# Lecture 4: Monte Carlo simulation - Urn Models

# This week's 538.com "Riddler" Questions - The hard one

A grasshopper lands somewhere randomly on your lawn, which has an area of 1 square meter.

As soon as it lands, it jumps 30 centimeters.

What shape should your lawn be to maximize the chances that the grasshopper will still be on the lawn after the 30-centimeter jump?

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(Hint: It's not a circle.)

*Extra credit:* What if the grasshopper jumps *X* centimeters instead?

# Notes for Assignment 1

- Github working ok!
- Write-ups were too short. Don't just give us results. It doesn't need to look like a paper, but there needs to be some text to tell me what is going on. Explain what you are doing, how you are doing it, what your results are and how you interpret your results. Write it so that it can be understood by someone who did not know what the assignment was. We will deduct points if this is not true for Assignment 2.

- Given that the estimated p-values are quite small, use lots of simulations (e.g. ~10K)
- p-value depends upon the statistic being used - compare to concept of "sufficient statistic".
- There is a theory of "most powerful" tests. Chi-squared and variance (sd) are 'best' statistics (they boil down to the same thing). It doesn't follow that the these tests will necessarily give the smallest p-value for a particular test (but you can say that they will "on average" if the null is false).
- You can't really do 2-tailed tests here, because they assume the distribution is symmetric (c.f. literature on doubled p-values, which is what some of you actually did).
- No need to assume normality.
- Simulation is good, but shouldn't replace calculation when the latter is possible.
- Must choose your statistic, or test, *and what you are going to test*, before you look at the data.. (otherwise you are *data mining*)

- Assignment 2 due next Monday (March 1st) **at 11am.** Commit your answers to the Assignment repos created by accepting your GitHub invites.

- Last week's lab tasks…

# Examinable Assignment 2a

1. Write a program to implement the Secant method.
2. Test it, using $x_0=1$ and $x_1=2$ on:
    1. cos(x)-x
    2. log(x)-exp(-x)

   Compare it with the performance of Newton-Raphson on the same functions.
3. Write a function to show a plot of the iterations of the algorithm.
4. Turn it in as a knitted PDF document + RMD file.

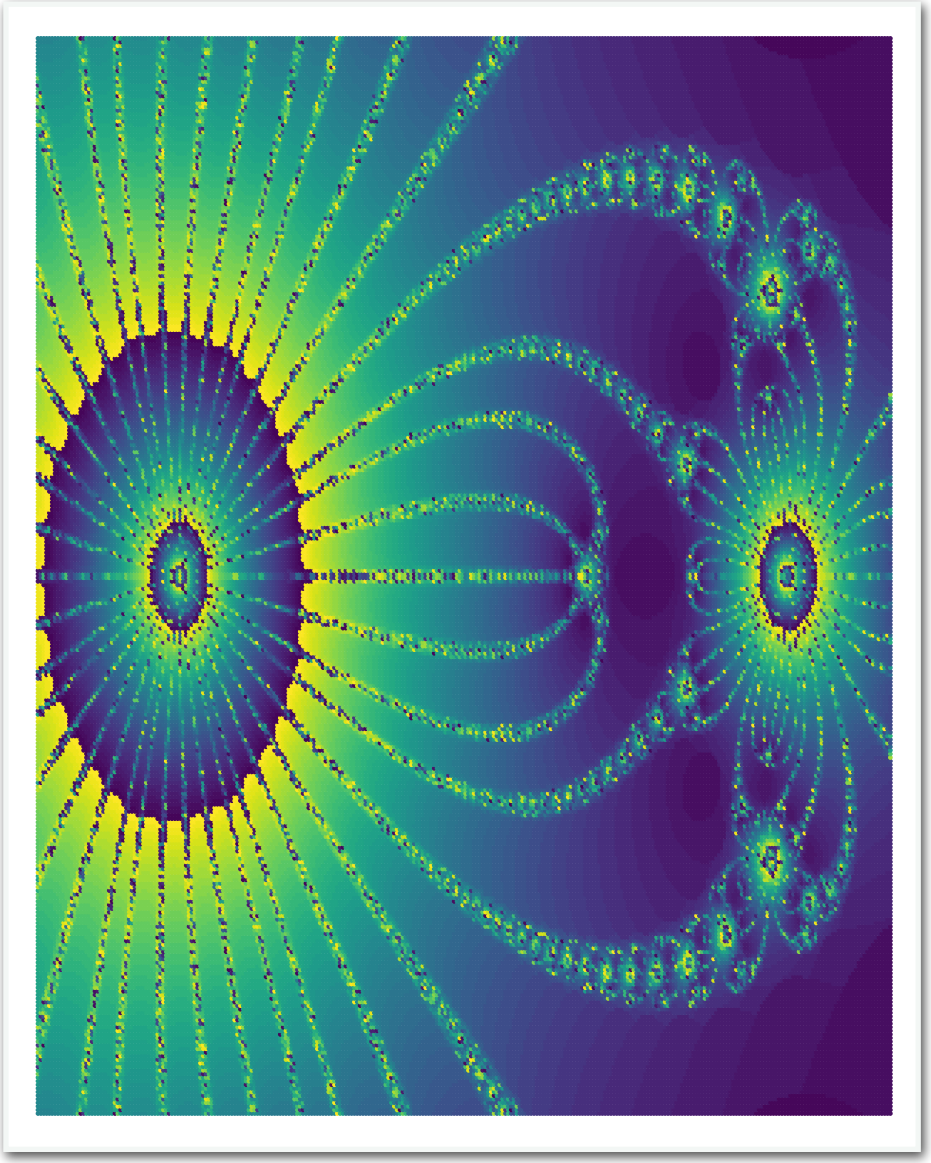
- **Deadline: Monday March 1st, 11am**

# Examinable assignment 2b: The Art Show

- Take the code for Newton-Raphson on the complex plain from Github:
  - Basic [slow] code: https://github.com/PM520-Spring-2020/Week3-NewtonRaphsonFractals. [do not use this].
  - Better [faster] code: https://github.com/PM520-Spring-2020/Week3-FasterFractals [use this!]
- Use it to draw some fractals for some functions **other than those we saw in class**. [Alternatively, Google "Julia Sets" and explore those]
- Write a report in Rmarkdown including your R code, and describing the functions you tried, but also including some pretty pictures for display at the start of class in three week's time. **They must be drawn by your R code**. (R's ggplot package is your friend here.)
- Each person should also commit one picture. You are strongly encouraged to give it a pretentious/artsy name. We will vote on who has the best (and I will provide some sort of prize for the best picture).
- **Deadline: Monday March 1st, 11am**

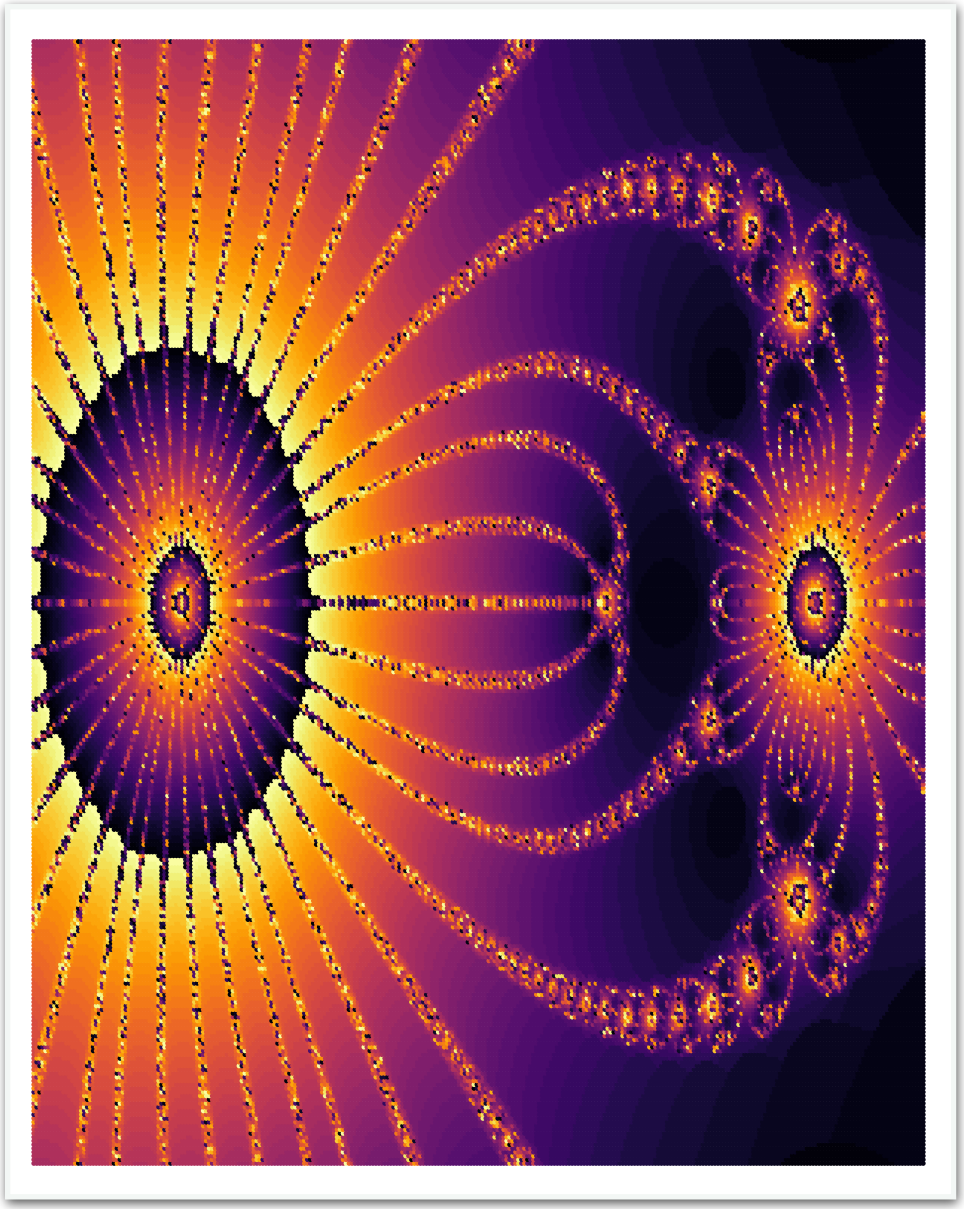
# Assignment submission

- Please upload your assignment answers to your clones of the repos called:
  - Week3-Assignment2a-SecantMethod
  - Week3--Assignment2b-ArtShow



```
library(viridis)
palette(viridis(256))
```

# This is the function 35z^9 - 180z^7 + 378z^5 - 420z^3 + 315z  (taken from http://www.chiark.greenend.org.uk/~sgtatham/newton/)



```
library(viridis)
palette(inferno(256))
```

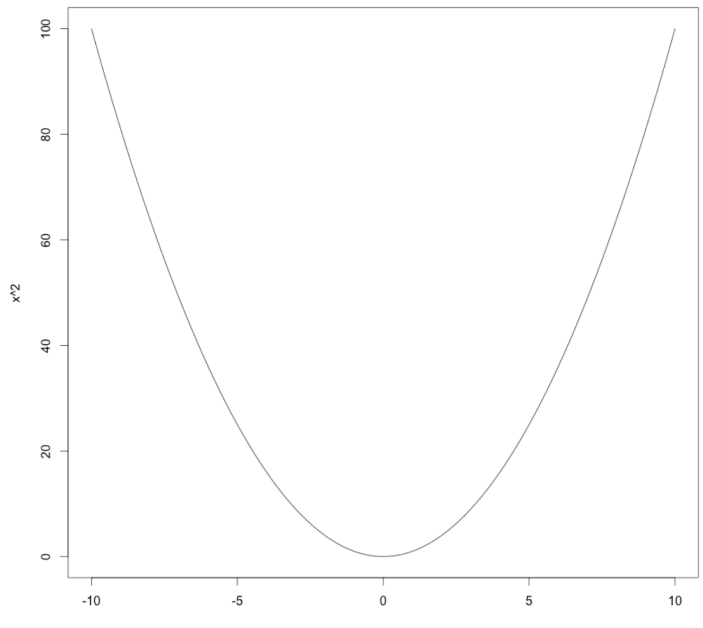
# This is the function 35z^9 - 180z^7 + 378z^5 - 420z^3 + 315z (taken from http://www.chiark.greenend.org.uk/~sgtatham/newton/)

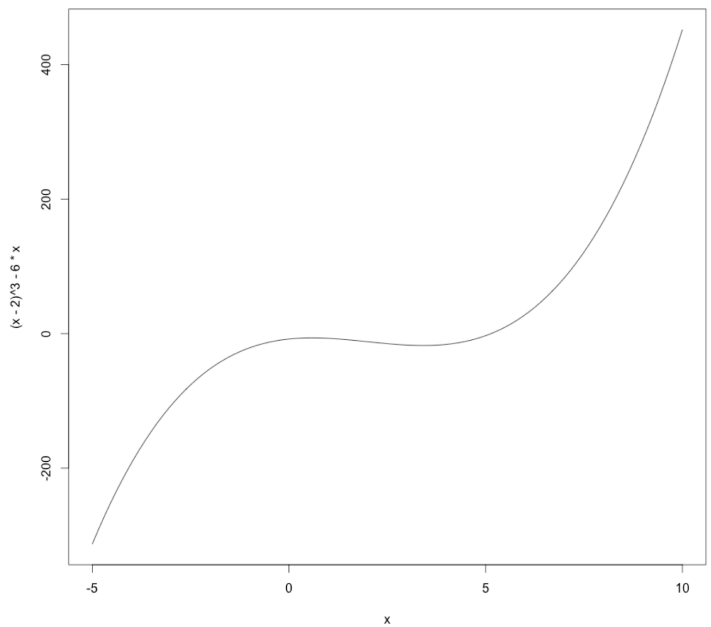
# Lab task (30 minutes)

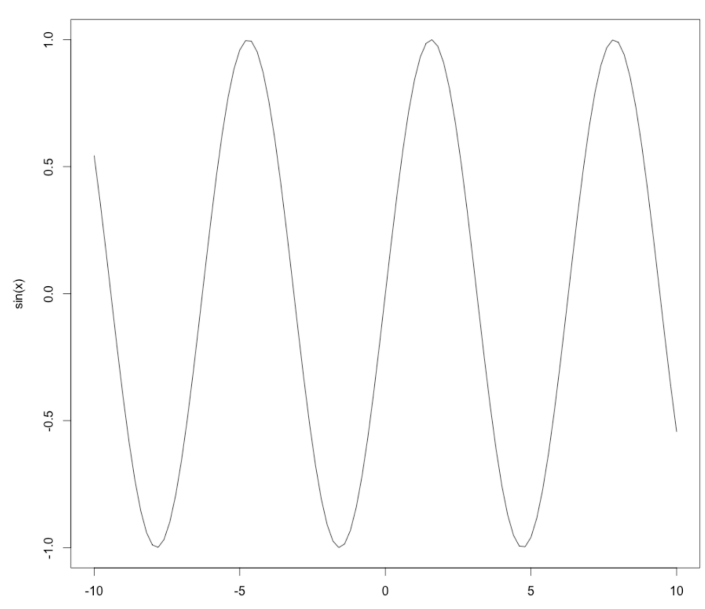
- Implement Newton-Raphson in R

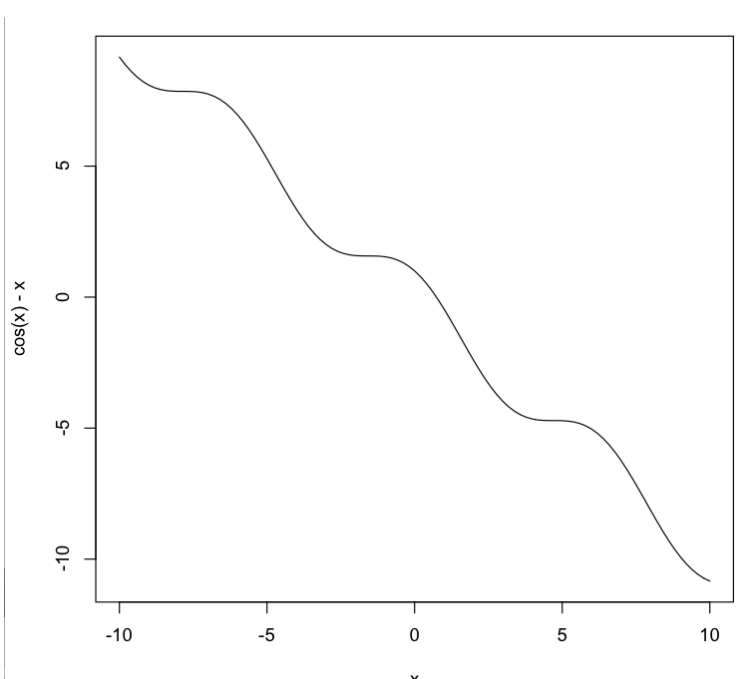
- Test it on the following functions:
  - $f(x)=x^2$
  - $f(x)=(x-2)^3-6x$
  - $f(x)=\sin(x)$
  - $f(x)=\cos(x)-x$

- Draw plots showing progress of the algorithm
- When you have something that works, or that is giving you problems, push it to GitHub with @pmarjora in your commit message.


x^2
x
0
20
40
60
80
100
-10
-5
0
5
10


(x - 2)^3 - 6 * x
x
-200
0
200
400
-5
0
5
10


sin(x)
x
-1.0
-0.5
0.0
0.5
1.0
-10
-5
0
5
10


cos(x) - x
x
-10
-5
0
5
-10
-5
0
5
10

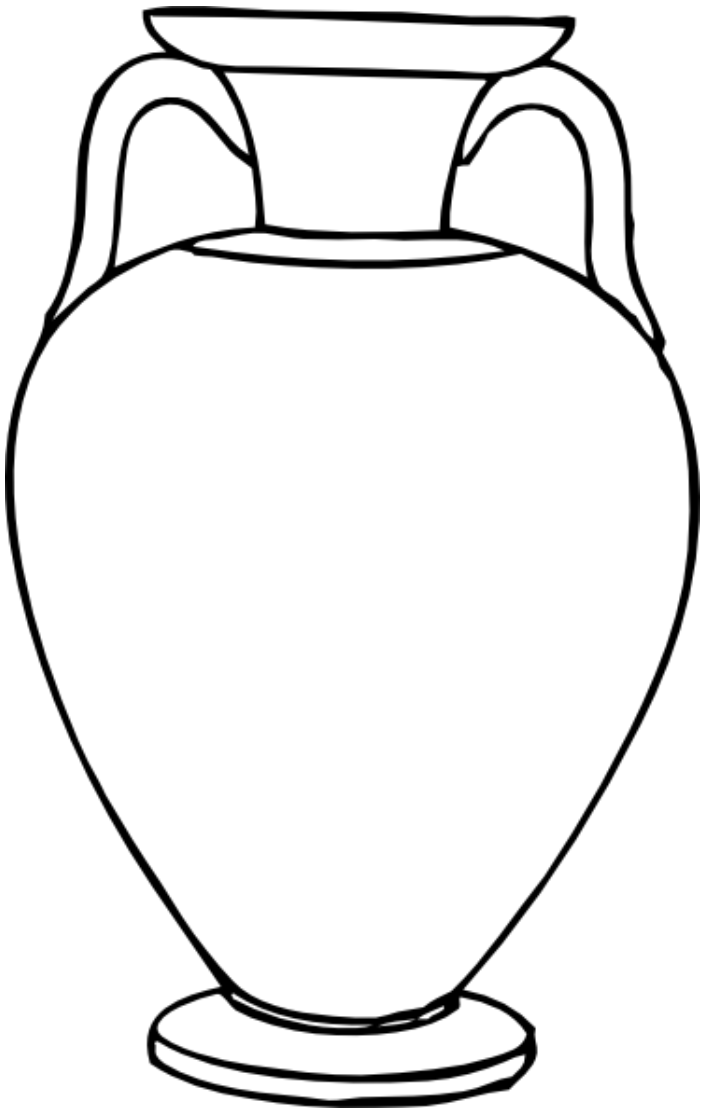
# Extended Monte Carlo Simulation example: Urn models


www.hiwtc.com

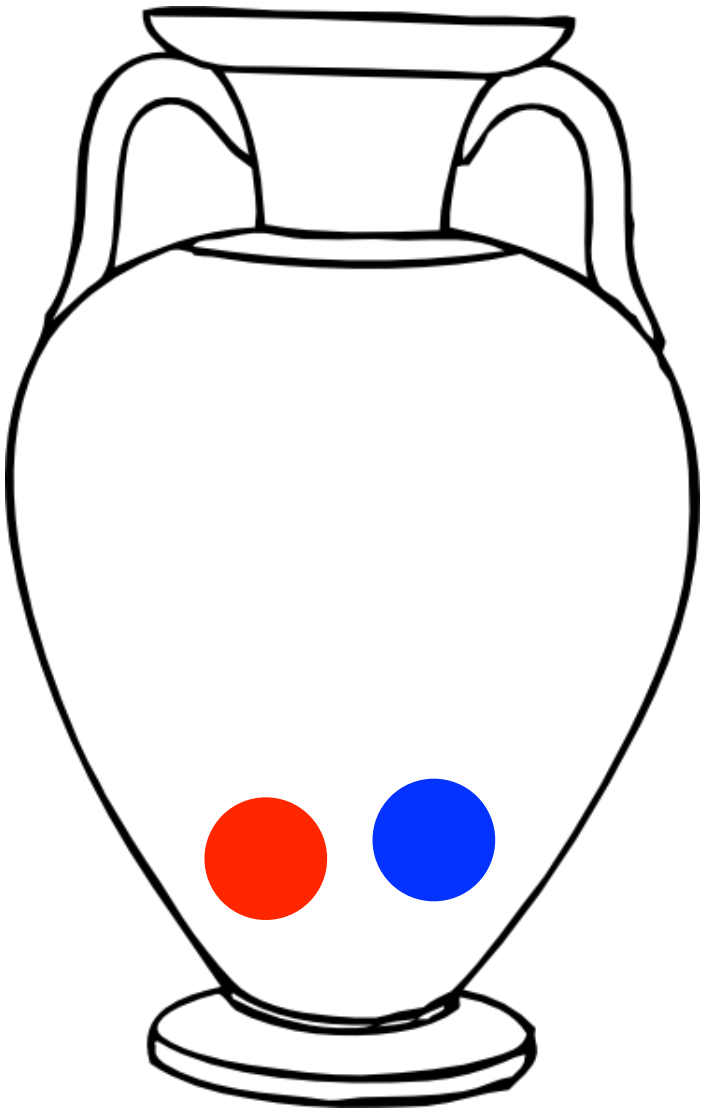
1. Draw balls from an urn
2. Take actions that depend upon what you drew
3. Ask questions about what happens after $n$ draws, say.

