软件分析与架构设计

踪迹语义

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踪迹语义（trace semantics）

- **syntax: rules to write programs of the language;**
- **semantics: defines the runtime behavior of programs that is what and how they compute when executed:**
  - trace: sequence of events recording the actions executed during a program execution,
  - partial trace: finite observation of an execution; this observation can stop at any time,
  - finite trace: partial trace that ends upon execution termination,
  - infinite trace: infinite observation of an execution that never terminates,
  - maximal trace: finite or infinite execution trace.

Finite traces

- **Program P:**

```
ℓ1 x = x + 1 ;
   while ℓ2 (tt) {
     ℓ3 x = x + 1 ;
     if ℓ4 (x > 2) ℓ5 break ;}ℓ6;ℓ7
```

- **Prefix traces (initially x = 0)**
  - ℓ_1
  - $\ell_1 \xrightarrow{\texttt{x = x + 1 =}\ 1} \ell_2 \xrightarrow{\mathtt{tt}} \ell_3 \xrightarrow{\texttt{x = x + 1 =}\ 2} \ell_4 \xrightarrow{\neg(\texttt{x > 2})} \ell_2 \xrightarrow{\mathtt{tt}} \ell_3$

- **Finite (maximal) traces:**

$$\ell_1 \xrightarrow{\texttt{x = x + 1 =}\ 1} \ell_2 \xrightarrow{\mathtt{tt}} \ell_3 \xrightarrow{\texttt{x = x + 1 =}\ 2} \ell_4 \xrightarrow{\neg(\texttt{x > 2})} \ell_2 \xrightarrow{\mathtt{tt}} \ell_3 \xrightarrow{\texttt{x = x + 1 =}\ 3}$$
$$\ell_4 \xrightarrow{\texttt{x > 2}} \ell_5 \xrightarrow{\texttt{break}} \ell_6 \xrightarrow{\texttt{skip}} \ell_7$$

Infinite traces

- **Program P:**

$$\ell_1\ \texttt{x = 0 ; while}\ \ell_2\ (\mathtt{tt})\ \{\ \ell_3\ \texttt{x = x+1 ;}\ \}\ \ell_4$$

- **Infinite trace:**

$$\ell_1 \xrightarrow{\texttt{x = 0} = 0} \ell_2 \xrightarrow{\mathtt{tt}} \ell_3 \xrightarrow{\texttt{x = x + 1} = 1} \ell_2 \xrightarrow{\mathtt{tt}} \ell_3 \xrightarrow{\texttt{x = x + 1} = 2} \ell_2 \ldots \ell_2 \xrightarrow{\mathtt{tt}} \ell_3 \xrightarrow{\texttt{x = x + 1} = n} \ell_2 \xrightarrow{\mathtt{tt}} \ell_3 \xrightarrow{\texttt{x = x + 1} = n+1} \ell_2 \ldots$$

Traces and Trace concatenation

- **Traces:**
  - T^+: the set of all finite traces,
  - T^∞: the set of all infinite traces,
  - $T^{+\infty}$: the set of all finite or infinite traces.
- **Conventions:**
  - $\pi = {}^{l}\pi'$: trace π is assumed to start with the program label l
  - $\pi = \pi'^{l}$: trace π is finite and ends with label l
  - Note, π' is not itself a properly formed trace)
- **Trace concatenation:**

$$
\begin{array}{rcll}
\pi_1\ell_1 \frown \ell_2\pi_2 & & & \text{undefined if } \ell_1 \neq \ell_2 \\
\pi_1\ell_1 \frown \ell_1\pi_2 & \triangleq & \pi_1\ell_1\pi_2 & \text{if } \pi_1 \text{ is finite} \\
\pi_1 \frown \pi_2 & \triangleq & \pi_1 & \text{if } \pi_1 \text{ is infinite}
\end{array}
$$

  - Empty trace $\ni$: e.g., ${}^{l}\pi^{l'} = l$, then $\pi = \ni$ and $l = l'$

Values of variables on a trace

- $\varrho(\pi)\mathrm{x}$: **the value of variable x at the end of trace π**
  - is the last value assigned to x (or 0 at initialization)

$$
\begin{aligned}
\varrho(\pi^{\ell} \xrightarrow{\mathtt{x = A} = v} \ell')\mathtt{x} &\triangleq v \\
\varrho(\pi^{\ell} \xrightarrow{\cdots} \ell')\mathtt{x} &\triangleq \varrho(\pi^{\ell}) \quad \text{otherwise} \\
\varrho(\ell)\mathtt{x} &\triangleq 0
\end{aligned}
$$

