda g}{s\lambda g} = \frac{\int_a^b xds}{s}.$$

The length ds can be computed as

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

3.2. Proving the First Theorem

Theorem 3.2 (Pappus's Centroid Theorem, Part 1)**.** *The surface area of a solid of revolution is the arc length of the generating curve multiplied by the distance traveled by the centroid of the curve.*

As we derived in §3.1.2, the y-coordinate of the centroid of a given curve $y = f(x), x \in [a, b]$ can be computed by the formula

$$\bar{y} = \frac{\int_a^b x ds}{s}$$

while

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

Plug in ds, we can get

$$\bar{y} = \frac{\int y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy}{s},$$

so the distance traveled by the centroid of the curve is

$$\text{distance} = 2\pi \frac{\int y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy}{s}.$$

At the same time, the surface area of a solid of revolution is

$$S = \int 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

So we have

$$
\begin{aligned}
LHS &= \text{surface area} \\
&= \int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= s \int \frac{2\pi y \sqrt{1 + (\frac{dy}{dx})^2} dx}{s} \\
&= s \cdot 2\pi \cdot \frac{\int y \sqrt{1 + (\frac{dx}{dy})^2} dy}{s} \\
&= \text{arc length} \cdot \text{distance traveled by centroid} \\
&= RHS.
\end{aligned}
$$

3.3. Proving the Second Theorem

Theorem 3.3 (Pappus's Centroid Theorem, Part 2)**.** *The volume of a solid of revolution generated by rotating a plane figure about an external axis is equal to the product of the area of the plane and the distance traveled by the geometric centroid of the plane.*

The area bounded by the two functions:

$$
\begin{aligned}
&y = f(x), y \geq 0 \\
&y = g(x), g(x) \geq f(x)
\end{aligned}
$$

and bounded by the interval $x \in [a, b]$ is given by:

$$A = \int_a^b dA = \int_a^b [f(x) - g(x)] dx$$

With reference to §3.1.1, x component of the centroid of this area is given by

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]dx.$$

If this area is rotated about the y-axis, the volume generated can be calculated using the shell method. It is given by:

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx$$

Using the last two equations to eliminate the integral we have:

$$V = 2\pi \bar{x} A$$

4. Application: Estimation of the Volume of an Apple

4.1. Measurement

To estimate the volume of a randomly selected apple, we employed the water displacement method. This approach is outlined as follows:

1. A cup, large enough to fully submerge the apple, is filled with water.

2. The total mass of the cup filled with water is measured and recorded as m_0.

3. The apple is gently placed into the cup until it is completely submerged.

4. The mass of the cup and the remaining water (i.e., the water not displaced by the apple) is measured and recorded as m_1.

5. The mass of the displaced water, corresponding to the volume of the apple, is calculated as $m_0 - m_1$. Given the density of water (ρ_{water}), the volume of the apple V_{apple} is determined

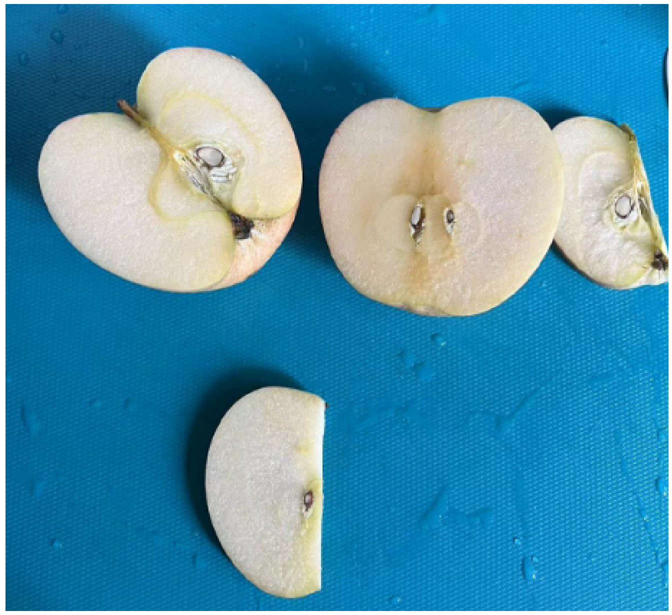


Figure 2: The slice of apple on the bottom of the photo is the sample we take as the representative section

by the equation:

