Calculus Crash Course

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Contents

§1 December 13 - Integration and Its Application

§1.1 What is Integration?

In this section, we will first find the relationship between area and definite integral. Then, we will delve into indefinite integral and some calculating techniques related to it. Finally, we will go back to some complex indefinite integration and its applications.

§1.1.1 Riemann Sums

Definition 1.1. A Riemann sum is an approximation of a region's area, obtained by adding up the areas of multiple simplified slices of the region.

The Riemann sums can be described in this formula:

$$S_n = \sum_{k=1}^{n} f(a + k \cdot \frac{b-a}{n})(\frac{b-a}{n})$$

where we can express $\frac{b-a}{n}$ as Δx

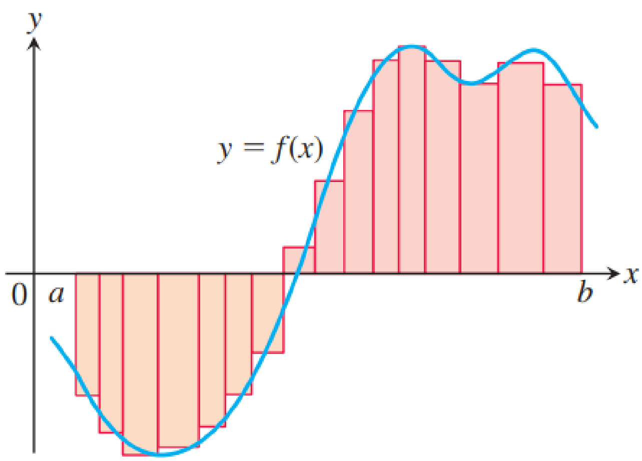

Figure 1: An Example of Riemann Sum

§1.1.2 Definite Integral

Definition 1.2.

- If limit $J = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(a + k\Delta x) \cdot \Delta x$ exists, we say that the definite integral exists.
- We express definite integral as $\int_a^b f(x)dx$, where a is the lower limit, b is the upper limit, $f(x)$ is the integrand, and dx is the variable of the integration.

Definition 1.3. If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, the definite integral $\int_a^b f(x)dx$ exists and f is the integrable over $[a, b]$.

§1.1.3 Definite Integral and Area

Example 1.4

The definite integral, or Riemann sum, of $f(x) = x^2$ over $[0,2]$ can be calculated through these procedures:

$$\begin{aligned}
A_{\text{smaller}} &= \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(k-1) \cdot \Delta x \\
&= \lim_{\Delta x \to 0} \sum_{k=1}^{n} \left((k-1)\frac{2}{n} \right)^2 \cdot \frac{2}{n} \\
&= 8 \lim_{\Delta x \to 0} \sum_{k=1}^{n} (k-1)^2 \cdot \left(\frac{1}{n} \right)^3 \\
&= 8 \lim_{n \to \infty} \frac{(n-1)(n)(2n-1)}{6n^3} \\
&= \frac{4}{3} \lim_{n \to \infty} \left(2 - \frac{3}{n} + \frac{1}{n^2} \right) \\
&= \frac{8}{3}
\end{aligned}$$

$$\begin{aligned}
A_{\text{larger}} &= \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(k) \cdot \Delta x \\
&= \lim_{\Delta x \to 0} \sum_{k=1}^{n} \left((k\frac{2}{n} \right)^2 \cdot \frac{2}{n} \\
&= 8 \lim_{\Delta x \to 0} \sum_{k=1}^{n} k^2 \cdot \left(\frac{1}{n} \right)^3 \\
&= 8 \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3} \\
&= \frac{4}{3} \lim_{n \to \infty} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) \\
&= \frac{8}{3}
\end{aligned}$$

