PERSONAL NOTE ON THE PARAMETERS AND THETA LIFTS (Sp-O)

par

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Résumé. — In this paper, we study, following Paul [45]'s work on the Howe correspondence for symplectic-orthogonal dual pairs, the Harish-Chandra paramters and the Langlands-Vogan paramters of both sides, and translate the statement in Paul [45] to the Langlands-Vogan parameter version. The result is more convenient to relate the local root numbers, which is used by Gan–Gross–Prasad to state the precise local Gan–Gross–Prasad conjecture in [20].

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1. Introduction

The explicit theta correspondences for symplectic-orthogonal real pair and real metaplectic-orthogonal pair using Harish-Chandra parameters are established by Moeglin [40] (completed by Paul [45]) and Adams-Barbash [2] respectively. The goal of this paper is to translate their results for limit of discrete series representations into Langlands-Vogan parameters.

Remark 1.1. — Initially it is an brief appendix of an upcoming paper by Zhe Li and Shanwen Wang, focusing on the Local Gan–Gross–Prasad Conjucture of generic L-packet in real symplectic-metaplectic case. They decided to use the theta correspondence to transfer the tempered Bessel orthogonal result, to the tempered symplectic-metaplectic problem. During this procedure they need an appropriate description about the theta lift using Langlands-Vogan parameter. Then they shared this topic to me, and left this translation to me as an exercise. Since we can choose concrete quasi-split orthogonal groups for later calcuation, we decide to focus on the orthogonal group $\mathrm{O}(p, q)$ when p, q are all even, although in Paul [45] he reformulated the both odd case. The corresponding translation for unitary groups case has been done in [27, Section 5.3.1].

The main part of the theory is written by first two authors. My note is bascically want to fullfill all the details and explain the theories as explicitly as I can.

Since the author did not decide whether to add any parts of this note into the formal paper, all contents of this paper is for reference and study purposes only, please do not spread temporarily.

Throughout this paper, we set $\Gamma = \{1, \sigma\}$ the Galois group of $\mathbb{C}/\mathbb{R}$ and by a representation, we mean a smooth Fréchet representation of moderate growth. Equivalently we can view it as a $(\mathfrak{g}, K)$ module, where $\mathfrak{g}$ is the Lie algebra of G, and K is one of the maximal compact subgroup. We begin by giving a precise description of the main ingredients.

1.1. Generic packets. — Let G be a quasi-split real reductive group and let G^d be its complex dual group. We denote by $\mathbf{G}$ the complexification of G and by σ_G the action of σ on $\mathbf{G}$ associated to G.

1.1.1. *Weil group.* — The Weil group $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ of $\mathbb{R}$ is the normalizer of $\mathbb{C}^\times$ in the multiplicative group $\mathbb{H}^\times$ of Hamilton's quaternions. The irreducible representations of $W_{\mathbb{R}}$ have dimension either 1 or 2 as $W_{\mathbb{R}}$ has an abelian subgroup of index 2.

1. The 1-dimensional representations of $W_{\mathbb{R}}$ are the quasi-characters of $W_{\mathbb{R}}^{\mathrm{ab}} = \mathbb{R}^\times$. Since the character of $\mathbb{R}^\times$ is the trivial or the sign character, any quasi-character of $W_{\mathbb{R}}$ is of the form $\chi_{\varepsilon,s}(x) = \mathbf{sgn}(x)^{\frac{\varepsilon-1}{2}}|x|^s$, for $\varepsilon \in \{\pm 1\}$ and $s \in \mathbb{C}$.
2. For $m \in \mathbb{Z}$, let $\rho_m = \mathrm{Ind}_{\mathbb{C}^\times}^{W_{\mathbb{R}}} w_m$ be the induced representation of a conjugate self-dual character $w_m(z) = z^m \cdot (z\bar{z})^{-m/2}$ of $\mathbb{C}^\times$ to $W_{\mathbb{R}}$. We know that $\rho_m \cong \rho_{-m}$ and ρ_m is irreducible unless $m = 0$ (in this case, $\rho_0 = 1 \oplus \mathrm{sgn}$). The irreducible representation ρ_m is self-dual. Moreover, it is symplectic (resp. orthogonal) if m is odd (resp. even). In general, the irreducible representations of $W_{\mathbb{R}}$ of dimension 2 are of the form
$$\rho_{m,s} = \mathrm{Ind}_{\mathbb{C}^\times}^{W_{\mathbb{R}}} \chi_{m,s},$$
for $\chi_{m,s}(z) = z^m \cdot (z\bar{z})^{-m/2} \cdot |z|_{\mathbb{C}}^s$ a quasi-character of $\mathbb{C}^\times$ with $m \in \mathbb{Z}$ and $s \in \mathbb{C}$.

1.1.2. *L-packets.* —

Definition 1.2. — [41, Definition 5.1] Let $(\mathbf{G}, \sigma_G)$ be a real form with complex reductive group $\mathbf{G}$. A Borel pair $(\mathbf{B}_*, \mathbf{T}_*)$ of a complex reductive group $\mathbf{G}$ is called fundamental if the following conditions are satisfied:

(i) $T_* = \mathbf{T}_*^{\sigma_G}$ is a maximal compact subgroup of $G = \mathbf{G}^{\sigma_G}$;

(ii) The set of roots of $\mathbf{T}_*$ in $\mathbf{B}_*$ is stable under $-\sigma_G$.

Moreover, a fundamental Borel pair $(\mathbf{B}_*, \mathbf{T}_*)$ of $\mathbf{G}$ is called of Whittaker type if all the imaginary simple roots of $\mathbf{T}_*$ in $\mathbf{B}_*$ are non-compact.

By [4, Prop. 6.24], a real form $(\mathbf{G}, \sigma_G)$ has a fundamental Borel pair of Whittaker type if and only if $(\mathbf{G}, \sigma_G)$ is quasi-split. This applies to our quasi-split group G and we fix one of the fundamental Borel pair of Whittaker type $(\mathbf{B}, \mathbf{T}_*)$ of $\mathbf{G}$ in the following.

By the identification of the simple roots of $\mathbf{T}_*$ in $\mathbf{B}$ and the simple coroots in dual groups, we get a pinning of the complex dual group G^d

$$\mathbf{Spl}_{G^d} = (\mathscr{B}, \mathscr{T}_*, \{\mathscr{X}_\alpha\}),$$

where $\{\mathscr{X}_\alpha\}$ is a set of root vectors for the simple roots of torus $\mathscr{T}_*$ in Borel subgroup $\mathscr{B}$ of G^d correspinding to $(\mathbf{B}, \mathbf{T}_*)$. The Langlands group ${}^LG$ of G associated to the pinning $\mathbf{Spl}_{G^d}$ is the semi-direct product $G^d \rtimes W_\mathbb{R}$, where the action of $W_\mathbb{R}$ on G^d factors through the projection $p_{W_\mathbb{R}} : W_\mathbb{R} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ stablizing $\mathbf{Spl}_{G^d}$. We remark that the L-group of G only depends on its inner class.

Definition 1.3. — A Langlands parameter of a real reductive group G is a continuous morphism $\varphi : W_\mathbb{R} \to {}^LG$ satisfying the following two conditions:

1. $p_\mathbb{R} \circ \varphi = \mathrm{Id}_{W_\mathbb{R}}$, where $p_\mathbb{R} : {}^LG \to W_\mathbb{R}$;
2. the image of the restriction map $\varphi_{|_{\mathbb{C}^\times}} : \mathbb{C}^\times \to G^d \times \mathbb{C}^\times$ consists of the elements which is semi-simple in G^d.

In particular, a Langlands parameter of G is called tempered if its image is bounded.

The complex dual group G^d acts by conjugation on the set of Langlands parameters of G. We denote by $\Phi(G)$ the set of G^d-conjugacy classes of the Langlands parameters of G and by $\Phi_{\mathrm{temp}}(G)$ the set of conjugacy classes of the tempered Langlands parameters. The classification theorem of Langlands says that there exists a partition of the set $\Pi(G)$ of the equivalent classes of irreducible representations of G:

$$\Pi(G) = \coprod_{\varphi \in \Phi(G)} \Pi(\varphi, G),$$

where $\Pi(\varphi, G)$'s are finite sets of irreducible representations of G, called the L-packets (or Langlands packets). Similarly, for classical group G which comes from the underlying space V, we denote $\Pi(\varphi, V)$ as the (finite) set of irreducible representation of G. This partition satisfies a number of properties and among them, the most important ones for us are the followings:

1. all the element of a L-packet admits the same infinitesimal character;
2. for all tempered representations, we have $\Pi_{\mathrm{temp}}(G) = \coprod_{\varphi\in\Phi_{\mathrm{temp}}(G)} \Pi(\varphi, G)$.

On the other hand, if G is a real form of equal rank (cf. Def. 2.5), we can consider a finite set $\Pi_\varphi(G)$ of irreducible representations of G and its pure inner forms $G^{'}$, called the Vogan L-packet: given a Langlands parameter $\varphi : W_{\mathbb{R}} \to {}^LG$, let Π_φ be the finite set of irreducible representation of G' with G' running over all pure inner forms of G. For generality reason, we introduce the strong real form of equal rank real group instead of pure inner form in this note. Let C_φ be the centralizer of the image of φ in G^d and we denote by C_φ^0 its connected component of identity. We set

$$A_\varphi = \pi_0(C_\varphi) = C_\varphi/C_\varphi^0,$$

called the component group of φ. It is known that the cardinality of the finite set $\Pi_\varphi(G)$ is equal to the number of irreducible representations of the finite group A_φ. Details can be found in Vogan [**55**]: 6. Langlands parameters: real case.

1.1.3. *Generic packets.* — In this subsection we will give a concrete example of what a generic L-packet looks like. The parabolic induction procedure partially explains why we consider the limits of discrete series at this stage.

Let $T = G \cap \mathbf{T}_*$, $B = G \cap \mathbf{B}$ and let N be the unipotent radical of B. The torus T acts on the group $\mathrm{Hom}(N, \mathbb{C}^\times)$. A character $\theta \in \mathrm{Hom}(N, \mathbb{C}^\times)$ is called *generic* if its stabilizer in T is equal to the center Z. An irreducible representation π of G is generic if there exists a generic character θ such that $\dim_{\mathbb{C}} \mathrm{Hom}_N(\pi, \theta) = 1$. By [**53**, Theorem 6.2], a generic representation is always an irreducible parabolic induction in the bijection as above. Note that the definition of generic representation depends on the choice of fundamental Borel pair of Whittaker type.

Definition 1.4. — An L-parameter φ of G is *generic* if the L-packet $\Pi(\varphi, G)$ contains a generic representation.

If φ is generic, the L-packet $\Pi(\varphi, G)$ consists of irreducible parabolically induced representations of the form

$$\mathrm{Ind}_P^G \pi_0 \otimes \tau \otimes \chi \otimes 1,$$

where π_0 is a limit of discrete series representation and ranges over Π_{φ_0}, the induction is given by the Levi decomposition of parabolic subgroup $P = MAN$.

Example 1.5. — For symplectic group, considering cuspidal parabolic subgroups (i.e., those of the form $P = MAN$ such that the Lie algebra $\mathfrak{m}_0$ of M has a theta stable Cartan subalgebra in $\mathfrak{t}_0$) of $\mathrm{Sp}_{2n}(\mathbb{R})$ are of the form $P = MAN$ with

$$MA \cong \mathrm{Sp}_{2v}(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})^s \times \mathrm{GL}_1(\mathbb{R})^t$$

and $n = v + 2s + t$. Let V be the underlying space of the group $\mathrm{Sp}_{2n}(\mathbb{R})$. Let $V_0 \subset V$ be a quadratic space of the same type of V so that its orthogonal complement is a symplectic space of rank $a + 2b$. We may take a parabolic subgroup P of G so that its Levi subgroup is isomorphic to

$$\mathrm{GL}_1(\mathbb{R})^a \times \mathrm{GL}_2(\mathbb{R})^b \times \mathrm{Sp}(V_0).$$

Let $\Pi_{\varphi_0}^{V_0}$ be the finite set of the irreducible limit of discrete series representations of $\mathrm{Sp}(V_0)$ with L-parameter φ_0. The parabolically induced representations

$$\mathrm{Ind}_P^G \pi_0 \otimes \tau \otimes \chi \otimes 1, \text{ with } \tau = \otimes_{i=1}^b \tau(m_i, s_i) \text{ and } \chi = \otimes_{i=1}^a \chi_{\varepsilon_i, \kappa_i},$$

have a unique irreducible Langlands quotient. Here $\tau(\mu, \nu)$ is limits of discrete series of $\mathrm{GL}_2(\mathbb{R})$ are parametrized by pairs (μ, ν), where μ is a non-negative integer and ν a complex number. The representation has infinitesimal character $(\frac{1}{2}(\mu + \nu), \frac{1}{2}(-\mu + \nu))$. The character $x \mapsto \mathrm{sgn}(x)^{\frac{\varepsilon-1}{2}} |x|^\kappa$ of $\mathrm{GL}_1(\mathbb{R})$ is denoted as $\chi_{\varepsilon,\kappa}$.

Then Π_φ^V is the collection of all these Langlands quotients where π_0 ranges over $\Pi_{\varphi_0}^{V_0}$. In short, taking Langlands quotients of parabolic inductions gives a bijection between Π_φ^V and $\Pi_{\varphi_0}^{V_0}$.

In this case, the component group A_φ of the L-parameter φ is the same as A_{φ_0}. To each representation $\pi \in \Pi_\varphi$, there is a unique character $\eta : A_\varphi \to \langle \pm 1 \rangle$ attached to it and this defines a bijection between Π_φ and the set of characters of A_φ. There is a similar bijection between Π_{φ_0} and A_{φ_0}. The following diagram commutes

$$\begin{tikzcd}
\Pi_{\varphi_0} \arrow[r] \arrow[d] & \mathrm{Hom}(A_{\varphi_0}, \langle \pm 1 \rangle) \arrow[d, equal] \\
\Pi_\varphi \arrow[r] & \mathrm{Hom}(A_\varphi, \langle \pm 1 \rangle)
\end{tikzcd},$$

where the left arrow is the bijection given by the parabolic induction as before.

1.2. Weil representation and theta lifts. — Fix $\psi : \mathbb{R} \to \mathbb{C}^\times$ an additive character throughout this paper. Let (V, q_V) be a real symplectic space. It is well-known that the oscillating representation of the Heisenberg group $H(V) =$

$V \oplus \mathbb{R}$ extends to a representation ω_ψ of $\widehat{\mathrm{Sp}}(V)$, depending on the choice of ψ, which is called the Weil representation of $\widehat{\mathrm{Sp}}(V)$(cf. [**35**, §I.1]).

Then we recall the reductive dual pairs described in [**21**, §3]. Let V and V' be the real vector spaces equipped with non-degenerate bilinear forms

$$(\cdot,\cdot) : V \times V \to \mathbb{R} \quad \text{and} \quad \langle\cdot,\cdot\rangle : V' \times V' \to \mathbb{R}$$

of opposite signs. Put $n = \dim V$ and $n' = \dim V'$. We will consider the following cases:

(A) $(\cdot,\cdot)$ is symplectic and $\langle\cdot,\cdot\rangle$ is symmetric with n' even;
(Â) $(\cdot,\cdot)$ is symmetric with n even and $\langle\cdot,\cdot\rangle$ is symplectic;
(B) $(\cdot,\cdot)$ is symplectic and $\langle\cdot,\cdot\rangle$ is symmetric with n' odd;
(B̂) $(\cdot,\cdot)$ is symmetric with n odd and $\langle\cdot,\cdot\rangle$ is symplectic.

Let $G(V)$ (resp. $G(V')$) be the isometry group of V (resp. V'). We will only describe Weil representation of the dual pair in the cases (A) and (B), and the description for the other cases can be obtained parallelly.

Let $\mathbb{V} = V \otimes V'$ equipped with a symplectic form $(\cdot,\cdot) \otimes \langle\cdot,\cdot\rangle$. Then $(G(V), G(V'))$ is a reductive dual pair in the symplectic group $\mathrm{Sp}(\mathbb{V})$ and there is a natural map

$$\iota : G(V) \times G(V') \to \mathrm{Sp}(\mathbb{V}).$$

Note that this map is not necessarily injective.

