Langlands Classification, Discrete Series (1)

Example: $GL(2,\mathbb{R})$, $Sp(4,\mathbb{R})$, $SU(2,2)$ ← second note (not here)

← this topic

and unitary dual question: $\{\text{unitary repn}\}/\sim$

$\{\text{irr. temper}\} \subseteq \{\text{irr. unitary}\} \subseteq \{\text{irr. admissible}\}$

(1) Normalized Induction:

G = linear reductive gp, Q = parabolic sub, $Q = \underset{\text{abelian}}{M}\overset{\text{cpt}}{A}\overset{\text{unipotent}}{N}$

$\Delta_Q(man) = \det \mathrm{Ad}(a)|_{\mathfrak{n}}$

$\Delta_G(man) = 1$ (since G is reductive)

(rmk: $d_\ell(\cdot t) = \Delta_{G'}(t)^{-1} d_\ell(\cdot)$, for G', d_ℓ: left invariant Haar measure)

Let $2\rho = \sum_{\lambda\in\Sigma^+} m_\lambda \lambda$: sum of restricted roots.

then $\Delta_Q(man) = e^{2\rho\log(a)}$

$$w(man) := \Delta_G(man)^{-1}\Delta_Q(man) = e^{2\rho\log(a)}$$

take $(\sigma, V) \in \mathrm{Rep}(M)$, $\nu \in i\mathfrak{a}' \rightsquigarrow e^\nu$: unitary multi character of A.

define $\sigma \otimes e^\nu \otimes 1 : man \mapsto e^\nu \sigma(m)$.

(if we write $M_P A_P N_P$, it means "minimal".)

Then induce:

$$\{F \in C(G,V) \mid F(xman) = w(man)^{-\frac{1}{2}} e^{-\nu}\sigma(m)^{-1}F(x)\}$$
$$= e^{-(\rho+\nu)\log(a)}\sigma(m)^{-1}F(x).$$

$$(g.F)(x) = F(g^{-1}x)$$

denoted as $U^{\sigma,\nu}$. It carries unitary σ (V上带norm) to unitary repn of G w.r.t. $\|F\|^2 = \int_{G/Q}|F|_V^2$, i.e.

$$\int_{G/Q}|U^{\sigma,\nu}(g)F|_V^2 = \int_{G/Q}|F|_V^2, \quad \forall g\in G.$$

norm preserving

(2)

Going from G-repn to $(\mathfrak{g},K)$-mod

Φ: a G-repn on a Hilbert space $V/\mathbb{C}$. $C^\infty(V)$, $C^\omega(V)$ (real) analytic function

<Thm>: $C^\infty(V) = \mathrm{span}\langle \int_G f(x)\Phi(x).v\,dx,\ f\in C^\infty_{com}(G)\rangle$ ← cpt. supported.

In particular, by identity approximation, $C^\infty(V) \subseteq V$ densely.

<Thm> (HC, Nelson) $C^\omega(V)$ is dense in V.

Tool: unitary case: invariant elliptic differential operators on G.

<rmk> for S an $U(\mathfrak{g})$-invariant subspace of $C^\infty(V)$, $\bar{S}$ need not to be G-

invariant. For ... of $C^{\omega}(V)$, $\bar{S}$ is G-invarant. ($\mathfrak{g}$-side to G-side).

Before to reductive gp G, we study cpt. K. An corollary of Peter-Weyl thm: every (unitary) repn of K, Φ, can be decomposed as a direct sum of $\hat{K}$ with finite copies: i.e. $V = \sum_{\tau \in \hat{K}} V_\tau$, where $\dim \mathrm{Hom}_K(V_\tau, \tau) = \frac{\dim(V_\tau)}{\dim(\tau)} < +\infty$

The projection operator is $E_\tau v = d_\tau \int_K \overline{\chi_\tau(k)} \Phi_k(v)\, dk$

where $d_\tau = \dim(\tau)$, $\chi_\tau(x) = \mathrm{Tr}(\tau(x)) = \sum_i (\tau(x) u_i, u_i)$

To reductive group: WLOG, assume $K \curvearrowright V$ unitarily.

<Thm>: If (Φ, V) an irr. unitary repn of G, every $\tau \in \hat{K}$ has fin. multi. in V.

A repn says <u>admissible</u> if every $\tau \in \hat{K}$ has finite multi. in V.

