Notes on Classification of Repn, example: $GL_2(\mathbb{R})$

Goal: $\boxed{\text{classification}}$ → Langlands Correspondence, D-module, Beilinson-Bernstein Localization

▸ LLC for $GL_2(\mathbb{Q}_p)$

notation: F: NA local field, $\mathcal{O}_F$, $\mathfrak{m}_F$, k_F, $\mathrm{ch}\, k_F = p > 0$, $\# k_F = q$

(1) study of field extension:

unramified: $\forall n \in \mathbb{N}$, the splitting field$/F$ of $x^{q^n} - x$, F_n, is the unique unramified ext'n of deg $= n$, with

$$\mathrm{Gal}(F^n/F) \cong \mathrm{Gal}(k_{q^n}/k_F) \cong \mathbb{Z}/n\mathbb{Z} \quad \text{(unramified)}$$

$$\rightsquigarrow \mathrm{Gal}(F^{nr}/F) = \varprojlim_n \mathrm{Gal}(F^n/F) = \hat{\mathbb{Z}}$$

For $m \geq 1$, $(p,m) = 1$, the unique ext'n of deg $= m$, is the totally tamely ramified ext'n E_m/F^{nr} which is given by $E^m = F^{nr}[t]/(t^m - \pi_{F^{nr}})$, so it is cyclic, with

$$\mathrm{Gal}(E^m/F^{nr}) \xrightarrow{\sim} \mu_m$$
$$\sigma \longmapsto \sigma(t)/t$$

$\rightsquigarrow E_\infty := \bigcup E_n$, $\quad \mathrm{Gal}(E_\infty/F^{nr}) \cong \varprojlim_{(n,p)=1} \mu_n = \prod_{\ell \neq p} \mathbb{Z}_\ell$

thus:

wild inertia $=: P_F$, $\quad \prod_{\ell \neq p} \mathbb{Z}_\ell$, $\quad \hat{\mathbb{Z}}$

$$\bar{F} - E_\infty - E^{nr} - F$$

$\longrightarrow$ the inertia gp of F: I_F

<Def>: Geometric Frobenius: $x \mapsto x^{-q}$ lift: Φ_F.

<Def>: Weil group W_F of F is the inverse image of $\langle \Phi_F \rangle$ in $\Omega_F = \mathrm{Gal}(\bar{F}/F)$ + locally cpt topology

<Def>: $W_F^{ab} = W_F / W_F^{der}$ → closure of commutator of W_F

<Thm> (Local Reciprocity Law)

For NA local F, $\exists!$ cont. homo:

$$\theta_F : F^\times \to \mathrm{Gal}(F^{ab}/F)$$

s.t. for $\forall$ unramified ext'n L/F, the composition θ_F with restriction $\mathrm{Gal}(F^{ab}/F) \twoheadrightarrow \mathrm{Gal}(L/F)$ denoted by

$$\theta_{L/F} : F^\times \to \mathrm{Gal}(L/F)$$

satisfies: $\theta_{L/F}(\pi_F) = \mathrm{Frob}_{L/F} : x \mapsto x^q$, for $\forall$ uniformizer

(2) $\theta_{L/F}$ is a surjection, kernel $= N_{L/F}(L^\times)$,

(3) (Functorial) L/F finite extn,

$$\begin{tikzcd} L^\times \arrow[r, "\theta_L"] \arrow[d, "N_{L/K}"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Gal}(L^{ab}/L) \arrow[d, "\mathrm{res}"] \\ K^\times \arrow[r, "\theta_K"'] & \mathrm{Gal}(K^{ab}/K) \end{tikzcd}$$

$\Leftarrow$ (2)

(4) $\{K$ fin. abel extn$\} \longleftrightarrow \{\mathrm{Gal}(K^{ab}/K)$ open subgp$\} \longleftrightarrow \{K^\times$ index fin. open subgp$\}$

$L \longleftrightarrow \ker(\mathrm{Gal}(K^{ab}/K) \to \mathrm{Gal}(L/K)) \longleftrightarrow N_{L/K} L^\times$ $\square$

---

(2) LLC for GL_1:

$\mathcal{A}_1(F) := \{$irr. sm. repn (π, V) of $F^\times\}/\sim$ (complex)