Prefix trace semantics

- $\pi_1 at[\![S]\!]$:
  - an initialization trace ending on entry $at[\![S]\!]$ of statement S.
- $\mathcal{S}^*[\![S]\!](\pi_1 at[\![S]\!])$:
  - the set of prefix traces $at[\![S]\!]\pi_2^\ell$ of S continuing the trace $\pi_1 at[\![S]\!]$ and reaching some program label $\ell \in labx[\![S]\!]$.

$$\xrightarrow{\quad\pi_1\quad} \underbrace{\mathrm{at}[\![S]\!] \xrightarrow{\quad\pi_2\quad} \ell}_{\in\ \mathcal{S}^*[\![S]\!](\pi_1 \mathrm{at}[\![S]\!])}$$

  - By convention $\mathcal{S}^*[\![S]\!](\pi_1^\ell) = \emptyset$ when $\ell \neq at[\![S]\!]$.

Maximal finite trace semantics

- $\pi_1 at[\![S]\!]$:
  - an initialization trace ending on entry $at[\![S]\!]$ of statement S
- $\mathcal{S}^+[\![S]\!](\pi_1 at[\![S]\!])$:
  - is the set of maximal finite traces $at[\![S]\!]\pi_2\mathrm{after}[\![S]\!]$ of S continuing the trace $\pi_1 at[\![S]\!]$ and reaching $\mathrm{after}[\![S]\!]$

$$\xrightarrow{\quad\pi_1\quad} \underbrace{\mathsf{at}[\![\mathsf{S}]\!] \xrightarrow{\quad\pi_2\quad} \mathsf{after}[\![\mathsf{S}]\!]}_{\in\ \mathcal{S}^+[\![\mathsf{S}]\!](\pi_1\mathsf{at}[\![\mathsf{S}]\!])}$$

- Formally, $\mathcal{S}^+[\![\mathsf{s}]\!](\pi_1\mathsf{at}[\![\mathsf{s}]\!]) \triangleq \{\pi_2\ell \in \mathcal{S}^*[\![\mathsf{s}]\!](\pi_1\mathsf{at}[\![\mathsf{s}]\!]) \mid \ell = \mathsf{after}[\![\mathsf{s}]\!]\}$

Structural prefix trace semantics

- **Structural prefix trace semantics $\widehat{\mathcal{S}}^*[\![\mathrm{S}]\!]$:**
  - S is a program component

$$\pi_2 \in \widehat{\mathcal{S}}^*[\![\mathtt{S}]\!](\pi_1)$$

Prefix trace at a program component S

- $\dfrac{}{\mathrm{at}[\![\mathtt{S}]\!] \in \widehat{\mathcal{S}}^*[\![\mathtt{S}]\!](\pi_1\mathrm{at}[\![\mathtt{S}]\!])}$

  - the prologue trace π_1 terminates at $at[\![\mathrm{S}]\!]$
  - the continuation trace π_2 starts at $\mathrm{at}[\![\mathrm{S}]\!]$

Prefix traces of an empty statement list $\mathtt{Sl} ::= \epsilon$

- $\dfrac{}{\mathrm{at}[\![\mathtt{Sl}]\!] \in \widehat{\mathcal{S}}^*[\![\mathtt{Sl}]\!](\pi\mathrm{at}[\![\mathtt{Sl}]\!])}$

Structural prefix trace semantics of stmt

Prefix traces of an empty statement list `Sl ::= ϵ`

- $$\frac{}{\mathsf{at}[\![\mathtt{Sl}]\!] \in \widehat{\mathcal{S}}^{*}[\![\mathtt{Sl}]\!](\pi\mathsf{at}[\![\mathtt{Sl}]\!])}$$

Prefix traces of a break statement `S ::=` ℓ `break ;`

- $$\frac{}{\ell \xrightarrow{\mathtt{break}} \mathsf{break\text{-}to}[\![\mathtt{S}]\!] \in \widehat{\mathcal{S}}^{*}[\![\mathtt{S}]\!](\pi\ell)}$$