<Prop>: $MAN = Q \subseteq G$, take $\sigma \in \hat{M}$ (an unitary) repn, $\mathrm{Ind}_{MAN}^{G}(\sigma \otimes V \otimes 1)$ is an adm. repn of G (proof: by Frobenius reciprocity).

<Prop> each $\tau \in \hat{K}$, $C^{\infty}(V) \cap V_\tau$, $C^{\omega}(V) \cap V_\tau$ both dense in V_τ.

write $V_K = \oplus_{\tau \in \hat{K}} V_\tau$ for each algebraic direct sum of V_τ. Then $C^{\infty}(V) \cap V_K$, $C^{\omega}(V) \cap V_K$ are both invariant under $U(\mathfrak{g})$. V_K: K-finite vector.

<Cor> admissible $\Rightarrow$ every K-fin. vector is analytic.

<Cor> (Φ, V) adm, then $\{$closed G-invariant subspace W of $V\}$

$$\updownarrow \ 1:1$$

$$\{U(\mathfrak{g})\text{-invariant subspace of } V_K\}$$

through $W \mapsto W_K$, $\bar{S} \longleftarrow S$.

(Lie alg. repn capture full information of group repn in admissible case).

thus we defin the $(\mathfrak{g}, K)$-mod.

irr. uni. repn of G (unitarily equivalent) $\Leftrightarrow$ <u>$(\mathfrak{g}, K)$-mod infinitesimally equiv.</u>

algebraically

(3) Cartan subgp and Global Character

① compact case: $G \subseteq U(n)$ with Lie alg $\mathfrak{g}_0$, $\mathfrak{t}_0 \subseteq \mathfrak{g}_0$: maxl abelian.

$T \subseteq G$: maxl torus. <u>Goal</u>: study χ_Φ

<Thm>: <u>two maxl abelian subspace (tori) of $\mathfrak{g}_0$ are conjugate</u>. Since $\forall$ member of $\mathfrak{g}_0$ is contained in a maxl abelian subspace $\Rightarrow$ $\forall$ member of $\mathfrak{g}_0$ is conjugate to one in a fixed maxl abelian subspace.

$(\mathfrak{g} \subset C_{\mathfrak{g}}(S))$

<Thm>: Let S be a torus of G. If $g \in G$ centralized S, there is a torus S' in G containing both S, g.

<Cor 1>: The centralizer in G of a torus is connected.

$\boxed{\text{<Cor 2>}}$: $Z_G(\mathfrak{t}_0) := \{g \in G \mid Ad(g)H = H \text{ for all } H \in \mathfrak{t}_0\} \subseteq G$ is connected. In other words, $T = Z_G(\mathfrak{t}_0)$ (called the Cartan subgp).

this is special property for compact case, this gives

<Thm 3>: Each element of G is conjugate to a member in T.

Benifits: $G = \bigcup gTg^{-1}$, $\Rightarrow$ only need to study $\chi_\Phi|_T$. (since it is a class function). For $SU(2)$, T is all $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. it acts with eigenvalue $e^{in\theta}, e^{i(n-2)\theta}, \dots, e^{-in\theta}$, each has multi $= 1$.

$$\text{character} = (e^{i(n+1)\theta} - e^{-i(n+1)\theta})(e^{i\theta} - e^{-i\theta})^{-1}$$

The generalized formula is <u>Weyl character formula</u>.

$$\chi_\lambda(t) = \frac{\sum_{w \in W} (\operatorname{sgn} w)\, e^{w(\lambda+\delta)\log(t)}}{\prod_{\alpha \in \Delta^+} (e^{\frac{1}{2}\alpha \log(t)} - e^{-\frac{1}{2}\alpha\log(t)})}$$

However, δ might not be <u>integral</u>. Thus rigorously,

$$\chi_\lambda(t) = \frac{\sum_{w \in W} (\operatorname{sgn} w)\, e^{(w(\lambda+\delta)-\delta)\log(t)}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha\log(t)})}$$

factor $e^{\delta \log(t)}$

λ is a differential of a multiplicative character of T.

② non-compact case

$G/\mathbb{R}$ reductive. θ: Cartan involution. $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. $\quad$ „ $\mathrm{Lie}(G)/\mathbb{R}$

<Def>: (Cartan subalg.) $\mathfrak{h}_0$ of $\mathfrak{g}_0$ ··· whose complexification in $\mathfrak{g}$ is a Cartan subalg. of $\mathfrak{g}$. Corresponding Cartan gp: $H = Z_G(\mathfrak{g}_0)$

<Thm>: Any CSA of $\mathfrak{g}_0$ is $Ad(G)$-conj to one that is stable under Cartan involution (i.e. is a θ-stable CSA).

