PROJECT # 3

[100/100] The Euler equations governing unsteady compressible inviscid flows can be expressed in conservative form as:

$$\frac{\partial Q}{\partial t} + \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 0 \tag{1}$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ E \end{bmatrix} \quad F_1 = \begin{bmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1 u_2 \\ u_1(E+p) \end{bmatrix} \quad F_2 = \begin{bmatrix} \rho u_2 \\ \rho u_1 u_2 \\ \rho u_2^2 + p \\ u_2(E+p) \end{bmatrix} \tag{2}$$

and the internal energy of the gas

$$E = \rho e_{int} + \frac{1}{2}\rho(u_1^2 + u_2^2) \tag{3}$$

The Euler equations are not complete without an equation of state. We choose an ideal gas for which

$$e_{int} = \frac{p}{\rho(\gamma - 1)} \tag{4}$$

The moving shock enters from the left inlet at $M_s = 2$ and free stream conditions are $\rho = 1$, $u = 0$, $v = 0$ and $p = 1/\gamma$. Use the first order Euler explicit with AUSM$^+$-up scheme [1], to solve the problem given in Figure 1 at $t = 0.3$. The state variables Q are defined at the cell centers.

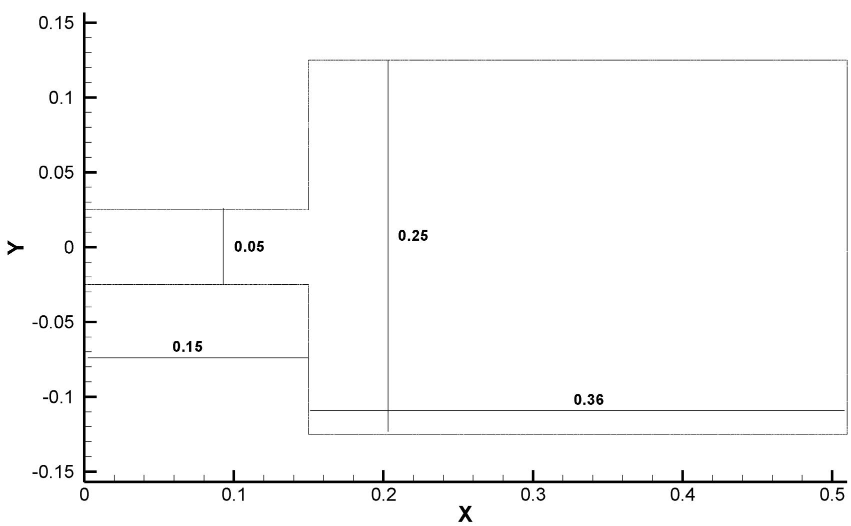

Figure 1: The 1:5 expansion problem with moving shock at $M_s = 2$ entering at inlet.

[0]ISTANBUL TECHNICAL UNIVERSITY, UUM601E, Return Date: 23 January 2025

The AUSM$^+$-up Flux

The AUSM$^+$-up for all speeds

$$M_{L/R} = \frac{\mathbf{n}\cdot\mathbf{u}_{L/R}}{a_{1/2}} \tag{5}$$

where $a_{1/2} = (a_L + a_R)/2$

$$\bar{M}^2 = \frac{|\mathbf{u}_L|^2 + |\mathbf{u}_R|^2}{2a_{1/2}^2} \tag{6}$$

$$M_0^2 = min(1, max(\bar{M}^2, M_\infty^2)) \tag{7}$$

$$f_a(M_0) = M_0(2 - M_0) \tag{8}$$

$$\rho_{1/2} = \frac{\rho_L + \rho_R}{2} \tag{9}$$

$$M_{1/2} = M^+_{(4)}(M_L) + M^-_{(4)}(M_R) - \frac{K_p}{f_a} max(1 - \sigma\bar{M}^2, 0)\frac{p_R - p_L}{\rho_{1/2}a_{1/2}^2} \tag{10}$$

where

$$M^\pm_{(4)}(M) = \begin{cases} \frac{1}{2}(M \pm |M|) & \text{if} \quad |M| \geq 1 \\ \pm\frac{1}{4}(M \pm 1)^2(1 + 16\beta\frac{1}{4}(M \mp 1)^2) & \text{otherwise} \end{cases} \tag{11}$$

and $\rho_{1/2} = (\rho_L + \rho_R)/2$, $K_p = 0.25$ and $\sigma = 1$. Then the mass flux

