Paul Jones
Rutgers University
CS206: Introduction to Discrete Structures II, Spring 2013
Professor David Cash

Homework 4

1. (10 points total) Two teams x and y are playing each other in the World Series, which is a best-of-seven-game match that ends when one team wins 4 games. Assume that team x wins each game with probability p, and that the outcome of each game constitutes an independent trial.

   (a) (0.5 points) What is the probability that x wins the first four games?

   - There is one way for x to win in four, and that's winning four in a row.
   - By the multiplicity principle, we can multiply the chance of winning each game to get the probability of winning all games.

$$p^4$$

   (b) (2 points) What is the probability that x wins four games after at most five game have been played?

   - Sample space, strings, x character means x won, y character means y won.

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xxxx
yxxxx
xyxxx
xxyxx
xxxyx
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   - The team represented by y cannot win at the end because after 4 x wins, the games halt.
   - P(x wins in less than five games) $= P(\{\text{xxxx}, \text{yxxxx}, \text{xyxxx}, \text{xxyxx}, \text{xxxyx}\})$
   - There is a $(1 - p)$ chance of y winning.
   - The x team still needs to win 4 games, which has a probability of p^4.
   - Multiplying these two values gets you the likelihood that this happens for any given instance of x winning in less than five games.
   - To get all of the possible less than 5 games combinations, multiply by the number there are, which is 4.
   - Then, add the probability of just 4 straight victories.

$$(1 - p) \times p^4 \times 4 + p^4$$

   (c) (2 points) What is the probability that x will win four games before y wins four games? (i.e., What is the probability that x wins the Series?)

- To begin, lets enumerate some possibilities using strings.

$$\{\text{xxxx}\}$$

$$\{\text{yxxxx, xyxxx, xxyxx, xxxyx}\}$$

$$\{\text{yyxxxx, yxyxxx, ..., xxxyyx}\}$$

$$\{\text{yyyxxxx, yyxyxxx, ..., xxxyyyx}\}$$

- These are all the possibilities for x winning. Name these X_1 through X_4 and notice the sum of their probabilities to be the probability that x wins four games.
- For X_3, there are $\binom{5}{2,3}$ ways of ordering x and y, y has to win twice, and x still has to win four times.
- You have to subtract off the cases where there is a y at the end,

$$\binom{4}{1,3} \times (1-y)^2 \times p^4$$

- For X_4, there are $\binom{6}{3,3}$ ways of ordering x and y, y has to win thrice, and x still has to win four times.

$$\binom{5}{2,3} \times (1-y)^3 \times p^4$$

- Now add 4 straight victories:

$$\binom{4}{1,3} \times (1-y)^2 \times p^4 + \binom{5}{2,3} \times (1-y)^3 \times p^4 + p^4$$

(d) (0.5 points) Calculate and simplify your answer in part (c) when $p = 1/2$ and when $p = 2/3$.

- $p = 1/2 : 1/2$
- $p = 2/3 : 1808/2187$

(e) (1 point) Let X be the random variable that counts the number of games that are played. What is Range(X)?

$$R(X) = \{4, 5, 6, 7\}$$

(f) (2 points) What is $P(X = 7)$?

$$\binom{6}{3}(1-p)^3p^4 + \binom{6}{3}p^3(1-p)^3p^4$$

(g) (2 points) What is $P(X \geq 6)$?

$$\binom{5}{2}(1-p)^2p^4 + \binom{5}{2}p^2(1-p)^4 + \binom{6}{3}(1-p)^3p^4 + \binom{6}{3}p^3(1-p)^4$$

2. (4 points) Suppose we roll two fair dice. Let the random variable $X =$ "the minimum of the two dice" and $Y =$ "the absolute value of the difference of the two dice". Find $E(X)$ and $E(Y)$.

   - $E(X)$
     - The formula to solve this is

$$E(X) = \sum_{a_i \in R(X)} a_i P(X = a_1)$$

     - The sample space has a cardinality of 36, as there 6 choices for each of the two choices.
     - All rolls are equally likely in a fair dice.
     - Starting with the highest element in the range (which is the set containing 1 through 6), there is only one way 6 can be the minimum.

$$P(x = 6) = P(\{(6,6)\}) = \frac{1}{36}$$

     - There are 3 ways of "getting 5", and that's rolling two fives, and then both variations of a five and a six.

$$P(x = 5) = P(\{(5,5),(5,6),(6,5)\}) = \frac{3}{36}$$

     - Apply the same pattern,

$$P(x = 4) = P(\{(4,4),(4,5),(5,4),(4,6),(6,4)\}) = \frac{5}{36}$$

$$P(x = 3) = P(\{(3,3),(3,4),(4,3),(3,5),(5,3),(3,6),(6,3)\}) = \frac{7}{36}$$

$$P(x = 2) = P(\{(2,2),(2,3),(3,2),(2,4),(4,2),(2,5),(5,2),(2,6),(6,2)) = \frac{9}{36}$$