Prefix traces of an assignment statement `S ::=` ℓ `x = A ;`

- $$\frac{v = \mathcal{A}[\![\mathtt{A}]\!]\varrho(\pi\ell)}{\ell \xrightarrow{\mathtt{x = A} = v} \mathsf{after}[\![\mathtt{S}]\!] \in \widehat{\mathcal{S}}^{*}[\![\mathtt{S}]\!](\pi\ell)}$$

Prefix traces of a skip statement `S ::=` ℓ`;`

- $$\frac{}{\ell \xrightarrow{\mathtt{skip}} \mathsf{after}[\![\mathtt{S}]\!] \in \widehat{\mathcal{S}}^{*}[\![\mathtt{S}]\!](\pi\ell)}$$

Structural inference rules

Prefix traces of a program `P ::= Sl` ℓ

- $$\frac{\pi_2 \in \widehat{\mathcal{S}}^*[\![\mathtt{Sl}]\!](\pi_1 \mathsf{at}[\![\mathtt{Sl}]\!])}{\pi_2 \in \widehat{\mathcal{S}}^*[\![\mathtt{P}]\!](\pi_1 \mathsf{at}[\![\mathtt{P}]\!])}$$

Prefix traces of a compound statement `S ::= { Sl }`

- $$\frac{\pi_2 \in \widehat{\mathcal{S}}^*[\![\mathtt{Sl}]\!](\pi_1)}{\pi_2 \in \widehat{\mathcal{S}}^*[\![\mathtt{S}]\!](\pi_1)}$$

Prefix traces of a conditional statement `S ::= if` ℓ `(B)` $\mathtt{S}_t$

- $$\frac{\mathcal{B}[\![\mathtt{B}]\!]\varrho(\pi_1\ell) = \mathrm{ff}}{\ell \xrightarrow{\neg(\mathtt{B})} \mathsf{after}[\![\mathtt{S}]\!] \in \widehat{\mathcal{S}}^*[\![\mathtt{S}]\!](\pi_1\ell)}$$
- $$\frac{\mathcal{B}[\![\mathtt{B}]\!]\varrho(\pi_1\ell) = \mathrm{tt}, \quad \pi_2 \in \widehat{\mathcal{S}}^*[\![\mathtt{S}_t]\!](\pi_1\ell \xrightarrow{\mathtt{B}} \mathsf{at}[\![\mathtt{S}_t]\!])}{\ell \xrightarrow{\mathtt{B}} \mathsf{at}[\![\mathtt{S}_t]\!] \frown \pi_2 \in \widehat{\mathcal{S}}^*[\![\mathtt{S}]\!](\pi_1\ell)}$$

Prefix traces of a conditional statement `S ::= if` ℓ `(B)` $\mathtt{S}_t$ `else` $\mathtt{S}_f$

- $$\frac{\mathcal{B}[\![\mathtt{B}]\!]\varrho(\pi_1\ell) = \mathrm{tt}, \quad \pi_2 \in \widehat{\mathcal{S}}^*[\![\mathtt{S}_t]\!](\pi_1\ell \xrightarrow{\mathtt{B}} \mathsf{at}[\![\mathtt{S}_t]\!])}{\ell \xrightarrow{\mathtt{B}} \mathsf{at}[\![\mathtt{S}_t]\!] \frown \pi_2 \in \widehat{\mathcal{S}}^*[\![\mathtt{S}]\!](\pi_1\ell)}$$
- $$\frac{\mathcal{B}[\![\mathtt{B}]\!]\varrho(\pi_1\ell) = \mathrm{ff}, \quad \pi_2 \in \widehat{\mathcal{S}}^*[\![\mathtt{S}_f]\!](\pi_1\ell \xrightarrow{\neg(\mathtt{B})} \mathsf{at}[\![\mathtt{S}_f]\!])}{\ell \xrightarrow{\neg(\mathtt{B})} \mathsf{at}[\![\mathtt{S}_f]\!] \frown \pi_2 \in \widehat{\mathcal{S}}^*[\![\mathtt{S}]\!](\pi_1\ell)}$$

