Contents

This note is intended to rekindle my memories about real reductive groups which have been lost at the seat 730. Thus it's mainly a collection of definitions, theorems and their brief proofs. In the introduction section of Wallach's book, He said that the serious students should approach this work with an ample supply of paper and pencils. **Be patient and it will be yours**.

1 Background Material

1.1 Invariant measures on homogenerous space

Let G be a LCH group.

If dg is a left invariant measure and if $x \in G$ then we can define a new left invariant measure on G, μ_x, as follows:

$$\mu_x(f) = \int_G f(gx)dg.$$

The uniqueness of left invariant measure (Haar's theorem) implies that

$$\mu_x(f) = \delta(x) \int_G f(g)dg$$

with δ a function (a continuous homomorphism: $G \to \mathbb{R}_+^\times$) of x which is usually called the modular function of G. Define *unimodular*. This implies that if G is cpt. then G is unimodular.

For G a Lie group,

$$\delta(x) = |\det \mathrm{Ad}(x)|$$

Ad is the adjoint representation of G, i.e. fix $g \in G$, $f_g(h) = ghg^{-1}$. $\mathrm{d}f : \mathfrak{g} \to \mathfrak{g}$ is denoted $\mathrm{Ad}(g)$.

$$\begin{tikzcd} G \arrow[r, "f_g"] & G \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{Ad}(g)"'] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd}$$

We should confirmed that Ad is a smooth homomorphism from G into $GL(\mathfrak{g})$, and the differential of Ad is ad (ad $x(y) = [x, y],\ x, y \in \mathfrak{g}$):

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & \mathrm{GL}(\mathfrak{g}) \\ T_e(G) = \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"'] & \mathrm{End}(\mathfrak{g}) \arrow[u, "\exp"'] \end{tikzcd}$$

We need some integration formulas.

Theorem 1. *Let G be a LCH group, and let H be a cpt. subgrp.. Let $\mathrm{d}\mu_G$ and $\mathrm{d}\mu_H$ be left Haar measures on G and H, respectively. Then there exists a regular Borel measure $\mathrm{d}\mu_{G/H}$ on G/H which is invariant under the action of G by left translation. The measure $\mathrm{d}\mu_{G/H}$ may be normalized so that, for $f \in C_c(G)$, we have*

$$\int_G f(g)\mathrm{d}\mu_G(g) = \int_{G/H} \int_H f(gh)\mathrm{d}\mu_H(h)\mathrm{d}\mu_{G/H}(gH).$$

Here the function $g \mapsto \int_H f(gh)\mathrm{d}\mu_H$ is constant on the cosets gH, and we are therefore identifying it with a function on G/H.

Theorem 2. *suppose that A and B are two closed subgroups of G such that*

$$AB = \{ab \mid a \in A, b \in B\} = G$$

and $A \cap B$ is cpt.. Then if dg denotes invariant measure on G, da denotes left invariant measure on A and db right invariant measure on B then up to constants of normalization

$$\int_G f(g)dg = \int_A \int_B f(ab)dadb.$$

In other words if $T : C_c(G) \to C_c(A \times B)$ is given by $T(f)(a, b) = f(ab)$, if μ is left invariant on A, ν is right invariant on B then $(\mu \times \nu) \circ T$ is invariant on G.

1.2 Structure theory of reductive Lie algeras

If $X \in \mathfrak{g}$ then define the polynomials D_j on $\mathfrak{g}$ by

$$\det(tI - ad\ X) = \sum t^j D_j(X),$$

here $n = \dim g$. Let r be the smallest index such that D_r is not identically zero. Set $D = D_r$. $X \in \mathfrak{g}$ is said to be regular if$D(X)$ is nonzero.

Proposition 1.1. *If X is regular then ad X is semi-simple. Futhermore, the centralizer in $\mathfrak{g}$ of a regular element is a Cartan subalgebra of $\mathfrak{g}$.*

Remark. We can also construct the Cartan subalgebra as the minimal Engel subalgebra.

Let $\mathfrak{g}$ a reductive Lie alg..Fix a $\mathfrak{h}$, a CSA of $\mathfrak{g}$. If $\alpha \in \mathfrak{h}^*$ then we set

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X \quad \text{for all } H \in \mathfrak{h}\}$$

If α and $\mathfrak{g}_\alpha$ are non-zero then we call α a *root* of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and $\mathfrak{g}_\alpha$ is called the *root space* corresponding to α. The set of all roots of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$, $\Phi(\mathfrak{g}, \mathfrak{h})$, called the *root system*. Then we have the decomposition and relations:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_\alpha, \quad \ldots$$

Through the non-degenerate invariant form B, we can define: for $\mu \in \mathfrak{h}^*$ then we can define $H_\mu \in \mathfrak{h}$ by

$$B(H, H_\mu) = \mu(H) \quad \text{for } H \in \mathfrak{h}$$

We can then define a non-degenerate symmetric bilinear form $(\quad , \quad)$ on $\mathfrak{h}^*$ by

$$(\mu, \tau) = B(H_\mu, H_\tau) \quad \text{for } \mu, \tau \in \mathfrak{h}^*.$$

Let $\mathfrak{h}_\mathbb{R}$ denote the real subspace of $\mathfrak{h}$ spanned by the H_α, for $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$. We define the $Weyl\ reflection$ as follows:

$$s_\alpha H = H - (2\alpha(H)/(\alpha, \alpha))H_\alpha \quad \text{for } H \in \mathfrak{h}$$

and $W(\mathfrak{g}, \mathfrak{h})$ the group generated by the reflections, called the $Weyl\ group$. Define the $Weyl\ Chamber$. Specifying a Weyl chamber is the same as specifying a system of *positive roots*. Define *simple system*.

1.3 Structure theory of compact connect Lie groups

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_\mathbb{C}$ denote the complexification of g. Then $\mathfrak{g}_\mathbb{C}$(resp. $\mathfrak{g}$) is a reductive Lie algebra over $\mathbb{C}(\mathbb{R})$.

Theorem 3. *If $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$ with negative definite Killing form (i.e. semisimple) then any connected Lie group with Lie algebra $\mathfrak{g}$ is cpt..*

A commutative cpt., connected Lie group is called a *torus*. It's Lie alg., $\mathfrak{t}$, is a addictive group, then exp is a covering homomorphism of $\mathfrak{t}$ to torus, T. The kernel of the map is a lattice, L, a free $\mathbb{Z}$ module of rank equal to dim $\mathfrak{t}$.

Let $T^\wedge$ denote the set of all continuous homomorphisms of T into the circle. If $\mu \in T^\wedge$ then the differential of μ(which we will also denote by μ) is a linear map of the Lie alg. into $i\mathbb{R}$ such that $\mu(L) \subset 2\pi i\mathbb{Z}$. If μ is a linear map of $\mathfrak{t}$ into $i\mathbb{R}$ such that $\mu(L) \subset 2\pi i\mathbb{Z}$ then μ is called *integral*. If μ is an integral linear form on $\mathfrak{t}$ then we define for $t = \exp(X), t^\mu = \exp(\mu(X))$. This sets up an identification of integral linear forms on $\mathfrak{t}$ and characters of T.

For cpt. G, define *maximal torus*, T.Then $\mathfrak{t}_\mathbb{C}$ is a CSA of $\mathfrak{g}_\mathbb{C}$. The elements of $\Phi\left(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}\right)$ are integral on $\mathfrak{t}$ and thus define elements of $T^\wedge$.

Here are some useful properties.

- The maximal torus is unique up to conjugate.
- Every element of G is contained in a maximal torus of G. That is, the exponential map of G is surjective (Btw, I know a proof which use the theory of riemannian geometry).
- G/T is simply connected.
- Let $N(T)$ denote the normalizer of T in G, $W(G,T)$ denote the group $N(T)/T$. Set

  $$sH = \mathrm{Ad}(g)H, \quad H \in \mathfrak{t},\ g \subset s \in W(G,T)$$

  Then under this action, $W(G,T) = W(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$.

We will talk about $compact\ form$ in Chapter 2.

1.4 Universal enveloping algebra (PBW theorem)

The universal enveloping alg. for $\mathfrak{g}$ is a pair (A,j) ...with the following universal mapping property:....

Theorem 4 (PBW)**.** *Let $\{X_i\}_{i\in A}$ be a basis of $\mathfrak{g}$, and suppose a simple ordering has been imposed on the index set A. Then the set of all monomials*

$$(\iota X_{i_1})^{j_1} \cdots (\iota X_{i_n})^{j_n}$$

with $i_1 < \cdots < i_n$ and with all $j_k \geq 0$, is a basis of $(\iota, U(\mathfrak{g}))$. Particularly, $(\iota, U(\mathfrak{g}))$ is countable dim..

I never read the whole proof lol. Here are some really useful tools.

Proposition 1.2. *The linear map $\tilde{\psi} : T(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$ respects multiplication and annihilates the defining ideal I for $S(\mathfrak{g})$. Therefore ψ descends to an algebra homomorphism*

$$\psi : S(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$$

that respects the grading. PBW tells that this homomorphism is an isomorphism.

Proposition 1.3. *Let W be a subspace of $T^n(\mathfrak{g})$, and suppose that the quotient map $T^n(\mathfrak{g}) \to S^n(\mathfrak{g})$ sends W isomorphically onto $S^n(\mathfrak{g})$. Then the image of W in $U_n(\mathfrak{g})$ is a vector-space complement to $U_{n-1}(\mathfrak{g})$.*

If $X_1, \ldots, X_k$ are in $\mathfrak{g}$ then set

$$\operatorname{symm}(X_1 \cdots X_k) = (1/k!) \sum_{\sigma} X_{\sigma 1} \cdots X_{\sigma k}$$

the sum over all permutations σ of k letters. Then symm extends to a linear map of $S(\mathfrak{g})$ to $U(\mathfrak{g})$ which is called *symmetrization mapping*. Then we see that

$$U_n(\mathfrak{g}) = \operatorname{symm}(S^n(\mathfrak{g})) \oplus U_{n-1}(\mathfrak{g})$$

We note that if $\mathfrak{a}$ the Lie algebra (0) then $U(\mathfrak{a}) = F$. Let ε be the Lie algebra homomorphism of $\mathfrak{g}$ onto $\mathfrak{a}$ given by $\varepsilon(X) = 0$. Then ε extends to a homomorphism of $U(\mathfrak{g})$ onto F which we also denote by ε. ε is called the *augmentation homomorphism*. Define $\mathfrak{g}^{\mathrm{opp}}$.

Let $\mathfrak{b}$ be a subalgebra of $\mathfrak{g}$. PBW implies that the canonical map of $U(\mathfrak{b})$ into $U(\mathfrak{g})$ is injective. We can thus identify $U(\mathfrak{b})$ with the associative subalgebra of $U(\mathfrak{g})$ generated by 1 and $\mathfrak{b}$. Let V be a subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{b} \oplus V$. Then P-B-W implies that the linear map

$$U(\mathfrak{b}) \otimes S(V) \to U(\mathfrak{g})$$

Given by $b \otimes v \mapsto b \operatorname{symm}(v)$ is a surjective linear isomorphism.

1.5 sl(2,$\mathbb{C}$) and Schur lemma

Theorem 5. *Suppose that V is countable dimensional and that $S \subset \operatorname{End}(V)$ acts irreducibly. If $T \in \operatorname{End}(V)$ commutes with every element of S then T is a scalar multiple of the identity operator. Particularly, for irr. $\mathfrak{g}-$mod V, $\operatorname{Hom}_{\mathfrak{g}}(V, V) = \mathbb{C}I$.*

1.6 Modules over the universal enveloping algebra

$U(\mathfrak{g})$ is Noetherian.

We can construct a large family of highest weight modules by exploiting the technique of induction. We start with the Borel subalgebra $\mathfrak{b}$ corresponding to a *fixed* choice of positive roots. Any $\lambda \in \mathfrak{h}^*$ then defines a 1-dimensional $\mathfrak{b}$-module with trivial $\mathfrak{n}$-action, denoted $\mathbb{C}_\lambda$. Now set $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, which has a natural structure of left $U(\mathfrak{g})$-module. This is called a *Verma module* and may also be written as $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$ to emphasize the functorial nature of induction. By the PBW Theorem, $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{b})$, hence we can see $M(\lambda) \cong U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$ as a left $U(\mathfrak{n}^-)$-module, makes $M(\lambda)$ a free $U(\mathfrak{n}^-)$-module of rank one. In particular, the vector $v^+ := 1 \otimes 1$ in the definition of $M(\lambda)$ is nonzero and is acted on freely by $U(\mathfrak{n}^-)$, while $\mathfrak{n} \cdot v^+ = 0$ and $h \cdot v^+ = \lambda(h)v^+$ for all $h \in \mathfrak{h}$. Thus v^+ is a maximal vector also a generator. Moreover, the set of *weights* of $M(\lambda)$ is visibly $\lambda - \Gamma$. It follows that $M(\lambda)$ lies in the BGG category $\mathcal{O}$ (i.e.).

Remark. We can alternatively describe $M(\lambda)$ by generations and relations.

We can write $L(\lambda)$ (resp. $N(\lambda)$) for the unique simple quotient (resp. unique maximal submodule) of $M(\lambda)$. Since every nonzero module in $\mathcal{O}$ has at least one maximal vector,

Theorem 6. *Every simple module in $\mathcal{O}$ is isomorphic to a module $L(\lambda)$ with$\lambda \in \mathfrak{h}^*$ and is therefore determined uniquely up to isomorphism by its highest weight. Moreover,* $\dim \operatorname{Hom}_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\lambda\mu}$.

Remark. For any M in $\mathcal{O}$,

$$\operatorname{Hom}_{U(\mathfrak{g})}(M(\lambda), M) = \operatorname{Hom}_{U(\mathfrak{g})}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M\right) \cong \operatorname{Hom}_{U(\mathfrak{b})}\left(\mathbb{C}_\lambda, \operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}} M\right)$$

where $\operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}}$ is the restriction functor. This adjointness property is known as Frobenius reciprocity which we will explain later.

