Contents

This note answers the question"How does the classical theory of modular forms connect with the theory of automorphic forms on GL_2?" and aims to use the representation theory to study them. At last we give some results about Whittaker models and the L-functions. This is essentially the material in the first ten sections of Jacquet and Langland [JL70]. Other standard references for some or all of this material include books by Bump [Bum97] and Gelbart [Gel75].

1 Classical modular forms

Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index. For such a Γ, it acts on $\mathcal{H}$ ina properly discontinuous way. The quotient $\Gamma\backslash\mathcal{H}$ will possesses a fundamental domain $\mathcal{F}$ which has finite volume (under the $\mathrm{SL}_2(\mathbb{R})$-invariant measure $\frac{\mathrm{d}x\mathrm{d}y}{y^2}$).

We will focus on two cases in parallel: $\Gamma = \Gamma_0(N)(N > 1)$ *and* $\mathrm{SL}_2(\mathbb{Z})$.

Definition 1 (cusp of Γ)**.** A cusp of Γ is a Γ-orbit in $\mathbb{Q} \cup \{\infty\}$.

Example 1. Because $\mathrm{SL}_2(\mathbb{Z})$ acts transtively on $\mathbb{Q} \cup \infty = \mathrm{SL}_2(\mathbb{Q})/B(\mathbb{Q})$, there is one cusp when $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. For $N = 2$, it has 3 cusps.

More generally, the number of cusps of Γ is finite: $\#\Gamma\backslash\mathrm{SL}_2(\mathbb{Q})/B(\mathbb{Q})$, and $\Gamma\backslash\mathcal{H}$ can be compactified by adding these cusps:

$$\overline{\Gamma\backslash\mathcal{H}} = \Gamma\backslash(\mathcal{H} \cup \mathbb{Q} \cup \{\infty\})$$

Definition 2 (**Holomorphic Modular Forms**)**.** Let χ be a finite order character of Γ. Let $f(z)$ be a holomorphic modular form of level Γ and weight k with character χ, which is equivalent to the following:

- f is a holomorphic function on the upper half plane $\mathcal{H} = \{z : \mathrm{Im}(z) > 0\}$.
- (automorphy condition) For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have

  $$f(\gamma \cdot z) = \chi(\gamma)(cz + d)^k f(z)$$

  For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$, one can define a j-cocycle: $j(\gamma, z) = (cz + d)$, and the operator by $(g|_k \gamma)(z) = \det(\gamma)^{k/2} j(\gamma, z)^{-k} g(\gamma \cdot z)$. So this condition can be rewritten as $f|_k \gamma = \chi(d) f$ for $\gamma \in \Gamma$.
- (cusp condition) f is holomorphic at the cusps. It requires that $f|_k \gamma$ be holomorphic at infinity for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. f is called a cusp form if $f = 0$ at all cusps.

Example 2. Let χ be a Dirichlet character modulo N: it defines a character on $\Gamma_0(N)$, by evaluating χ at the upper left entry.

We denote the $M_k(N)$ as the $\mathbb{C}$-vector space of the holomorphic modular forms with level $\Gamma_0(N)$ and weight k, $S_k(N)$ as the $\mathbb{C}$-vector space of the cupidal modular forms with level $\Gamma_0(N)$ and weight k.

1.1 Basic Properties

1.1.1 Fourier Expansion

because f is a function on the strip $\{x + iy : -1/2 \leq x < 1/2,\ y > 0\}$, the map $z \mapsto q = e^{2\pi iz}$ sends f to a holomorphic function $\tilde{f}(q)$ on the puncturted disc which has a Laurent expansion about 0. The holomorphic property means $a_n = 0$ if $n < 0$, and f is cupidal if and only if the zeroth Fourier coefficient $a_0(f)$ in the Fourier expansion of f at every cusp is zero.

1.1.2 Finite Dimensionality

$M_k(N)$ and $S_k(N)$ are finite-dimensional as vector spaces (proof uses the Riemann-Roch theorem). For example:

$$\dim M_k(1) = \begin{cases} \frac{k}{12} + 1, \text{ if } k \neq 2 \bmod 12 \\ \frac{k}{12}, \text{ if } k = 2 \bmod 12 \end{cases}$$

This is because given a Eisenstein series E_k,

$$M_k(1) = \mathbb{C} \cdot E_k \oplus S_k(1)$$

We will come back to this later.

1.1.3 Ring Structure

If $f_i \in M_{k_i}(N)$, then $f_1 \cdot f_2 \in M_{k_1+k_2}(N)$. Thus $\oplus_k M_k(N)$ has a graded ring structure. Moreover, if one of f_i's is cuspidal, so is $f_1 \cdot f_2$.

1.1.4 Petersson Inner Product

The space $S_k(N)$ is equipped with a natural inner product:

$$\langle f_1, f_2 \rangle_k = \int_{\Gamma\backslash\mathcal{H}} f_1(z)\overline{f_2(z)} y^k \cdot \frac{dxdy}{y^2}.$$

It remains convergent as long as one of the functions is cuspidal.

1.1.5 Hecke Operator for $\mathrm{SL}_2(\mathbb{Z})$

Let us first assume that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Let $g \in \mathrm{GL}_2^+(\mathbb{Q})$ and consider the double coset $\Gamma g \Gamma$ as a finite union:

$$\Gamma g \Gamma = \bigcup_i \Gamma a_i$$

Then we define an operator $M_k(\Gamma) \to M_k(\Gamma)$:

$$f|_k[g] = \sum_i f|_k a_i$$

It is well defined (i.e. independent of the choice of a_i).

Let $M(n)$ be the determinant n integral matrix. We have

$$M(n) = \bigcup_{d|a, ad=n} \Gamma t(a,d)\Gamma, \ where\ t(a,d) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

We set

$$T_n f = n^{k/2-1} \sum_{d|a, ad=n} f|_k [t(a,d)].$$

For example, T_p is the operator defined by the double coset $t(p,1)$. Then we have explicitly

$$\begin{aligned} M(p) &= \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma \\ &= \bigcup_{k=0}^{p-1} \Gamma \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix} \cup \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

we have the definition of the hecke operator as following:

$$(T_p f)(z) = p^{k-1} f(pz) + \frac{1}{p} \sum_{k=0}^{p-1} f\left(\frac{z+k}{p}\right).$$

Proposition 1.1.

- *Effects on Fourier coefficients:*

$$a_n(T_p f) = a_{pn}(f) + p^{k-1} a_{n/p}(f)$$

  where the second summand is interpreted to be 0 *if* $p \nmid n$.

- *if* $(n,m) = 1$, *then* $T_n T_m = T_{nm} = T_{mn}$. *Moreover,*

$$T_p T_{p^r} = T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}$$

- T_n *is self adjoint with respect to the Petersson inner product.* T_n *preserves* S_k, *and the action of* T_n *on* S_k *can be simultaneously diagonalized.*

Thus we see that the linear span of the T_n's form an commutative algebra which generated by T_p's. Thanks to the last property, we can define an **eigenform** as a modular form which is an eigenvector for all Hecke operators T_n. If f is a cuspidal Hecke eigenform with eigenvalues λ_n for T_n, then

$$a_n(f) = \lambda_n \cdot a_1(f)$$

Theorem 1 (Multiplicity One)**.** *If f is a normalized cuspidal eigenform (i.e. $a_1(f) = 1$), then f is completely determined by its Hecke eigenvalues.*

1.1.6 Euler Products

The fact that the Fourier coefficients of f, a normalized cupidal eigenform, are multiplicative implies that $L(f, s)$ has an Euler product:

$$\begin{aligned} L(f,s) &= \prod_p (\sum_k a_{p^k} p^{-ks}) \\ &= \prod_p \frac{1}{1 - a_p(f)p^{-s} + p^{k-1-2s}} \end{aligned}$$

It has an analytic continuation, satisfies appropriate functional equation.

