Rutgers University
CS206: Introduction to Discrete Structures II, Spring 2013
Professor David Cash

Homework 6

Due at the beginning of class on Wednesday, April 3

Instructions: Point values for each problem are listed. Write your solutions neatly or type them up. Typed solutions will also be accepted via Sakai.

1. (5 points) Let X be a random variable on some sample space S with expected value μ and variance σ^2. Let X_1 and X_2 be *identical independent copies* of X. Find $E((X_1 - X_2)^2)$ in terms of μ and σ^2. Give some intuition for why your answer makes sense.

2. (3 points) Prove that $V(cX) = c^2 V(X)$ for any random variable X and real number c.

3. (5 points) Let X be the random variable that is the sum of two independent fair die rolls. Let Y be the outcome of the first roll minus the outcome of the second roll. Calculate $\mathrm{Cov}(X, Y)$. Are X and Y correlated?

4. (4 points) Let C be the random variable that is the number of heads in 100 independent fair coin flips. What are $E(C)$ and $V(C)$? Find upper bounds on $P(C > 75)$, using Markov's Inequality and Tchebychev's Inequality.

5. (4 points) Find an example where Markov's inequality is tight in the follow sense: For each positive integer a, find a non-negative random variable X such that $P(X \geq a) = E(X)/a$.

6. (4 points) Prove the following: If X is a non-negative random variable with $E(X) = \mu$, then for every k, $P(X \geq k\mu) \leq 1/k$. Hint: Apply Markov's inequality.

7. **Extra credit (4 points):** If X is a non-negative random variable with $E(X) = \mu$, then Markov's inequality tells us that for every a, $P(X \geq a) \leq \mu/a$. As we saw in class, this bound is sometimes very loose. In this problem we'll look at a situation in which extra information can be used to tighten the bound.

   Suppose that we are told X is *bounded from below by some number b*, meaning $P(X \geq b) = 1$. Use this information to find a tighter bound on $P(X \geq a)$. Hint: Apply Markov's inequality to the r.v. $Y = X - b$.

   For concreteness, suppose $E(X) = 1000$ and $P(X \geq 500) = 1$. Markov's says that $P(X \geq 2000) \leq 1000/2000 = 1/2$. Use your method find a tighter bound on $P(X \geq 2000)$.