Structural inference rules

Prefix traces of a statement list $\texttt{sl} ::= \texttt{sl}'\ \texttt{s}$

- $$\frac{\pi_2 \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{sl}' \rrbracket(\pi_1)}{\pi_2 \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{sl} \rrbracket(\pi_1)}$$
- $$\frac{\pi_2 \in \widehat{\mathcal{S}}^{+}\llbracket \texttt{sl}' \rrbracket(\pi_1), \quad \pi_3 \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{s} \rrbracket(\pi_1 \smallfrown \pi_2)}{\pi_2 \smallfrown \pi_3 \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{sl} \rrbracket(\pi_1)}$$

$$\xrightarrow{\pi_1} \underbrace{\underbrace{\begin{matrix}\mathsf{at}\llbracket \texttt{sl} \rrbracket \\ \mathsf{at}\llbracket \texttt{sl}' \rrbracket\end{matrix} \xrightarrow{\pi_2} \begin{matrix}\mathsf{after}\llbracket \texttt{sl}' \rrbracket \\ \mathsf{at}\llbracket \texttt{s} \rrbracket\end{matrix}}_{\in\ \widehat{\mathcal{S}}^{+}\llbracket \texttt{sl}' \rrbracket(\pi_1 \mathsf{at}\llbracket \texttt{sl}' \rrbracket)} \underbrace{\xrightarrow{\pi_3} \ell}_{\in\ \widehat{\mathcal{S}}^{*}\llbracket \texttt{s} \rrbracket(\pi_1 \mathsf{at}\llbracket \texttt{sl}' \rrbracket \pi_2 \mathsf{at}\llbracket \texttt{s} \rrbracket)}}_{\in\ \widehat{\mathcal{S}}^{*}\llbracket \texttt{sl} \rrbracket(\pi_1 \mathsf{at}\llbracket \texttt{sl} \rrbracket)}$$

Prefix traces of an iteration statement $\texttt{S} ::= \texttt{while}\ \ell\ \texttt{(B)}\ \texttt{S}_b$

- $$\frac{}{\ell \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{S} \rrbracket(\pi_1\ell)}$$
- $$\frac{\ell\pi_2\ell \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{S} \rrbracket(\pi_1\ell), \quad \mathcal{B}\llbracket \texttt{B} \rrbracket\varrho(\pi_1\ell\pi_2\ell) = \mathrm{ff}}{\ell\pi_2\ell \xrightarrow{\neg(\texttt{B})} \mathsf{after}\llbracket \texttt{S} \rrbracket \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{S} \rrbracket(\pi_1\ell)}$$
- $$\frac{\ell\pi_2\ell \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{S} \rrbracket(\pi_1\ell), \quad \mathcal{B}\llbracket \texttt{B} \rrbracket\varrho(\pi_1\ell\pi_2\ell) = \mathrm{tt}, \quad \pi_3 \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{S}_b \rrbracket(\pi_1\ell\pi_2\ell \xrightarrow{\texttt{B}} \mathsf{at}\llbracket \texttt{S}_b \rrbracket)}{\ell\pi_2\ell \xrightarrow{\texttt{B}} \mathsf{at}\llbracket \texttt{S}_b \rrbracket \smallfrown \pi_3 \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{S} \rrbracket(\pi_1\ell)}$$

$$\frac{\ell\pi_2\ell \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{S} \rrbracket(\pi_1\ell), \quad \mathcal{B}\llbracket \texttt{B} \rrbracket\varrho(\pi_1\ell\pi_2\ell) = \mathrm{tt}, \quad \pi_3 \xrightarrow{\texttt{break}} \mathsf{break\text{-}to}\llbracket \texttt{S} \rrbracket \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{S}_b \rrbracket(\pi_1\ell\pi_2\ell \xrightarrow{\texttt{B}} \mathsf{at}\llbracket \texttt{S}_b \rrbracket)}{\ell\pi_2\ell \xrightarrow{\texttt{B}} \mathsf{at}\llbracket \texttt{S}_b \rrbracket \smallfrown \pi_3 \xrightarrow{\texttt{break}} \mathsf{break\text{-}to}\llbracket \texttt{S} \rrbracket \in \widehat{\mathcal{S}}^{*}\llbracket \texttt{S} \rrbracket(\pi_1\ell)}$$

Prefix trace semantics

- The prefix trace semantics is defined structurally:

$$\mathcal{S}^*[\![\mathsf{S}]\!] \triangleq \hat{\mathcal{S}}^*[\![\mathsf{S}]\!]$$

- The prefix traces starting from a set $\mathcal{R}_0$ of initial traces are

$$\mathcal{S}^*[\![\mathsf{S}]\!]\mathcal{R}_0 \triangleq \bigcup\{\mathcal{S}^*[\![\mathsf{S}]\!](\pi^{\ell}) \mid \pi^{\ell} \in \mathcal{R}_0\}$$