$$V_{\text{apple}} = \frac{m_0 - m_1}{\rho_{\text{water}}}$$

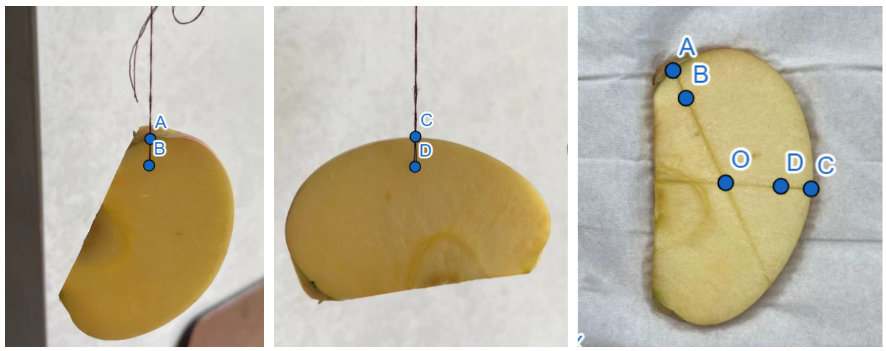
Using this method, the measured volume of the apple was found to be $211.58\,\text{cm}^3$.

4.2. Estimation

4.2.1. Determination of the Centroid for an Apple Slice

To estimate the theoretical volume, the apple was sliced in half along the cross-section perpendicular to the z-axis. This apple slice was treated as a representative section, which, when rotated around the z-axis, forms the volume of the apple.

Since we assume that the apple processes a uniform density throughout its volume, so the centroid will be the same as the center of mass (and the center of gravity). To locate the centroid of this cross-section, the slice was suspended from a point B, and the intersection of the edge of the slice and string is marked as A. At equilibrium, the gravitational force acting on the slice aligns vertically with the tension in the supporting string, indicating that the centroid must lie along the vertical line AB. The slice was then suspended from another point D, C was marked as the other intersection between the slice edge and the string, and the vertical line CD was marked. The intersection of lines AB and CD, denoted as O, represents the centroid of the apple slice. The distance from the centroid to the z-axis was measured to be 1.6 cm.



(a) (b) (c)

Figure 3: (a), (b): The apple slice is hung from different points and the four points, A, B, C, D are marked. (c): The intersection of AB and CD is marked as O, which is the center of gravity, equivalent to the centroid by our assumption.

The area of the cross-section was then determined using a photograph taken with the camera positioned parallel to the plane of the slice. The image was imported into GeoGebra, where the length scale was calibrated using a ruler visible in the photograph. The polygon tool was used to trace the slice's outline, and the area tool calculated the area of the polygon, yielding an approximate area of $A \approx 20.21\,\text{cm}^2$.

4.2.2. Calculation of the Theoretical Volume of the Apple

With the centroid's distance from the z-axis and the cross-sectional area determined, the Pappus Centroid Theorem was applied to estimate the theoretical volume of the apple. The theorem states:

$$V_{\text{theoretical}} = 2\pi\bar{x}A$$

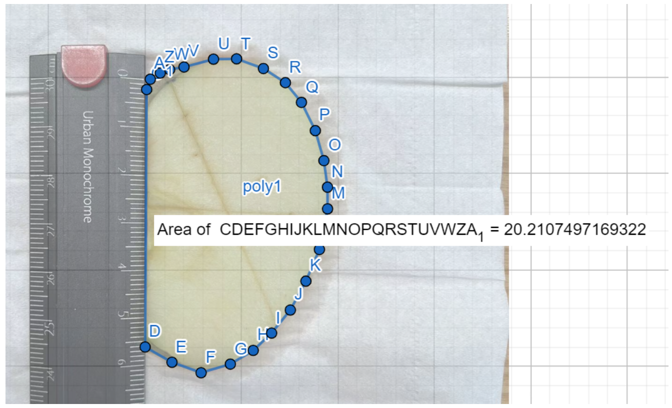


Figure 4: The area of the apple slice, A, as measured using GeoGebra, [3], [3], was approximately $20.21\,\text{cm}^2$.

Substituting the measured values:

$$V_{\text{theoretical}} = 2\pi \times 1.60\,\text{cm} \times 20.21\,\text{cm}^2 \approx 203.2\,\text{cm}^3$$

Comparing this theoretical volume with the measured volume of $211.58\,\text{cm}^3$, the experimental error is calculated as:

$$\Delta V = \frac{V_{\text{experimental}} - V_{\text{theoretical}}}{V_{\text{experimental}}} = \frac{211.58 - 203.2}{211.58} \approx 3.97\%$$

This error is within a reasonable range, demonstrating the reliability of the estimation method.

