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CS206: Introduction to Discrete Structures II, Spring 2013
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Homework 6

1. (5 points) Let X be a random variable on some sample space S with expected value μ and variance σ^2. Let X_1 and X_2 be *identical independent copies* of X. Find $E((X_1 - X_2)^2)$ in terms of μ and σ^2. Give some intuition for why your answer makes sense.

   - Definition of variance:
   $$V(X) = E(X^2) - E(X)^2$$
   - Substitute:
   $$\sigma^2 = E(X^2) - \mu^2$$
   - Rearrange:
   $$\sigma^2 + \mu^2 = E(X^2)$$
   - Now for the algebra:
   $$\begin{aligned} &E((X_1 - X_2)^2) \\ &= E(X_1^2 - 2X_1X_2 + X_2^2) \\ &= E(X_1^2) - 2E(X_1X_2) + E(X_2^2) \\ &= E(X_1^2) - 2E(X_1)E(X_2) + E(X_2^2) \\ &= \sigma^2 + \mu^2 - 2\mu^2 + \sigma^2 + \mu^2 \end{aligned}$$
   - Simplify:
   $$\begin{aligned} &= 2\sigma^2 + 2\mu^2 - 2\mu^2 \\ &= 2\sigma^2 \end{aligned}$$
   - This is so because we actually calculated the variance, because,
   $$V(X_1 - X_2) = E((X_1 - X_2)^2) - E(X_1 - X_2)^2$$
   $$= E((X_1 - X_2)^2) - 0$$

2. (3 points) Prove that $V(cX) = c^2V(X)$ for any random variable X and real number c.

$$V(cX) = E((cX - \mu)^2)$$
$$= E((cX)^2) - (E(cX))^2$$
$$= E(c^2(X - \mu)^2)$$
$$= c^2E((X - \mu)^2)$$
$$= c^2V(X)$$

3. (5 points) Let X be the random variable that is the sum of two independent fair die rolls. Let Y be the outcome of the first roll minus the outcome of the second roll. Calculate $\text{Cov}(X, Y)$. Are X and Y correlated?

   - Definition of covariance

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

   - Let Z_1 be a random variable representing the first roll, and Z_2 equal the second roll.

$$X = Z_1 + Z_2$$
$$Y = Z_1 - Z_2$$
$$XY = Z_1^2 - Z_2^2$$
$$= E((Z_1 + Z_2)(Z_1 - Z_2)) - E(Z_1 + Z_2)E(Z_1 - Z_2)$$

   - Notice that X and Y are independent, by linearity of expectation, $E(Z_1^2 - Z_2^2) = E(Z_1^2) - E(Z_2^2)$.
   - Because of the definition of Z_1 and Z_2, the expectations are equal (they're both die rolls).
   - If you subtract any squared value from that same squared value, you get zero.

$$E((Z_1 + Z_2)(Z_1 - Z_2)) = 0$$

   - Which leaves us with:

$$= 0 - E(Z_1 + Z_2)E(Z_1 - Z_2)$$
$$= 0 - (E(Z_1) + E(Z_2))(E(Z_1) - E(Z_2))$$

   - The expectation of any die roll is about 3, but by linearity of expectation, it's always the same thing.

$$= 0 - (3 + 3)(3 - 3)$$
$$= 0$$

4. (4 points) Let C be the random variable that is the number of heads in 100 independent fair coin flips. What are $E(C)$ and $V(C)$? Find upper bounds on $P(C > 75)$, using Markov's Inequality and Tchebychev's Inequality.

- Begin with expected value

$$E(C) = E(X_1 + X_2 + X_3 + ... + X_{100})$$

- Notice linearity of expectation

$$= \sum_{i=1}^{100} E(X_i) = 50$$

- Now for variance

$$V(C) = (100)\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right)$$

- Markov

$$P(C \geq 70) \leq \frac{5}{7}$$

- Tchebychev

$$P(C \geq 70) \leq \frac{25}{70^2}$$

5. (4 points) Find an example where Markov's inequality is tight in the follow sense: For each positive integer a, find a non-negative random variable X such that $P(X \geq a) = E(X)/a$.

   - Define X to be a non-negative random variable with expectation, $E(X) = ac$.
   - The probability that X is equal to a is defined as c, for any $a \geq c$.
   - Markov's inequality is therefore,

$$\Pr(X \geq a) \leq \frac{ac}{a}$$

$$\frac{ac}{a} = c$$

   - Because Markov's caps off the probability at the probability, this is tightly bound.

6. (4 points) Prove the following: If X is a non-negative random variable with $E(X) = \mu$, then for every k, $P(X \geq k\mu) \leq 1/k$.

   - Markov's

$$\Pr(X \geq a) \leq \frac{\mathrm{E}(X)}{a}$$

   - Let $a = k\mu$, where $k \geq 0$

$$\Pr(X \geq k\mu) \leq \frac{\mathrm{E}(X)}{k\mu}$$

$$\Pr(X \geq k\mu) \leq \frac{\mu}{k\mu}$$

$$\Pr(X \geq k\mu) \leq \frac{1}{k}$$