2 Elementary Representation Theory

2.1 Definitions

Let G be a separable, locally cpt. group with left invariant measure. Let V be a *topological* vector space over $\mathbb{C}$. We denote by, $\operatorname{End}(V)$, the space of *continuous* endomorphisms of V and by $GL(V)$ the group of all invertible elements of $\operatorname{End}(V)$. Then a representation of G on V is a *homomorphism*, π, of G into $GL(V)$ such that the map $G \times V \to V$ given by $g, v \mapsto \pi(g)v$ is *continuous*.

If G is a Lie group and if V is a *Fréchet* space (i.e. complete metrizable locally convex space) then a representation (π, V) of G is said to be *smooth* if the maps of G to V are C^∞ for all $v \in V$.

A representation (π, H) where H is a (separable) Hilbert space is called a *Hilbert* representation. If (π, H) is a Hilbert representation and if $\pi(g)$ is a unitary operator for all $g \in G$ then we call (π, H) a *unitary* representation. Let $|\cdots|$ denotes the operator norm on $\mathrm{End}(H)$. Then:

(1) For a compact subset Ω of G, there is a constant, C_Ω, s.t.

$$|\pi(g)| \leq C_\Omega, \quad for\ all\ g \in \Omega$$

(see uniform boundedness thm.). The definition also implies:

(2)$\forall v, w \in H$, $g \mapsto \langle \pi(g)v, w\rangle$ is continuous.

Proposition 2.1. *Let H be a Hilbert space and let π be a homomorphism of G into $GL(H)$. If (π, H) satisfies (1) and (2) above then (π, H) is a representation of G.*

Remark. We can substitute $v \in H$ to any dense subspace of H.

The proof is classic, here are some useful operators (which are also appeared in the proof).

If $f \in C_c(G)$ then we define for $v, w \in H$ the sesquilinear form $\mu_f(v, w)$ by

$$\mu_f(v, w) = \int_G f(g)\langle \pi(g)v, w\rangle dg.$$

By riesz thm., there is an operator $\pi(f)$ in $\mathrm{End}(H)$ s.t.

$$|\pi(f)| \leq C_\Omega \|f\|_1 \quad and \quad \mu_f(v, w) = \langle \pi(f)v, w\rangle \quad for\ v, w \in H.$$

If f is a function on G we set $L(g)f(x) = f\left(g^{-1}x\right)$ for $g, x \in G$. Then

$$\pi(L(x)f) = \pi(x)\pi(f) \quad for\ f \in C_c(G), g \in G.$$

If (π, H) is a Hilbert representation of G then we set $\pi^*(g) = \left(\pi(g^{-1})\right)^*$ (* represents the conjugate transpose). Then (π^*, H) is a representation of G which is called the *conjugate dual* representation of (π, H). Clearly, one has

$$\langle \pi(g)v, \pi^*(g)w\rangle = \langle v, w\rangle \quad \text{for } v, w \in H, g \in G$$

2.2 Schur Lemma and functional results

The Schur Lemma (the simplest form) declare that ofr irr. uni. rep.(π, H), $\mathrm{Hom}_G(H, H) = \mathbb{C}I$. We will rephrase this result in the context of operator algebras. Let $A \subset \mathrm{Hom}(H, H)$ be a subalgebra. Then it is called a $^*algebra$ if whenever $T \in A$, $T^* \in A$. We say that A is an irr. subalg. if whenever $V \subset H$ is a closed subspace invariant under all the elements of A, $V = \{0\}$ or $V = H$. If A is a subset of $\mathrm{Hom}(H, H)$ then we denote by A' the set $\{T \in \mathrm{Hom}(H, H) \mid Ta = aT, a \in A\}$. A' is called the commutant of A. Then

Theorem 7 (Schur Lemma)**.** *A $^*algebra$ $A \subset \mathrm{Hom}(H, H)$ is irr. iff. $A' = \mathbb{C}I$.*

Using $Von\ Neumann\ density\ theorem$, we will have

Theorem 8. *Let A be a $^*subalgebra$ of $\mathrm{Hom}(H, H)$ then if I is in the closure of A w.r.t the strong operator topology then the alg. $(A')'$ is the closure of A in the strong operator topology. Particularly, if A acts irreducibly on H, then the closure of A in the strong operator topology is $\mathrm{Hom}(H, H)$.*

A $^*subalgebra$ of $\mathrm{Hom}(H, H)$ is called a $Von\ Neumann$ algebra if it is closed in the strong operator topology and contains the identity. The above results imply A $^*subalgebra$ of $\mathrm{Hom}(H, H)$ is a Von Neumann algebra iff. $(A')' = A$.

Then we get a useful version of the Schur Lemma.

Theorem 9. *Let (π, H) be an irr. uni. representation of G. Let D be a dense subspace of H such that $\pi(g)D \subset D$ for all $g \in G$ and let T be a linear map of D to H such that $T\pi(g)v = \pi(g)Tv$ for all $g \in G, v \in D$. Assume that there exists a dense subspace D' in H and a linear map S from D' to H such that*

$$\langle Tv, w\rangle = \langle v, Sw\rangle, \quad \forall v \in D, w \in D'$$

Then $T = \lambda I_D$ for some $\lambda \in \mathbb{C}$.

2.3 Square integrable representations, schur orthogonality

We will consider $L^2(G)$ as a uni. representation under the right regular action. Here we write $R_g f(x) = f(xg)$, called the *right regular* representation. For (π, H), $v, w \in H$, denote $c_{v,w}$ for the function

$$g \mapsto \langle \pi(g)v, w\rangle$$

The functions are called *matrix* $coefficients$ of π. Then we say that a irr. uni. representation (π, H) is *square integrable* if it has *a* nonzero, square integrable matrix coefficient.

Lemma 2.1. *Let (π, H) be a square integrable representation of G. Then every matrix entry $(c_{v,w}, v, w \in H)$ is square integrable. Furthermore, there exists an element $T \in L_G(H, L^2(G))$ with closed range consisting of continuous functions that is a unitary bijection onto its range. The map T can be implemented as follows: fix v_0 in H a unit vector then $T(w) = c_{w,v_0}$.*

Theorem 10 (Schur orthogonality relations)**.** *Let (π, H) and (ρ, V) be square integrable representations of G. If π and ρ are inequivalent then their matrix coefficients are orthogonal. There exists a positive real number $d(\pi)$ (which depends only on π and the normalization of Haar measure, called the $formal$ degree) such that if $v_1, v_2, w_1, w_2 \in H$ then*

$$\int_G \langle \pi(g)v_1, w_1 \rangle \overline{\langle \pi(g)v_2, w_2 \rangle} dg = \frac{1}{d(\pi)} \langle v_1, v_2 \rangle \langle w_2, w_1 \rangle$$

Particularly, if G is compact, then $d(\pi) = \mathrm{dim} H < \infty$.

2.4 Peter-Weyl theorem

Using the injective intertwining operator: $T \to C(G) \cap L^2(G)$, we get

Theorem 11. *(π, H) be an irr. Hilbert representation of G. Then $\dim H < \infty$. If (π, H) is unitary and we normalize the Haar measure, μ, on G such that $\mu(1) = 1$ then $d(\pi) = \dim H$.*

Recall that if $T : H_1 \to H_2$ is a continuous linear map of Hilbert spaces then T is said to be completely continuous (or compact, since completely continuous + reflective$\to$ cpt.) if the image of a bounded set has compact closure. If H is a Hilbert space then we denote by $CC(H)$ the space of all completely continuous operators from H to H. a unitary representation (π, H) of G is said to be of *class CC* if $\pi(f)$ is completely continuous for all $f \in C_c(G)$. We say that G is a CCR group if every irr. uni. representation of G is of *class* CC. One of Harish-Chandra's basic theorems is that all real reductive groups are CCR groups.

Theorem 12. *Let (π, H) be a unitary representation of G of class CC. Then (π, H) is unitarily equivalent with a unitary direct sum of irreducible representations of G.*

Let G a cpt. group. What we mainly concern is the unitary representation. Since the Unitarian trick (for cpt. group):

Theorem 13 (Unitarian Trick)**.** *(π, H) is a Hilbert representation of cpt. K. Let $\langle \ldots, \ldots \rangle$ be the Hilbert space structure on H. Then there is an inner product $(\ldots, \ldots)$ on H such that $(\pi(k)v, \pi(k)w) = (v, w)$ for all $k \in K, v, w \in H$ and such that the topology on H induced by $(\ldots, \ldots)$ is the same as the original topology.*

Consider:

$$(v, w) = \int_K \langle \pi(k)v, \pi(k)w \rangle dk, \quad v, w \in H$$

is enough. Let $\widehat{G}$ denote the set of equivalence classes of irreducible finite dimensional representations of G. For each $\gamma \in \widehat{G}$ we fix $(\tau_\gamma, V_\gamma) \in \gamma$ which we assume is unitary. If (π, V) is a representation of G then we set $V(\gamma)$ equal to the sum of the closed, G-invariant, irreducible subspaces in the class of γ, called the γ-*isotypic component* of V. Then

$$\dim L^2(G)(\gamma) < \infty \quad L^2(G) = \oplus L^2(G)(\gamma)$$

If (τ, V) is a finite dimensional representation of G then its character is defined to be the function $\chi_V(g) = \operatorname{tr}(\tau(g))$. We note that $\chi_V \in C(G)$ and that $\chi_V\left(xgx^{-1}\right) = \chi(g)$ for all $x, g \in G$. We claim that (τ_1, V_1) and (τ_2, V_2) are equivalent iff. $\chi_{V_1} = \chi_{V_2}$. We set $\alpha_\gamma = d(\gamma)\bar{\chi}_\gamma$ (complex conjugate).

Proposition 2.2. *The orthogonal projection of $L^2(G)$ onto $L^2(G)(\gamma)$ is the operator $P_\gamma = \pi\left(\alpha_\gamma\right)$.*

Let (π, H) be a uni. representation of G.If $\gamma \in \widehat{G}$ then we set $E_\gamma = \pi\left(\alpha_\gamma\right)$. Then if $v, w \in H$ we have

$$\langle E_\gamma \pi(g)v, w \rangle = \left(P_\gamma c_{v,w}\right)(g).$$

We see that E_γ is the orthogonal projection of H onto $H(\gamma)$. We conclude

Theorem 14. *Let (π, H) be a Hilbert representation of G then the algebraic sum of the spaces $H(\gamma), \gamma \in \widehat{G}$ is dense in H. Furthermore, if (π, H) is unitary then H is the Hilbert space direct sum of the spaces $H(\gamma), \gamma \in \widehat{G}$.*

2.5 A class of induced representations

Let G be a unimodular, LC group. Let K and P be close subgrps of G s.t. K is cpt. and $G = PK$. Let (σ, H_σ) be a Hilbert representation of P. We assume that it is unitary when restricted to $K \cap P$ (this is no real assumption in light of unitarian trick). Let H_0^σ denote the space of all continuous functions

$$f : G \to H_\sigma$$

such that $f(pg) = \delta(p)^{\frac{1}{2}}\sigma(p)f(g)$ for $p \in Pandg \in G$. We note that if $f \in H_0^\sigma$ and $f_{|K} = 0$ then $f = 0$. We endow H_0^σ with a pre-Hilbert space structure by taking

$$\langle f_1, f_2 \rangle = \int_K \langle f_1(k), f_2(k) \rangle \, dk$$

for $f_1, f_2 \in H_0^\sigma$ (here the inner product inside the integral is that of H_σ). Let H^σ denote the Hilbert space completion of H_0^σ. If $g \in G$ then we define the operator $\pi_\sigma(g)$ on H_0^σ by $\pi_\sigma(g)f(x) = f(xg)$.

Lemma 2.2. *If $g \in G$ then $\pi(g)$ extends to a bounded operator on H^σ. Furthermore, (π_σ, H^σ) defines a Hilbert representation of G which is unitary if (σ, H_σ) is unitary.*

The representation (π_σ, H^σ) is usually denoted $\mathrm{Ind}_P^G(\sigma)$ and called an *parabolically induced representation.*

Remark. If the parabolic subgroup is *minimal* and if the representation σ is *one dimensional* and is given by $p \longmapsto |p|^{i\nu}$ for $\nu \in \mathbb{R}$ then we say that $\mathrm{Ind}_P^G(\sigma)$ is a spherical principal series representation.

Theorem 15 (Frobenius reciprocity)**.**

2.6 C^∞ vectors and infinitesimal character

G is a Lie group with finite number of connected components. Let (π, H) be a Hilbert representation of G. if $v \in H$ is s.t. the function $\Phi(g) = \pi(g)v$ is of class C^∞ then v is called a C^∞ vector or smooth vector.

Proposition 2.3. *If $f \in C_c^\infty(G)$ then $\pi(f)v$ is a smooth vector for (π, H). Particularly, smooth vectors of H is dense in H (using identity approximation).*

Let H^∞ denote the space of all C^∞ vectors. Then we define for $X \in \mathrm{Lie}(G)$

$$d\pi(X)v = \frac{d}{dt}\pi(\exp tX)v_{|t=0}$$

We have

- $d\pi(X)H^\infty \subset H^\infty$ for all $X \in \mathrm{Lie}(G)$, $\pi(g)H^\infty \subset H^\infty$ for $g \in G$.
- $d\pi(aX + bY) = ad\pi(X) + bd\pi(Y),\ a, b \in \mathbb{R},\ X, Y \in \mathrm{Lie}(G)$.
- $d\pi([X, Y]) = d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X)$, for all $X, Y \in \mathrm{Lie}(G)$.
- If $g \in G, X \in \mathrm{Lie}(G)$ then $\pi(g)\pi(X)v = \pi(\mathrm{Ad}(g)X)\pi(g)v$ (compatibility condition).

Hence (π, H^∞) defines a representation of $\mathfrak{g}$, then extends to $U(\mathfrak{g})$ (or $U(\mathfrak{g}_\mathbb{C})$).

Theorem 16. *Let (π, H) be an irr. uni. representation of G then there exists an algebra homomorphism $\eta_\pi : Z_G(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}$ such that $\pi(z)v = \eta_\pi(z)v$ for all $v \in H^\infty$.*

The homomorphism η_π is called the infinitesimal character of (π, H).
We give H^∞ a semi norm:

$$\text{If } D \in U(\mathfrak{g}) \text{ then we set } p_D(v) = ||\pi(D)v||$$

We give H^∞ the corresponding locally convex topology.