1.2.1. *Duality Correspondance.* — Consider the double cover of $\mathrm{Sp}(\mathbb{V})$:

$$1 \to \{\pm 1\} \to \widehat{\mathrm{Sp}}(\mathbb{V}) \to \mathrm{Sp}(\mathbb{V}) \to 1.$$

Case (A): The natural map $\iota : \mathrm{Sp}(V) \times \mathrm{O}(V') \to \mathrm{Sp}(\mathbb{V})$ can be lifted to a homomorphism:

$$\iota_{V,V'} : \mathrm{Sp}(V) \times \mathrm{O}(V') \to \widehat{\mathrm{Sp}}(\mathbb{V}). \tag{1.1}$$

Such a splitting is not unique, which depends on some auxiliary data described in [**21**, §3.2]. We call the pullback of the Weil representation of $\widehat{\mathrm{Sp}}(\mathbb{V})$ to $\mathrm{Sp}(V) \times \mathrm{O}(V')$ via the splitting $\iota_{V,V'}$ and obtain the Weil representation of $\mathrm{Sp}(V) \times \mathrm{O}(V')$, denoted by $\omega_{V,V'}$.

Case (B): In this case, we consider the dual pair $(\widehat{\mathrm{Sp}}(V), \widehat{\mathrm{O}}(V'))$, where $\widehat{\mathrm{Sp}}(V)$ and $\widehat{\mathrm{O}}(V')$ are the double covers of $\mathrm{Sp}(V)$ and $\mathrm{O}(V')$ respectively. The map

$$\iota : \mathrm{Sp}(V) \times \mathrm{O}(V') \to \mathrm{Sp}(\mathbb{V})$$

extends to a homomorphism

$$\iota_{V,V'} : \widehat{\mathrm{Sp}}(V) \times \widehat{\mathrm{O}}(V') \to \widehat{\mathrm{Sp}}(\mathbb{V}). \tag{1.2}$$

We call the pullback of the Weil representation of $\widehat{\mathrm{Sp}}(\mathbb{V})$ to $\widehat{\mathrm{Sp}}(V) \times \widehat{\mathrm{O}}(V')$ via $\iota_{V,V'}$ and obtain the Weil representation of $\widehat{\mathrm{Sp}}(V) \times \widehat{\mathrm{O}}(V')$, denoted by $\omega_{V,V'}$.

In both cases, if π and π' are irreducible admissible representation of $G(V)$ and $G(V')$ (or their two-fold cover, depending on the splitting) respectively, we say π and π' correspond if $\pi \otimes \pi'$ is a quotient of $\omega_{V,V'}$, restricted to their product. Howe [**35**] showed that this defines a one-to-one correspondence between subsets of the admissible duals. Our interest is to compute this correspondence explicitly using the Langlands-Vogan parameter.

1.3. Application: Local Gan–Gross–Prasad Conjecture. — The Gan–Gross–Prasad conjecture has been a focal point of study for the past two decades. We briefly recall the previous work on this conjecture in this remark. The first major breakthrough was made by Waldspurger [**56, 57**], and Moeglin–Waldspurger [**43**], where the case of p-adic special orthogonal groups were completely settled. Their basic strategy was to use a local trace formula to express the multiplicities and local root numbers in terms of certain integrals, and then to compare these integrals using theory of twisted endoscopy. Their work was extended to the case of Bessel models for p-adic unitary groups by Beuzart-Plessis in his thesis [**11**]. Soon afterwards, Beuzart-Plessis developed a new local trace formula to establish integral formulae for the multiplicities, which includes both p-adic and real unitary groups. In this way, he proved the "multiplicity one in tempered packets" for real unitary groups, which is a slightly weaker but very useful conjecture stated in [**20**, Conjecture 17.1]. Roughly around the same time, H. He [**26**] established the Gan–Gross–Prasad conjecture for discrete series packets for real unitary groups. He used theta correspondences instead of the local trace formula to attack the conjecture. Inspired by He's method, Xue [**60, 61**] completely settled the conjecture for real unitary groups. Recently Chen and Luo claimed a proof of the conjecture for real special orthogonal groups in a series of paper [**15, 16, 39**].

In the other direction, it has already been observed in [**20**] that Fourier–Jacobi models are connected with Bessel models via theta correspondences. It is thus well-expected that the Gan–Gross–Prasad conjecture for Fourier–Jacobi models can be deduced from the case of Bessel models via theta correspondences. However various ingredients are missing which prevent people to apply theta correspondences directly. Gan and Ichino [**21**] supplied a crucial missing ingredient in the theory of theta correspondences and as a result proved the Gan–Gross–Prasad conjecture in the case of Fourier–Jacobi models for p-adic unitary groups. This work was supplemented by the recent work of Xue [**62**] which prove the same result for real unitary groups. The method of Gan and Ichino was extended by Atobe [5] to prove the Gan–Gross–Prasad conjecture for p-adic symplectic-metaplectic groups.

Thus the only remaining case of the local Gan–Gross–Prasad conjecture is the reason for writting this paper: the case of real symplectic-metaplectic groups.

1.4. Structure. — We now give a brief description of the contents of each chapter. In the 2. Preliminary part, we recall some basic structure theories of the real classical groups and their representations, and then introduce two parameterization methods: the Harish-Chandra-Langlands parameter and the Langlands-Vogan parameters. We focus more on the precise choice of these correspondences for later use, and give two concrete examples: the case of real symplectic groups and the case of real orthogonal groups. In the 3. Theta lifts part, we first describe the work of Paul [45] on the explicit theta correspondence of the equal rank Sp-O case via Harish-Chandra-Langlands paramters. Then we do the translation work, using the construction we made in the first part. Most of the work we did was to make sure that the parameters are well settled to the right place.

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2. Preliminary

In this paragraph, we recall the Harish-Chandra-Langlands parametrization of representations of real reductive group of equal rank (cf. definition 2.5). In particular, we will consider the following real classical groups of rank n with $n \geq 1$:

1. The symplectic group Sp_{2n}. Its Langlands dual group is $\mathrm{SO}_{2n+1}(\mathbb{C})$ and its L-group is the direct product $\mathrm{SO}_{2n+1}(\mathbb{C}) \times W_{\mathbb{R}}$.

2. The odd special orthogonal group SO_{2n+1}. Its Langlands dual group is $\mathrm{Sp}_{2n}(\mathbb{C})$ and its L-group is the direct product $\mathrm{Sp}_{2n}(\mathbb{C}) \times W_{\mathbb{R}}$.
3. The even split special orthogonal group $\mathrm{SO}^{\mathrm{s}}_{2n}$, $n \geq 2$. Its Langlands dual group is $\mathrm{SO}_{2n}(\mathbb{C})$ and its L-group is the direct product $\mathrm{SO}_{2n}(\mathbb{C}) \times W_{\mathbb{R}}$.
4. The quasi-split even special orthogonal group $\mathrm{SO}^{\mathrm{qs}}_{2n}$, $n \geq 1$. Its Langlands dual is $\mathrm{SO}_{2n}(\mathbb{C})$ and its L-group is the semi-direct product $\mathrm{SO}_{2n}(\mathbb{C}) \rtimes W_{\mathbb{R}}$.

2.1. Basic setup.

— Let $\mathbf{G}$ be a complex reductive group. We begin with basic root data and the automorphisms.

2.1.1. *Based Root Datum.* — A Borel pair $(\mathbf{B}, \mathbf{T})$ of $\mathbf{G}$ is a pair consisting of a maximal torus $\mathbf{T}$ of $\mathbf{G}$ and a Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. For any Borel pair $(\mathbf{B}, \mathbf{T})$ of $\mathbf{G}$, let $X(\mathbf{T})$ be the group of characters of $\mathbf{T}$, Φ the set of roots, $X^\vee(\mathbf{T})$ the group of cocharacters of $\mathbf{T}$ and $\Phi^\vee$ the set of coroots. We denote the natural pairing $X(\mathbf{T}) \times X^\vee(\mathbf{T}) \to \mathbb{Z}$ by $\langle \cdot, \cdot \rangle$. In this article we fix the canonical isomorphisms:

$$\mathfrak{t} \cong X^\vee(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{C}, \quad \mathfrak{t}^\vee \cong X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

We define the 4-tuple $R(\mathbf{G}, \mathbf{T}) = (X, \Phi, X^\vee, \Phi^\vee)$ together with the pairing $\langle \cdot, \cdot \rangle$ a based root datum of $\mathbf{G}$. By [18, Prop. 5.1.6], this 4-tuple is a reduced root datum in the sense of [18, Def. 1.3.3]. Moreover, the 6-tuple $D_b = (X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$ is the based root datum (attached to the Borel pair $(\mathbf{B}, \mathbf{T})$), where Δ is the set of positive simple roots corresponding to $\mathbf{B}$, $\Delta^\vee$ is the set of coroots $\alpha^\vee = \frac{2\langle \alpha, - \rangle}{\langle \alpha, \alpha \rangle}$ for $\alpha \in \Delta$, see [18, §1.5]. Up to a canonical isomorphism, the 6-tuple is independent of the choice of Borel pair: If $(\mathbf{B}', \mathbf{T}')$ is another Borel pair, then there exists $g \in \mathbf{G}$ such that $\mathrm{ad}(g)$ carries $(\mathbf{B}', \mathbf{T}')$ to $(\mathbf{B}, \mathbf{T})$. This induced isomorphism on root data is independent of g.

For $\alpha \in \Delta$, we define the reflection $s_\alpha \in \mathrm{Hom}(X, X)$:

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad (x \in X),$$

and define $s_{\alpha^\vee} \in \mathrm{Hom}(X^\vee, X^\vee)$ respectively.

It is well known that if we define the based root datum using the basic axioms, it is always the root datum of a reductive algebraic group, which is determined uniquely up to isomorphism.

2.2. Root systems of real reductive groups.

— Let $\mathbf{G}$ be a quasi-split reductive group with dual group G^d. As discussed in the introduction, we need to fix a fundamental Borel pair $(\mathbf{B}_*, \mathbf{T}_*)$ of $\mathbf{G}$ of Whittaker type. It is equivalent to provide $D_b = (X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$ the based root datum attached to the fundamental Borel pair $(\mathbf{B}_*, \mathbf{T}_*)$. In particular, the imaginary simple roots in Δ are non-compact since $\mathbf{G}$ is quasi-split.

We identifies the Cartan subgroup $\mathbf{T}_*$ with $\mathbf{U}(1)^n$ via Δ and the elements in $\mathbf{T}_*[2]$ will be denoted by $t = (\pm 1, \cdots, \pm 1)$. Let Ψ_* be the system of positive roots generated by Δ and let $\Psi_{c,*} \subset \Psi_*$ be the subset of compact positive roots. We set

$$\rho(\Psi_*) = \frac{1}{2}\sum_{\alpha\in\Psi_*}\alpha \quad \text{and} \quad \rho_c(\Psi_*) = \frac{1}{2}\sum_{\alpha\in\Psi_{c,*}}\alpha.$$

Note that $\rho(\Psi_*)$ and $\rho_c(\Psi_*)$ are independent of choice of set of positive roots and we simply denote them by ρ and ρ_c respectively. Similary, we set $\rho^\vee = \frac{1}{2}\sum_{\alpha\in\Psi_*}\alpha^\vee$.

In this paragraph, we gives the explicit description of the choices of root systems of real classical groups appearing in our applications.

2.2.1. *Symplectic case.* — Let $G = \mathrm{Sp}_{2n}$ and let σ_G be the complex conjugation σ acting on the complex points $\mathbf{G}$ of the real reductive group G such that $\mathbf{G}^{\sigma_G} = G$. We fix a maximal torus

$$T_* = \begin{pmatrix} \mathrm{diag}(\cos\theta_1,\cdots,\cos\theta_n) & \mathrm{diag}(\sin\theta_1,\cdots,\sin\theta_n) \\ \mathrm{diag}(-\sin\theta_1,\cdots,-\sin\theta_n) & \mathrm{diag}(\cos\theta_1,\cdots,\cos\theta_n)\end{pmatrix}$$

of G, whose Lie algebra $\mathfrak{t}_0$ is:

$$\mathfrak{t}_0 = \{\begin{pmatrix} 0_n & \mathrm{diag}(t_1,\cdots,t_n) \\ \mathrm{diag}(-t_1,\cdots,-t_n) & 0_n\end{pmatrix} : t_i \in \mathbb{R}, 1 \le i \le n\}. \tag{2.1}$$

Let $\mathfrak{t}$ be the complexification of $\mathfrak{t}_0$. For $1 \le i \le n$, we set

$$e_i : \begin{pmatrix} 0_n & \mathrm{diag}(t_1,\cdots,t_n) \\ \mathrm{diag}(-t_1,\cdots,-t_n) & 0_n\end{pmatrix} \in \mathfrak{t} \mapsto 2it_i \in \mathbb{C}.$$

Let $\mathfrak{g}_0$ be the Lie algebra of G, and $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$. Then the set $\Phi(\mathfrak{g},\mathfrak{t})$ of roots is

$$\Phi(\mathfrak{g},\mathfrak{t}) = \{\pm e_i \pm e_j : 1 \le i < j \le n\} \cup \{\pm 2e_i : 1 \le i \le n\}.$$

Fix a set of simple roots

$$\Delta = \{(-1)^{i-1}(e_i + e_{i+1}), (-1)^{n-1}2e_n, i = 1,\cdots,n-1\},$$

and we denote by Ψ the set of positive roots generated by Δ. Moreover, the set of compact roots is $\Phi_c = \{e_i - e_j : 1 \le i,j \le n\}$, the set of the non compact roots are $\Phi_n = \{\pm(e_i + e_j) : 1 \le i,j \le n\}$. The choice of root systems above gives a fundamental Borel pair of Whittaker type $(\mathbf{B}_*, \mathbf{T}_*)$ of $\mathbf{G}$ and an identification between $\mathbf{T}_*$ and the complexe torus $\mathbf{U}(1)^n$ via the simple roots of $\mathbf{T}_*$ in $\mathbf{B}_*$.

2.2.2. *Orthogonal case (1).* — First let $G = \mathrm{O}(p,q)$ with $p \geq q$ even. Let $\mathfrak{g}_0$ be the Lie algebra of $\mathrm{O}(p,q)$, and $\mathfrak{g}$ its complexification. Let $K \cong \mathrm{O}(p) \times \mathrm{O}(q)$ which is a maximal compact subgroup of $\mathrm{O}(p,q)$. Let $\mathfrak{k}_0$ be the Lie algebra of K and $\mathfrak{k}$ its complexification. We choose a Cartan subgroup T of K with Lie algebra $\mathfrak{t}_0$ as follows:

$$\mathfrak{t}_0 = \{\mathrm{diag}(g(t_1), \cdots, g(t_{p_0}), g(s_1), \cdots, g(s_{q_0})) : t_i, s_i \in \mathbb{R}\}$$

with $g(t) = \left(\begin{smallmatrix} 0 & t \\ -t & 0 \end{smallmatrix}\right)$ for all $t \in \mathbb{R}$, $p_0 = \frac{p}{2}$ and $q_0 = \frac{q}{2}$. For $1 \leq i \leq p_0$ and $1 \leq j \leq q_0$, we set

$$e_i : (\mathrm{diag}(g(t_1), \cdots, g(t_{p_0}), g(s_1), \cdots, g(s_{q_0})) : t_i, s_i \in \mathbb{R}\} \mapsto 2it_i \in \mathbb{C},$$

$$f_j : (\mathrm{diag}(g(t_1), \cdots, g(t_{p_0}), g(s_1), \cdots, g(s_{q_0})) : t_i, s_j \in \mathbb{R}\} \mapsto 2is_j \in \mathbb{C}.$$

Let $\mathfrak{t}$ be the complexification of $\mathfrak{t}_0$. The set $\Phi(\mathfrak{g}, \mathfrak{t})$ of roots is

$$\begin{aligned}\Phi(\mathfrak{g}, \mathfrak{t}) =& \{\pm e_i \pm e_j : 1 \leq i < j \leq p_0\} \cup \{\pm f_i \pm f_j : 1 \leq i < j \leq q_0\} \\ & \cup \{\pm e_i \pm f_j : 1 \leq i \leq p_0, 1 \leq j \leq q_0\}\end{aligned}$$

We fix a set Δ of simple roots as follows:

1. If $p = q$ or $p = q+2$, then $\Delta = \{e_i - f_i, f_i - e_{i+1} | 1 \leq i \leq q_0\} \cup \{e_{p_0} + f_{q_0}\}$;
2. If $p > q+2$, then

$$\Delta = \{e_i - f_i, f_i - e_{i+1} | 1 \leq i \leq q_0\} \cup \{e_j - e_{j+1} | q_0 + 1 \leq j \leq p_0\} \cup \{e_{p_0 - 1} + e_{p_0}\}.$$

Moreover, fix the set of compact roots is $\Phi_c = \{\pm e_i \pm e_j : 1 \leq i \neq j \leq p_0\} \cup \{\pm f_i \pm f_j : 1 \leq i \neq j \leq q_0\}$, the set of the non compact roots is $\Phi_n = \{\pm e_i \pm f_j : 1 \leq i \leq p_0, 1 \leq j \leq q_0\}$.