- The prefix traces starting from a set $\mathcal{R}_0$ of initial traces and arriving at program label ℓ are:

$$\begin{array}{rcl} \dot{\mathcal{S}}^*[\![\mathsf{s}]\!] & \in & \wp(\mathbb{T}^+) \xrightarrow{\nearrow} (\mathbb{L} \to \wp(\mathbb{T}^+)) \\ \dot{\mathcal{S}}^*[\![\mathsf{s}]\!]\,\mathcal{R}_0\,\ell & \triangleq & \{\pi_0\ell_0\pi_1\ell_1 \mid \pi_0\ell_0 \in \mathcal{R}_0 \wedge \ell_0\pi_1\ell_1 \in \mathcal{S}^*[\![\mathsf{s}]\!](\pi_0\ell_0) \wedge \ell_1 = \ell\} \end{array}$$

Example of prefix trace semantics

- $\texttt{S} = \texttt{while}\ ^{\ell_1}\ (\texttt{tt})\ ^{\ell_2}\ \texttt{x = x + 1 ;}^{\ell_3}$.
- $\widehat{\mathcal{S}}^*[\![\texttt{S}]\!](\ell_1) = \left\{\left(\ell_1 \xrightarrow{\texttt{tt}} \ell_2 \xrightarrow{\texttt{x} = i} \ell_1\right)_{i=1}^{n}, \left(\ell_1 \xrightarrow{\texttt{tt}} \ell_2 \xrightarrow{\texttt{x} = i} \ell_1\right)_{i=1}^{n} \xrightarrow{\texttt{tt}} \ell_2 \;\middle|\; n \in \mathbb{N}\right\}$

  (reduced to ℓ_1 for $n = 0$).

❑Notation:

- $\left(^{\ell}\pi(i)^{\ell}\right)_{i=1}^{n}$ denotes the finite trace $^{\ell}\pi(1)^{\ell}\pi(2)^{\ell}\ldots\pi(n)^{\ell}$. This is the trace $^{\ell}$ for $n = 0$.
- $\left(^{\ell}\pi(i)^{\ell}\right)_{i=1}^{\infty}$ denotes the infinite trace $^{\ell}\pi(1)^{\ell}\pi(2)^{\ell}\ldots\pi(n)^{\ell}\pi(n+1)^{\ell}\ldots$.

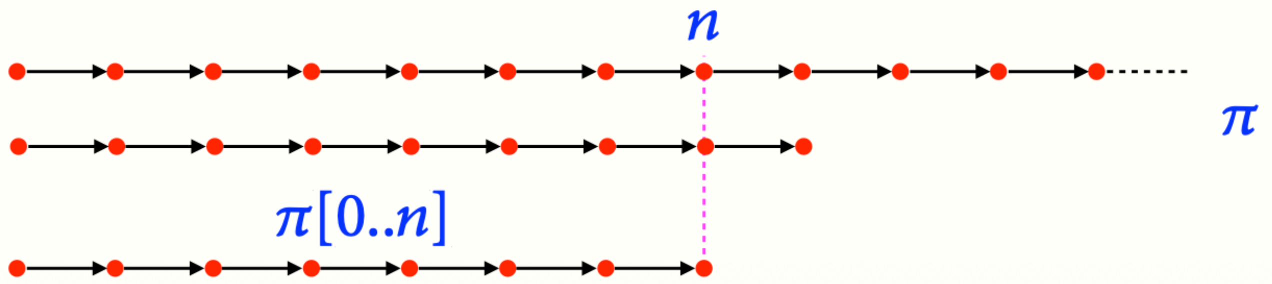
Finite maximal trace semantics

- $\mathcal{S}^{+}[\![\mathrm{S}]\!](\boldsymbol{\pi_1}\mathrm{at}[\![\mathrm{S}]\!])$:
  - The set of maximal finite traces $\mathrm{at}[\![\mathrm{S}]\!]\pi_2\mathrm{after}[\![\mathrm{S}]\!]$ of S continuing the trace $\pi_1\mathrm{at}[\![\mathrm{S}]\!]$ and reaching $\mathrm{after}[\![\mathrm{S}]\!]$
    - $\mathcal{S}^{+}[\![\mathsf{S}]\!](\pi_1\mathsf{at}[\![\mathsf{S}]\!]) \triangleq \{\pi_2\ell \in \mathcal{S}^{*}[\![\mathsf{S}]\!](\pi_1\mathsf{at}[\![\mathsf{S}]\!]) \mid \ell = \mathsf{after}[\![\mathsf{S}]\!]\}$
    - $\mathcal{S}^{+}[\![\mathsf{S}]\!](\pi_1\ell) = \varnothing$ when $\ell \neq \mathsf{at}[\![\mathsf{S}]\!]$
  - Schematically,