1.1.7 Hecke Operators for $\Gamma_0(N)$

We can still define the operators T_n as before, the algebra is still commutative and generated by all the T_p. But T_n is self-adjoint only if $(n, N) = 1$. So we can only simultaneously diagonalize the actions of T_n with $(n, N) = 1$. To be precise, Let $\Delta_0(N) = \{\gamma \in \mathrm{M}_2(\mathbb{Z}) : \det(\gamma) > 0, N \mid c, (a, N) = 1\}$ For $\alpha \in \Delta_0(N)$, we defines:

$$T_\alpha(f)(z) = \det(\alpha)^{k-1}\chi(\alpha^{-1})(cz + d)^{-k} f(\frac{az + b}{cz + d})$$

Since

$$\{\alpha \in \Delta_0(N) : \det(\alpha) = n\} = \bigcup_{ad=n, a>0, (a,q)=1} \bigcup_{0\leq b\leq d-1} \Gamma_0(N) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

We define similarly:

$$T_n(f)(z) := \sum_j (T_{\alpha_j} f)(z) = n^{k-1} \sum_{ad=n, a>0, 0\leq b\leq d-1} \chi(a)d^{-k} f(\frac{az + b}{d})$$

With this definition, one sees easily that the T_n's preserve the modularity and cupsidality. One can give the action of the T_p, for p prime, on the fourier expansion:

- If $(p, N) = 1$, $T_p(f)(z) = \sum_n a_{pn}(f)q^n + \chi(p)p^{k-1} \sum_n a_n(f)q^{pn}$. We call p is a good prime.
- If $p \mid N$, $T_p(f)(z) = \sum_n a_{pn}(f)q^n$. We call p a bad prime.

The algebra is still commutative. An important observation is T_n is self adjoint only if $(n, N) = 1$, so we can only simultaneously diagonalize all hecke operator at good primes. In particular, if f is such an eigenfunction with eigenvalues $\{\lambda_f\}$, one has $a_p(f) = \lambda_p a_1(f)$ at good p.

1.1.8 Newforms and Oldforms

We want to establish the Euler Product and Multiplicity One theorem for $M_k(N)$. Some simple examples show that it will not always exist for all modular forms. Thus we will introduce new forms.

Suppose χ defines a Dirichlet character modulo N', for $N' \mid N$. For any cusp form g in $\mathcal{S}_k(N', \chi)$, one checks easily that $z \mapsto g(dz)$ defines an element of $\mathcal{S}_k(N, \chi)$, for any $d \mid (N/N')$. Let

$$\mathcal{S}_k^{\text{old}}(N, \chi) = \bigcup_{\chi \text{ factors through } N'|N,\ d|(N/N')} \{z \mapsto g(dz) : g \in \mathcal{S}_k(N', \chi)\}$$

be the space of **oldforms**, and let

$$\mathcal{S}_k^{\text{new}}(N, \chi) = \mathcal{S}_k^{\text{old}}(N, \chi)^{\perp}$$

be the space of **newforms** (it may be zero). Then it can be shown that the **whole** Hecke algebra can be diagonalized on the space of newforms. Then we can define the normalized Hecke eigenforms. Their L-series have an Euler product, which is absolutely convergent if $\Re(s) > 1 + k/2$:

$$L(s, f) := \sum_n \frac{a_n(f)}{n^s} = \prod_p L(s, f_p)$$

with

$$\begin{aligned} L(s, f_p) &= \left(1 - a_p(f)p^{-s} + \chi(p)p^{k-1-2s}\right)^{-1} \\ &= \left(1 - \alpha_1(p, f)p^{-s}\right)^{-1} \left(1 - \alpha_2(p, f)p^{-s}\right)^{-1} \end{aligned}$$

at a good prime p, and

$$L(s, f_p) = \left(1 - a_p(f)p^{-s}\right)^{-1}$$

at a bad prime, along with an analytic continuation and functional equation.

Similarly, we have the multiplicity one theorem: the newforms can be distinguished from one another by their eigenvalues with respect to the T_p's with $(p, N) = 1$.

2 Automorphic Forms

One can generalize the factor of automorphy to certain general group G (in place of SL_2), namely those real semisimple group G such that the symmetric space G/K has a complex structure. In that case, G/K is a **hermitian symmetric domain**.

An example is the symplectic group $G = \mathrm{Sp}_{2n}$, where

$$G/K = \left\{ Z = X + iY \in M_n(\mathbb{C}) : Z^t = Z, Y > 0 \right\}$$

is the **Siegel upper half space**. In this case, one has the theory of **Siegel modular forms**, with

$$j(g, Z) = CZ + D, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R}), Z \in G/K$$

Here $\mathrm{Sp}_2 n$ is the symplectic group which lies in $M_{2n \times 2n}$.

2.1 Automorphic Forms on $\mathrm{GL}_2(\mathbb{R})^+$

Observe that $\mathrm{GL}_2(\mathbb{R})^+$ acts on $\mathcal{H}$ with stabilizer $K = \mathrm{SO}_2(\mathbb{R})$, we have the following definition:

Definition 3 (**Automorphic Forms for** $\mathrm{GL}_2(\mathbb{R}^+)$)**.** Given a holomorphic modular form f, we consider the function defined on $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$ by

$$F(g) := \left(f|_k \, g \right)(i) = (\det(\gamma))^{k/2} j(\gamma, i)^{-k} f \left(\frac{ai + b}{ci + d} \right).$$

This is the automorphic form for $\mathrm{GL}_2(\mathbb{R})$ associated to f. It has the following properties.

- (Γ action) For $\gamma \in \Gamma$, it satisfies
$$F(\gamma g) = (f|_k \gamma g)(i) = \chi(d)(f|_k g)(i) = \chi(d) f(g)$$
- (K finite) For $\kappa = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in K = \mathrm{SO}_2(\mathbb{R})$. we have
$$F(g\kappa) = e^{-ik\theta} F(g)$$
- $F(\mathrm{diag}(\lambda, \lambda) g) = \omega(\lambda) F(g)$, where where $\omega(\lambda)$ is 1 when $\lambda > 0$ and is $\chi(-1)$ when $\lambda < 0$.
- (Growth condition) For any norm $|| \cdot ||$ on $\mathrm{GL}_2(\mathbb{A})$, there exists a real number $A > 0$, such that $\phi_f(g) \lesssim ||g||^A$. In other words, ϕ_f is of moderate growth. When f is cuspidal, it is bounded.

2.1.1 Cusp Form

A cusp form is defined by the vanishing of the 0th Fourier coefficient at each cusp. At the cusp $i\infty$,

$$a_0(f) = \int_0^1 f(x+iy)\mathrm{d}x \quad \text{for any } y.$$

We see that $a_0(f) = 0$ if and only if

$$\phi_N(g) := \int_{\mathbb{Z}\backslash\mathbb{R}} \phi_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\mathrm{d}x = 0$$

for all g. Recall that the cusps of Γ are in bijection with $\Gamma\backslash\mathrm{SL}_2(\mathbb{Q})/B(\mathbb{Q})$. If x is a cuspidal point, its stabilizer in SL_2 is a Borel subgroup B_x defined over $\mathbb{Q}$. Then the 0th coefficient of f at x vanishes if and only if

$$\int_{(\Gamma\cap N_x)\backslash N_x} \phi_f(ng)\mathrm{d}n = 0.$$

Thus f is cuspidal if and only if the above integral is 0 for any Borel subgroup defined over $\mathbb{Q}$.

2.1.2 Differential Operator: Lie Algebra

The differential operators of the smooth functions on $\Gamma\backslash\mathrm{GL}_2(\mathbb{R})$ is the complexified Lie algebra $\mathfrak{gl}_2(\mathbb{C})$, acting by right infinitesimal translation: if $X \in \mathfrak{g}_0 = \mathfrak{gl}_2(\mathbb{R})$, then

$$(X\phi)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(g\cdot\exp(tX))\bigg|_{t=0}.$$

This defines a left-invariant first-order differential operator on smooth functions on $\mathrm{SL}_2(\mathbb{R})$. To see this, if we write:

$$\mathrm{SL}_2(\mathbb{R}) = N\cdot A\cdot K \cong \mathbb{R}\times\mathbb{R}_+^\times\times S^1.$$

Explicitly,

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\cdot\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Thus we can regard ϕ_f as a funciton of (x, y, θ):

$$\phi_f(x, y, \theta) = e^{ik\theta}y^{k/2}f(x+iy).$$

Lemma 2.1. *f is holomorphic on $\mathcal{H}$ if and only if*

$$L\phi_f = 0$$

where

$$L = -2iy\frac{\partial}{\partial \bar{z}} + \frac{i}{2}\frac{\partial}{\partial \theta}.$$

In the lie algebra perspective, we have these basis:

$$H = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{k} = \mathrm{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C}, E = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, F = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

They satisfy:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H$$

Thus F lowers eigenvalues of H by 2, whereas E increases it by 2.

