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Introduction to Discrete Structures II (01:198:206)

Homework 1

1. (8 points) Prove that $P(A \cup B) \leq P(A) + P(B)$ for any events A and B. Prove the general version by induction, which says that if $A_1, ..., A_n$ are events then $P(\bigcup_{i=1}^{n} A_i) \leq P(\sum_{i=1}^{n} A_i)$. When does this inequality become an equality?

   - Base case

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$$

   - Assume it holds for a general case $(n = k)$

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

$$P(A_1 \cup A_2 \cup ... \cup A_k) \leq P(A_1) + P(A_2) + ... + P(A_k)$$

   - Prove that it hold for next case using your assumption.

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) \leq \sum_{i=1}^{k+1} P(A_i)$$

$$P(A_1 \cup A_2 \cup ... \cup A_k \cup A_{k+1}) \leq P(A_1) + P(A_2) + ... + P(A_k) + P(A_{k+1})$$

$$(A_1 \cup A_2 \cup ... \cup A_k) = C$$

$$P(A_1) + P(A_2) + ... + P(A_k) = D$$

$$P(C \cup A_{k+1}) \leq P(D) + P(A_{k+1})$$

   By the base case, this is true, and expression is proven for all events.

2. (4 points) If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{5}$, and $P(A \cup B) = 3/5$, what are $P(A \cap B)$, $P(A^c \cup B)$, and $P(A^c \cap B)$?

   - $P(A \cap B) = \frac{1}{10}$
   - $P(A^c \cup B) = 3/5$ because the probability of A is the same as A^c
   - $P(A^c \cap B) = \frac{1}{10}$

3. (3 points) How many elements are there in the set
$\{x : 10^7 \leq x \leq 10^8$, and the base 10 representation of x has no digit used twice$\}$?

- $10^7 = 10000000$, and $10^8 = 100000000$
- The smallest number possible is 12345678, the greatest number possible is 98765432
- For the first number in any element, the first option is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
- Then for the every second number, it's every number besides the first choice plus 0.
- The "tree" looks as follows, where $i \neq j \neq k \neq l \neq m \neq n \neq o \neq p$:
  (a) $\{1ijklmno\}$ where $\{i, j, k, l, m, n, o\} \in \{0, 2, 3, 4, 5, 6, 7, 8, 9\}$
  (b) $\{2ijklmno\}$ where $\{i, j, k, l, m, n, o\} \in \{0, 1, 3, 4, 5, 6, 7, 8, 9\}$
  (c) $\{3ijklmno\}$ where $\{i, j, k, l, m, n, o\} \in \{0, 1, 2, 4, 5, 6, 7, 8, 9\}$
  (d) $\{4ijklmno\}$ where $\{i, j, k, l, m, n, o\} \in \{0, 1, 2, 3, 5, 6, 7, 8, 9\}$
  (e) $\{5ijklmno\}$ where $\{i, j, k, l, m, n, o\} \in \{0, 1, 2, 3, 4, 6, 7, 8, 9\}$
  (f) $\{6ijklmno\}$ where $\{i, j, k, l, m, n, o\} \in \{0, 1, 2, 3, 4, 5, 7, 8, 9\}$
  (g) $\{7ijklmno\}$ where $\{i, j, k, l, m, n, o\} \in \{0, 1, 2, 3, 4, 5, 6, 8, 9\}$
  (h) $\{8ijklmno\}$ where $\{i, j, k, l, m, n, o\} \in \{0, 1, 2, 3, 4, 5, 6, 7, 9\}$
  (i) $\{9ijklmno\}$ where $\{i, j, k, l, m, n, o\} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
- Every "node" (a) through (i) is $\binom{9}{1}$.
- When you pick the first number, you still have 9 choices because zero is added. When you choose the second number, you have 8 choices because a number is taken out and non are put back into the set of choices. When you pick your third number, you have 7 choices because a number is taken out and none put back in. ...
- So for integer strings i through o:

$$[i] \qquad [j] \qquad [k] \qquad [l] \qquad [m] \qquad [n] \qquad [o] \qquad [p]$$

- The number of elements is equal to

$$\binom{9}{1} \times \binom{9}{1} \times \binom{8}{1} \times \binom{7}{1} \times \binom{6}{1} \times \binom{5}{1} \times \binom{4}{1} \times \binom{3}{1}$$

4. (3 points) An army output has 19 posts to staff using 30 indistinguishable guards. How many ways are there to distribute the guards if no post is left empty?