$$\xrightarrow{\pi_1} \underbrace{\mathsf{at}[\![\mathsf{S}]\!] \xrightarrow{\pi_2} \mathsf{after}[\![\mathsf{S}]\!]}_{\in\ \mathcal{S}^{+}[\![\mathsf{S}]\!](\pi_1\mathsf{at}[\![\mathsf{S}]\!])}$$

Prefixes of a trace

- If $\pi = \ell_0 \xrightarrow{e_0} \ldots \ell_i \xrightarrow{e_i} \ldots \ell_n$ is a finite trace then its prefix $\pi[0..p]$ at p is
  - π when $p \geqslant n$
  - $\ell_0 \xrightarrow{e_0} \ldots \ell_j \xrightarrow{e_j} \ldots \ell_p$ when $0 \leqslant p \leqslant n$.
- If $\pi = \ell_0 \xrightarrow{e_0} \ldots \ell_i \xrightarrow{e_i} \ldots$ is an infinite trace then its prefix $\pi[0..p]$ at p is $\ell_0 \xrightarrow{e_0} \ldots \ell_j \xrightarrow{e_j} \ldots \ell_p$.

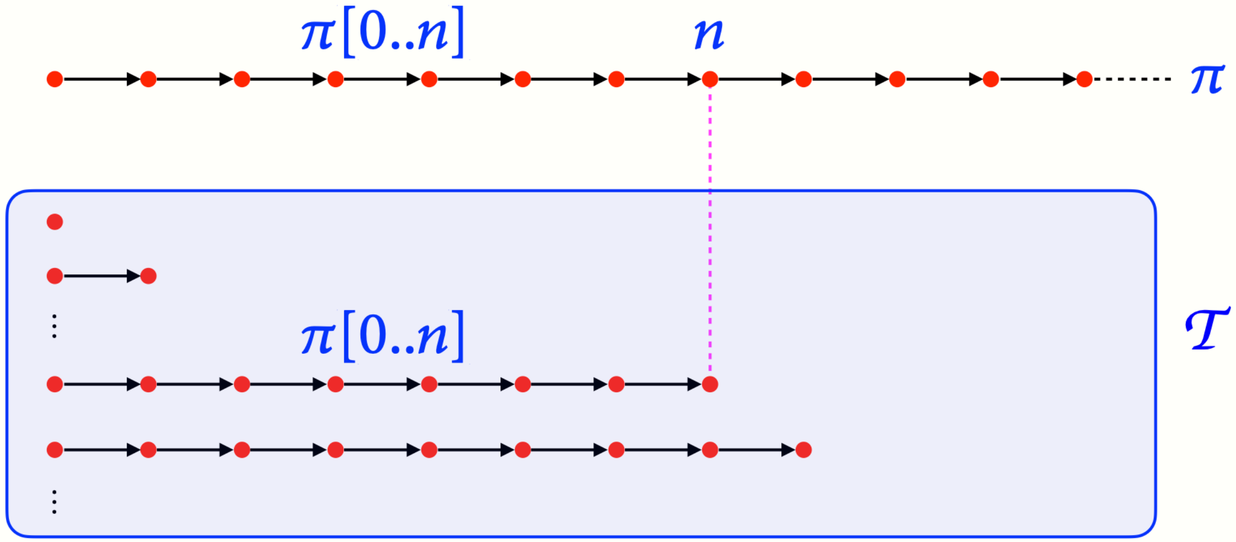


Limit of prefix traces

- **Limit of a set of finite traces:**
  - Is the set of infinite traces which prefixes are traces in this set

$$\lim \mathcal{T} \triangleq \{\pi \in \mathbb{T}^{\infty} \mid \forall n \in \mathbb{N} \,.\, \pi[0..n] \in \mathcal{T}\} \qquad \lim \emptyset = \emptyset$$