Theorem 17. *(π, H^∞) is a smooth Fréchet representation (i.e. if $v \in H^\infty$ then the map $g \mapsto \pi(g)v$ defines a C^∞ map from G to H^∞) .*

Let (π, H) be a Hilbert representation of G. Then we say that $v \in H$ is an analytic vector for (π, H) if the function

$$g \mapsto \langle \pi(g)v, w\rangle$$

is real analytic for all $w \in H$. We use the notation H^ω for this space.We can also see that H^ω is a representation of $\mathfrak{g}$. The main reason for the introduction of analytic vectors is the following result:

Proposition 2.4. *Let G be connected. If V is a $\mathfrak{g}$-invariant subspace of H^ω, then $Cl(V)$ (in H) is a G-invariant subspace of H.*

We need this result to show that an irr. uni. representation is admissible in Chapter 4, which is a main algebraic result of real reductive group by Harish Chandra.

Dixmier-Malliavan have proved that if (π, H) is a Hilbert representation of G then H^∞ is the span of the space $\pi(f)H$ with f a sm. cpt. supp. function on G. That is,

$$H^\infty = \text{span of } \{\int_G f(g)\pi(g)hdg,\ f \in C_c^\infty(G),\ h \in H\}$$

2.7 Representations of compact Lie groups

I'm too familier with GTM9...So I'll skip more.

Theorem 18 (classification of irr. finite representations of $\mathfrak{g}$)**.**

- If V is an irr., finite dim $\mathfrak{g}$-module then V has a unique highest weight (i.e., maximal weight), which we write as Λ_V. Furthermore, the Λ_V weight space is one dimensional.
- If V and W are irr. finite dim. $\mathfrak{g}$-modules then V and W are equivalent if and only if $\Lambda_V = \Lambda_W$.
- If Λ is a dominant integral linear form on $\mathfrak{h}$ then there exists an irr. finite dim. $\mathfrak{g}$-module, V, such that $\Lambda_V = \Lambda$, and its set of weight $\Pi(\Lambda)$ is permuted by $\mathscr{W}$, and dim $V_\mu =$ dim $V_{\sigma\mu}$ for $\sigma \in \mathscr{W}$.

Then we focus on G a connect cpt. Lie group with maximal torus T. For the rest of this section we will use the notation $\mathfrak{g}$ for the complexification of the Lie algebra of G. We will also write $\mathfrak{h}$ for $\mathfrak{t}_{\mathbb{C}}$. Then $\mathfrak{g}$ is a reductive Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let (π, H) be an irreducible (unitary) representation of G. Then an isotypic component for T is a weight space for $\mathfrak{h}$. We will thus use the notation $H(\mu)$ for the μ weight space and also think of μ as a character of T. Let $G^{\sim}$ be the simply connected covering group of G. Let μ be a dominant integral functional on $\mathfrak{h}$ that is also T-integral. Then there is a representation π of $G^{\sim}$ on $L(\mu)$ whose differential gives the action of $\mathfrak{g}$. Let Z denote the kernel of the covering homomorphism of $G^{\sim}$ onto G. We assert that Z is contained in Ker π. Hence:

Theorem 19. *Let μ be a dominant integral, T-integral form on $\mathfrak{h}$. Then there exists an irreducible unitary representation (π_μ, F^μ) of G whose differential is equivalent to the $\mathfrak{g}$-module $L(\mu)$. Let γ_μ denote the equivalence class of π_μ. Then $G^\wedge = \{\gamma_\mu : \mu$ dominant integral and T-integral $\}$.*

3 Real Reductive Groups

We start from some Lie alg. with simpler structure. The review part before the review refers to Knapp *Lie Groups Beyond an Introdunction* and Bump *Lie Groups* (GTM225).

Definitions: complexification, real form ($W^{i.e.\mathbb{R}} = V \oplus iV$), split real form, compact form (i.e. the Killing form is negative definite), involution, Cartan involution (i.e. $B_\theta(X, Y) = -B(X, \theta Y)$ is positive definite). Our main results are as following: Let $\mathfrak{g}$ be a complex semisimple Lie algebra

- Split real form and compact form are exist.
- let $\mathfrak{u}_0$ be a compact real form of $\mathfrak{g}$, and let τ be the conjugation of $\mathfrak{g}$ w.r.t $\mathfrak{u}_0$. If $\mathfrak{g}$ is regarded as a real Lie alg. $\mathfrak{g}^{\mathbb{R}}$, then τ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$.
- Let θ be a Cartan involution, and let σ be any involution. Then there exists $\varphi \in \mathrm{Int}\mathfrak{g}_0$ such that $\varphi\theta\varphi^{-1}$ commutes with σ.
- Any two Cartan involutions of $\mathfrak{g}_0$ are conjugate via $\mathrm{Int}\mathfrak{g}_0$ (resp. compact real forms, $\mathfrak{g}$).

Thus we can give any real semisimple Lie alg. a decomposition through the Cartan involution (using its eigenvalue). Meanwhile,

Proposition 3.1. *If $\mathfrak{g}_0$ is a real ss. Lie alg., then $\mathfrak{g}_0$ is isomorphic to a Lie alg. of real matrices that is closed under transpose. If a Cartan involution θ of $\mathfrak{g}_0$ has been specified, then the isomorphism may be chosen so that θ is carried to negative transpose.*

(What a relief that we can only focus on matrices). Then we can lift it to a decomposition on the Lie group level. Let G be a semisimple Lie group, let θ be a Cartan involution of its Lie algebra $\mathfrak{g}_0$, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}_0$. Then

- there exists a Lie group automorphism Θ of G with differential θ, and Θ has $\Theta^2 = 1$,
- the subgroup of G fixed by Θ is K which is closed,
- the mapping $K \times \mathfrak{p}_0 \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism onto,
- K contains the center Z of G and K is cpt. iff. Z is finite. When Z is finite, K is a *maximal compact subgroup* of G.

Remark. Unforturnately, not every semisimple Lie group can be realized as a group of matrices.

Iwasawa decomposition. We first give an example of $\mathrm{GL}(n, \mathbb{C})$. The specific construction will be given later. The Borel subgroup B of a (noncompact) Lie group G is a maximal closed and connected solvable subgroup. Let $G = \mathrm{GL}(n, \mathbb{C})$. It is the complexification of $K = U(n)$, which is a maximal compact subgroup. Let T be the maximal torus of K consisting of diagonal matrices with eigenvalues that have absolute value 1 . The complexification $T_{\mathbb{C}}$ of T can be factored as TA, where A is the group of diagonal matrices with eigenvalues that are positive real numbers. Let B be the group of upper triangular matrices in G, and let B_0 be the subgroup of elements of B whose diagonal entries are positive real numbers. Finally, let N be the subgroup of unipotent elements of B. Recalling that a matrix is called unipotent if its only eigenvalue is 1, the elements of N are upper triangular matrices with diagonal entries that are all equal to 1. We may factor $B = TN$ and $B_0 = AN$. The subgroup N is normal in B and B_0, so these decompositions are semidirect products.

Proposition 3.2 (Iwasawa decomposition of $G = \mathrm{GL}(n, \mathbb{C})$)**.** *Notations as above, every element of $g \in G$ can be factored uniquely as bk where $b \in B_0$ and $k \in K$, or as $a\nu k$, where $a \in A$, $\nu \in N$, and $k \in K$. The multiplication maps $N \times A \times K \longrightarrow G$ and $A \times N \times K \longrightarrow G$ are diffeomorphism onto.*

As a beginner of real reductive group theory, I prefer the easier definition. Knapp's version: (or we can say a reductive Lie group) is actually a 4-tuple (G, K, θ, B) consisting of a Lie group G, a compact subgroup K of G, a Lie algebra involution θ of the Lie algebra $\mathfrak{g}_0$ of G, and a nondegenerate, $\mathrm{Ad}(G)$ invariant, θ invariant, bilinear form B on $\mathfrak{g}_0$ such that (Some of the following are not assumptions, just use them directly.)

- $\mathfrak{g}_0$ is a reductive Lie algebra,
- the decomposition of $\mathfrak{g}_0$ into $+1$ and -1 eigenspaces under θ is $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, where $\mathfrak{k}_0$ is the Lie algebra of K,
- $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal under B, and B is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$,
- multiplication, as a map from $K \times \exp \mathfrak{p}_0$ into G, is a diffeomorphism onto, and
- every automorphism $\mathrm{Ad}(g)$ of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ is *inner* for $g \in G$, i.e., is given by some x in $\mathrm{Int}\mathfrak{g}$.

Lots of examples can be found in Knapp *Lie Groups Beyond an Introdunction* (if you have enough patience).

3.1 The definition and basic structure

(Wallach's version) Let $F = \mathbb{R}$ or $\mathbb{C}$. Let $M_n(F)$ denote the space of all $n \times n$ matrices over F. Let $\mathrm{GL}(n, F)$ denote (as usual) the group of all invertible elements of $M_n(F)$. Let $f_1, \ldots, f_m$ be complex polynomials on $M_n(\mathbb{C})$ such that each f_j is *real valued* on $M_n(\mathbb{R})$ and such that the set of simultaneous zeros of the f_j in $\mathrm{GL}(n, \mathbb{C})$ is a subgroup, $G_{\mathbb{C}}$. Then $G_{\mathbb{C}}$ is called an affine algebraic group defined over $\mathbb{R}$. The subgroup, $G_{\mathbb{R}} = G_{\mathbb{C}} \cap \mathrm{GL}(n, \mathbb{R})$ is called the group of *real points*. If in addition, $g^* \in G_{\mathbb{C}}$ (conjugate transpose) for $g \in G_{\mathbb{C}}$ then $G_{\mathbb{C}}$ is called a *symmetric subgroup* of $\mathrm{GL}(n, \mathbb{C})$. We define an automorphism θ of $G_{\mathbb{R}}$ by $\theta(g) = (g^{-1})^*$.

Let $G_{\mathbb{C}}$ be a symmetric subgroup of $\mathrm{GL}(n, \mathbb{C})$ with real points $G_{\mathbb{R}}$. By a *real reductive group* we will mean a *finite covering*, G, of an open subgroup G_0 of $G_{\mathbb{R}}$. Denote $p : G \to G_0$. We will identify the Lie algebra of G with that of $G_{\mathbb{R}}$ (why?). Thus we can define on $\mathfrak{g}$, the Lie algebra of G, an involutive automorphism, θ, given by $\theta(X) = -X^*$. This automorphism is usually called *a Cartan involution*. (I have to admit that polynomials point of view is really convenient... Although I can't complete all the details.)

We can define $B(X,Y) = tr(XY)$, $\langle X, Y \rangle = -B(X, \theta Y)$, then Cartan decomposition for $\mathfrak{g}$ as the same.

Proposition 3.3. *The Lie alg. of a real reductive group is reductive.*

We now study the global structure of a real reductive group, G. We first look at G_0. Set

$$K_0 = \{g \in G_0 : \theta(g) = g\}.$$

Then $K_0 = G_0 \cap O(n)$. Hence it is compact. We can prove that:

Lemma 3.1 (Cartan decomposition)**.** *The map $K_0 \times \mathfrak{p} \to G_0$ given by $k, X \mapsto k \exp X$ is a surjective diffeomorphism. As its corollary, $\theta(G_0) = G_0$, and G_0/K_0 is connected and simply connected.*

Let $\mathfrak{a}$ be a subspace of $\mathfrak{p}$ that is maximal subject to the condition that it is an *abelian* subalgebra of $\mathfrak{g}$. If $H \in \mathfrak{a}$ then since H is self-adjoint H is diagonalizable. Thus $\mathrm{ad}H$ is diagonalizable. That is, $\mathfrak{a}$ acts *semisimply* on $\mathfrak{g}$ under ad. (And we can prove that maximal abelian subalg., $\mathfrak{a}$, is unique up to $\mathrm{Ad}(K_0)$.) If $\mu \in \mathfrak{a}^*$ we set $\mathfrak{g}^\mu = \{X \in \mathfrak{g} : [H, X] = \mu(H)X\}$ for $H \in \mathfrak{a}$. Set $\Phi(\mathfrak{g}, \mathfrak{a}) = \{\mu \in \mathfrak{a}^* : \mu \neq 0 \text{ and } \mathfrak{g}^\mu \neq 0\}$.

$\mathfrak{g}^0$ is θ-invariant. Thus $\mathfrak{g}^0 = \mathfrak{k} \cap \mathfrak{g}^0 \oplus \mathfrak{p} \cap \mathfrak{g}^0$. Now, $\mathfrak{p} \cap \mathfrak{g}^0 = \mathfrak{a}$ by the choice of $\mathfrak{a}$. We set ${}^0\mathfrak{m} = \mathfrak{k} \cap \mathfrak{g}^0$. Then

$$\mathfrak{g} =^0 \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\mu \in \Phi} \mathfrak{g}^\mu$$

let $\mathfrak{n} = \bigoplus_{\mu \in P} \mathfrak{g}^\mu$, $\overline{\mathfrak{n}} = \theta \mathfrak{n}$ where P is a set of positive roots. Obviously, $\mathfrak{n}$ is nilpotent, $\mathfrak{n} \oplus \mathfrak{a}$ is solvable, and $[\mathfrak{n} \oplus \mathfrak{a}, \mathfrak{n} \oplus \mathfrak{a}] = \mathfrak{n}$. Since the decomposition:

$$\begin{aligned} X &= H + X_0 + \sum_{\mu \in \Phi} X_\mu \\ &= \left(X_0 + \sum_{\mu \in P} (X_{-\mu} + \theta X_{-\mu}) \right) + H + \left(\sum_{\mu \in P} (X_\mu - \theta X_{-\mu}) \right) \\ &\in \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}. \end{aligned}$$

We now give the Iwasawa decomposition of group. Let N_1 and let A_1 be respectively the connected subgroups of G_0 corresponding to $\mathfrak{n}$ and $\mathfrak{a}$. Then the map $A_1 \times N_1 \times K_0 \to G_0$ given by $a, n, k \mapsto nak$ is a surjective diffeomorphism.

