

Homework 4

1. (10 points total) Two teams x and y are playing each other in the World Series, which is a best-of-seven-game match that ends when one team wins 4 games. Assume that team x wins each game with probability p , and that the outcome of each game constitutes an independent trial.

- (a) (0.5 points) What is the probability that x wins the first four games?

- There is one way for x to win in four, and that's winning four in a row.
- By the multiplicity principle, we can multiply the chance of winning each game to get the probability of winning all games.

$$p^4$$

- (b) (2 points) What is the probability that x wins four games after at most five game have been played?

- Sample space, strings, x character means x won, y character means y won.

```
xxxx
yxxxx
xyxxx
xxyxx
xxxyx
```

- The team represented by y cannot win at the end because after 4 x wins, the games halt.
- $P(x \text{ wins in less than five games}) = P(\{xxxx, yxxxx, xyxxx, xxyxx, xxxyx\})$
- There is a $(1 - p)$ chance of y winning.
- The x team still needs to win 4 games, which has a probability of p^4 .
- Multiplying these two values gets you the likelihood that this happens for any given instance of x winning in less than five games.
- To get all of the possible less than 5 games combinations, multiply by the number there are, which is 4.
- Then, add the probability of just 4 straight victories.

$$(1 - p) \times p^4 \times 4 + p^4$$

- (c) (2 points) What is the probability that x will win four games before y wins four games? (i.e., What is the probability that x wins the Series?)

- To begin, let's enumerate some possibilities using strings.

$$\{xxxx\}$$

$$\{yxxxx, xyxxx, xxyxx, xxxyx\}$$

$$\{yyxxxx, yxyxxx, \dots, xxxyyx\}$$

$$\{yyyxxx, yyxyxx, \dots, xxxyyx\}$$

- These are all the possibilities for x winning. Name these X_1 through X_4 and notice the sum of their probabilities to be the probability that x wins four games.
- For X_3 , there are $\binom{5}{2,3}$ ways of ordering x and y, y has to win twice, and x still has to win four times.
- You have to subtract off the cases where there is a y at the end,

$$\binom{4}{1,3} \times (1-y)^2 \times p^4$$

- For X_4 , there are $\binom{6}{3,3}$ ways of ordering x and y, y has to win thrice, and x still has to win four times.

$$\binom{5}{2,3} \times (1-y)^3 \times p^4$$

- Now add 4 straight victories:

$$\binom{4}{1,3} \times (1-y)^2 \times p^4 + \binom{5}{2,3} \times (1-y)^3 \times p^4 + p^4$$

- (d) (0.5 points) Calculate and simplify your answer in part (c) when $p = 1/2$ and when $p = 2/3$.

- $p = 1/2 : 1/2$
- $p = 2/3 : 1808/2187$

- (e) (1 point) Let X be the random variable that counts the number of games that are played. What is $\text{Range}(X)$?

$$R(X) = \{4, 5, 6, 7\}$$

- (f) (2 points) What is $P(X = 7)$?

$$\binom{6}{3} (1-p)^3 p^4 + \binom{6}{3} p^3 (1-p)^3 p^4$$

- (g) (2 points) What is $P(X \geq 6)$?

$$\binom{5}{2} (1-p)^2 p^4 + \binom{5}{2} p^2 (1-p)^4 + \binom{6}{3} (1-p)^3 p^4 + \binom{6}{3} p^3 (1-p)^4$$

2. (4 points) Suppose we roll two fair dice. Let the random variable X = “the minimum of the two dice” and Y = “the absolute value of the difference of the two dice”. Find $E(X)$ and $E(Y)$.

- $E(X)$

- The formula to solve this is

$$E(X) = \sum_{a_i \in R(X)} a_i P(X = a_i)$$

- The sample space has a cardinality of 36, as there 6 choices for each of the two choices.
- All rolls are equally likely in a fair dice.
- Starting with the highest element in the range (which is the set containing 1 through 6), there is only one way 6 can be the minimum.

$$P(x = 6) = P(\{(6, 6)\}) = \frac{1}{36}$$

- There are 3 ways of “getting 5”, and that’s rolling two fives, and then both variations of a five and a six.

$$P(x = 5) = P(\{(5, 5), (5, 6), (6, 5)\}) = \frac{3}{36}$$

- Apply the same pattern,

$$P(x = 4) = P(\{(4, 4), (4, 5), (5, 4), (4, 6), (6, 4)\}) = \frac{5}{36}$$

$$P(x = 3) = P(\{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3), (3, 6), (6, 3)\}) = \frac{7}{36}$$

$$P(x = 2) = P(\{(2, 2), (2, 3), (3, 2), (2, 4), (4, 2), (2, 5), (5, 2), (2, 6), (6, 2)\}) = \frac{9}{36}$$