Used as very early model of epidemics.

# In pictures

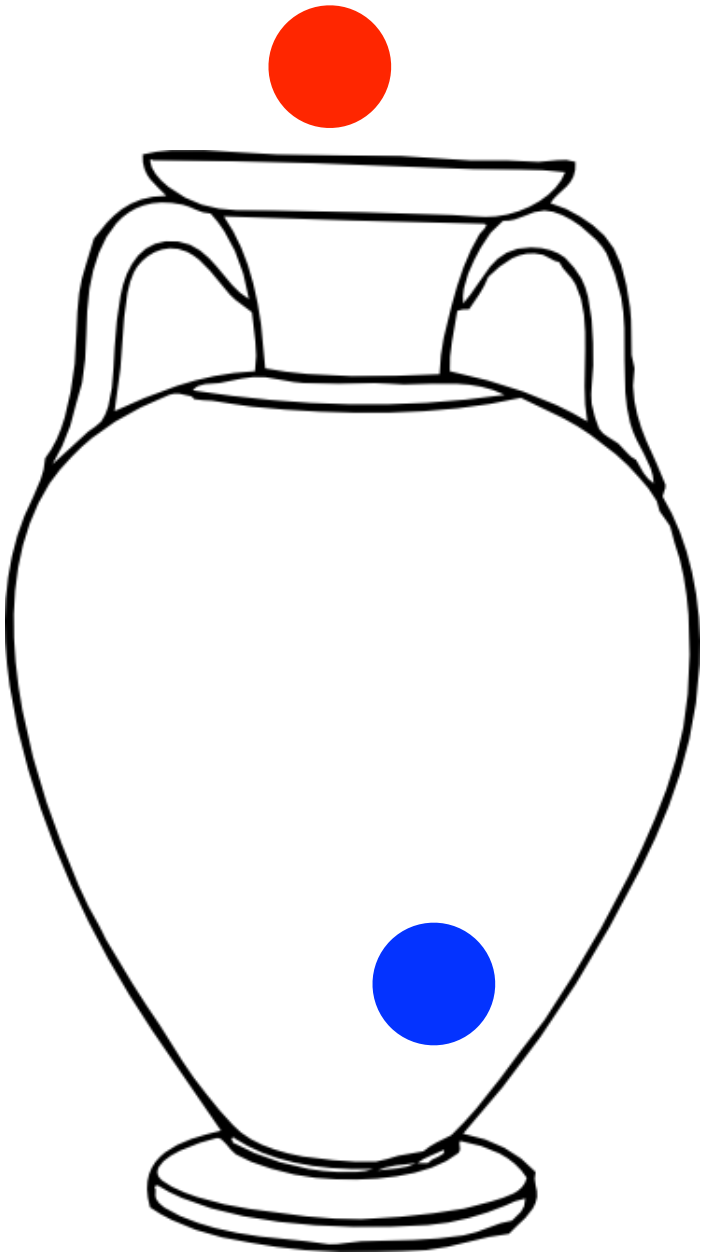


# Simplest form



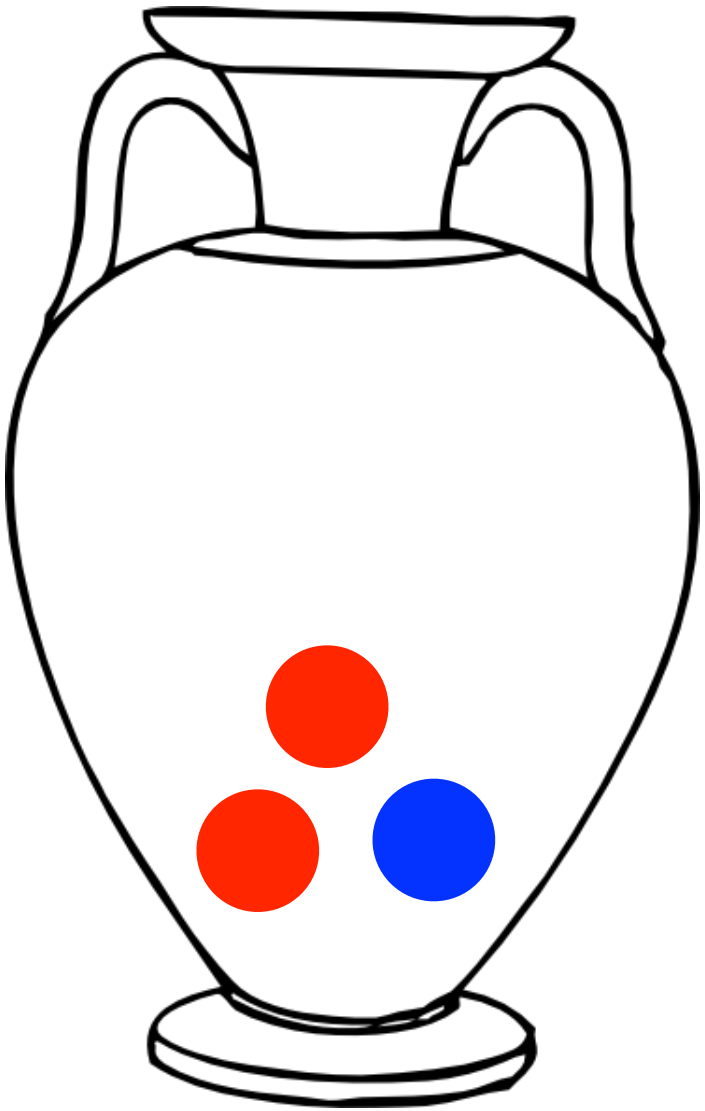
1. Start with two balls, of different colors.

# Simplest form



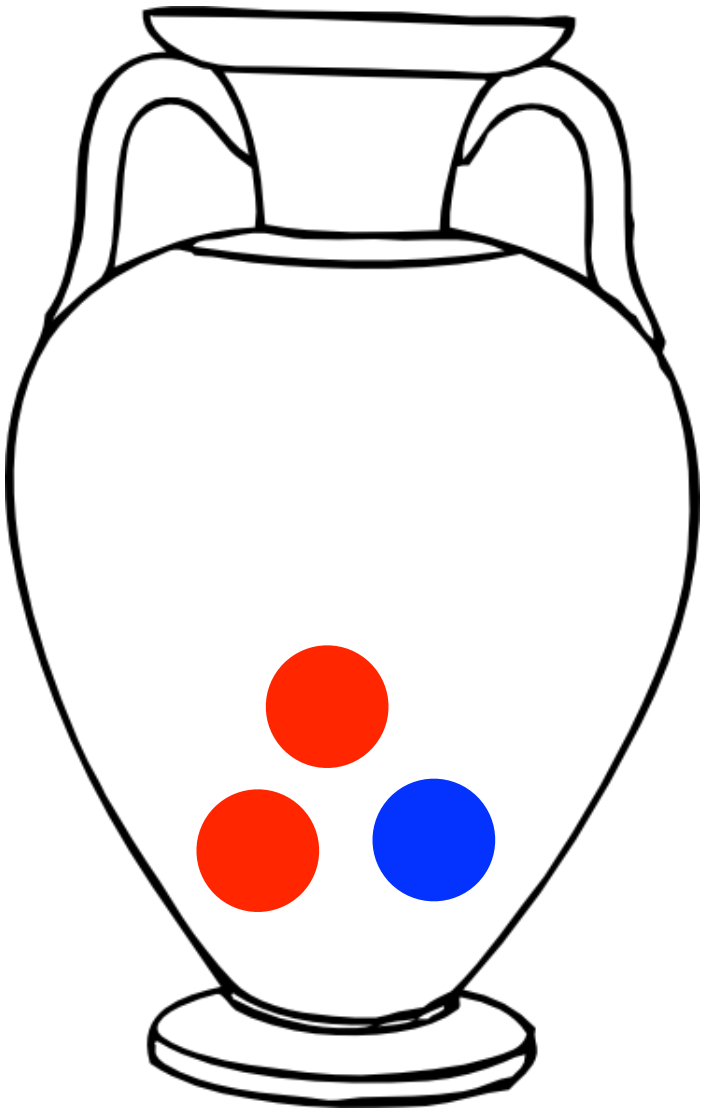
1. Start with two balls, of different colors.
2. Draw a ball from the urn.

# Simplest form



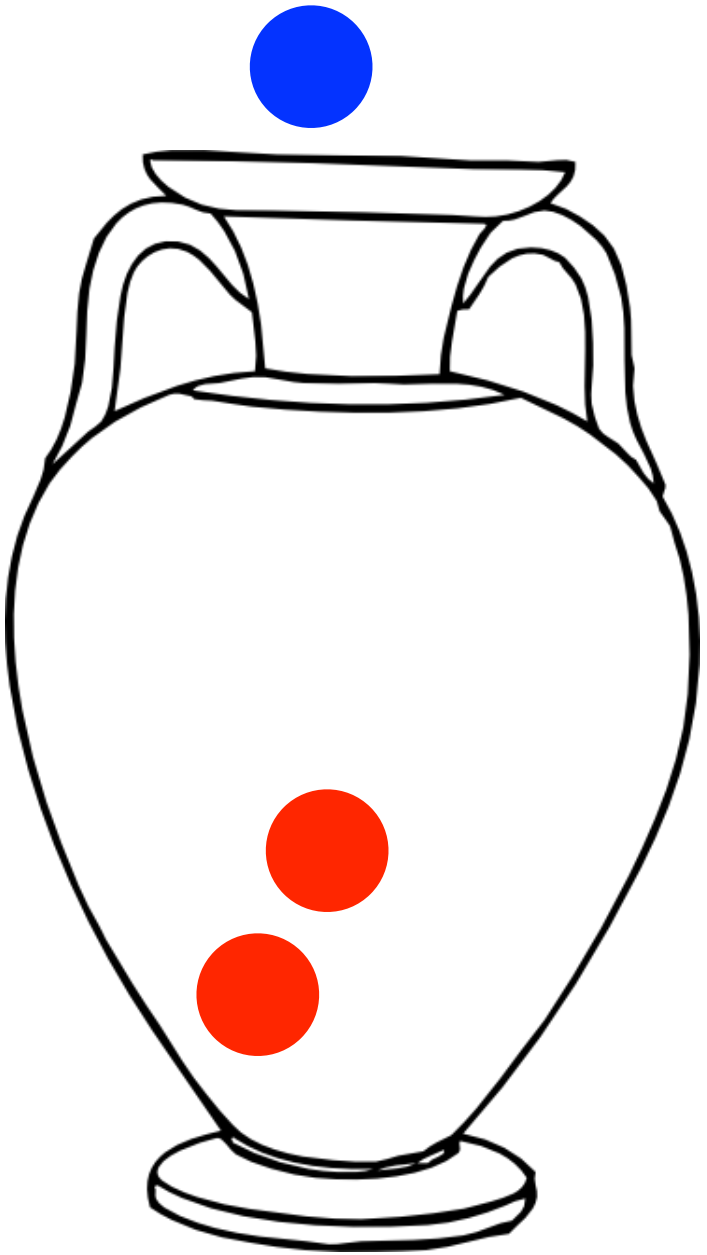
1. Start with two balls, of different colors.
2. Draw a ball from the urn.
3. Put the ball back into the urn *and add another ball of the same color.*

# Simplest form



1. Start with two balls, of different colors.
2. Draw a ball from the urn.
3. Put the ball back into the urn *and add another ball of the same color.*
4. Iterate.

# Simplest form



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2. Draw a ball from the urn.
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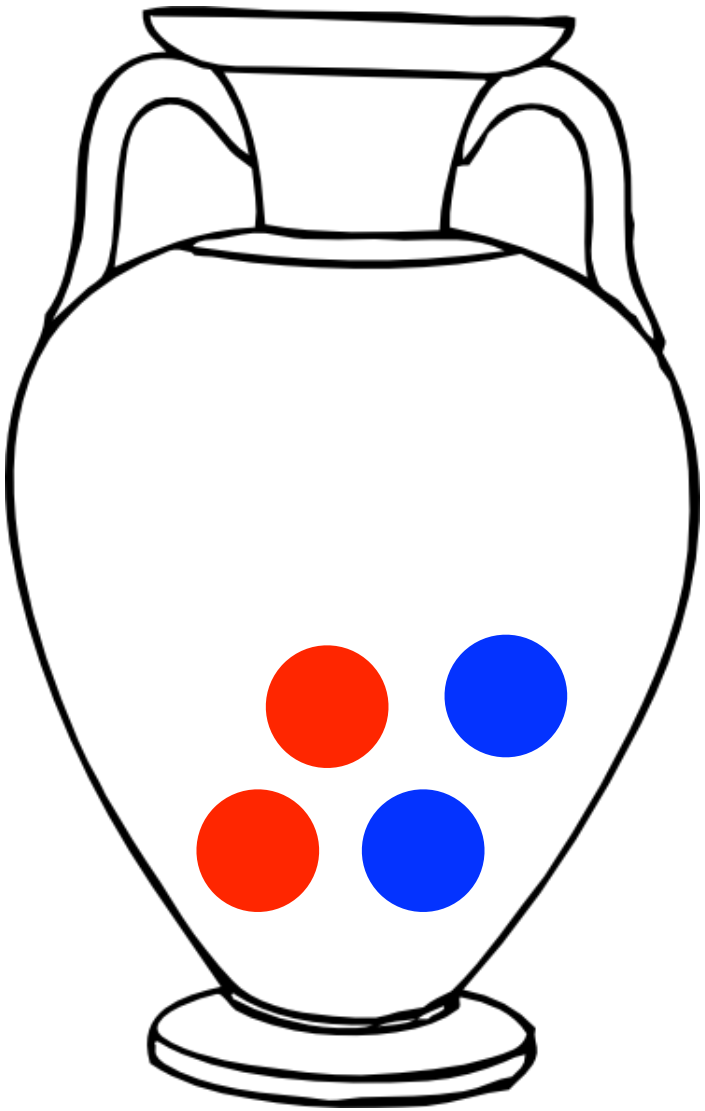

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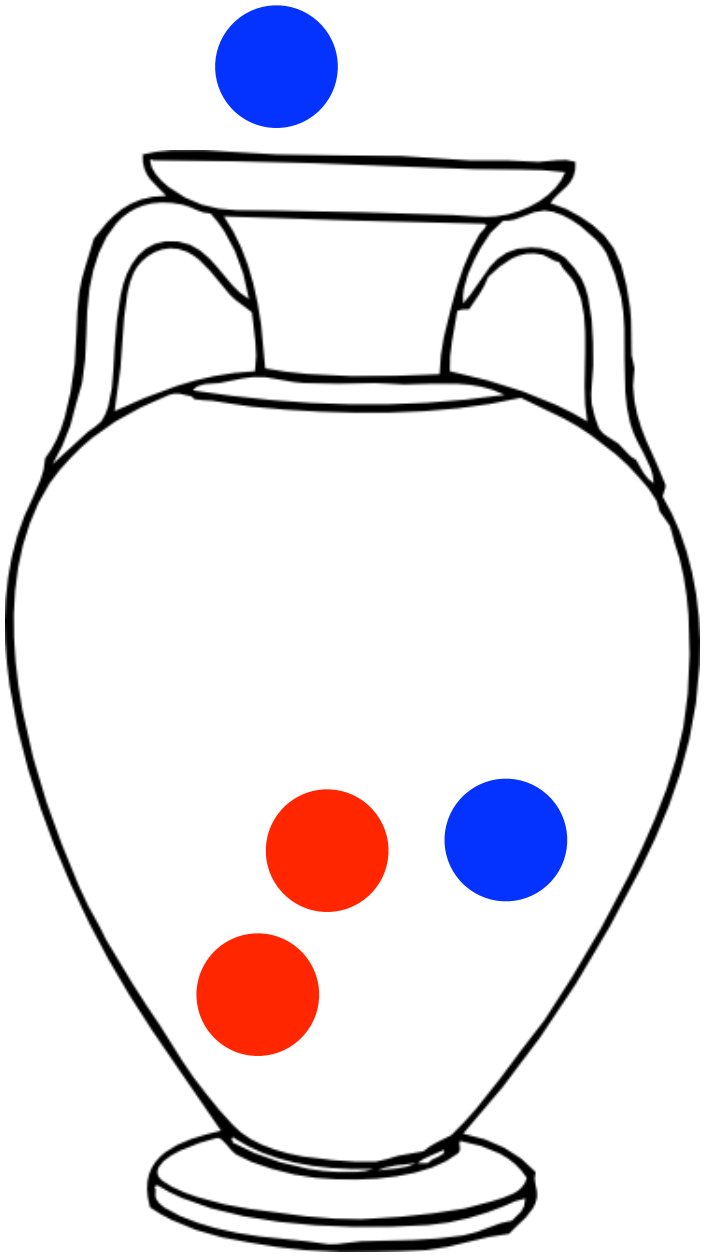

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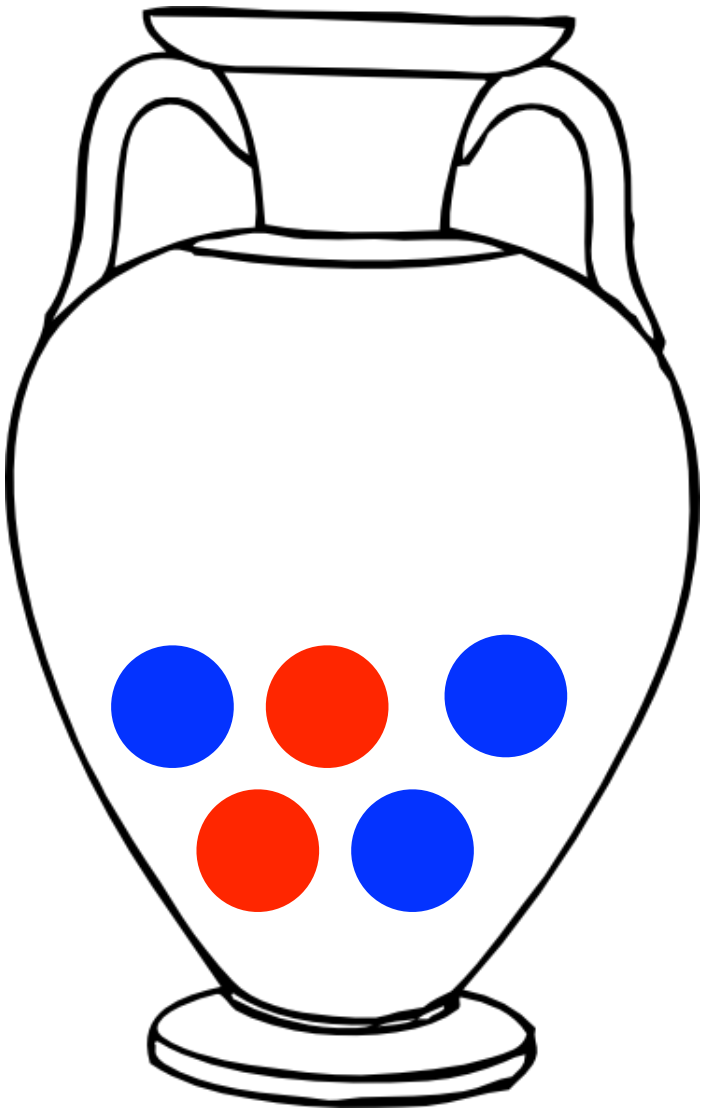
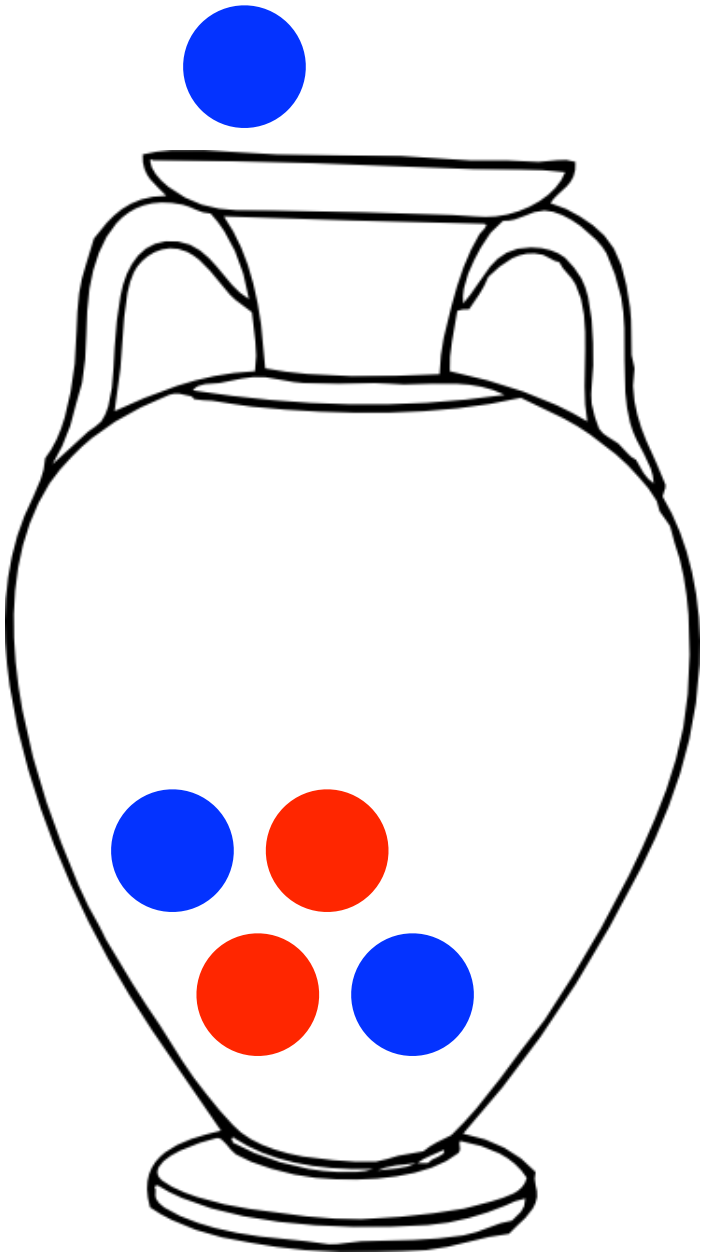


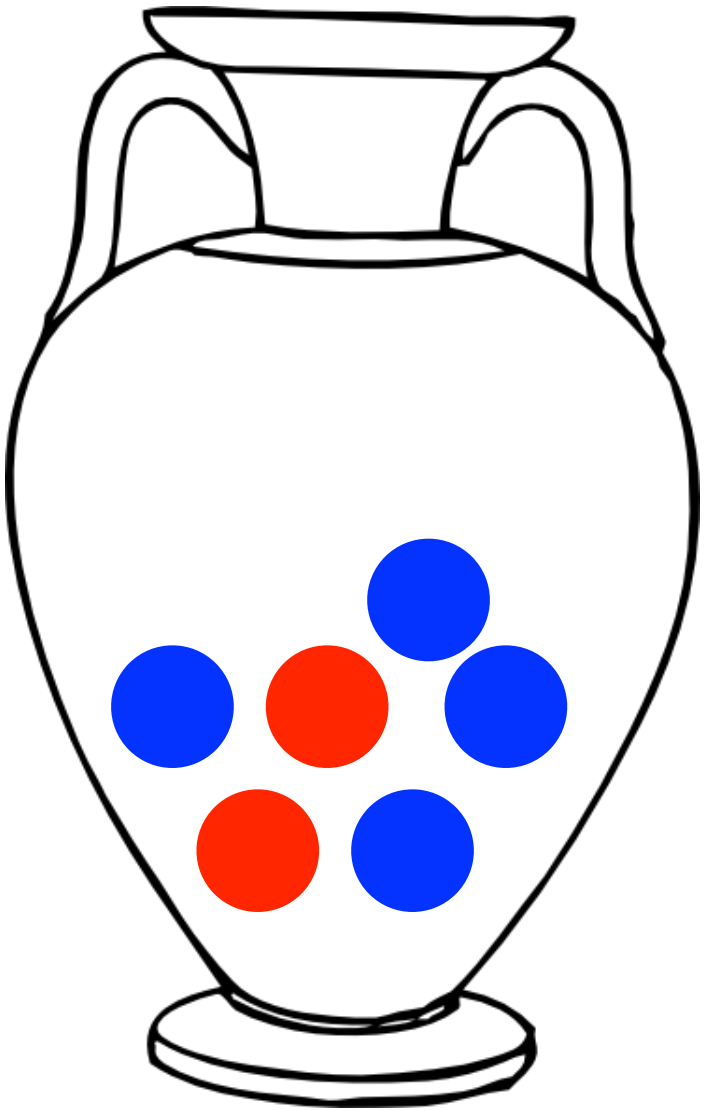
1. Start with two balls, of different colors.
2. Draw a ball from the urn.
3. Put the ball back into the urn *and add another ball of the same color.*
4. Iterate.