The correspondence is if we think of F as a differential operator, then

$$F = e^{-2i\theta} \cdot L$$

2.1.3 Casimir Operator

The action of $\mathfrak{sl}_2(\mathbb{C})$ on the smooth functions of $\mathrm{SL}_2(\mathbb{R})$ as a left-invariant differential operators extends to an action of the universal enveloping algebra $U(\mathfrak{sl}_2)$. It is well known that is a canonical element in $Z(\mathfrak{g})$ called the Casimir operator Δ. In the case of SL_2, one has:

$$\Delta = -\frac{1}{4}H^2 + \frac{1}{2}H - 2EF, \quad Z(\mathfrak{g}) = \mathbb{C}[\Delta]$$

If write this as a differential operator, we have:

$$\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + y\frac{\partial^2}{\partial x \partial \theta}$$

If f a holomorphic modular form, then

$$\Delta\phi_f = \frac{k}{2}(1 - \frac{k}{2})\phi_f$$

2.1.4 Passage from SL_2 to GL_2

It relies on the identification:

$$\Gamma \backslash \mathrm{SL}_2(\mathbb{R}) \cong Z(\mathbb{R})\Gamma' \backslash \mathrm{GL}_2(\mathbb{R})$$

Here $\Gamma = \Gamma_0(N)$ and

$$\Gamma' = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{GL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

Proposition 2.1 ([Gel75], Prop. 3.1)**.** *The map $f \mapsto \phi_f$ defines an isomorphism of $M_k(\Gamma)$ to the space $V_k(\Gamma')$ of smooth functions ϕ of $Z(\mathbb{R})\Gamma' \backslash \mathrm{GL}_2(\mathbb{R})$ satisfying:*

- *ϕ is smooth;*
- *$\phi(gr_\theta) = e^{ik\theta}\phi(g)$;*
- *$F\phi = 0$ (F is lowering operator)*
- *ϕ is of moderate growth.*

Moreover, the image of the space of cusp forms consists of those functions ϕ such that for ANY Borel $\mathbb{Q}$-subgroup $B = T \cdot N$, the constant term ϕ_N along the unipotent radical N is zero. Further, the image of cusp forms is contained in $L^2\left(Z(\mathbb{R})\Gamma' \backslash \mathrm{GL}_2(\mathbb{R})\right)$.

2.1.5 Hecke Operator

For $\alpha \in \mathrm{GL}_2(\mathbb{Q})$, we have the Hecke operator T_α on the space of functions on $\Gamma \backslash \mathrm{GL}_2(\mathbb{R})$ by:

$$(T_\alpha \phi)(g) = \sum_{i=1}^{r} \phi(a_i g)$$

if

$$\Gamma \alpha \Gamma = \bigcup_{i=1}^{r} \Gamma a_i$$

The definition is independent of the choice of representatives a_i. The reason for left Γ-invariance is preserved is that if $\gamma \in \Gamma$, then $\{\Gamma a_i \gamma\}$ is a permutation of $\{\Gamma a_i\}$. Let α_p denote the diagonal matrix $\mathrm{diag}(p, 1)$. Earlier, we have defined an action of $\Gamma \alpha_p \Gamma$ on a modular form f:

$$T_{\alpha_p} f := f|_k [\alpha_p] = \sum_i f|_k a_i$$

if $\Gamma\alpha_p\Gamma = \cup_i \Gamma a_i$. This operator is the Hecke operator T_p:

$$T_p = p^{k/2-1} T_{\alpha_p}.$$

Proposition 2.2. *The isomorphism $M_k(\Gamma) \longrightarrow V_k(\Gamma')$ is an isomorphism of Hecke modules, i.e. for any prime p,*

$$\phi_{T_\alpha f} = T_{\alpha_p}\phi_f.$$

Proof.

$$\begin{aligned}\phi_{T_\alpha f}(g) &= \left((T_\alpha f)|_k\, g\right)(i) = ((\sum_j f|a_j)|g)(i) \\ &= \sum_j (f|(a_j g))(i) = \left(T_{\alpha_p}\phi_f\right)(g).\end{aligned}$$

□

2.2 Automorphic Forms (local)

Wee can give an general description of the automorphic forms.

Let G be a reductive linear algebraic group defined over $\mathbb{Q}$, and let Γ be an *arithmetic group*. We shall assume for simplicity that $\Gamma \subset G(\mathbb{Q})$. By an automorphic form on G with respect to an arithmetic group Γ, we mean a function ϕ on $\Gamma\backslash G(\mathbb{R})$ satisfying:

- ϕ is smooth;
- ϕ is of moderate growth;
- ϕ is right K-finite;
- ϕ is $Z(\mathfrak{g})$-finite, i.e., $\dim(Z(\mathfrak{g}(\phi))) < \infty$. Equivalently, ϕ is annihilated by an ideal of finite codimension in $Z(\mathfrak{g})$.

Remark. We can give a description about the $Z(\mathfrak{g})$. The following theorem belongs to Harish-Chandra:

Theorem 2. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. There is a universal enveloping algebra homomorphism $\psi : Z(\mathfrak{g}) \to U(\mathfrak{h})$, satisfies:*

- *ψ is an isomorphism of $Z(\mathfrak{g})$ onto $U(\mathfrak{h})^W$, where $U(\mathfrak{h})^W$ denotes the subalgebra which is invariant under the action of the Weyl group W.*
- *For all $\lambda, \mu \in \mathfrak{h}^*$, we have $\chi_\lambda = \chi_\mu$ if and onlt if they are W-linked.*

- *Every central character* $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ *is of the form* χ_λ *for some* $\lambda \in \mathfrak{h}^*$.

Let $\mathcal{A}(\Gamma\backslash G)$ denote the space of automorphic forms on G (sometimes $\mathcal{A}(G,\Gamma)$). Choose $\rho \in \widehat{K}$ a finite set of irreducible representations of K and J is an ideal of finite codimension in $Z(\mathfrak{g})$, then we let:

$\mathcal{A}(\Gamma\backslash \mathrm{GL}_2(\mathbb{R}), J)$ be the subspace consisting of functions which are killed by J;

$\mathcal{A}(\Gamma\backslash \mathrm{GL}_2(\mathbb{R}), J, \rho)$ be the subspace (of $\mathcal{A}(\Gamma\backslash G, J)$) consisting of function ϕ such that the finite dimensional representation of K generated by ϕ is supported on ρ.

Example 3.

$$M_k(\Gamma) \subset \mathcal{A}(\Gamma\backslash \mathrm{GL}_2(\mathbb{R}), J = \langle \Delta - \frac{k}{2}(\frac{k}{2} - 1)\rangle, \rho : r_\theta \mapsto e^{ik\theta})$$

2.2.1 Cusp

Definition 4. If f is automorphic, then f is cuspidal if for any parabolic $\mathbb{Q}$-subgroup $P = MN$ (Levi decomposition) of G, we have

$$f_N(g) := \int_{(\Gamma\cap N)\backslash N} f(ng)\mathrm{d}n = 0.$$

The function f_N on G is called the *constant term of f along N*.

To check for cuspidality, it suffices to check for a set of representatives for the Γ-orbits of maximal parabolic $\mathbb{Q}$-subgroups. We let $\mathcal{A}_0(G,\Gamma)$ be the space of cusp forms.

2.2.2 Fourier coefficients

For any unitary character χ of N which is left invariant under $\Gamma\backslash N$, we set:

$$f_{N,\chi}(g) = \int_{(\Gamma\cap N)\backslash N} f(ng) \cdot \overline{\chi(n)}\mathrm{d}n.$$

This is the χ-th Fourier coefficient of f along N.

If N is abelian, then

$$f(g) = \sum_\chi f_{N,\chi}(g)$$

so that f can be recovered from its Fourier coefficients along N.

