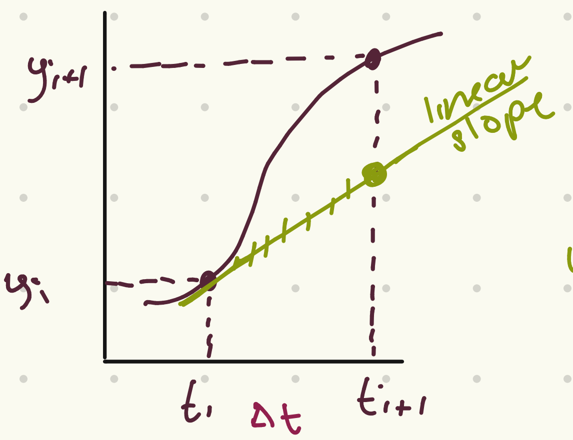
Higher Order Methods (of solving Diff Eqns)



$$y_{t+1} = y_t + \Delta t\,(\text{slope}) \leftarrow$$

What slope should we use?

$$\dot{f}_t = \frac{df}{dt}$$

$$f'_x = \frac{df}{dx}$$

Review: Taylor Series Expansion: $f(t)$ @ $f(t+\Delta t)$

in general: $f(t+\Delta t) = f(t) + \sum_{i=1}^{n} f^{n}(t)\frac{\Delta t^n}{n!}$

first 3 terms:

$$f(t+\Delta t) \approx f(t) + \Delta t\,\dot{f}(t) + \frac{\Delta t^2}{2}\ddot{f}(t) + \mathcal{O}(\cancelto{0}{\Delta t^3})$$

2nd order accurate if $\Delta t \sim 0.1$ accurate to 0.01

$\cdot)\ \dot{y}(t) = G(y,t)$ Euler: $y_{t+1} = y_t + \underbrace{\Delta t\, G(y,t)}_{\text{step}}$

taylor expanding: $y_{t+1} = y_t + \underline{\Delta t\, G_t + \frac{\Delta t^2}{2}\dot{G}_t + \frac{\Delta t^3}{6}\ddot{G}_t + \frac{\Delta t^4}{24}\dddot{G}_t} + \mathcal{O}(\cancelto{0}{\Delta t^5})$

$$\Rightarrow\ = y_t + \Delta t\underbrace{\left[G_t + \frac{\Delta t}{2}\dot{G}_t + \frac{\Delta t^2}{6}\ddot{G}_t + \frac{\Delta t^3}{24}\dddot{G}_t\right]}_{\text{slope}}$$

Runge-Kutta Methods:

$$y_{t+1} = y_t + \Delta t\left(G_t + \frac{\Delta t}{2}\dot{G}_t + \frac{\Delta t^2}{6}\ddot{G}_t + \frac{\Delta t^3}{24}\dddot{G}_t\right) \Rightarrow$$

$$y_{t+1} = y_t + \Delta t \sum_{i=1}^{N} a_i k_i \quad , \quad a_i\text{'s} = \text{weights}, \ \sum_i a_i = 1$$

k_i's = slopes

4th order: $y_{t+1} = y_t + \Delta t\left(a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4\right)$

Runge-Kutta Order 1:

$$y_{t+1} = y_t + \Delta t(a_1 k_1) \quad , \ a_1 = 1 \ , \ k_1 = G$$

$$y_{t+1} = y_t + \Delta t(G_t)$$ (Euler's Method)

1st order accurate if $\Delta t = 0.1$ we're accurate to 0.1

Runge-Kutta 2:

$$y_{t+1} = y_t + \Delta t\left(G_t + \frac{\Delta t}{2}\dot{G}_t\right)$$

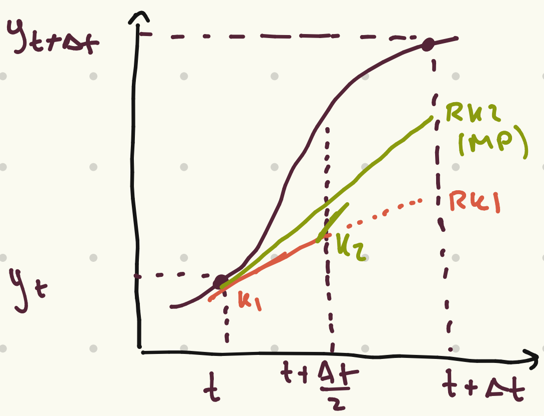
$$= y_t + \Delta t(a_1k_1 + a_2k_2)$$ need $\sum a_i = 1$