Property: (1) not necessarily all conjugate

(2) finite many. up to conjugacy.

(3) Any two have same dimension.

(4) $Ad(K)$-conjugacy $\Leftrightarrow$ $Ad(G)$-conjugacy.

Example: $G = SL_2(\mathbb{R})$. $\mathfrak{h}_0$: $\mathbb{R}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad$ not connected.

H: $\left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right\}$ or $\left\{\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, r \in \mathbb{R}^\times\right\}$

<Prop>: (Structure): For θ-stable Cartan $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$. decompose $\mathfrak{h}_0$ w.r.t. θ-action gives $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$

(1) $\mathfrak{h}_0$ is maxl cpt $\Rightarrow$ H is connected.

(2) $H = TA$, $T = Z_K(\mathfrak{h}_0)$ $A = \exp(\mathfrak{a}_0)$

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regular element (where trace function is meaningful)

Lie gp level: Consider

$$\det((\lambda+1)1 - Ad(x)) = \lambda^n + \sum_{j=0}^{n-1} D_j(x)\lambda^j$$

For a single element x, the minimal index r with $D_r(x) \neq 0$ is the multiplicity of 1 as generalized eigenvalue for $Ad(x)$. Let l be the minimum index with $D_l(x) \neq 0$, varing $x \in G$. then l is the common dimension of all Cartan subalg. A <u>regular element</u> of G is an $x \in G$ s.t. $D_l(x) \neq 0$. (G_{reg} or G')

this is open (w.r.t. Zariski top.) and dense

<Thm>: Denote $\{H_i\}_{i=1}^r$ all θ-stable CSG. Then

(1) $G' \subseteq \bigcup_{i=1}^r \bigcup_{g \in G} gH_ig^{-1}$

(2) Each member of G' lies in just one CSG.

▸ Global Character

G: linear connected reductive with $\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0$ ($\mathfrak{t}_0 \to K$), θ, $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$.

take (π, V), V Hilbert, $\pi|_K$ unitary.

Question: Can't expect $\sum_i (\pi(x)v_i, v_i)$ converge

Solution: realize it as a linear functional (locally integrable).

for $f \in C_{comp}(G) \mapsto \pi(f): v \mapsto \int_G f(x)\pi(x)v\, dx$

<Def> (Trace Class) A trace class operator L on a Hilbert space V, is a bounded linear operator for which $\sum |(B^{-1}LBv_i, v_i)| < \infty$, for every orthonormal basis $\{v_i\}$ and every bounded invertible linear B. This sum is independent of B, called <u>trace of L</u>.

<Def> (Global Character) An adm. rep'n π of G has a <u>global character</u> if $\pi(f)$ is of trace class for all $f \in C_{comp}(G)$, and if $\mathrm{Tr}\,\pi(f) =: \Theta(f)$ a <u>distribution</u>.

<Thm> Every adm. rep'n π of G whose decomp. $\pi|_K = \sum_{\tau\in\hat{K}} n_\tau \tau$ has $n_\tau \le C\dim(\tau)$ has a global character.

<Thm>: π, π' are infinitesimally equiv. adm. rep'n and both have global characters $\Rightarrow$ their characters are equal.

<Thm> $\pi_1, \cdots, \pi_n$ are infinitesimally inequivalent, irr. adm. rep'n with global character $\Theta_1, \cdots, \Theta_n \Rightarrow \Theta_1, \cdots, \Theta_n$ are linearly independent.

(proof: Knapp: Rep'n ⋯ of semisimple gp. based on examples.)

Example: Character Formulas for $SL_2(\mathbb{R})$.

fix Haar measure: Iwasawa decomp $G = KN_pA_p$:

$$x = k_\theta n_s a_t = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad dx = \frac{d\theta}{2\pi}\, ds\, dt$$

$M_p = \{\pm I\}$, measure: total mass one.

$T = M_pA_p$. $B = K$, take $d(\pm)\,da$, $\frac{d\theta}{2\pi}$ respectively.