To see this, consider the function on $N(\mathbb{R})$:

$$\Phi_g(x) = f(xg)$$

It is in fact a function on

$$(\Gamma \cap N)\backslash N \cong (\mathbb{Z}\backslash\mathbb{R})^r.$$

So we can expand this in a Fourier series:

$$\Phi_g(x) = \sum_{\chi} a_\chi(g)\chi(x)$$

where

$$a_\chi(g) = \int_{(\Gamma\cap N)\backslash N} \overline{\chi(x)} \cdot f(xg)\mathrm{d}x = f_{N,\chi}(g)$$

Putting $x = 1$ in the Fourier series gives the assertion.

2.2.3 $(\mathfrak{g}, K)$-module structure

Definition 5. Let V be a $\mathfrak{g}$-module that is also a module for K (for the moment we ignore the topology of K). Then V is called a $(\mathfrak{g}, K)$-module if the following three conditions are satisfied:

(1) $k \cdot X \cdot v = \mathrm{Ad}(k)X \cdot k \cdot v \quad$ for $v \in V, k \in K, X \in \mathfrak{g}$.

(2) If $v \in V$ then Kv spans a $finite$ dim. vector subspace of V, W_v, such that the action of K on W_v is continuous.

(3) If $Y \in \mathfrak{k}$ and if $v \in V$ then $d/dt_{t=0} \exp(tY)v = Yv$.

If V and W are $(\mathfrak{g}, K)$-modules then we denote by $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$ the space of all $\mathfrak{g}$-homomorphisms that are *also* K homomorphisms. V and W are said to be equivalent if there is an invertible element in $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$. We denote by $\mathrm{C}(\mathfrak{g}, K)$ the category of all $(\mathfrak{g}, K)$-modules with $\mathrm{Hom}_{\mathfrak{g},K}$ as morphism set.

Theorem 3. *$\mathcal{A}(G, \Gamma)$ is a $(\mathfrak{g}, K)$-module.*

Theorem 4. *Fix an ideal $J$of finite codimension in $Z(\mathfrak{g})$. Then $\mathcal{A}(G, \Gamma, J)$ is an admissible $(\mathfrak{g}, K)$-module. In particular, for any irreducible $(\mathfrak{g}, K)$-mod π,*

$$\dim \mathrm{Hom}_{\mathfrak{g},K}(\pi, \mathcal{A}(G, \Gamma)) < \infty$$

Thus we see the entrance of representation theory.

2.2.4 Hecke Algebra

Beside the structure of $(\mathfrak{g}, K)$-module, $\mathcal{A}(G, \Gamma)$ also has a Hecke algebra module:

The Hecke operator is defined as before, We can equivalently think of $\Gamma\alpha\Gamma$ as the characteristic function of this double set, and the Hecke algebra for Γ is the algebra of bi-Γ-invariant functions on $G(\mathbb{Q})$ which supported on finitely many cosets. The multiplication is by **convolution**.

Since the $(\mathfrak{g}, K)$-action is by right translation, while the Hecke operator is a sum of left translation, they commutes. Thus, if π is an irreducible $(\mathfrak{g}, K)$-submodule, then the Hecke algebra acts on:

$$\mathcal{H}(G, \Gamma) \curvearrowright \mathrm{Hom}_{\mathfrak{g},K}(\pi, \mathcal{A}(G, \Gamma)), \quad ([\Gamma\alpha\Gamma]f)(\pi) := [\Gamma\alpha\Gamma](f(\pi))$$

By the admissibility, this Hom-space is finite dimensional.

From this point of view, we can corresponde the representation theory to the modular forms.

2.2.5 Representation Theory

We will define one type of infinite irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$. The $\mathrm{GL}_2(\mathbb{R})$ case is similar to it. For details, see [Bum97], Proposition 2.5.2. There is also a classification of irreducible $(\mathfrak{g}, K)$-modules in [Bum97], Theorem 2.5.5 or [JL70], Section 5.

Let $n \geq 2$ is an integer, $\mathcal{H}$ is the upper half plane. Consider:

$$\mathscr{D}_n^+ = \left\{ f : \mathcal{H} \to \mathbb{C} \text{ holomorphic} \middle| ||f||^2 = \int_{\mathcal{H}} |f(z)|^2 y^{n-2} \, \mathrm{d}x \, \mathrm{d}y < \infty \right\}.$$

Define the $\mathrm{SL}_2(\mathbb{R})$ action on $\mathscr{D}_n^+$ by

$$\pi_n \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) f(z) = (-bz + d)^{-n} f\left(\frac{az - c}{-bz + d} \right)$$

The norm $\| \cdot \|$ gives $\mathscr{D}_n^+$ a Hilbert space structure. It is the holomorphic discrete representation of $\mathrm{SL}_2(\mathbb{R})$, $(\pi_n, \mathscr{D}_n^+)$. Moreover, define $\mathrm{SL}_2(\mathbb{R})$ action on $\mathscr{D}_n^- = \left\{ \bar{f} \mid f \in \mathscr{D}_n^+ \right\}$ by $\pi_{-k}(g)\bar{f} = \overline{\pi_k(g)f}$. These representations are called the **discrete series representation**.

For a classical modular form f corresponding to the automorphic form ϕ on SL_2, we proved that ϕ is annihilated by the lowering operator F (which is a lie algebra action). Then the set:

$$\{\phi, E\phi, E^2\phi, ...\}$$

are eigenfunctions of K with eigenvalues $k, k+2, \ldots$. It is a $(\mathfrak{g}, K)$-module.

We conclude that ϕ generates the holomorphic discrete series $\mathcal{D}_{|k|}^{\mathrm{sign}(k)}$ of minimal weight k, and

$$M_k(\Gamma) \cong \mathrm{Hom}_{\mathfrak{g},K}(\mathcal{D}_{|k|}^{\mathrm{sign}(k)}, \mathcal{A}(G,\Gamma))$$

This is an isomorphism of the Hecke algebra modules.

Given $l \in \mathrm{Hom}_{\mathfrak{g},K}(\mathcal{D}_{|k|}^{\mathrm{sign}(k)}, \mathcal{A}(G,\Gamma))$, one can take the lowest weight vector in $l(\mathcal{D}_{|k|}^{\mathrm{sign}(k)})$, as the converse construction.

Remark. they are the component at infinity for the automorphic representation associated to a cuspidal modular form.

Remark. Similarly, for a **principal series representation** π_s:

$$\pi_s = \mathrm{Ind}_B^{\mathrm{SL}_2} \delta_B^{1/2+s}$$

Then the space of **Maass Forms** of Γ, parameter s, is isomorphic (in the Hecke algebra module sense) to

$$\mathrm{Hom}_{\mathfrak{g},K}(\pi_s, \mathcal{A}(G,\Gamma))$$

2.3 Automorphic Forms of Adele Groups

We saw that the classical modular forms correspond to different ways of embedding the irreducible, which is generated from the representation theory, into $\mathcal{A}(G,\Gamma)$:

$$M_k(N) \cong \mathrm{Hom}_{\mathfrak{g},K}(\mathcal{D}_{|k|}^{\mathrm{sign}(k)}, \mathcal{A}(\mathrm{PGL}_2, \Gamma_0'(N)))$$

where π_k is the discrete series of $\mathrm{PGL}_2(\mathbb{R})$ with lowest weight k.

Thus we are interested in how $\mathcal{A}(G,\Gamma)$ decomposes as a $(\mathfrak{g}, K) \times \mathcal{H}(G,\Gamma)$-module. The adelic setting describes them in parallel. This is one of the reasons to formulate adelic automorphic forms.

Define the adele ring of $\mathbb{Q}$:

$$\mathbb{A} \subset \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p$$

consisting of those $x = (x_v)$ such that for almost all primes p, $x_p \in \mathbb{Z}_p$.

The ring $\mathbb{A}$ has a natural topology, the topological basis consists of:

$$\prod_{v \in S} U_v \times \prod_{v \notin S} \mathbb{Z}_v$$

where S is a finite set of places of $\mathbb{Q}$ including the archimedean place. This topology makes $\mathbb{Q}$ is discrete in $\mathbb{A}$ with $\mathbb{Q}\backslash\mathbb{A}$ compact.

