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CS206: Introduction to Discrete Structures II, Spring 2013
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Homework 3

1. (6 points) What the probability that a 5 card hand contains exactly 3 spades? What if we condition on the hand containing at least 1 spade?

    - A deck of cards has 52 cards, 4 suits, 13 ranks, one of each suit/rank combination.
    - There are $\binom{52}{5}$ possible hands.
    - There are 13 cards which are spades.
    - Let E_i = "a spade was drawn" and F_i = "anything except a spade was drawn."
    - We want $P(E_1 \cap E_2 \cap E_3 \cap F_1 \cap F_2)$.
    - The probability of E_1 is $\frac{13}{52}$, because thirteen of the cards are spades.
    - Assuming E_1, the probability of E_2 is $\frac{12}{51}$, because there is one less spade and one less card in general.
    - Assuming E_2, the probability of E_3 is $\frac{11}{50}$, because there are now two less spades and two less cards in general.
    - There are now 49 cards in all, 10 of which *are* spades.
    - Assuming E_1 through E_3, the probability of $F_1 = \frac{39}{49}$.
    - Assuming F_1, the probability of $F_2 = \frac{38}{48}$.

    $$\frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{39}{49} \times \frac{38}{48} = 0.008154261704$$

    - Alternatively, there are 13 choose 3 ways of picking a spade, 39 choose 2 way of picking a "not spade," and there are 52 choose 5 possible options:

    $$\frac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}} = \frac{286 \times 741}{2598960} = 0.008154261704$$

    - For part two, you only have to pick four cards out of 51, because one is a space already.

    $$\frac{\binom{13}{2}\binom{39}{2}}{\binom{51}{4}}$$

2. (7 points) Suppose n people each throw a six-sided die. Let A_n be the event that at least two distinct people roll the same number. Calculate $P(A_n)$ for $n = 1, 2, 3, 4, 5, 6, 7$.

- For A_1, It is impossible for two distinct people to roll the same number if only person rolls.
$$P(A_1) = 0$$
- For A_2, there are 36 possible throws, but only 6 of could contain two distinct people rolling the same number.
$$P(A_2) = \frac{6}{6^2} = \frac{1}{6} = 0.1666666667$$
- For A_3 consider the complement, which can be described as "No two distinct people roll the same number." The probability that all three people roll unique numbers is $1 \times \frac{5}{6} \times \frac{4}{6} \times$,
$$P(A_3) = 1 - P(A_3^c) = 1 - \frac{5}{6} \times \frac{4}{6} = 0.4444444444$$
- For A_4, consider the complement again, and apply the same reasoning.
$$P(A_4) = 1 - 1 \times \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} = \frac{13}{18} = 0.7222222222$$
- For A_5, consider the complement again, and apply the same reasoning.
$$P(A_5) = 1 - \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} = 0.9074074074$$
- For A_6, consider the complement again. There are 6! ways of arranging the integer elements in the string "123456." This is out of a 6^6 ways of writing a string with integers from one to six.
$$P(A_6) = 1 - \frac{6!}{6^6} = 0.98456790123$$
- If 7 people roll a dice with 6 sides, it is inevitable that at least two distinct people roll the same number.
$$P(A_7) = 1$$

3. (3 points) Suppose we draw 2 balls at random from an urn that contains 5 distinct balls, each with a different number from $\{1, 2, 3, 4, 5\}$, and define the events A and B as

$$A = \text{“5 is drawn at least once”} \quad \text{and} \quad B = \text{“5 is drawn twice”}$$

Compute $P(A)$ and $P(B)$.

- $P(B) = 0$
- $P(A) = \frac{4}{\binom{5}{2}} = 0.4$

4. (4 points) In the previous problem, suppose we place the first ball back in the urn before drawing the second. Compute $P(A), P(B), P(A|B), P(B|A)$ in this version of the experiment.