We need Weyl integration formula:

$$\int_G f(x)\,dx = \int_B (e^{-i\theta} - e^{i\theta}) F_f^B(\theta)\frac{d\theta}{2\pi} \pm \frac{1}{2}\int_T |e^t - e^{-t}| F_f^T(\pm a_t)\, d(\pm)\, dt$$

where $F_f^B(\theta) = (e^{i\theta} - e^{-i\theta})\int_{G/B} f(x k_\theta x^{-1})\, dx^*$ for $\sin\theta \neq 0$

$F_f^T(\pm a_t) = \pm|e^t - e^{-t}| \int_{G/T} f(x(\pm a_t)x^{-1})\, dx^*$ for $t \neq 0$

<Lem>: $f \in L^\infty_{cpt}(SL_2(\mathbb{R})) \Rightarrow F_f^B, F_f^T$ both L^∞ of cpt. supp.

If π has a global character Θ_π given by a local integrable func.

$$\begin{aligned} \mathrm{Tr}\,\pi(f) &= \int_G \Theta(x) f(x)\, dx \\ &= \int_B (e^{-i\theta} - e^{i\theta})\Theta(\theta) F_f^B(\theta)\frac{d\theta}{2\pi} \\ &\pm \frac{1}{2}\int_T |e^t - e^{-t}|\, \Theta(\pm a_t) F_f^T(\pm a_t)\, d(\pm)\, dt \end{aligned} \quad (*)$$

<Prop> If $\Theta(\theta)$, $\Theta(\pm a_t)$ are functions on B, T, s.t.

(1) $(e^{i\theta} - e^{-i\theta})\Theta(\theta)$, $\pm|e^t - e^{-t}|\Theta(\pm a_t)$ locally integrable

(2) $\Theta(a_t)$ is invariant under $t \mapsto -t$

$\Rightarrow$ function $\Theta(x) = \begin{cases} \Theta(\theta) & \text{if } x \text{ conj. to } k_\theta \\ \Theta(a_t) & \text{if } x \text{ conj to } \pm a_t \end{cases}$

is a well-defined locally integrable function on G.

And if $\mathrm{Tr}\,\pi(f)$ given by right side $(*)$ $\Rightarrow$ character of $\pi = \Theta(x)$.

<Lem> (How to calculate trace:)

X: $\boxed{\text{cpt } C^\infty \text{ mfld}}$, dx. $L \in C^\infty(X \times X)$ define a bounded operator T, on $L^2(X, dx)$: $\quad (Tf)(x) = \int_X L(x,y) f(y)\, dy$

Then T is of trace class with $\mathrm{Tr}\,T = \int_X L(x,x)\, dx$ — means for general ν

<Prop>: The character of $\boxed{\text{nonunitary}}$ principal series rep'n $U^{\sigma,\nu}$ of $SL_2(\mathbb{R})$ is given by

$$\Theta_{\sigma,\nu}(x) = \begin{cases} 0 & x \text{ conj to } k_\theta \\ \dfrac{\sigma(\pm)\left(e^{\nu \log(a_t)} + e^{-\nu \log(a_t)}\right)}{|e^t - e^{-t}|} & x \text{ conj to } \pm a_t \end{cases}$$

General Induced Character is easy to be calculated!

Proof: idea: express $U^{\sigma,\nu}$ as a kernel operator.

$$U^{\sigma,\nu}(\varphi)(k) = \int_K L(k,k')\, \varphi(k')\, dk'$$

on K with $L \in C^\infty(K \times K)$. then trace $= \int_K L(k,k)\, dk$

$$U^{\sigma,\nu}(x)(\varphi)(k) = e^{-(\nu+\rho)H(x^{-1}k)}\, \varphi(\kappa(x^{-1}k)) \qquad x \in G$$

where κ, H assign the M, A part of $B = MAN$.

$$\rightsquigarrow U^{\sigma,\nu}(x)(f)(k) = \int_G e^{-(\nu+\rho)H(x^{-1}k)} f(x)\, \varphi(\kappa(x^{-1}k))\, dx$$

$$= \int_G e^{-(\nu+\rho)H(x)} f(kx^{-1})\, \varphi(\kappa(x))\, dx \quad \text{under } x^{-1}k \mapsto x$$

$$= \int_{KN_pA_p} e^{(\nu+\rho)\log(a)} f(ka^{-1}n^{-1}k'^{-1})\, \varphi(k')\, dk'\, dn\, da$$