# Simplest form



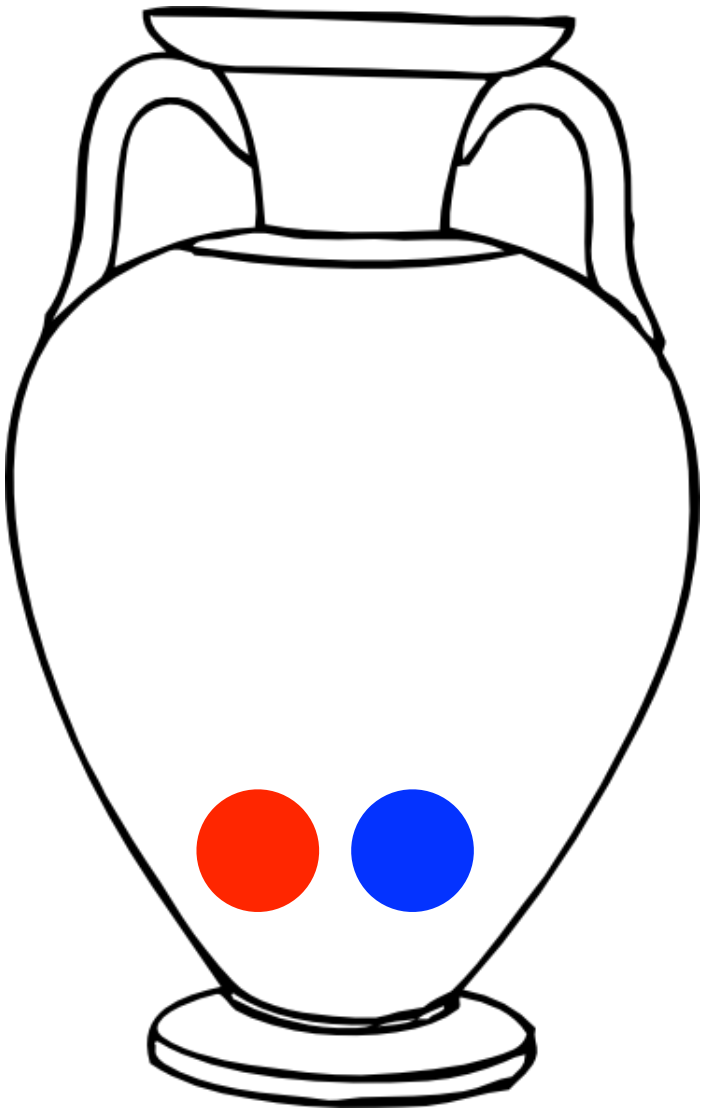
1. Start with two balls, of different colors.
2. Draw a ball from the urn.
3. Put the ball back into the urn *and add another ball of the same color.*
4. Iterate.

# Simplest form



1. Start with two balls, of different colors.
2. Draw a ball from the urn.
3. Put the ball back into the urn *and add another ball of the same color.*
4. Iterate.

# Full statement:



1. Start with two balls, of different colors.
2. While there are less than n balls in the urn, repeat the following:
    1. Draw a ball from the urn.
    2. Put the ball back into the urn *and add another ball of the same color.*

# Pseudocode for Polya Urn model -Version 1

[UrnPseudoCode.Rmd in Week4-UrnModels ]

```
---
title: "UrnPseudocode"
author: "Paul M"
date: "12/9/2020"
output: html_document
---


## Pseudocde for Urn problem (first part)

As ever, set the random number seed for reproducibility
```{r GoodHabits}
set.seed(16)
```

Define your variables
```{r}
# How many balls do we start with
InitialNumberOfBalls <- 2
# How many balls do we need eventually
TargetNumberOfBalls <- 10
```
```

# How to draw a ball...

```
# Here's how I declared my urn and set its initial state
Urn<-mat.or.vec(1,TargetNumberOfBalls)
# What is the initial state
Urn[1]<-"blue"
Urn[2]<-"red"


# choose the ball
WhichBall<-sample(1:NumberOfBalls,1) #NumberOfBalls is how many #balls
there are now, so it starts at 2 and increases by one each time

# what color is that ball?
WhatColorIsIt<-Urn[WhichBall]
# add another ball of the same color
Urn[NumberOfBalls+1]<-WhatColorIsIt
```

# Lab Task #1

- Working in groups if you prefer - code up the Urn model.
- Start with 1 red and one blue ball
- Continue until you have 50 balls
  1. What is the expected number of red balls at the end? (Intuition?)
  2. What is the distribution of the number of red balls? (Plot a histogram - what properties do you think the distribution will have?)
- Suggest: Use **at least** 10000 replicates.
- We will then look at the results.
  - Empirical estimate of expected value of a distribution is just the mean of the observed values (i.e. mean number of red balls across the 10000(say) reps.)
  - Empirical estimate of a distribution is the histogram.

# Function to count the red balls

```
CountBalls<-function(Urn,ColorToCount)
{
    # works with the urn "Urn", and returns the number of balls of color ColorToCount
    # we use the fact that the function "==" will be applied to each element of the
     vector Urn
    return (sum(Urn==ColorToCount))
}
```

# Lab Task #2

- Code up the Urn model
- Start with any number, n, of red and blue balls
- Assume n-1 of the starting balls are red (and so 1 is blue).
- Continue until you have 50 balls

1. What is the expected number of red balls at the end, as a function of the number we start with? (Draw a plot of mean # of reds, versus #reds at start). (Also draw a plot of mean #reds versus proportion of reds at start.)

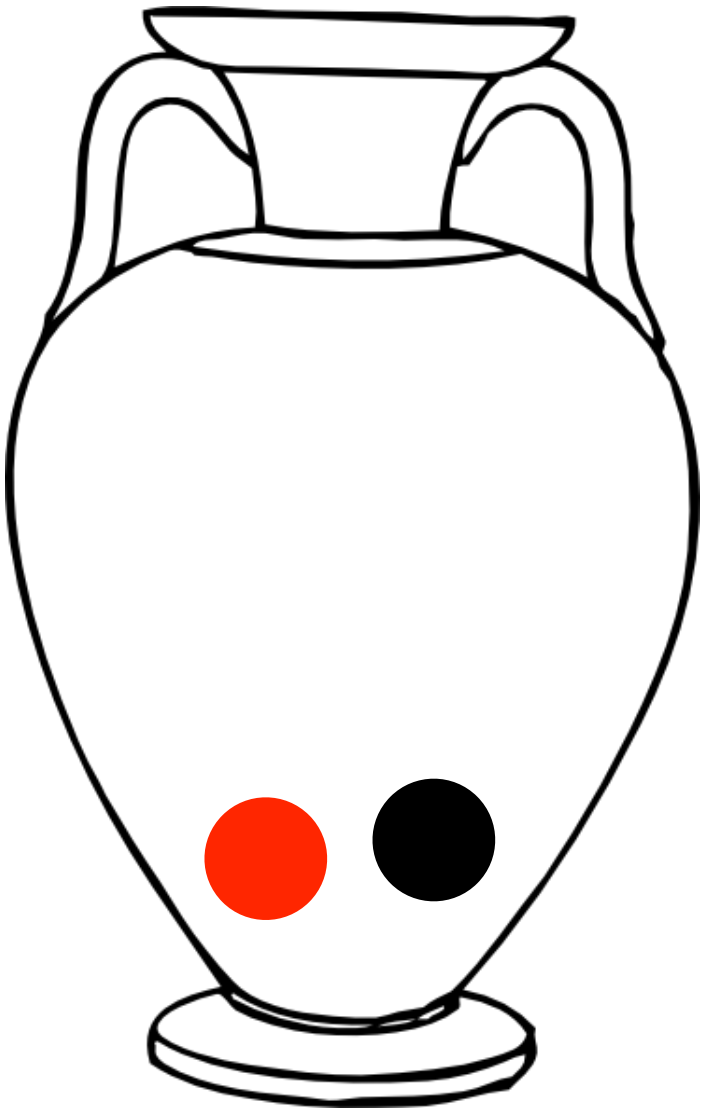
2. Is this described by a simple relationship?

# Lab Task #3 - Generalized Urn

- Use the urn model in which we start with any number of balls, and in which when we remove a ball, instead of returning it and adding one more ball of the same color, we return it and add B balls of the same color, for some B.
- Use two starting balls (one red, one blue).
- Try B=1,2,3,4,5,...10, and drawing until you have at least 100 balls each time.
- What does this do to:

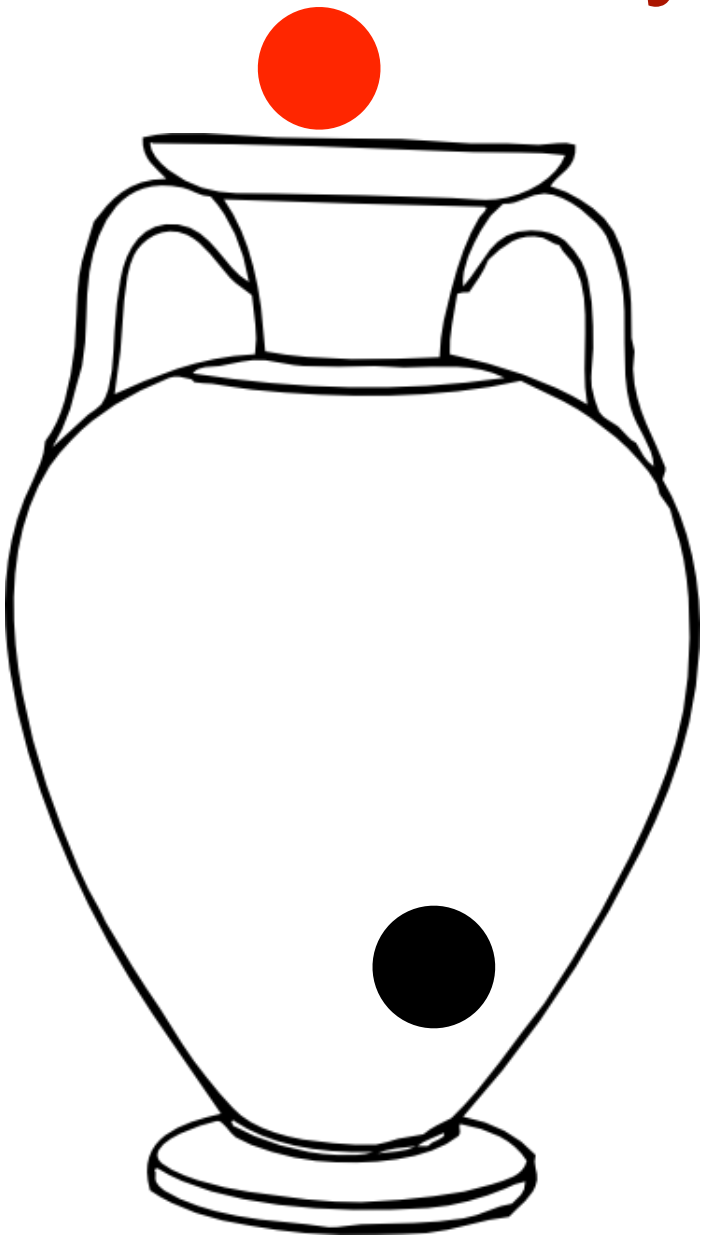
1. The Expected number/fraction of red (say) balls at the end?
2. The distribution of the number/fraction of red balls at the end?

# The Polya Urn version 2



Add a black (special color) ball.
Start with two balls, one of which is black.

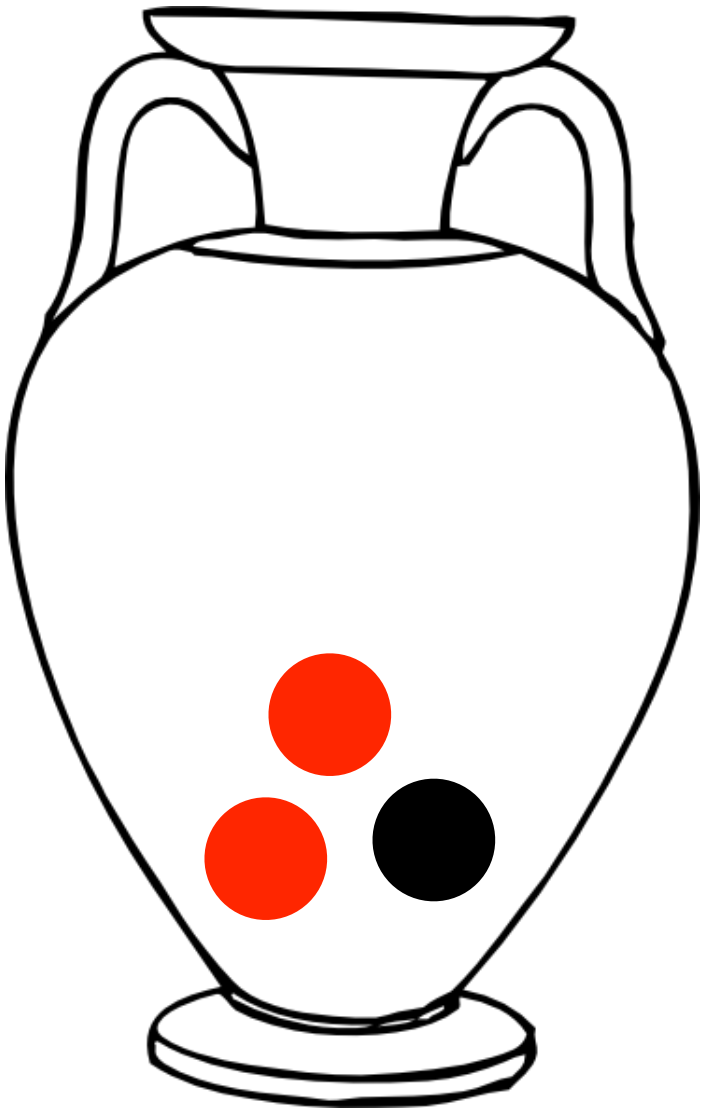
# The Polya Urn version 2



Add a black (special color) ball.

If you draw a non-black ball, proceed as before (add an additional ball of the same color)

# The Polya Urn version 2



Add a black (special color) ball.

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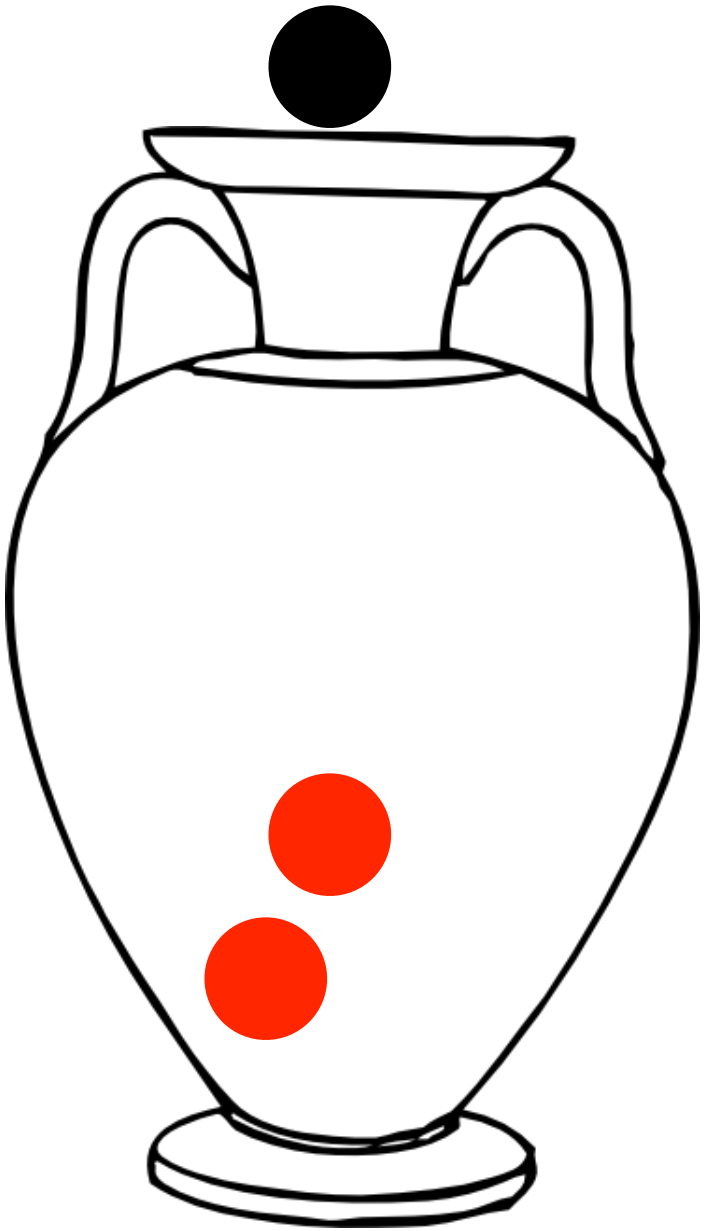
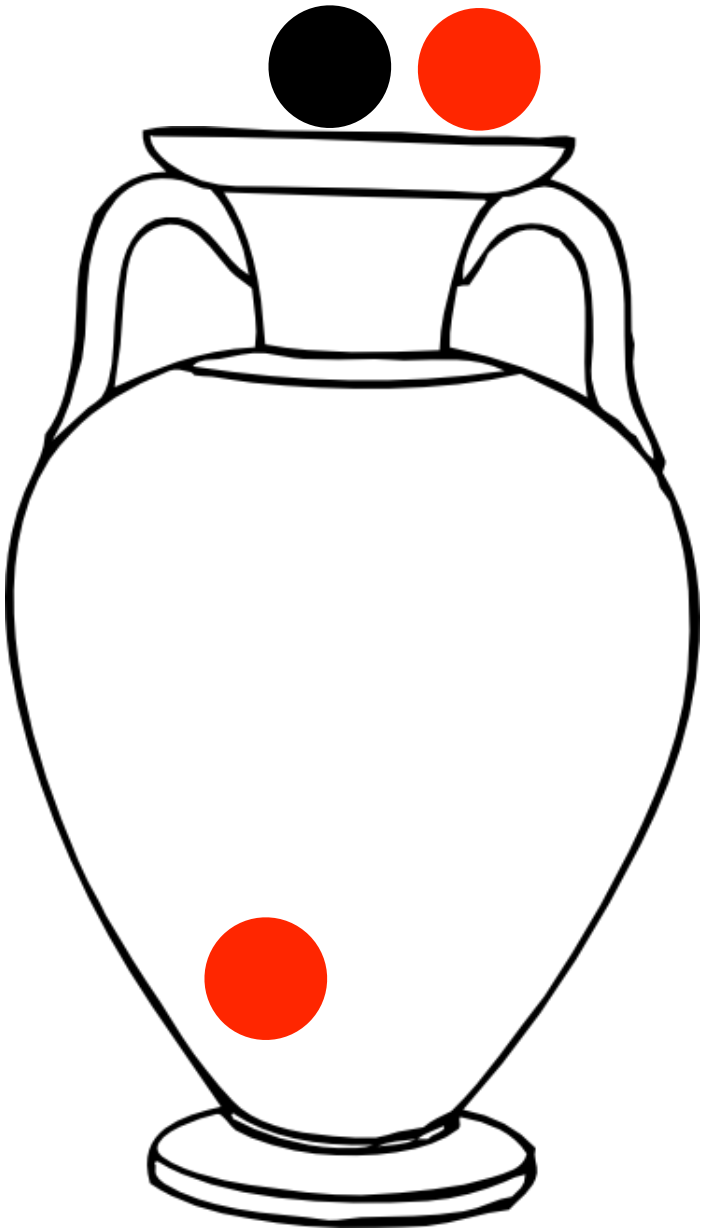
# The Polya Urn version 2



Add a black (special color) ball.

If you draw a black ball do the following:
1.Keep the black ball for now.

