

Homework 4

1. (10 points total) Two teams x and y are playing each other in the World Series, which is a best-of-seven-game match that ends when one team wins 4 games. Assume that team x wins each game with probability p , and that the outcome of each game constitutes an independent trial.

- (a) (0.5 points) What is the probability that x wins the first four games?

- There is one way for x to win in four, and that's winning four in a row.

$$p^4$$

- (b) (2 points) What is the probability that x wins four games after at most five game have been played?

- Sample space, strings, x character means x won, y character means y won.

xxxx
 yxxxx
 xyxxx
 xxyxx
 xxxyx

$$(1 - p)p^4 \times 4$$

- $P(x \text{ wins in less than five games}) = P(\{xxxx, yxxxx, xyxxx, xxyxx, xxxyx\})$
- $P(x \text{ wins in 6}) = \binom{5}{2}(1 - p)^2 p^4$

- (c) (2 points) What is the probability that x will win four games before y wins four games? (i.e., What is the probability that x wins the Series?)
 - (d) (0.5 points) Calculate and simplify your answer in part (c) when $p = 1/2$ and when $p = 2/3$.
 - (e) (1 point) Let X be the random variable that counts the number of games that are played. What is $\text{Range}(X)$?
 - (f) (2 points) What is $P(X = 7)$?
 - (g) (2 points) What is $P(X \geq 6)$?
2. (4 points) Suppose we roll two fair dice. Let the random variable $X =$ “the minimum of the two dice” and $Y =$ “the absolute value of the difference of the two dice”. Find $E(X)$ and $E(Y)$.

- $E(X)$

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- $E(Y)$

– The range is $\{0, 1, 2, 3, 4, 5\}$.

– There are 6 ways of getting 0, $\{6 - 6, 5 - 5, \dots\}$

$$\frac{6}{36}$$

– There are 6 ways of getting 1, $\{6 - 5, 5 - 6, 4 - 3, 3 - 4, 3 - 2, 2 - 3, 2 - 1, 1 - 2\}$.

$$\frac{8}{36}$$

- (4 points) Suppose boxes of cereal are filled with a random prize, each drawn from independently and uniformly from 6 possible prizes. If we buy N boxes of cereal, what is the expected number of distinct prizes we will collect? Hint: Consider the indicator random variables I_{E_i} for the event $E_i =$ “the i -th prize was in some box”.
- (4 points) A group of m men and w randomly sit in a single row at a theater. If a man and woman are seated next to each other we say they form a couple. (Couples can overlap, meaning that one person can be a member of two couples.) What is the expected number of couples? Hint: Use indicator random variables for each possible couple forming.
- (3 points) Suppose an experiment tosses a fair coin twice; the experiment “succeeds” if both tosses were Heads. We repeat this experiment for 12 independent trials. Let N be the random variable that counts the fraction of trials that are successful (so $N = S/12$, where S is the number of successful trials). Find $E(N)$.
- Extra Credit: (4 points)** Consider the experiment where n balls are to be placed randomly into n boxes. Let N_1 count the number of boxes with exactly one ball, and let N_2 count the number of boxes with exactly two balls. Find the probability of the events “ $N_1 = n$ ” and “ $N_1 = n - 1$ ”. Use the indicator technique to find $E(N_1)$ and $E(N_2)$.