Here is a various construction. If S is a finite set of places of $\mathbb{Q}$, we let:

$$\mathbb{Q}_S = \prod_{v\in S} \mathbb{Q}_v,\ \mathbb{A}^S = \{(x_v) \in \prod_{v\notin S} \mathbb{Q}_v : x_v \in \mathbb{Z}_v \text{ for almost all } v\}$$

We call $\mathbb{A}^S$ the S-Adeles. If S consists only of the place ∞, then we call $\mathbb{A}^S$ the finite adeles and denote it by $\mathbb{A}_f$.

We can define $G(\mathbb{A})$ for general linear algebraic group $G/\mathbb{Q}$. For example, when $G = \mathrm{GL}_1$,

$$\mathrm{GL}_1(\mathbb{A}) = \{x = (x_v) \in \prod_v \mathbb{Q}_v^\times,\ x_p \in \mathbb{Z}_p^\times \text{ for almost all } p\}$$

This is the idele group of $\mathbb{Q}$. Similarly, $G(\mathbb{Q})$ is discrete in $G(\mathbb{A})$.

The following approximation theorem allows one to relate the adelic picture to the case $\Gamma\backslash G(\mathbb{R})$:

Theorem 5. *Assume that G is simply-connected and S is a finite set of places of $\mathbb{Q}$ such that $G(\mathbb{Q}_S)$ is not compact, then $G(\mathbb{Q})$ is dense in $G\left(\mathbb{A}^S\right)$.*

Here is a reformulation. Given any open compact subgroup $U^S \subset G\left(\mathbb{A}^S\right)$, we have:

$$G(\mathbb{A}) = G(\mathbb{Q}) \cdot G(\mathbb{Q}_S) \cdot U^S.$$

Thus under the assumtions above, if we let $\Gamma = G(\mathbb{Q}) \cap U^S$, then

$$G(\mathbb{Q})\backslash G(\mathbb{A})/U^S \cong \Gamma\backslash G(\mathbb{Q}_S).$$

Example 4. Consider the case when $G = \mathrm{SL}_2$ and $S = \{\infty\}$. Then

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})/U_f \cong \Gamma\backslash\mathrm{SL}_2(\mathbb{R})$$

where U_f is any open compact subgroup of $G(\mathbb{A}_f)$ and $\Gamma = G(\mathbb{Q}) \cap U_f$. Let's take U_f to be the group

$$K_0(N) = \prod_{p\mid N} I_p \cdot \prod_{(p,N)=1} \mathrm{SL}_2(\mathbb{Z}_p)$$

where I_p is an Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$:

$$I_p = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) : c \equiv 0 \bmod p \right\}$$

Then it is clear that

$$\Gamma_0(N) = K_0(N) \cap \mathrm{SL}_2(\mathbb{Q}).$$

So we have:

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})/K_0(N) \cong \Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{R})$$

This isomorphism allows us to regard an automorphic form f on $\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{R})$ as a function on $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})$, which is right invariant under $K_0(N)$.

Therefore, We define that Γ is a **congruence subgroup** of G if $\Gamma = G(\mathbb{Q}) \cap U_\Gamma$ for some open compact subgroup U_Γ of $G\left(\mathbb{A}_f\right)$. Thus if Γ is congruence, and G satisfies strong approximation, we have:

$$\Gamma\backslash G(\mathbb{R}) \cong G(\mathbb{Q})\backslash G(\mathbb{A})/U_\Gamma$$

and we can regard an automorphic form on $\Gamma\backslash G(\mathbb{R})$ as a function on $G(\mathbb{Q})\backslash G(\mathbb{A})$ which is right-invariant under U_Γ.

We now describe how to associate to f and F an automorphic form ϕ_f on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$.

Definition 6 (**Automorphic Forms for** $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$)**.** The strong approximation gives the following product:

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2(\mathbb{R})K_0(N),$$

where $K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : c \equiv 0 \bmod N \right\}$. Let f be a modular form of weight k, character χ and level N. define:

$$\phi_f\left(\gamma g_\infty k_0\right) := F\left(g_\infty\right)\lambda\left(k_0\right) = \left(f|_k\, g_\infty\right)(i)\lambda\left(k_0\right).$$

where the function λ is an adelization of χ. For example, the Dirichelet character χ' is associated with a finite order idele class character. It can be extended to a character of $K_0(N)$.

2.3.1 Basic Properties

- This is a well-defined smooth function (i.e. C^∞ on the archimedean place and locally constant on the finite adeles).
- It is left invariant under $\mathrm{GL}_2(\mathbb{Q})$.
- ($K = K_0(N)\mathrm{SO}_2(\mathbb{R})$ finiteness): In the adelic setting, the condition of K-finiteness on ϕ_f means that the subspace $\mathrm{span}\{R(g)\phi)f\}$ is finite-dimensional, since

$$\phi_f(gk_\infty k_f) = \omega(k_f)\mathrm{exp}(2\pi ik\theta)\phi_f(g)$$

- (Center) For any $z \in \mathbb{A}$, $g \in \mathrm{GL}_2(\mathbb{A})$, $\phi_f(zg) = \omega_\chi(z)\phi_f(g)$.

- The Casimir operator Δ acts on the infinite component. One have:

$$\Delta\phi_f = \frac{k}{2}(1-\frac{k}{2})\phi_f$$

  This implies that ϕ_f is $Z(\mathfrak{g})$-finite.

- (Growth condition) For any norm $||\cdot||$ on $\mathrm{GL}_2(\mathbb{A})$, there exists a real number $A>0$, such that $\phi_f(g) \lesssim ||g||^A$. In other words, ϕ_f is of moderate growth.

We let $\mathcal{A}(G)$ denote the spaces of automorphic forms on G.

An automrophic form f on G is called a cusp form if, for any parabolic $\mathbb{Q}$-subgroup $P=MN$ of G, the constant term

$$f_N(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng)\mathrm{d}n$$

is zero as a function on $G(\mathbb{A})$.

It suffices to check this vanishing on a set of representatives of G-conjugacy classes of maximal parabolic subgroups. We let $\mathcal{A}_0(G)$ denote the space of cusp forms on G. In fact, $\mathcal{A}_0(G) \subset L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

2.4 Automorphic Representations

The space $\mathcal{A}(G)$ possesses the structure of a $(\mathfrak{g},K)$-module as before. In addition, for each prime p, the group $G\left(\mathbb{Q}_p\right)$ acts on $\mathcal{A}(G)$ by right translation. Thus, $\mathcal{A}(G)$ has the structure of a representation of

$$(\mathfrak{g},K)\times G\left(\mathbb{A}_f\right).$$

Moreover, as a representation of $G\left(\mathbb{A}_f\right)$, it is a smooth representation. We can abuse terminology, and say that $\mathcal{A}(G)$ is a smooth representation of $G(\mathbb{A})$.

Definition 7. An irreducible smooth representation π of $G(\mathbb{A})$ is called an **automorphic representation** if π is a subquotient of $\mathcal{A}(G)$.

Theorem 6. *An automorphic representation π is admissible, i.e. given any irreducible representation ρ of K, the multiplicity with which ρ occurs in π is finite.*

2.4.1 Restricted Tensor Product

We usually expect an irreducible representation of a direct product of groups G_i to be the tensor product of irreducible representations V_i of G_i.

Definition 8. Suppose we have a family (W_v) of vector spaces, and for almost all v, we are given a non-zero vector $u_v^0 \in W_v$. The restricted tensor product $\otimes_v' W_v$ of the W_v's with respect to (u_v^0) is the direct limit of $\{W_S = \otimes_{v\in S} W_v\}$, where for $S \subset S'$, one has $W_S \longrightarrow W_{S'}$ defined by

$$\otimes_{v\in S} u_v \mapsto (\otimes_{v\in S} u_v) \otimes \left(\otimes_{v\in S'\setminus S} u_v^0\right).$$

We think of $\otimes_v' W_v$ as the vector space generated by the elements

$$u = \otimes_v u_v \quad \text{with } u_v = u_v^0 \text{ for almost all } v,$$

with the usual linearity conditions in the definition of the usual tensor product.

Now if each W_v is a representation of $G(\mathbb{Q}_v)$, and for almost all v, the distinguished vector u_v^0 is fixed by the maximal compact K_v, then the restricted tensor product inherits an action of $G(\mathbb{A})$: if $g = (g_v)$, then

$$g(\otimes_v u_v) = \otimes_v g_v u_v.$$

Because $g_v \in K_v$ and $u_v = u_v^0$ for almost all v, the resulting vector still has the property that almost all its local components are equal to the distinguished vector u_v^0.