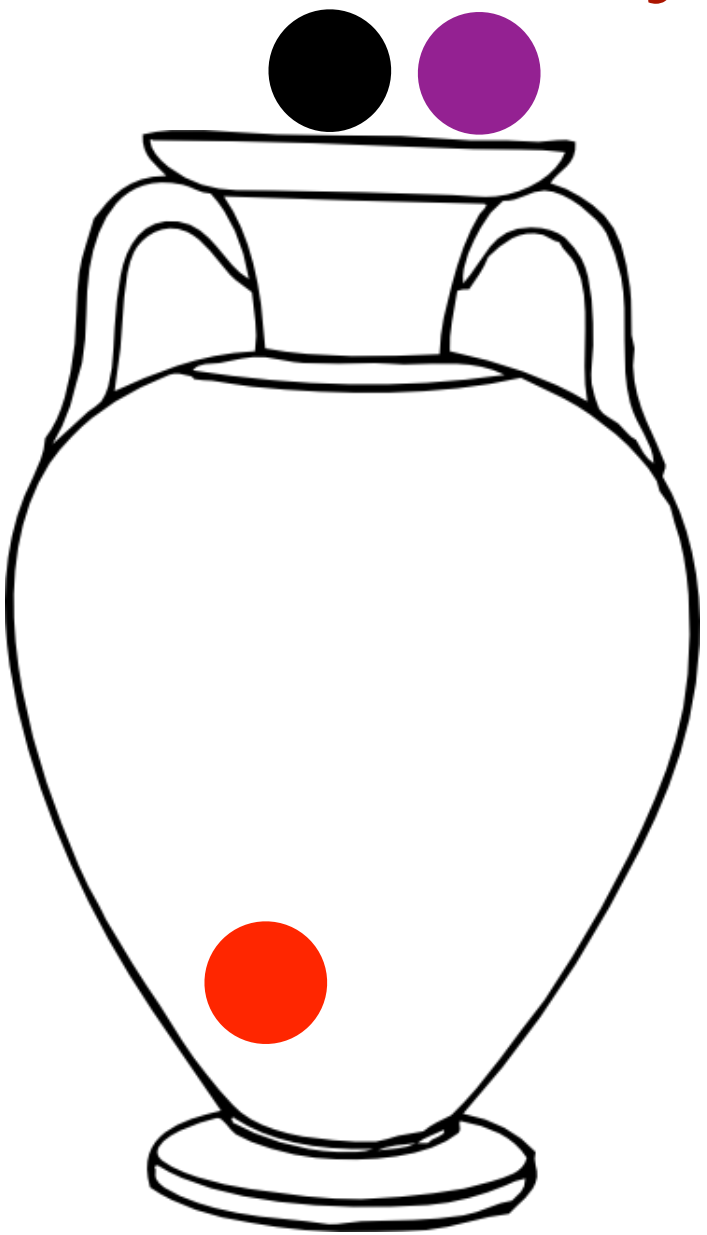
# The Polya Urn version 2



Add a black (special color) ball.

If you draw a black ball do the following:
1.Keep the black ball for now.
2.Draw another ball from the urn

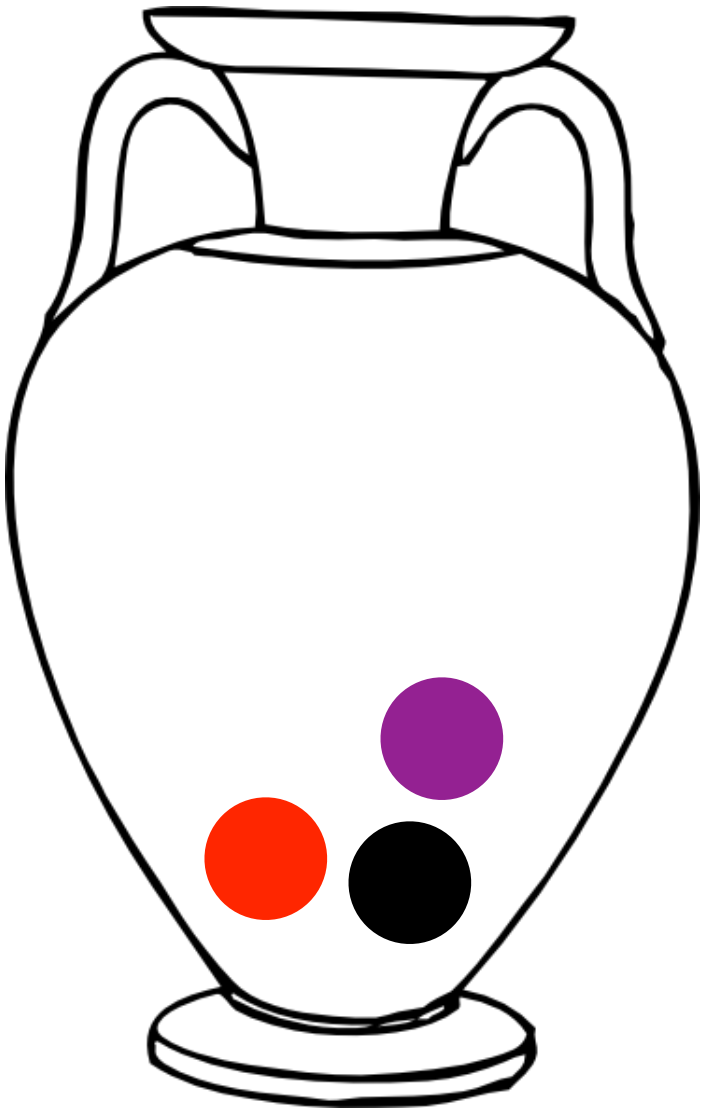
# The Polya Urn version 2



Add a black (special color) ball.

If you draw a black ball do the following:
1.Keep the black ball for now.
2.Draw another ball from the urn.
3.Change the color of the second ball, making it a color that is not currently in the urn.

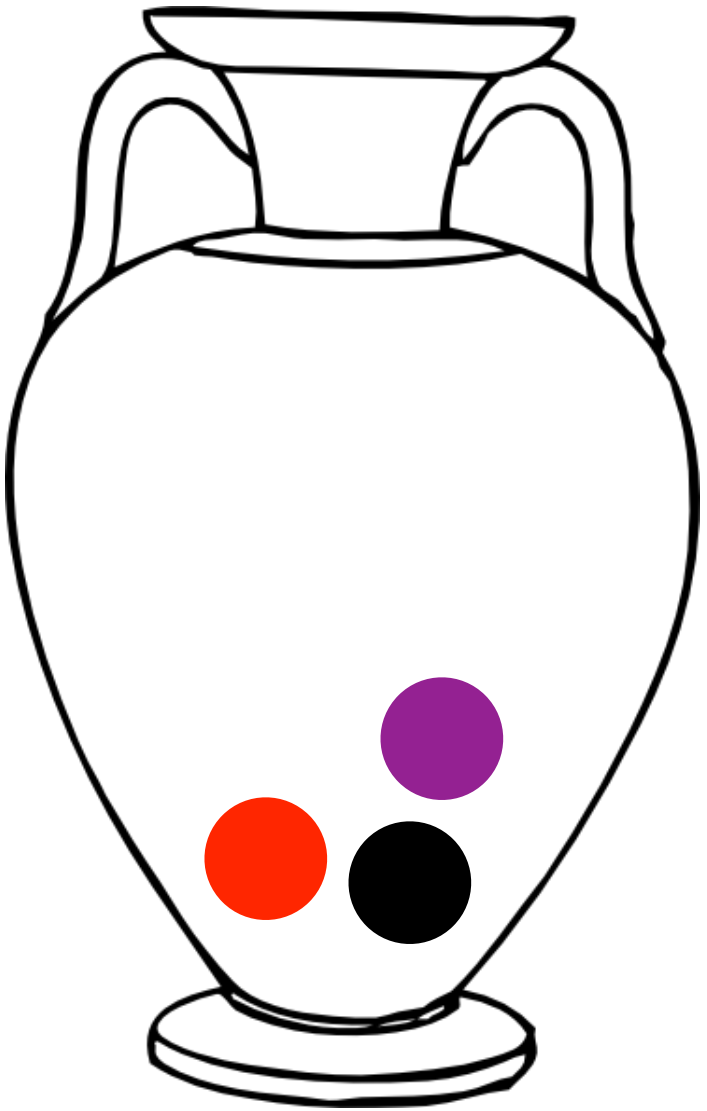
# The Polya Urn version 2



Add a black (special color) ball.

If you draw a black ball do the following:
1.Keep the black ball for now.
2.Draw another ball from the urn.
3.Change the color of the second ball, making it a color that is not currently in the urn.
4.Put both balls back in the urn.

# The Polya Urn version 2

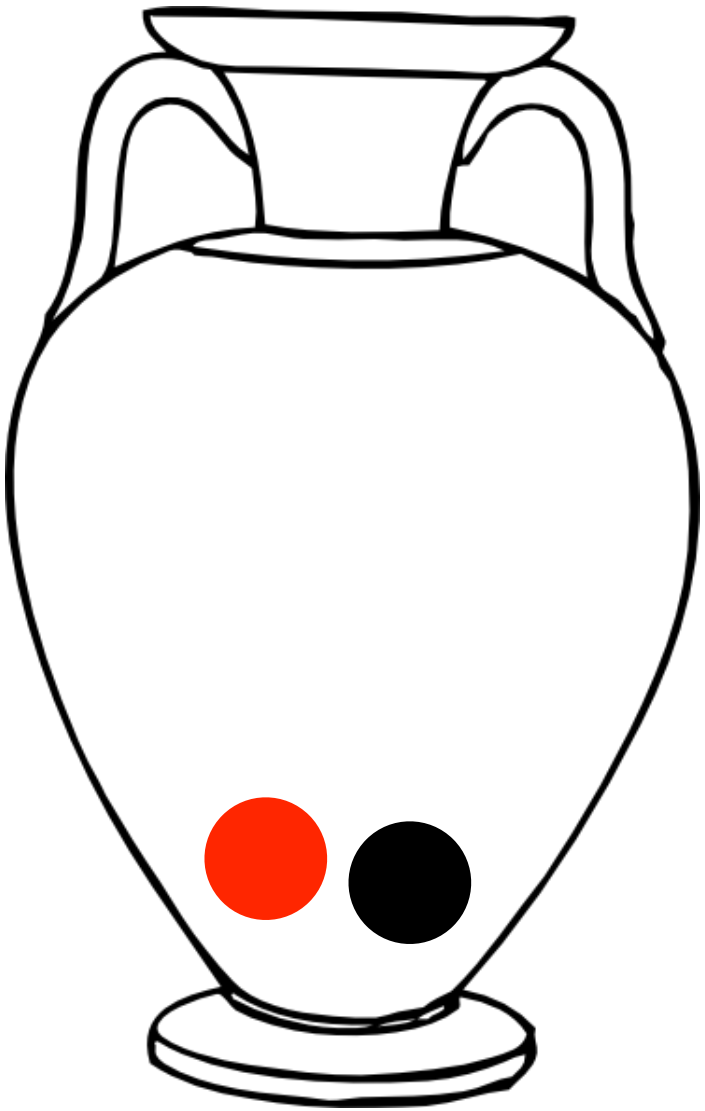


Add a black (special color) ball.

**So:**
black ball -> draw another ball, change its color, return both balls to the urn.
non-black ball -> return it to the urn along with another ball of the same color.

**Notes:**
The number of black balls never changes.
The number of iterations you need to reach N (non-black) balls in the urn is a random variable.
Colors can disappear.

# The Polya Urn version 2 - example



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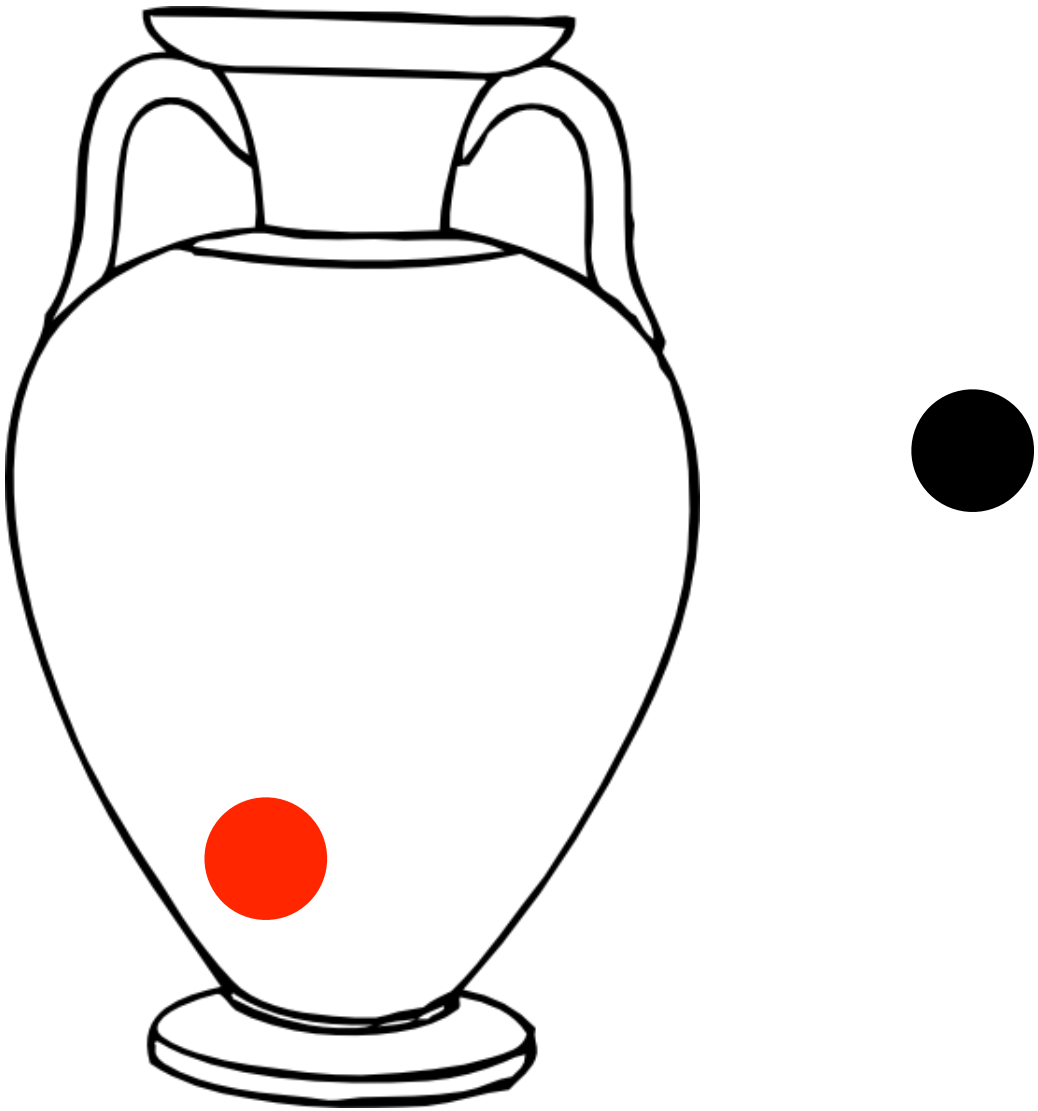
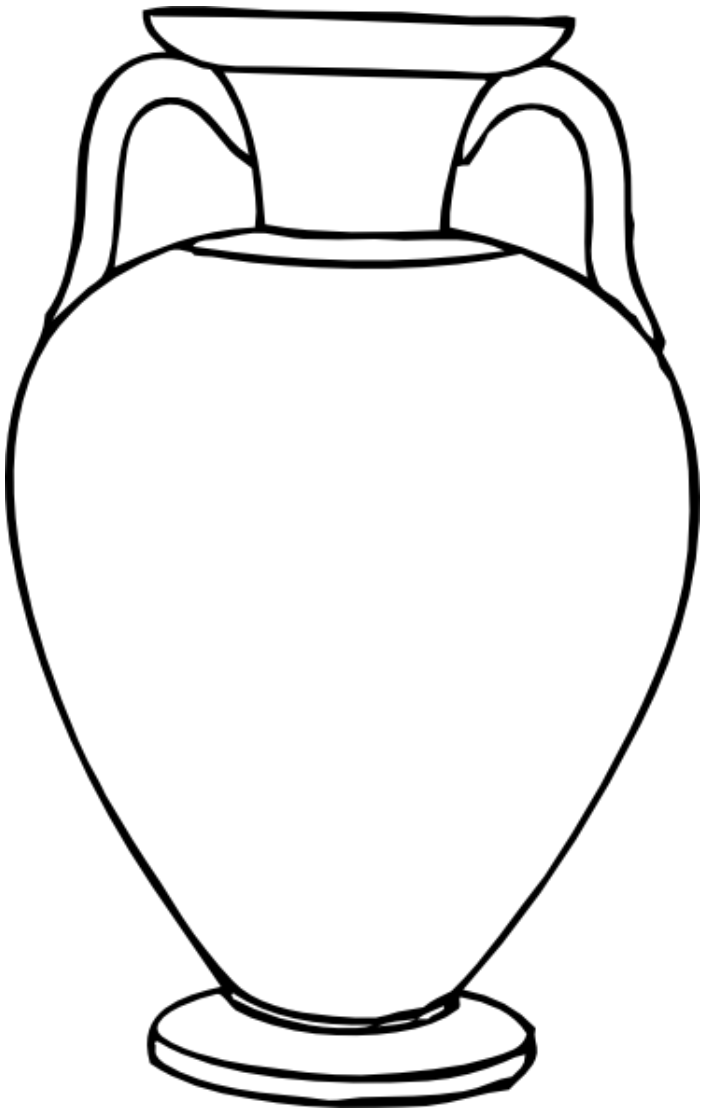
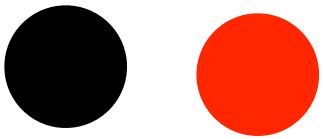
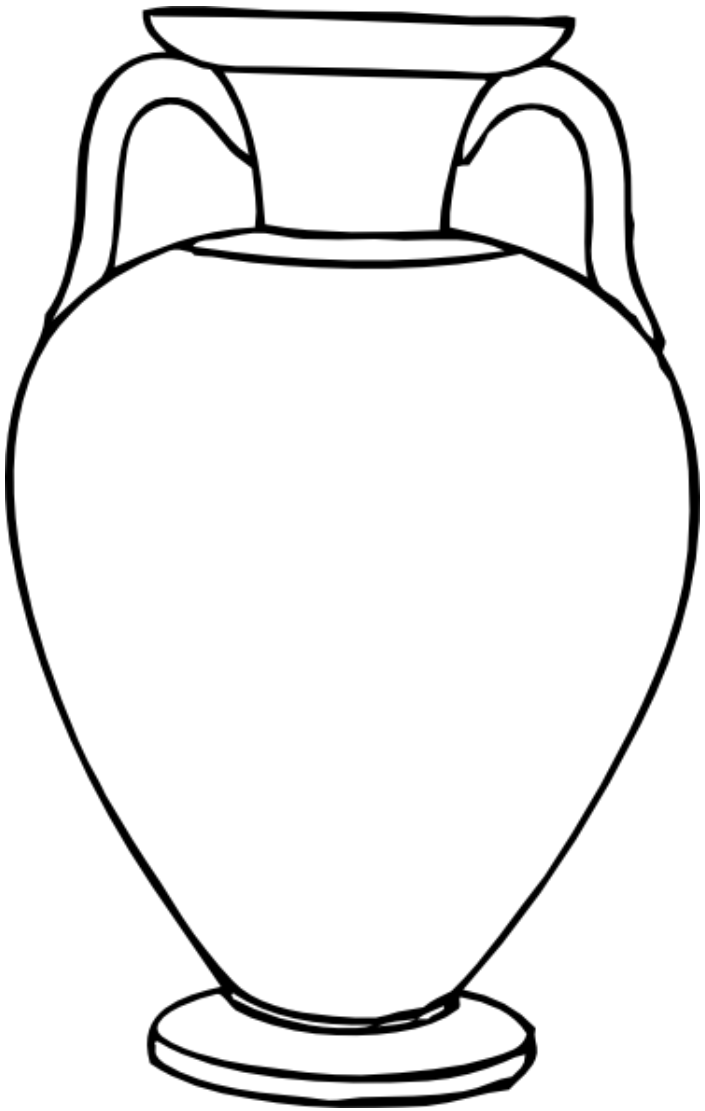


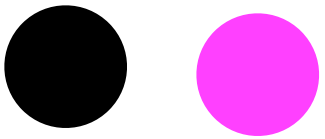
# The Polya Urn version 2 - example





# The Polya Urn version 2 - example





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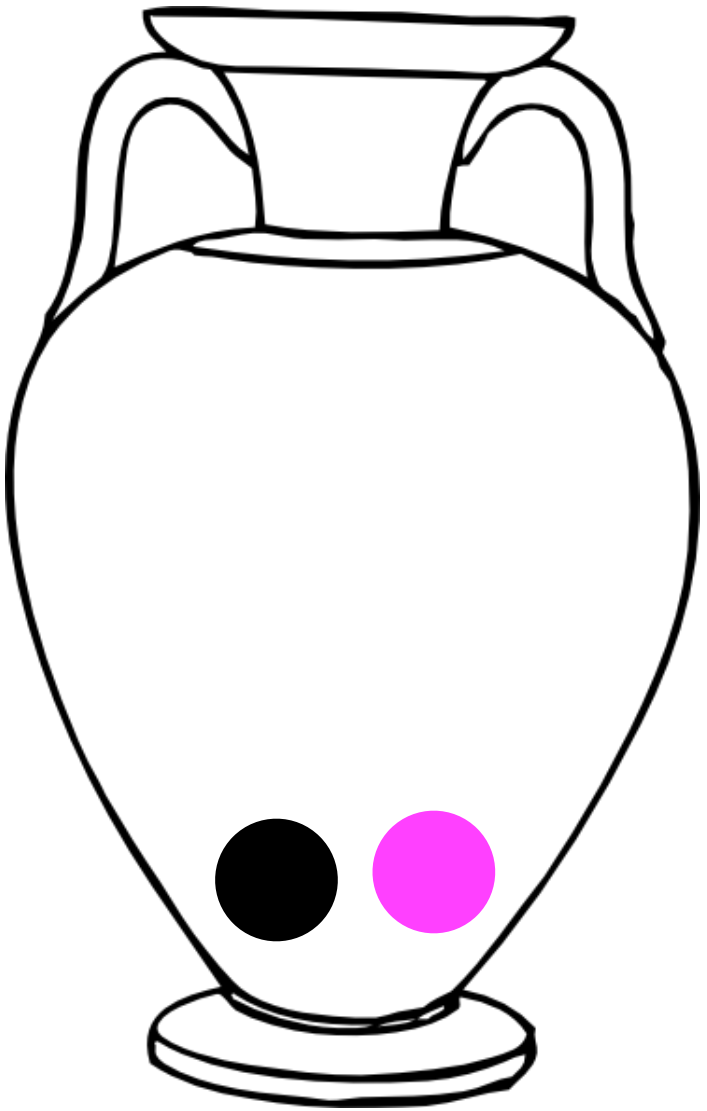