2.2.3. *Orthogonal case (2).* — Then let $G = \mathrm{O}(p,q)$ with $p \geq q$ odd. We follow the above notation with only a few changes. We choose a Cartan subgroup T of K with Lie algebra $\mathfrak{t}_0$ as follows:

$$\mathfrak{t}_0 = \{\mathrm{diag}(g(t_1), \cdots, g(t_{p_0}), 1, 1, g(s_1), \cdots, g(s_{q_0})) : t_i, s_i \in \mathbb{R}\}$$

with $g(t) = \left(\begin{smallmatrix} 0 & t \\ -t & 0 \end{smallmatrix}\right)$ for all $t \in \mathbb{R}$, $p_0 = \frac{p-1}{2}$ and $q_0 = \frac{q-1}{2}$. Now the set $\Phi(\mathfrak{g}, \mathfrak{t})$ of roots is

$$\begin{aligned}\Phi(\mathfrak{g}, \mathfrak{t}) =& \{\pm e_i \pm e_j : 1 \leq i < j \leq p_0\} \cup \{\pm f_i \pm f_j : 1 \leq i < j \leq q_0\} \\ & \cup \{\pm e_i \pm f_j, \pm e_i, \pm f_j : 1 \leq i \leq p_0, 1 \leq j \leq q_0\}\end{aligned}$$

with the roots of the form $\{\pm e_i\}$ and $\{f_j\}$ each occurring twice here. We fix a set Δ of simple roots as follows:

1. If $p = q$ or $p = q+2$, then $\Delta = \{e_i - f_i, f_i - e_{i+1} | 1 \leq i \leq q_0\} \cup \{e_{p_0} + f_{q_0}\} \cup \{e_i, f_j : 1 \leq i \leq p_0, 1 \leq j \leq q_0\}$;

2. If $p > q+2$, then $\Delta = \{e_i - f_i, f_i - e_{i+1} | 1 \leq i \leq q_0\} \cup \{e_j - e_{j+1} | q_0 + 1 \leq j \leq p_0\} \cup \{e_{p_0-1} + e_{p_0}\} \cup \{e_i, f_j : 1 \leq i \leq p_0, 1 \leq j \leq q_0\}$.

Moreover, fix the set of compact roots is $\Phi_c = \{\pm e_i \pm e_j : 1 \leq i \neq j \leq p_0\} \cup \{\pm f_i \pm f_j : 1 \leq i \neq j \leq q_0\} \cup \{e_i, f_j : 1 \leq i \leq p_0, 1 \leq j \leq q_0\}$, and the set of the non compact roots is the same.

2.2.4. *Automorphisms.* — Again let $\mathbf{G}$ be a complex reductive group. We denote by $\mathrm{Inn}(\mathbf{G})$, $\mathrm{Aut}(\mathbf{G})$ and $\mathrm{Out}(\mathbf{G})$, the group of inner automorphisms of $\mathbf{G}$, the group of (holomorphic) automorphisms of $\mathbf{G}$ and the group of outer automorphisms of $\mathbf{G}$ respectively. We have a short exact sequence of groups

$$1 \to \mathrm{Inn}(\mathbf{G}) \to \mathrm{Aut}(\mathbf{G}) \to \mathrm{Out}(\mathbf{G}) \to 1. \tag{2.2}$$

The group $\mathrm{Aut}(\mathbf{G})$ acts on the set of Borel pairs with maximal split torus $\mathbf{T}$. If $\sigma \in \mathrm{Aut}(\mathbf{G})$, the Borel pairs $(\sigma(\mathbf{B}), \sigma(\mathbf{T}))$ and $(\mathbf{B}, \mathbf{T})$ are conjugate by an element $g_\sigma \in \mathbf{G}$, which is uniquely determined by σ up to an element of $\mathbf{T}$ (cf. [**18**, Prop. 6.2.11 (2)]). This induces a group homomorphism $\mathrm{Aut}(\mathbf{G}) \to \mathrm{Aut}(R(\mathbf{G}, \mathbf{T}), \Delta)$ defined by $\sigma \mapsto \mathrm{Ad}(g_\sigma) \circ \sigma$, where $\mathrm{Aut}(R(\mathbf{G}, \mathbf{T}), \Delta)$ is the group consisting of automorphisms of $R(\mathbf{G}, \mathbf{T})$, which is an isomorphism $\phi \in \mathrm{Aut}(X)$ satisfies $\phi(\Delta) = \Delta$ and $\phi^t(\Delta^\vee) = \Delta^\vee$. Here $\phi^t \in \mathrm{Aut}(X^\vee)$ is defined as the adjoint of ϕ which is compatible with the pairing $\langle \cdot, \cdot \rangle$. By [**18**, Prop. 7.1.6], we have an exact sequence

$$1 \to \mathrm{Inn}(\mathbf{G}) \to \mathrm{Aut}(\mathbf{G}) \to \mathrm{Aut}(R(\mathbf{G}, \mathbf{T}), \Delta) \to 1, \tag{2.3}$$

identifying $\mathrm{Out}(\mathbf{G})$ with $\mathrm{Aut}(R(\mathbf{G}, \mathbf{T}), \Delta)$.

2.2.5. *Pinning, L-group.* — A pinning for $\mathbf{G}$ is a triple $\mathbf{Spl_G} = (\mathbf{B}, \mathbf{T}, \{X_\alpha\}_{\alpha \in \Delta})$, where $(\mathbf{B}, \mathbf{T})$ is a Borel pair of $\mathbf{G}$ and where, for α a simple root, X_α is a α-root vector of $\mathbf{T}$ in $\mathrm{Lie}(\mathbf{B})$. Then $\mathbf{G}$ acts transitively on the set of pinnings by conjugation, the stabilizer of any pinning is the center $Z(\mathbf{G})$ of $\mathbf{G}$. Given a pinning $\mathbf{Spl_G}$, this gives an isomorphism

$$s_{\mathbf{Spl_G}} : \mathrm{Out}(\mathbf{G}) \cong \mathrm{Stab}_{\mathrm{Aut}(\mathbf{G})}(\mathbf{Spl_G}) \subset \mathrm{Aut}(\mathbf{G})$$

and this is a splitting of the exact sequence 2.2. We call a splitting distinguished if it fixes a pinning data.

The dual group $\mathbf{G}^d$ of $\mathbf{G}$ is the complex, connected reductive group whose root data is isomorphic to $R(\mathbf{G}, \mathbf{T})^\vee$. Fix such an isomorphism of $R(\mathbf{G}, \mathbf{T})^\vee$ with $R(\mathbf{G}^d, \mathbf{T}^d)$. Through the natural Galois action Γ on $R(\mathbf{G}, \mathbf{T})$, we obtain a homomorphism $\Gamma \to \mathrm{Out}(\mathbf{G}^d)$, using the identification of two short sequences 2.2 and 2.3. Then composite with the section $s_{\mathbf{Spl}_{\mathbf{G}^d}}$ defined by a pinning $\mathbf{Spl}_{\mathbf{G}^d}$, we obtain an action of Γ on $\mathbf{G}^d$ which preserves the pinning $\mathbf{Spl}_{\mathbf{G}^d}$.

The Langlands group ${}^L\mathbf{G}$ of $\mathbf{G}$ associated to the pinning $\mathbf{Spl}_{\mathbf{G}^d}$ is the semi-direct product $\mathbf{G}^d \rtimes W_{\mathbb{R}}$, where the action of $W_{\mathbb{R}}$ on $\mathbf{G}^d$ factors through the projection $p_{W_{\mathbb{R}}} : W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ stablizing $\mathbf{Spl}_{\mathbf{G}^d}$.

We remark that the L-group of $\mathbf{G}$ only depends on its inner class.

2.3. Generalities on real forms and Cartan involutions.

— We recall some structure theories of real reductive groups and the parametrization of real forms. Our main reference is [3].

Definition 2.1. — Let $\mathbf{G}$ be a complex reductive group.

1. A real form of a complex reductive group $\mathbf{G}$ is an antiholomorphic involutive automorphism σ of $\mathbf{G}$.
2. We say that two real forms σ_1, σ_2 are inner to each other, or in the same *inner class*, if $\sigma_1\sigma_2^{-1}$ is an inner automorphism of $\mathbf{G}$.
3. We say that two real forms σ_1, σ_2 are equivalent, if they are conjugate by an inner automorphism of $\mathbf{G}$.
4. A real form σ of $\mathbf{G}$ is said to be a compact real form if $\mathbf{G}^\sigma$ is compact and meets every component of $\mathbf{G}$.

Given a real reductive group G, it is equivalent to provide a real form σ_G, which satisfies $(G \otimes_{\mathbb{R}} \mathbb{C})^{\sigma_G} = G$.

Remark 2.2. — The standard definition of equivalence of real forms (cf. [46, Section III.1]) allows conjugation by $\mathrm{Aut}(\mathbf{G})$. But since we are interested in the inner class, we follow the definition of Adams and Taïbi in [3]. Moreover, for a real form σ of $\mathbf{G}$, by [3, Lemma 8.1], the set of equivalent classes of real forms in the inner class of σ is parametrized by $H^1(\sigma, \mathbf{G}_{\mathrm{ad}})$, where $\mathbf{G}_{\mathrm{ad}}$ is the adjoint group. Explicitly, the map is $\mathrm{cl}(h) \mapsto [\mathrm{int}(h) \circ \sigma]$. By [3, Lemma 2.4], for equivalence classes $[\sigma]$, we have a well-defined pointed set $H^1([\sigma], \mathbf{G}_{\mathrm{ad}}) = H^1(\sigma, \mathbf{G}_{\mathrm{ad}})$.

Remark 2.3. — By [3, Lemma 3.4], our definition of compact real form is equivalent to the definition of Mostow [42, Section 2], which defines a compact real form to be a compact subgroup G_K such that $\mathrm{Lie}(\mathbf{G}) = \mathrm{Lie}(G_K) \oplus i\mathrm{Lie}(G_K)$, and G_K meets every component of G. The bijection is given by $\sigma \mapsto \mathbf{G}^\sigma$. Every complex reductive group has a compact real form (Weyl, Chevalley, Mastow [42, Lemma 6.1]), and the uniqueness is up to the $\mathbf{G}^0$-conjugation (Cartan, Hochschild, Mostow [42, Chapter XV]).

Cartan involution provides a description of real forms in terms of holomorphic involutions which is better suited to our purposes.

Definition 2.4. — A *Cartan involution* for $(\mathbf{G},\sigma)$, where σ is a real form of complex reductive group $\mathbf{G}$, is a holomorphic involutive automorphism θ of $\mathbf{G}$, commuting with σ, such that $\theta\sigma$ is a compact real form of $\mathbf{G}$.

The existence and uniqueness (up to conjugation by $\mathrm{Inn}(\mathbf{G}^0)$) of Cartan involution, and the correspondence between

$$\{\text{antiholomorphic involutive automorphisms of } \mathbf{G}\}/\mathrm{Inn}(\mathbf{G}^0) \tag{2.4}$$

and

$$\{\text{holomorphic involutive automorphisms of } \mathbf{G}\}/\mathrm{Inn}(\mathbf{G}^0) \tag{2.5}$$

induced by the correspondence between real forms and Cartan involutions, are given by [**3**, Theorem 3.13], based on Remark 2.3. The following construction also gives an explanation.

Fix a Borel pair $(\mathbf{B},\mathbf{T})$ of $\mathbf{G}$. If σ is a real form of $\mathbf{G}$ and θ is a Cartan involution for $(\mathbf{G},\sigma)$, then both σ and θ naturally act on the based root datum D_b attached to $(\mathbf{B},\mathbf{T})$, giving rise to involutions $\overline{\sigma},\overline{\theta}\in\mathrm{Aut}(R(\mathbf{G},\mathbf{T}),\Delta)$, which are also seen as elements of the subgroup $\mathrm{Out}(\mathbf{G})[2]$ of order 2 elements in $\mathrm{Out}(\mathbf{G})$. They are related by $\overline{\sigma}\overline{\theta}=-w_0$, where w_0 is the longest element of the Weyl group of D_b and -1 is the inversion automorphism of $\mathbf{T}$. Note that w_0 is invariant under $\mathrm{Aut}(R(\mathbf{G},\mathbf{T}),\Delta)$, and so $\iota:=-w_0$ is a central involution in $\mathrm{Out}(\mathbf{G})$. As a result, we see that the set of inner classes of real forms of $\mathbf{G}$ can be parameterized by $\mathrm{Out}(\mathbf{G})[2]$, and we say that a real form σ lies in the inner class defined by $\delta\in\mathrm{Out}(\mathbf{G})[2]$ if $\iota\delta=\overline{\sigma}$.

Definition 2.5. — The real forms in the inner class defined by $1\in\mathrm{Out}(\mathbf{G})[2]$ are called *of equal rank*.

Every real form $\mathbf{G}(\mathbb{R})$ in this inner class contains a compact Cartan subgroup, and every Cartan involution in this inner class is contained in $\mathrm{Inn}(\mathbf{G})$. The reason why this inner class is for our interest is, a necessary and sufficient condition for $\mathbf{G}(\mathbb{R})$ to admit discrete series is that it has a compact Cartan subgroup, by [**24**, Theorem 13].

For an inner class $\delta\in\mathrm{Out}(\mathbf{G})[2]$ and a pinning $\mathbf{Spl_G}=(\mathbf{B},\mathbf{T},\{X_\alpha\}_{\alpha\in\Delta})$, there is a unique real form $\sigma_{\mathrm{qs}}(\delta,\mathbf{Spl_G})$ of $\mathbf{G}$ preserving $\mathbf{Spl_G}$ and such that $\overline{\sigma_{\mathrm{qs}}(\delta,\mathbf{Spl_G})}=\iota\delta$, and it is naturally a quasisplit real form[(1)]. Since for $g\in\mathbf{G}_{\mathrm{ad}}$, we have $\sigma_{\mathrm{qs}}(\delta,\mathrm{int}(g)(\mathbf{Spl_G}))=\mathrm{int}(g)\circ\sigma_{\mathrm{qs}}(\delta,\mathbf{Spl_G})\circ\mathrm{int}(g)^{-1}$, the equivalence class $\sigma_{\mathrm{qs}}(\delta,\mathbf{Spl_G})$ does not depends on the choice of $\mathbf{Spl_G}$ and denote this unique class of quasi-split forms in the inner class defined by δ as $[\sigma_{\mathrm{qs}}(\delta)]$. Conversely, the above construction contains all quasisplit real forms of $\mathbf{G}$. The

(1) A real form is called quasi-split if it preserves a Borel subgroup of $\mathbf{G}$.

conjugacy class of quasi-split real forms has a particular interest as they can be characterized by Borel pairs with special properties.

2.4. Harish-Chandra-Langlands parameters. — Let G be a real reductive group of equal rank. We fix a fundamental Borel pair $(\mathbf{B}_*, \mathbf{T}_*)$ and the based root datum $(X, \Phi, \Delta_*, X^\vee, \Phi^\vee, \Delta_*^\vee)$ associated to $(\mathbf{B}_*, \mathbf{T}_*)$. Let Ψ_* be the set of positive roots generated by Δ_* and $\Psi_{c,*}$ its subset of compact positive roots.

Let $\mathrm{Rep}(G)$ be the set of representations of G. Since we are considering the real group, by a representation we will view it as an $(\mathfrak{g}, K)$-mod. Then there is an "infinitesimal character map"

$$\mathrm{Rep}(G) \to \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(Z(\mathfrak{g}), \mathbb{C}), \pi \mapsto \lambda_\pi,$$

and by Harish-Chandra's finiteness theorem this map has finite fibres. Let π be a representation of G with infinitesimal character λ_π. By Harish-Chandra homomorphism(2), we can lift λ_π to a character of $\mathfrak{t}$, denoted still by λ_π. We say λ_π is regular (resp. integral) if $\langle \lambda_\pi, \alpha^\vee \rangle \neq 0$ (resp. is in $\mathbb{Z}$) for all $\alpha \in \Delta$. Note that, if λ_π is regular (resp. integral), then π is a discrete series (resp. limits of discrete series). Note that the lifting of the infinitesimal characters is not unique and but we can determine a unique lifting by the representation π via its standard representation Thus, we still denote by λ_π its lifting obtained in this way. Some concrete examples can be found below.