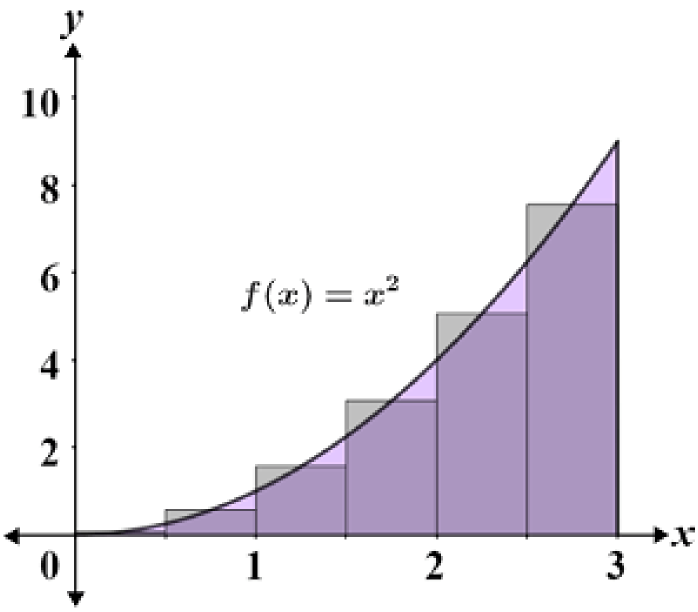

Figure 2: $y = x^2$ Diagram

But how is this related to the derivative we discussed before? In fact, integration is the inverse operation of derivative, and applying integration can help us to calculate the value of definite integral in a very simple way.

Remark 1.5. If f is continuous over $[a,b]$ and F is any antiderivative of f on $[a,b]$ ($F'(x) = f(x)$), then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof. Suppose $f(t)$ is continuous over $[a,b]$. Looking at this $y = f(t)$ at $t = x$, we want to find the rate of the increasing rate of the Area $A(x)$ under the curve if an infinitely small amount $h = dt$ is added to x.

$$\frac{dA(x)}{dt} = \frac{f(x)\cdot h}{dt} = f(x)$$

Since integration is the inverse operation of derivative, we conclude that $\int_a^x f(t)dx = F(x) - F(a)$,where $F'(x) = f(x)$ □

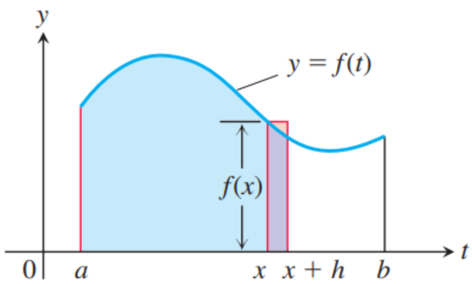

Figure 3: $y = f(t)$ at $t = x$

Example 1.6

We can apply this approach to compute the area under the curve of $f(x) = x^2$ over $[0,2]$:

$$\int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$$

We can see this aligns with the result we obtain in Example 1.4.

Example 1.7

Compute the area under the curve $f(x) = \sin x$ over $[0,\pi]$:

$$\int_0^\pi \sin x dx = [-\cos x]_0^\pi = 1 + 1 = 2$$

And the area under the curve $f(x) = \sin x$ over $[0,2\pi]$:

$$\int_0^{2\pi} \sin x dx = [-\cos x]_0^{2\pi} = -1 + 1 = 0$$

How could the area become zero? This is because the approach considers the area under x-axis as negative, so it cancels out.

Example 1.8

Compute the definite integral

$$\int_0^{\frac{12}{13}} \frac{\sec^2(\frac{1}{2}\arcsin x)}{2\sqrt{1-x^2}}dx.$$

Solution. Let $u = \sin x$; then $du = \cos x dx$, and $u|_{x=\frac{12}{13}} = \arcsin\frac{12}{13}$.

$$I = \int_0^{\arcsin\frac{12}{13}} \frac{\sec^2\left(\frac{1}{2}u\right)}{2}du = \tan\left(\frac{1}{2}\arcsin\frac{12}{13}\right) = \frac{12}{18} = \frac{2}{3}.$$

□

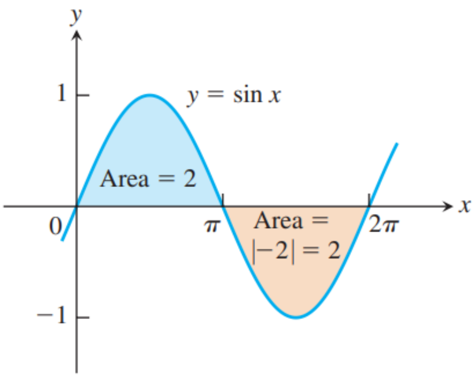

Figure 4: $y = \sin x$

§1.1.4 Properties of Definite Integral

Some definite integral rules are listed below.