$$\dot{m}_{1/2} = \begin{cases} a_{1/2}M_{1/2}\rho_L & \text{if} \quad M_{1/2} > 0 \\ a_{1/2}M_{1/2}\rho_R & \text{otherwise} \end{cases} \tag{12}$$

the pressure flux

$$p_{1/2} = P^+_{(5)}(M_L)p_L + P^-_{(5)}(M_R)p_R - K_u P^+_{(5)}(M_L)P^-_{(5)}(M_R)(\rho_L + \rho_R)(f_a a_{1/2}^2)(M_R - M_L) \tag{13}$$

and

$$P^\pm_{(5)}(M) = \begin{cases} \frac{1}{2M}(M \pm |M|) & \text{if} \quad |M| \geq 1 \\ \pm\frac{1}{4}(M \pm 1)^2\left[(\pm 2 - M) + 16\alpha M\frac{1}{4}(M \mp 1)^2\right] & \text{otherwise} \end{cases} \tag{14}$$

using the parameters

$$\alpha = \frac{3}{16}(-4 + 5f_a^2) \qquad \beta = \frac{1}{8} \qquad K_u = 0.75 \tag{15}$$

The whole flux

$$\mathbf{n}\cdot\mathbf{F} = \dot{m}_{1/2}\begin{bmatrix} 1 \\ u_1 \\ u_2 \\ H \end{bmatrix}_{\substack{L \quad \text{if} \quad M_{1/2} > 0 \\ R \quad \text{otherwise}}} + \begin{bmatrix} 0 \\ p_{1/2}n_1 \\ p_{1/2}n_2 \\ 0 \end{bmatrix} \tag{16}$$

where the local enthalpy is given by

$$H = \frac{E + p}{\rho} \tag{17}$$

[130/100] This more advanced option in three dimension and it is highly recommended to use this approach to solve the above problem. In this approach use three-dimensional OpenFOAM mesh files to solve the above problem. For this part of the project you may use the *VTK_mesh.F* FORTRAN source code to read OpenFOAM mesh files and plot the results in TECPLOT polyhedral format. The OPENFOAM mesh includes of the following files:

1. "./constant/polyMesh/points" file includes the coordinates of vertices (x_i, y_i, z_i).
2. "./constant/polyMesh/faces" file includes the total vertex number for each face and the corresponding global vertex numbers.
3. "./constant/polyMesh/owner" file incudes the owner element number for each face (left element number).
4. "./constant/polyMesh/neighbour" file incudes the neighbour element number for each face (right element number).
5. "./constant/polyMesh/boundary" file includes the boundary condition for the exterior faces which includes the number of faces for that boundary condition and the first global face number.

For cell centered finite volume formulation you should evaluate the inviscid fluxes for each faces and add their contribution to correct left and right elements.

References

[1] Meng-Sing Liou, A sequel to AUSM, Part II: AUSM$^+$-up for all speeds. *J. Comput. Phys.* **214**, (2006), 137–170.

Read Mesh Vetices

```
for i ← 1 to np do
  | read x[i],y[i],z[i]
end
```

Read Element Connectivity

```
for i ← 1 to ne do
  | read nec[i,1],nec[i,2],nec[i,3],nec[i,4]
end
```

Read Neighbouring Element Numbers

```
for i ← 1 to ne do
  | read fec[i,1],fec[i,2],fec[i,3],fec[i,4]
end
```

Initial Q Value

$Q^1 := Q_\infty$

Time integration

```
for Time ← 0 to 4 do
  |
  | Compute Q^{n+1}
  | for i ← 1 to ne do
  |   RHS := 0
  |   for n ← 1 to 4 do
  |     ....
  |     ....
  |     ....
  |     if Solid Wall Boundary Condition then
  |       ...
  |       RHS := RHS + ...
  |     else if Inflow Boundary Condition then
  |       ...
  |       RHS := RHS + ...
  |     else if Outflow Boundary Condition then
  |       ....
  |       RHS := RHS + ...
  |     else
  |       Q_L = ...
  |       Q_R = ...
  |       n_x = ...
  |       n_x = ...
  |       call AUSM_Flux(Q_L, Q_R, n_x, n_y, n · F)
  |       RHS := RHS + n · F...
  |     end
  |     ΔQ :=
  |   end
  |   Q^{n+1} := Q^n + ΔQ
  |
  | end
end
```

Table 1: The structure of an unstructured FVM code.