Take E the orthogonal projection: $E\varphi(k) = \int_{M_p} \sigma(m)\, \varphi(km)\, dm$

this projects to σ-factor (recall σ is a rep'n of M_p). Since $U^{\sigma,\nu}$ on the subspace that transforms according to σ, then

$$\mathrm{Trace}(U^{\sigma,\nu}) = \mathrm{Trace}(U^{\sigma,\nu} E), \text{ Thus } \varphi \in L^2(K) \text{ implies}$$

$$(U^{\sigma,\nu}E)(f)(k) = \int_{\boxed{KN_pA_p \times M_p}} e^{-(\nu+\rho)\log(a)} \sigma(m) f(ka^{-1}n^{-1}mk'^{-1})\, \varphi(k')\, dm\, dk'\, dn\, da$$

$$= \int_K \Big[\int_{\boxed{M_pA_pN_p}} e^{(\nu+\rho)\log a} \sigma(m) f(kmank'^{-1})\, dm\, da\, dn\Big] \varphi(k')\, dk'$$

$$\Rightarrow \mathrm{Tr}(U^{\sigma,\nu})(f) = \int_{K \times M_pA_pN_p} e^{(\nu+\rho)\log(a)} \sigma(m) f(kmank^{-1})\, dm\, da\, dn\, dk$$

$$= \int_T e^{(\nu+\rho)\log(a)} \sigma(\pm) \Big[\int_{K \times N_p} f(k(\pm a_t)nk^{-1})\, dk\, dn\Big] d(\pm)\, dt$$

$$= \int_T e^{\nu \log(a)} \sigma(\pm) \Big[|e^t - e^{-t}| \int_{G/T} f(x(\pm a_t)x^{-1})\, dx^*\Big] d(\pm)\, dt$$

Here we use one lemma: For $f \in C_{com}^{\infty}(G)$, $t \neq 0$

$$F_f^T(\pm a_t) = \pm|e^t - e^{-t}| \int_{G/T} f(x(\pm a_t)x^{-1})\,dx^* = \pm e^t \int_{K\times N} f(k(\pm a_t)nk^{-1})\,dk\,dn$$

proof: assume function $f>0$, function $h \geq 0$ on N_p, $\rightsquigarrow$

$$\int_{-\infty}^{\infty} h\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix} ds = |1-e^{-2t}| \int_{-\infty}^{\infty} h\begin{pmatrix}1 & (e^{-2t}-1)s\\ 0 & 1\end{pmatrix} ds$$

$$\Rightarrow \int_N h(n)\,dn = |1-e^{-2t}| \int_N h(a_t^{-1} n a_t n^{-1})\,dn$$

put $h(n) = f(k(\pm a_t)nk^{-1})$, integrate over K, gives

$$\int_{K\times N} f(k(\pm a_t)nk^{-1})\,dk\,dn = |1-e^{-2t}| \int_{K\times N} f(kn(\pm a_t)n^{-1}k^{-1})\,dk\,dn$$

$$= |1-e^{-2t}| \int_{G/T} f(x(\pm a_t)x^{-1})\,dx^* \quad \square$$

On the right side, change x to xw, $w = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ $\because$ without affecting $\forall$ integrals, then we can change variables on T to bring w out, so that is conjugates a_t. Averaging gives:

$$\mathrm{Tr}(U^{\sigma,\nu})(f) = \pm\frac{1}{2}\int_T \frac{\sigma(\pm)\left(e^{\nu\log(a_t)} + e^{-\nu\log(a_t)}\right)}{\Theta(\pm a_t)} F_f^T(\pm a_t)\,d(\pm)\,dt$$

by comparing with Weyl integration. $\square$

<Cor> use $\boxed{\mathcal{D}_n^+ \oplus \mathcal{D}_n^-} \subseteq \begin{cases} U^{+,(n-1)\rho} & \text{if } n \text{ even} \\ U^{-,(n-1)\rho} & \text{if } n \text{ odd} \end{cases}$, $n \geq 1$

for $n \geq 1$

$$\Rightarrow \Theta_n(x) = \begin{cases} \dfrac{-e^{i(n-1)\theta} + e^{-i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}} & \text{if } x \text{ conj to } k_\theta \\[2ex] (\pm)^n \dfrac{e^{(n-1)t}(1-\mathrm{sgn}(t)) + e^{-(n-1)t}(1+\mathrm{sgn}(t))}{|e^t - e^{-t}|} & \text{if } x \text{ conj to } \pm a_t \end{cases}$$