  - Requires the set to be prefix closed



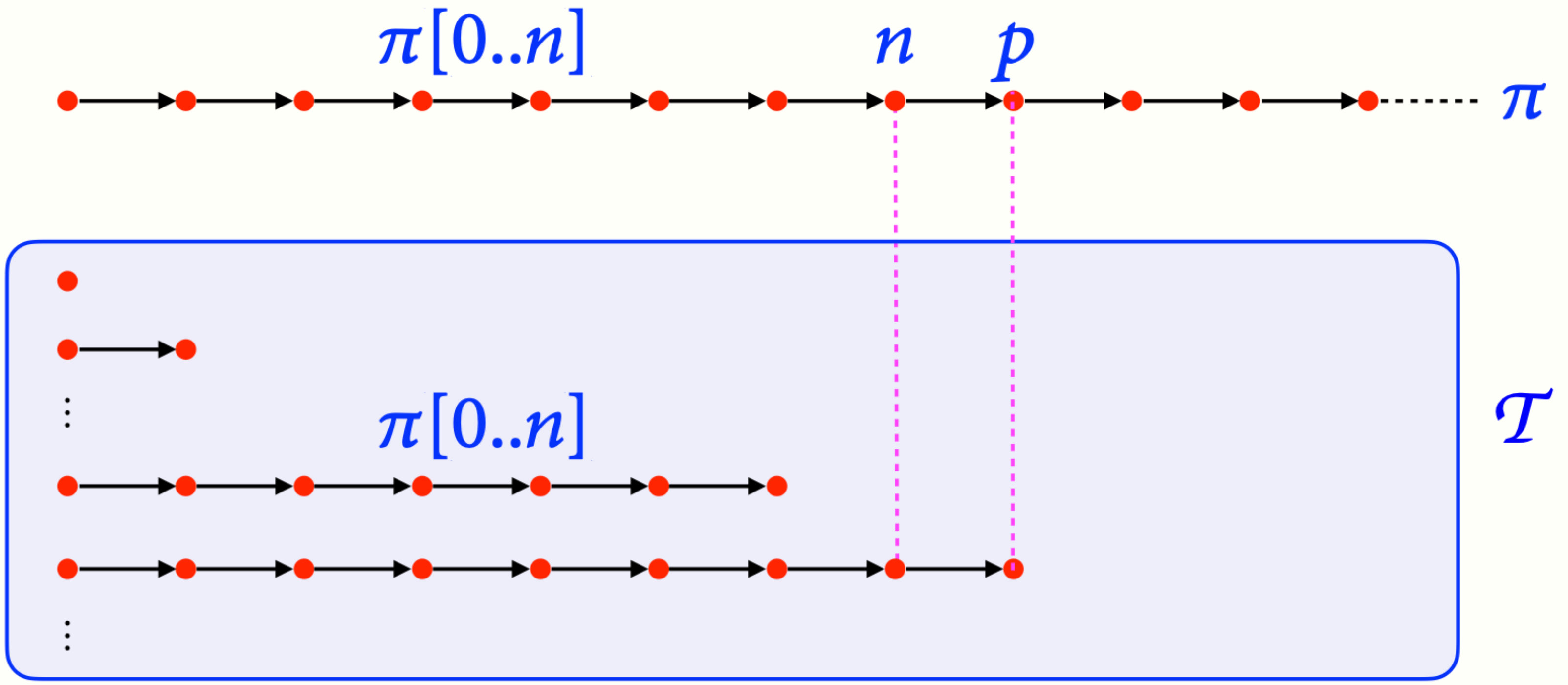
Example I of limit of prefix traces

- The prefix semantics of the program $\mathtt{S} = \mathtt{while}\ \ell_1\ (\mathtt{tt})\ \ell_2\ \mathtt{x = x + 1\ ;}\ \ell_3$ is $\mathcal{S}^*[\![\mathtt{S}]\!](\ell_1) = \left\{\left(\ell_1 \xrightarrow{\mathtt{tt}} \ell_2 \xrightarrow{\mathtt{x} = i} \ell_1\right)_{i=1}^{n}, \left(\ell_1 \xrightarrow{\mathtt{tt}} \ell_2 \xrightarrow{\mathtt{x} = i} \ell_1\right)_{i=1}^{n} \xrightarrow{\mathtt{tt}} \ell_2 \;\middle|\; n \in \mathbb{N}\right\}$.
- Its limit is $\lim(\mathcal{S}^*[\![\mathtt{S}]\!](\ell_1)) = \{\pi\}$ where the infinite trace is $\pi = \left(\ell_1 \xrightarrow{\mathtt{tt}} \ell_2 \xrightarrow{\mathtt{x} = i}\right)_{i=1}^{\infty}$.
- All prefixes of π belong to $\mathcal{S}^*[\![\mathtt{S}]\!](\ell_1)$.