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$
- $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

§1.2 Indefinite Integral

Definition 1.9. We defined the **indefinite integral** of the function with respect to x as the set of all antiderivatives of , symbolized by $\int (x)dx$. We need to add a constant because the derivative of a constant is 0:

$$\int (x)dx = F(x) + C$$

§1.3 Integration Techniques

§1.3.1 Substitution

Suppose

$$\frac{dF}{du} = f(u)$$

$$\int f(u)du = F(u) + c$$

Recall chain rule

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx}$$

$$\int f(u)\frac{du}{dx}dx = F(u) + c$$

comparing the two integration

$$\int f(u)\frac{du}{dx}dx = \int f(u)du$$

Example 1.10

Compute

$$\int_1^2 \frac{\ln x}{x}dx$$

Solution. Let $u = \ln x$, then $du = \frac{1}{x}dx, u|_{x=1} = 0, u|_{x=2} = \ln 2$

$$\int_1^2 \frac{\ln x}{x}dx = \int_0^{\ln 2} udu = \frac{(\ln 2)^2}{2}$$

□

§1.3.2 Integration by Parts

Recall product rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x),$$

in the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + f'(x)g(x)]dx = f(x)g(x)$$

rearranging the equation, it becomes

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

This is the **formula for integration by parts**.

Example 1.11

Compute

$$\int_0^{\pi/2} e^x \sin x dx$$

Solution.

$$\int e^x \sin x dx = e^x(-\cos x) - \int e^x(-\cos x)dx = -e^x \cos x + \int e^x \cos x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$$

$$\int_0^{\pi/2} e^x \sin x dx = \left[\frac{-e^x \cos x + e^x \sin x}{2}\right]_0^{\pi/2} = \frac{1+e^{\pi/2}}{2}$$

□

§1.3.3 Trig Substitution

Recall basic trigonometric formula:

$$1 - \sin^2(x) = \cos^2(x)$$

$$1 + \tan^2(x) = \sec^2(x)$$

Thus we can have the following trigonometric substitution:

for	$\sqrt{a^2 - x^2}$	we can have	$x = a\sin\theta$
	$\sqrt{a^2 + x^2}$		$x = a\tan\theta$
	$\sqrt{x^2 - a^2}$		$x = a\sec\theta$

Table 1: Trigonometric substitution

Example 1.12

Compute

$$\int \frac{\sqrt{9-x^2}}{x^2}dx$$

Solution. Let $x = 3\sin\theta$, then $dx = 3\cos\theta d\theta, \theta = \arcsin\frac{x}{3}$

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{9\cos^2\theta}{9\sin^2\theta}d\theta = \int (\csc^2\theta - 1)d\theta = -\cot\theta - \theta + c = -\frac{\sqrt{9-x^2}}{x} - \arcsin\frac{x}{3} + c$$

□

§1.3.4 Partial Fractions

$$\int \frac{dx+e}{ax^2+bx+c}dx = \int (\frac{A}{mx+n} + \frac{B}{px+q})dx = \frac{A}{m}\ln|mx+n| + \frac{B}{p}\ln|px+q| + c$$

Example 1.13

Compute

$$\int \frac{7x-2}{x^2-x-2}dx$$

Solution:

$$\int \frac{7x-2}{x^2-x-2}dx = \int (\frac{4}{x-2} + \frac{3}{x+1})dx = 4\ln|x-2| + 3\ln|x+1| + c$$

§1.4 Common Applications of Integration

§1.4.1 Area between Curves

Consider the region S shown in the figure that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$.

We divide S into n strips of equal width and then we approximate the ith strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$, we could take all of the sample points to be right endpoints, in which case $x_i^* = x_i$. The Riemann sum

$$\sum_{i=1}^{n}[f(x_i^*) - g(x_i^*)]\Delta x$$

is therefore an approximation to what we intuitively think of as the area of S.

When we divide the area to $n \to \infty$ pieces,

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i*) - g(x_i*)]\Delta x$$

We recognize the limit is equivalent to the definite integral of $f - g$, therefore, the area formula is

$$A = \int_a^b [f(x) - g(x)]dx$$

Example 1.14

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution. It is trivial that they intersect at $(0, 0)$ and $(1, 1)$, so the total area is

$$A = \int_0^1 (2x - 2x^2)dx = 2\int_0^1 (x - x^2)dx$$
$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right] = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \boxed{\frac{1}{3}}$$

□

§1.4.2 Volumes

Consider a random non-cylindrical geometry S.

We can cut it into n slices by approximating each slice as a column. Use a plane to intersect the geometric body S to obtain a planar region called the plane region of the cross section of S. Let $A(x)$ be the area of cross section of S on plane P_x that lies perpendicular to the x-axis that passes through point x which $a \leq x \leq b$.