$\mathcal{G}_1(F) := \{$cont. homo. $W_K \to \mathbb{C}^\times\}$, $\mathbb{C}^\times$ here is equipped with Euclidean topology (Recall that: Weil gp. $W_F = \varprojlim \langle \Phi_F \rangle$ with topology: there $\exists!$ topo of W_F s.t. proj. $W_F \to \langle \Phi_F \rangle \cong \mathbb{Z}$ is cont., here $\mathbb{Z}$ with discrete topo, we have I_F as its sub: $1 \to I_F \to W_F \to \mathbb{Z} = \langle \Phi_F \rangle \to 1$).

thus such cont. homo. iff restriction to I_F is. But I_F is profinite, $\rightsquigarrow$ its image in $\mathbb{C}^\times$ is finite. So cont. here also holds respect to discrete topo. Then the Local Reciprocity law gives LLC for GL_1:

$\theta_F(a)|_{F^{nr} \cap F^\times} = \mathrm{Frob}^{v_F(a)} \Rightarrow \theta_F : F^\times \xrightarrow{\sim} W_F^{ab} \Rightarrow \mathcal{A}_1(F) \xrightarrow{\sim} \mathcal{G}_1(F)$. $\square$

---

(3) Preparation for LLC

① Deligne repn

<Def>: A Deligne repn of W_F is a tuple (ρ, V, n), where (ρ, V) is a fin. dim'l sm. repn of W_F and $n \in \mathrm{End}_{\mathbb{C}}(V)$ is a nilpotent endomorphism s.t. $\forall x \in W_F$. $\rho(x) n \rho(x)^{-1} = |\theta_K^{-1}(x)|_F\, n$

<Def>: semisimple Deligne repn if (ρ, V) is.

<rk>: (1) n is nilpotent is a natural condition:

Let $\Gamma \in W_F$ s.t. $\theta_F^{-1}(\Gamma) = \pi_F \rightsquigarrow \rho(\Gamma) n \rho(\Gamma)^{-1} = q n$

$\Rightarrow n$ has eigenvalue 0, thus nilpotent.

(2) $\ker(n)$ is stable under $W_F \rightsquigarrow$ if (ρ, V) is irr., n is equal to 0.

Therefore, irr. Deligne repn of W_F are simply irr. repn of W_F

(③) A Deligne rep'n (ρ, V, n) is equivalent to a complex s.s. rep'n of $W_K \times SL_2(\mathbb{C})$. If η is a Deligne rep'n, $\rightsquigarrow$ a (r, n) s.t.

$$r(w) = \eta\left(w, \begin{pmatrix} |\theta_F^{-1}(w)|_F^{1/2} & 0 \\ 0 & |\theta_F^{-1}(w)|_F^{-1/2} \end{pmatrix}\right), \quad \exp(n) = \eta\left(1, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right).$$

② L-factors and ε-factors

Goal: irr. sm. rep'n of $GL_2(F)$ is classified by L and ε-factors

<u>GL1 case</u>: μ^*: fixed Haar measure on $GL_1(F) = F^\times$

$\Phi \in C_c^\infty(F)$, ϖ: a uniformizer of F, $\mathcal{U}_F = \mathcal{O}_F^\times$ open and compact

$\forall \Phi \in C_c^\infty(F)$, $\Phi \cdot 1_{\varpi^m \mathcal{U}_F} \in C_c^\infty(F^\times)$, $\rightsquigarrow$ $\boxed{\text{fix } \chi: F^\times \to \mathbb{C}^\times}$,

def'n: $Z_m = Z_m(\Phi, \chi) = \int_{\varpi^m \mathcal{U}_F} \Phi(x)\chi(x)\, d\mu^*(x)$

B/c $\Phi \cdot 1_{\varpi^m \mathcal{U}_F} = 0$ when $m \ll 0$, we can define:

$$\boxed{\Phi \in C_c^\infty(F) \longmapsto Z(\Phi, \chi, X) = \sum_{m \in \mathbb{Z}} Z_m X^m \in \mathbb{C}((X))}.$$

Denote the image by $Z(\chi) \triangleq \{Z(\Phi, \chi, X) : \Phi \in C_c^\infty(F)\}$

For $a \in F^\times$, we define $a\Phi : x \mapsto \Phi(a^{-1}x)$, then

$$Z(a\Phi, \chi, X) = \chi(a) X^{v_F(a)} Z(\Phi, \chi, X)$$

So we can view $Z(\chi)$ as $\mathbb{C}[X, X^{-1}]$-mod. More precisely:

<prop> $Z(\chi) = Z(\chi, X) = P_\chi(X)^{-1}\mathbb{C}[X, X^{-1}] \subseteq \mathbb{C}((X))$,

where $P_\chi(X) = \begin{cases} 1 - \chi(\varpi)X, & \chi \text{ is unramified (i.e. } \chi|_{\mathcal{U}_F} \text{ trivial)} \\ 1, & \text{otherwise} \end{cases}$

<Def> (Fourier Transform)

For a fixed character ψ of F, $\psi \neq 1$, and a Haar measure μ on F, define, for $\Phi \in C_c^\infty(F)$, $\hat{\Phi}(x) = \int_F \Phi(y)\psi(xy)\, d\mu(y)$, $x \in F$

<rk>: $\hat{\Phi}(x)$ is locally const and reduces to a finite sum.

<prop>: $\exists\, c = c(\psi, \mu) \in \mathbb{R}_{>0}$, s.t. $\hat{\hat{\Phi}} = c\Phi(-x)$; For a given ψ, $\exists!$ Haar measure μ_ψ, s.t. $c(\psi, \mu_\psi) = 1$.

(Moreover, For $\psi: F \to \mathbb{C}^\times$, we call the least integer d, s.t. $\varpi^d \mathcal{O}_F \subseteq \ker\psi$. Then $\mu_\psi(\mathcal{O}_F) = q^{d/2}$; $\mu_{a\psi} = |a|_F^{1/2}\mu_\psi$. We call μ_ψ the selfdual measure (w.r.t. ψ) on F. In the following, Fourier Transform is under (ψ, μ_ψ).)

(Thm) For character $\chi: F^\times \to \mathbb{C}$, there $\exists$ a rational function

$c(\chi,\psi,X)\in\mathbb{C}((X))$, s.t. $Z(\hat{\Phi},\check{\chi},\frac{1}{qX}) = c(\chi,\psi,X)Z(\Phi,\chi,X)$ (8)

holds for $\forall \Phi\in C_c^\infty(F)$.

$\boxed{\text{proof}}$: Consider the space Λ of linear maps:

$$\lambda: C_c^\infty(F)\to\mathbb{C}(X)$$

s.t. $\lambda(a\Phi)=\chi(a)X^{v_F(a)}\lambda(\Phi)$, $\forall\Phi\in C_c^\infty(F)$ and $a\in F^\times$.

This is non-empty since $\boxed{(\Phi\mapsto Z(\Phi,\chi,X))\in\Lambda.}$ By calculation,

$\widehat{a\Phi}=|a|_F a^{-1}\hat{\Phi}\Rightarrow$ map $\boxed{\lambda:\Phi\mapsto Z(\hat{\Phi},\check{\chi},\frac{1}{qX})\in\Lambda}$

thus we only need: $\boxed{\dim_{\mathbb{C}(X)}\Lambda=1.}$

By continuity of χ, $\exists n\in\mathbb{N}$, s.t. $\underline{U_F^n=1+\varpi^n\mathcal{O}_F\in\ker\chi.}$

Denote $\Phi_n=1_{U_F^n}$. consider $L_n:\Lambda\to\mathbb{C}(X)$, $\lambda\mapsto\lambda(\Phi_n)$

injectivity of L_n will imply the claim. (since $\mathbb{C}(X)$ is PID.)

Suppose $\lambda(\Phi_n)=0$, for $k\geq n$, $a\in U_F^n$. we have

$$\lambda(a\Phi_k)=\chi(a)X^{v_F(a)}\lambda(\Phi_k)=\lambda(\Phi_k)$$

Hence $$\lambda(\Phi_k)=q^{n-k}\lambda(\Phi_n)=0,$$

by using U_F^n can be covered by U_F^k disjointly (q^{n-k} copies) $\Rightarrow\lambda(\Phi)=0$ $\forall\Phi\in C_c^\infty(F^\times)$.

Therefore, $\lambda(\Phi)$ for $\Phi\in C_c^\infty(F)$ only depends on $\boxed{\Phi(0)}$.