Limit of prefix traces (II)

- ❑Limit **lim** T as the set of infinite traces which prefixes can be extended to a trace in T:

$$\lim \mathcal{T} \triangleq \{\pi \in \mathbb{T}^\infty \mid \forall n \in \mathbb{N} \,.\, \exists p \geqslant n \,.\, \pi[0..p] \in \mathcal{T}\}$$



Maximal trace semantics

- Infinite maximal trace semantics

$$\mathcal{S}^{\infty}[\![\mathsf{S}]\!](\pi^{\ell}) \quad \triangleq \quad \lim(\mathcal{S}^{*}[\![\mathsf{S}]\!](\pi^{\ell}))$$

- Maximal finite trace semantics

$$\mathcal{S}^{+}[\![\mathsf{S}]\!](\pi_1\mathsf{at}[\![\mathsf{S}]\!]) \quad \triangleq \quad \{\pi_2{}^{\ell} \in \mathcal{S}^{*}[\![\mathsf{S}]\!](\pi_1\mathsf{at}[\![\mathsf{S}]\!]) \mid \ell = \mathsf{after}[\![\mathsf{S}]\!]\}$$

- Maximal infinite trace semantics

$$\begin{aligned}
\mathcal{S}^{+\infty}[\![\mathsf{S}]\!](\pi^{\ell}) &\quad \triangleq \quad \mathcal{S}^{+}[\![\mathsf{S}]\!](\pi^{\ell}) \cup \mathcal{S}^{\infty}[\![\mathsf{S}]\!](\pi^{\ell}) \\
\mathcal{S}^{+\infty}[\![\mathsf{S}]\!]\Pi &\quad \triangleq \quad \bigcup\{\mathcal{S}^{+\infty}[\![\mathsf{S}]\!](\pi^{\ell}) \mid \pi^{\ell} \in \Pi\} \\
\mathcal{S}^{+\infty}[\![\mathsf{S}]\!] &\quad \triangleq \quad \mathcal{S}^{+\infty}[\![\mathsf{S}]\!](\mathbb{T}^{+}) \\
\mathcal{S}^{+\infty}[\![\mathsf{P}]\!] &\quad \triangleq \quad \mathcal{S}^{+\infty}[\![\mathsf{P}]\!](\{\mathsf{at}[\![\mathsf{P}]\!]\}).
\end{aligned}$$

Example II of limit of prefix traces

- $\lim\left\{\left(\ell_1 \xrightarrow{\texttt{tt}} \ell_2 \xrightarrow{\mathtt{x} = i} \ell_1\right)_{i=1}^{n} \;\middle|\; n \in \mathbb{N}\right\} = \{\pi\}$ where $\pi = \pi = \left(\ell_1 \xrightarrow{\texttt{tt}} \ell_2 \xrightarrow{\mathtt{x} = i}\right)_{i=1}^{\infty}$.
- All prefixes of π are of the form $\left(\ell_1 \xrightarrow{\texttt{tt}} \ell_2 \xrightarrow{\mathtt{x} = i} \ell_1\right)_{i=1}^{n}$ or $\left(\ell_1 \xrightarrow{\texttt{tt}} \ell_2 \xrightarrow{\mathtt{x} = i} \ell_1\right)_{i=1}^{n} \xrightarrow{\texttt{tt}} \ell_2$ and this last one can be extended to a finite trace $\left(\ell_1 \xrightarrow{\texttt{tt}} \ell_2 \xrightarrow{\mathtt{x} = i} \ell_1\right)_{i=1}^{n+1}$.
- The maximal trace semantics of the program $\mathtt{S} = \mathtt{while}\,\ell_1\ (\texttt{tt})\ \ell_2\ \mathtt{x = x + 1}\ ;\ell_3$ is $\mathcal{S}^{+\infty}[\![\mathtt{S}]\!](\ell_1) = \left\{\left(\ell_1 \xrightarrow{\texttt{tt}} \ell_2 \xrightarrow{\mathtt{x} = i}\right)_{i=1}^{\infty}\right\}$.

(Optional) 作业：

- ❑阅读