Definition 2.6. — 1. A limit of discrete series Harish-Chandra parameter of G is an integral element

$$\lambda_d \in i\mathfrak{t}_0^* \subset \mathfrak{t}^*.$$

In particular, if λ_d is regular, then we say λ_d is a discrete series Harish-Chandra parameter.

(2)The Poincare-Birkhoff-Witt theorem implies that we have a decomposition of $U(\mathfrak{g})$:

$$U(\mathfrak{g}) = U(\mathfrak{t}) \oplus (U(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^- U(\mathfrak{g})),$$

and the projection of $Z(\mathfrak{g})$ to the second factor lies in $U(\mathfrak{g})\mathfrak{n}^+ \cap \mathfrak{n}^- U(\mathfrak{g})$. Let $\gamma' : Z(\mathfrak{g}) \to Z(\mathfrak{t})$ be the projection on to the first factor. Let ρ be the half the sum of the positive roots. Let $t_\rho : U(\mathfrak{t}) \to U(\mathfrak{t})$ be the translation operator $t_\rho(\phi(\lambda)) = \phi(\lambda - \rho)$, where $\phi \in \mathbb{C}[\mathfrak{t}^*]$ and $\lambda \in \mathfrak{t}^*$. The composition $t_\rho \circ \gamma' : Z(\mathfrak{g}) \to U(\mathfrak{t}) = S(\mathfrak{t}) = \mathbb{C}[\mathfrak{t}^*]$ is a homomorphism, known as the Harish-Chandra homomorphism. The image of the Harish-Chandra homomorphism is invariant under the action of Weyl group, and the map is actually an isomorphism $\gamma : Z(\mathfrak{g}) \to \mathbb{C}[\mathfrak{t}^*]^W = \mathbb{C}[\mathfrak{t}^*/W]$.

2. A limit of discrete series Harish-Chandra-Langlands parameter is a pair (λ_d, Ψ), where λ_d is a limit of discrete series Harish-Chandra paramter of G, and $\Psi \subset \Phi$ is the set of positive roots with respect to λ_d satisfying:
   (a) $\Psi_{c,*} \subset \Psi$;
   (b) λ_d is dominant with respect to Ψ;
   (c) if a simple root $\alpha \in \Psi$ satisfies $\langle \lambda_d, \alpha \rangle = 0$, then α is non-compact.

Remark 2.7. — If λ_d is a discrete series Harish-Chandra parameter, then conditions (a), (b) and (c) uniquely determines the set Ψ, thus it is also a Harish-Chandra-Langlands parameter. For a limits of discrete series Harish-Chandra parameter λ_d, there are possible several set Ψ's admitting the same conditions (a), (b) and (c). We will also do a concrete calcuation later.

If an infinitesimal character is specified, then by [**31**, Theorem 10.23], there are finitely many irreducible representations of K such that every irreducible representation π of G with that infinitesimal character contains one of these K-types as its minimal K-type(3) , equivalently, has the root system Ψ as its Harish-Chandra-Langlands parameter.

2.4.1. *Symplectic case.* — The explicit description of the Harish-Chandra parameters of the representation of symplectic groups can be given as follows. Let $\Phi = \Delta(\mathfrak{g}, \mathfrak{t})$ be the set of roots of $\mathrm{Sp}_{2n}(\mathbb{R})$ and let π be a limit of discrete series representation of $\mathrm{Sp}_{2n}(\mathbb{R})$. Then π can be parametrized by pairs (λ_d, Ψ), where

(3)To better understand the classification of the limit of discrete series representations of G, one can use an equivalent way which is the so-called minimal K-type condition. This is exactly the description used in [**45**]. To save the complexity of the main article, we would like to leave this notation as an remark here.

Let $\pi \in \Pi(\varphi, G)$. The restriction of π to K can be decomposed into a completed direct sum of irreducible representations of K. Note that a *K-type* is an irreducible representation τ of K which can be parametrized by its highest weight $\mu_\tau \in i\mathfrak{t}_0^* \subset \mathfrak{t}^*$. Vogan defined a norm

$$\|\tau\| := \sqrt{\langle \mu_\tau + 2\rho_c, \mu_\tau + 2\rho_c \rangle}$$

on the set of K-types.

Definition 2.8. — [**52**, Definition 5.1] A minimal K-type of a $(\mathfrak{g}, K)$-module π is a K-type that has minimal norm among all K-types occurring in $\pi_{|K}$. More precisely, by calcuation, the minimal K-type of $(\mathfrak{g}, K)$-module π is given by

$$\lambda_d + \rho_n - \rho_c,$$

where ρ_n and ρ_c are one-half the sums of the non-compact and compact roots of Ψ.

- The parameter λ_d is of the form

$$\lambda_d = (\underbrace{\lambda_1, \cdots, \lambda_1}_{p_1}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{p_k}, \underbrace{0, \cdots, 0}_{z}, \underbrace{-\lambda_k, \cdots, -\lambda_k}_{q_k}, \cdots, \underbrace{-\lambda_1, \cdots, -\lambda_1}_{q_1}),$$

  with $\lambda_i \in \mathbb{Z}$, $\lambda_1 > \cdots > \lambda_k > 0$, $|p_i - q_i| \leq 1$;
- $\Psi \subset \Phi$ is a root system containing all the positive compact roots, such that λ_d is dominant with respect to Ψ, and for all simple roots $\alpha \in \Psi$ we have that if $\langle \lambda_d, \alpha \rangle = 0$, then α is non-compact.
- The minimal K-type Λ is given by $\lambda_d + \rho_n - \rho_c$.
- If $z = 0$ and $p_i + q_i = 1$ for all i, then the representation associated to (λ_d, Ψ) is a discrete series.

Consider the real group $G = \mathrm{Sp}_4(\mathbb{R})$ with the based root datum described in §2.2.1.

Let $\lambda_d = (2, 1)$ be an infinitesimal character. Then there are four Harish-Chandra parameters $(2, 1), (2, -1), (1, -2), (-1, -2)$ as liftings of λ_d, each of which determines a unique discrete series representations of G with infinitesimal character λ_d.

Let $\lambda_d = (2, 0)$ be an infinitesimal character. Then there are two Harish-Chandra parameters $(2, 0), (0, -2)$ as liftings of λ_d, each of which determines two limit of discrete series of G. We use the minimal K-types to distinguish them. For example, to the Harish-Chandra parameters $(2, 0)$, we can associate two sets of simple roots $\{e_1 - e_2, 2e_2\}$ with positive non-compact roots $\{2e_1, 2e_2, e_1 + e_2\}$ and $\{e_1 + e_2, -2e_2\}$ with positive non-compact roots $\{2e_1, -2e_2, e_1 + e_2\}$[(4)].

2.4.2. *Orthogonal case.* — In the following, we recall the Harish-Chandra-Langlands parametrization for representations of the orthogonal group of equal rank $\mathrm{O}(p, q)$, with p and q two non-negative integers. Let $n = [\frac{p+q}{2}]$. The maximal compact subgroup of $\mathrm{O}(p, q)$ is $K = \mathrm{O}(p) \times \mathrm{O}(q)$. Thus, the equal rank condition implies that p and q are even if $p + q$ is even.

Set $p_0 = [\frac{p}{2}]$ and $q_0 = [\frac{q}{2}]$. We can parametrize an irreducible representation of compact group $\mathrm{O}(p)$ by $(\lambda_0; \varepsilon)$, here $\lambda_0 = (a_1, \cdots, a_{p_0})$ is the usual highest weight of a finite dimensional representation of $\mathrm{SO}(p)$ and $\varepsilon \in \{\pm 1\}$. If p is even and $a_{p_0} > 0$, $(\lambda_0; 1)$ and $(\lambda_0; -1)$ correspond to the same representation of $\mathrm{O}(p)$. If p is odd then $-\mathrm{Id}$ acts by $(-1)^{\sum_{i=1}^{p_0} a_i} \varepsilon$. If p is even, the parameter of the trivial representation of $\mathrm{O}(p)$ corresponds to $(0, \cdots, 0; 1)$, the sign representation of $\mathrm{O}(p)$ corresponds to $(0, \cdots, 0; -1)$, and we have $(a_1, \cdots, a_{[\frac{p}{2}]}; \varepsilon) \otimes \mathrm{sgn} = (a_1, \cdots, a_{[\frac{p}{2}]}; -\varepsilon)$. The representations of $\mathrm{O}(q)$ can be parameterized in the

(4)The corresponding minimal K-types $\lambda_d + \rho_n - \rho_c$ are $(3, 2)$ and $(3, 0)$ respectively.

same way. Hence an irreducible finite dimensional representation of K is parametrized by $(a_1, \cdots, a_{p_0}; \varepsilon) \otimes (b_1, \cdots, b_{q_0}; \varepsilon')$. The explicit description of the Harish-Chandra parameters of the representation of even orthogonal groups can be given as follows. Let Φ be the set of roots of $\mathrm{O}(p,q)$ and let π be a limit of discrete series representation of $\mathrm{O}(p,q)$. Then π can be parametrized by the triple (λ_d, ξ, Ψ), where

- The representation π is one of the irreducible representations of $\mathrm{O}(p,q)$ whose restriction to $\mathrm{SO}(p,q)$ contains the limit of discrete series with parameter (λ_d, Ψ), where

$$\lambda_d = (\underbrace{\lambda_1, \cdots, \lambda_1}_{p_1}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{p_k}, \underbrace{0, \cdots, 0}_{z}, \underbrace{\lambda_1, \cdots, \lambda_1}_{q_1}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{q_k}, \underbrace{0, \cdots, 0}_{z'}),$$

  with
  (a) $\lambda_k \in \mathbb{Z}$, $\lambda_1 > \cdots > \lambda_k > 0$, $|p_i - q_i| \leq 1$,
  (b) $|z - z'| \leq 1$,
  (c) $p_0 = \sum_{i=1}^k p_i + z$ and $q_0 = \sum_{i=1}^k q_i + z'$;
- $\Psi \subset \Phi$ is a root system containing all the positive compact roots, such that λ_d is dominant with respect to Ψ;
- The minimal K-type Λ is given by $\lambda_d + \rho_n - \rho_c$. If $z + z' = 0$ then the entries of Λ are all non-zero, so the signs of Λ are arbitrary; in this case choose $\xi = 1$ and there is only one limit of discrete series of $\mathrm{O}(p,q)$ corresponding to (λ_d, Ψ). If $z + z' > 0$ then there are two possible minimal K-types corresponding to (λ_d, Ψ); in this case choose $\xi = 1$ for the representation whose minimal K-type Λ has signs $(1; 1)$, and $\xi = -1$ for the other one.
- If $z + z' \leq 1$ and $p_i + q_i = 1$ for all i, then the representation associated to (λ_d, ξ, Ψ) is a discrete series.

2.5. Real classical groups of equal rank. — We are interested in the real classical groups of equal rank[(5)] since the real forms of equal rank admit discrete series representations. Note that, if n is even (resp. odd), then the split (resp. quasi split) orthogonal groups of rank $n+1$ is not of equal rank. Thus for even special orthogonal group, we will take the quasi-split (resp. split) orthogonal groups of rank $n+1$ when n is even (resp. odd).

(5) The real form in the equal rank case is called pure inner form in the sense of Kaletha (cf. [29]).

2.5.1. *Central invariants of strong inner forms.* — Fix an inner class $\delta \in \mathrm{Out}(\mathbf{G})[2]$. Consider the action of $\mathrm{Aut}(\mathbf{G})$ on the center $\mathbf{Z} := Z(\mathbf{G})$ of $\mathbf{G}$, which factors through the quotient $\mathrm{Out}(\mathbf{G})$. We denote by $\mathbf{Z}^\delta$ the subgroup of $\mathbf{Z}$ fixed by δ. To any real form σ of $\mathbf{G}$ in the inner class defined by δ, one identify $[\sigma]$ with a class in $H^1([\sigma_{\mathrm{qs}}(\delta)], \mathbf{G}_{\mathrm{ad}})$, by Remark 2.2, and define a central invariant $\mathrm{Inv}([\sigma]) \in \mathbf{Z}^\delta/(1+\delta)\mathbf{Z}$ of the equivalence class $[\sigma]$, using the map

$$H^1([\sigma_{\mathrm{qs}}(\delta)], \mathbf{G}_{\mathrm{ad}}) \to H^2(\sigma_{\mathrm{qs}}(\delta), \mathbf{Z}) \cong \widehat{H}^0(\sigma, \mathbf{Z}) \cong \mathbf{Z}^\sigma/(1+\sigma)\mathbf{Z} \cong \mathbf{Z}^\delta/(1+\delta)\mathbf{Z}.$$

The first map is from the connecting homomorphism of group cohomology for the exact sequence

$$1 \to \mathbf{Z} \to \mathbf{G} \to \mathbf{G}_{\mathrm{ad}} \to 1.$$

The second and the third arrow are from properties of Tate cohomology, and the last one is from [**3**, Lemma 8.6]. If two real forms σ_1, σ_2 of $\mathbf{G}$ live in the same inner class and have the same central invariant, then $H^1(\sigma_1, \mathbf{G}) \cong H^1(\sigma_2, \mathbf{G})$ (cf. [**3**, Lemma 8.10]).

Definition 2.9. — [**3**, Definition 8.11] Let $\mathbf{Z}_{\mathrm{tor}}$ be the subgroup of torsion elements of $\mathbf{Z}$. Fix $\delta \in \mathrm{Out}(\mathbf{G})[2]$ and a quasi-split real form σ_{qs} in the inner class defined by δ.

1. A strong real form in the inner class of σ_{qs} is an element of

$$\mathrm{SRF}_{\sigma_{\mathrm{qs}}}(\mathbf{G}) := Z^1(\sigma_{\mathrm{qs}}, \mathbf{G}; \mathbf{Z}_{\mathrm{tor}})/(1+\sigma_{\mathrm{qs}})\mathbf{Z},$$

   where $Z^1(\sigma_{\mathrm{qs}}, \mathbf{G}; \mathbf{Z}_{\mathrm{tor}}) := \{g \in \mathbf{G} : g\sigma_{\mathrm{qs}}(g) \in \mathbf{Z}_{\mathrm{tor}}\}$. Moreover, to a strong real form $g \in \mathrm{SRF}_{\sigma_{\mathrm{qs}}}(\mathbf{G})$, we can associate a central invariant

$$\mathrm{Inv}(g) = g\sigma_{\mathrm{qs}}(g) \in \mathbf{Z}^\delta_{\mathrm{tor}}.$$

2. Two strong real forms g, h are said to be equivalent if they map to the same element of $H^1(\sigma_{\mathrm{qs}}, \mathbf{G}; \mathbf{Z}_{\mathrm{tor}}) := Z^1(\sigma_{\mathrm{qs}}, \mathbf{G}; \mathbf{Z}_{\mathrm{tor}})/[g \sim tg\sigma_{\mathrm{qs}}(t^{-1}), t \in \mathbf{G}]$.

Note that two equivalent strong real forms g, h have the same central invariant. Since the choice of quasi-split real form in an inner class depends on the choice of the pinning of $\mathbf{G}$, we define $\mathrm{SRF}_\delta = \lim_{\mathbf{Spl}_\mathbf{G}} \mathrm{SRF}_{\sigma(\delta,\mathbf{Spl}_\mathbf{G})}(\mathbf{G})$, where the (projective or injective) limit is taken over all pinning of $\mathbf{G}$ and similarly define the set $[\mathrm{SRF}_\delta]$ of equivalence classes of strong real forms in the inner class defined by δ together with a central invariant map $\mathrm{Inv} : [\mathrm{SRF}_\delta] \to Z^\delta_{\mathrm{tor}}$.