Midpoint Method:

$$y_{t+1} = y_t + \Delta t\, K_2$$

$$K_1 = G(y_t, t)$$

$$K_2 = G\left(y_t + k_1\frac{\Delta t}{2}, t + \frac{\Delta t}{2}\right)$$



HEUN'S METHOD:

$$y_{t+1} = y_t + \Delta t\,\frac{1}{2}(k_1 + k_2)$$

$$K_1 = G(y_t, t)$$

$$K_2 = G(y_t + k_1\Delta t, t + \Delta t)$$

RK-4: most popular

$$y_{t+1} = y_t + \Delta t\left[G_t + \frac{\Delta t}{2}\dot{G}_t + \frac{\Delta t^2}{6}\ddot{G}_t + \frac{\Delta t^3}{24}\dddot{G}_t\right] + \mathcal{O}(\Delta t^5)$$

$$= y_t + \Delta t\left[a_1 k_1 + a_2 k_2 + a_3 k_3 + a_4 k_4\right]$$

$\Rightarrow$ tedious algebra $\Rightarrow$

$$y_{t+1} = y_t + \Delta t \underbrace{\left[\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4\right]}_{\text{average slope} \rightarrow 4 \text{ values of } G}$$

RK4

$$\boxed{y_{t+1} = y_t + \frac{\Delta t}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]}$$

WHERE:

$k_1 = G(y_t, t)$

$k_2 = G\left(y_t + k_1\frac{\Delta t}{2}, t + \frac{\Delta t}{2}\right)$

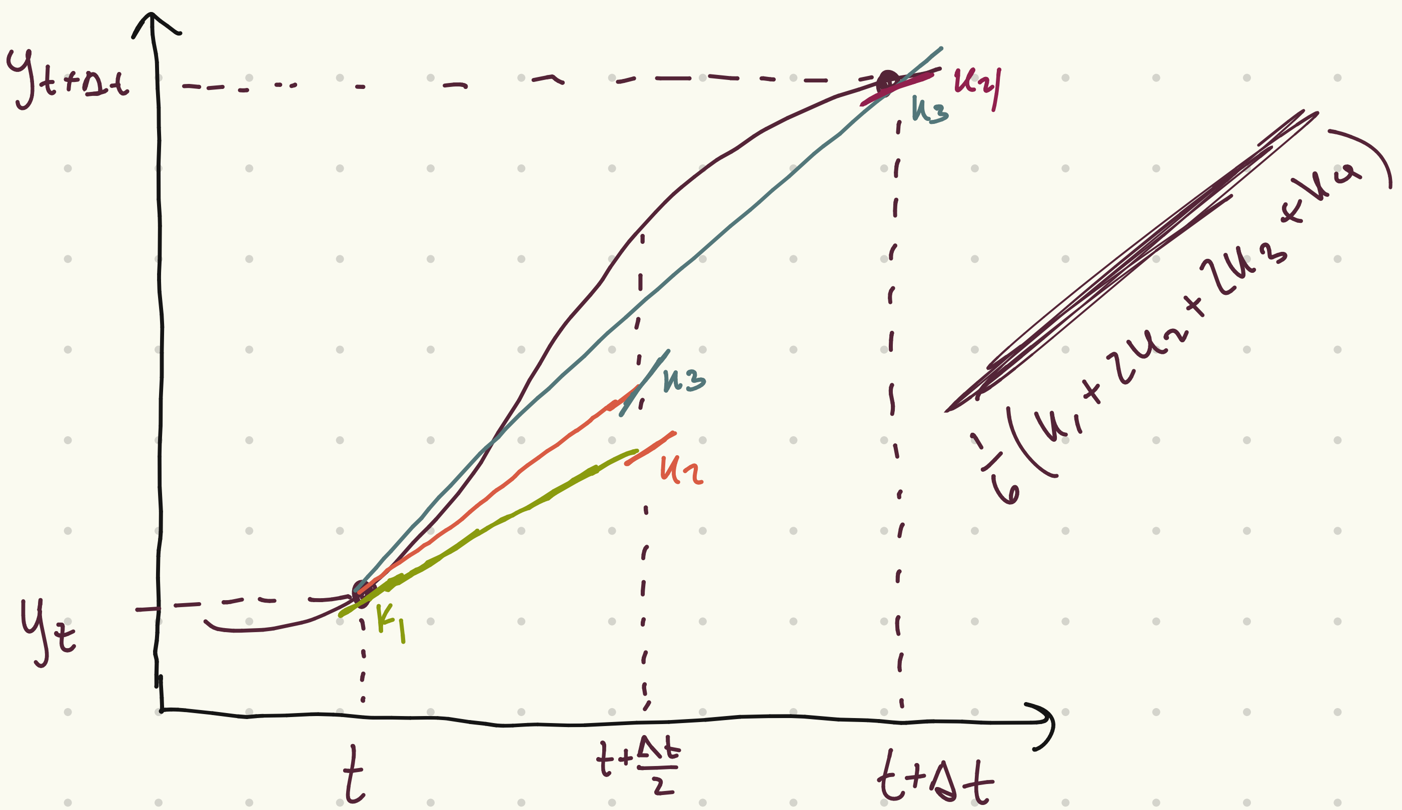
$k_3 = G\left(y_t + k_2\frac{\Delta t}{2}, t + \frac{\Delta t}{2}\right)$