Separately,

Proposition 10.14. For $n \geq 1$ in $\mathrm{SL}(2,\mathbb{R})$, the character of $\mathcal{D}_n^+$ is given by the locally integrable function

see Knapp's book!

$$\Theta_n^+(x) = \begin{cases} \dfrac{-e^{i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}} & \text{if } x \text{ conjugate to } k_\theta \\[2ex] (\pm)^n \dfrac{e^{(n-1)t}(1-\mathrm{sgn}\,t) + e^{-(n-1)t}(1+\mathrm{sgn}\,t)}{2|e^t - e^{-t}|} & \text{if } x \text{ conjugate to } \pm a_t. \end{cases}$$

The character of $\mathcal{D}_n^-$ is given by the locally integrable function

$$\Theta_n^-(x) = \begin{cases} \dfrac{e^{-i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}} & \text{if } x \text{ conjugate to } k_\theta \\[2ex] (\pm)^n \dfrac{e^{(n-1)t}(1-\mathrm{sgn}\,t) + e^{-(n-1)t}(1+\mathrm{sgn}\,t)}{2|e^t - e^{-t}|} & \text{if } x \text{ conjugate to } \pm a_t. \end{cases}$$

▷ Differential Equations

<Def> (transposed operator D^t): Let D a linear partial differential operator on a connected open set $U \subseteq \mathbb{R}^n$. def'n $D^t : D \mapsto D^t$ is $\mathbb{C}$-linear, it reverses order in composition, it sends $a(x)I$ into itself, and it sends $\partial/\partial x_j$ to $-\partial/\partial x_j$. From integration by parts, it follows that

$$\int_U (Df_1)(x) f_2(x)\, dx = \int_U f_1(x)\, D^t f_2(x)\, dx$$

where f_1, f_2 are sm. at least one has cpt. supp..

We use this to motivate the def. of $D\Theta$ when Θ is a distribution.

(given by $(D\Theta)(f) = \Theta(D^t f)$ for $f \in C_{cpt}^\infty(G)$).

For left-invariant differential operator D, compute D^t as:

<Lem>: Let X in $\mathfrak{g} \curvearrowright$ as left differentiation, and let f_1, f_2 sm. on G, with at least one of them cpt. supp.

$$\Rightarrow \int_G (Xf_1)(x) f_2(x)\, dx = -\int_G f_1(x)(Xf_2)(x)\, dx$$

proof: $\int_G (Xf_1)(x) f_2(x)\, dx = \int_G \frac{d}{dt}[f_1(x\exp(tX))]_{t=0} f_2(x)\, dx$

$$= \frac{d}{dt}\left[\int_G f_1(x\exp(tX)) f_2(x)\, dx\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\int_G f_1(x) f_2(x\exp(tX)^{-1})\, dx\right]_{t=0}$$

$$= \int_G f_1(x)(-Xf_2)\, dx$$

<Cor> The restriction $D \mapsto D^t$ to $U(\mathfrak{g})$ is an associate alg. anti-automorphism of $U(\mathfrak{g})$, that extends from Lie alg antiauto.: $X \mapsto -X$. The map $D \mapsto D^t$ carries $Z(\mathfrak{g})$ itself.

<Prop>: π: irr. adm. rep'n st $\pi(z) = \chi(z)I$ for $z \in Z(\mathfrak{g}^{\mathbb{C}})$, Θ_π. If z is considered as a left-invariant differential operator,

$$z\Theta = \chi(z)\Theta.$$

Proof: $f \in C_{comp}^\infty(G) \Rightarrow (z\Theta)(f) = \Theta(z^t f)$. Let $\{v_i\}$: orthogonal basis of K-fin. vector for π. Then

$$(\pi(z^t f)v_i, v_i) = \int_G (\pi(x)v_i, v_i)(z^t f)(x)\, dx$$

$$= \int_G z(\pi(x)v_i, v_i) f(x)\, dx \quad \text{(by lemma)}$$

$$= \int_G (\pi(x)\pi(z)v_i, v_i) f(x)\, dx$$

by $X(\pi(g)u, v) = \frac{d}{dt}(\pi(g\exp(tX))u, v)\big|_{t=0}$
$= \frac{d}{dt}(\pi(\exp(tX))u, \pi(g)^* v)\big|_{t=0}$
$= (\pi(g)\pi(X)u, v)$