Transfer everythin to G. Set $K = p^{-1}(K_0)$ and $A,\ N$. Then

Theorem 20. *Both the map $\mathfrak{p} \times K \to G$ given by $X, k \mapsto \exp Xk$ and $A \times N \times K \to G$ given by $a, n, k \mapsto ank$ are surj. diffeomorphisms.*

Then we try to define the *Weyl* group. Fix P, let $\mu \in P$. Let $X \in \mathfrak{g}^\mu$ be such that $\langle X, X \rangle = 1$. Then

$$[X, \theta X] = -H_\mu \text{ where } B\left(H, H_\mu\right) = \mu(H),\ \forall H \in \mathfrak{a}$$

Thus if $x = (2/\mu\left(H_\mu\right)) X,\ y = -\theta X,\ h = (2/\mu\left(H_\mu\right)) H_\mu$. Then x, y, h spans a TDS over $\mathbb{R}$. There is thus a Lie homomorphism, σ, of $\mathrm{SL}(2, \mathbb{R})$ into G_0 which compatible with cartan involution (i.e $\sigma\left(g^*\right) = (\theta\sigma(g))^{-1}$). Let k be the image of

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

under σ. Then it is *easily* checked that if s_μ is defined by $s_\mu H = H - B(H, h)H_\mu$ for $H \in \mathfrak{a}$ then $\mathrm{Ad}(k)H = s_\mu H$. Let $N(\mathfrak{a}) = \{u \in K^0 : \mathrm{Ad}(u)\mathfrak{a} = \mathfrak{a}\}$. Set $W(\mathfrak{g}, \mathfrak{a}) = \{\mathrm{Ad}(u)|_{\mathfrak{a}} \mid u \in N(\mathfrak{a})\}$. Then for all $\mu \in P$, $s_\mu \in W(\mathfrak{g}, \mathfrak{a})$.

Let $\mathfrak{a}'$ be the set of all $H \in \mathfrak{a}$ s.t. $\mu(H)$ is non-zero for all $\mu \in \Phi$. A connected component of $\mathfrak{a}'$ is called a *Weyl chamber* of $\mathfrak{a}$. If C is a Weyl chamber then the set of all $\mu \in \Phi$ such that μ is positive on C, denoted P_C, is called a system of positive roots. If$\mu \in P$ and if μ cannot be written as a sum of two elements of P then μ is called *simple* in P. Then as usual, $W(\mathfrak{g}, \mathfrak{a})$ is generated by the s_μ for μ simple, and $W(\mathfrak{g}, \mathfrak{a})$ acts transitively on the Weyl chambers.

3.2 Parabolic pairs (P_F, A_F)

Let $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}$ be the center of $\mathfrak{g}$. Then $\theta(\mathfrak{z}) = \mathfrak{z}$. Hence $\mathfrak{z} = \mathfrak{k} \cap \mathfrak{z} \oplus \mathfrak{p} \cap \mathfrak{z}$ (Meanwhile, $Z(G) = (K \cap Z(G))\exp(\mathfrak{p} \cap \mathfrak{z})$). We set $\mathfrak{s} = \mathfrak{p} \cap \mathfrak{z}$. Then $\mathfrak{s}$ is called a *standard split component* of $\mathfrak{g}$. Here standard is relative to a choice of θ. Set $G^+ = \{g \in G | \operatorname{Ad}(g)|_{\mathfrak{z}} = I\}$. Then it is easily seen that G^+ is a real reductive group in our sense.

We set We set $X(G)$ equal to the set of all continuous homomorphisms of G into $\mathbb{R}^\times$. We set

$$^0G = \left\{g \in G : \chi(g)^2 = 1 \quad \text{for all } \chi \in X(G)\right\}.$$

(I just fix χ as $|\det|$ most of the time.) Put $S = \exp(\mathfrak{s})$. Then S is called a standard split component of G. We have

Lemma 3.2. *The map $S \times {}^0G^+ \to G^+$ given by $s, g \mapsto sg$ is a surjective Lie group iso..*

(I forget whether the notations appeared in the proof are important or not... Just copy them. Prove when $G = G^+$. Set ${}^0\mathfrak{a} = \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]$, ${}^0A = \exp^0 \mathfrak{a}$. Let G^0 be the identity component of G. Let ${}^0\mathfrak{g}$ denote the orthogonal complement to $\mathfrak{s}$ in $\mathfrak{g}$ relative to B. Let G^1 be the connected subgroup of G with Lie algebra ${}^0\mathfrak{g}$.)

Example 1. $G = \mathrm{GSp}(n, \mathbb{R})$. Check that $\mathrm{GSp}(n, \mathbb{R})$ is a real reductive group. Then $S = \{aI | a > 0\}$. $G = G^+$. ${}^0G = \{g \in G \mid gJg^* = \pm J\}$.

Let $\mathfrak{a}, \Phi = \Phi(\mathfrak{g}, \mathfrak{a})$, etc as usual. Let $\mathfrak{m} = \{X \in \mathfrak{g} \mid [X, \mathfrak{a}] = 0\}$ $(\mathfrak{g}^0)$. Set M equal to the set of all $g \in G$ such that $\operatorname{Ad}(g)$ is I on $\mathfrak{a}$. Then M is a real reductive group. The standard split component of M is A since $\mathfrak{m} = \mathfrak{a} \oplus {}^0\mathfrak{m}$. It is also clear that ${}^0M = M \cap K$. Let $\mathfrak{t}$ (torus!) be a maximal abelian subalgebra of ${}^0\mathfrak{m}$. Set $\mathfrak{h}_0 = \mathfrak{t} \oplus \mathfrak{a}$. Set $\mathfrak{h}$ equal to the complexification of $\mathfrak{h}_0$.

- $\mathfrak{h}$ is a Cartan subalg. of $\mathfrak{g}_{\mathbb{C}}$.
- $\Phi(\mathfrak{g}, \mathfrak{a}) = \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})|_{\mathfrak{a}} - 0$.(since $\mathfrak{h}_{\mathbb{R}} = (i\mathfrak{t} + \mathfrak{a}) \cap [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$?)
- Let H_1 be an element of $\mathfrak{a}' \cap \mathfrak{h}_{\mathbb{R}}$ ($\mathfrak{a}' = \{H \in \mathfrak{a} \mid \mu(H) \neq 0,\ \mu \in \Phi(\mathfrak{g}, \mathfrak{a})\}$). Let $H_1, \ldots, H_r$ be a basis of $\mathfrak{h}_{\mathbb{R}}$. Let R denote the corresponding positive root system. Let R_0 be the set of all $\mu \in \Phi(\mathfrak{g}, \mathfrak{a})$ such that $\mu(H_1) > 0$. Then R_0 is a system of positive roots for $\Phi(\mathfrak{g}, \mathfrak{a})$. Then it is clear that $R|_{\mathfrak{a}} - \{0\} = R_0$. Let Δ (resp. Δ_0) be the corresponding system of *simple* roots for R (resp. R_0). Set $F_0 = \{\alpha \in \Delta | \alpha|_{\mathfrak{a}} = 0\}$. Then

$$(\Delta - F_0)|_{\mathfrak{a}} = \Delta_0$$

- Δ_0 is a linearly independent subset of $\mathfrak{a}^*$.

Let F be a subset of Δ_0. We set $\mathfrak{a}_F = \{H \in a \mid \mu(H) = 0 \text{ for } \mu \in F\}$. Set $\mathfrak{m}_F = \{X \in \mathfrak{g} \mid [X, \mathfrak{a}_F] = 0\}$. Put $M_F = \{g \in G \mid \mathrm{Ad}(g)H = H \text{ for } H \in \mathfrak{a}_F\}$ and $A_F = \exp \mathfrak{a}_F$. Then M_F is a real reductive group and relative to θ the split component of M_F is A_F. Let R_F be the subset of those roots in R_0 whose restriction to $\mathfrak{a}_F$ is non-zero. Set

$$\mathfrak{n}_F = \bigoplus_{\mu \in R_F} \mathfrak{g}^\mu.$$

Let N_F denote the connected subgroup of G with Lie alg. $\mathfrak{n}_F$. Then it is a nilpotent subalg.. Set $P_F = M_F N_F$. Then P_F is called a standard parabolic subgroup of G. The word standard has to do with the choices of $\mathfrak{a}$ and R_0. The pair (P_F, A_F) will be called a parabolic pair (p-pair). Lemma 3.2 implies that under the multiplication mapping M_F is isomorphic with $A_F \times {}^0M_F$. We have

Proposition 3.4. *The map $M_F \times N_F \to P_F$ given by $m, n \mapsto mn$ is a surjective diffeomorphism. Particularly, ${}^0M_F \times A_F \times N_F \to P_F$ is a surjective diffeomorphism as well.*

The decomposition in the cor. is called a *Langlands decomposition* of P_F. $P_\varnothing$ is called a minimal parabolic subgroup of G.

We say that a real reductive group is of *inner type* if $\mathrm{Ad}(G)$ is a subgroup of $\mathrm{Int}\,(\mathfrak{g}_\mathbb{C})$.

Lemma 3.3. *Let G be a real reductive group of inner type. Let (P_F, A_F) be as above. Then M_F is a real reductive group of inner type.*

(Set $K_F = K \cap M_F$. Then $M_F = K_F(M_F)^0$. It is enough to show that $\mathrm{Ad}(K_F)$ is contained in $\mathrm{Int}((\mathfrak{m}_F)_\mathbb{C})$.)

$\mathrm{GL}(n, \mathbb{R})$ case is important. $\mathfrak{g} = M_n(\mathbb{R})$. We take $\mathfrak{a}$ to be the diagonal matrices. Let $E_{i,j}$ be the matrix with 1 in the i, j position and 0's everywhere else. If H is the diagonal matrix with $h_1, \ldots, h_n$ on the main diagonal we set $\varepsilon_j(H) = h_j$. Then $\Phi(\mathfrak{g}, \mathfrak{a}) = (\varepsilon_i - \varepsilon_j \mid i \neq j\}$. We take $R_0 = \{\varepsilon_i - \varepsilon_j \mid i < j\}$. $\Delta_0 = \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n$. If $m_1, \ldots, m_p$ are positive integers adding up to n then we set $P\,(m_1, \ldots, m_p)$ equal to the subgroup of all matrices in the following block form: First we write every matrix in the form $[A_{i,j}]$ with $A_{i,j}$ an $m_i \times m_j$ matrix. Then the form of the elements of $P\,(m_1, \ldots, m_p)$ is $A_{i,j} = 0$ for $i > j$. This describes all standard parabolic subgroups of $\mathrm{GL}(n, \mathbb{R})$.

Then we talk about the Bruhat decomposition, which gives us a convenient method to construct all the parabolic pairs (Assign different).

Remark. I'll share a result which can be deduced from Bruhat decomposition and impressed me. The *Borel-Weil* theorem realizes an irreducible representation of a compact Lie group or its complexification as an action on the space of sections of a holomorphic line bundle on the flag variety. More precisely, with a delicately constructed $\mathcal{L}_\lambda$,

Theorem 21. *The space $\Gamma(\mathcal{L}_\lambda)$ is zero unless λ is dominant. If λ is dominant, then $\Gamma(\mathcal{L}_\lambda)$ is irreducible as a G-module, with highest weight λ.*

Recall that M equal to the set of all $g \in G$ such that $\mathrm{Ad}(g)$ is I on $\mathfrak{a}$ (centralizer). Fix $(P_\varnothing, A_\varnothing) = (P, A)$. Let $N_G(A) = \{g \in G \mid \mathrm{Ad}(g)\mathfrak{a} = \mathfrak{a}\}$ (normalizer). Set $W(G, A) = N_G(A)/M$. We look upon $W(G, A)$ as a group of linear automorphisms of $\mathfrak{a}$. Then if G is of inner type then $W(G, A) = W(\mathfrak{g}, \mathfrak{a})$.

Theorem 22. *Assume that G is of inner type. Then G is the disjoint union of the sets Ps^*P, for $s \in W(G, A)$ and s^* as a representative element in $N_G(A)$.*

Remark. In $\mathrm{SL}_2(R)$ case, $|W(G, A)| = 2$. Hence the Bruhat decomposition will be:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}$$

I've been used it in constructing $\mathrm{GL}_2(\mathbb{F}_q)(\mathrm{SL}_2(\mathbb{F}_q))$ parabolically induced representation once. It's worthwhile to try somthing much more non-trivial.

We will now apply this result to prove the *Gelfand-Naimark* decomposition. Let F be a subset of Δ_0 and let (P_F, A_F) be the corresponding p-pair. Let $P_F = M_F N_F$ as usual.

Proposition 3.5. *Assume that $G = G^+$ (i.e. $\mathrm{Ad}(g)|_{\mathfrak{z}} = I,\ \forall g \in G$). The map of $\theta(N_F) \times P_F$ to G given by $x, p \mapsto xp$ defines a diffeomorphism onto an open subset of G whose complement has measure 0 relative to dg.*

3.3 Cartan subalg. (subgroup) and their classification

We define the polynomials D_j on $\mathfrak{g}$ by

$$\det(tI - \mathrm{ad}\, X) = \sum t^j D_j(X).$$

Let l be the dimension of *a* Cartan subalgebra of $\mathfrak{g}_C$ (why this is well defined?). Then $D_k = 0$ for $k < l$. We set $D = D_l$. Then D is a non-zero polynomial function on $\mathfrak{g}$. Set $\mathfrak{g}' = \{X \in \mathfrak{g} \mid D(X) \neq 0\}$. Then $\mathfrak{g}'$ is *open* and *dense* in $\mathfrak{g}$ (w.r.t Zariski topology). Let $\mathrm{Int}(\mathfrak{g})$ denote the group of automorphisms of $\mathfrak{g}$ generated by the automorphisms of the form $\exp(\mathrm{ad}\,X)$ for $X \in \mathfrak{g}$. As is well known $\mathrm{Int}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})^0$. If $X \in \mathfrak{g}$ is such that $\mathrm{ad}X$ is a semisimple endomorphism of g_C then we say that X is *semisimple*. By classical Lie alg. theory,

Lemma 3.4. *If $X \in \mathfrak{g}'$ then X is semisimple, $C_{\mathfrak{g}}(X) = \{Y \in \mathfrak{g} \mid [X, Y] = 0\}$ is a Cartan subalgebra of $\mathfrak{g}$. If X is a semisimple element of $\mathfrak{g}$ then $C_{\mathfrak{g}}(X)$ is a reductive subalgebra of $\mathfrak{g}$ that contains a Cartan subalgebra.*

Remark. We can define a regular element of a Lie alg. or Lie group as an element whose centralizer has dimension as small as possible. it corresponds to the initial definition (why?). When char $\mathbb{F} = 0$, the smallest dimension is the rank of $\mathfrak{g}$ or G. In the special case of $\mathfrak{gl}_n(\mathbb{F})$, a regular element is an element whose geometric multiplicity of each eigenvalue is 1 (i.e. Jordan normal form contains a single Jordan block for each eigenvalue). Take it as an linear algebra exercise.