By $\lambda(a\Phi)=\lambda(\Phi)$, for all $a\in F^\times$ + $\lambda(\Phi)=0$, $\forall\Phi\in C_c^\infty(F)\Rightarrow\lambda=0$ $\square$

For $s\in\mathbb{C}$, defn

$$\mathcal{Z}(\Phi,\chi,s)=Z(\Phi,\chi,q^{-s})$$

$$L(\chi,s)=P_\chi(q^{-s})^{-1}=\begin{cases}(1-\chi(\varpi)q^{-s})^{-1}, & \text{unramified}\\ 1, & \text{otherwise}\end{cases}$$

$$\gamma(\chi,s,\psi)=c(\chi,\psi,q^{-s})$$

In particular, $\mathcal{Z}(\Phi,\chi,s)=\int_{F^\times}\underline{\Phi(x)\chi(x)|x|_F^s}\,d\mu^*(x)$ (**)

$\hookrightarrow$ not always cpt. supported.

in the sense that

$\mathcal{Z}(\Phi,\chi,s)$ is a limit of sum, absolutely and uniformly in the half plane $Re(s)>s_0$ for some $s_0\in\mathbb{R}$ $\rightsquigarrow$ then it is rati function in q^{-s}, admits a analytic continuation to whole...

<rmk> for unramified χ_1,χ_2, $L(\chi_1,s)$, $L(\chi_2,s)$ have same pole $\Leftrightarrow\chi_1=\chi_2$.

<Def> (ε-factor)

$$\mathbb{C}(q^{-s})\ni\varepsilon(\chi,s,\psi)=\gamma(\chi,s,\psi)\,L(\chi,s)/L(\check{\chi},1-s)$$

Basic Property:

$\mathcal{Z}(\hat{\Phi}, \check{\chi}, 1-s) = \gamma(\chi, s, \psi)\mathcal{Z}(\Phi, \chi, s)$, apply it twice,

$\mathcal{Z}(\hat{\hat{\Phi}}, \check{\chi}, s) = \gamma(\check{\chi}, 1-s, \psi)\gamma(\chi, s, \psi)\mathcal{Z}(\Phi, \chi, s)$

since $\hat{\hat{\Phi}} = (-1)\Phi \Rightarrow \chi(-1) = \gamma(\check{\chi}, 1-s, \psi)\gamma(\chi, s, \psi)$

$= \varepsilon(\check{\chi}, 1-s, \psi)\varepsilon(\chi, s, \psi)$ $\square$

GL_2 case:

$K = M_2(\mathcal{O}_F)$, $A = M_2(F)$, $G = GL_2(F)$

<Def>: Let (π, V) be a sm. repn of G, for vector $v \in V$, $\check{v} \in V^\vee$,

we get a sm. function: $\gamma_{v \otimes v^\vee} : g \longmapsto \langle v^\vee, \pi(g)(v) \rangle$

Denote $C(\pi) = \mathrm{Span}_{\mathbb{C}}\{\gamma_{v \otimes v^\vee} ; v \otimes v^\vee \in V \otimes V^\vee\}$,

$f \in C(\pi)$: matrix coefficients of π.

Define $\mathcal{Z}(\Phi, f, s) = \int_G \underline{\Phi(x) f(x) |\det x|_F^s} \, d\mu^*(x)$ (**)

$\hookrightarrow$ locally cpt. const. supp. on A

<Thm> (π, V) irr. sm. repn on G.

① $\exists s_0 \in \mathbb{R}$, s.t. (**) converge $s > s_0$ + is a rational function in q^{-s}.

② $Z(\pi) := \{\mathcal{Z}(\Phi, f, s + \frac{1}{2}) ; \Phi \in C_c^\infty(A), f \in C(\pi)\}$,

$\rightsquigarrow \exists!$ poly. $P_\pi(x) \in \mathbb{C}[x]$ s.t. $P_\pi(0) = 1$, $Z(\pi) = P_\pi(q^{-s})^{-1}\mathbb{C}[q^s, q^{-s}]$.

Again, denote $L(\pi, s) = P_\pi(q^{-s})^{-1}$.

Finally, define Fourier Transform $\hat{\Phi}$ of $\Phi \in C_c^\infty(A)$:

$$\hat{\Phi}(x) = \int_A \Phi(y) \underbrace{\psi_A(xy)}_{:= \psi \circ \mathrm{tr}_A} d\mu(y)$$

Similarly to GL_1 case, we need the dual of $f \in C(\pi)$:

$$f^\vee : g \mapsto f(g^{-1}) \in C(\pi^\vee)$$

induces $C(\pi) \cong C(\pi^\vee)$.