Theorem 7. *An irreducible admissible representation of $G(\mathbb{A})$ is a restricted tensor product of irreducible admissible representations π_v of $G(\mathbb{Q}_v)$ with respect to a family of vectors (u_v^0) such that $u_v^0 \in \pi_v^{K_v}$, $\dim \pi_v^{K_v} = 1$, for almost all v. Meanwhile, we have the relationship between local and global (if denote W, W_v as the representation space):*

$$\begin{tikzcd}
((\mathfrak{g}, K) \times G(\mathbb{A}_f)) \times W \arrow[r] \arrow[d, "\cong"] & W \arrow[d, "\cong"] \\
\otimes'_{v\nmid\infty} G(\mathbb{Q}_v) \times \otimes'_{v\nmid\infty} W_V \arrow[r] & \otimes'_{v\nmid\infty} W_v
\end{tikzcd}$$

In particular, an automorphic representation π has a restricted tensor product decomposition: $\pi \cong \otimes_v' \pi_v$, where for almost all v, $\pi_v^{K_v} \neq 0$.

We call an irreducible representation of $G(\mathbb{Q}_{\shortmid})$ **unramified** (or spherical) with respect to K_p if $\dim \pi_p^{K_p} = 1$. These has been classifed, using **Satake isomorphism**.

2.4.2 Cuspidal Automorphic Representations

The space $\mathcal{A}_0(G)$ of cusp forms is a submodule of $\mathcal{A}(G)$ under $G(\mathbb{A})$. When G is reductive with center Z, we usually specify a central character χ for $Z(\mathbb{A})$. Namely, if χ is a character of $Z(\mathbb{Q})\backslash Z(\mathbb{A})$, then we let $\mathcal{A}(G)_\chi$ be the subspace of automorphic forms f which satisfy:

$$f(zg) = \chi(z) \cdot f(g)$$

We let $\mathcal{A}_0(G)_\chi$ be the subspace of cuspidal functions in $\mathcal{A}(G)_\chi$. Then the basic functional analysis says that $\mathcal{A}_0(G)_\chi$ decomposes as the direct sum of irreducible representations of $G(\mathbb{A})$, each occurring with finite multiplicities.

Definition 9. A representation π of $G(\mathbb{A})$ is **cuspidal** if it occurs as a submodule of $\mathcal{A}_0(G)_\chi$.

If f is a classical cuspidal Hecke eigenform on $\Gamma_0(N)$, we have seen that f gives rise to an automorphic form ϕ_f on $\Gamma_0'(N)\backslash \mathrm{PGL}_2(\mathbb{R})$ which generates an irreducible $(\mathfrak{g}, K)$-module isomorphic to the discrete series representation of lowest weight k.

Now if we then transfer ϕ_f to a cusp form Φ_f on $\mathrm{PGL}_2(\mathbb{Q})\backslash \mathrm{PGL}_2(\mathbb{A})$, we can consider the subrepresentation π_f of $\mathcal{A}_0\left(\mathrm{PGL}_2\right)$ generated by Φ_f. It turns out that this is an **irreducible** representation of $G(\mathbb{A})$ if f is a newform. Thus a Hecke eigen-newform in $S_k(N)$ corresponds to a cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$. Moreover, if $\pi_f \cong \otimes_v' \pi_v$, then π_p is unramified for all p not dividing N.

2.4.3 Adelic Hecke Algebras

Recall the representation theroy, if V is a smooth representation of a locally profinite group G and $U \subset G$ is an open compact subgroup, then the map $V \mapsto V^U$ defines a functor from the category of smooth representatioons of G to the category of modules for the Hecke agebra $\mathcal{H}(G//U)$, which is the ring of bi-U-invariant functions in $C_c^\infty(G)$, and the product is given by **convolution** of functions.

A basis for $\mathcal{H}(G//U)$ is given by the characteristic functions $f_\alpha = 1_{U\alpha U}$. The action of this on a vector in V^U is:

$$f_\alpha \cdot v = \int_G f_\alpha(g)(g.v)\mathrm{d}g = \int_{U\alpha U} v\mathrm{d}g = \sum_i a_i v$$

if $U\alpha U = \bigcup a_i U$ (and dg gives U volume 1). The adelic Hecke algebra $\mathcal{H}\left(G\left(\mathbb{A}_f\right)//U_\Gamma\right)$ acts on $\mathcal{A}(G)^{U_\Gamma}$ as the following: Since $U_\Gamma \alpha^{-1} U_\Gamma = \bigcup_i a_i^{-1} U_\Gamma$, the characteristic function of $U_\Gamma \alpha^{-1} U_\Gamma$ acts by

$$(T_\alpha f)(g) = \sum_i \left(a_i^{-1} f\right)(g) = \sum_i f\left(g a_i^{-1}\right)$$

We can calculate that For $f \in \mathcal{A}(G)^{U_\Gamma}$, then identity f with a function on $\Gamma \backslash G(\mathbb{R})$ given by restriction. The above definition makes

$$(T'_\alpha(f)) \mid_{G(\mathbb{R})} = \tilde{T}_\alpha(f \mid_{G(\mathbb{R})})$$

where $\tilde{T}_\alpha$ is the usual Hecke operator. In conclusion, we see that the action of $\mathcal{H}(G, \Gamma)$ on $\mathcal{A}(G, \Gamma)$ gets translated to an action of the adelic Hecke algebra $\mathcal{H}\left(G\left(\mathbb{A}_f\right) // U_\Gamma\right)$ on $\mathcal{A}(G)^{U_\Gamma}$.

2.4.4 Local Hecke Algebras

Because $G\left(\mathbb{A}_f\right)$ is a restricted direct product, we have in fact

$$\mathcal{H}\left(G\left(\mathbb{A}_f\right) // U\right) \cong \otimes'_v \mathcal{H}\left(G\left(\mathbb{Q}_p\right) // U_p\right)$$

if $U = \prod_p U_p$. So the structure of $\mathcal{H}\left(G\left(\mathbb{A}_f\right) // U\right)$ is known once we understand the local Hecke algebras $\mathcal{H}\left(G\left(\mathbb{Q}_p\right) // U_p\right)$

For almost all p, however, we know that $U_p = K_p$ is a maximal compact subgroup. In that case, the structure of the local Hecke algebra is known, by the **Satake isomorphism**. In particular, $\mathcal{H}\left(G\left(\mathbb{Q}_p\right) // K_p\right)$ is commutative and its irreducible modules are classified.

Example 5. For $G = \mathrm{GL}_2(\mathbb{Q}_p)$, the satake transform:

$$Sat : \mathcal{H}\left(\mathrm{GL}_2\left(\mathbb{Q}_p\right) // K_p\right) \to \mathcal{H}(T // T \cap K_p) = \mathbb{C}[T/T \cap K_p]$$

is the morphism of algebras (after normalize the Haar measures), by

$$Sat(f)(t) = \delta_B^{1/2}(t) \int_N f(tn) \mathrm{d}n$$

In fact, it induces the isomorphism:

$$\mathcal{H}\left(\mathrm{GL}_2\left(\mathbb{Q}_p\right) // K_p\right) \cong \mathbb{C}[T/T \cap K_p]^W$$

In particular, $\mathcal{H}\left(\mathrm{GL}_2\left(\mathbb{Q}_p\right) // K_p\right)$ is commutative. More explicitly, Let T_p and R_p be the characteristic functions of $K\mathrm{diag}(p,1)K$ and $K\mathrm{diag}(p,p)K$. Then $\mathcal{H}\left(\mathrm{GL}_2\left(\mathbb{Q}_p\right) // K_p\right)$ is a polynomial algebra generated by T_p, R_p, and R_p^{-1}.

Because $V \mapsto V^{K_p}$ induces a bijection of irreduible unramified representations with simple modules of $\mathcal{H}\left(G\left(\mathbb{Q}_p\right)//K_p\right)$. By commutative property, simple finite dimensional $\mathcal{H}\left(G\left(\mathbb{Q}_p\right)//K_p\right)$-module is one-dimentional. we get in this way the classification of irreducible unramified representations of $G\left(\mathbb{Q}_p\right)$:

Let us assume for simplicity that G is a split group (e.g. $G = \mathrm{GL}_n$). Let $B = T \cdot N$ be a Borel subgroup of G. So $T \cong (\mathrm{GL}_1)^r$ and $T\left(\mathbb{Q}_p\right) \cong \left(\mathbb{Q}_p^\times\right)^r$. We let $W := N_G(T)/T$ be the Weyl group of G.