Since the main results are mentioned in the preface part of section 3, I'll only list the new definitions (which will occur later). Fix *a* Cartan involution θ, of $\mathfrak{g}$. Let B be a non-degenerate θ and $\mathfrak{g}$ invariant form on $\mathfrak{g}$ such that $\langle X, Y\rangle = -B(X, \theta Y)$ defines an inner product on $\mathfrak{g}$. We say that θ *is associated with* B. Then

- If θ_1 is another Cartan involution of $\mathfrak{g}$ that is associated with B then there exists $x \in \mathrm{Int}(\mathfrak{g})$ so that $x\theta x^{-1} = \theta_1$.
- Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then there exists $x \in \mathrm{Int}(\mathfrak{g})$ such that $x\mathfrak{h}$ is θ-invariant.

(They can help us use better structures without extra assumptions. the following results are all depends on them.)

Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. Then we say that $\mathfrak{h}$ is a maximally *split* Cartan subalgebra of $\mathfrak{g}$ if $\mathfrak{h} \cap \mathfrak{p}$ is *maximal abelian* in $\mathfrak{p}$. We say that $\mathfrak{h}$ is $fundamental$ if $\mathfrak{h} \cap \mathfrak{k}$ is maximal abelian in $\mathfrak{k}$. Then we claim that fundamental and maximally split Cartan subalgs. exist, and any two fundamental (resp. maximally split) Cartan subalgs. are conjugated under $\mathrm{Int}(\mathfrak{g})$.

Let $\mathfrak{a}$ be maximal abelian in $\mathfrak{p}$. Fix $G = ANK$ an Iwasawa decomposition of G. Then a standard p-pair, (P_F, A_F), is said to be *cuspidal* if ${}^0\mathfrak{m}_F$ $(C(\mathfrak{a}_F))$ has a Cartan subalgebra, $\mathfrak{t}_F$, completely contained in $\mathfrak{k}$. Set $\mathfrak{h}_F = \mathfrak{t}_F + \mathfrak{a}_F$. Then it's a cartan subalg.. Similarly, we define these in the group level. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then a subgroup of the form $C_G(\mathfrak{h}) =$

$\left\{g \in G \mid \operatorname{Ad}(g)|_{\mathfrak{h}} = I\right\}$ will be called a *Cartan subgroup* of G (i.e. exists a cartan subalg. $\mathfrak{h}$). A standard p-pair, (P_F, A_F), is cuspidal if and only if ${}^0M_F$ has a compact Cartan subgroup, T_F. In this case $H_F = T_F A_F$ is a *Cartan subgroup.* Then we claim that discussion of p-pair is fundamental and enough, since

Proposition 3.6 (conjugate theorem)**.**

- Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then there exists a standard, cuspidal, p-pair, (P_F, A_F), and $x \in \operatorname{Int}(\mathfrak{g})$ such that $x\mathfrak{h} = \mathfrak{h}_F$.
- If H a Cartan subgroup of G then there exists a standard cuspidal p-pair, (P_F, A_F), and $g \in G^0$ (i.e. identity component) such that $gHg^{-1} = H_F$.

If H is a Cartan subgroup of G then we call H fundamental (resp. maximally split) if $\mathfrak{h}$ is fundamental (resp. maximally split).

A parabolic subgroup of G is said to be maximal if it is proper (non-trivial) and is not properly contained in any parabolic subgroup of G. The maximal parabolic subgroups of G are conjugate to the subgroups, P_F, with F of the form $\Delta_0 - \{\alpha\}$ with α a simple root.

3.4 Integration fomulas

Recall: Let G be a real reductive group. Fix θ, a Cartan involution, and $G = NAK$, an Iwasawa decomposition of G. Let (P_F, A_F) be a standard p-pair. If $\mu \in (\mathfrak{a}_F)^*$ and if $H \in \mathfrak{a}_F$ we write $a^\mu = \exp \mu(H)$ if $a = \exp H$. We define $\rho_F \in (\mathfrak{a}_F)^*$ by $\rho_F(H) = (1/2)\operatorname{tr}\left(\operatorname{ad} H|_{\mathfrak{n}_F}\right)$.

Lemma 3.5. *Let dn, da, dm be respectively invariant measures on N_F, A_F, ${}^0M_F$. Let dk be the normalized invariant measure on K. Then we can choose an invariant measure dg on G such that*

$$\int_G f(g)dg = \int_{N_F \times A_F \times {}^0M_F \times K_F} f(namk)a^{-2\rho_F} dndadmdk,$$

for $f \in C_c(G)$. Also if $u \in C(K)$ then

$$\int_K u(k)dk = \int_{K \times K_F} u\left(k_F k(kg)\right) a(kg)^{2\rho_F} dk_F dk$$

here if $g \in G$ and if $g = nak$, $n \in N$, $a \in A$, $k \in K$ then $a(g) = a$ and $k(g) = k$.

The proof is direct, since

$$\int_G = f(g)dg = \int_{P_F \times K} f(pk)dpdk$$

and the Jacobian of the action $n \mapsto ana^{-1}$, is

$$\det(\mathrm{Ad}(a)|_{\mathfrak{n}_F}) = \det(\exp(\mathrm{ad}H|_{\mathfrak{n}_F})) = \exp(\mathrm{tr}(\mathrm{ad}H|_{\mathfrak{n}_F})) = a^{2\rho_F}$$

Assume G is of inner type. Let R be the system of positive roots for $\Phi(\mathfrak{g}, \mathfrak{a})$ corresponding to the choice of $\mathfrak{n}$. Set $\mathfrak{a}^+$ equal to the Weyl chamber corresponding to R. Set $A^+ = \exp(\mathfrak{a}^+)$. If $a \in A, a = \exp H$, we set $\gamma(a) = \Pi_{\alpha \in R} \sinh(\alpha(H))$.

Lemma 3.6. *dg can be normalized so that*

$$\int_G f(g)dg = \int_{K \times A^+ \times K} \gamma(a) f(k_1 a k_2)\, dk_1 da dk_2$$

Remark. In KAK decomposition, $a \in A$ is uniquely determined up to conj. by $W(G, A)$.

Proposition 3.6 implies that there exist θ-stable Cartan subalgebras $\mathfrak{h}_1, \ldots,$ $\mathfrak{h}_r$ that are mutually non-conjugate and such that every Cartan subalgebra of $\mathfrak{g}$ is conjugate to one of them (relative to $\mathrm{Ad}(G)$). Let $H_1, \ldots, H_r$ be the corresponding Cartan subgroups. Set $N_j = \{g \in G \mid \mathrm{Ad}(g)\mathfrak{h}_j = \mathfrak{h}_j\}$. Then $W_j = N_j/H_j$ is a finite group. Let P_j be a system of positive roots for $\Phi\left(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{h}_j)_{\mathbb{C}}\right)$. Set $\pi_j(H) = \Pi_{\alpha \in P}\alpha(H)$ for$H \in \mathfrak{h}_j$. Let D be as usual. Then $|D(H)| = |\pi_j(H)|^2$ (consider the degree and the possible roots). Since G and each H_j is unimodular, each coset space G/H_j has a G-invariant measure, dx_j.

Proposition 3.7 (Weyl integral formula for $\mathfrak{g}$)**.** *There exist positive constants $c_j, j = 1, \ldots, r$ and normalizations of Lebesgue measure on $\mathfrak{g}$ and the $\mathfrak{h}_j$ such that*

$$\int_{\mathfrak{g}} f(X)dX = \sum c_j \int_{h_j} |D(H)| \left(\int_{G/H_j} f(\mathrm{Ad}\, xH) dx_j \right) dH, \qquad \textit{for } f \in C_c(\mathfrak{g})$$

Only need to notice:

$$d\mu_{gH,h}(X, Z) = \mathrm{Ad}(g)(\mathrm{ad}\, Xh + Z)$$

for fixed j, $\mu : G/H \times (\mathfrak{h} \cap \mathfrak{g}') \to \mathfrak{g}$ which is defined by $\mu(gH, h) = \mathrm{Ad}(g)h$ and $X \in \mathfrak{n}^+ + \mathfrak{n}^-, Z \in \mathfrak{h}$. Hence the Jacobian of μ at (gH, h) is $|D(h)|$.

We now derive the Weyl integral formula for G. On the level of Lie algebras, we have concentrated on eigenvalue 0 for ad$X$. On the level of reductive Lie groups, the analogous procedure is to concentrate on eigenvalue 1 for $\mathrm{Ad}(g)$. We define d_j on G by

$$\det(tI - (\mathrm{Ad}\, g - I)) = \sum t^j d_j(g)$$

Here $n = \dim G$. Set $d = d_j$ for $j = \mathrm{rank}\,(\mathfrak{g}_{\mathbb{C}})$. We set $G' = \{g \in G \mid d(g) \neq 0\}$. Then G' is open, dense with complement of measure 0 in G.

Remark. The similar calculation gives us: Let $\delta \in \mathfrak{h}^*$ be half the sum of the elements of R, then

$$|t^\delta \Pi_{\alpha \in R}(1 - t^{-\alpha})|^2 = d(t)$$

Proposition 3.8. *There exist positive constants m_j so that if dg and dh_j are respectively invariant measure on G and H_j, then for $f \in C_c(G)$*

$$\int_G f(g)dg = \sum m_j \int_{H_j} |d(h_j)| \left(\int_{G/H_j} f\left(gh_jg^{-1}\right) d\left(gH_j\right) \right) dh_j.$$

Only need to notice:

$$d\sigma_{gH,h}(X, Z) = \left(\mathrm{Ad}(g)\left(\left(\mathrm{Ad}\left(h^{-1}\right) - I\right) X + Z\right)\right)$$

for fixed j, $\sigma : G/H \times (H \cap G') \to G$ which is defined by $\sigma(gH, h) = ghg^{-1}$ and $X \in \mathfrak{n}^+ + \mathfrak{n}^-, Z \in \mathfrak{h}$. Hence the Jacobian of σ at (gH, h) is $|d(h)|$.

Remark. Using character function, we have $m_j = |W(H_j : G)|^{-1}$.

We now derive the integration formula that are related to the Gelfand-Naimark decomposition. We set $V_F = \theta N_F$. Fix invariant measures dn, dm,d a, dv respectively on $N_F, {}^0M_F, A_F$ and V_F.

Lemma 3.7. *The invariant measure dg can be normalized so that*

$$\int_G f(g)dg = \int_{N_F \times {}^0M_F \times A_F \times V_F} a^{-2\rho_F} f(nmav) dndmdadv$$

for $f \in C_c(G)$. If $u \in C(K)$ then

$$\int_K u(k)dk = \int_{K_F \times V} a(v)^{2\rho_F} u\left(k_F k(v)\right) dk_F dv.$$

Remark. I never check the last one.

3.5 Weyl (integral, character, dimension) formula

Let G be a cpt. Lie group. It is a real reductive group (just accept it).Assume that G is connected. Let T be a maximal torus of G. Then T is the only Cartan subgroup of G up to conjugacy (w.r.t. G). Let $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$. Fix R a system of positive roots for $\Phi\left(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}\right)$. Define δ as usual. Fix $\langle\ ,\ \rangle$ an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ as usual, and $(\ ,\ )$. Let Δ be the simple root system of R. First, we have

$$2(\delta, \alpha)/(\alpha, \alpha) = 1 \quad \forall \alpha \in \Delta$$

This implies that δ is dominant integral. Hence there is a finite covering $G^{\sim}$ of G so that if $T^{\sim}$ is the corresponding maximal torus then δ is $T^{\sim}$ integral. Define on $T^{\sim}$ by $\Delta(t) = t^{\delta}\Pi_{\alpha\in R}\left(1 - t^{-\alpha}\right)$.

Proposition 3.9 (Weyl integral formula)**.** *Let dg and dt be normalized invariant measure on G and T respectively. Then*

$$\int_G f(g)dg = (1/W(G,T)) \int_T |\Delta(T)|^2 \int_G f\left(gtg^{-1}\right) dgdt$$

We give a example here. Let $G = \mathrm{U}(n)$, T be the diagonal torus. Writing $t = \operatorname{diag}\{t_1, ..., t_n\} \in T$ and dt normalized. Since $N(T)/T \cong S_n$, we have

$$\int_G f(g)dg = \frac{1}{n!}\int_T f\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \prod_{i<j} |t_i - t_j|^2 \, dt$$

If $\mu \in T^{\wedge}$ we set $A(\mu)(t) = \Sigma_{s\in W}\det(s)t^{s\mu}(W = W(G,T))$. We say that μ is regular if $s\mu \neq \mu$ for $s \in W - \{1\}$. It is easy to see that $A(\mu) = 0$ if μ is not regular (symmetric property). If μ is regular than there exists $s \in W$ such that $s\mu$ is *dominant integral*. Let μ and β be integral, dominant integral and regular then

- $\int_T A(\mu)(t)\operatorname{conj}(A(\beta))(t)dt = w\delta_{\mu,\beta}$.
- $\Delta = A(\delta)$.
- (Weyl character formula) Let $\gamma \in G^{\wedge}$ and let Λ be the highest weight of γ relative to R. Let χ_γ be the character of γ, i.e.

  $$\chi_\gamma = \sum_{\alpha\in R}(\mathrm{mult}_\gamma \alpha)e^{\alpha}$$

  Then

  $$A(\delta)\chi_\gamma = A(\Lambda + \delta)$$

- (Weyl dimension formula) Let γ and Λ as usual. Then

$$d(\gamma)=\prod_{\alpha\in R}(\Lambda+\delta,\alpha)/(\alpha,\alpha).$$

Remark. Let $\mathcal{E}(R)$ denote the free R-module on the set of symbols $\left\{e^\lambda \mid \lambda\in\Lambda\right\}$. It consists of all formal sums $\sum_{\lambda\in\Lambda} n_\lambda e^\lambda$ with $n_\lambda\in R$ such that $n_\lambda=0$ for all but finitely many λ. It is a ring with the multiplication

$$\left(\sum_{\lambda\in\Lambda} n_\lambda\cdot e^\lambda\right)\left(\sum_{\mu\in\Lambda} m_\mu\cdot e^\mu\right)=\sum_{\nu\in\Lambda}\left(\sum_{\lambda+\mu=\nu} n_\lambda m_\mu\right)\cdot e^\nu.$$

This makes sense because only finitely many n_λ and only finitely many m_μ are nonzero. Of course, $\mathcal{E}(R)$ is also the group algebra over R of Λ. The Weyl group acts on $\mathcal{E}(R)$, and we will denote by $\mathcal{E}(R)^W$ the subring of W-invariant elements. Usually, we are interested in the case $R=\mathbb{Z}$, and we will denote $\mathcal{E}=\mathcal{E}(\mathbb{Z})$, $\mathcal{E}^W=\mathcal{E}(\mathbb{Z})^W$.