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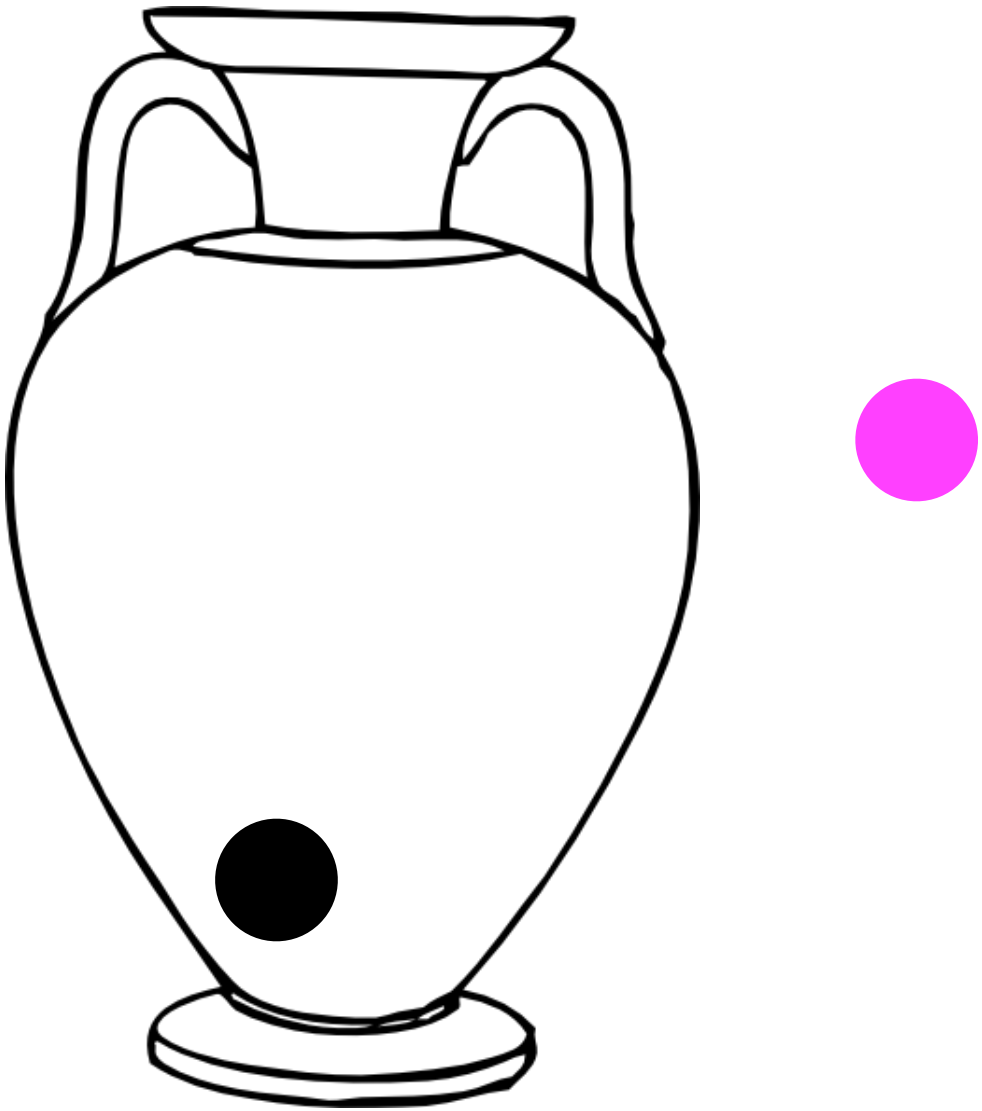

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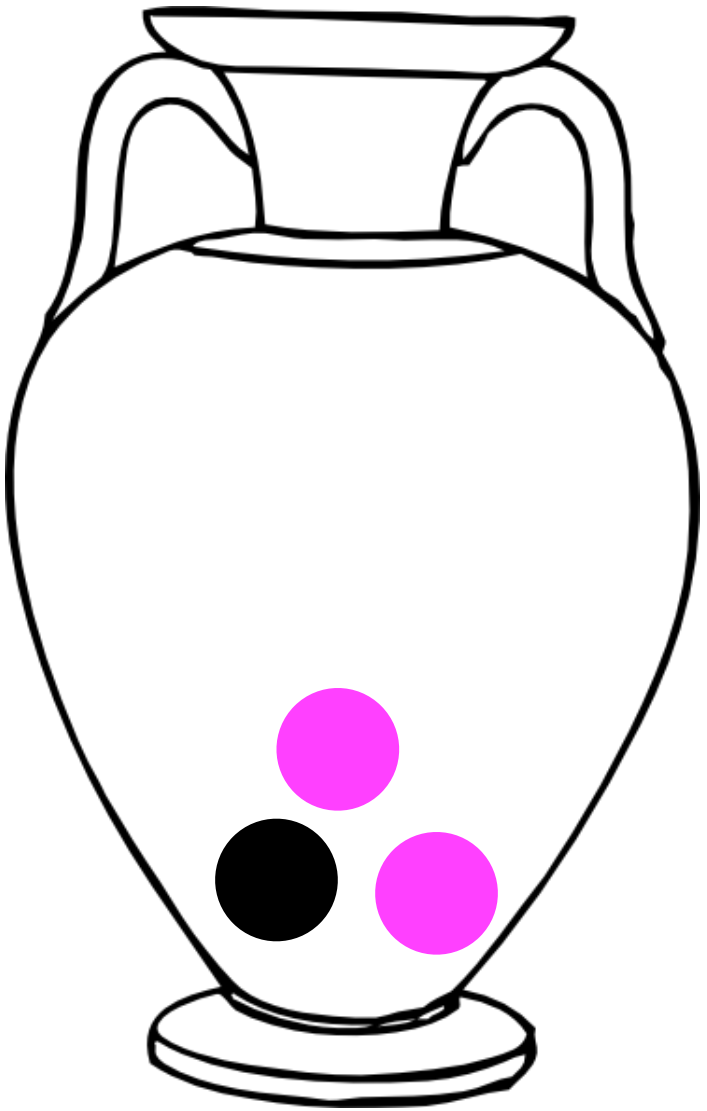

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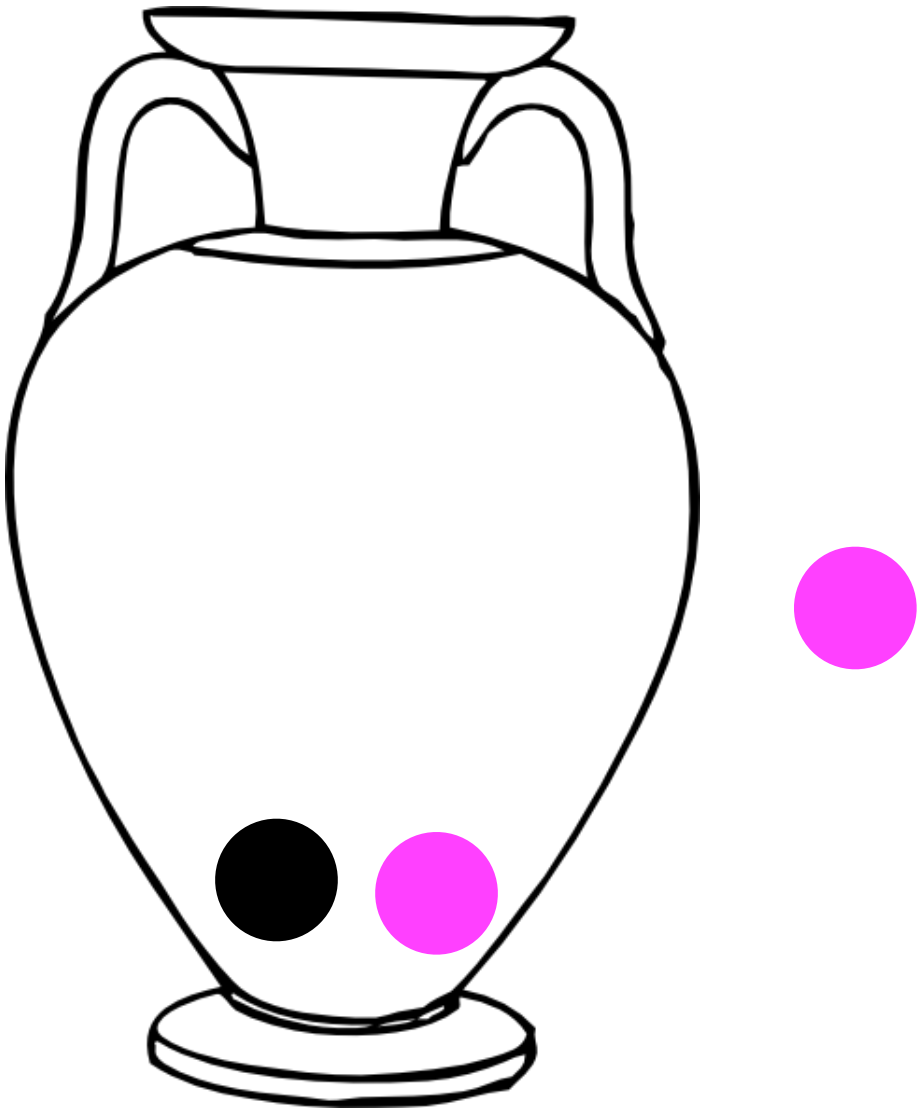
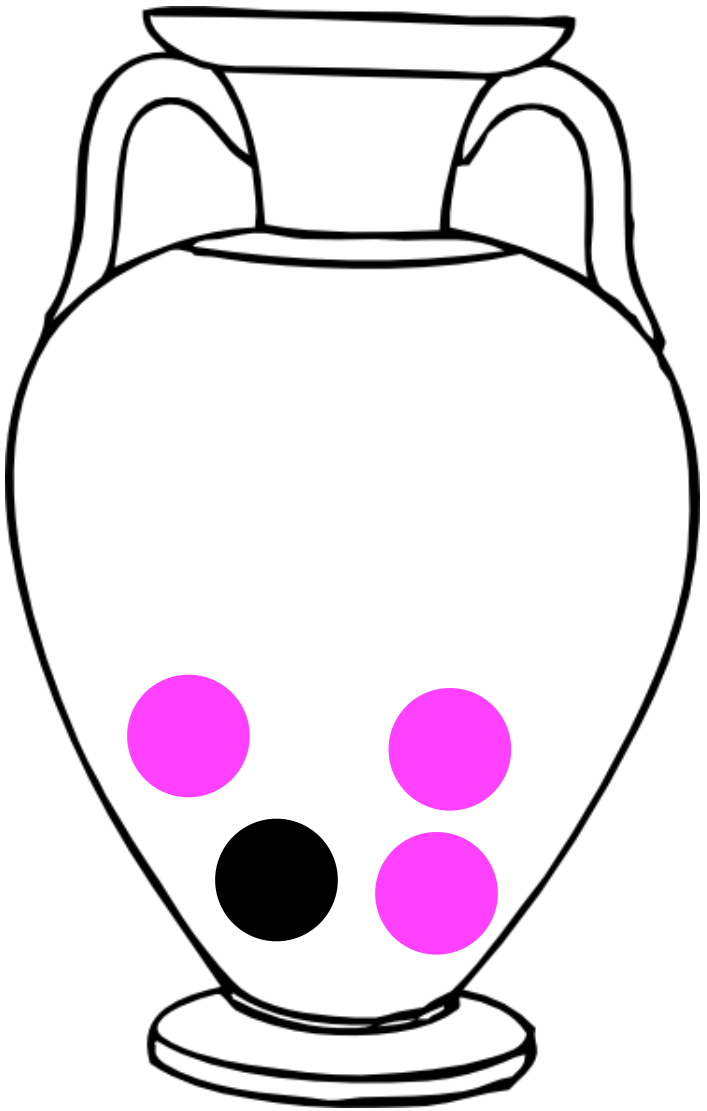
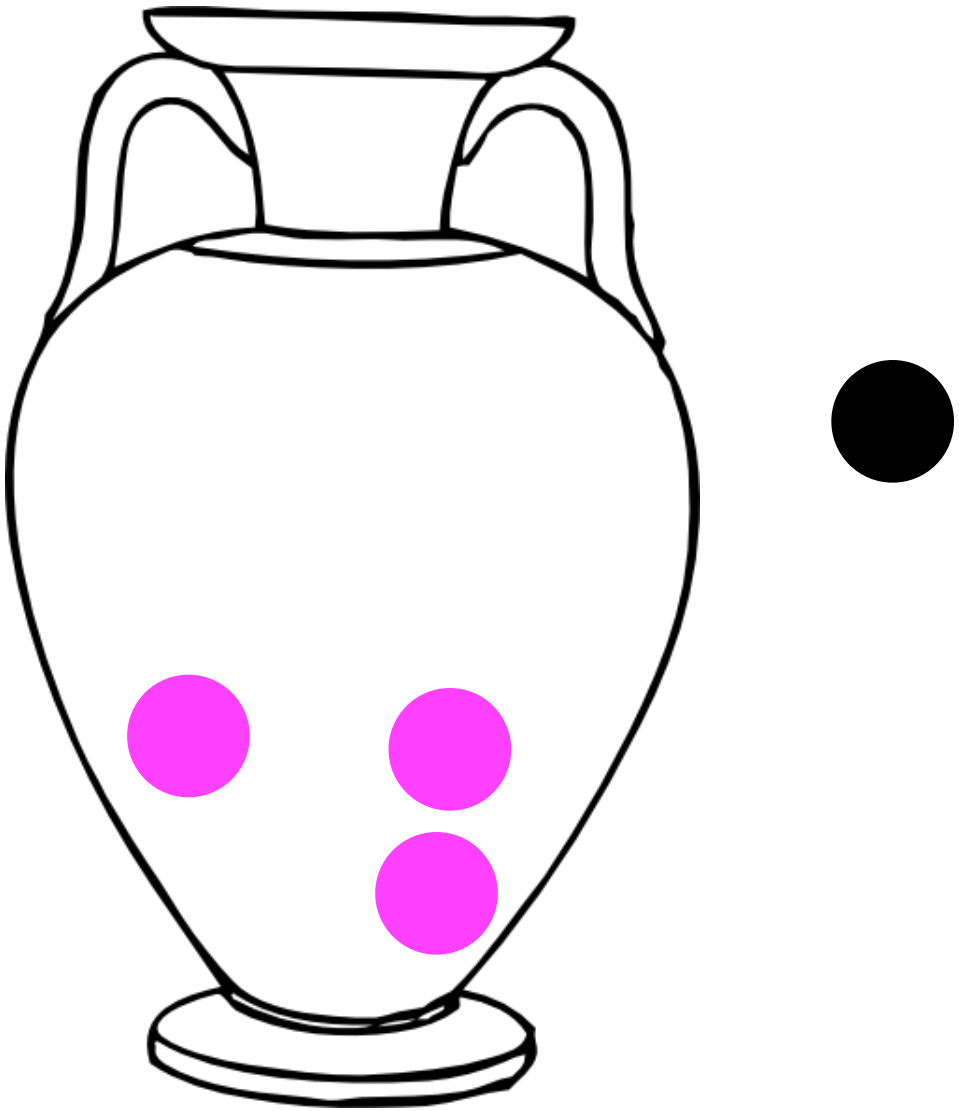


# The Polya Urn version 2 - example



# The Polya Urn version 2 - example



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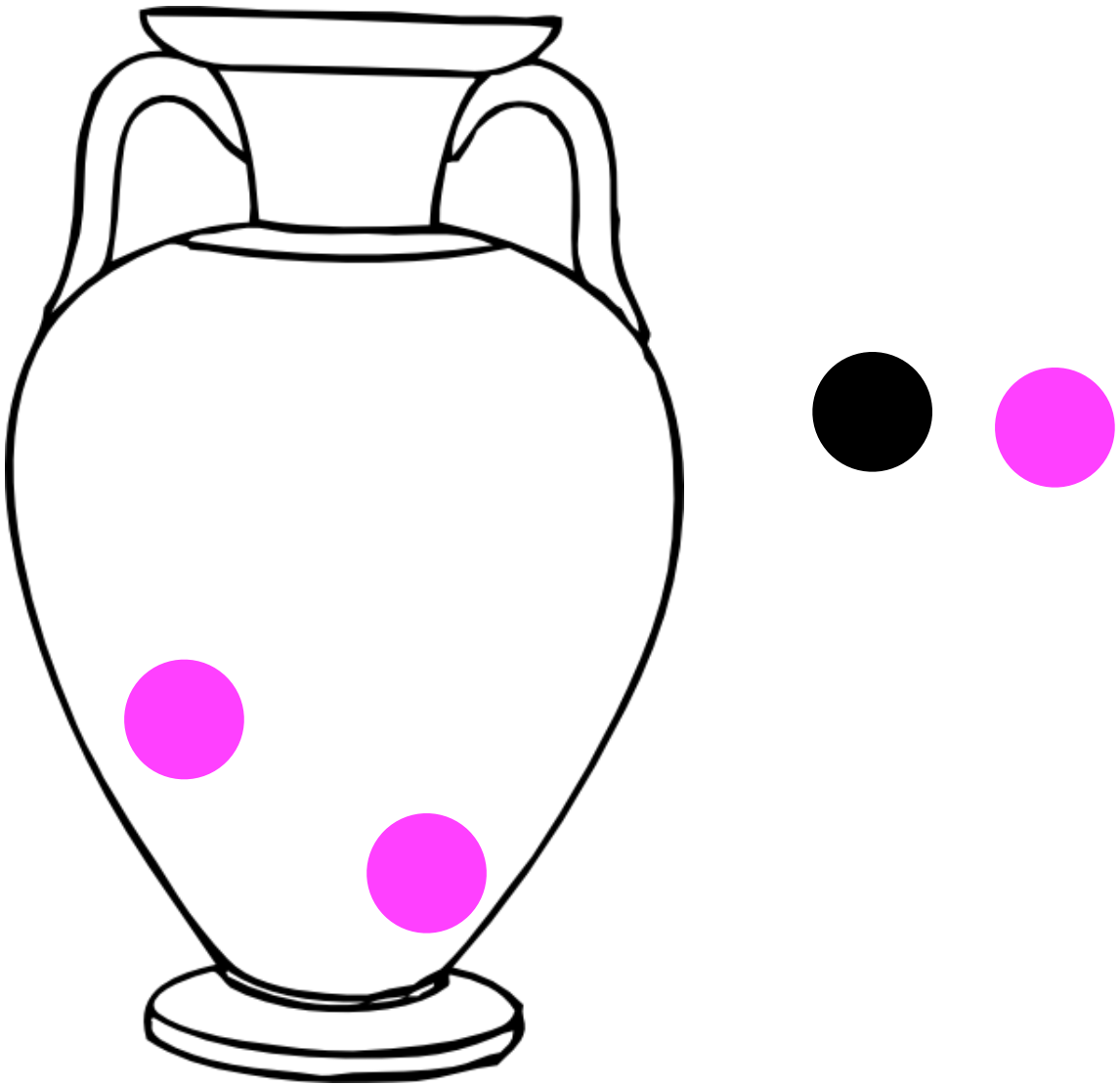