<Thm> (Godement – Jacquet functional equation)

$\exists!$ rational function $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$, s.t. $\forall \Phi \in C_c^\infty(A)$, $f \in C(\pi)$,

$$\mathcal{Z}(\hat{\Phi}, \check{f}, \tfrac{3}{2} - s) = \gamma(\pi, s, \psi)\mathcal{Z}(\Phi, f, \tfrac{1}{2} + s).$$

thus defn $\varepsilon(\pi, s, \psi) = \gamma(\pi, s, \psi) L(\pi, s) / L(\check{\pi}, 1-s)$, similarly,

$$\varepsilon(\pi, s, \psi)\varepsilon(\check{\pi}, 1-s, \psi) = \omega_\pi(-1)$$

where ω_π is the central character of π.

(3) Converse Theorem: use L-factor, ε-factor classify repn of G_n.

<Thm>: For char $\chi: F^\times \to \mathbb{C}^\times$, (π, V) irr. sm. repn of G_n. denote $\chi_\pi: g \mapsto \chi(\det(g))\pi(g)$. Fix a non-trivial $\psi: F \to \mathbb{C}$, then two irr. sm. repn $\pi_1 \cong \pi_2$ iff for $\forall$ char $\chi: F^\times \to \mathbb{C}^\times$

$$L(\chi\pi_1, s) = L(\chi\pi_2, s) \text{ and } \varepsilon(\chi\pi_1, s, \psi) = \varepsilon(\chi\pi_2, s, \psi)$$

(4) L-factor and ε-factor for Weil group

Let (ρ, V, n) be a s.s. Deligne repn. Denote

$$V_n = \ker(n), \quad V_n^{I_F} = \{v \in V_n : \rho(i)v = v, \forall i \in I_F\}$$

$$L(\rho, s) = \det(1 - q^{-s} \cdot \Phi_F|_{V_n^{I_F}})^{-1}$$

where $\Phi_F \in W_F$ is the geometric Frobenius. ← standard Artin L-function.

For non-trivial $\psi: F \to \mathbb{C}^\times$, let $l(\psi)$ be a max'l integer l, s.t. $\psi(\mathfrak{p}_F^{-m}\mathcal{O}_F) = 1$ and du: additive Haar measure of F.

construction of ε-factor for repn (ρ, V):

simple case: ρ is an unramified quasi-character χ, i.e. $\chi = 1 \cdot |\cdot|_F^s$ for some $s \in \mathbb{C}$. ⇝

$$\varepsilon(\chi, \psi, du) := \chi(w) q^{l(\psi)(1-s)} \mu(\mathcal{O}_F)$$

where $w \in W_F$ is an element whose valuation is $l(\psi)$.

If χ is ramified, Let $f(\chi)$ be the conductor (i.e. smallest integer f. s.t. $\chi(\mathcal{O}_F(U_F^f)) = 1$, and let $c \in F^\times$, s.t. it's valuation is $n(\psi) + f(\chi)$.

Then set $\varepsilon(\chi, \psi, du) = \int_{c^{-1}U_F} \chi^{-1}(\mathcal{O}_F(x))\psi(x)\,dx$

when $\dim V > 1$, the following thm. def'ns ε-factor:

<Thm> (Langlands, Deligne) $\exists!\ \varepsilon(\rho, \psi, du) \in \mathbb{C}^\times$, s.t. some "good" properties hold.

⇝ define $\varepsilon(\rho, \psi, s) = \varepsilon(1 \cdot |\cdot|^s \rho, \psi, du) \det(-\Phi_F|_{V^{I_K}/V_n^{I_K}})$.

For two Deligne repn (ρ_i, n_i, V_i) $i = 1, 2$. ← choose self-dual du

define tensor product $(\rho_1 \otimes \rho_2, V_1 \otimes V_2, n_1 \otimes n_2)$ by

$$(\rho_1 \otimes \rho_2)(w)(V_1 \otimes V_2) = \rho_1(w)V_1 \otimes \rho_2(w)V_2$$

$$(n_1 \otimes n_2)(V_1 \otimes V_2) = n_1 V_1 \otimes V_2 + V_1 \otimes n_2 V_2$$

if $\dim \rho_i = m_i$, ρ_i are irr. Deligne repn ⇝

$$L(\rho_1 \otimes \rho_2, s) = \prod_\chi L(\chi, s)$$

where χ runs all unramified quasi-character of $F^\times \backslash W^{ab}$ s.t.

where χ runs all unramified quasi-character of $F^\times$ s.t. $\chi \otimes \rho_1^\vee = \rho_2$. In particular, $L(\rho_1 \otimes \rho_2, s) = 1$, for $m_1 \neq m_2$.