Remark. The proof in Wallach's book is standard using Weyl integral formula (which is the same as Knapp, *representation theory of semisimple groups* Vol.1). Humphreys GTM9(also an excellent textbook by Yucai Su) gives a readable proof through the language of group algebra using the most basic method (by count the multiplicity).

4 The Basic Theory of $(\mathfrak{g},K)$-Modules

4.1 $\mathrm{Res}_{\mathfrak{p}/\mathfrak{a}}(P(\mathfrak{p}))^K\cong P(\mathfrak{a})^W,\ \mathrm{Res}_{\mathfrak{g}/\mathfrak{h}}(I(\mathfrak{g}))\cong I(\mathfrak{h})$

Let G be a real reductive group. Let θ be a Cartan involution for G and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the corresponding Cartan decomposition. Let $K=p^{-1}(G_0\cap O(n))$. If V is a real vector space then we denote by $P(V)$ the space of complex valued polynomial functions on V. Let K act on $P(\mathfrak{p})$ by $kf(X)=f\left(\mathrm{Ad}\left(k^{-1}\right)X\right)$ for $k\in K$, $X\in\mathfrak{p}$ and $f\in P(\mathfrak{p})$. We denote by $P(\mathfrak{p})^K$ the space of all $f\in P(\mathfrak{p})$ such that $kf=f$ for all $k\in K$.

Let $\mathfrak{a}$ be as usual (i.e. a subspace of $\mathfrak{p}$ that is maximal subject to the condition that it is an abelian subalgebra of $\mathfrak{g}$). Denote $W=W(\mathfrak{g},\mathfrak{a})$ as $\{\mathrm{Ad}(u)|_{\mathfrak{a}}\mid u\in N(\mathfrak{a})\}$. Let W act on $P(\mathfrak{a})$ by $sf(H)=f\left(s^{-1}H\right)$ for $s\in W$, $H\in\mathfrak{a}$, $f\in P(\mathfrak{a})$. Let $P(\mathfrak{a})^W$ denote the set of all $f\in P(\mathfrak{a})$ such that $sf=f$ for all $s\in W$. Define for $f\in P(V)$, $\mathrm{Res}_{V/W}(f)$ to be the restriction of f to $W\subset V$ where W and V are both vector space.

Theorem 23. *Assume that G is of inner type. Then* $\operatorname{Res}_{\mathfrak{p}/\mathfrak{a}}$ *is an algebra isomorphism of* $P(\mathfrak{p})^K$ *onto* $P(\mathfrak{a})^W$.

Since the proof is classic and instructive, I'll list the main steps here.

- $\operatorname{Res}_{\mathfrak{p}/\mathfrak{a}}\left(P(\mathfrak{p})^K\right)$ is contained in $P(\mathfrak{a})^W$.
- $\operatorname{Res}_{\mathfrak{p}/\mathfrak{a}}$ is injective on $P(\mathfrak{p})^K$ (Since any two $\mathfrak{a}$ are conjugate w.r.t. $\operatorname{Ad}(K_0)$).
- $H_j \in a, j=1,2$ and if $WH_1 \cap WH_2 = \varnothing$ then there exists a continuous function f on $\mathfrak{p}$ such that $f(\operatorname{Ad}(k)X) = f(X)$ for all $k \in K$ and $X \in \mathfrak{p}$ and $f(H_1)=0, f(H_2)=1$.
- Let $H_j, j=1,2$ be the same as above. Then there exists $p \in P(\mathfrak{p})^K$ such that $p(H_1) \neq p(H_2)$.
- Let F denote the quotient field of $P(\mathfrak{a})$. Let L be the quotient field of $J = \operatorname{Res}_{\mathfrak{p}/\mathfrak{a}}\left(P(\mathfrak{p})^K\right)$ Then we see that $L = \{f \in F : \sigma f = f \text{ for all } \sigma \in \operatorname{Gal}(F/L)\}$, hence $\sigma P(\mathfrak{a})$ is contained in $P(\mathfrak{a})$ for all $\sigma \in \operatorname{Gal}(F/L)$.

Then comes our final approach. Denote by U the group of all automorphisms of $P(\mathfrak{a})$ that are equal to I on J. Then we have shown that $J = \{f \in P(\mathfrak{a}) \mid \sigma f = f \text{ for all } \sigma \in U\}$. If $\sigma \in U$ and if $H \in \mathfrak{a}$ then $\delta(f) = (\sigma f)(H)$ defines a homorphism of $P(\mathfrak{a})$ into $\mathbb{C}$. Then there exists H_1 such that $\delta(f) = f(H_1)$ for all $f \in P(\mathfrak{a})$ (why?). Now, $\sigma f = f$ for $f \in J$, so there exists $s \in W$ such that $H_1 = sH$. We have therefore shown that if $f \in P(\mathfrak{a})^W$ then $\sigma f = f$ for all $\sigma \in U$. Hence $P(\mathfrak{a})^W$ is contained in $J = \operatorname{Res}_{\mathfrak{p}/\mathfrak{a}}\left(P(\mathfrak{p})^K\right)$.

Remark. If we define $N_K(A) = \{k \in K \mid \operatorname{Ad}(k)\mathfrak{a} = \mathfrak{a}\}$ and $W = N_K(A)/{}^0M$ then the conclusion of the above theorem is still true for G which is not necessarily of inner type .

The Chevalley restriction theorem can be seen in a different perspective. Let $\mathfrak{g}$ be a reductive Lie alg. over $\mathbb{C}$. We define an action of $\mathfrak{g}$ on $P(\mathfrak{g})$ by $Xf(Y) = d/dt_{t=0} f(\exp(-t\operatorname{ad} X)Y)$. Set $I(\mathfrak{g}) = \{f \in \mathfrak{g} \mid Xf = 0 \text{ for all } X \in \mathfrak{g}\}$ (It's sort of equivalent to invariant under Int $\mathfrak{g}$). Let $W = W(\mathfrak{g}, \mathfrak{h})$. We let W act on $P(\mathfrak{h})$ by $sf(H) = f(s^{-1}H)$. Let $I(\mathfrak{h})$ denote the W-invariants in $P(\mathfrak{h})$.

Set $\mathfrak{g}_u$ be a compact form of $\mathfrak{g}$ such that $\mathfrak{g}_u \cap \mathfrak{h}$ is maximal abelian in $\mathfrak{g}_u$. Let θ denote the conjugation relative to $\mathfrak{g}_u$. Then θ is a Cartan involution of $\mathfrak{g}$. Then we set $\mathfrak{k} = \mathfrak{g}_u$ and $\mathfrak{p} = i\mathfrak{g}_u$ as the corresponding Cartan decomposition. Hence $\operatorname{Res}_{\mathfrak{g}/\mathfrak{p}}$ is an isomorphism of $I(\mathfrak{g})$ onto $P(\mathfrak{p})^K$, $\operatorname{Res}_{\mathfrak{h}/\mathfrak{h}\cap\mathfrak{p}}$ is an isomorphism of $I(\mathfrak{h})$ onto $P(\mathfrak{h} \cap \mathfrak{p})^W$. Then

Theorem 24. $\mathrm{Res}_{\mathfrak{g}/\mathfrak{h}}$ *is an isomorphism of* $I(\mathfrak{g})$ *onto* $I(\mathfrak{h})$.

Remark. The collection of all Sym λ^k, $\lambda \in \Lambda, k \in \mathbb{Z}^+$ spans $I(\mathfrak{h})$, and $I(\mathfrak{g})$ can be constructed via representation theory. For a irr. finite dim. representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$,

$$x \mapsto \mathrm{Tr}(\phi(x)^k)$$

To prove $\mathrm{Res}_{\mathfrak{g}/\mathfrak{h}} : I(\mathfrak{g}) \hookrightarrow I(\mathfrak{h})$, we only need to prove that $\mathrm{Res}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{Sym}\ \lambda^k)$ lies in the combination of the form $x \mapsto \mathrm{Tr}(\phi(x)^i)$. And $\mathrm{Res}_{\mathfrak{g}/\mathfrak{h}} : I(\mathfrak{g}) \twoheadrightarrow I(\mathfrak{h})$ is equivalent to prove that the set of all regular semisimple elements is dense w.r.t. Zariski topology.

4.2 $\gamma(Z(\mathfrak{g})) \cong U(\mathfrak{h})^W$

I'll change the order a little bit (which I think is more comfortable).

For $z \in Z(\mathfrak{g})$ and $h \in \mathfrak{h}$, we have

$$h \cdot \left(z \cdot v^+\right) = z \cdot \left(h \cdot v^+\right) = z \cdot \left(\lambda(h) v^+\right) = \lambda(h) z \cdot v^+.$$

where M is a (finite dim.) highest weight module which is generated by v^+ of weight λ. Since $\dim M_\lambda = 1$, this forces $z \cdot v^+ = \chi_\lambda(z) v^+$ for some scalar $\chi_\lambda(z) \in \mathbb{C}$. In turn, z acts on any elements in M by the same scalar.

For fixed λ, we call the algebra homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ the *central character* associated with λ.

We write $z \in Z(\mathfrak{g})$ as a linear combination of PBW monomials based on the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. We see that $z \cdot v^+$ depends just on the PBW monomials with factors in $\mathfrak{h}$. That is, when denoting $\mathrm{pr} : U(\mathfrak{g}) \to U(\mathfrak{h})$ the projection onto the subspace $U(\mathfrak{h})$, we have

$$\chi_\lambda(z) = \lambda(\mathrm{pr}(z)) \text{ for all } z \in Z(\mathfrak{g}).$$

The restriction of pr to $Z(\mathfrak{g})$, denoted by ξ, is called the $Harish - Chandra$ *homomorphism*.

Remark. To understand more intrinsically why ξ a *homomorphism* (more generally), we need to prove:

- $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+)$. Let q denote the projection of $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ corresponding to this direct sum decomposition. Let $U(\mathfrak{g})^{\mathfrak{h}}$ be centralizer of $U(\mathfrak{h})$.
- $U(\mathfrak{g})^{\mathfrak{h}} \cap (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+) = U(\mathfrak{g})^{\mathfrak{h}} \cap \mathfrak{n}^- U(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{h}} \cap U(\mathfrak{g})\mathfrak{n}^+$.
- $\forall u_1, u_2 \in U(\mathfrak{g})^{\mathfrak{h}},\ u_1 u_2 \equiv q\left(u_1\right) q\left(u_2\right) (\bmod\ (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+)$.

Let $\rho \in \mathfrak{h}^*$ be half the sum of the elements of R. For $w \in W$ and $\lambda \in \mathfrak{h}^*$, define a shifted action of W (called the *dot action*) by $w \cdot \lambda = w(\lambda + \rho) - \rho$. If $\lambda, \mu \in \mathfrak{h}^*$, we say that λ and μ are linked (or W-linked) if for some $w \in W$, we have $\mu = w \cdot \lambda$. Then linkage is an equivalence relation on $\mathfrak{h}^*$. The orbit $\{w \cdot \lambda \mid w \in W\}$ of λ under the dot action is called the linkage class (or W-linkage class) of λ. By the highest weight results of the finite dim. representation $M(\lambda)$ (uniqueness), we will have

Proposition 4.1. *If $\lambda \in \mathfrak{h}^*$ and μ is linked to λ, then $\chi_\lambda = \chi_\mu$.*

Fix an invariant from, B, on $\mathfrak{g}$ as usual. We define a mapping $X \mapsto X^\#$ of $\mathfrak{g}$ onto $\mathfrak{g}^*$ by $B(Y, X) = X^\#(Y)$ for $Y \in \mathfrak{g}$. Then $X \mapsto X^\#$ induces an algebra isomorphism of $S(\mathfrak{g})$ (i.e. symmetric algebra) onto $P(\mathfrak{g})$. Meanwhile, ad induces an action of $\mathfrak{g}$ on $S(\mathfrak{g})$ as derivations. Under $X \mapsto X^\#$ this corresponds to the action of $\mathfrak{g}$ on $P(\mathfrak{g})$ (which is used to define $I(\mathfrak{g})$). We may thus identify $S(\mathfrak{g})$ and $P(\mathfrak{g})$ as $\mathfrak{g}$-modules.

We define an isomorphism, μ, of $S(\mathfrak{h})(U(\mathfrak{h}))$ given by $\mu(H) = H - \rho(H)$ on $\mathfrak{h}$ and extended to $S(\mathfrak{h})$ by the universal mapping property. Call the composite homomorphism $\psi = \mu \circ \xi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ the *twisted* $Harish-Chandra$ *homomorphism*. Calculate directly, we have

$$\chi_\lambda(z) = (\lambda + \rho)(\psi(z)) \text{ for all } z \in Z(\mathfrak{g})$$

The important observation is that the image of ψ lies in $S(\mathfrak{h})^W$.