5. Reflection

5.1. Reflection on the Theory

In the theoretical proof presented, the scope was restricted to cases where the shape of the plate can be represented as a function. This assumption simplifies the analysis by ensuring that for each value of x, there is a unique corresponding value of y. However, it is important to acknowledge that there are cases where the plate's shape is better described as a relation rather than a function, particularly when a single x value corresponds to multiple y values. These cases introduce additional complexity into the analysis, as they require a more nuanced approach to determine the centroid and, subsequently, the volume of the solid of revolution.

Moreover, Pappus's Centroid Theorem, despite its theoretical elegance, is not widely utilized in practice, likely due to the challenge of accurately determining the centroid for irregular shapes. For many complex forms, the process of finding the centroid does not offer significant simplification compared to traditional methods involving direct integration. Nevertheless, Pappus's Theorem remains a powerful and efficient tool for calculating the volume of more regular solids of revolution, such as cylinders, cones, and tori, where the centroid can be determined more straightforwardly. In these cases, the theorem provides a streamlined alternative to more cumbersome volumetric calculations.

5.2. Evaluation of Assumptions and Error Analysis in Application

In this estimation, several key assumptions were made:

- The density of the apple is assumed to be uniform throughout its volume.
- The shape of the apple is approximated as a solid of revolution, generated by rotating half of its cross-section about the z-axis.

- The density of water is taken as $1\,\mathrm{g/cm^3}$, a standard assumption in such volumetric measurements.
- In the water displacement method used for volume measurement, any water adhering to the apple upon removal was disregarded, assuming it had negligible impact on the volume calculation.

The discrepancy observed between the experimental volume and the theoretical volume calculated via the Pappus Centroid Theorem may be attributed to challenges in accurately measuring the apple's volume. These challenges include:

- The lack of an appropriately sized graduated cylinder to fully immerse the apple necessitated the use of the water displacement method.
- During the water displacement procedure, the apple, having a density less than that of water, tended to float. To ensure complete submersion, a stick was used to push the apple below the water surface. However, the volume of the submerged portion of the stick was not accounted for in the calculation.
- Upon removing the apple from the water, some water adhered to my fingers. This excess water was not included in the final volume measurement, potentially leading to minor inaccuracies.
- The term "cup filled with water" is not entirely accurate due to the presence of water's surface tension, which can cause the water to form a meniscus. This meniscus allows for a wider range of possible volumes within the cup, depending on the curvature of the water surface. Such variability can introduce additional uncertainty in the initial mass measurement m_0, thereby affecting the calculated volume.

These factors contribute to the observed error and highlight the practical challenges encountered during the experimental determination of the apple's volume.

6. Conclusion

This paper explored the application of geometric principles, particularly Pappus's Centroid Theorem, in estimating the volume of irregular objects, using an apple as a case study. We began with an introduction to the fundamental concepts and provided a clear example through the derivation of the volume of a torus, which served as a foundation for understanding the theorem's application.

The core of our analysis centered on Pappus's Centroid Theorem, for which we rigorously deduced the formula for the centroid and provided proofs for both the first and second theorems. These theoretical frameworks enabled us to apply the theorem effectively to real-world problems, specifically the estimation of an apple's volume.

The experimental measurement was conducted using the water displacement method, a practical technique to approximate the volume of irregular objects. We then applied Pappus's Centroid Theorem to calculate a theoretical volume of the apple, using its centroid and cross-sectional area as key inputs. The comparison between the experimentally measured volume and the theoretically calculated volume allowed us to assess the accuracy and validity of our assumptions. To evaluate, we identified several factors contributing to the observed discrepancies, such as the challenges in accurately measuring the apple's volume due to its buoyancy, the potential inaccuracies introduced by surface tension in the water, and the limitations of the displacement method. These factors might have caused practical difficulties in applying theoretical principles to physical measurements and underscored the need for careful consideration of experimental conditions.

In conclusion, this study demonstrates the utility of Pappus's Centroid Theorem in estimating the volume of complex shapes. While the application of the theorem provided a reasonably accurate estimate, the evaluation of errors underscores the importance of experimental precision and the limitations inherent in practical measurement methods. Future studies could explore alternative methods to further refine the accuracy of volume estimation for irregular objects, or extend the application of Pappus's Theorem to other geometric forms and practical contexts.

References

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