We can use planes P_{x1}, P_{x2}, ... to divide S into equal width of slices. We approximate the ith slice by a cylindrical with base $A(\overline{x_i})$ and height Δx.

The Riemann sum

$$\sum_{i=1}^{n} A(x_i)\Delta x$$

is therefore an approximation to the volume of S.

When we divide S to $n \to \infty$ slices,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x$$

Thus, the volume formula is

$$V = \int_a^b A(x)dx$$

Example 1.15

Derive the formula $V = \frac{1}{3}a^2h$ for the volume of a pyramid with a square base.

Solution. According to figure below, $A(x)$ would be $A(x) = s^2 = (\frac{ax}{h})^2$

So the volume is

$$V = \int_0^h A(x)dx = \frac{a^2}{h^2}\int_0^h x^2dx = [\frac{a^2}{h^2}(\frac{1}{3}x^3)]_0^h = \frac{1}{3}a^2h$$

□

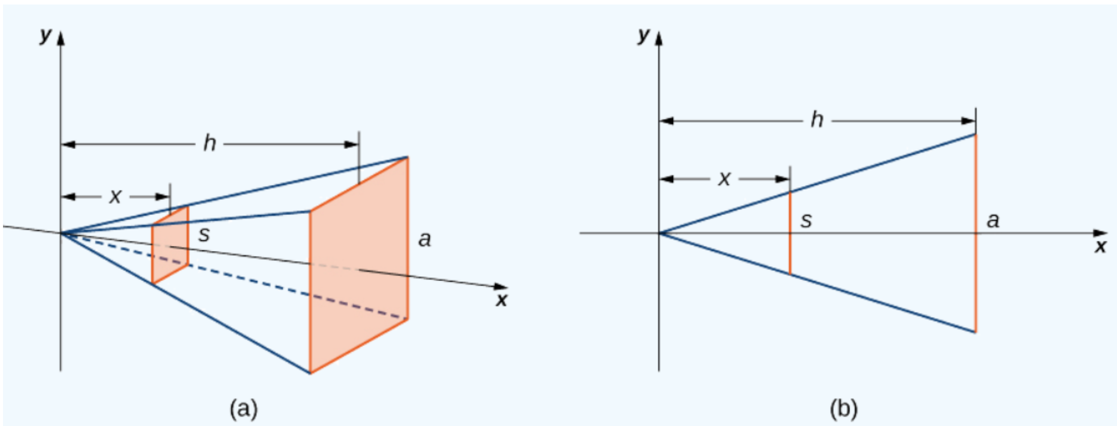

Figure 5: (a)A pyramid with a square base (b) A two-dimensional view of the pyramid

§1.4.3 Volumes by Cylindrical Shells

Again, consider the region S shown in the figure that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$.

Now suppose this region S rotate around the y-axis and consider the shape V obtained by the revolution.

Again, we divide V into n rings with equal width and we approximate the ith ring by a cylindrical shell with mean radius $\overline{x_i}$, height $f(\overline{x_i}) - g(\overline{x_i})$, and width Δx.

The Riemann sum

$$\sum_{i=1}^{n} 2\pi\overline{x_i}[f(\overline{x_i}) - g(\overline{x_i})]\Delta x$$

is therefore an approximation to the volume of V.

When we divide V to $n \to \infty$ pieces,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\overline{x_i}[f(\overline{x_i}) - g(\overline{x_i})]\Delta x$$

Therefore, the volume formula is

$$V = \int_a^b 2\pi x[f(x) - g(x)]dx$$

With the same method, we can also deduct the following formula:
When the region is rotating around the x-axis:

$$V = \int_a^b \pi[f(x)^2 - g(x)^2]dx$$

Example 1.16

Find the volume of a sphere, radius of 1.

Solution. In this case, $f(x) = \sqrt{1-x^2}$ and $g(x) = 0$.
So the volume is

$$V = \int_{-1}^{1} \pi(\sqrt{1-x^2})^2 dx = \pi[x - \frac{1}{3}x^3]_{-1}^{1} = \frac{4}{3}\pi$$

□

§1.4.4 Arc Length

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$.

We divide the curve into n segments with equal width and we approximate the ith segment by a line segment with horizontal change Δx and vertical change $\Delta y_i = f(x_i + \Delta x) - f(x_i)$.