Theorem 25. *Let $\psi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ be the homomorphism above.*

- *ψ is an isomorphism of $Z(\mathfrak{g})$ onto $S(\mathfrak{h})^W$.*
- *For all $\lambda, \mu \in \mathfrak{h}^*$, we have $\chi_\lambda = \chi_\mu$ iff. they are W-linked.*
- *Every central character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is of the form χ_λ for some $\lambda \in \mathfrak{h}^*$.*

Remark. The first time I used it was when I tried to obtain several formulas for the characters and multiplicities of finite dimensional $\mathfrak{g}$-modules (GTM9). However, we will use them from a different perspective. The proof of the third dot relies on standard commutative algebra: Note that any $f \in S(\mathfrak{h})$ is a root of $\Pi_{w \in W}(t - wf)$ where the coefficients lie in $S(\mathfrak{h})^W$, so it's an integral extension of the ring of invariants. Meanwhile, any homomorphismm $S(\mathfrak{h}) \to \mathbb{C}$ is given by a point valuation. To porve the first dot, we need to "chase" the following diagram:

$$\begin{tikzcd} S(\mathfrak{g})^G \arrow[r, "\pi"] \arrow[d, "\mathrm{Id}"'] & Z(\mathfrak{g}) \arrow[r, "\psi"] & S(\mathfrak{h})^W \arrow[d, "\mathrm{Id}"] \\ I(\mathfrak{g}) \arrow[rr, "\mathrm{Res}_{\mathfrak{g}/\mathfrak{h}}"] & & I(\mathfrak{h}) \end{tikzcd}$$

In fact, this diagram is not quite commutative, however, it is enough to see that ψ is a bijective. The trick is to see that for the "highest degree", the diagram commutes. $\mathfrak{sl}(2, \mathbb{C})$ is fundamental:

Let $L = \mathfrak{sl}(2, \mathbb{C})$, with standard basis (x, y, h). The dual basis (x^*, y^*, h^*) are identified with $\left(\frac{1}{4}y, \frac{1}{4}x, \frac{1}{8}h\right)$ via the Killing form. If λ is the fundamental dominant weight $\left(\lambda = \frac{1}{2}\alpha\right)$, then λ identifies here with h^*, and $1, \lambda^2$ generates $I(\mathfrak{h})$. λ being the highest weight of the irr. representation of L, an easy trace polynomial calculation shows that $h^{*2} + x^*y^*$ equals $\mathrm{Res}^{-1}_{\mathfrak{g}/\mathfrak{h}}(\lambda^2)$. Under the identification, this becomes the symmetric tensor $(1/64)(h \otimes h) + (1/32)(x \otimes y + y \otimes x)$. π maps this element to $(1/64)h^2 + (1/32)xy + (1/32)yx \in Z(\mathfrak{g})$. We rewrite this element in the PBW basis as $(1/64)h^2 + (2/32)yx + (1/32)h$ (to calculate $\mathfrak{h}$ component). Hence ξ sends this to $(1/64)\left(h^2 + 2h\right)$, and μ to the W-invariant $(1/64)\left(h^2 - 1\right)$. Reverting to $I(\mathfrak{h})$, this yields $\lambda^2 - 1/64$. Therefore, after "mod (lower degree)", it commutes.

Remark. Example: We look at the case when $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. We take for $\mathfrak{h}$ the space of diagonal matrices. If $H \in \mathfrak{h}$ and if H has diagonal entries $h_1, \ldots, h_n$, then define $\varepsilon_j(H) = h_j$. Then $\Phi(\mathfrak{g}, \mathfrak{h})$ is the set of all $\varepsilon_j - \varepsilon_k$ for distinct j, k. Take Φ^+ to be the set of all $\varepsilon_j - \varepsilon_k$ for $j < k$. We claim that W is the set of all permutations of the diagonal entries, $I(\mathfrak{h})$ is equal to $\mathbb{C}[\sigma_1, \ldots, \sigma_n]$, where σ_j is the j-th elementary symmetric function in the diagonal entries of H that are defined by

$$\prod_{1 \le j \le n} (t + h_j) = \sum t^{n-j} \sigma_j(H)$$

Define for $X \in M_n(\mathbb{C})$ the polynomials p_j by

$$\det(tI + X) = \sum t^{n-j} p_j(X)$$

Then it is clear that $\mathrm{Res}_{\mathfrak{g}/\mathfrak{h}}\, p_j = \sigma_j$. Meanwhile, all p_j generates $I(\mathfrak{g})$.

If A is an associative algebra over $\mathbb{C}$ and if $[a_{j,k}]$ is an $n \times n$ matrix over A, we set

$$\det\left([a_{j,k}]\right) = \sum \mathrm{sgn}(\sigma) \prod_{j=1}^{n} a_{\sigma j, j},$$

the sum over all permutations of n-letters. We take $E_{j,k}$ to be the standard basis of $\mathfrak{gl}(n, \mathbb{C})$ and treat them as an element in $U(\mathfrak{g})$ (which is a differential operator $D_{x_j x_k}$). That is, for a polynomial $f((x)) = f(x_1, \cdots, x_n)$ of n variables x_i, we can write the expansion:

$$f((x + t \cdot y)) = f((x)) + tf_1((x, y)) + \cdots$$

Then denote the coefficient $f_1((x, y))$ as D_{yx}, called the polarized polynomial. Let t be an indeterminate and set $a_{j,k}(t) = D_{x_j x_k} + (j - 1 + t)\delta_{j,k}$. Write $\det([a_{j,k}(t)]) = \Sigma t^{n-j} u_j$. Then by the capelli identity (which is really complicated), we have $u_j \in Z(\mathfrak{g})$. One computes that $\psi(u_j) = \sigma_j$.

4.3 $(\mathfrak{g}, K)$-modules

Everything in this subsection is fundamental and instructive, so I'm not ready to skip too many sentences.

Let V be a $\mathfrak{g}$-module that is also a module for K (for the moment we ignore the topology of K). Then V is called a $(\mathfrak{g}, K)$-module if the following three conditions are satisfied:

(1) $k \cdot X \cdot v = \mathrm{Ad}(k)X \cdot k \cdot v \quad$ for $v \in V, k \in K, X \in \mathfrak{g}$.

(2) If $v \in V$ then Kv spans a *finite* dim. vector subspace of V, W_v, such that the action of K on W_v is continuous.

(3) If $Y \in \mathfrak{k}$ and if $v \in V$ then $d/dt_{t=0} \exp(tY)v = Yv$.

If V and W are $(\mathfrak{g}, K)$-modules then we denote by $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$ the space of all $\mathfrak{g}$-homomorphisms that are *also* K homomorphisms. V and W are said to be equivalent if there is an invertible element in $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$. We denote by $\mathrm{C}(\mathfrak{g}, K)$ the category of all $(\mathfrak{g}, K)$-modules with $\mathrm{Hom}_{\mathfrak{g},K}$ as morphism set.

A $(\mathfrak{g}, K)$-mod V is said to be *finitely generated* if it is finitely generated as a $U(\mathfrak{g})$-module. V is said to be irreducible if the only $\mathfrak{g}$ *and* K invariant subspaces of V are trivial. Then by Schur lemma, for irr. V, $\mathrm{Hom}_{\mathfrak{g},K}(V, W) = \mathbb{C}I$. Let v be a nonzero element of irr. V. Then $U(\mathfrak{g})W_v$ is a $\mathfrak{g}$ and a K-invariant subspace of V. Hence, $V = U(\mathfrak{g})W_v$. In particular, this implies that V is countable dimensional.

Let V be a $(\mathfrak{g}, K)$-module. Let $\gamma \in K^\wedge$. Then we set $V(\gamma)$ equal to the *sum* of all the K-invariant, finite dimensional, subspaces of V that are in the class of γ (called it the γ-isotypic component). The Peter-Weyl theorem implies that as a K-mod,

$$V = \bigoplus_{\gamma \in K^\wedge} V(\gamma)$$

We say that V is *admissible* if $\dim\ V(\gamma) < \infty$ for all $\gamma \in K^\wedge$. Let $\mathscr{H}$ denotes the category of all finitely generated, admissible, $(\mathfrak{g}, K)$-modules.

Lemma 4.1. *Let V be a $(\mathfrak{g}, K)$-module. Then V is admissible if and only if* $\dim_{H_K}(W, V) < \infty$ *for all finite dimensional K-modules, W.*

Let (π, H) be a Hilbert representation of K. Then it's a Hilbert space direct sum of the $H(\gamma)$ for $\gamma \in K^\wedge$. Here we are assuming (again, isn't restrictive) that $\pi|_K$ is unitary. Peter-Weyl theorem also implies that $H(\gamma) \cap H^\infty$ is dense in $H(\gamma)$. We set H_K equal to the algebraic direct sum of the $H(\gamma) \cap H^\infty$ for $\gamma \in K^\wedge$. Clearly, H_K is a dense subspace of H(resp. H^∞). Considering the finiteness and the structure of H^∞ (the compatibility condition), we have

Lemma 4.2. *H_K is a $\mathfrak{g}$-invariant subspace of H^∞. H_K is a $(\mathfrak{g}, K)$-module.*

H_K is called the space of C^∞, K-finite vectors of H or the *underlying* $(\mathfrak{g}, K)$-module of H. We say that H is *admissible* if H_K is admissible. H is said to be *infinitesimally irreducible* if H_K is irreducible as a $(\mathfrak{g}, K)$-module. If (π, H) and (σ, V) are Hilbert representations of G then π is infinitesimally equivalent with σ if the $(\mathfrak{g}, K)$-modules H_K and V_K are equivalent.

Remark. Let $V \in C(\mathfrak{g}, K)$. If $\mu \in V^*$ then we write $X \cdot \mu$ (resp. $k \cdot \mu$) for the functional $X \cdot \mu(v) = -\mu(Xv)$ (resp. $k \cdot \mu(v) = \mu(k^{-1}v)$). Relative to these actions V^* is a $\mathfrak{g}$ and a K-module that satisfies the compatibility condition. We set $V^\sim = \{\mu \in V^* \mid K\mu \text{ spans a finite dimensional subspace}\}$. Hence $V^\sim$ is a $(\mathfrak{g}, K)$-module which is called the $(\mathfrak{g}, K)$-dual module of V.

Remark. We set $V^\#$ equal to the space of all conjugate-linear functionals on V with $\mathfrak{g}$ and K acting on V^* as above. Let V be a subset of $V^\#$ consisting of those elements μ such that $K\mu$ spans a fin. dim. subspace. Then as above V is a $(\mathfrak{g}, K)$-module which is called the conjugate dual $(\mathfrak{g}, K)$-module of V.

Then we explain the relationship between $(\mathfrak{g}, K)$-module and the representation we mentioned before. Denote $Z_G(\mathfrak{g})$ as the subalgebra of $U(\mathfrak{g})$ consisting of those $\mathrm{Ad}(G)$-invariant elements.

Theorem 26. *Let V be a finitely generated $(\mathfrak{g}, K)$-module. If $\gamma \in K^\wedge$ then $V(\gamma)$ is finitely generated as a $Z_G(\mathfrak{g})$-module.*

Proof. Let W be a finite dimensional K-invariant subspace of V such that $V = U(\mathfrak{g})W$. Then $V = \mathrm{symm}(S(\mathfrak{p}))W$ (since $U(\mathfrak{g})$ can be viewed as $U(\mathfrak{k})$ right module which is generated by $\mathrm{symm}(S(\mathfrak{p}))$. Then apply the compatibility condition). We define $V_0 = W$ and $V_{j+1} = \mathfrak{p}V_j + V_j$ for $j = 0, 1, \ldots$. Set $\mathrm{Gr}(V)$ equals to the direct sum of the spaces (V_j/V_{j-1}), here $V_{-1} = (0)$. Then $\mathrm{Gr}(V)$ is equivalent with V as a K-module.

Let p_j be the natural projection of V_j into V_j/V_{j-1}. Define an action of each $X \in \mathfrak{p}$ on $\mathrm{Gr}(V)$ by $Xp_j(v) = p_{j+1}(Xv)$ for $v \in V_j$ (well-defined).

Then we define a new Lie algebra structure on $\mathfrak{p}$ as follows:

If $X, Y \in \mathfrak{k}$ or if $X \in \mathfrak{k}, Y \in \mathfrak{p}$ then $[X, Y]$ has the same meaning as it did in $\mathfrak{g}$. If $X, Y \in \mathfrak{p}$ then $[X, Y] = 0$. We denote by $\mathfrak{g}^C$ the Lie algebra $\mathfrak{k} \oplus \mathfrak{p}$ with commutation relations given as above. Then form a Lie group G^C with total space $K \times p$ and multiplication given by:

$$(k, X)(u, Y) = \left(ku, \mathrm{Ad}\left(u^{-1}\right) X + Y\right), \quad k, u \in K,\ X, Y \in \mathfrak{p}$$

Then G^C is a Lie group with Lie algebra $\mathfrak{g}^C$. Then we can prove the theorem by steps:

- $\mathrm{Gr}(V)$ is a finitely generated $\left(\mathfrak{g}^C, K\right)$-module.
- If $\gamma \in K^\wedge$ then $\mathrm{Gr}(V)(\gamma)$ is finitely generated as a $S(\mathfrak{p})^K$-module.
- $P(\mathfrak{p})^K$ is finitely generated as a $\mathrm{Res}_{\mathfrak{g}/\mathfrak{p}}\left(P(\mathfrak{g})^G\right)$-module.
- Then treat $\mathrm{Gr}(V)(\gamma)$ as a finitely generated $\mathrm{Res}_{\mathfrak{g}/\mathfrak{p}}\left(P(\mathfrak{g})^G\right)$ module. Let $\bar{v}_1, \ldots, \bar{v}_d$ be homogeneous generators with $\bar{v}_j$ homogeneous of degree k_j. Let $v_j \in V_{k_j}$ project onto $\bar{v}_j$. We can prove that

  $$\mathrm{Gr}(V)(\gamma) = \sum I(\mathfrak{p})\bar{v}_j = \bigoplus p_k\left(\left(\sum Z_G(\mathfrak{g})v_j\right) \cap V_k\right)$$

  hence $\Sigma Z_G(\mathfrak{g})v_j = V(\gamma)$.

