Calculus Crash Course

Hechen Sha, Suni Yao, Yuyang Wang, Xinyan Huang

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Contents

§1 November 19 - Taylor Series and Prerequisites

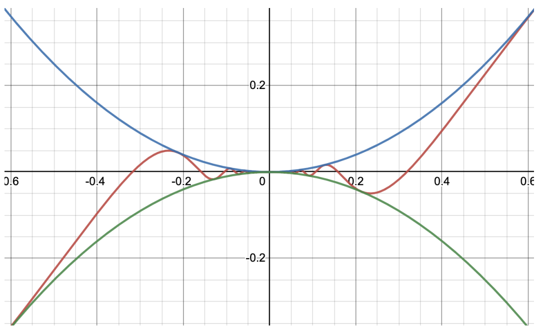
§1.1 Infinitesimals

Definition 1.1.
- Suppose the function f defined on $U^\circ(x_0)$ satisfies the limit $\lim_{x\to x_0} f(x) = 0$. Then, f is the infinitesimal when $x \to x_0$. (The infinitesimal when x goes to ∞, or reaches a right or left limit is defined analogously.)
- We call g a bounded quantity when $x \to x_0$ if the function g is bounded on some $U^\circ(x_0)$.

Example 1.2 (Miscellaneous Calculations)
Prove that

- The sum, difference, and product of two infinitesimal amounts of the same type are all infinitesimal amounts.
- The product of an infinitesimal quantity and a bounded quantity is an infinitesimal quantity.

Proof to the second observation:
Suppose that $\forall x \in U^\circ(x_0)$, $|g(x)| < M$. Then, we must have $|f(x)g(x) - 0| \le |f(x)||g(x)| < M|f(x)|$. By the definition of the infinitesimal, we know that $\forall \epsilon' > 0$, $\exists \delta > 0$ such that $|f(x)| < \epsilon' = \frac{\epsilon}{M}$ for all $x \in U^\circ(x_0, \delta)$ by the arbitrary nature of ϵ'. This naturally ends the proof.



$x^2 \sin \frac{1}{x}$

Next, we would like to introduce a method to compare the speed at which infinitesimals approach 0.

Definition 1.3. If $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$, then f is an infinitesimal of higher order when $x \to x_0$. We note this as

$$f(x) = o(g(x)) \ (x \to x_0).$$

Specifically, $f(x) = o(1)$ $(x \to x_0)$ stands for "f is the infinitesimal as $x \to x_0$."

Example 1.4
It is obvious that $x^{k+1} = o(x^k),\ x \to 0$ for $k \in \mathbb{Z}^+$.

Example 1.5
Show that $1 - \cos x = o(\sin x),\ x \to 0$.

Note: it should be clarified that $o(g(x))$ refers to $\{f | \lim_{x\to x_0} \frac{f(x)}{g(x)} = 0\}$. The following definitions address other outcomes of convergence speed comparison. (We will not cover them in the lecture.)

Definition 1.6.

- If $\exists K, L > 0$ such that
$$K \le \left|\frac{f(x)}{g(x)}\right| \le L,$$
then f and g are infinitesimals of the same order when $x \to x_0$. Specifically, $\lim_{x\to x_0} \frac{f(x)}{g(x)} = c \ne 0$ ensures the holding of this condition.
- If $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$, f and g are equivalent infinitesimals noted as $f(x) \sim g(x)$, $(x \to x_0)$.

The readers should be able to prove that equivalent infinitesimals can be used interchangeably in limit computations.

§1.2 Differentials

Example 1.7

The side length (x_0) of a square is increased by Δx. What is the square's change in area ΔS?

$$\Delta S = 2x_0\Delta x + (\Delta x)^2$$

Remark 1.8. By Example 1.4, $(\Delta x)^2 = o(\Delta x)$.

Definition 1.9. Some function f defined near x_0 is differentiable at x_0 if $\exists A \in \mathbb{C}$ such that

$$\Delta y = A\Delta x + o(\Delta x).$$

Remark 1.10. After taking the limit as $x \to x_0$, we see that $A = f'(x_0)$.

§1.3 Taylor Series

Example 1.11

Let's look at an example of how polynomials approximate function $f(x) = \sin x$.