(b) The Local Langlands correspondence

Denote $\mathcal{A}_2(F) = \{\text{irr. sm. repn of } G\}/\simeq$

$\mathcal{G}_2(F) = \{\text{2-dim ss. Deligne repn of } W_F\}/\simeq$

<Thm> fix $\psi: F \to \mathbb{C}^\times$. $\exists!$ map $\pi: \mathcal{G}_2(F) \to \mathcal{A}_2(F)$.

s.t. $\forall \rho \in \mathcal{G}_2(F)$, $\chi \in \widehat{F^\times}$,

$$\begin{cases} L(\chi \otimes \rho, s) = L(\chi_{\pi(\rho)}, s) \\ \varepsilon(\chi \otimes \rho, s) = \varepsilon(\chi_{\pi(\rho)}, s, \psi) \end{cases}$$

Moreover, π is bijective. equations hold for $\forall \psi \neq 1$.

$\mathcal{A}_2(F) \begin{cases} \mathcal{A}_2^0(F) = \{\text{cusp. repn}\}/\simeq \\ \mathcal{A}_2^1(F) = \{\text{non-cusp. repn}\}/\simeq \end{cases}$ $\mathcal{G}_2(F) \begin{cases} \mathcal{G}_2^0(F) = \{\text{irr. ss. 2-dim Deligne}\} \\ \mathcal{G}_2^1(F) = \{\text{reducible} \cdots\} \end{cases}$

<prop>: ① $\pi \in \mathcal{A}_2(F) \Rightarrow \pi \in \mathcal{A}_2^0(F) \Leftrightarrow L(\chi\pi, s) = 1$. $\forall \chi \in \widehat{F^\times}$.

② $\rho \in \mathcal{G}_2(F) \Rightarrow \rho \in \mathcal{G}_2^0(F) \Leftrightarrow L(\chi \otimes \rho, s) = 1$. $\forall \chi \in \widehat{F^\times}$.

<Thm> $\exists!$ $\pi': \mathcal{G}_2^1(F) \xrightarrow{\sim} \mathcal{A}_2^1(F)$, s.t. $L(\chi \otimes \rho, s) = L(\chi \pi'(\rho), s)$,

$\forall \rho \in \mathcal{G}_2^1(F)$, $\forall \chi \in \widehat{F^\times}$. $\Rightarrow \pi'(\chi \otimes \rho) = \chi \pi'(\rho)$, $\varepsilon(\rho, s, \psi) = \varepsilon(\pi'(\rho), s, \psi)$

proof, Defn explicitly. Let $(\rho, V, n) \in \mathcal{G}_2^1(F)$, $\underline{\rho = \chi_1 \oplus \chi_2}$, $\chi_i \in \widehat{F^\times}$.

$\rightsquigarrow$ $\chi := \chi_1 \otimes \chi_2 \in \hat{T}$ (act diagonally)

Take normalized induction of G: $\pi = \mathrm{Ind}_B^G(\delta_B^{-1/2} \otimes \chi)$

$\hookrightarrow B = TN \to \mathbb{C}$, $tn \mapsto \|t_1/t_2\|_F$

π reducible iff $\chi_1\chi_2(x) = |x|_F^{\pm 1}$.

If π is irr. $\Rightarrow n = 0$. In fact, let $n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\Rightarrow (\chi_1 \oplus \chi_2)(x)\, n\, (\chi_1 \oplus \chi_2)(x)^{-1} = |x|_F\, n$$

$$\Rightarrow \begin{pmatrix} a & \chi_1(x)\chi_2(x)^{-1} b \\ \chi_1(x)^{-1}\chi_2(x)\, c & d \end{pmatrix} = |x|_F \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

since $\chi_1\chi_2(x) \neq |x|_F^{\pm 1} \Rightarrow b = c = 0$, since n is nilpotent $\Rightarrow n = 0$.

If π is reducible. Take $\chi_1(x) = \varphi(x)|x|_F^{-1/2}$ and $\chi_2(x) = \varphi(x)|x|^{1/2}$ for some $\varphi \in \widehat{F^\times}$.

We claim: there are exactly two Deligne repn $\in \mathcal{G}_2^1(F)$, s.t.