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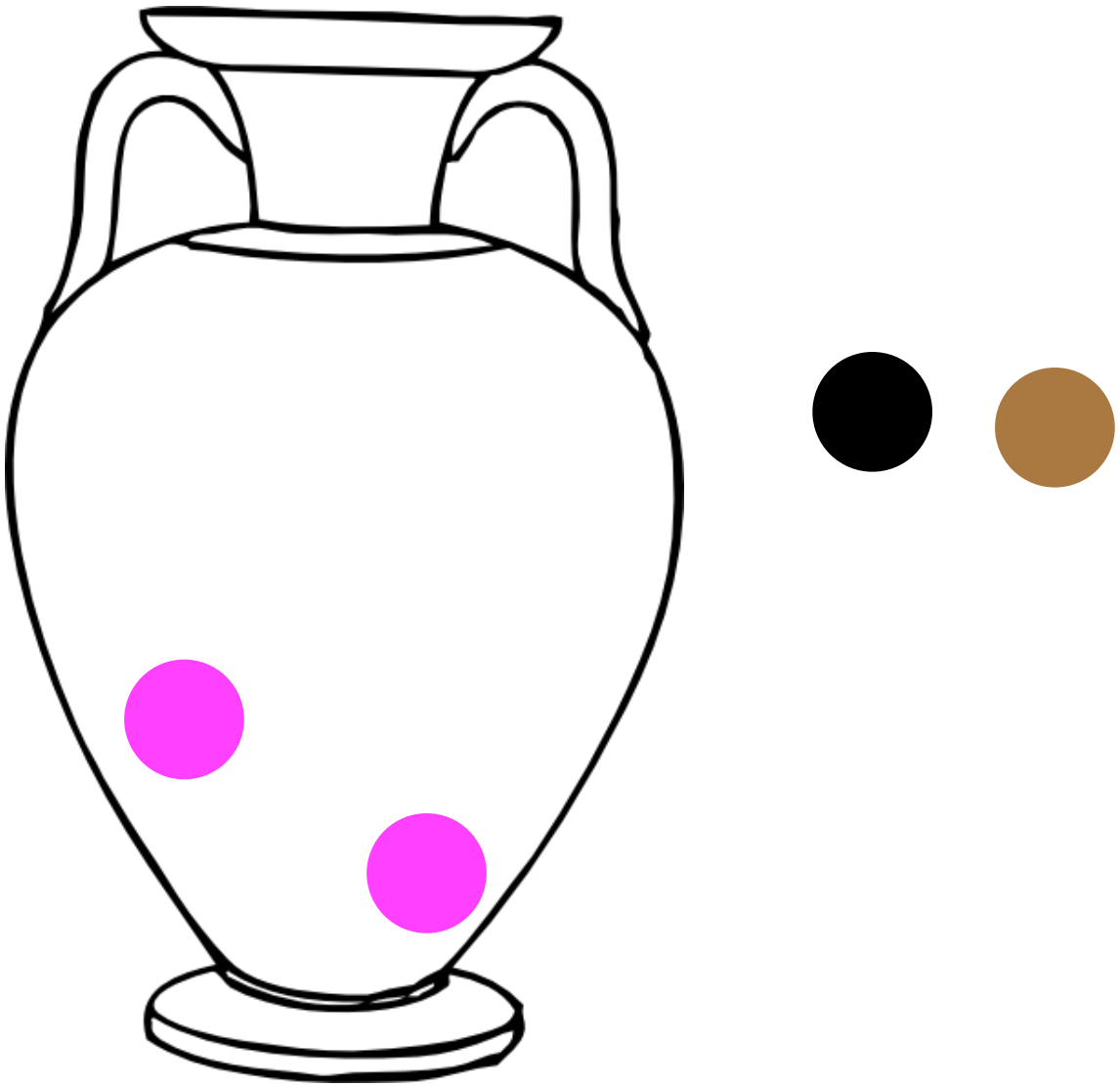
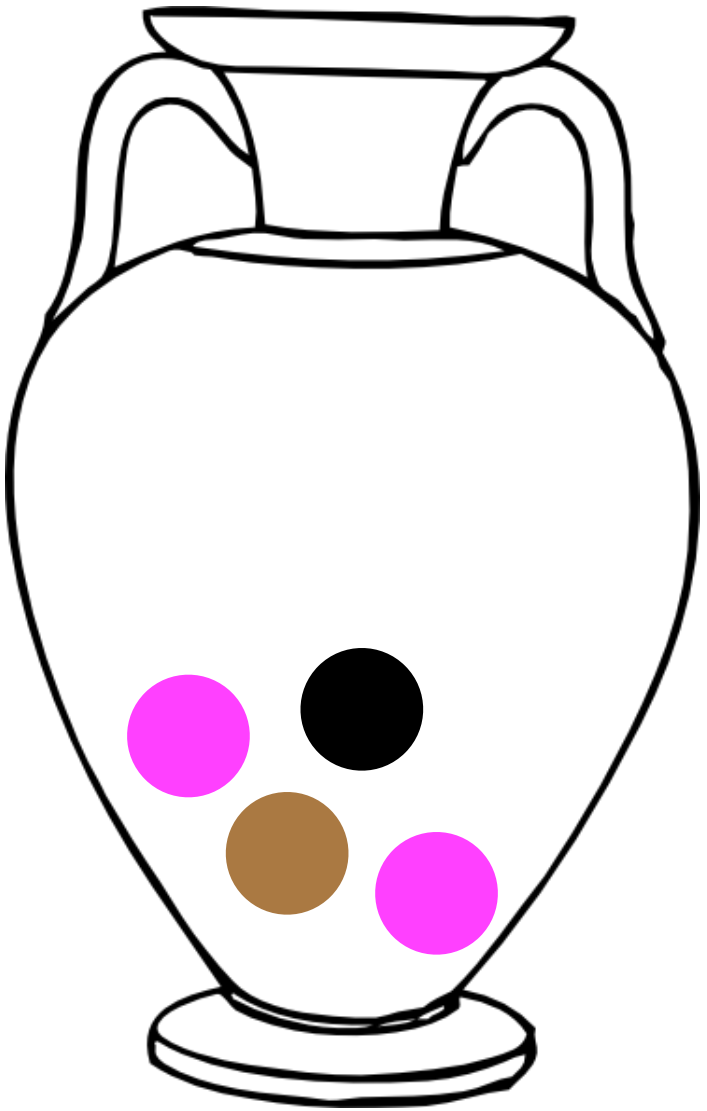
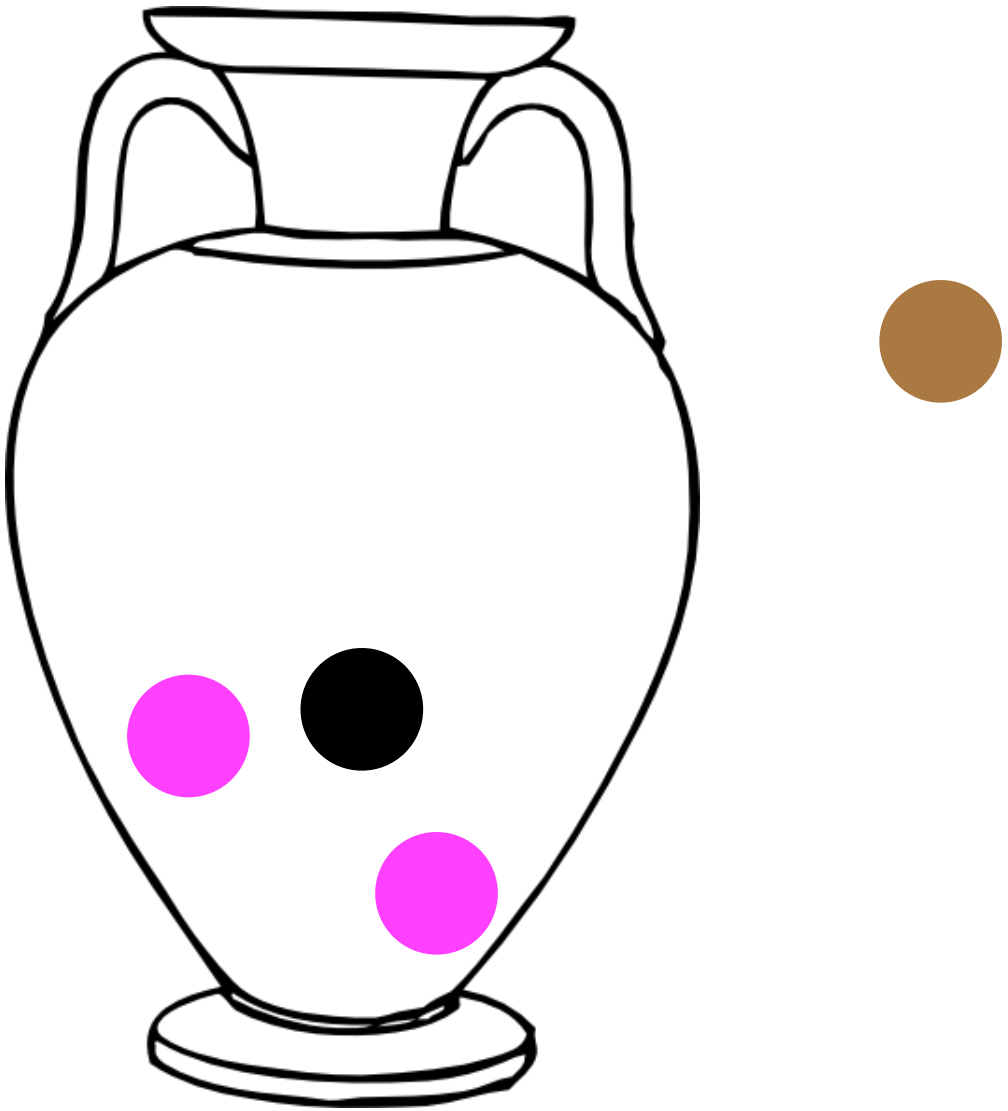


# The Polya Urn version 2 - example



# The Polya Urn version 2 - example



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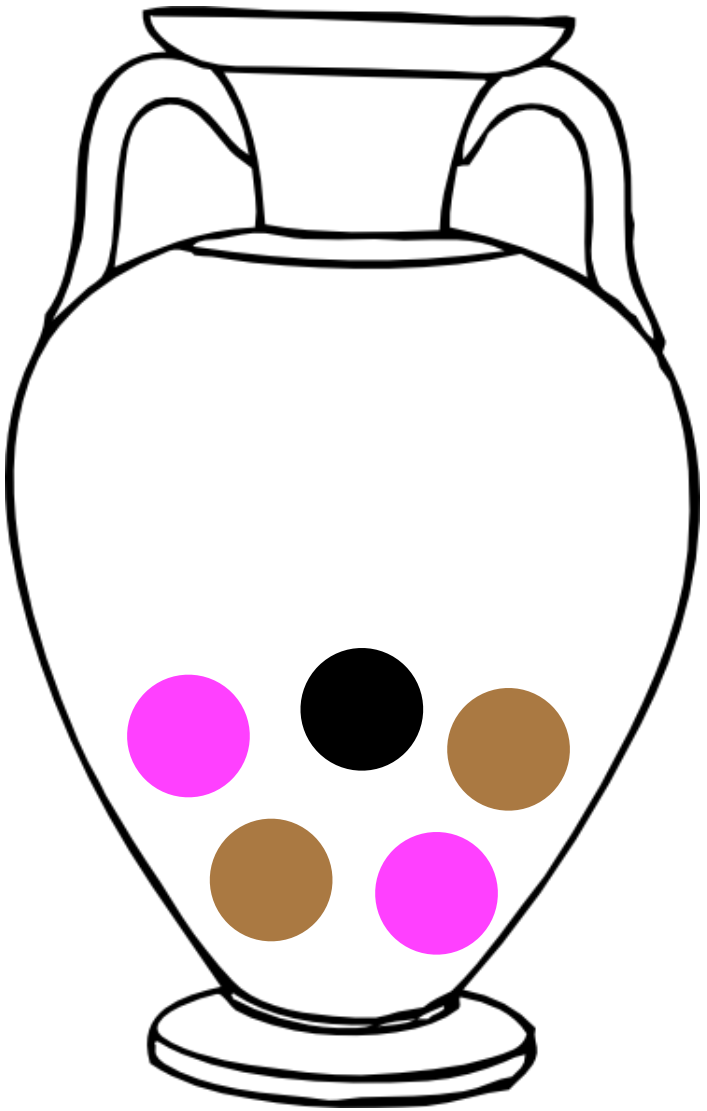
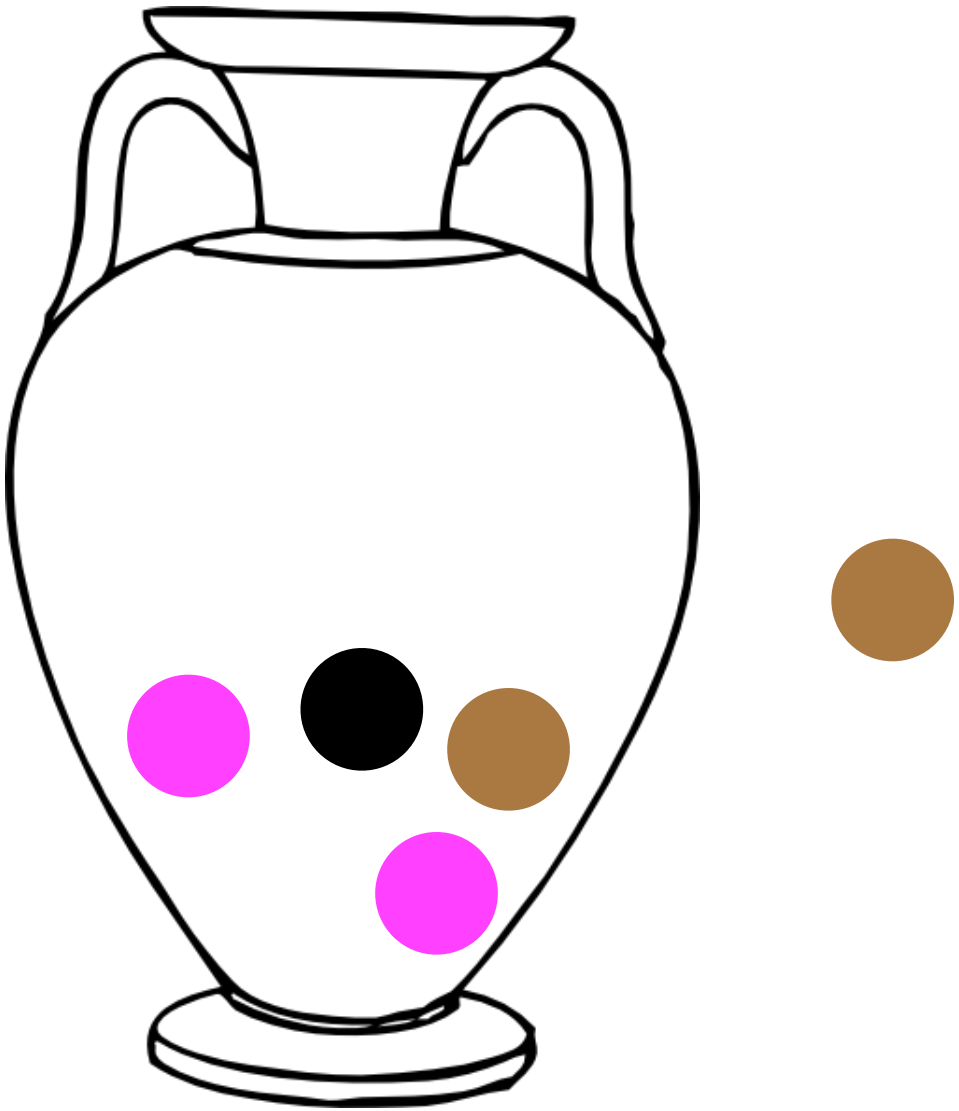
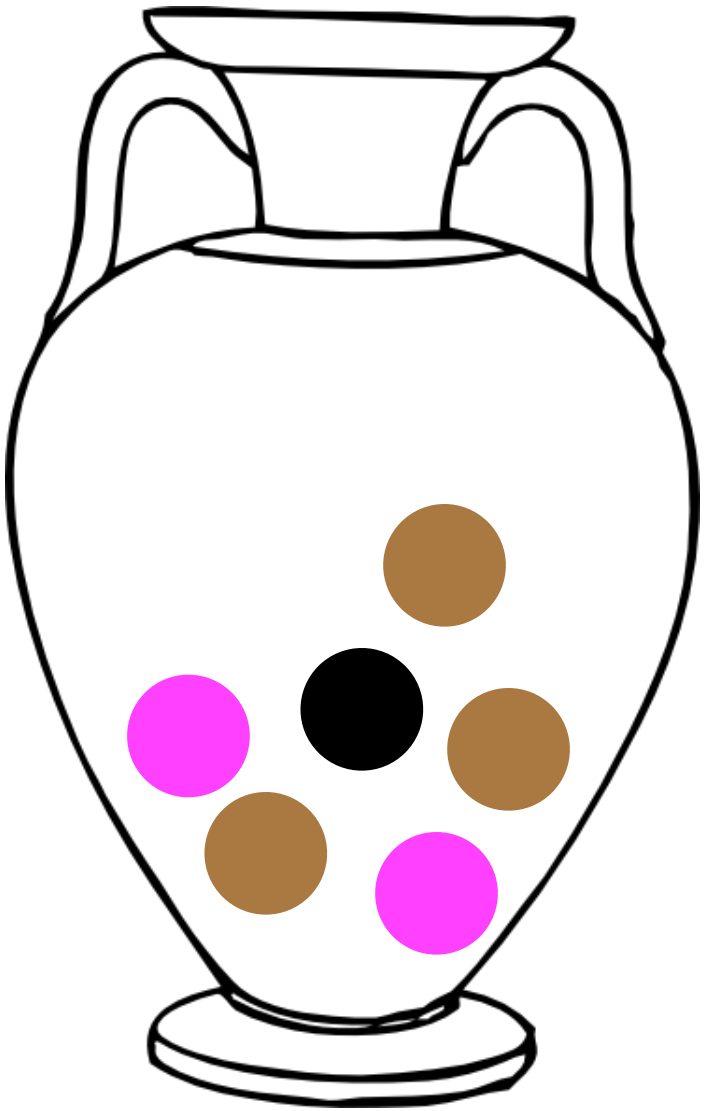