- $P(A) = 1 - \left(\frac{4}{5}\right)^2 = .36$
- $P(B) = \left(\frac{1}{5}\right)^2 = .04$
- $P(A|B) = 1$
- $P(B|A) = \frac{1}{5}$

5. (4 points) Suppose 5 percent of cyclists cheat by using illegal doping. The blood test for doping returns positive 98 percent of the people doping and 12 percent who do not. If Lance's test comes back positive, what the probability that he is doping? (Ignoring all other evidence, of course...)

- $P(C) =$ “the probability a cyclist cheated by doping” $= .05$
- $P(T|C) =$ “the probability a cyclist giving a positive test if they doped” $= .98$
- $P(T|C^c) =$ “the probability a cyclist giving a positive test if they *did not* dope” $= .12$
- We want the probability of a cyclist doping given a positive test, which is $P(C|T)$.

$$P(T|C) = .98 = \frac{P(T \cap C)}{P(C)} = \frac{P(T \cap C)}{.05}$$

$$P(T \cap C) = .049$$

$$P(T \cap C^c) = .12 = \frac{P(T \cap C^c)}{P(C^c)} = \frac{P(T \cap C^c)}{.95}$$

$$P(T \cap C^c) = .114$$

- For any event T and C,

$$P(T) = P(T \cap C) + P(T \cap C^c)$$

$$P(T) = .049 + .114 = .163$$

$$P(C|T) = \frac{P(C \cap T)}{P(T)} = \frac{.049}{.163} = 0.3006134969 \approx .30$$

- Intuition:
  - Take 100 cyclists, 5 of them actually doped according to these numbers.
  - If all 100 cyclists are tested, it's very, very likely the 5 will return positive.
  - Of the remaining 95 cyclists, 12 percent of them will also return positive, which makes a little less than 12 cyclists.
  - So for each of the 17 who tested positive, there are five who actually doped, which means each has a $\frac{5}{17} = 0.2941176471 \approx .30$ probability of doping.

6. (6 points) If $A \subseteq B$, can A and B be independent? What we if require that $P(A)$ and $P(B)$ both not equal 0 or 1?

   - A and B are independent if they satisfy the condition $P(A \cap B) = P(A)P(B)$
   - If $A \subseteq B$, it is true that $A \cap B = B$.
   - This means that $P(A \cap B) = P(B)$.
   - In order to be independent when $A \subseteq B$ is true, $P(B)$ must equal $P(A) \cdot P(B)$.
   - The only way anything multiplied by something can equal itself is if that something is one.
   - Therefore, when $A \subseteq B$, A and B can be independent when $P(A) = 1$.
   - Furthermore, being as anything multiplied by zero yields zero, when $A \subset B$, A and B can be independent if $P(B) = 0$.
   - So no.

7. **Extra credit (5 points)** Consider the experiment where two dice are thrown. Let A be the event that the sum of the two dice is 7. For each $i \in \{1, 2, 3, 4, 5, 6\}$ let B_i be the event that at least one i is thrown.

   (a) Compute $P(A)$ and $P(A|B_1)$.
   - $P(A) = \frac{6}{6^2} = \frac{1}{6}$
   - $P(A|B_1) = \frac{P(A \cap B_1)}{P(B_1)} = \frac{\frac{3}{\binom{6}{2}}}{\frac{1}{6}+\frac{1}{6}} = 0.6$

   (b) Prove that $P(A|B_i) = P(A|B_j)$ for all i and j.
   - Being as the sum has to be seven, there is one and only way to sum to seven for the integers 1 through 6.
   - So it doesn't matter what is rolled on the first roll, the probability "rides on" the second roll being the number the first roll needs to sum to seven.
   - The probability of rolling any given number on a fair die is always $\frac{1}{6}$.
   - Therefore, for all i and j, the probability cannot be anything but $\frac{1}{6}$.

   (c) Since you know that some B_i always occurs, does it make sense that $P(A) \neq P(A|B_i)$? (After all, if E is an event with $P(E) = 1$, then for any event F, $P(F|E) = P(F)$. What is going on? Does this seem paradoxical?)