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Introduction to Discrete Structures II (01:198:206)

Homework 1

1. Let $n \geq 2$ and $A_1, \ldots, A_n$ be sets in some universe S. In this problem we will give a proof by induction of the identity

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c.$$

   (a) (5 points) State and prove the base case for an inductive proof, meaning that the identity is true when $n = 2$.
      i. $\left(\bigcap_{i=1}^{2} A_i\right)^c = \bigcup_{i=1}^{2} A_i^c$
      ii. $(A_1 \bigcap A_2)^c = A_1^c \bigcup A_2^c$ is true by DeMorgan's

   (b) (5 points) State and prove the inductive step, where one shows that the identity is true for general $n > 2$, assuming it is true for $n - 1$.
      i. $(\bigcap_{i=1}^{n} A_i)^c = \bigcup_{i=1}^{n} A_i^c$ assume true for n
      ii. $\left(\bigcap_{i=1}^{n-1} A_i\right)^c = \bigcup_{i=1}^{n-1} A_i^c$ induction for n - 1
      iii. $\bigcup_{i=1}^{n} A_i^c \cup A_{n+1}^c = (\bigcap_{i=1}^{n} A_i) \cup A_{n+1}^c$
      iv. $(\bigcap_{i=1}^{n} A_i)^c = (\bigcap_{i=1}^{n} A_i) \cup A_{n+1}^c$

2. Give sample spaces that model the outcomes for the following experiments. You may use a regular expression or other formalisms that you find convenient. (2 points each)

   (a) Rolling 3 dice:
   $S = \{(i, j, k) \mid i, j, k \in \{1, 2, 3, 4, 5, 6\}\}$

   (b) Rolling a die until an even result comes up, or the die is rolled three times:
   $S = \{(i), (j, i), (j, j, \{i, j\}) \mid i \in \{1, 2, 4\}, j \in \{1, 3, 5\}\}$

   (c) Tossing a pair of coins until they both come up tails.
   $S = \{(e_1), (e_2, e_1), (e_2, e_2, e_1), (e_2, e_2, e_2, ..., e_1) \mid e_1 = TT, e_2 \in \{HT, TH, HH\}\}$

   (d) Draw 2 balls from an urn which contains 6 balls, each with a distinct label from $\{1, 2, 3, 4, 5, 6\}$.
   $S = \{i, j \mid i, j \in \{1, 2, 3, 4, 5, 6\} \wedge i \neq j\}$

   (e) Draw 1 ball from the same urn, then replace it and draw a ball again.
   $S = \{(i, j) \mid i, j \in \{1, 2, 3, 4, 5, 6\}\}$

3. For each of the sample space, describe the events (as sets) $A \cup B$ and $A \cap B$, when A and B are as follows. (2 points each)

   (a) $A =$ "5 is rolled exactly twice" and $B =$ "dice values add to an odd number".
      i. $A \cup B = \{\{i, i, i\}, \{j, j, i\} \mid i \in \{1, 3, 5\}, j \in \{2, 4, 6\}\}$

ii. $A \cap B = \{\{5, 5, i\} \mid i \in \{1, 3\}\}$

(b) $A =$ "1 comes up exactly twice" and $B =$ "3 comes up exactly twice".

i. $A \cup B = \{(1, 1, i), (j, 1, 1), (1, j, 1) \mid i \in \{2, 3, 4, 5, 6\} \wedge j \in \{3, 5\}\} \cup \{(3, 3, i), (j, 3, 3), (3, j, 3) \mid i \in \{1, 2, 4, 5, 6\} \wedge j \in \{1, 5\}\}$

ii. $A \cap B = \emptyset$

(c) $A =$ "both coins come up heads at the same time at some point" and $B =$ "both coins come up tails at the same time at some point"

i. $A \cup B = S = \{(e_1), (e_2, e_1), (e_2, e_2, e_1), (e_2, e_2, e_2, ..., e_1) \mid e_1 = TT, e_2 \in \{HT, TH, HH\}\}$

ii. $A \cap B = A = \{(e_1, e_2, e_3, ..., e_n) \mid e_n = TT \wedge e_i \in \{HT, TH, HH\} \wedge 1 \leq i \leq n-1 \wedge n \in N \wedge \exists j e_j = HH \wedge 1 \leq j \leq n-1\}$

(d) $A =$ "1 is drawn at least once" and $B =$ "1 is drawn twice".

i. $A \cup B = A = \{\{i, j\} \mid i, j \in \{1, 2, 3, 4, 5, 6\} \wedge (i \neq j) \wedge (i = 1 \oplus j = 1)\}$

ii. $A \cap B = \emptyset$

(e) $A =$ "1 is drawn at least once" and $B =$ "1 is drawn twice".

i. $A \cup B = \{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}\}$

ii. $A \cap B = \{\{1, 1\}\}$