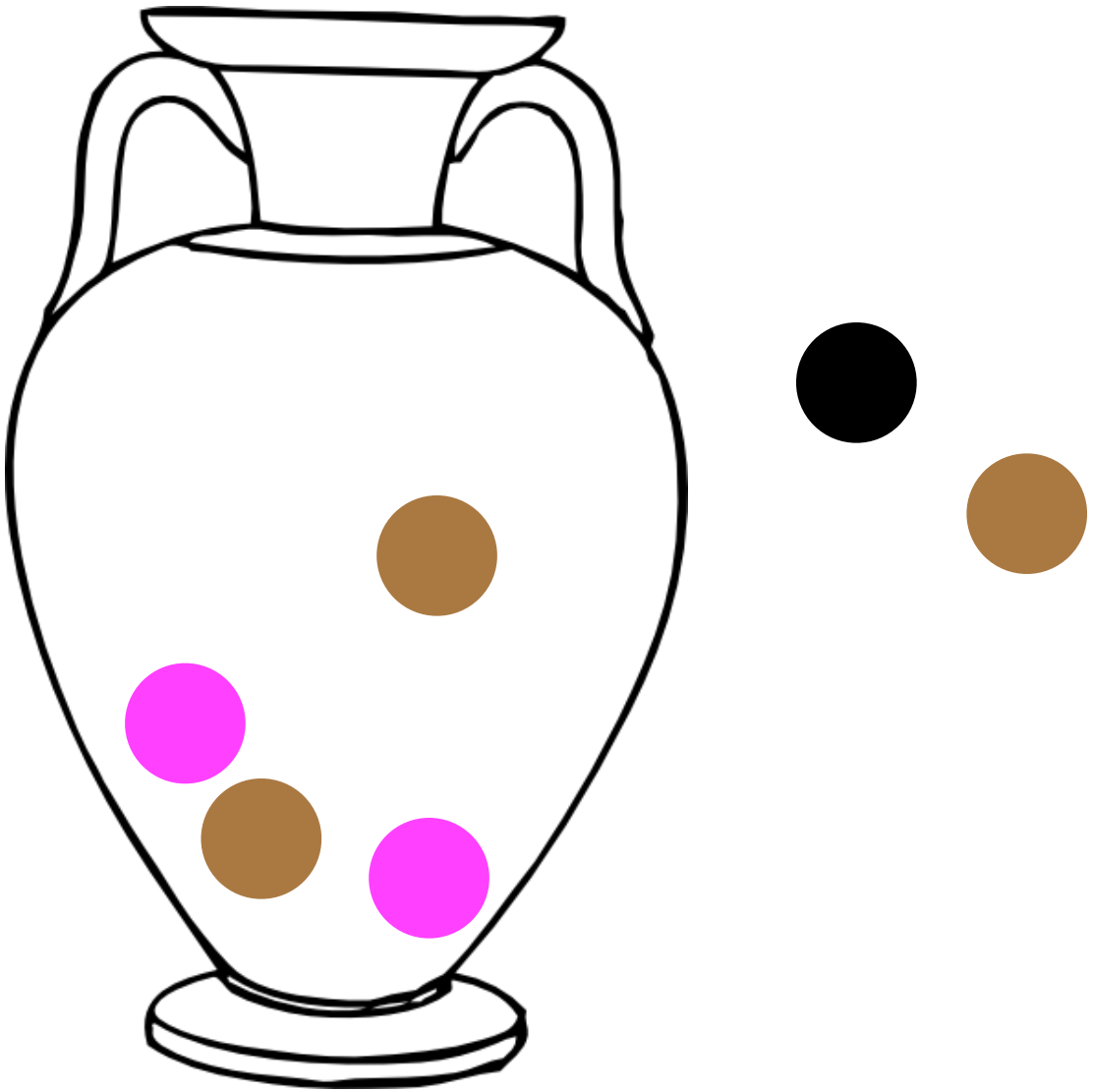
# The Polya Urn version 2 - example



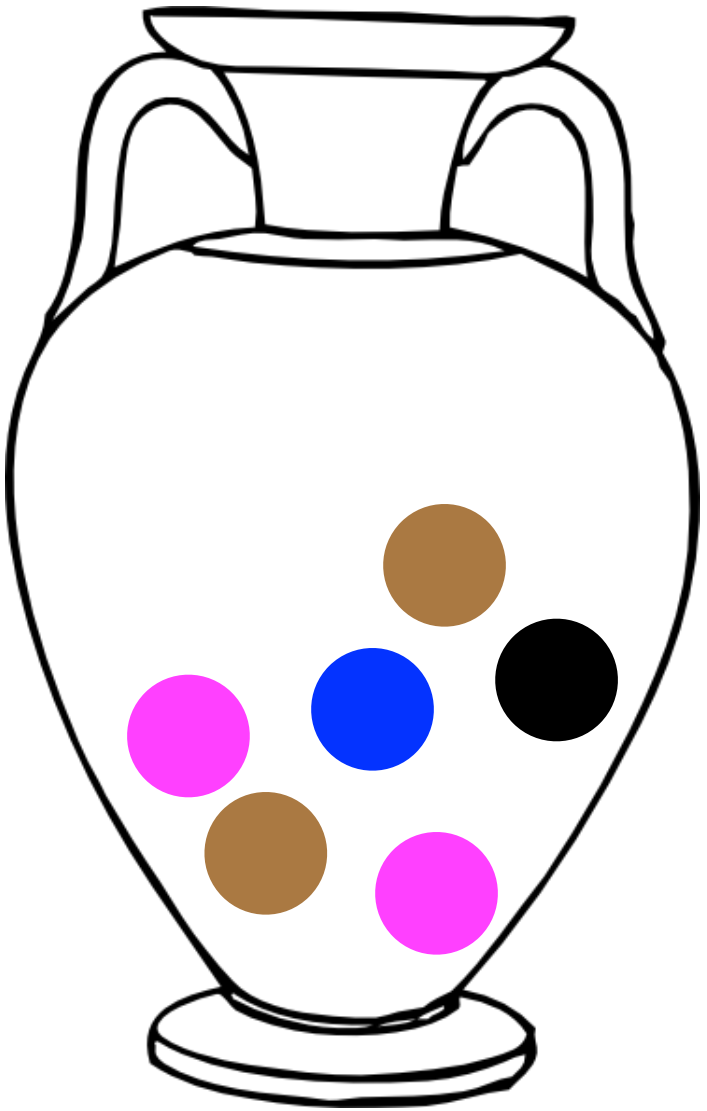
# The Polya Urn version 2 - example



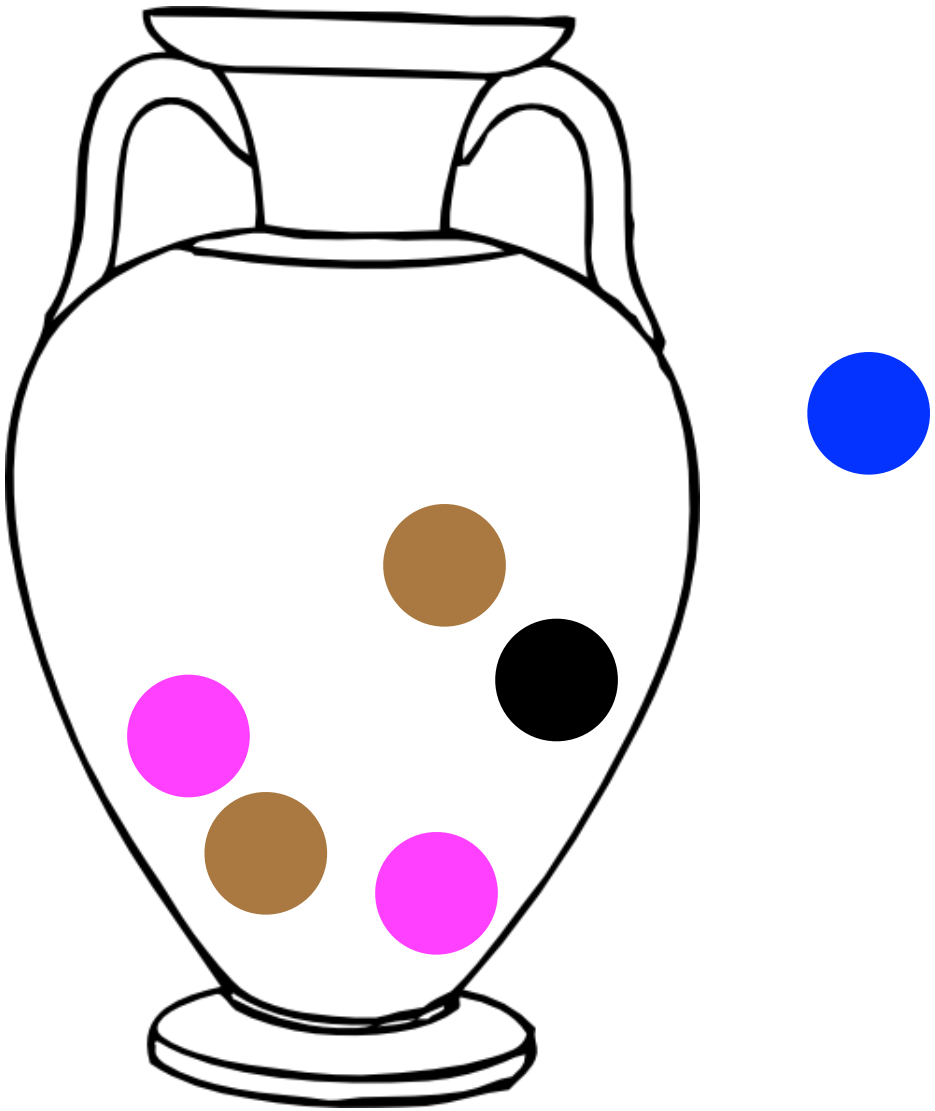
# The Polya Urn version 2 - example



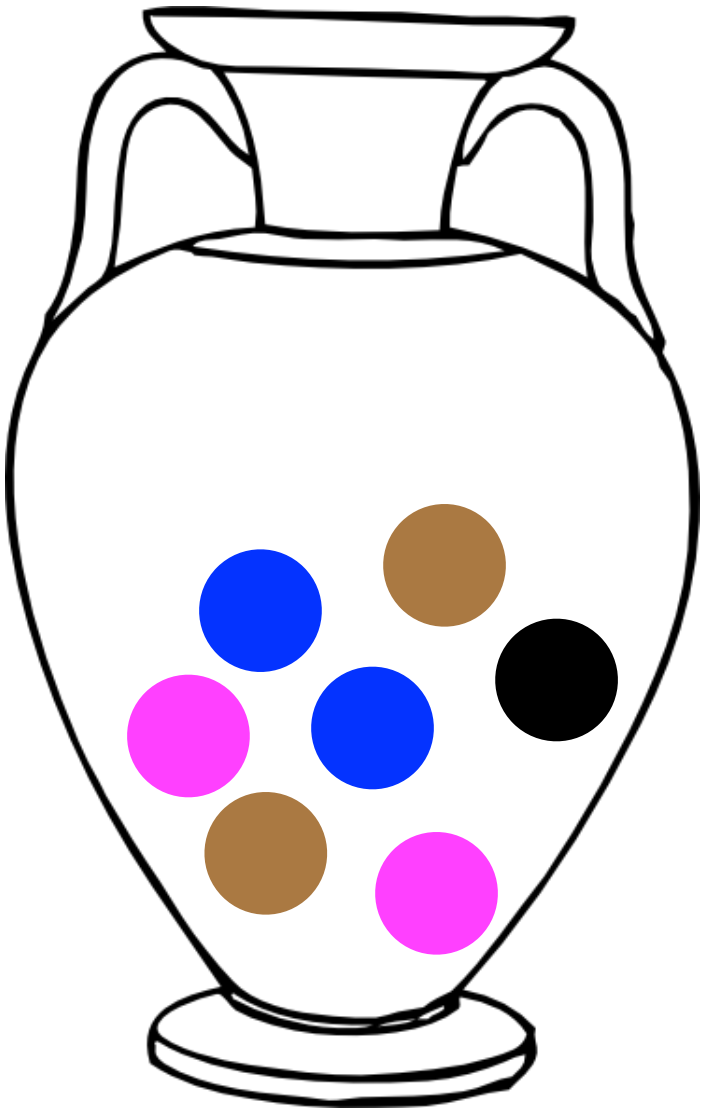
# The Polya Urn version 2 - example



# The Polya Urn version 2 - example



# The Polya Urn version 2 - example



If we want to generate a sample of N non-black balls, we will do it two, slightly different ways:

1. Stop as soon as you have N non-black balls.

In your R-script, how are you going to represent the balls?

# Pseudo-code (UrnWithBlackBall_Pseudocode.Rmd)

**## Pseudcode for the version of the Urn in which there is a "black" ball**

The idea here is to build it on your existing code (i.e., define variables in mostly the same way, etc.)
But then you will need to change the function for drawing a ball...
Since you don't know how many colors you will eventually need, it is simpler to notate colors with numbers, so black==0, other colors== 1,2,3...
that way every time you add a new 'color' to the Urn, you can label it with a number that is one higher than the last number you used.

Your new code for drawing from the Urn will look something like this

```
```{r outline}
# NumberOfColorsUsed<-NumberOfColorsInUrnAtStart
# while NumberOfBalls<(HowManyBallsWeNeed+1){
  # draw a ball
  # if (Ball is black){
    # pick another ball
    # change color of ball to new color (i.e. new number)
    NumberOfColorsUsed<-NumberOfColorsUsed+1
```

[etc.]
```

# Lab Task #4 - The Polya Urn

- Code up this version of the Urn model. Then try the following:
- Start with one 'red' ball and the black ball (but code the colors as numbers, for simplicity's sake, so Ball1 <- 1, Ball2 <- 2, say).
- Repeat until you have 50 non-black balls; replicate this process multiple times.

1. What is the distribution of the number of (non-black) colors at the end?
2. What is the distribution of the number of balls of the commonest color at the end?
3. How many replicates do you need to do to answer these questions? How did you decide upon this number?
4. Repeat 1. and 2., but start with two balls of a same color+black ball each time. Compare your results to those from 1. and 2. Can you explain what you see?

# Lab Task 5 - The Generalized Polya Urn (note: you will need this version later in the course, so it better work!)

- Code up a further generalized version of the Polya Urn model
  - Imagine that each color has a 'weight'
  - When we draw a ball, a given ball is chosen with probability equal to (Weight of that ball)/(total weight of all balls in the Urn).
  - [So, our previous version had implicit weights of 1 for each color]
- Start with two 'red' balls and the black ball (but, again, code the colors as numbers, for simplicity's sake).
- Repeat until you have 50 non-black balls; replicate this process multiple times.
- Assume all, non-black colors have weight 1
  1. What is the expected number of different (non-black) colors in the Urn at the end as a function of the weight of the black ball ?
  2. What is the distribution of the number of (non-black) colors at the end, as a function of the weight of the black ball? (Try weight=1, or 2, or, ... , or 10.

# And if we have time…

# The Restaurant Process

Customers arrive one by one in an empty restaurant which has an unlimited number of round tables.

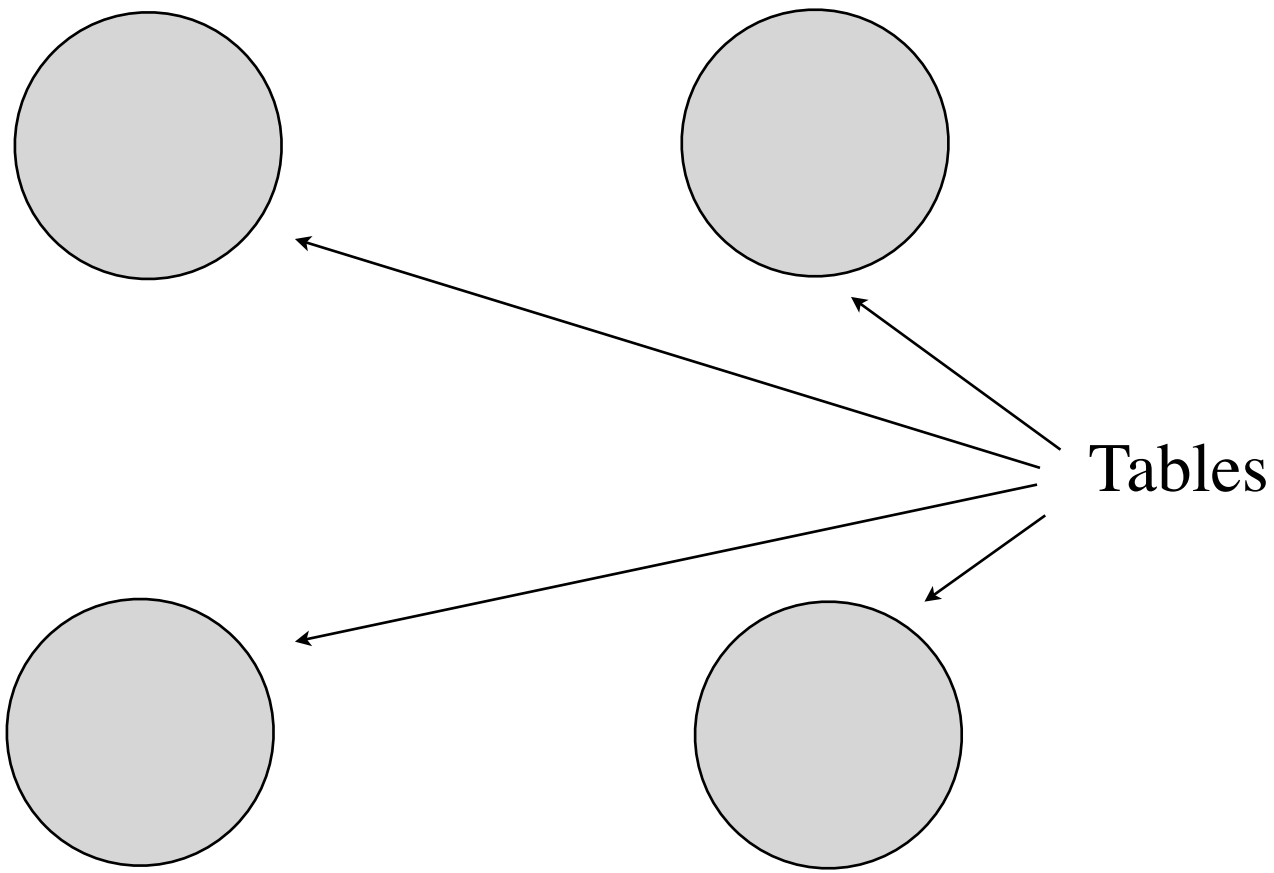
Initially, customer 1 sits by themself.

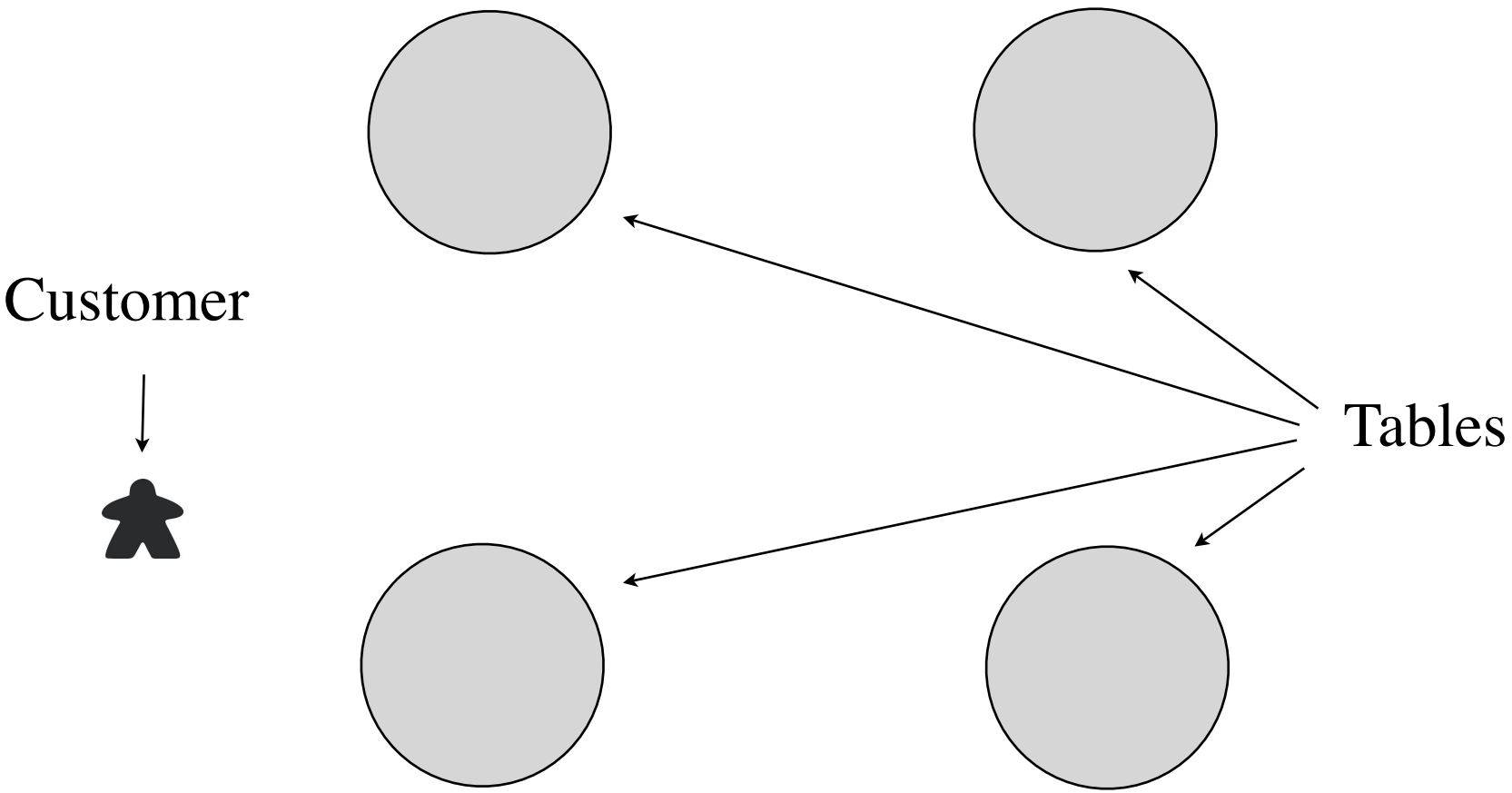
When the $(n+1)^{th}$ customer arrives, they sit down as follows:

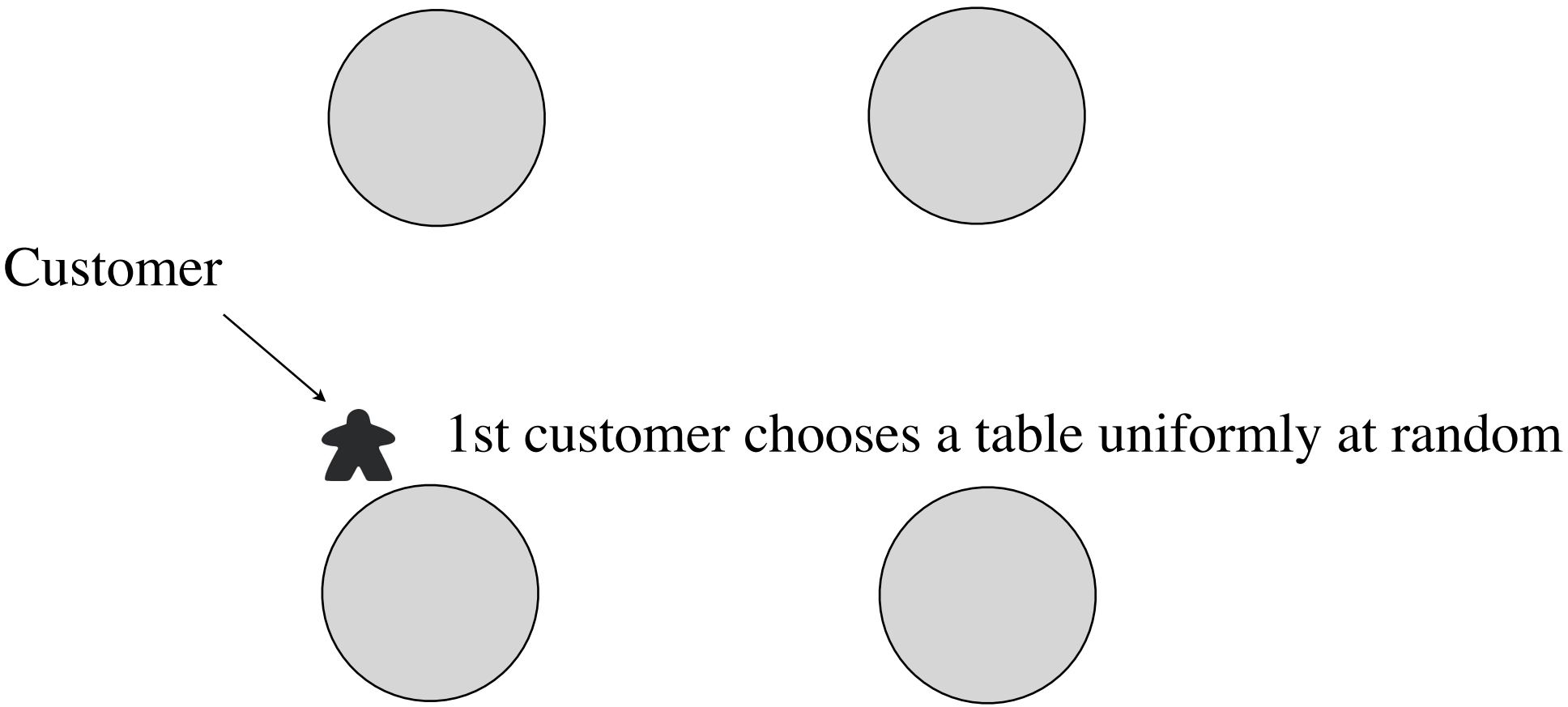
i) with probability $1/(n+1)$ they sit at a new table

ii) otherwise (i.e. with prob. $n/(n+1)$) they sit to the right of an existing diner; choosing to sit to the right of a **customer** chosen uniformly at random from among those already seated.

N.B. "customer", not "table"

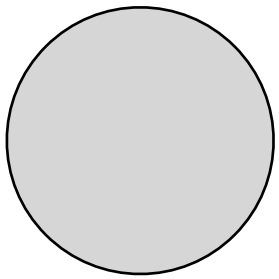
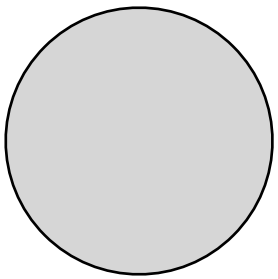

Tables

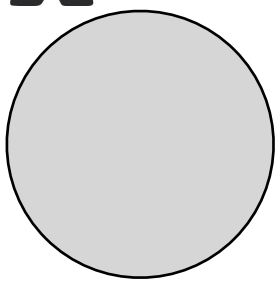
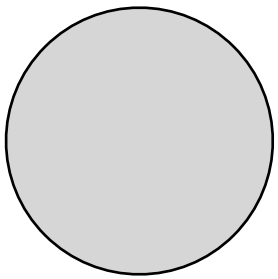

Customer
Tables


Customer
1st customer chooses a table uniformly at random

Customer n+1 does one of two things:

1. With probability 1/(n+1) they sit at an unoccupied table
2. Otherwise, they pick a person, uniformly at random, and sits at the same table as them, to their right.

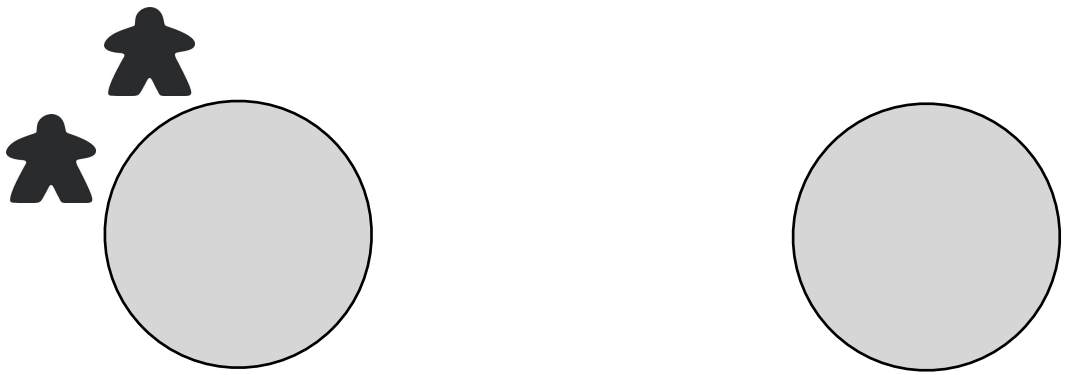



Customer n+1 does one of two things:

1. With probability 1/(n+1) they sit at an unoccupied table
2. Otherwise, they pick a person, uniformly at random, and sits at the same table as them, to their right.