- Notice that there 2 more each time.

$$P(x = 1) = \frac{11}{36}$$

- Now sum them and multiply them by their value (a_i) to find expected value,

$$E(X) = \left(6 \times \frac{1}{36}\right) + \left(5 \times \frac{3}{36}\right) + \left(4 \times \frac{5}{36}\right) + \left(3 \times \frac{7}{36}\right) + \left(2 \times \frac{9}{36}\right) + \frac{11}{36} = 2.5277777778$$

- $E(Y)$

- The range is $\{0, 1, 2, 3, 4, 5\}$.
- This is another application of the formula

$$E(X) = \sum_{a_i \in R(X)} a_i P(X = a_i)$$

- There are 6 ways of getting 0, $\{6-6, 5-5, \dots\}$. For $i = 0$,

$$0 \times P(X = 0) = 0 \times \frac{6}{36} = 0$$

- There are 10 ways of getting 1, $\{6-5, 5-6, 5-4, 4-5, 4-3, 3-4, 3-2, 2-3, 2-1, 1-2\}$. For $i = 1$,

$$1 \times P(X = 1) = 1 \times \frac{10}{36}$$

- There are 8 ways of getting 2, $\{6-4, 4-6, 5-3, 3-5, 4-2, 2-4, 1-5, 5-1\}$. For $i = 2$,

$$2 \times P(X = 2) = 2 \times \frac{8}{36} = \frac{4}{9}$$

- There are 6 ways of getting 3, $\{6-3, 3-6, 5-2, 2-5, 4-1, 1-4\}$. For $i = 3$,

$$3 \times P(X = 3) = 3 \times \frac{6}{36} = \frac{2}{3}$$

- There are 4 ways of getting 4, $\{6-2, 2-6, 5-1, 1-5\}$. For $i = 4$,

$$4 \times P(X = 4) = 4 \times \frac{4}{36} = \frac{4}{9}$$

- There are 2 ways of getting 5, $\{6-1, 1-6\}$. For $i = 5$,

$$5 \times P(X = 5) = 5 \times \frac{2}{36} = \frac{5}{18}$$

- We know we've covered the sample space because the sum of the specific instances is the same as the cardinality of the sample space.
- The expected value,

$$E(X) = 0 + \frac{2}{9} + \frac{4}{9} + \frac{2}{3} + \frac{4}{9} + \frac{5}{18} = 1.94 \approx 2$$

- (4 points) Suppose boxes of cereal are filled with a random prize, each drawn from independently and uniformly from 6 possible prizes. If we buy N boxes of cereal, what is the expected number of distinct prizes we will collect? Hint: Consider the indicator random variables I_{E_i} for the event $E_i =$ "the i -th prize was in some box".
- (4 points) A group of m men and w randomly sit in a single row at a theater. If a man and woman are seated next to each other we say they form a couple. (Couples can overlap, meaning that one person can be a member of two couples.) What is the expected number of couples? Hint: Use indicator random variables for each possible couple forming.
 - Use linearity of expectation for each pair of seats.
 - Let x equal the number of seats.
 - Then, $E(X_1) = P(\text{couple in a seat 1 and 2})$, $E(X_2) = P(\text{couple in a seat 2 and 3})$, etc.
 - For any given seat, there is a $\frac{1}{2}$ chance of their being a man or a woman in the seat.

- The four possibilities are $\{mw, wm, mm, ww\}$.

$$\sum_i^x$$

5. (3 points) Suppose an experiment tosses a fair coin twice; the experiment “succeeds” if both tosses were Heads. We repeat this experiment for 12 independent trials. Let N be the random variable that counts the fraction of trials that are successful (so $N = S/12$, where S is the number of successful trials). Find $E(N)$.

$$\begin{aligned} E(N) &= E\left(\frac{S}{12}\right) \\ S &= \{x_1 + \dots + x_{12}\} \\ \frac{1}{12}E(S) \\ E(S) &= \sum_{a_i \in R(S)} a_i \times P(X_i) \\ P(X_i) &= \frac{1}{4} \\ E(S) &= 12 \times E(X_i) \\ \frac{1}{12}E(S) &= 12 \times \frac{1}{4} \\ E(S) &= \frac{1}{4} \end{aligned}$$

6. **Extra Credit: (4 points)** Consider the experiment where n balls are to be placed randomly into n boxes. Let N_1 count the number of boxes with exactly one ball, and let N_2 count the number of boxes with exactly two balls. Find the probability of the events “ $N_1 = n$ ” and “ $N_1 = n - 1$ ”. Use the indicator technique to find $E(N_1)$ and $E(N_2)$.