Calculus Crash Course

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Contents

§1 October 8 - Derivative

§1.1 Derivative Function

Definition 1.1 (Derivative Function)**.** Gradient function, gradient of the tangent for the original function, of $y = f(x)$ is called its derivative function and is labelled $f'(x)$ or $\frac{dy}{dx}$

Exercise 1.2. What is the derivative function of $y = 3$ and $y = 2x$?

§1.2 First principle

Question 1.3. What is the gradient of a line if A $(a, f(a))$ and B $(a + h, f(a + h))$ are on the line?

Claim 1.4 — When A and B gets infinitely close, the gradient is the gradient of the tangent for $y = f(x)$ where $x = a$.

Definition 1.5 (First principle)**.** The derivative function is defined as: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

Exercise 1.6. Compute $y = 2x$, $y = 3x^2$ using first principle.

Exercise 1.7. Prove that $\frac{d}{dx}x^n = nx^{n-1}$ using first principle.

Exercise 1.8. Prove that if $f(x) = cu(x)$, then $f'(x) = cu'(x)$ using first principle.

Exercise 1.9. Prove that if $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$ using first principle.

§1.3 Differentiability

Definition 1.10. If the limit $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists, $f(x)$ is differentiable at $x = a$.

Claim 1.11 — If f is differentiable at $x = a$, then f is also continuous at $x = a$.

Proof.

$$
\begin{aligned}
&\lim_{h\to 0} f(a+h) - f(a) \\
=&\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \times h \\
=&\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \times \lim_{h\to 0} h \qquad \{\text{by the limit laws, since both limits exist}\} \\
=&f'(a) \times 0 \\
=&0
\end{aligned}
$$

Therefore, $\lim_{h\to 0} f(a+h) = f(a)$

Letting $x = a + h$, this is equivalent to $\lim_{x\to a} f(x) = f(a)$.

Therefore, f is continuous at $x = a$. □

So we can conclude the way to test for differentiability:

Proposition 1.12 (Test for Differentiability)
A function f with domain D is **differentiable at $\boldsymbol{x = a}$**, $a \in D$, if:

1. f is continuous at $x = a$, and
2. $f'_-(a) = \lim_{h \to 0^-} \dfrac{f(a+h) - f(a)}{h}$ and $f'_+(a) = \lim_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h}$ both exist and are equal.

§1.4 Fundamental rules of differentiation

We have learned from former exercise that if $f(x) = cu(x)$, then $f'(x) = cu'(x)$, and if $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$.

Then we can start thinking about the $f'(x)$ when $f(x) = u(x)v(x)$ or $f(x) = \dfrac{u(x)}{v(x)}$. Try to deduce the formula by using first principle.

Theorem 1.13 (The Product Rule)
If $f(x) = u(x)v(x)$, then $f'(x) = u'(x)v(x) + u(x)v'(x)$. Alternatively, if $y = uv$ where u and v are functions of x, then

$$\frac{dy}{dx} = u'v + uv' = \frac{du}{dx}v + u\frac{dv}{dx}$$

Theorem 1.14 (The Quotient Rule)
If $Q(x) = \dfrac{u(x)}{v(x)}$, then $Q'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$. Alternatively, if $y = \dfrac{u}{v}$ where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$

The rules about calculations between simple functions are all listed and the next and maybe the most important rule is the chain rule.

Definition 1.15 (Chain rule)**.** Version 1: If $y = g(u)$ where $u = f(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Version 2: If $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$

Proof.

$$\begin{aligned}
\frac{dy}{du} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta u}\frac{\delta u}{\delta x} \\
&= (\lim_{\delta x \to 0} \frac{\delta y}{\delta u})(\lim_{\delta x \to 0} \frac{\delta u}{\delta x}) \\
&= (\lim_{\delta u \to 0} \frac{\delta y}{\delta u})(\lim_{\delta x \to 0} \frac{\delta u}{\delta x}) \\
&= \frac{dy}{du}\frac{du}{dx}
\end{aligned}$$

□

§1.5 Derivative of different functions

§1.5.1 Derivative of logarithmic functions

Exercise 1.16. Prove that $(\log_a(x))' = \dfrac{1}{x \ln a}$ by using first principle.