As the pinning varies, the map $g \in \mathrm{SRF}_{\sigma_{\mathrm{qs}}(\delta,\mathbf{Spl}_\mathbf{G})} \mapsto \mathrm{int}(g) \circ \sigma_{\mathrm{qs}}(\delta, \mathbf{Spl}_\mathbf{G})$ are compatible and induce a surjection from SRF_δ to the set of real forms in the inner class defined by δ. This induces a surjection from $[\mathrm{SRF}_\delta]$ to $H^1([\sigma_{\mathrm{qs}}(\delta)], \mathbf{G}_{\mathrm{ad}})$. Moreover, we have the following commutative diagram:

$$\begin{tikzcd} {[\mathrm{SRF}_\delta]} \arrow[r] \arrow[d, "\mathrm{Inv}"] & H^1([\sigma_{\mathrm{qs}}(\delta)], \mathbf{G}_{\mathrm{ad}}) \arrow[d, "\mathrm{Inv}"] \\ \mathbf{Z}^\delta_{\mathrm{tor}} \arrow[r] & \mathbf{Z}^\delta/(1+\delta)\mathbf{Z} \end{tikzcd} .$$

By [**3**, Proposition 8.14, 8.16], for equal rank real form of $\mathbf{G}$, we have the following classification of the equivalent classes of strong real forms with fixed central invariant.

Proposition 2.10. — *Suppose that σ is a real form of $\mathbf{G}$ in the inner class defined by δ. Choose a representative $z \in \mathbf{Z}^\delta_{\mathrm{tor}}$ of $\mathrm{inv}([\sigma]) \in \mathbf{Z}^\delta/(1+\delta)\mathbf{Z}$. Then there is a bijection:*

$H^1(\sigma, \mathbf{G}) \leftrightarrow$ {*equivalent classes of strong real forms with central invariant* z}.

furthermore, suppose that σ is an equal rank real form of $\mathbf{G}$ (i.e. $\delta = 1 \in \mathrm{Out}(\mathbf{G})$). Choose $x \in \mathbf{G}$ such that $\mathrm{int}(x)$ is a Cartan involution for σ, and $z = x^2 \in \mathbf{Z}$. Then we have an explicit bijection

$$H^1(\sigma, \mathbf{G}) \leftrightarrow S(z),$$

where $S(z)$ is the set of conjugacy classes of $\mathbf{G}$ with square equal to z. If $\mathbf{H}$ is a Cartan subgroup of $\mathbf{G}$ with Weyl group W, then $S(z)$ is equal to $\{h \in \mathbf{H} : h^2 = z\}/W$.

Example 2.11. — The set $\{h \in \mathbf{H} : h^2 \in \mathbf{Z}\}/W$ can be identified with the set $(\frac{1}{2}P^\vee/X^\vee(\mathbf{H}))/W$, where $P^\vee$ is the coweight lattice for $\mathbf{G}$:

$P^\vee = \{\lambda^\vee \in X^\vee(\mathbf{H}) \otimes_{\mathbb{Z}} \mathbb{C} | \langle \alpha, \lambda^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\} \cong \{\lambda^\vee \in \mathfrak{h} | \exp(2\pi i \lambda^\vee) \in \mathbf{Z}\}$

For $\mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{C})$, fix the isomorphism $X^\vee(\mathbf{H}) = \Delta^\vee \cong \mathbb{Z}^n$, and the strong real forms are parametrized by

$$(\frac{1}{2}[\mathbb{Z}^n \cup (\mathbb{Z} + \frac{1}{2})^n]/\mathbb{Z}^n)/W.$$

For representatives we choose $\frac{1}{4}(1, ..., 1)$ and $\frac{1}{2}(\overbrace{1, ..., 1}^{p}, \overbrace{0, ..., 0}^{q})$, with $0 \le p \le n$. The central invariant z of $\frac{1}{4}(1, ..., 1)$ is $-I$, which corresponds to the split real symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ and the cohomology set $\left|H^1(\sigma_{\mathrm{Sp}_{2n}(\mathbb{R})}, \mathrm{Sp}(2n)\right| = 1$; the central invariant z of $\frac{1}{2}(\overbrace{1, ..., 1}^{p}, \overbrace{0, ..., 0}^{q})$, with $0 \le p \le n$, is I, which corresponds to the quaternionic symplectic group $\mathrm{Sp}(p, q)$, with $0 \le p \le n$, $p + q = n$, and the cohomology set $\left|H^1(\sigma_{\mathrm{Sp}(p,q)}, \mathrm{Sp}(2n))\right| = p + q + 1$. Meanwhile, we can calculate that $\left|H^1(\sigma_{\mathrm{Sp}_{2n}(\mathbb{R})}, \mathrm{Sp}(2n)_{\mathrm{ad}})\right| = \left|H^1(\sigma_{\mathrm{Sp}(p,q)}, \mathrm{Sp}(2n)_{\mathrm{ad}})\right| = \lfloor \frac{n}{2} \rfloor + 2$, since all the (equivalent) equal rank real form are listed above and are inner to each other, the strong real form gives us a more sophisticated parametrization.

Remark 2.12. — Here the strong real forms are defined in terms of the Galois action, as opposed to the Cartan involution as in some other papers. By [**3**, Theorem 3.13], these two theories are equivalent: the Galois version takes the quasi-split form as the "basepoint", and the Cartan involution version takes the so-called "quasi-compact" form as the "basepoint". Note that the set of pure inner forms which is parametrized by $H^1(\sigma, \mathbf{G})$ includes the quasisplit form, but may not includes the quasicompact one.

2.5.2. *Representation of strong involutions.* —

Definition 2.13. — [**1**, Definition 2.1] A strong involution of $\mathbf{G}$ for the equal rank inner class is an elliptic element(6) $\tilde{x} \in \mathbf{G}$ such that $\tilde{x}^2 \in Z$.

Let $\tilde{S}$ be the set of strong involutions of $\mathbf{G}$ and we have a stratification $\tilde{S} = \cup_{z \in Z} \tilde{S}(z)$, where $\tilde{S}(z) = \{\tilde{x} \in \mathbf{G}_* : \tilde{x}^2 = z\}$, called the set of strong involutions of type z. By proposition 2.10, we identify $S(z) = \tilde{S}(z)/W$ with the set of conjugacy classes of strong real forms with central invariant z. Thus the natural surjection $\tilde{S}(z) \to S(z)$ induces a surjection from $\tilde{S}(z)$ to the set of conjugacy classes of strong real forms with central invariant z.

For a strong involution $\tilde{x} \in \tilde{S}$ of $\mathbf{G}$, we set $\theta_{\tilde{x}} = \mathrm{int}(\tilde{x})$ and $\mathbf{K}_{\tilde{x}} = \mathbf{G}^{\theta_{\tilde{x}}} = \mathrm{Cent}_{\mathbf{G}}(\tilde{x})$.

Definition 2.14. — A representation of a strong involution $\tilde{x} \in \tilde{S}$ is a pair $(\tilde{x}, \pi)$ where π is a $(\mathfrak{g}, \mathbf{K}_{\tilde{x}})$-module.

Two representations $(\tilde{x}, \pi)$ and $(\tilde{x}', \pi')$ of strong involutions are equivalent if there exists $g \in \mathbf{G}$ such that $g\tilde{x}g^{-1} = \tilde{x}'$ and $\pi' \cong \pi^g$. For a strong involution $\tilde{x} \in \tilde{S}$ and a $(\mathfrak{g}, \mathbf{K}_{\tilde{x}})$-module π, we denote by $[\tilde{x}, \pi]$ the equivalent class of representation $(\tilde{x}, \pi)$. Let R be the set of representations $(\tilde{x}, \pi)$ of strong involutions and let E be the set of equivalent classes of representations of strong involutions. The group $\mathbf{G}$ acts on the sets R and E by conjugation.

Since the subgroup $\mathbf{T}_*$ acts trivially on R, it induces an action of Weyl group $W = \mathrm{Norm}_{\mathbf{G}}(\mathbf{T}_*)/\mathbf{T}_*$ on R. On the other hand, the Weyl group W acts trivially on E and this allows us to give the following notion of representation of strong real forms.

(6) An element of G is called elliptic if the closure in the analytic topology of the cyclic subgroup generated by g is compact.

Definition 2.15. — A representation of a strong real form corresponding to $x \in S(z)$ is the equivalent class $[\tilde{x}, \pi]$ defined by a representation $(\tilde{x}, \pi)$ of a strong involution $\tilde{x} \in \tilde{S}(z)$ lifting x.

For two representations $[\tilde{x}, \pi]$ and $[\tilde{x}, \pi^{'}]$ of a strong real form x, $[\tilde{x}, \pi] = [\tilde{x}, \pi^{'}]$ if and only if there exists $g \in \mathbf{G}$ such that $g\tilde{x}g^{-1} = \tilde{x}$ and $\pi^g \cong \pi^{'}$. This holds if and only if $g \in \mathbf{K}_{\tilde{x}}$. Since $S(z) = \tilde{S}(z)/W$, the set of representations of a strong real form with central invariant z is stable under the action of W. For any strong real form $x \in S(z)$ with lifting $\tilde{x} \in \tilde{S}(z)$, a representation $[\tilde{x}, \pi_{\tilde{x}}(\lambda)]$ of x with Harish-Chandra parameter λ and $w \in W$, we have

$$[\tilde{x}, \pi_{\tilde{x}}(\lambda)] = w[\tilde{x}, \pi_{\tilde{x}}(\lambda)] = [w\tilde{x}, \pi_{w\tilde{x}}(w\lambda)].$$

2.5.3. *L-packet associated to discrete Langlands parameter and strong involution.* —

Definition 2.16. — If a Langlands parameter φ of $\mathbf{G}$ has finite centralizer, then φ is called a *discrete series parameter.*

Let φ be a discrete series L-parameter of $\mathbf{G}$ with infinitesimal character λ. Let $\tilde{x} \in \tilde{S}(z)$ be a strong involution with $z \in Z$. We denote by $\Pi(\tilde{x}, \varphi)$ the L-packet associate to φ and the strong involution $\tilde{x}$, i.e. the set of discrete series $(\mathfrak{g}, \mathbf{K}_{\tilde{x}})$-modules with infinitesimal character λ. In other words, $\Pi(\tilde{x}, \varphi)$ is exactly the L-packet of φ and the real form of $\mathbf{G}$ determined by $\tilde{x}$.

For any Harish-Chandra parameter μ lifting λ, there exists a $w \in W$ such that $\mu = w^{-1} \cdot \lambda$ and a unique $(\mathfrak{g}, \mathbf{K}_{\tilde{x}})$-module associated to the Harish-Chandra-Langlands parameter $(\mu, w^{-1} \cdot \Psi_*)$. As a result, the L-packet $\Pi(\tilde{x}, \varphi)$ can be parametrized as following

$$\Pi(\tilde{x}, \varphi) = \{\pi_{\tilde{x}}(\omega^{-1}\lambda) : \omega \in W/W(\mathbf{T}_*, \mathbf{K}_{\tilde{x}})\},$$

where $W(\mathbf{T}_*, \mathbf{K}_{\tilde{x}})$ is the Weyl group of $\mathbf{T}_*$ in $\mathbf{K}_{\tilde{x}}$ and $\pi_{\tilde{x}}(\omega^{-1}\lambda)$ is a discrete series $(\mathfrak{g}, \mathbf{K}_{\tilde{x}})$-modules with Harish-Chandra parameter $w^{-1} \cdot \lambda$.

Define $\Pi(\varphi)$ be set of all the representations of strong real forms with infinitesimal character λ. The L-packet $\Pi(\tilde{x}, \varphi)$ can be embedded into $\Pi(\varphi)$ as

$$\Pi(\tilde{x}, \varphi) = \{[\tilde{x}, \pi_{\tilde{x}}(w^{-1}\lambda)] : w \in W/W(\mathbf{T}_*, \mathbf{K}_{\tilde{x}})\},$$

which induces a W-equivariant bijection $\Pi(\tilde{x}, \varphi) \leftrightarrow \{wx | w \in W/W(\mathbf{T}_*, \mathbf{K}_{\tilde{x}})\}$ using the equality (2.5.2). To make the bijection more explicit, choose a set S' of representatives of $\tilde{S}/W$ which naturally parametrizes the strong real forms. Thus, we have a W-equivariant bijection

$$\Pi(\varphi) = \coprod_{\tilde{x} \in S'} \Pi(x, \varphi) \leftrightarrow \coprod_{\tilde{x} \in S'} \{wx | w \in W/W(\mathbf{T}_*, \mathbf{K}_{\tilde{x}})\} \cong \tilde{S}$$

Thus we obtain a W-equivariant bijection.

$$\tilde{S} \leftrightarrow \Pi(\varphi), \quad \tilde{x} \mapsto [\tilde{x}, \pi_{\tilde{x}}(\lambda)].$$

We regroup the L-packets of φ for inner forms using the central invariant: for $z \in Z$, we set

$$\Pi_z(\varphi) := \cup_{\tilde{x}\in\tilde{S}(z)}\Pi(\tilde{x}, \varphi).$$

Proposition 2.17. — [1, Prop. 5.3] *For any $z \in Z$ and any discrete series Langlands parameter φ of $\mathbf{G}$, we have a W-equivariant bijection between $\tilde{S}(z)$ and $\Pi_z(\varphi)$, given by $\tilde{x} \mapsto [\tilde{x}, \pi_{\tilde{x}}(\lambda)]$.*

By [53, Theorem 6.2(f)], a discrete series representation $\pi(\lambda)$ is generic if and only if every simple root in the chamber defined by λ is non-compact. By definition, a root α is compact with respect to the Cartan involution θ_x with $x \in \tilde{S}$, if $\alpha(x) = 1$, and is non-compact if $\alpha(x) = -1$. Thus,

Lemma 2.18. — *$x \in \tilde{S}$ corresponds to a generic discrete series representation through the bijection $\tilde{S} \leftrightarrow \Pi(\varphi)$ if and only if $\alpha(x) = -1$ for all simple roots α.*

The element $\rho \in X \otimes \mathbb{R}$ produces a basepoint of the strong involutions: let

$$x_b = \exp(i\pi\rho^\vee) \in \tilde{S} = \cup_{z\in Z}\tilde{S}(z) \subset \mathbf{T}_*,$$

which is independent of choice of the set of positive roots. Note for any simple root $\alpha \in \Delta$, we have $\langle \alpha, \rho^\vee \rangle = 1$. We can deduce that $\alpha(x_b) = \exp(i\pi\langle \alpha, \rho^\vee \rangle) = -1$, for any simple root $\alpha \in \Delta$.

By prop. 2.17, for $z(\rho^\vee) = x_b^2$ and a discrete Langlands parameter φ of $\mathbf{G}$, we have a W-equivariant bijection

$$\Pi_{z(\rho^\vee)}(\varphi) \cong \tilde{S}(z(\rho^\vee)).$$

Example 2.19. — Let G be a real classical groups of equal rank with Borel pair $(\mathbf{B}, \mathbf{T})$. Fix a based root datum $D_b = (X, \Phi, \Delta, X^\vee, \Phi, \Delta^\vee)$ attached to $(\mathbf{B}, \mathbf{T})$. We compute the basepoints for G following the above discussion.

1. Suppose $G^d = \mathrm{Sp}_{2n}(\mathbb{C})$. Then, as in example 2.2.1,

   $$\Psi^\vee = \{e_i \pm e_j | 1 \le i < j \le n\} \cup \{2e_i | 1 \le i \le n\}$$

   is a positive root system. The half sum of these positive roots is

   $$\rho^\vee = \frac{1}{2}((2n)e_1 + 2(n-1)e_2 + \cdots + 2e_n) \in X^\vee = \mathrm{Hom}(\mathbb{C}^*, \mathbf{T}_*).$$

   Thus, the basepoint for the complexe group $\mathbf{G} = G \otimes \mathbb{C}$ is

   $$x_b = (\cos(n\pi), \cdots, \cos(\pi)) = ((-1)^n, \cdots, 1, -1) \in \mathbf{U}(1)^n$$

   and $z(\rho^\vee) = x_b^2 = I$.

2. Suppose $G^d = \mathrm{O}(2n+1, \mathbb{C})$. Then, as in example 2.2.2,
$$\Psi^\vee = \{e_i \pm e_j | 1 \le i < j \le n\} \cup \{e_i | 1 \le i \le n\}$$
is a positive root system. The half sum of these positive roots is
$$\rho^\vee = \frac{1}{2}((2n-1)e_1 + (2n-3)e_2 + \cdots + 3e_{n-1} + e_n) = (\frac{2n-1}{2}, \cdots, \frac{3}{2}, \frac{1}{2}).$$
Thus, the basepoint for $\mathbf{G}$ is
$$x_{b,\mathrm{Sp}} = (i\sin(\frac{2n-1}{2}\pi), \cdots, i\sin(\frac{1}{2}\pi)) = i((-1)^{n-1}, \cdots, -1, 1) \in \mathbf{U}(1)^n$$
and $z(\rho^\vee) = x_b^2 = -I$.
3. Suppose $G^d = \mathrm{O}(2n, \mathbb{C})$. Then
$$\Psi^\vee = \{e_i \pm e_j | 1 \le i < j \le n\}$$
is a positive root system. The half sum of these positive roots is
$$\rho^\vee = \frac{1}{2}(2(n-1)e_1 + 2(n-2)e_2 + \cdots + 2e_{n_1}) = (n-1, \cdots, 1, 0).$$
Thus, the basepoint is
$$x_{b,\mathrm{O}} = (\cos((n-1)\pi), \cdots, \cos(0)) = ((-1)^{n-1}, \cdots, -1, 1) \in \mathbf{U}(1)^n$$
and $z(\rho^\vee) = x_b^2 = I$.