Let $\chi : T\left(\mathbb{Q}_p\right) \to \mathbb{C}^\times$ be a (smooth) character of $T\left(\mathbb{Q}_p\right)$. We say that χ is an unramifed character if χ is trivial when restricted to $T\left(\mathbb{Z}_p\right) \cong \left(\mathbb{Z}_p^\times\right)^r$. Thus it is of the form

$$\chi\left(a_i, \ldots, a_r\right) = t_1^{\mathrm{ord}_p(a_1)} \cdot \ldots \cdot t_r^{\mathrm{ord}_p(a_r)}, \quad a_i \in \mathbb{Q}_p^\times$$

for some $s_i \in \mathbb{C}^\times$.

We may regard χ as a character of $B\left(\mathbb{Q}_p\right)$ using the projection $B\left(\mathbb{Q}_p\right) \to N\left(\mathbb{Q}_p\right) \backslash B\left(\mathbb{Q}_p\right) \cong T\left(\mathbb{Q}_p\right)$.

Given an unramified character χ of $T\left(\mathbb{Q}_p\right)$, we may form the induced representation

$$I_B(\chi) := \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \delta_B^{1/2} \cdot \chi.$$

Here, δ_B is the modulus character of B, defined by:

$$\delta_B(b) = |\det(\mathrm{Ad}(b)\mid_{\mathrm{Lie}(N)})|_p.$$

The space of $I_B(\chi)$ is the subspace of $C^\infty\left(G\left(\mathbb{Q}_p\right)\right)$ satisfying:

- $f(bg) = \delta(b)^{1/2} \cdot \chi(b) \cdot f(g)$ for any $b \in B\left(\mathbb{Q}_p\right)$ and $g \in G\left(\mathbb{Q}_p\right)$.
- f is right-invariant under some open compact subgroup U_f of $G\left(\mathbb{Q}_p\right)$.

Then $I_B(\chi)$ is an admissible representation of $G\left(\mathrm{Q}_p\right)$ (possibly reducible). The representations $I_B(\chi)$ are called the **principal series representations**.

Theorem 8. *$I_B(\chi)$ has a unique irreducible subquotient π_χ with the property that $\pi_\chi^{K_p} \neq 0$, and* ***any*** *irreducible unramified representation of $G\left(\mathbb{Q}_p\right)$ is of the form π_χ for some unramfied character χ of $T\left(\mathbb{Q}_p\right)$.*

The Weyl group W acts naturally on $T(\mathbb{Q}_p)$ and use this to acts on $\widehat{T(\mathbb{Q})_p}$: For $w \in W$,

$$(w\chi)(t) = \chi(w^{-1}tw)$$

Proposition 2.3. *$\pi_\chi \cong \pi_{\chi'}$ if and only if $\chi = w\chi'$ for some $w \in W$.*

Thus, the irreducible unramified representations are classified by W-orbits of unramified characters of $T(\mathbb{Q}_p)$.

3 Multiplicity One Theorem for GL_n

Theorem 9. *The multiplicity $m_0(\pi)$ of an irreducible representation π of $\mathrm{GL}_n(\mathbb{A})$ in $\mathcal{A}_0(G)$ is ≤ 1.*

One may have a stronger edition:

Theorem 10. *Let $\pi_1, \pi_2 \subset \mathcal{A}_0(G)$ are such that $\pi_{1,v}, \pi_{2,v}$ are isomorphic for almost all place v. Then $\pi_1 \cong \pi_2$ as two irreducible cuspidal representations.*

The proof of the first multiplicity one theorem has two ingredients, one of which is global and the other local. Details may be found in [JL70], Section 11 and [Bum97], Chapter 3. The stronger edition needs the trace formula which is far beyond our motivation.

Remark. We can use the strong edition to prove one main theorem: Let f be a holomorphic modular form of level N and character χ. Suppose further that it is a cusp form and an eigenfunction for the Hecke operators T_p for $p \nmid N$. We have associated an adelic automorphic form ϕ_f.

Since the the space of cuspidal L_2 functions decomposes as a direct sum, One can let (π, V) be one irreducible factor such that the projection of ϕ_f is non-zero. We can show that all of the local components of π at places not dividing N or infinity are determined by f. Thus using the multiplicity one theorem,

Theorem 11. *The automorphic form ϕ_f lies in an unique irreducible admissible automorphic representation $\pi_f \subset \mathcal{A}_0(\mathrm{GL}_2(\mathbb{A}))$.*

3.1 Generic Character

Let f be an automorphic form on $G = \mathrm{GL}_n$. If $N \subset G$ is a unipotent subgroup, say the unipotent radical of a parabolic subgroup, one can consider the Fourier coefficients of f along N: If χ is a unitary character of $N(\mathbb{A})$ which is trivial on $N(\mathbb{Q})$, we have

$$f_{N,\chi}(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \overline{\chi(n)} \cdot f(ng)\mathrm{d}n$$

Note that if N is abelian, then we have:

$$f(g) = \sum_{\chi} f_{N,\chi}(g)$$

We apply the above to the unipotent radical N of the Borel subgroup B of upper triangular matrices.

Definition 10. A character χ of $N(\mathbb{A})$ is **generic** if the stabilizer of χ in $T(\mathbb{A})$ is the center $Z(\mathbb{A})$ of $\mathrm{GL}_n(\mathbb{A})$.

Example 6. When $G = \mathrm{GL}_2$, a generic character of $N(\mathbb{Q})\backslash N(\mathbb{A})$ means a non-trivial character of $\mathbb{Q}\backslash\mathbb{A}$. If we fix a character ψ of $\mathbb{Q}\backslash\mathbb{A}$, then all others are of the form

$$\chi_\lambda(x) = \psi(\lambda x)$$

for some $\lambda \in \mathbf{Q}$.

When $G = \mathrm{GL}_3$, a character of $N(\mathbb{A})$ trivial on $N(\mathbb{Q})$ has the form

$$\chi_{\lambda_1,\lambda_2}\begin{pmatrix} 1 & a_1 & * \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} = \psi\left(\lambda_1 a_1 + \lambda_2 a_2\right)$$

for some λ_1 and $\lambda_2 \in \mathbb{Q}$. Saying that $\chi_{\lambda_1,\lambda_2}$ is generic means that λ_1 and λ_2 are non-zero.

Since $Z(\mathbb{Q})\backslash T(\mathbb{Q})$ acts transitively on the generic characters of $N(\mathbb{A})$ trivial on $N(\mathbb{Q})$, and if $t \cdot \chi = \chi'$ with $t \in T(\mathbb{Q})$, then

$$f_{N,\chi'}(g) = f_{N,\chi}(t^{-1}g)$$

we will define:

Definition 11. A representation $\pi \subset \mathcal{A}(G)$ is said to be **globally generic** if there exists $f \in \pi$ whose Fourier-Whittaker coefficient $f_{N,\chi} \neq 0$ for some (hence all) generic character χ.

equivalently, consider the linear map:

$$l_\chi : \mathcal{A}(G) \to \mathbb{C},\ l_\chi(f) = f_{N,\chi}(1)$$

when χ is generic. Then π is globally generic if $l_\chi \neq 0$ when restricted to π.

Example 7. $G = \mathrm{GL}_2,\ \pi \subset \mathcal{A}_0(G)$ is an irreducible cuspidal representation. Take any non-zero $f \in \pi$. Then use the expansion:

$$f(g) = \sum_\chi f_{N,\chi}(g)$$

Since f cuspidal, $f_N = 0$. So some $f_{N,\chi} \neq 0$.

More generally, we have the global genericity:

Theorem 12. *Let $\pi \subset \mathcal{A}(G)$ be an irreducible cuspidal representation. Then π is globally generic.*

Remark. Similar to the GL_2 case, we need to show the expansion:

$$f(g) = \sum_{\gamma \in N_{n-1}(\mathbb{Q}) \backslash \mathrm{GL_{n-1}}(\mathbb{Q})} f_{N,\chi_0}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Here N_{n-1} is the unipotent radical of the Borel subgroup of $\mathrm{GL_{n-1}}$.