Proof.

$$
\begin{aligned}
(\log_a(x))' &= \lim_{\delta x \to 0} \frac{\log_a(x+\delta x) - \log_a(x)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\log_a(\frac{x+\delta x}{x})}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\log_a(1+\frac{\delta x}{x})}{x} \cdot \frac{x}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\log_a(1+\frac{\delta x}{x})^{\frac{x}{\delta x}}}{x} \\
&= \frac{\log_a(e)}{x} \\
&= \frac{1}{x \cdot \ln a}
\end{aligned}
$$

□

Exercise 1.17. Show that $(\ln f(x))' = \dfrac{f'(x)}{f(x)}$

§1.5.2 Derivative of exponential functions

Exercise 1.18. Using $x = \ln e^x$, find $(e^x)'$

Exercise 1.19. Show that $(a^x)' = \ln a \cdot a^x$

Exercise 1.20. Compute $(x^x)'$

§1.5.3 Derivative of trigonometric functions

Exercise 1.21. Show that $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

Proof.

$$
\begin{aligned}
(\sin x)' &= \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\sin x \cos \delta x + \sin \delta x \cos x - \sin x}{\delta x} \\
&= \cos x \\
(\cos x)' &= \lim_{\delta x \to 0} \frac{\cos(x+\delta x) - \cos x}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\cos x \cos \delta x - \sin \delta x \sin x - \cos x}{\delta x} \\
&= -\sin x
\end{aligned}
$$

□

Try to prove the following derivatives by using product rule and quotient rule:

$$\begin{aligned}
(\sin x)' &= \cos x \\
(\cos x)' &= -\sin x \\
(\tan x)' &= \sec^2 x \\
(\cot x)' &= -\csc^2 x \\
(\sec x)' &= \tan x \cdot \sec x \\
(\csc x)' &= -\cot x \cdot \csc x
\end{aligned}$$

Proof.

$$\begin{aligned}
(\tan x)' &= (\frac{\sin x}{\cos x})' \\
&= \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

□

§1.5.4 Derivative of inverse trigonometric functions

Exercise 1.22. Show that $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$, $(\arctan x)' = \frac{1}{1+x^2}$

Proof.

$$\begin{aligned}
y = \arcsin x&, x = \sin y \\
\frac{dx}{dy} &= \cos y \\
\frac{dy}{dx} &= \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1-\sin y^2}} \\
&= \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

□

§2 October 16 - Applications of Derivative

§2.1 Sketching Graph by Derivative

Definition 2.1. Suppose S is an interval in the domain of $f(x)$ such that $f(x)$ is defined for all x in S

- $f(x)$ is increasing on S $\longleftrightarrow f(a) \leqslant f(b)$ for all $a, b \in S$ and $a < b \longleftrightarrow f'(x) \geqslant 0$
- $f(x)$ is decreasing on S $\longleftrightarrow f(a) \geqslant f(b)$ for all $a, b \in S$ and $a < b \longleftrightarrow f'(x) \leqslant 0$

Example 2.2

Prove that lnx is an increasing function when $x > 0$
Traditional Way:

$$\begin{gathered} \forall x_1 > x_2 > 0 \\ f(x_1) - f(x_2) = lnx_1 - lnx_2 = ln\tfrac{x_1}{x_2} > 0 \\ \therefore f(x_1) > f(x_2) \\ \therefore lnx \text{ is an increasing function when } x > 0 \end{gathered}$$

Using Derivative:

$$\begin{gathered} (lnx)' = 1/x \\ \because x > 0 \therefore 1/x > 0 \\ \therefore lnx \text{ is an increasing function when } x > 0 \end{gathered}$$

Theorem 2.3 (Fermat's Theorem)

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Example 2.4

Find the maximum and minimum value of $sinx + cos2x$
Traditional way:

$$\begin{gathered} \sin x + \cos 2x = \sin x + (1 - 2\sin^2 x) = -2\sin^2 x + \sin x + 1 = -2(\sin x - \tfrac{1}{4})^2 + \tfrac{9}{8} \\ \because -1 \leqslant \sin x \leqslant 1 \\ \therefore -2 \leqslant -2(sinx - \tfrac{1}{4})^2 + \tfrac{9}{8} \leqslant \tfrac{9}{8} \end{gathered}$$

Using Derivative:

$$(\sin x + \cos 2x)' = \cos x - 2\sin 2x$$

The original function $f(x)$ reaches its maximum when $(\sin x + \cos 2x)' = 0$, solving the equation and we can get $\sin x = \frac{1}{4}$, $\cos 2x = 1 - 2 \times \frac{1}{4}^2 = \frac{7}{8}$ or $\cos x = 0, x = \pi/2 + k\pi (k \in \mathbb{Z})$. Therefore, the maximum of function is $\frac{9}{8}$ while the minimum of function is -2.