Let $\lambda_0 = (\lambda_1, \cdots, \lambda_n)$ be an infinitesimal character of G with $\lambda_1 > \cdots > \lambda_n > 0$. There are two generic discrete series representation associated to λ_0. If we pose the condition λ_1 is positive on the Harish-Chandra lifting of the infinitesimal character, then we have a unique generic discrete series representation associated to λ_0 and the basepoint $x_{b,G}$.

Example 2.20. — Let $G = \mathrm{Sp}_{2n}(\mathbb{R})$. Let λ_0 be an infinitesmal character of G as above.

1. If n is even, then we set $t_{b,\mathrm{Sp}} = ix_b$ and we take the Harish-Chandra lifting
$$\lambda_{d,\mathrm{Sp}} = ix_b \cdot \lambda_0 = (\lambda_1, -\lambda_2, \cdots, -\lambda_n)$$
of λ_0. By the dominant condition, the corresponding positive root system $\Psi_{b,\mathrm{Sp}}$ is generated by the simple roots $\{e_1 + e_2, -e_2 - e_3, \cdots, e_{n-1} + e_n, -2e_n\}$.
2. If n is odd, then we take the Harish-Chandra lifting
$$\lambda_{d,\mathrm{Sp}} = -ix_b \cdot \lambda_0 = (\lambda_1, -\lambda_2, \cdots, \lambda_n)$$
of λ_0. By the dominant condition, the corresponding positive root system $\Psi_{b,\mathrm{Sp}}$ is generated by the simple roots $\{e_1 + e_2, -e_2 - e_3, \cdots, -e_{n-1} - e_n, 2e_n\}$.

In both cases, the root system $\Psi_{b,\mathrm{Sp}}$ is exactly that one we fixed in example 2.2.1 and the unique generic representation in the L-packet is determined by the Harish-Chandra-Langlands parameter $(\lambda_{d,\mathrm{Sp}}, \Psi_{b,\mathrm{Sp}})$.

Example 2.21. — Let G be the even orthogonal group of rank n associated to the basepoint $x_{b,\mathrm{O}}$. Thus G has signature (n,n) (resp. $(n+1,n-1)$) if n is even (resp. odd). Let λ_0 be an infinitesimal character of G as above.

1. If n is even, then we take the Harish-Chandra lifting $-x_{b,\mathrm{O}}\lambda_0$ of λ_0 and the corresponding Harish-Chandra parameter is
$$\lambda_{d,\mathrm{O}} = (\lambda_1, \lambda_3, \cdots, \lambda_{n-1}; \lambda_2, \cdots, \lambda_n).$$
By the dominant condition, the corresponding positive root system $\Psi_{b,\mathrm{O}}$ is generated by the simple roots $\{e_1 - f_1, f_1 - e_2, \cdots, e_{\frac{n}{2}} - f_{\frac{n}{2}}, e_{\frac{n}{2}} + f_{\frac{n}{2}}\}$.
2. If n is odd, then we take the Harish-Chandra lifting $x_{b,\mathrm{O}}\lambda_0$ and the corresponding Harish-Chandra lifting is
$$(\lambda_1, \lambda_3, \cdots, \lambda_n; \lambda_2, \cdots, \lambda_{n-1}).$$
By the dominant condition, the corresponding positive root system $\Psi_{b,\mathrm{O}}$ is generated by the simple roots $\{e_1 - f_1, f_1 - e_2, \cdots, f_{\frac{n-1}{2}} - e_{\frac{n+1}{2}}, e_{\frac{n+1}{2}} + f_{\frac{n-1}{2}}\}$.

In both cases, the root system $\Psi_{b,\mathrm{O}}$ is exactly that one we fixed in example 2.2.2 and the generic discrete series representation is determined by the Harish-Chandra-Langlands parameter $(\lambda_{d,\mathrm{O}}, \Psi_{b,\mathrm{O}})$.

2.6. Langlands-Vogan parameters for real classical groups of equal rank. — Let $G = (\mathbf{G}, \sigma_G)$ be a quasi split real classical groups of equal rank. Let G^d be the dual group of $\mathbf{G}$ and φ be a limit of discrete L-parameter of G. We fix a fundamental Borel pair of Whittaker type $(\mathbf{B}_*, \mathbf{T}_*)$ of $\mathbf{G}$. Let $D_b = (X, \Phi, \Delta, X^\vee, \Phi^\vee, \Delta^\vee)$ be the based root datum associated to $(\mathbf{B}_*, \mathbf{T}_*)$. We denote by Ψ_* the set of positive roots generated by Δ. Set $\rho = \frac{1}{2}\sum_{\alpha\in\Psi_*}\alpha$ and $\rho^\vee = \frac{1}{2}\sum_{\alpha^\vee\in\Psi_*^\vee}\alpha^\vee$. We identifies the Cartan subgroup $\mathbf{T}_*$ with $\mathbf{U}(1)^n$ via Δ and the elements in $\mathbf{T}_*[2]$ will be denoted by $t = (\pm1, \cdots, \pm1)$.

Let φ be the Langlands parameter of a limit of discrete series representation of an equal rank real form of $\mathbf{G}$. The restriction of φ to $\mathbb{C}^\times$ determines an infinitesimal character λ (cf. [4, Proposition 2.10]). More precisely, after conjugating by $G^\vee$ we may assume $\varphi(\mathbb{C}^\times) \in \mathbf{T}_*^\vee$. The discrete property implies that $\varphi(j)h\varphi(j)^{-1} = h^{-1}$, for $h \in \mathbf{T}_*^\vee$, and for $\mathfrak{z} \in \mathbb{C}^\times$, we have $\varphi(\mathfrak{z}) = (\mathfrak{z}/\bar{\mathfrak{z}})^\lambda$, with $\lambda \in \rho + X$ a $\mathbf{B}_*$-dominant regular element. This also implies $A_\varphi = \{h \in \mathbf{T}_*^\vee : h^2 = 1\}$.

In this section, we will give the explicit description of the component groups of these groups (cf. Example 2.22) and the correspondence between their Langlands-Vogan parameters and Harish-Chandra parameters (cf. Example 2.24 for symplectic groups and Example 2.25 for orthogonal groups).

In the above examples, we send the base point x_b to a canonical element

$$t_* = (1, -1, \cdots, (-1)^{n-1}) \in T_*,$$

which correspond to the generic representation of the quasi-split real form G of equal rank with respect to the fundamental Borel pair of Whittaker type $(\mathbf{B}_*, \mathbf{T}_*)$.

Example 2.22. — (1) Let $G = \mathrm{Sp}_{2n}(\mathbb{R})$. Let us write the limit of discrete series L-parameter φ of G as

$$\varphi = \bigoplus_{i=1}^{k} c_i \rho_{\lambda_i} \bigoplus (2z+1)\mathbf{1}, \tag{2.6}$$

where

- $\mathbf{1}$ is the trivial character of $W_\mathbb{R}$, and z is a positive integer;
- $\rho_{\lambda_1}, \cdots, \rho_{\lambda_k}$ are self-dual irreducible representations of $W_\mathbb{R}$ of dimension 2 with λ_i even natural number and $c_i > 0$ an integer.

Note that if moreover φ is a discrete series Langlands parameter, then all the $c_i = 1$ and $z = 0$.

The component group A_φ of φ is

$$A_\varphi = \begin{cases} \oplus_{i=1}^{k} (\mathbb{Z}/2\mathbb{Z}) a_i, & \text{if } z = 0, \\ \oplus_{i=1}^{k} (\mathbb{Z}/2\mathbb{Z}) a_i \oplus (\mathbb{Z}/2\mathbb{Z}) b, & \text{if } z > 0, \end{cases}$$

where a_i is a symbol corresponding to ρ_{λ_i} and b is a symbol corresponding to $\mathbf{1}$.

(2) Let $G = O(p, q)$ with p, q even. Let φ be a limit of discrete series L-parameter with decomposition

$$\varphi = \bigoplus_{i=1}^{k} c_i \rho_{\lambda_i} \bigoplus 2z\mathbf{1}, \tag{2.7}$$

- $\mathbf{1}$ is the trivial character of $W_\mathbb{R}$, and z is a positive integer;
- $\rho_{\lambda_1}, \cdots, \rho_{\lambda_k}$ are self-dual irreducible representations of $W_\mathbb{R}$ of dimension 2 with λ_i even natural number and $c_i \in \mathbb{N}_{>0}$.

Note that if moreover φ is a discrete series Langlands parameter, then we have $c_i = 1$ for all i and $z = 0$.

Definition 2.23. — A Langlands-Vogan parameter of G is a pair (φ, η), where φ is a limit of discrete series L-parameter of G and η is a character of the component group $\widehat{A}_\varphi$. The group G^d acts by conjugation on the set of these paramters. Conjugate parameters are said to be equivalent, and we write $\Pi_{\mathrm{pure}}(G/\mathbb{R})$ for the set of equivalence classes.

If φ is a discrete series parameter of G, we have $A_\varphi = \{h \in \mathbf{T}_*^\vee : h^2 = 1\}$ This gives a canonical perfect pairing

$$\mathbf{T}_*[2] \times A_\varphi \to \mathbb{C}^\times.$$

This perfect pairing induces an identification $\widehat{A_\varphi} = \mathbf{T}_*[2] \cong \tilde{S}(z(\rho^\vee))$. The precise map is given as the following: first we identify $A_\varphi = \mathbf{T}_*^\vee[2]$ with $X(\mathbf{T}_*^\vee)/2X(\mathbf{T}_*^\vee) = X^\vee(\mathbf{T}_*)/2X^\vee(\mathbf{T}_*)$, denote this isomorphism $s : X^\vee(\mathbf{T}_*)/2X^\vee(\mathbf{T}_*) \cong A_\varphi$. Then

$$\tilde{S}(z(\rho^\vee)) \to \widehat{A_\varphi}, \quad x \mapsto [s(\lambda) \mapsto \lambda(x x_b^{-1})],$$

which takes $x = \exp(\pi i \gamma^\vee)$, $\gamma^\vee \in X(\mathbf{T}_*)$, to $[s(\lambda) \mapsto \exp(\pi i \langle \gamma^\vee - \rho^\vee, \lambda \rangle)] \in \widehat{A_\varphi}$, and the identity of $\widehat{A_\varphi}$ to the strong involution $x_b \in \tilde{S}(z(\rho^\vee)) \subset \mathbf{T}_*$. Here $\rho^\vee$ is the half sum of the positive coroots and $z(\rho^\vee) = \exp(2\pi i \rho^\vee))$.

For any $t \in \mathbf{T}_*[2]$, we can associate a unique element $\eta_t = (\eta_1, \cdots, \eta_n)$ the corresponding character of A_φ with $\eta_i \in \{\pm 1\}$ for all $1 \leq i \leq n$, and an element $x_t = t x_b \in \tilde{S}(z(\rho^\vee))$. Thus there is an element $w_t \in W$ such that $w_t \cdot x_b = x_t$. This implies the element t corresponds to an equivalent class of representation $[x_t, \pi_{x_t}(\lambda)]$ of strong involution x_t. Note that, $t = 1$ corresponds to the generic representation $[x_b, \pi_{x_b}(\lambda)]$ with respect to the fundamental Borel pair of Whittaker type $(\mathbf{B}_*, \mathbf{T}_*)$. Thus, for a discrete series Langlands-Vogan parameter (φ, η_t), we get a Harish-Chandra-Langlands parameter (λ_t, Ψ_t), where $\Psi_t = w_t \Psi_*$.

Following [**41**, Remarque 5.4], we can modify the above discussion to relate the limit of discrete series Langlands-Vogan parameter (φ, η) to its Harish-Chandra-Langlands parameter. In fact, since our classical group are of equal rank, we can realize the component group A_φ as a quotient group of the component group of $A_{\varphi^{\mathrm{reg}}}$ with φ^{reg} a discrete series parameter. Thus, η can be viewed as an element of $\widehat{A_{\varphi^{\mathrm{reg}}}}$.

Using the notation and the choice we made in §2.2.1 and 2.2.2, fix Φ be the set of roots and Φ_c (resp. Φ_n) be the set of compact (resp. non-compact) roots. In the following examples, we establish the explicit correspondence of Langlands-Vogan parameters and Harish-Chandra-Langlands parameters for symplectic groups and orthogonal groups.

Example 2.24. — Let (φ, η) be a limit of discrete series Langlands-Vogan parameter of $G = \mathrm{Sp}_{2n}(\mathbb{R})$ with a decomposition of the L-parameter into irreducible representations as in Example 2.22 (1)

$$\varphi = \bigoplus_{i=1}^{k} c_i \rho_{\lambda_i} \bigoplus (2z+1)\mathbf{1},$$

where $z \in \mathbb{N}, \lambda_i \in 2\mathbb{N}$ and $\lambda_1 > \cdots > \lambda_k > 0$. The component group A_φ is a quotient of

$$A_{\varphi^{\mathrm{reg}}} = \oplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z}) a_i \bigoplus \oplus_{j=1}^{z} (\mathbb{Z}/2\mathbb{Z}) b_j.$$

where $r = c_1 + \cdots + c_k$. Then, the character $\eta \in \widehat{A_\varphi}$ can be identified as an element $(\eta_1, \cdots, \eta_n) \in \widehat{A_{\varphi^{\mathrm{reg}}}} \cong \mathbf{T}_*[2]$. Let λ be the infinitesimal character determined by φ and let

$$\lambda_0 = (\underbrace{\lambda_1, \cdots, \lambda_1}_{p_1}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{p_k}, \underbrace{0, \cdots, 0}_{z}, \underbrace{-\lambda_k, \cdots, -\lambda_k}_{q_k}, \cdots, \underbrace{-\lambda_1, \cdots, -\lambda_1}_{q_1})$$

be the Harish-Chandra lifting of λ associated to the basepoint x_b.

Thus the Harish-Chandra lifting λ_η of λ associated to ηx_b is the reordering $(\lambda_{\eta,i})_{1 \le i \le n}$ of $\eta \cdot \lambda_0$ which is of the form

$$\lambda_\eta = (\underbrace{\lambda_1, \cdots, \lambda_1}_{p_{\eta,1}}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{p_{\eta,k}}, \underbrace{0, \cdots, 0}_{z}, \underbrace{-\lambda_k, \cdots, -\lambda_k}_{q_{\eta,k}}, \cdots, \underbrace{-\lambda_1, \cdots, -\lambda_1}_{q_{\eta,1}}).$$

It is left to determine the root system Ψ_η associated to the Langlands-Vogan parameter (φ, η).

1. If φ is a discrete series parameter (equivalently, we have $z = 0$ and $p_{\eta,i} + q_{\eta,i} = 1$ with $p_{\eta,i} \neq q_{\eta,i}$ for all i), then λ_η uniquely determines the root system Ψ_η. More precisely, we have
   (a) For $1 \le i \neq j \le n$, $e_i - e_j \in \Psi_\eta$ if $|\lambda_i| > |\lambda_j|$,
   (b) $\{e_i + e_j | 1 \le i < j \le \sum_{l=1}^{k} p_{\eta,l}\} \subset \Psi_\eta$ and $\{-e_i - e_j | \sum_{l=1}^{k} p_{\eta,l} + 1 \le i < j \le n\} \subset \Psi_\eta$,
   (c) $2e_i \in \Psi_\eta$ (resp. $-2e_i \in \Psi_\eta$) if $1 \le i \le \sum_{l=1}^{k} p_{\eta,l}$ (resp. $\sum_{l=1}^{k} p_{\eta,l} + 1 \le i \le n$),
   (d) For i, j with $p_{\eta,i} = q_{\eta,j} = 1$, we have Ψ_η contains $e_i + e_j$ (resp. $-e_i - e_j$) if $|\lambda_i| > |\lambda_j|$ (resp. $|\lambda_i| < |\lambda_j|$).
2. If φ is a limit of discrete series parameter (equivalently, we have $z > 0$ or $p_{\eta,i} = q_{\eta,i}$ for some i) and we denote $p_\eta = p_{\eta,1} + \cdots + p_{\eta,k}$, $q_\eta = q_{\eta,1} + \cdots + q_{\eta,k}$, then Ψ_η can be generated by the roots

(a) $\{e_i - e_j | 1 \leq i < j \leq p_\eta \text{ or } 1 \leq i \leq p_\eta, p_\eta + z + 1 \leq j \leq n\} \subset \Psi_\eta$, $\{-e_i + e_j | p_\eta + z + 1 \leq i \leq n, 1 \leq j \leq p_\eta \text{ or } p_\eta + z + 1 \leq i < j \leq n\} \subset \Psi_\eta$

(b) $\{2e_i | 1 \leq i \leq p_\eta\} \subset \Psi_\eta$ and $\{-2e_j | n - q_\eta + 1 \leq j \leq n\} \subset \Psi_\eta$,

(c) $\{e_i + e_j | 1 \leq i, j \leq p_\eta + z\} \subset \Psi_\eta$ and $\{-e_i - e_j | n - q_\eta + 1 \leq i, j \leq n\} \subset \Psi_\eta$.