$k_4 = G\left(y_t + k_3\Delta t, t + \Delta t\right)$

4th order accurate

$\Delta t = 10^{-2}$

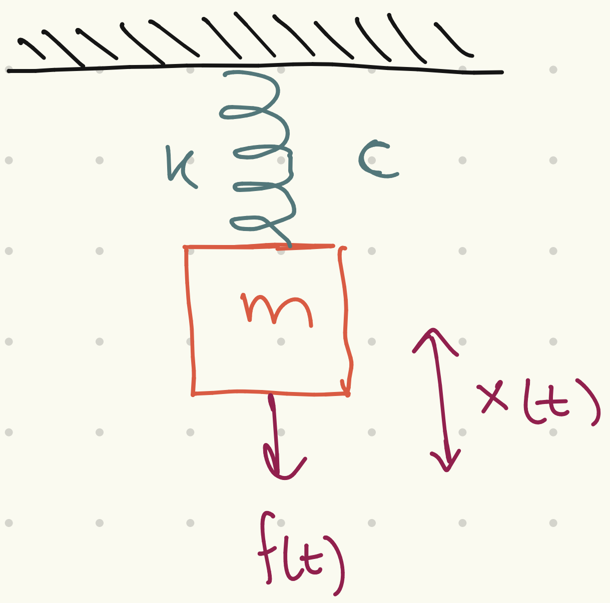
accurate to 10^{-8}



Working Example: Damped Harmonic Oscillator

$\dot{x} = \dfrac{dx}{dt}$

$\ddot{x} = \dfrac{d^2x}{dt^2}$



$f(t) = kx + c\dot{x} + m\ddot{x}$

2nd order, → 1st order

let:

$\vec{y} = \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \qquad \dot{\vec{y}} = \begin{bmatrix} \ddot{x} \\ \dot{x} \end{bmatrix}$

$$\vec{F} = \begin{bmatrix} f(t) \\ 0 \end{bmatrix} = \overset{A}{\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} \ddot{x} \\ \dot{x} \end{bmatrix} + \overset{B}{\begin{bmatrix} c & k \\ -1 & 0 \end{bmatrix}} \begin{bmatrix} \dot{x} \\ x \end{bmatrix}$$

1st order diff eq: $\vec{F} = A\dot{\vec{y}} + B\vec{y}$

$$\dot{\vec{y}} = \underbrace{A^{-1} \cdot (\vec{F} - B\vec{y})}_{G} \Rightarrow y_{t+1} = y_t + \Delta t\, G$$

i.c. $\vec{y}_0 = \begin{bmatrix} \dot{x}(t=0) \\ x(t=0) \end{bmatrix}$

$KE = \frac{1}{2} m \dot{x}^2$, $PE = \frac{1}{2} k x^2$ $\qquad KE + PE = \text{const.}$