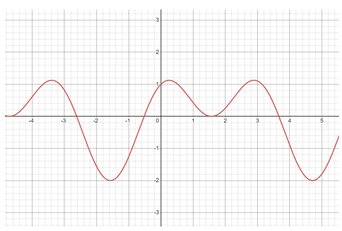

Figure 1: $\sin x + \cos 2x$

Definition 2.5. The second derivative, or the second-order derivative, of a function f is the derivative of the derivative of f. It can be written as:

$$\frac{d^2y}{dx^2} = f''(x)$$

Definition 2.6. If the graph of lies above all of its tangents on an interval , then it is called **concave upward on** $(f''(x) > 0)$**.** If the graph of lies below all of its tangents on I,it is called **concave downward on** $(f''(x) < 0)$**.**

This is because $f''(x)$ represents rate of change of$f'(x)$, namely the slope of a function.

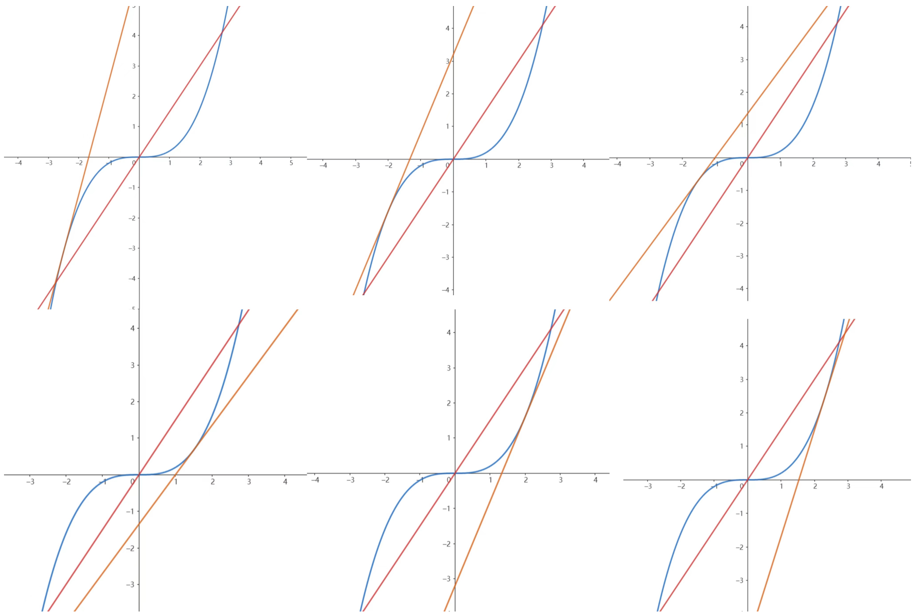

Figure 2: Example of how rate of change of slope effect function's shape

Definition 2.7. A point P on a curve $f(x)$ is called **an inflection point** if $f(x)$ is continuous there, the curve changes from concave upward to concave downward or from concave downward to concave upward at P $(f'' = 0)$.

Remark 2.8. Are the gradient of a function at an inflection point necessarily equal to 0?
The answer is NO. There is no relationship between $y'' = 0$ and $y' = 0$.

Theorem 2.9 (The Second Derivative Test)
For $f(x)$ continuous near a:
If $f'(a) = 0$ and $f''(a) > 0$, $f(x)$ has a local minimum at a.
If $f'(a) = 0$ and $f''(a) < 0$, $f(x)$ has a local maximum at a.

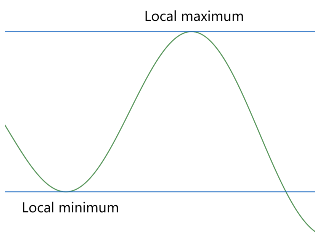

Figure 3: Example of the second derivative test

Now we can sketch almost all elementary functions. Let's try!

Exercise 2.10. Sketch the graph of $y = x^4 - 3x^3 + 1$.

$$y' = 4x^3 - 9x^2, \qquad \text{when } y' = 0, x = 0 \text{ or } \frac{9}{4},$$
$$y'' = 12x^2 - 18x, \qquad \text{when } y' = 0,\ x = 0 \text{ or } \frac{3}{2}.$$

Exercise 2.11. Sketch the graph of $y = \dfrac{x^2}{\sqrt{x+1}}$

Exercise 2.12. Sketch the graph of $y = \sin(2x) + \cos(x)$

Answers are in the shared GeoGebra File.