(d) If $z > 0$, we have $\eta_{r+1} = \cdots = \eta_n$. Then Ψ_η contains

$$\{2e_i | p_\eta + 1 \leq i \leq p_\eta + z\} \quad (\text{resp.} \{-2e_i | p_\eta + 1 \leq i \leq p_\eta + z\})$$

in the case $\eta_{r+1} = 1$ (resp.$\eta_{r+1} = -1$).

(e) If $p_{\eta,i} = q_{\eta,i}$, then there exists s, s' such that $\eta_s = \cdots = \eta_{s'}$. For $\eta_s = \pm 1$, then Ψ_η contains $\pm(e_{p'_i+1} + e_{n-q'_i})$, where $p'_i = \sum_{j=1}^{i-1} p_{\eta,j}$ and $q'_i = \sum_{j=1}^{i-1} q_{\eta,j}$.

Example 2.25. — Let (φ, η) be a limit of discrete series Langlands-Vogan parameter of $G = \mathrm{O}(2n, \mathbb{R})$ with the decomposition of L-parameter as in example 2.22 (2)

$$\varphi = \bigoplus_{i=1}^{k} c_i \rho_{\lambda_i} \bigoplus 2z\mathbf{1}.$$

Let $t_{b,\mathrm{O}} = (1, (-1), \cdots, (-1)^{n-1}) \in \mathbf{T}_*[2]$.

The component group A_φ is a quotient of

$$A_{\varphi^{\mathrm{reg}}} = \oplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z}) a_i \bigoplus \oplus_{j=1}^{z} (\mathbb{Z}/2\mathbb{Z}) b_j,$$

where $r = c_1 + \cdots + c_k$. Then, $\eta \in \widehat{A_\varphi}$ can be identified with an element $(\eta_1, \cdots, \eta_{r+z}) \in \widehat{A_{\varphi^{\mathrm{reg}}}}$ with

$$\eta_1 = \cdots = \eta_{c_1}, \cdots, \eta_{\sum_{i=1}^{k-1} c_i + 1} = \cdots = \eta_r, \eta_{r+1} = \cdots = \eta_{r+z}.$$

Let p_0 be the number of $+1$ in $\eta t_{b,\mathrm{O}}$ and q_0 be the number of -1 in $\eta t_{b,\mathrm{O}}$. Then the corresponding real form of equal rank n is $\mathrm{O}(p, q)$ with $p = 2p_0$ and $q = 2q_0$. Let λ be the infinitesimal character determined by φ and let $\lambda_0 = (\lambda_1, \cdots, \lambda_{p_0}; \lambda_{p_0+1}, \cdots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_{p_0}$ and $\lambda_{p_0+1} \geq \cdots \geq \lambda_{p_0+q_0}$, which is the Harish-Chandra parameter λ_η of λ associated to the Harish-Chandra lifting $\eta t_{b,\mathrm{O}}$. We may write the Harish-Chandra parameter λ_η as

$$\lambda_\eta = (\underbrace{\lambda_1, \cdots, \lambda_1}_{p_{\eta,1}}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{p_{\eta,k}}, \underbrace{0, \cdots, 0}_{z_\eta}, \underbrace{\lambda_1, \cdots, \lambda_k}_{q_{\eta,1}}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{q_{\eta,k}}, \underbrace{0, \cdots, 0}_{z'_\eta}), \tag{2.8}$$

where $p_{\eta,1} + \cdots + p_{\eta,k} + z_\eta = p_0$, $q_{\eta,1} + \cdots + q_{\eta,k} + z'_t = q_0$.

1. If φ is a discrete parameter (equivalently, we have $z + z' \leq 1$ or $p_{\eta,i} \neq q_{\eta,i}$ for all i), then λ_η determines completely the root system Ψ_η as follows:
   (a) $\{e_i \pm e_j | 1 \leq i < j \leq p_{\eta,0}\} \in \Psi_\eta$,
   (b) $\{f_i \pm f_j | 1 \leq i < j \leq q_{\eta,0}\} \in \Psi_\eta$,
   (c) If $p_{\eta,i} = q_{\eta,j} = 1$ with $\lambda_i > \lambda_j$, then $e_i \pm f_j \in \Psi_\eta$,
   (d) If $p_{\eta,i} = q_{\eta,j} = 1$ with $\lambda_i < \lambda_j$, then $f_i \pm e_j \in \Psi_\eta$.
2. If φ is a limit of discrete parameter (equivalently, we have $z = z' > 1$ and $p_{\eta,i} = q_{\eta,i}$ for some i), then Ψ_η contains
   (a) $\{e_i \pm e_j | 1 \leq i < j \leq p_{\eta,0} - z\} \in \Psi_\eta$,
   (b) $\{f_i \pm f_j | 1 \leq i < j \leq q_{\eta,0} - z'\} \in \Psi_\eta$,
   (c) If $\lambda_i > \lambda'_j$ for $1 \leq i \leq p_{\eta,0} - z$, $1 \leq j \leq q_{\eta,0} - z'$, then $e_i \pm f_j \in \Psi_\eta$,
   (d) If $\lambda_i < \lambda'_j$ for $1 \leq i \leq p_{\eta,0} - z$, $1 \leq j \leq q_{\eta,0} - z'$, then $f_i \pm e_j \in \Psi_\eta$,
   (e) If $z = z' > 0$, which implies $w > 1$. Then Ψ_η contains the roots:
$$\begin{cases} \{e_{p_{\eta,0}-z+1} \pm f_{q_{\eta,0}-z'+1}\}, & \text{if } \eta_{r+1} = 1, \\ \{-e_{p_{\eta,0}-z+1} + f_{q_{\eta,0}-z'+1}\}, & \text{if } \eta_{r+1} = -1. \end{cases}$$
   (f) If $p_{\eta,i} = q_{\eta,j}$, which implies there exists some s, s' such that $\eta_s = \cdots = \eta_{s'}$. Then we set $p'_{\eta,i} = \sum_{j=1}^{i-1} p_{\eta,j}$ and $q'_{\eta,i} = \sum_{j=1}^{i-1} q_{\eta,j}$, Ψ_η contains the roots:
$$\begin{cases} \{e_{p'_{\eta,i}+1} \pm f_{q'_{\eta,i}+1}\}, & \text{if } \eta_s = 1, \\ \{f_{q'_{\eta,i}+1} \pm e_{p'_{\eta,i}+1}\}, & \text{if } \eta_s = -1. \end{cases}$$
   (g) If $z + z' > 0$, which implies $w > 0$. Then Ψ_η contains
$$\begin{cases} \{e_{p_{\eta,0}} \pm f_{q_{\eta,0}}\}, & \text{if } \eta_{r+1} = 1, \\ \{-e_{p_{\eta,0}} + f_{q_{\eta,0}}\}, & \text{if } \eta_{r+1} = -1. \end{cases}$$

3. Theta lifts

Tempered representations are irreducible parabolic inductions from limit of discrete series representations. By the Langlands-Vogan parameterization of parabolic inductions (cf. [20]) and the induction principle of theta lifts (cf. [40] and [2]), the crux of the matter is the case when φ is an Langlands parameter of limit of discrete series representations. Assume from now on that this is the case.

3.1. Parameters of theta lift for symplectic-orthogonal dual pairs. — In this paragraph, we describe explicit theta correspondence of equal rank groups via Langlands-Vogan parameter. Such a description has been done by Paul ([45] for case (A)) and Adams-Barbasch ([2] for case (B)) respectively in terms of Harish-Chandra-Langlands parameters. Thus, we only need to translate their results into Langlands-Vogan parametrization. Note that, in [20, §11], assuming the Langlands-Vogan parametrization of odd orthogonal group over $\mathbb{R}$, the authors define the normalized Langlands-Vogan parametrization of genuine representation of metaplectic group using the Langlands-Vogan parameters for odd orthogonal group via theta correspondence. As a result, there is nothing to translate in this case and we only need to treat the case (A).

3.1.1. *Result of Paul* [45]. — We now recall the result of Paul and in the statement. Here we will replace the notation $\theta_{V,V'}$ by $\theta_{p,q}$, where (p,q) is the signature of V'.

Theorem 3.1. — [45, Theorem 15] *Let π be a limit of discrete series representation of $\mathrm{Sp}_{2n}(\mathbb{R})$ and (λ,Ψ) be the Harish-Chandra parameter of π, where*

$$\lambda=(\underbrace{\lambda_1,\cdots,\lambda_1}_{p_1},\cdots,\underbrace{\lambda_k,\cdots,\lambda_k}_{p_k},\underbrace{0,\cdots,0}_{z},\underbrace{-\lambda_k,\cdots,-\lambda_k}_{q_k},\cdots,\underbrace{-\lambda_1,\cdots,-\lambda_1}_{q_1}).$$

Let $w=[\frac{z}{2}]$, $p_0=\sum_{i=1}^k p_i+w$ and $q_0=\sum_{i=1}^k q_i+w$. There are exactly four pairs of integers (p,q) with $p+q=2n$ or $2n+2$ such that $\theta_{p,q}(\pi)$ is a non-zero limit of discrete series representation of $\mathrm{O}(p,q)$ [(7)].

1. *$z=2w$: $\theta_{2p_0,2q_0}(\pi)\neq 0$ with the Harish-Chandra parameter $(\lambda_{0,0},1,\Psi_{0,0})$, where*

$$\begin{aligned}\lambda_{0,0}=(&\underbrace{\lambda_1,\cdots,\lambda_1}_{p_1},\cdots,\underbrace{\lambda_k,\cdots,\lambda_k}_{p_k},\underbrace{0,\cdots,0}_{w},\\&\underbrace{\lambda_1,\cdots,\lambda_1}_{q_1},\cdots,\underbrace{\lambda_k,\cdots,\lambda_k}_{q_k},\underbrace{0,\cdots,0}_{w}),\end{aligned}\tag{3.1}$$

   and $\Psi_{0,0}$ is obtained from Ψ as follows: for $1\leq i\leq p_0$ and $1\leq j\leq q_0$, the root $e_i-f_j\in\Psi_{0,0}$ if and only if $e_i+e_{n-j+1}\in\Psi$. (This determines $\Psi_{0,0}$ completely.)
2. *$z=2w>0$:*

[(7)] In the statement here, we only remains the case when p,q are all even. The main reason is that since we allow the quasi-split group $\mathrm{O}(2k,2k+2)$, and its Langlands-Vogan parameter translation will be more 'clean' than $\mathrm{O}(2k+1,2k+1)$'s (i guess...). Four lifts will ensure us to get the appropriate group we want.

– *If* $e_{k+1} + e_{k+z} \in \Psi$, $\theta_{2p_0+2,2q_0}(\pi) \neq 0$ *with the parameter* $(\lambda_{2,0}, 1, \Psi_{2,0})$, *where* $\lambda_{2,0}$ *is obtained from* $\lambda_{0,0}$ *by adding a zero on the left and* $\Psi_{0,0} \subset \Psi_{2,0}$.
– *If* $-e_{k+1} - e_{k+z} \in \Psi$, $\theta_{2p_0,2q_0+2}(\pi) \neq 0$ *with the parameter* $(\lambda_{0,2}, 1, \Psi_{0,2})$, *where* $\lambda_{0,2}$ *is obtained from* $\lambda_{0,0}$ *by adding a zero on the right and* $\Psi_{0,0} \subset \Psi_{0,2}$.

3. $z = w = 0$: $\theta_{2p_0+2,2q_0}(\pi) \neq 0$ *with parameter* $(\lambda_{2,0}, 1, \Psi_{2,0})$ *and* $\theta_{2p_0,2q_0+2}(\pi) \neq 0$ *with parameter* $(\lambda_{0,2}, 1, \Psi_{0,2})$, *where* $\lambda_{2,0}$ *and* $\lambda_{0,2}$ *are obtained from* $\lambda_{0,0}$ *by adding a zero on the left and right respectively, and* $\Psi_{0,0} \subset \Psi_{2,0}, \Psi_{0,2}$.
4. $z = 2w + 1$:
   – *If* $e_{k+1} + e_{k+z} \in \Psi$, *then* $\theta_{2p_0+2,2q_0+2}(\pi) \neq 0$ *with the parameter* $(\lambda_{1,1}, 1, \Psi_{1,1})$, *where* $\lambda_{1,1}$ *is obtained from* $\lambda_{0,0}$ *by adding a zero on each side of the semicolon, and* $\Psi_{0,0} \cup \{e_{p_0+1} - f_{q_0+1}\} \subset \Psi_{1,1}$. *Moreover,* $\theta_{2p_0+2,2q_0}(\pi) \neq 0$ *with parameter* $(\lambda_{1,0}, 1, \Psi_{1,0})$, *where* $\lambda_{1,0}$ *is obtained from* $\lambda_{0,0}$ *by adding a zero on the left, and* $\Psi_{0,0} \subset \Psi_{1,0}$.
   – *If* $-e_{k+1} - e_{k+z} \in \Psi$, *then* $\theta_{2p_0+2,2q_0+2}(\pi) \neq 0$ *with the parameter* $(\lambda_{1,1}, 1, \Psi_{1,1})$, *where* $\lambda_{1,1}$ *is obtained from* $\lambda_{0,0}$ *by adding a zero on each side of the semicolon, and* $\Psi_{0,0} \cup \{-e_{p_0+1} + f_{q_0+1}\} \subset \Psi_{1,1}$. $\theta_{2p_0+2,2q_0}(\pi) \neq 0$ *with parameter* $(\lambda_{0,1}, 1, \Psi_{0,1})$, *where* $\lambda_{0,1}$ *is obtained from* $\lambda_{0,0}$ *by adding a zero on the right, and* $\Psi_{0,0} \subset \Psi_{0,1}$.

3.1.2. *Translation.* — Let $(V, (,))$ be a symplectic space and let $\varphi : W_{\mathbb{R}} \to O(M)$ be a limit of discrete series L-parameter of $\mathrm{Sp}(V)$, where M is a $(2n+1)$-dimensional orthogonal space. Let λ_d be the infinitesimal character associated to φ.

Let $\pi \in \Pi_\varphi$ be a limit of discrete series representation of $\mathrm{Sp}(V)$ with Langlands-Vogan parameter (φ, η).