- This is a "stars and bars" problem, where the number of "stars" k is equal to 30, and the number of "bars," "bins," or "posts" n is equal to 19.

$$\binom{k-1}{n-1} = \binom{30-1}{19-1} = \binom{29}{18} = 34597290$$

5. (1 point) What is the coefficient of $x^{10}y^{13}$ when $(x + y)^{23}$ is expanded?

- This problem requires the binomial theorem, where $n = 23$:

$$(x+y)^{23} = \sum_{k=0}^{23} \binom{23}{k} x^k y^{23-k}$$

- When $k = 10$, $23 - k$ will equal 13.

$$\binom{23}{10} x^{10} y^{13}$$

- So the coefficient for $x^{10}y^{13}$ will be

$$\binom{23}{10} = 1144066$$

6. (4 points) What is the coefficient of $w^9x^{31}y^4z^{19}$ when $(w+x+y+z)^{63}$ is expanded? How many monomials appear in the expansion?

- This is a multinomial coefficient problem where $k = 4$ and $n = 63$:

$$(a_1 + a_2 + ... + a_k)^n = \sum_{\substack{n_1, n_2, ..., n_k \geq 0 \\ n_1 + n_2 + ... + n_k = n}} \frac{n!}{n_1! n_2! ... n_k!} a_1^{n_1} a_2^{n_2} ... a_k^{n_k}$$

- Set $n_1 = 9$, $n_2 = 31$, $n_3 = 4$, and $n_4 = 19$.

$$\left(\frac{63!}{9!31!4!19!}\right) \times w^9 x^{31} y^4 z^{19}$$

- Therefore, the coefficient is

$$\frac{63!}{9!31!4!19!}$$

- The number of monomials in a multinomial coefficient can be expressed as a "stars and bars" problem.

$$\binom{66}{3}$$

7. (6 points) Let p be a prime number and $1 \leq k \leq p-1$. Prove that $\binom{p}{k}$ is a multiple of p. Show that this is not true if p is not prime.

$$\binom{p}{k} = \frac{p(p-1) \times ... \times (p-k+1)}{1 \times 2 \times ... \times (k-1) \times k} = p\binom{p-1}{k}$$

$$\binom{4}{2} = 6$$

6 is not a multiple of 4 and is not prime.

8. Verify that for any $n \geq k \geq 1$

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$$

Then give a combinatorial argument for why this is true.

- Observe that the binomial coefficient with two for all reals:

$$\binom{r}{2} = \frac{r(r-1)}{2}$$

- Apply fact to both sides:

$$\frac{n(n-1)}{2} = \frac{k(k-1)}{2} + k(n-k) + \frac{(n-k)(n-k-1)}{2}$$

- Multiply everything by two

$$n(n-1) = k(k-1) + 2k(n-k) + (n-k)(n-k-1)$$

- Expand out

$$n(n-1) = k^2 - k + 2kn - 2k^2 + k + k^2 - n - 2kn + n^2$$

- Simplify

$$n(n-1) = -n + n^2$$

- Change form

$$n(n-1) = n(n-1)$$

- Combinatorial argument:
  - $\binom{n}{2}$ is the number of ways we can arrange n objects into groups of 2.
  - Splitting n into 2, one of those groups will be of size k, making the other set of size $n-k$
  - Using the new partitions k and $n-k$, arrange n "things" into groups of two.
  - The first partition is of length k, resulting in $\binom{k}{2}$ ways, and for the second part, we have $n-k$ choose 2 ways, explaining the first and last terms.
  - Now choose 2 "things", one from different groups of 2, we can choose 1 of the k objects and all of $n-k$ "things", which is equal to $k(n-k)$.