§2.2 Indeterminate Forms and L' Hopital Rule

§2.2.1 Indeterminate Form 0/0

Theorem 2.13 (L' Hopital[a] Rule)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

[a] L' Hopital should be pronounced as *lowpeetal* as its original pronunciation in French.

§2.2.2 Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an indeterminate form like $\infty/\infty, \infty \cdot 0, \infty - \infty$ instead of $0/0$. We first consider the form ∞/∞.

When we are trying to calculate $\lim_{x \to a} f(x)/g(x)$ while $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$$

and since $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, $1/f(x) \to 0$ and $1/g(x) \to 0$, therefore, we can apply L'Hopital Rule to it.

Similarly, for the $0 \cdot \infty$ case, just transform the ∞ to $1/0$ and therefore, the $0 \cdot \infty$ indeterminate case turns into $0/0$ form.

For the $\infty - \infty$ case, turn $f(x) - g(x)$ into fractional form, an example here will be more clear:

Example 2.14

Find the limit of this $\infty - \infty$ form:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

Solution.

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) &= \lim_{x\to 0}\frac{x-\sin x}{x\sin x} && \frac{0}{0}\\
&= \lim_{x\to 0}\frac{(x-\sin x)'}{(x\sin x)'}\\
&= \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} && \text{Still } \frac{0}{0}\\
&= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.
\end{aligned}$$

□

§2.2.3 Extension - Proof of L'Hopital Rule

Theorem 2.15 (The Rolle's Theorem)

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Proof. This is intuitively easy and is related to the local/global minima/maxima and interior points. Can you sketch a proof for it by yourself? This is left as an exercise for reader. □

Theorem 2.16 (The Mean Value Theorem)

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof. We picture the graph of f and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$. The secant line can be expressed by

$$g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

with point-slope equation. The vertical difference between the graphs of f and g at x is

$$\begin{aligned}
h(x) &= f(x) - g(x)\\
&= f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)
\end{aligned}$$

According to Rolle's Theorem, we know that there must exist at least one point c such that $h'(c) = 0$.

We differentiate both sides of the equation with respect to x and set $x = c$:

$$\begin{aligned}
h'(x) &= f'(x) - \frac{f(b)-f(a)}{b-a}\\
h'(c) &= f'(c) - \frac{f(b)-f(a)}{b-a}\\
0 &= f'(c) - \frac{f(b)-f(a)}{b-a}\\
f'(c) &= \frac{f(b)-f(a)}{b-a}
\end{aligned}$$

and therefore we are done. □

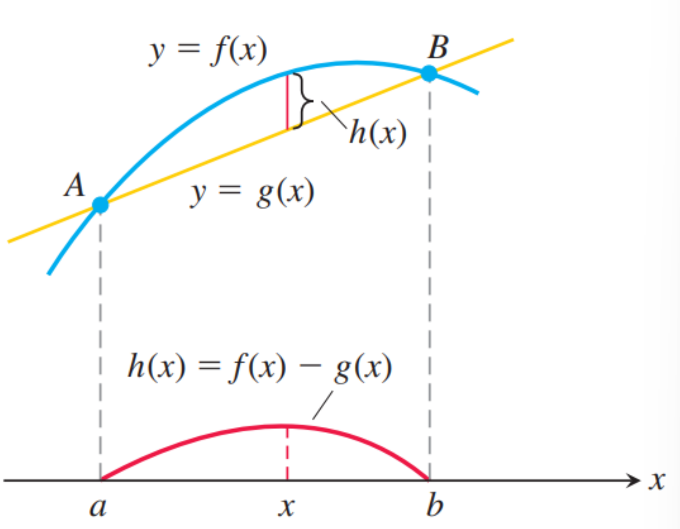

Figure 4: The secant AB is the graph of the function $g(x)$. THe function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x.

Theorem 2.17 (L' Hopital Rule)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

Proof. We first establish the limit equation for the case $x \to a^+$. The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.

Suppose that x lies to the right of a. Then $g'(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from a to x. This step produces a number c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

But $f(a) = g(a) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

As x approaches a, c approaches a because it always lies between a and x. Therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

which establishes L'Hopital's Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy's Mean Value Theorem to the closed interval $[x, a], x < a$. $\square$