3.2 Whittaker Functionals

One can define the notion of a "generic representation" locally. Let π_v be a representation of $G(\mathbb{Q}_v)$ and let

$$\chi_v : N(\mathbb{Q}_v) \to \mathbb{C}$$

be a generic unitary character.

Definition 12. Let p be a finite prime. Then π_p is an abstractly generic representation if, given any generic χ_p, there is a non-zero linear functional $l_p : \pi_p \to \mathbb{C}$ such that

$$l_p(n \cdot v) = \chi_p(n) \cdot l_p(v)$$

for all $n \in N(\mathbb{Q}_p)$ and $v \in \pi_p$. Such a functional is called a local Whittaker functional.

One can make the same definition at the infinite prime. Since π_∞ is a $(\mathfrak{g}, K)$-module and $N(\mathbb{R})$ does not act on π_∞. The definition is a bit more subtle. However, It suffices all the properties as nonarchimedean place does.

Now let $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of $G(\mathbb{A})$, one says that π is an abstractly generic representation if each of its local components π_v is abstractly generic.

Theorem 13 (Local uniqueness of Whittaker functionals)**.** *Let π_v be an irreducible smooth representation of $G(\mathbb{Q}_v)$. Then the space of (continuous) Whittaker fuctional on π_v is at most 1 dimensional.*

3.3 Proof of Multiplicity One

Proof. We need to show that for any ireducible admissible representation π of $G(\mathbb{A})$,

$$\dim \mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathcal{A}_0(G)) \leq 1.$$

Let χ be a generic character of $N(\mathbb{A})$ trivial on $N(\mathbb{Q})$. Denote $\mathbb{C}_\chi$ as the fucntional such that $l_p(n \cdot v) = \chi(n) \cdot l_p(v)$, then we have the map

$$l_\chi : \mathcal{A}(G) \longrightarrow \mathbb{C}_\chi$$

given by

$$l_\chi(\phi) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \overline{\chi(n)} \cdot \phi(n) \mathrm{d}n.$$

Now we have a map

$$\mathrm{Hom}_{G(\mathbb{A})}\left(\pi, \mathcal{A}_0(G)\right) \longrightarrow \mathrm{Hom}_{N(\mathbb{A})}\left(\pi, \mathbb{C}_\chi\right)$$

given by $f \mapsto l_\chi \circ f$.

By the global genericity, this map is injective. So it suffices to show that the RHS has dimension ≤ 1.

The generic character χ is of the form $\prod_v \chi_v$ for generic characters χ_v of $N\left(\mathbb{Q}_v\right)$.

Now if $L \in \mathrm{Hom}_{N(\mathbb{A})}\left(\pi, C_\chi\right)$ is non-zero, then for each v,

$$\dim \mathrm{Hom}_{N(\mathbb{Q}_v)}\left(\pi_v, \mathbb{C}_{\chi_v}\right) \neq 0$$

i.e. π is abstractly generic. By local uniqueness, the above dimenson is 1, and for almost all v, a non-zero local functional l_v is non-zero on $\pi_v^{K_v}$.

Let us choose $l_v \neq 0$ so that for almost all v, $l_v\left(u_v^0\right) = 1$, where u_v^0 is the distinguished K_v-fixed vector in π_v. Then one has, for some constant c,

$$L(u) = c \cdot \prod_v l_v\left(u_v\right) \qquad \text{for any } u = \otimes_v u_v.$$

This shows that

$$\dim\ \mathrm{Hom}_{N(\mathbb{A})}(\pi, \mathbb{C}_\chi) = 1$$

as desired. □

3.4 Whittaker Models and L-functions

I wrote some stories of "Hecke theory and Jacquet Langlands" before. Here are something correspond to today's topic. These examples give what the Whittaker model looks like and how they work. I know clearly that some notations are abused! But I am tired to change them, since our note actually ends at "Proof of Multiplicity One". This subsection mainly follows [**?**, Gel75]

Consider the case: $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ with trivial character for simplicity. For real number $y > 0$, write

$$L(f, s) = \int_{\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times} \phi_f\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) |y|^s \mathrm{d}^\times y.$$

For an additive character τ, $\phi_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)$ has a Fourier expansion (as a function of x):

$$\phi_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \sum_{\lambda\in\mathbb{Q}} \phi_{f,\lambda}(g)\tau(\lambda x),$$

where

$$\phi_{f,\lambda}(g) = \int_{\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times} \phi_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\overline{\tau(\lambda x)}\mathrm{d}x$$

is the λth Fourier coefficient which depends on g. (That is what we have discussed.) Let $x = 0$. define the 1st Fourier coefficient of ϕ_f:

$$W_{\phi_f}(g) = \int_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}} \phi_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\overline{\tau(x)}\ \mathrm{d}x$$

Consider the property of the adelic automorphic form, we have:

$$\phi_f\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \sum_{\lambda\in\mathbb{Q}^\times} \phi_{f,\lambda}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \sum_{\lambda\in\mathbb{Q}^\times} W_{\phi_f}\left(\begin{pmatrix} \lambda y & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Then for $L(f, s)$,

$$\begin{aligned} L(f,s) &= \int_{\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times} \sum_{\lambda\in\mathbb{Q}^\times} W_{\phi_f}\left(\begin{pmatrix} \lambda y & 0 \\ 0 & 1 \end{pmatrix}\right)|y|^s\mathrm{d}^*y \\ &= \int_{\mathbb{A}_{\mathbb{Q}}^\times} W_{\phi_f}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)|y|^s\mathrm{d}^*y. \end{aligned}$$

In words, $L(f, s)$ is the *adelic Mellin transform* along $\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$ of the 1st Fourier coefficient of ϕ_f.

In order to study this adelic Mellin transform, we list some properties of the one W_{ϕ_f} when f is a classical cusp form. The right convolution of W_{ϕ_f} generate a space $W(\pi_f)$ (since we can correspond unique irreducible automorphic representation for such f) of function W on $\mathrm{GL}_2(\mathbb{A})$ satisfying:

- For any $x \in \mathbb{A}_F$,

$$W\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] = \tau(x)W(g)$$

,

- For any $z \in \mathbb{A}_F^\times$,

$$W\left[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} g\right] = \psi(z)W(g)$$

,

- W is right K-finite. C^∞ and rapidly decreasing on the archimedean place, i.e. for any $x \in \mathbb{A}_F$,

$$W\left[\begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix}\right] = O(|x|^{-N}), \quad \text{for all } N$$

,

- The Global Hecke Algebra acts by right convolution is equivalent to π_f.

It is called the *Whittaker model* of π_f, and the space $W(\pi_f)$ called the *Whittaker space* (sometimes we can treat it as a functional). The uniqueness of the local *Whittaker model* follows the representation theory. Local uniqueness implies global uniqueness. This observation helps us to associate the L-function to such an automorhic representation π_f.

Let π_v be an irreducible admissible representation of $\mathrm{GL}_2(F_v)$, χ a unitary character of $F_v^\times$, $g \in \mathrm{GL}_2(F_v)$, $W \in W(\pi_v)$. Then define *local ζ-function*:

$$\zeta(g, \chi, W, s) = \int_{F_v^\times} W\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] \chi(a)|a|^{s-1/2}\mathrm{d}^\times a.$$

Theorem 14.

1. Local ζ function converges in some right half-plane.
2. There exist $W^0 \in W(\pi_v)$ such that $L(\chi \otimes \pi_v, s) = \zeta(1, \chi, W^0, s)$ is an Euler factor making $\frac{\zeta(g,\chi,W,s)}{L(\chi\otimes\pi_v,s)}$ is entire for any g, χ, W.
3. $\zeta(g, \chi, W, s)$ has an analytic continuation to the whole plane satisfying the functional equation

$$\frac{\zeta(g, \chi, W, s)}{L(\chi \otimes \pi_v, s)}\varepsilon(\pi_v, \chi, s) = \frac{\zeta(wg, \chi^{-1}\psi_v^{-1}, W, 1-s)}{L(\chi^{-1}\psi_v^{-1} \otimes \pi_v, 1-s)}$$

where ψ_v is the central character of π_v, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

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