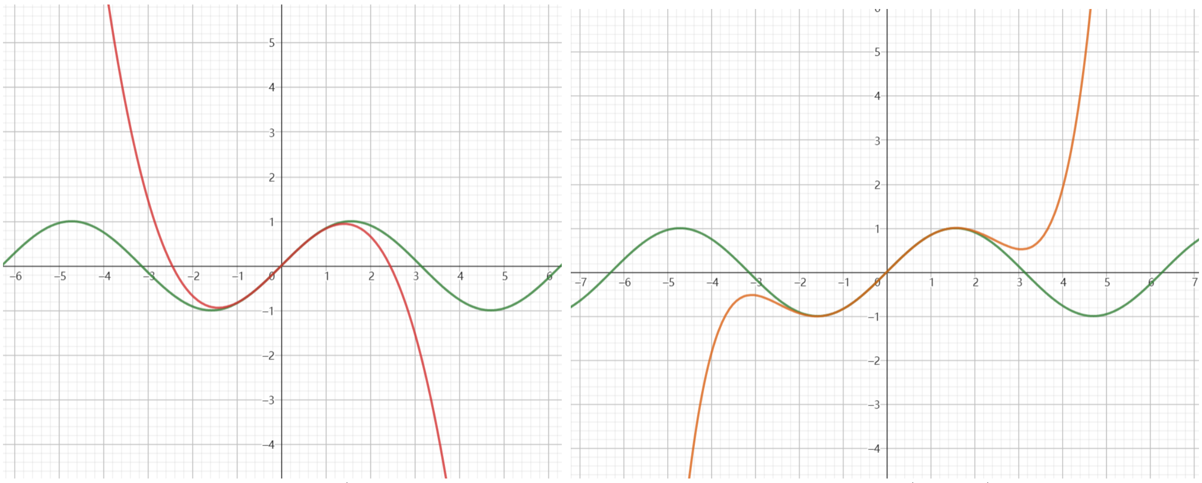
We start from $x = 0$. Suppose the polynomial to be $p(x)$. We need $f(0) = p(0)$, so $p(x) = 0$.

To approximate the trend of $f(x)$, we need $f'(0) = p'(0)$, so $p(x) = 0 + x$.

To approximate the concavity and convexity of $f(x)$, we need $f''(0) = p''(0)$, so $p(x) = 0 + x + 0 \cdot x^2$.

Following the same pattern, we need $f'''(0) = p'''(0)$, so $p(x) = x - \frac{1}{3!}x^3$.

Following the same pattern, we get $p(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{(-1)^{m-1}}{(2m-1)!} + \ldots$.



(a) $p(x) = x - \frac{1}{3!}x^3$ (b) $p(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

Example 1.12

We derive from the definition of differentials that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0).$$

This equation $f(x) = f(x_0) + f'(x_0)(x - x_0)$ can provide a decent approximation for $f(x)$ at x_0. However, in real-world approximations, $o(x - x_0)$ often prove to be not accurate enough: we would like to have an error of $o((x - x_0)^n)$ for any positive integer n. To do so, we consider the polynomial

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n.$$

Compute $p_n(x_0)$, $p_n'(x_0)$, ..., $p_n^{(n)}(x_0)$. What do you notice?
Solution: $p_n(x_0) = a_0$; $p_n'(x_0) = a_1$; ... $p_n^{(k)}(x_0) = k!a_k$; ...; $p_n^{(n)}(x_0) = n!a_n$.
We thus substitute these derivations into $p_n(x)$ and deduce that

$$p_n(x) = p_n(x_0) + p_n'(x_0)(x - x_0) + \frac{p_n''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{p_n^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Definition 1.13. For a general function f that is n^{th}-order differentiable at x_0, define its Taylor Series at x_0 as

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

We easily observe that $T_n(x_0) = f(x_0)$, $f^{(k)}(x_0) = T^{(k)}(x_0)$, $(k = 0, 1, \ldots, n)$.

Theorem 1.14

For a general function f satisfying the properties in the previous definition,

$$f(x) = T_n(x) + o((x - x_0)^n).$$

Proof. Let $R_n(x) = f(x) - T_n(x)$, $Q_n(x) = (x - x_0)^n$. We aim to show that $\lim_{x \to x_0} \frac{R_n(x)}{Q_n(x)} = 0$. By our previous observations,

$$R_n(x_0) = R_n'(x_0) = \cdots = R_n^{(n)}(x_0) = 0,$$

$$Q_n(x_0) = \cdots = Q_n^{(n-1)}(x_0) = 0,\ Q_n^{(n)}(x_0) = n!.$$

Since $f^{(n)}(x_0)$ exists, $f^{(n-1)}(x_0)$ exists $\forall x \in U^\circ(x_0)$. Therefore, L'Hopital's rule may be applied up to $n-1$ times. We thus have

$$\begin{aligned}\lim_{x\to x_0}\frac{R_n(x)}{Q_n(x)} = \lim_{x\to x_0}\frac{R_n^{(n-1)}(x)}{Q_n^{(n-1)}(x)} &= \lim_{x\to x_0}\frac{f^{(n-1)}(x)-f^{(n-1)}(x_0)-f^{(n)}(x_0)(x-x_0)}{n!(x-x_0)}\\ &= \frac{1}{n!}\lim_{x\to x_0}\left[\frac{f^{(n-1)}(x)-f^{n-1}(x_0)}{x-x_0}-f^{(n)}(x_0)\right]\\ &= 0.\end{aligned}$$

□

Remark 1.15. $o((x-x_0)^n)$ is the Peano remainder of the Taylor Series. We will introduce another type of remainder next week. (Lagrange's remainder)

Example 1.16

Some common Taylor Series:

$$e^x = 1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+o(x^3)$$

$$\ln(x+1) = x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)$$

$$\sin x = x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+o(x^5)$$

$$\cos x = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+o(x^4)$$

$$\frac{1}{1-x} = 1+x^2+x^3+o(x^3)$$

$$(1+x)^a = 1+\frac{a}{1!}x+\frac{a(a-1)}{2!}x^2+\frac{a(a-1)(a-2)}{3!}x^3+o(x^3)$$