Prob.= 1/2



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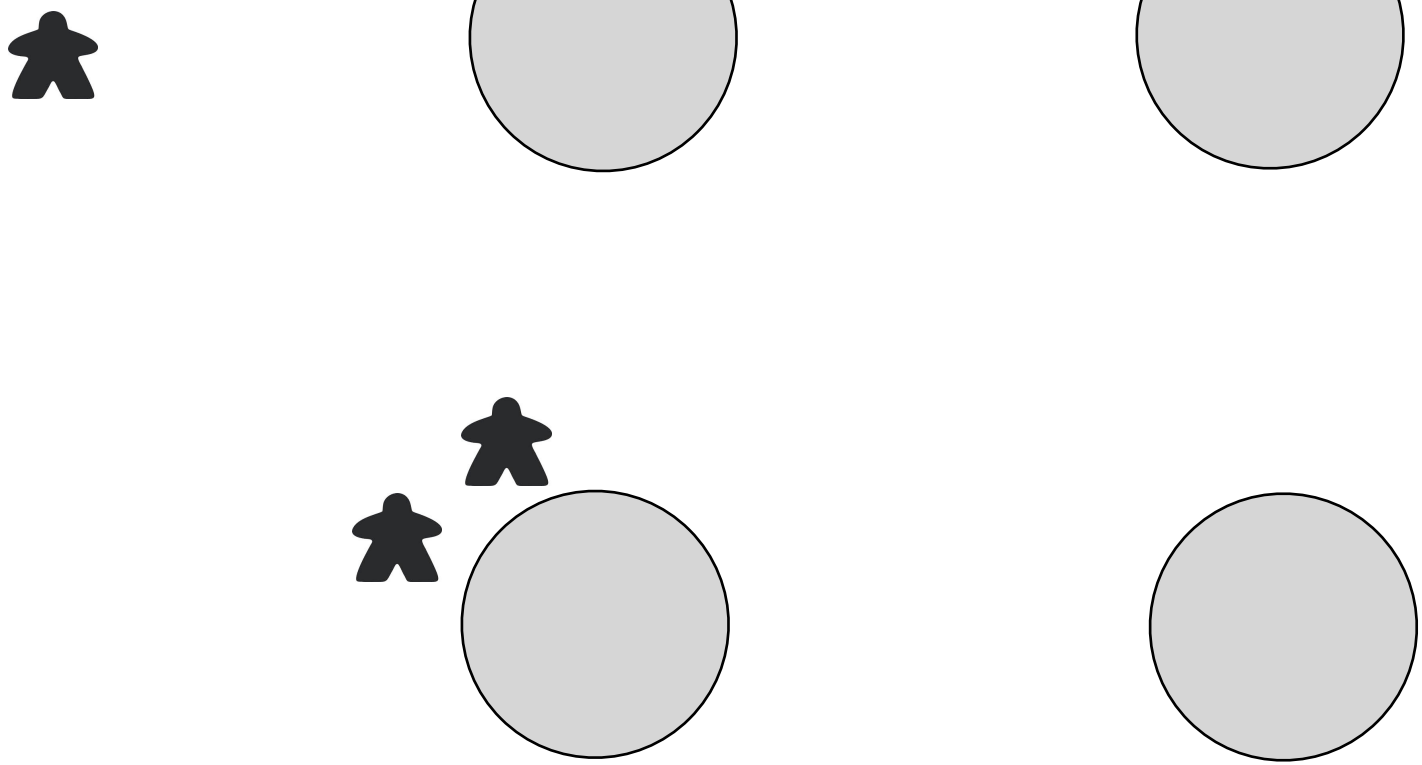


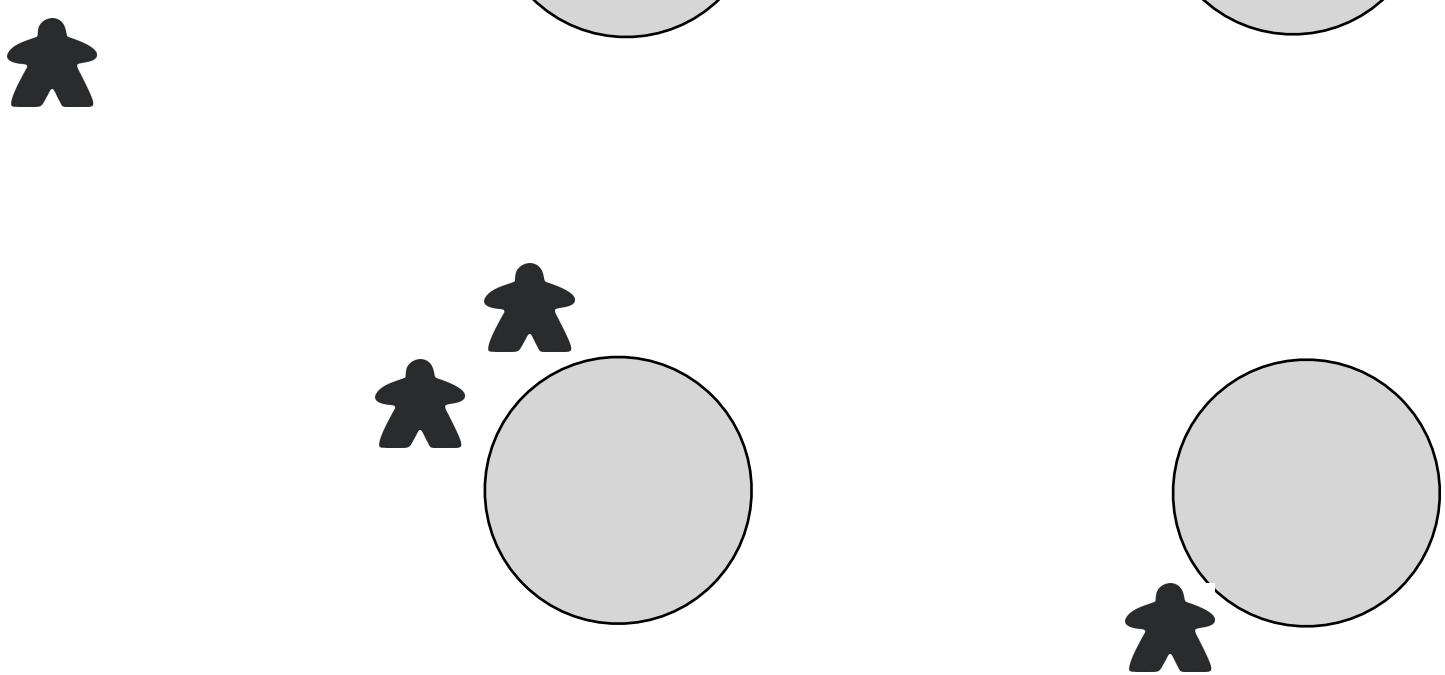
Customer n+1 does one of two things:

1. With probability 1/(n+1) they sit at an unoccupied table
2. Otherwise, they pick a person, uniformly at random, and sits at the same table as them, to their right.

Prob.= 1/3

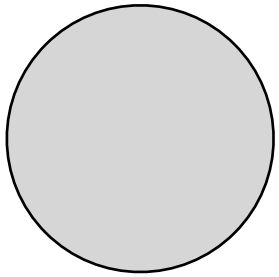
Customer n+1 does one of two things:

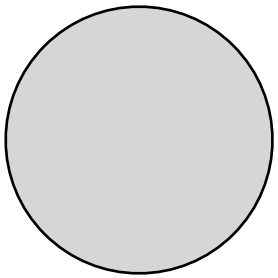
1. With probability 1/(n+1) they sit at an unoccupied table
2. Otherwise, they pick a person, uniformly at random, and sits at the same table as them, to their right.



Customer n+1 does one of two things:

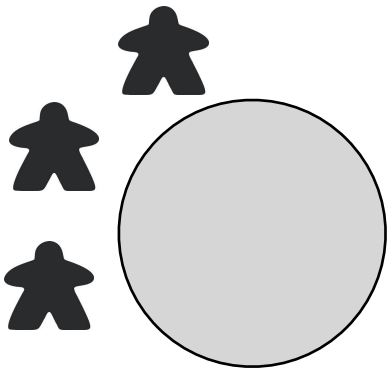
1. With probability 1/(n+1) they sit at an unoccupied table
2. Otherwise, they pick a person, uniformly at random, and sits at the same table as them, to their right.

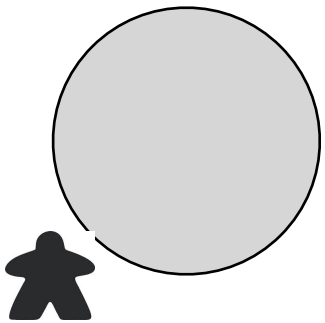




Prob.= 1/4

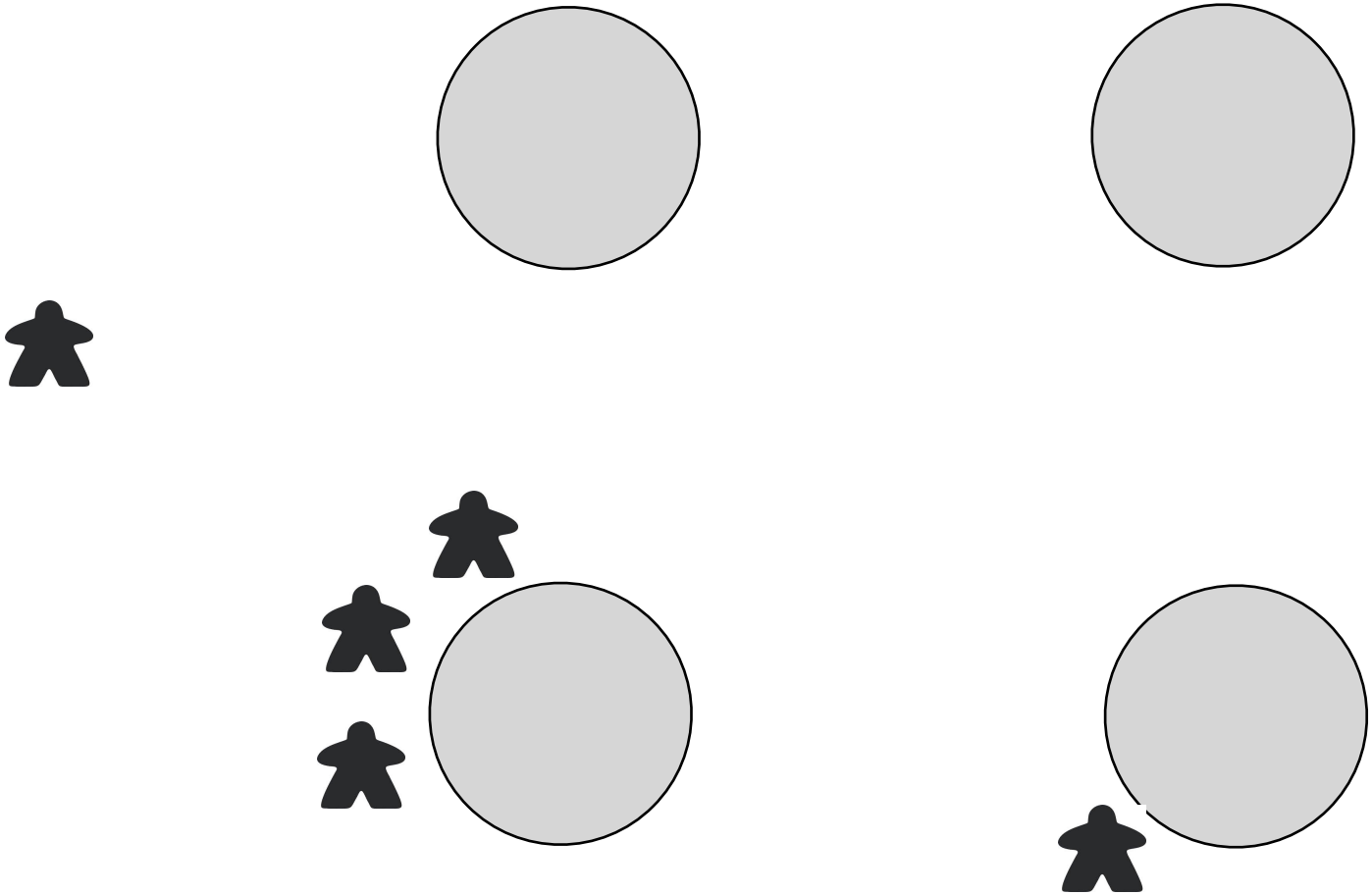
(Prob. sit somewhere at that table was 2/4)





Customer n+1 does one of two things:

1. With probability 1/(n+1) they sit at an unoccupied table
2. Otherwise, they pick a person, uniformly at random, and sits at the same table as them, to their right.



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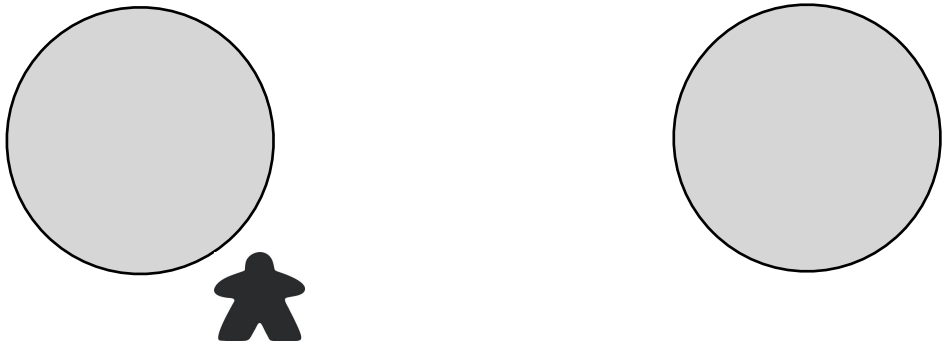
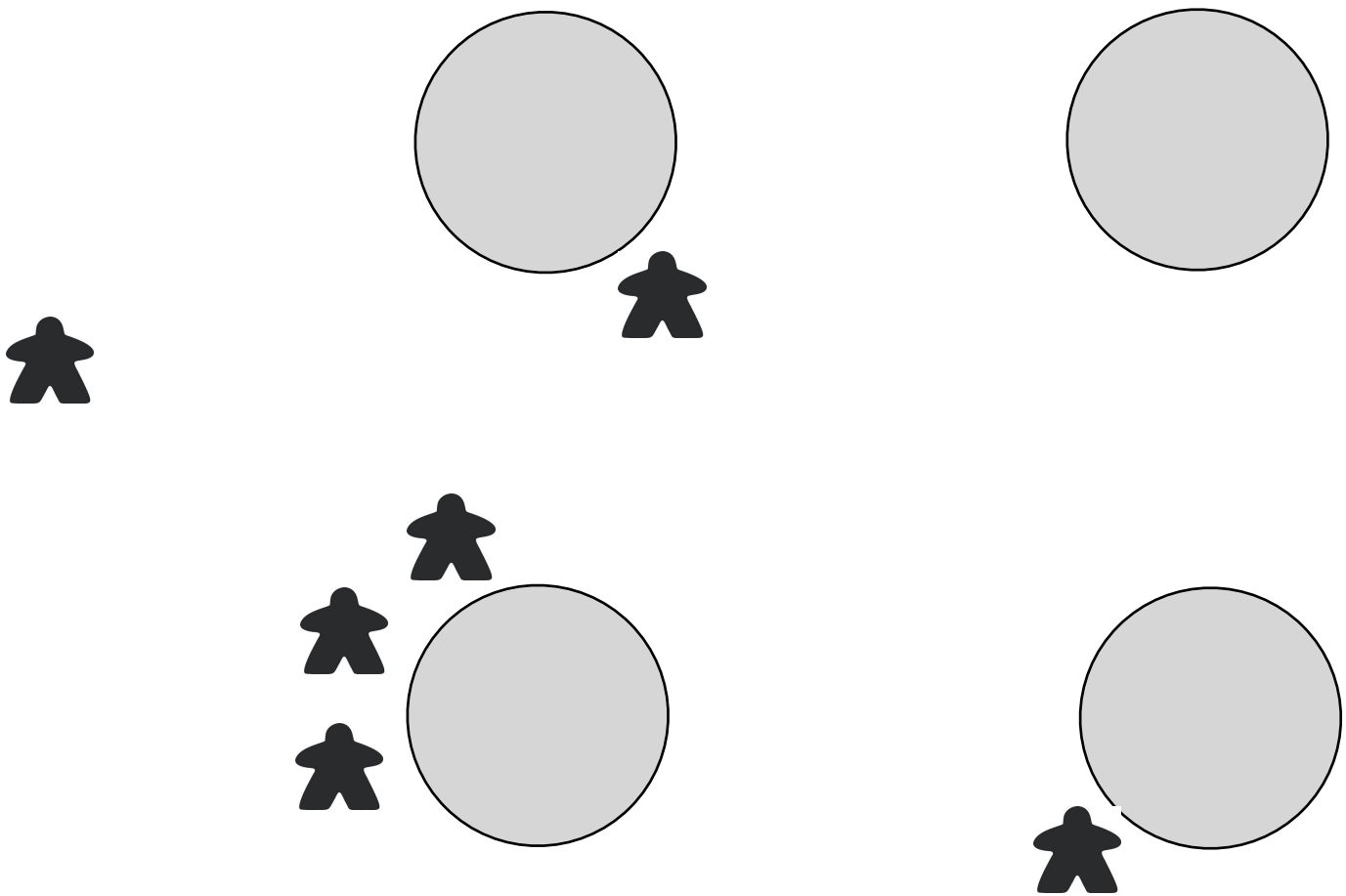


Prob.= 1/5

Customer n+1 does one of two things:

1. With probability 1/(n+1) they sit at an unoccupied table
2. Otherwise, they pick a person, uniformly at random, and sits at the same table as them, to their right.



Customer n+1 does one of two things:

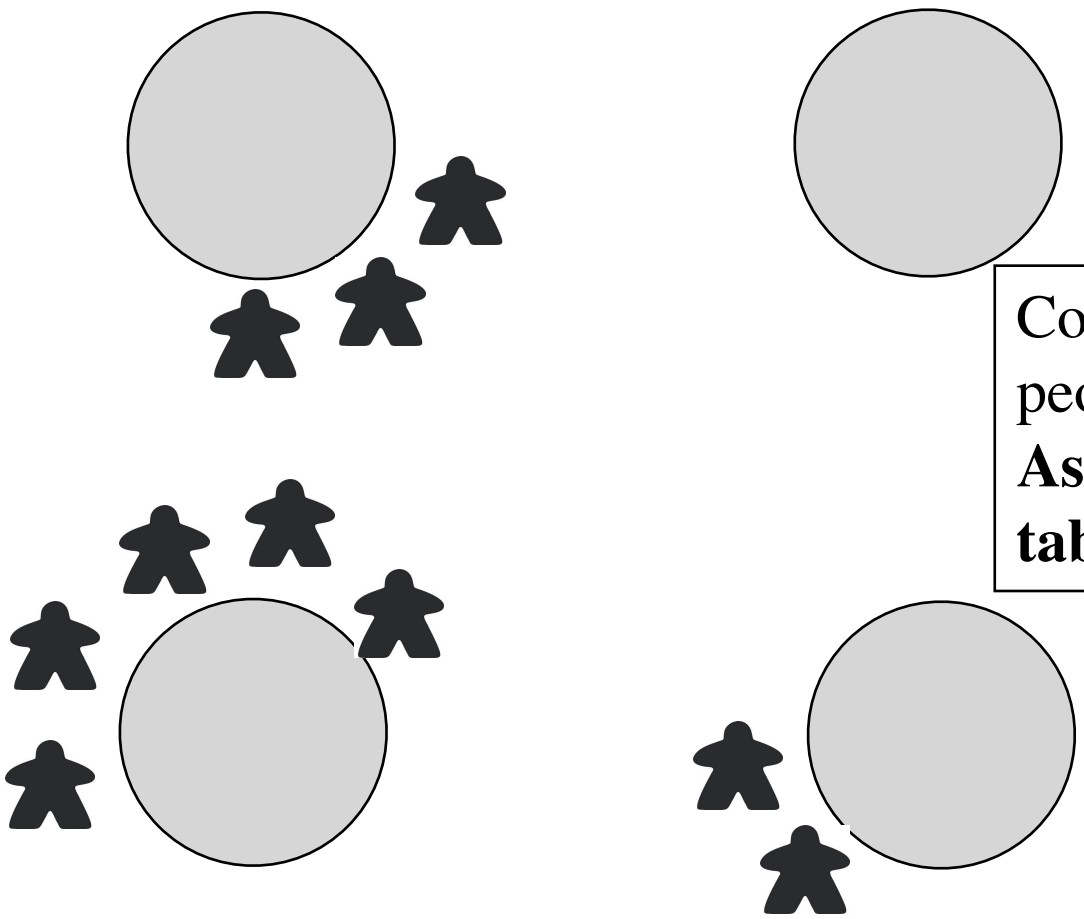
1. With probability 1/(n+1) they sit at an unoccupied table
2. Otherwise, they pick a person, uniformly at random, and sits at the same table as them, to their right.

Prob.= 1/6

Prob. sit at that table (anywhere) = 1/6

Customer n+1 does one of two things:

1. With probability 1/(n+1) they sit at an unoccupied table
2. Otherwise, they pick a person, uniformly at random, and sits at the same table as them, to their right.



Continue until we have n people seated.

**Assume the supply of tables is unlimited.**

# Task - the Restaurant Process [RP] 1

- Write a function to simulate the RP.
- Write another function to replicate the RP many times.
- Suppose we eventually have 50 customers.
  - What is the expected number of occupied tables?
  - What is the distribution of the number of occupied tables?

# Pseudocode for single RP

```
# define the number of customers (NumberToSeat)  and a vector of customers (CustVec <-
mat.or.vec(1,CustVec)
# Put customer 1 and table 1 (CustVec[1]<-1)
# initialize the count of occupied tables and customers (TablesOcc<-1, SeatedCustomers<-1)

# Now the customers arrive
while (SeatedCustomers < NumberToSeat){
    # does the next customer sit at a new table (with prob. 1/(SeatedCustomers+1). If so:
            # Increase number of seated customers and occupied tables by 1, and then
            CustVec[SeatedCustomers]<-TablesOcc
        # If not:
            # Increase number of seated customers by one
            # Choose a customer to sit to the right of (Neighbor)
            CustVec[SeatedCustomers]<-CustVec[Neighbor]
  }
```

["Week4—Restaurant Process" repo on Github]

# Task - the Restaurant Process 2

- Suppose that, conditional on there currently being N customers already seated, the prob. that the next customer sits at a new table is $\theta/(N+\theta)$, rather than $1/(N+1)$

- Suppose we eventually have 50 customers.
  - What is the expected number of occupied tables as a function of $\theta$? [Suggest try $\theta$=0.1,0.5,1,2,3,4,...,10,100]
  - How does the distribution of the number of occupied tables vary as a function of $\theta$?
  - How does the run time vary as a function of $\theta$?

- Do you see any connection with your answers to Urn Task 5 here, or between Urn Task 4 and the previous problem?

# END