By the Pythagorean theorem, the length of the line segment is

$$\sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{1 + (\frac{\Delta y_i}{\Delta x})^2}\Delta x$$

By the Mean Value Theorem, there is a point x_i^* which $x_i + \Delta x \geq x_i^* \geq x_i$ such that $f'(x_i^*) = \dfrac{\Delta y_i}{\Delta x}$
Then the length of the line segment is given by

$$\sqrt{1 + (f'(x_i^*))^2}\Delta x$$

The Riemann sum

$$\sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x$$

is therefore an approximation of the curve length.

Thus, the arc length formula is

$$l = \int_a^b \sqrt{1 + (f'(x_i^*))^2}dx$$

Example 1.17

Find the perimeter of a circle, radius of 1. We first consider one half of the circle, so the function is:

$$f(x) = \sqrt{1 - x^2}$$

$$f'(x) = \frac{x}{\sqrt{(1 - x^2)}}$$

Applying the formula:

$$l = \int_{-1}^{1} \sqrt{1 + \frac{x^2}{(1 - x^2)}} dx = \int_{-1}^{1} \sqrt{\frac{1}{1 - x^2}} dx = \arcsin x|_{-1}^{1} = \pi$$

(Note: This is just a display of this method's application. It is actually a circular proof.)

§1.4.5 Area of a Surface of Revolution

Again, consider $f(x)$ as a smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis.

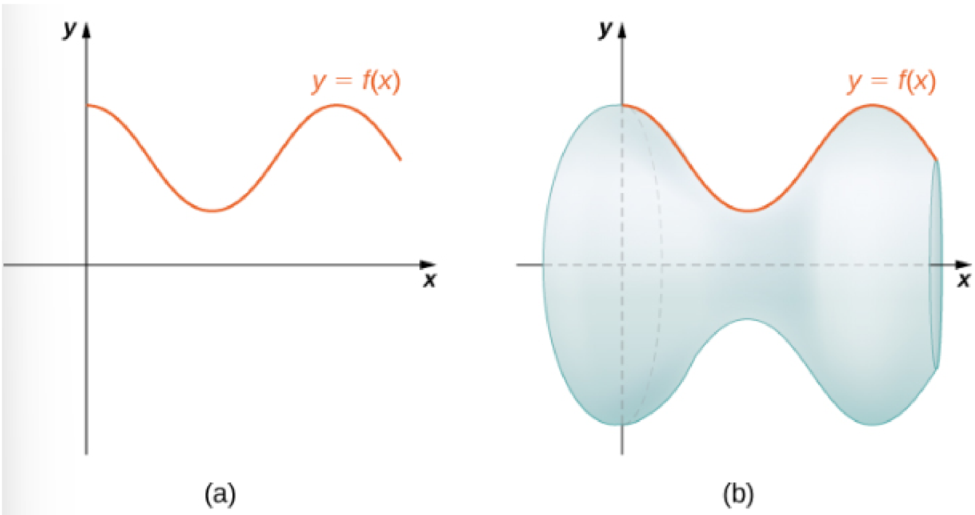

Figure 6: (a)A curve representing f(x).(b)The surface of revolution formed by revolving the graph of $f(x)$ around the x-axis

As we have done many times, we are going to divide it into n slices with equal width and we approximate the ith slice by a ring. We can unfold the ring and calculate the rectangle area with length $2\pi f(x_i^*)$ and width $\sqrt{1 + (f'(x_i^*))^2}\Delta x$.

The Riemann sum

$$\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + (f'(x_i^*))^2}\Delta x$$

is therefore an approximation of the area of the surface of revolution.

Thus, the area formula is

$$A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2} dx$$

Example 1.18

Find the surface area of a sphere, radius of 1.

Solution. The curve $f(x)$ would be $y = \sqrt{1 - x^2}$ in this case, and its derivative is $f'(x) = \dfrac{-x}{\sqrt{1 - x^2}}$.

Thus the surface area would be $S = \int_{-1}^{1} 2\pi\sqrt{1 - x^2}\sqrt{1 + (\dfrac{-x}{\sqrt{1 - x^2}})^2}dx = [2\pi x]_{-1}^{1} = 4\pi$ □

References and Extended Reading Materials

[1] Thomas' Calculus (George B. Thomas, Joel R. Hass etc.)

[2] Calculus (J. Stewart, D. Clegg, S. Watson) Metric Version — 9E Early Transcendentals