     - Notice that there 2 more each time.

$$P(x = 1) = \frac{11}{36}$$

     - Now sum them and multiply them by their value (a_i) to find expected value,

$$E(X) = \left(6 \times \frac{1}{36}\right) + \left(5 \times \frac{3}{36}\right) + \left(4 \times \frac{5}{36}\right) + \left(3 \times \frac{7}{36}\right) + \left(2 \times \frac{9}{36}\right) + \frac{11}{36} = 2.5277777778$$

   - $E(Y)$
     - The range is $\{0,1,2,3,4,5\}$.
     - This is another application of the formula

$$E(X) = \sum_{a_i \in R(X)} a_i P(X = a_1)$$

- There are 6 ways of getting 0, $\{6-6, 5-5, ...\}$. For $i = 0$,

$$0 \times P(X = 0) = 0 \times \frac{6}{36} = 0$$

- There are 10 ways of getting 1, $\{6{-}5, 5{-}6, 5{-}4, 4{-}5, 4{-}3, 3{-}4, 3{-}2, 2{-}3, 2{-}1, 1{-}2\}$. For $i = 1$,

$$1 \times P(X = 1) = 1 \times \frac{10}{36}$$

- There are 8 ways of getting 2, $\{6-4, 4-6, 5-3, 3-5, 4-2, 2-4, 1-5, 5-1\}$. For $i = 2$,

$$2 \times P(X = 2) = 2 \times \frac{8}{36} = \frac{4}{9}$$

- There are 6 ways of getting 3, $\{6-3, 3-6, 5-2, 2-5, 4-1, 1-4\}$. For $i = 3$,

$$3 \times P(X = 3) = 3 \times \frac{6}{36} = \frac{2}{3}$$

- There are 4 ways of getting 4, $\{6-2, 2-6, 5-1, 1-5\}$. For $i = 4$,

$$4 \times P(X = 4) = 4 \times \frac{4}{36} = \frac{4}{9}$$

- There are 2 ways of getting 5, $\{6-1, 1-6\}$. For $i = 5$,

$$5 \times P(X = 5) = 5 \times \frac{2}{36} = \frac{5}{18}$$

- We know we've covered the sample space because the sum of the specific instances is the same as the cardinality of the sample space.
- The expected value,

$$E(X) = 0 + \frac{2}{9} + \frac{4}{9} + \frac{2}{3} + \frac{4}{9} + \frac{5}{18} = 1.94 \approx 2$$

3. (4 points) Suppose boxes of cereal are filled with a random prize, each drawn from independently and uniformly from 6 possible prizes. If we buy N boxes of cereal, what is the expected number of distinct prizes we will collect? Hint: Consider the indicator random variables I_{E_i} for the event $E_i =$ "the i-th price was in some box".

- Let $X_1 ... X_6$ be the identifier for each of 6 unique toys

$$E\left(\sum I_{E_i}\right) = \sum_{i=1}^{6} E(I_{E_i}) = 6(1 - (5/6))^n$$

4. (4 points) A group of m men and w randomly sit in a single row at a theater. If a man and woman are seated next to each other we say they form a couple. (Couples can overlap, meaning that one person can be a member of two couples.) What is the expected number of couples? Hint: Use indicator random variables for each possible couple forming.

- Use linearity of expectation for each pair of seats.
- Let x equal the number of seats.
- Then, $E(X_1) = P(\text{couple in a seat 1 and 2})$, $E(X_2) = P(\text{couple in a seat 2 and 3})$, etc.
- For any given seat, there is a $\frac{1}{2}$ chance of their being a man or a woman in the seat.
- The four possibilities are {mw, wm, mm, ww}.

$$\sum_{i}^{x}$$

5. (3 points) Suppose an experiment tosses a fair coin twice; the experiment "succeeds" if both tosses were Heads. We repeat this experiment for 12 independent trials. Let N be the random variable that counts the fraction of trials that are successful (so $N = S/12$, where S is the number of successful trials). Find $E(N)$.

$$E(N) = E\left(\frac{S}{12}\right)$$

$$S = \{x_1 + ... + x_1 2\}$$

$$\frac{1}{12}E(S)$$

$$E(S) = \sum_{a_i \in R(S)} a_i \times P(X_i)$$

$$P(X_i) = \frac{1}{4}$$

$$E(S) = 12 \times E(X_i)$$

$$\frac{1}{12}E(S) = 12 \times \frac{1}{4}$$

$$E(S) = \frac{1}{4}$$

6. **Extra Credit: (4 points)** Consider the experiment where n balls are to be placed randomly into n boxes. Let N_1 count the number of boxes with exactly one ball, and let N_2 count the number of boxes with exactly two balls. Find the probability of the events "$N_1 = n$" and "$N_1 = n - 1$". Use the indicator technique to find $E(N_1)$ and $E(N_2)$.