ρ has composition factor χ_1, χ_2: $\begin{pmatrix} a & |x|_F b \\ |x|_F c & d \end{pmatrix} = |x|_F \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow a = b = d = 0$

$\Rightarrow c = 0$ or $c \neq 0$.

π_1 is 'defined' as: $\pi'((\rho, V, n)) = \begin{cases} \pi = n\text{-}\mathrm{Ind}_B^G \chi, & \chi \text{ irr.} \\ \varphi \circ \det & \chi \text{ reducible } n = 0 \\ \varphi \circ \mathrm{St}_G & \text{otherwise.} \end{cases}$

<Thm> $\exists!$ $\pi_0: \mathcal{G}_2^0(F) \xrightarrow{\sim} \mathcal{A}_2^0(F)$.

Real Story (LLC for $GL_2(\mathbb{R})$).

Prop. (Structure) Every parabolic subgp Q_p has a Langlands decomp:
$Q_p = M_p A_p N_p$, where M_p reductive, A_p abelian, N_p nilpotent.
$\subset$ cent$(\mathfrak{a}_p)$ in K_0 max'l cpt in $G_0/\mathbb{R}$

(Thm) (Langlands) $\{$irr adm. rep'n of $G_0\}/\{$infinitesimal equivalence$\}$
$\updownarrow$ 1:1
triple $(MAN, [\sigma], \nu)$ - satisfying

(a) MAN is parabolic of G

(b) σ is an irr. temper (unitary) rep'n of M. $[\sigma]$ is the class.

(c) ν is a member in $\mathfrak{a}'$, s.t. $\mathrm{Re}\langle \nu, \alpha\rangle > 0$. for every positive restricted root not vanishing on $\mathfrak{a}$.

In particular. $(MAN, [\sigma], \nu) \longmapsto \mathrm{Ind}_{MAN}^{G}(\sigma \otimes \nu \otimes 1)$ $\exists!$ irr. quotient.

▸ Rep'n of Weil Group $W_{\mathbb{R}}$

Def. ($W_{\mathbb{R}}$): non-split extension of $\mathbb{C}^\times$ by $\mathbb{Z}/2\mathbb{Z}$. given by
$$W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times,\ j^2 = 1,\ jcj^{-1} = \bar{c} \text{ for } \forall c \in \mathbb{C}^\times.$$

▸ classify n-dim complex rep'n of $W_{\mathbb{R}}$ whose image consists of semisimple elements

<Lem>: 1-dim rep'n of $\mathbb{C}^\times$ are of form $z \mapsto z^\mu \bar{z}^\nu$ with $\mu, \nu \in \mathbb{C}$, $\mu - \nu \in \mathbb{Z}$.

<Prop> 1-dim rep'n φ of $W_{\mathbb{R}}$ are of form
$$\langle +, t\rangle : \varphi(z) = |z|_{\mathbb{R}}^{t},\ \varphi(j) = 1$$
$$\langle -, t\rangle : \varphi(z) = |z|_{\mathbb{R}}^{t},\ \varphi(j) = -1$$
t can be any complex number.

<Prop> $\{$2-dim irr. ss. rep'n $\varphi\}/\simeq$ of $W_{\mathbb{R}}$ are of form
$$\langle l, t\rangle : \varphi(re^{i\theta})\alpha = r^{2t}e^{il\theta}\alpha,\ \varphi(j) = \beta$$
$$\varphi(re^{i\theta})\beta = r^{2t}e^{-il\theta}\beta,\ \varphi(j)\beta = (-1)^l\alpha$$
where α, β form a basis of 2-dim vect. space, $l \in \mathbb{Z}^+$, $t \in \mathbb{C}$.

proof: by semisimple, choose $\{a, b\}$ as basis. s.t. $\varphi(\mathbb{C}^\times)$ consists of diagonal matrices. By lemma, take $\varphi(z)a = z^{\mu_1}\bar{z}^{\nu_1}a$, $\varphi(z)b = z^{\mu_2}\bar{z}^{\nu_2}b$. by irr. $\Rightarrow \mu_1 \neq \mu_2$ or $\nu_1 \neq \nu_2$. Let $a' = \varphi(j)a$
$$\Rightarrow \quad \varphi(z)a' = \varphi(j\bar{z}j^{-1})a' = \varphi(j\bar{z})a = z^{\nu_1}\bar{z}^{\mu_1}a'$$