Rutgers University
CS206: Introduction to Discrete Structures II, Spring 2013
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Homework 2 - Solutions

1. (8 points) Prove that $P(A \cup B) \le P(A) + P(B)$ for any events A and B. Prove the general version by induction, which says that if $A_1, \ldots, A_n$ are events then $P(\bigcup_{i=1}^n A_i) \le \sum_{i=1}^n P(A_i)$. When does this inequality become an equality?

   Solution: For any events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, meaning that $P(A \cup B) \le P(A) + P(B)$ because $P(A \cap B)$ is non-negative.

   Next we prove the general version. The above serves as the base-case for induction (for $n = 2$, we could also verify that it works for $n = 1$). For the inductive step, let $A_1, \ldots, A_n, A_{n+1}$ be events, we assume that the claim is true for n, i.e. that $P\left(\bigcup_{i=1}^n A_i\right) \le \sum_{i=1}^n P(A_i)$. We need to show that

   $$P\left(\bigcup_{i=1}^{n+1} A_i\right) \le \sum_{i=1}^{n+1} P(A_i).$$

   The easiest way to do this is to let $X = \bigcup_{i=1}^n A_n$, and the observe that

   $$P\left(\bigcup_{i=1}^{n+1} A_i\right) = P\left(X \cup A_{n+1}\right) \le P(X) + P(A_{n+1}) \le \sum_{i=1}^{n} P(A_i) + P(A_{n+1}) = \sum_{i=1}^{n+1} P(A_i),$$

   where the first "$\le$" is by the base-case applied to the sets X and A_{n+1} and the second "$\le$" is by the inductive assumption.

   For the final part, we have equality when the events are mutually exclusive. (In fact, there can be equality when they are not: Observe what the inclusion-exclusion formula says about $P(\bigcup_{i=1}^n A_i)$. It expands this probability into $\sum_{i=1}^n P(A_i)$ plus/minus the sums of the intersections. If those sums all amount to zero, then we have equality. One can show that this happens when the pairwise intersections all have probability zero.)

2. (4 points) If $P(A) = 1/2$, $P(B) = 1/5$, and $P(A \cup B) = 3/5$, what are $P(A \cap B)$, $P(A^c \cup B)$, and $P(A^c \cap B)$?

   Solution: With this sort of problem I always draw a Venn diagram to see what it's asking and to help find the answer, but below I'll only put the formal solution.

   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, so

   $$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 1/10.$$

   For the next part, we write $A^c \cup B = A^c \cup (A \cap B)$. Since A^c and $A \cap B$ are disjoint,

   $$P(A^c \cup B) = P(A^c) + P(A \cap B) = (1 - 1/2) + 1/10 = 3/5.$$

For the final part, note that $B = (B \cap A) \cup (B \cap A^c)$ and that the sets in the union are mutually exclusive. This means $P(B) = P(B \cap A) + P(B \cap A^c)$, and rearranging gives

$$P(A^c \cap B) = P(B) - P(A \cap B) = 1/5 - 1/10 = 1/10.$$

3. (3 points) How many elements are there in the set

$$\{x : 10^7 \leq x \leq 10^8, \text{ and the base 10 representation of } x \text{ has no digit used twice}\}?$$

Solution: It is easier to think of the numbers in this set as strings. This is the set of all strings of 8 digits from $\{0, 1, \ldots, 9\}$ that do not start with a 0 and do not repeat a digit. Thus there are 9 choices for the first digit (i.e., everything but 0), 9 choices for the second (i.e., everything but the first digit, but now 0 is available), 8 for the third, and so on until the there are 3 choices for the eighth (final) digit. By the multiplication principle there are $9 \cdot 9 \cdot 8 \cdots 3 = 1632960$ such numbers.

4. (3 points) An army output has 19 posts to staff using 30 indistinguishable guards. How many ways are there to distribute the guards if no post is left empty?

Solution: This is a direct application of the formula for partitioning n balls into r non-empty groups, so the answer is $\binom{30-1}{19-1}$.

5. (1 point) What is the coefficient of $x^{10}y^{13}$ when $(x + y)^{23}$ is expanded?

Solution: By the binomial theorem this is $\binom{23}{10}$.

6. (4 points) What is the coefficient of $w^9x^{31}y^4z^{19}$ when $(w + x + y + z)^{63}$ is expanded? How many monomials appear in the expansion?

Solution: By the multinomial theorem this is $\binom{63}{9,31,4,19}$. There are $\binom{63+4-1}{4-1}$ terms in the sum.

7. (6 points) Let p be a prime number and $1 \leq k \leq p - 1$. Prove that $\binom{p}{k}$ is a multiple of p. Show that this is not true if p is not prime.

Solution: We have

$$\binom{p}{k} = \frac{p!}{k!(p-k)!} = p\frac{(p-1)!}{k!(p-k)!}.$$

Since p is prime and all of the factors of $k!(p - k)!$ are less than p, it does not cancel. This means that $\binom{p}{k}$ is a multiple of p.

For the counterexample we can take $\binom{4}{2} = 6$, which is not a multiple of 4.

8. (6 points) Verify that for any $n \geq k \geq 1$

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}.$$

Then give a combinatorial argument for why this is true.

Solution:

$$\binom{k}{2} + k(n-k) + \binom{n-k}{2} = \frac{k(k-1)}{2} + \frac{2k(n-k)}{2} + \frac{(n-k)(n-k-1)}{2}$$
$$= \frac{k^2 - k + 2kn - 2k^2 + n^2 - 2nk - n + k^2 + k}{2} = \frac{n^2 - n}{2} = \binom{n}{2}.$$

For a combinatorial argument, suppose we are to choose a 2-person committee from amongst n people. We can imagine that k of these people are men and $n-k$ of them are women. The possible committees either have 2 men, 2 women or one of each. Since there are $\binom{k}{2}$ committees with 2 men, $\binom{n-k}{2}$ with 2 women, and $k(n-k)$ with one of each, we get the identity.

9. **Extra credit (5 points).** Prove that

$$\sum_{k=0}^{n} k^2 \binom{n}{k} = n(n+1)2^{n-2}.$$

Solution: We will use the easy-to-verify fact that for $k > 0$

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

Using this fact we have

$$\begin{aligned}
\sum_{k=0}^{n} k^2 \binom{n}{k} &= \sum_{k=1}^{n} k^2 \frac{n}{k}\binom{n-1}{k-1} \\
&= n\sum_{k=1}^{n} k\binom{n-1}{k-1} \\
&= n\sum_{k=1}^{n} (k-1)\binom{n-1}{k-1} + n\sum_{k=1}^{n}\binom{n-1}{k-1}
\end{aligned}$$

We apply the fact again for the first sum, giving

$$\begin{aligned}
n\sum_{k=1}^{n} (k-1)\binom{n-1}{k-1} + n\sum_{k=1}^{n}\binom{n-1}{k-1} &= n\sum_{k=2}^{n} (k-1)\frac{n-1}{k-1}\binom{n-2}{k-2} + n\sum_{k=1}^{n}\binom{n-1}{k-1} \\
&= n(n-1)\sum_{k=2}^{n}\binom{n-2}{k-2} + n\sum_{k=1}^{n}\binom{n-1}{k-1} \\
&= n(n-1)2^{n-2} + n2^{n-1} \\
&= n(n+1)2^{n-2}.
\end{aligned}$$