Firstly, we suppose that φ is a discrete series Langlands parameter. In this case, we only need to determine the image of the generic representation associated to Langlands-Vogan parameter $(\varphi, \mathbf{1})$ under the theta correspondence. For the symplectic group $\mathrm{Sp}(V)$ of rank n, the corresponding base point of strong involution is $x_{b,\mathrm{Sp}} = i((-1)^{n-1}, \cdots, 1) \in \mathbf{T}_*[2]$ as in Example 2.19. If n is even (resp. odd), we set the generic Harish-Chandra lifting $t_{b,\mathrm{Sp}} = ix_{b,\mathrm{Sp}}$ (resp. $-ix_{b,\mathrm{Sp}}$) (cf. Example 2.20). In both cases, we have $t_{b,\mathrm{Sp}} = (1, \cdots, (-1)^{n-1})$ and the Harish-Chandra-Langlands parameter $(\lambda_{d,\mathrm{Sp}}, \Psi_{b,\mathrm{Sp}})$ of the generic discrete series representation in the L-packet $\Pi(\varphi)$ is described in Example 2.20. For general Vogan parameter η, we have the Harish-Chandra lifting

$\lambda_{d,\eta} = (\eta \cdot t_{b,\mathrm{Sp}}) \star \lambda_d$, where the operator $\star$ means the reordering the entries of the multiplication $(\eta \cdot t_{b,\mathrm{Sp}})\lambda_d$ from large to small. Note that, in the case of discrete series parameter, $\lambda_{d,\eta}$ uniquely determines the root system $\Psi_{\eta,\mathrm{Sp}}$ by the dominant condition. Let

$$\lambda_{b,0,0} = \begin{cases} (\lambda_1, \cdots, \lambda_{n-1}; \lambda_2, \cdots, \lambda_{2[\frac{n}{2}]}), & \text{if } n \equiv 0 \mod 2; \\ (\lambda_1, \cdots, \lambda_{2[\frac{n}{2}]+1}; \lambda_2, \cdots, \lambda_{2[\frac{n}{2}]}), & \text{if } n \equiv 1 \mod 2. \end{cases}$$

We only consider the orthogonal group of equal rank $n+1$ for the theta correspondence, i.e. case 2,3,4 in theorem 3.1. Thus, for the generic representation in the discrete series L-packet $\Pi(\varphi)$ of $\mathrm{Sp}(V)$ and n is even (resp. odd), we choose the quasi split (resp. split) even orthogonal group $\mathrm{O}(n+2, n)$ or $\mathrm{O}(n, n+2)$ (resp. $\mathrm{O}(n+1, n+1)$) and the corresponding Harish-Chandra parameter is $\lambda_{b,2,0}$ (resp. $\lambda_{b,0,2}$) obtained from $\lambda_{b,0,0}$ by adding a zero on the left (resp. right) appearing in the theta correspondence.

Proposition 3.2. — *Let π be a generic discrete series representation of* $\mathrm{Sp}(V)$ *corresponding to the base point $x_{b,\mathrm{Sp}}$. Then $\theta_{V,V'}(\pi)$ is a generic discrete series representation of* $\mathrm{O}(V')$ *with signature (p,q), which corresponds to the base point $x_{b,\mathrm{O}}$.*

Démonstration. — We have described the Harish-Chandra-Langlands parameter of the basepoint $x_{b,\mathrm{Sp}}$ and $x_{b,\mathrm{O}}$ in example 2.19. The proposition follows from our description of the Harish-Chandra-Langlands parameters of the basepoints $x_{b,\mathrm{Sp}}$ and $x_{b,\mathrm{O}}$ in Example 2.19 and the explicit theta correspondence in [**45**, Theorem 15].

In the following, we use the notation of the parameters and the root systems of symplectic groups as in Example 2.20.

Assume n is even. The Harish-Chandra parameter of π has the form

$$(\lambda_1, \lambda_3, \cdots, \lambda_{n-1}, -\lambda_n, \cdots, -\lambda_2),$$

and the corresponding root system is generated by the simple roots

$$\{e_1 + e_2, -e_2 - e_3, \cdots, e_{n-1} + e_n, -2e_n\}.$$

Hence by case 3 of theorem 3.1, the Harish-Chandra-Langlands parameter of $\theta_{V,V'}(\pi)$ is the pair $(\lambda_{2,0}, 1, \Psi_{2,0})$, where

$$\lambda_{2,0} = (\lambda_1, \lambda_3, \cdots, \lambda_{n-1}, 0; \lambda_2, \cdots, \lambda_n)$$

and the corresponding root system

$$\Psi_{2,0} = \langle e_1 - f_1, f_1 - e_2, \cdots, e_{\frac{n}{2}-1} - f_{\frac{n}{2}}, f_{\frac{n}{2}} - e_{\frac{n}{2}}, f_{\frac{n}{2}} + e_{\frac{n}{2}} \rangle.$$

Since $\lambda_1 > \cdots > \lambda_n > 0$ and the root system is generated by the non-compact simple roots, the parameter $(\lambda_{2,0}, 1, \Psi_{2,0})$ is exactly the Harish-Chandra-Langlands parameter of the generic discrete series representation of $\mathrm{O}(n+2,n)$ as in Example 2.21. As a result, $\theta_{V,V'}(\pi)$ is a generic discrete series of $\mathrm{O}(n+2,n)$.

Assume n is odd. The Harish-Chandra parameter of π has the form

$$(\lambda_1, \lambda_3, \cdots, \lambda_n, -\lambda_{n-1}, \cdots, -\lambda_2),$$

and the corresponding root system is generated by the simple roots

$$\Psi_{\mathrm{Sp}} = \langle e_1 + e_2, -e_2 - e_3, \cdots, -e_{n-1} - e_n, 2e_n \rangle.$$

Hence by case 4 of Theorem 3.1, the Harish-Chandra parameter of $\theta_{V,V'}(\pi)$ is $(\lambda_{1,1}, 1, \Psi_{1,1})$ where

$$\lambda_{1,1} = (\lambda_1, \lambda_3, \cdots, \lambda_n; \lambda_2, \cdots, \lambda_{n-1}, 0).$$

and the corresponding root system

$$\Psi_{1,1} = \langle e_1 - f_1, f_1 - e_2, \cdots, f_{\frac{n-1}{2}} - e_{\frac{n-1}{2}}, e_{\frac{n-1}{2}} - f_{\frac{n+1}{2}}, f_{\frac{n+1}{2}} + e_{\frac{n+1}{2}} \rangle.$$

Similarly, the parameter $(\lambda_{1,1}, \Psi_{1,1})$ is exactly the Harish-Chandra-Langlands parameter of the generic discrete series representation of $\mathrm{O}(n+1, n+1)$ as in Example 2.21. Hence $\theta_{V,V'}(\pi)$ is a generic discrete series of $\mathrm{O}(n+1, n+1)$. $\square$

Now, we consider the general limit of discrete series Langlands-Vogan (φ, η) of $\mathrm{Sp}(V)$. Without loss of generality, we may assume that the L-parameter φ of $\mathrm{Sp}(V)$ admits a decomposition into irreducible representations as in Example 2.22 (1). Then as in example 2.24, its corresponding Harish-Chandra parameter is of the form

$$\lambda_\eta = (\underbrace{\lambda_1, \cdots, \lambda_1}_{p_{\eta,1}}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{p_{\eta,k}}, \underbrace{0, \cdots, 0}_{z}, \underbrace{-\lambda_k, \cdots, -\lambda_k}_{q_{\eta,k}}, \cdots, \underbrace{-\lambda_1, \cdots, -\lambda_1}_{q_{\eta,1}}).$$

Let $p_{\eta,0} = \sum_{l=1}^{k} p_{\eta,l}$ and $q_{\eta,0} = \sum_{l=1}^{k} q_{\eta,l}$, and $w = [\frac{z}{2}]$. Moreover, if we set $c_{\eta,i} = \sum_{j=1}^{i}(p_{\eta,j} + q_{\eta,j})$, then the character η has the form

$$(\eta_1, \cdots, \eta_{c_{\eta,1}}, \cdots, \eta_{c_{\eta,i}+1}, \cdots, \eta_{c_{\eta,i+1}}, \cdots, \eta_{c_{\eta,k}+1}, \cdots, \eta_{c_{\eta,k}+z}),$$

where $\eta_1 = \cdots = \eta_{c_{\eta,1}}, \cdots, \eta_{c_{\eta,i+1}} = \cdots = \eta_{c_{\eta,i+1}}, \cdots, \eta_{c_{\eta,k}+1} = \cdots = \eta_{c_{\eta,k}+z}$. This character determines a positive root system Ψ_η.

We set $p_\eta = p_{\eta,0} + w$ and $q_\eta = q_{\eta,0} + w$. Note that

$$p_\eta + q_\eta = \begin{cases} n, & \text{if } z \equiv 0 \mod 2; \\ n-1, & \text{if } z \equiv 1 \mod 2. \end{cases}$$

We set

$$\lambda_{\eta,0,0} = (\underbrace{\lambda_1, \cdots, \lambda_1}_{p_{\eta,1}}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{p_{\eta,k}}, \underbrace{0, \cdots, 0}_{w}; \underbrace{\lambda_1, \cdots, \lambda_1}_{q_{\eta,1}}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{q_{\eta,k}}, \underbrace{0, \cdots, 0}_{w}),$$

and a root system $\Psi_{\eta,0,0}$ obtained from Ψ_η as follows: for $1 \le i \le p_\eta$ and $1 \le j \le q_\eta$, the root $e_i - f_j \in \Psi_{\eta,0,0}$ if and only if $e_i + e_{n-j+1} \in \Psi_\eta$.

1. If $z = w = 0$ (i.e. case 3 of Theorem 3.1), then we have $p = 2p_\eta + 2$ and $q = 2q_\eta$ and our normalization on generic theta correspondence for discrete series. The corresponding Harish-Chandra-Langlands parameter is $(\lambda_{\eta,2,0}, 1, \Psi_{\eta,2,0})$, where $\lambda_{\eta,2,0}$ is obtained from $\lambda_{\eta,0,0}$ by adding a zero on the left and $\Psi_{\eta,0,0} \subset \Psi_{\eta,2,0}$.
2. If $z = 2w > 0$ (i.e. case 2 of Theorem 3.1), there are two possible cases:
   (a) If $e_{p_{\eta,0}+1} + e_{p_{\eta,0}+z} \in \Psi_\eta$, then we have $p = 2p_\eta + 2$ and $q = 2q_\eta$. The corresponding Harish-Chandra-Langlands parameter is $(\lambda_{\eta,2,0}, 1, \Psi_{\eta,2,0})$, where $\lambda_{\eta,2,0}$ is obtained from $\lambda_{\eta,0,0}$ by adding a zero on the left side, and $\Psi_{\eta,2,0}$ contains $\Psi_{\eta,0,0}$.
   (b) If $-e_{p_{\eta,0}+1} - e_{p_{\eta,0}+z} \in \Psi_\eta$, then we have $p = 2p_\eta$ and $q = 2q_\eta + 2$. The corresponding Harish-Chandra-Langlands parameter is $(\lambda_{\eta,0,2}, 1, \Psi_{\eta,0,2})$, where $\lambda_{\eta,0,2}$ is obtained from $\lambda_{\eta,0,2}$ by adding a zero on the right side, and $\Psi_{\eta,0,2}$ contains $\Psi_{\eta,0,0}$.
3. If $z = 2w + 1$ (i.e. case 4 of Theorem 3.1), then we have $p = 2p_\eta + 2$ and $q = 2q_\eta + 2$. The corresponding Harish-Chandra-Langlands parameter is $(\lambda_{\eta,1,1}, 1, \Psi_{\eta,1,1})$, where $\lambda_{\eta,1,1}$ is obtained from $\lambda_{\eta,0,0}$ by adding a zero on each side of the semicolon. Moreover,
   (a) if $e_{p_{\eta,0}+1} + e_{p_{\eta,0}+z} \in \Psi_\eta$, then $e_{p_{\eta,0}+w+1} - f_{q_{\eta,0}+w+1} \in \Psi_{\eta,1,1}$.
   (b) If $-e_{p_{\eta,0}+1} - e_{p_{\eta,0}+z} \in \Psi_\eta$, then $-e_{p_{\eta,0}+w+1} + f_{q_{\eta,0}+w+1} \in \Psi_{\eta,1,1}$.

At last, we need to translate this Harish-Chandra-Langlands parameter $(\lambda_{\eta,a,b}, 1, \Psi_{\eta,a,b})$ of $\theta_{V,W}(\pi)$ into the Langlands-Vogan parameter of $\theta_{V,W}(\pi)$, where $(a, b) = (2, 0)$, $(0, 2)$ or $(1, 1)$, which depends on the Langlands-Vogan parameters $(\lambda_\eta, \Psi_\eta)$ of π. As in Example 2.25,

1. If $(a, b) = (2, 0)$, then corresponding Harish-Chandra parameter is

$$\lambda_{\eta,2,0} = (\underbrace{\lambda_1, \cdots, \lambda_1}_{p_{\eta,1}}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{p_{\eta,k}}, \underbrace{0, \cdots, 0}_{w+1}; \underbrace{\lambda_1, \cdots, \lambda_1}_{q_{\eta,1}}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{q_{\eta,k}}, \underbrace{0, \cdots, 0}_{w})$$

and a root system $\Psi_{\eta,2,0} \supset \Psi_{\eta,2,0}$.

2. If $(a,b)=(0,2)$, then corresponding Harish-Chandra parameter is

$$\lambda_{\eta,2,0} = (\underbrace{\lambda_1,\cdots,\lambda_1}_{p_{\eta,1}},\cdots,\underbrace{\lambda_k,\cdots,\lambda_k}_{p_{\eta,k}},\underbrace{0,\cdots,0}_{w};\underbrace{\lambda_1,\cdots,\lambda_1}_{q_{\eta,1}},\cdots,\underbrace{\lambda_k,\cdots,\lambda_k}_{q_{\eta,k}},\underbrace{0,\cdots,0}_{w+1})$$

and a root system $\Psi_{\eta,0,2} \supset \Psi_{\eta,0,0}$.

3. If $(a,b)=(1,1)$, then corresponding Harish-Chandra parameter is

$$\lambda_{\eta,2,0} = (\underbrace{\lambda_1,\cdots,\lambda_1}_{p_{\eta,1}},\cdots,\underbrace{\lambda_k,\cdots,\lambda_k}_{p_{\eta,k}},\underbrace{0,\cdots,0}_{w+1};\underbrace{\lambda_1,\cdots,\lambda_1}_{q_{\eta,1}},\cdots,\underbrace{\lambda_k,\cdots,\lambda_k}_{q_{\eta,k}},\underbrace{0,\cdots,0}_{w+1})$$

and a root system $\Psi_{\eta,1,1} \supset \Psi_{\eta,0,0}$.

Then the corresponding Langslands-Vogan parameter of $\theta_{V,W}(\pi)$ is given by

$$\theta_{V,W}(\varphi) = \oplus_{i=1}^{k}(p_{\eta,i}+q_{\eta,i})\rho_{\lambda_i}\bigoplus(2w+2)\mathbf{1}$$

and

$$\theta_{V,W}(\eta) = (\eta_1,\cdots,\eta_{c_{\eta,1}},\cdots,\eta_{c_{\eta,k}+1},\cdots,\eta_{c_{\eta,k}+z+1})$$

with $\eta_{c_{\eta,k}+1}=\cdots=\eta_{c_{\eta,k}+z}=\eta_{c_{\eta,k}+z+1}$.

We summary our description of theta correspondence for symplectic-orthogonal dual pairs in term of Langlands-Vogan parameters in the following theorem.

Theorem 3.3. — *If π is a limit of discrete series representation of $\mathrm{Sp}(V)$ with Langlands-Vogan parameter (φ,η), where*

$$\varphi = \oplus_{i=1}^{k}(p_{\eta,i}+q_{\eta,i})\rho_{\lambda_i}\bigoplus(z+1)\mathbf{1},$$

with $p_{\eta,i},q_{\eta,i},z\in\mathbb{N}$, $i=1,\cdots,k$, $\sum_{i=1}^{k}(p_{\eta,i}+q_{\eta,i})+z=n$ and ρ_{λ_i} is self-dual irreducible representation of the Weil group $W_{\mathbb{R}}$ of dimension 2. Then there exists a pair of even integers (p,q) such that $p+q=2n+2$, and $\theta_{V,W}(\pi)$ is a limit of discrete series representation of $\mathrm{O}(p,q)$ with Langlands parameter $\theta_{V,W}(\varphi)$, where

$$\theta_{V,W}(\varphi) = \oplus_{i=1}^{k}(p_{\eta,i}+q_{\eta,i})\rho_{\lambda_i}\bigoplus(z+2)\mathbf{1}.$$

Moveover, if

$$\eta = (\eta_1,\cdots,\eta_{c_{\eta,1}},\cdots,\eta_{c_{\eta,k}+1},\cdots,\eta_{c_{\eta,k}+z})$$

with $\eta_1=\cdots=\eta_{c_\eta 1},\cdots,\eta_{c_{\eta,k}+1}=\cdots=\eta_{c_{\eta,k}+z}$. , then we have

$$\theta_{V,V'}(\eta) = (\eta_1,\cdots,\eta_{c_{\eta,1}},\cdots,\eta_{c_{\eta,k}+1},\cdots,\eta_{c_{\eta,k}+z+1})$$

with $\eta_{c_{\eta,k}+1}=\cdots=\eta_{c_{\eta,k}+z}=\eta_{c_{\eta,k}+z+1}$.

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