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CS206: Introduction to Discrete Structures II, Spring 2013
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Homework 4

1. (3 points) In the card game bridge we deal 13-card hands to 4 players named North, South, East, and West (all 52 cards are dealt). What is the probability that East and West have no spades?

   - The total number of possibilities in the sample space for this situation is the total number of possible card orders over the number of cards each player is dealt factorial raised to the fourth.

   $$\frac{52!}{13!^4}$$

   - There are 13 Spades in a deck of cards, and then 39 non-Spade cards.
   - For the hands of East and West, who collectively represent 26 cards, and who are vying for 39 non-Spade cards, this is the number of possibilities:

   $$\binom{39}{26}$$

   - North and South also account for 26 total cards, and they are vying for the remaining cards including Spades. Twenty-six cards were taken out of the deck by East and West.

   $$\binom{26}{26} = 1$$

   - The probability is therefore:

   $$\frac{\binom{39}{13} \times \binom{26}{13} \times \binom{26}{13} \times \binom{13}{13}}{\frac{52!}{13!^4}}$$

2. (4 points) An urn contains $n > 0$ white balls and $m > 0$ black balls. Suppose we draw two balls without replacement. What is the probability that the balls are of the same color? What if we draw them with replacement? Show your work. Which of these probabilities is larger? Briefly explain some intuition for why one should be larger.

   - Without replacement
     - There are $n+m$ balls in all, and we're interested in choosing 2 of them. The sample space is:

     $$\binom{m+n}{2}$$

     - Call W "drawing two white balls" and B "drawing two black balls. Then $W \cup B$ is either of those events happening or both.

- The events are mutually exclusive, the drawings cannot all be the same for the same drawing.

$$P(W \cup B) = P(W) + P(B)$$
$$= \frac{\binom{m}{2} + \binom{n}{2}}{\binom{m+n}{2}}$$

- With replacement
  - The events are, again, mutually exclusive.
  - Being as the balls are placed back in the urn, each event is equally likely to occur.
  - Count the number of either color, place it over total balls, and multiply it by itself to represent that the event has to happen twice.
  - Do the same for the other color, add the two values.

$$\left(\frac{m}{m+n}\right)^2 + \left(\frac{n}{m+n}\right)^2$$

3. (4 points) Again consider an urn with $n > 0$ white balls and $m > 0$ black balls. Suppose we draw $r \geq 1$ balls from the urn without replacement. What is the probability that we draw exactly k white balls?

- The probability of getting a white, where i is the number of total balls already drawn, is

$$\left(\frac{n-i}{m+n-i}\right)$$

- And the probability of getting a black under the same conditions is

$$\left(\frac{m-i}{m+n-i}\right)$$

- So draw k white balls, and then transfer the counter to drawing the remaining balls to avoid over-counting

$$\left(\prod_{i=0}^{k-1} \frac{n-i}{m+n-i}\right) \times \left(\prod_{i=k}^{r-1} \frac{m-(i-k)}{m+n-i}\right)$$

4. (4 points) A box contains a mixture of cubes and spheres and any of these objects can be either white or black. Suppose the box contains 4 black cubes, 6 black spheres, 6 white cubes, and x white spheres. Consider the experiment of drawing a random object from the box, and let A be the event that a cube is drawn and B be the event that a black object is drawn. If A and B are independent, what is x?

- The probability of a cube being drawn is the total number of cubes over the total number of cubes and spheres.

$$\frac{4+6}{4+6+6+x}$$

- The probability of a black object being drawn is similarly the total number of black objects divided by the total number of objects

$$\frac{6+4}{4+6+6+x}$$

- Consider the complement of $A \cup B$, it is the event that a non-black and non-cube item is draw, which is a white sphere.

$$P(x) = P((A \cup B)^c)$$

5. (10 points total) Two fair dice are rolled. Define the random variables $X =$ the sum of the two rolls, $Y =$ the maximum of the two rolls, $Z =$ the absolute value of the difference of the two rolls and $W = XY$ (i.e., the product of X and Y).

   (a) (2 points) What are Range(X), Range(Y), Range(Z) and Range(W)?

   - Range(X) $= \{2, 3, 4, ..., 11, 12\}$
   - Range(Y) $= \{1, 2, 3, 4, 5, 6\}$
   - Range(Z) $= \{0, 1, 2, 3, 4, 5\}$
   - Range(W) $= \{i \times j \ : \ i \in \{2, 3, 4, ..., 11, 12\}, j \in \{1, 2, 3, 4, 5, 6\}\}$

   (b) (2 points) What are the partitions $\mathcal{A}_X$ and $\mathcal{A}_Z$?

   - $\mathcal{A}_X$
     - $2 = 1 + 1$
     - $3 = 1 + 2$
     - $4 = 1 + 3 = 2 + 2$
     - $5 = 1 + 4 = 3 + 2$
     - $6 = 1 + 5 = 2 + 4 = 3 + 3$
     - $7 = 1 + 6 = 2 + 5 = 3 + 4$
     - $8 = 2 + 6 = 3 + 5 = 4 + 4$
     - $9 = 3 + 6 = 4 + 5$
     - $10 = 4 + 6 = 5 + 5$
     - $11 = 5 + 6$
     - $12 = 6 + 6$
   - $\mathcal{A}_Z$
     - $0 = 1 - 1 = 2 - 2 = 3 - 3 = 4 - 4 = 5 - 5 = 6 - 6$
     - $1 = 6 - 5 = 5 - 4 = 4 - 3 = 3 - 2 = 3 - 2 = 2 - 1$
     - $2 = 5 - 4 = 5 - 3 = 4 - 2 = 3 - 1$
     - $3 = 6 - 3 = 5 - 2 = 4 - 1$
     - $4 = 6 - 2 = 5 - 1$
     - $5 = 6 - 1$

   (c) (3 points) Give tables showing the values of f_X, f_Y, f_Z, and f_W.

   (d) (3 points) Are the events $X = 7$ and $Z = 1$ independent?

   - No. It doesn't meet the definition.

Table 1: f_X

x	2	3	4	5	6	7	8	9	10	11	12
f(x)	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{3}{11}$	$\frac{3}{11}$	$\frac{3}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{1}{11}$

Table 2: f_Y

x	1	2	3	4	5	6
f(y)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$

Table 3: f_Z

x	0	1	2	3	4	6
f(z)	$\frac{6}{12}$	$\frac{5}{12}$	$\frac{4}{11}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{1}{12}$