□

Proposition 4.2. *Let V be a finitely generated $(\mathfrak{g}, K)$-module such that if $v \in V$ then $\dim Z_G(\mathfrak{g})v < \infty$. Then V is admissible. Particularly, V is admissible if it is irreducible.*

Before claim our main theorem ($(\mathfrak{g}, K)$-module has the same amount of information compared to (π, V)), we need a lemma first:

Lemma 4.3. *Let (π, H) be a Hilbert representation of G, denote $C \in Z_G(\mathfrak{g})$ as the casimir operator. Suppose that if $v \in H_K$ then $\dim \mathbb{C}[C]v < \infty$. Then H_K is a subspace of the space of analytic vectors for π.*

In the light of the previous results and Proposition 2.4, we have

Theorem 27. *Let (π, H) be an irreducible unitary representation of G. Then (π, H) is admissible.*

Theorem 28. *Let (π, H) be a uni. representation of G. Then (π, H) is irr. if and only if it is infinitesimally irreducible. If (π, H) and (σ, V) are irr. uni. representations of G then π and σ are unitarily equivalent if and only if they are infinitesimally equivalent.*

Remark. Lie group G^C with its alg. $\mathfrak{g}^C$ is called the *Cartan motion group* associated with G. If V is a finitely generated $(\mathfrak{g}, K)$-mod and a K-invariant filtration (with $U(\mathfrak{g}_{\mathbb{C}})V_0 = V$), $\mathrm{Gr}(V)$ is natually a $(\mathfrak{g}^C, K)$-module also a finitely generated and graded as a $S(\mathfrak{p}_{\mathbb{C}})$-module.

4.4 The subquotient theorem

I can't follow all the technical details of this subsection.

We now assume that G is a real reductive group. Let $P = {}^0MAN$ be a minimal parabolic subgroup of G with given standard Langlands decomposition. Let (σ, H_σ) be an irreducible unitary representation of ${}^0M$. If $\mu \in (\mathfrak{a}_{\mathbb{C}})^*$ then we denote by σ_μ the representation of P given by $\sigma_\mu(man) = a^{\mu+\rho}\sigma(m)$ for $m \in {}^0M$, $a \in A$, and $n \in N$. We set $\mathrm{Ind}_P^G(\sigma_\mu) = (\pi_{\sigma,\mu}, H^{\sigma,\mu})$. The representations $\pi_{\sigma,\mu}$ are called the principal series.

Lemma 4.4. *$(H^{\sigma,\mu})_K$ is an admissible $(\mathfrak{g}, K)$-module.*

Our main result is the following:

Theorem 29. *Assume that G is connected. Let V be an irreducible $(\mathfrak{g}, K)$-module. Then there exist $\sigma \in {}^0M^\wedge$ and $\mu \in (a_{\mathbb{C}})^*$ such that V is equivalent to a subquotient (that is, a quotient of subrepresentations on closed subspaces) of $(H^{\sigma,\mu})_K$.*

We will show the process also all the definitions which will be used in future. Let $U(\mathfrak{g})^K$ denote the centralizer of K in $U(\mathfrak{g})$ (which can be proved that it is noetherian), and let γ be an equivalence class of finite dim. irr.representations of K. Any member of γ naturally induces a representation of $U(\mathfrak{k})$, and we denote its kernel by I_γ. Let β_γ be the projection of $U(\mathfrak{k})$ to $U(\mathfrak{k})/I_\gamma$. Then $U(\mathfrak{g})^K \cap U(\mathfrak{g})I_\gamma$ is a two-sided ideal of the algebra $U(\mathfrak{g})^K$, and the quotient has considerable significance, since its irreducible representations correspond exactly to the irreducible representations of $U(\mathfrak{g})$ whose restriction to $U(\mathfrak{t})$ contains members of γ with positive multiplicity.

We note that PBW implies that the map $U(\mathfrak{n}) \otimes U(\mathfrak{a}) \otimes U(\mathfrak{k}) \to U(\mathfrak{g})$ given by $n, a, k \mapsto nak$ is a linear bijection. We identify $U(\mathfrak{a}) \otimes U(\mathfrak{k})$ with its image under this mapping and have

$$U(\mathfrak{g}) = U(\mathfrak{a}) \otimes U(\mathfrak{k}) + \mathfrak{n}U(\mathfrak{g}).$$

We give $U(\mathfrak{a}) \otimes U(\mathfrak{k})$ the tensor product algebra structure. Let p denote the linear projection of $U(\mathfrak{g})$ onto $U(\mathfrak{a}) \otimes U(\mathfrak{k})$ corresponding to the decomposition above. And let

$$p_\gamma = (1 \otimes \beta_\gamma) \circ p : U(\mathfrak{g}) \to U(\mathfrak{a}) \otimes U(\mathfrak{k})/I_\gamma$$

Also, let $U(\mathfrak{k})^M$ denote the centralizer of M in K w.r.t. the adjoint action. Then we have

Proposition 4.3.

- The restriction of p to $U(\mathfrak{g})^K$ is an algebra antihomomorphism which injects $U(\mathfrak{g})^K$ into $U(\mathfrak{a}) \otimes U(\mathfrak{k})^M$.
- The restriction of p_γ to $U(\mathfrak{g})^K$ is an algebra antihomomorphism with kernel precisely $U(\mathfrak{g})^K \cap U(\mathfrak{g}) I_\gamma = U(\mathfrak{g})^K \cap I_\gamma U(\mathfrak{g})$, and p_γ takes $U(\mathfrak{g})^K$ into $U(\mathfrak{a}) \otimes (U(\mathfrak{k})^M / U(\mathfrak{k})^M \cap I_\gamma)$.
- In particular, p_γ induces an injection:

  $$U(\mathfrak{g})^K / U(\mathfrak{g})^K \cap U(\mathfrak{g}) I_\gamma \hookrightarrow U(\mathfrak{a}) \otimes (U(\mathfrak{k})^M / U(\mathfrak{k})^M \cap I_\gamma)$$

  which is an algebra antihomomorphism.

Remark. We now use p to compute the action of $U(\mathfrak{g})^K$ on $H^{\sigma,\mu}(\gamma)$. Frobenius reciprocity says that the map $T \mapsto T^\wedge$ defines an iso. of $\operatorname{Hom}_K(V_\gamma, (H^{\sigma,\mu})_K)$ onto $\operatorname{Hom}_M(V_\gamma, H_\sigma)$. Here $T^\wedge(v) = T(v)(1)$.

$$(\mathfrak{n}\,(H^{\sigma,\mu})_K)\,(1) = 0.$$

Indeed, if $f \in (H^{\sigma,\mu})_K$ and $X \in \mathfrak{n}$ then $Xf(1) = d/\,dt|_{t=0}\, f(\exp tX) = 0$ since $f(N) = f(1)$.

We can apply these map structure to representation theory:

Proposition 4.4. *Let V be a $(\mathfrak{g}, K)$-module. Let $\gamma \in \hat{K}$, and let $X \in \gamma$. Then*

$$\dim \operatorname{Hom}_K(X, V) = m([V], \gamma),$$

and if $V_\gamma \neq 0$, then the $U(\mathfrak{g})^K$-module structure on $\operatorname{Hom}_K(X, V)$ naturally induces and irreducible $U(\mathfrak{g})^K \cap U(\mathfrak{g}) I_\gamma$-module structure on $\operatorname{Hom}_K(X, V)$.

Remark. The following result is a corollary of the subquotient theorem, but in fact it can be proved directly, much more easily:

Theorem 30. *Let V be a irreducible $(\mathfrak{g}, K)$-module. Then for all $\gamma \in \hat{K}$,*

$$m([V], \gamma) \leq \max_{\beta \in \hat{M}} m(\gamma, \beta) \quad (\leq d(\gamma) < \infty).$$

and this estimate is the best possible.

Then we need to apply the above preperations to the principal series representation. Unfortunately, I can't comprehand/calculate more details (actually I do know how they work seperately but konw nothing when they are combined). However, definitions and related results are worth listing here (Maybe I'll realize it someday in a sudden).

Identify $U(\mathfrak{k})/I_\gamma$ with $\mathrm{End}(V_\gamma)$. If $\sigma \in {}^0M^\wedge$ we fix $H_\sigma \in \sigma$. Let P_σ be the projection of V_γ onto $V_\gamma(\sigma)$. Let $\beta_{\gamma,\sigma}(k) = P_\sigma \beta_\gamma(k)$, $k \in U(\mathfrak{k})$. We set $p_\gamma = (I \otimes \beta_\gamma)\, p$ and $p_{\gamma,\sigma} = (I \otimes \beta_{\gamma,\sigma})\, p$. We also set for $\mu \in (a_\mathbb{C})^*$, $p_{\gamma,\sigma,\mu} = ((\mu+\rho) \otimes I) p_{\gamma,\sigma}$. Let T be a maximal torus in ${}^0M^0$. Then $\mathfrak{h} = (\mathrm{t}+\mathfrak{a})_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. Fix a positive root system, Φ^+, in $\Phi\,(m_\mathbb{C}, \mathfrak{h})$, let ρ_m be half the sum of the elements of Φ^+. If $\alpha \in (\mathfrak{a}_\mathbb{C})^*$ (resp. $\beta \in (\mathfrak{k}_\mathbb{C})^*$) then we extend α (resp. β) to $\mathfrak{h}$ by setting $\alpha|_\mathrm{t} = 0$ (resp. $\beta|_\mathfrak{a} = 0$). If $\sigma \in {}^0M^\wedge$ let Ω_σ denote the *lowest* weight of σ with respect to Φ^+. If $\mu \in (\mathfrak{a}_\mathbb{C})^*$ and if $\sigma \in {}^0M^\wedge$ then we set $\Omega(\sigma, \mu) = \Lambda_\sigma + \mu - \rho_m$. We have:

- If $z \in Z(\mathfrak{g})$ then $p_{\gamma,\sigma,\mu}(z) = \chi_{\Omega(\sigma,\mu)}(z) P_\sigma$ for $\mu \in (a_\mathbb{C})^*$ and $\sigma \in {}^0M^\wedge$. Here χ_Λ is the central character.
- If $T \in \mathrm{Hom}_K\left(V_\gamma, (H^{\sigma,\mu})_K\right)$ and $u \in U(\mathfrak{g})^K$ then $(uT)^\wedge = T^\wedge p_{\gamma,\sigma,\mu}(u)$. In particular, by the $\cdot$ above, $(H^{\sigma,\mu})_K$ has infinitesimal character $\chi_{\Omega(\sigma,\mu)}$.
- If $u \in U(\mathfrak{g})^K$ and if $p_\gamma(u) = 0$ then $u \in U(\mathfrak{g})^K \cap U(\mathfrak{g}) I_\gamma$.
- Lots of conclusion I can't comprehand/calculate.

Then we finish the proof.

4.5 The spherical principal series

We set $\mathrm{Ind}_P^G(\sigma_\mu) = (\pi_{\sigma,\mu}, H^{\sigma,\mu})$ as usual. Denote $\pi_{\sigma,\mu}$ by π_μ and $H^{\sigma,\mu}$ by H^μ. If $f \in L^2\left({}^0M \backslash K\right)$ then we set $f_\mu(nak) = a^{\mu+\rho} f(k)$ for $n \in N$, $a \in A$, $k \in K$. We can check that $f_\mu \in H^\mu$.

If $g \in G$ and $g = nak$ as the NAK decomposition then we write $n(g) = n$, $a(g) = a$, $k(g) = k$. These functions on G are smooth. Let 1_K denote the function on K that is identically equal to 1. Let γ_0 denote the class of the trivial representation of K. Then we have

$$(H^\mu)_K(\gamma_0) = \mathbb{C}1_K.$$

Denote $(1_K)_\mu(g) = a(g)^{\mu+\rho}$ as 1_μ. Note that $1_\mu(k) = 1$ for $g \in G$, $k \in K$, we have (define Ξ_μ by)

$$\Xi_\mu(g) = \langle \pi_\mu(g)1_\mu, 1_\mu \rangle = \int_K a(kg)^{\mu+\rho} dk \quad \text{for } g \in G.$$

Proposition 4.5. *If $s \in W(\mathfrak{g}, \mathfrak{a})$ then $\Xi_{s\mu} = \Xi_\mu$ for all $\mu \in (\mathfrak{a}_\mathbb{C})^*$.*

(Sketch of proof) To prove the proposition it is enough to show that if $f \in C_c(K\backslash G/K)$ then

$$\int_G f(g)\Xi_\mu(g)dg = \int_G f(g)\Xi_{s\mu}(g)dg.$$

After calculation, we can show that

$$\int_G f(g)\Xi_\mu(g)dg = \int_A F_f(a)a^\mu da \text{ where } F_f(a) = a^{-\rho}\int_N f(na)dn$$

Thus to prove the proposition it is enough to show

$$F_f(\exp H) = F_f(\exp sH) \text{ for all } H \in \mathfrak{a}, f \in C_c^\infty(K\backslash G/K), s \in W(\mathfrak{g}, \mathfrak{a}).$$

Let Φ^+ be the positive root system in $\Phi(\mathfrak{g}, \mathfrak{a})$ corresponding to $\mathfrak{n}$. Let Δ_0 be the corresponding set of *simple* roots. We can simplify the prop. to the case when $\Delta_0 = \{\alpha\}$ (not easy). Set $p_0 = p_\gamma$ for γ the class of the trivial representation. Let β be the automorphism of $U(\mathfrak{a})$ defined by $\beta(H) = H + \rho(H)1$ for $H \in \mathfrak{a}$. Set $\gamma_0 = \beta \cdot p_0$.

Remark. I don't know how it works actually. In this case I_γ is the kernel of the trivial representation. It will be $U(\mathfrak{k})$?

$$p_\gamma = (1 \otimes \beta_\gamma) \circ p : U(\mathfrak{g}) \to U(\mathfrak{a}) \otimes U(\mathfrak{k})/I_\gamma$$

How to describe $\mathrm{Aut}(U(\mathfrak{a})) \curvearrowright U(\mathfrak{a}) \otimes U(\mathfrak{k})$? Why not acts on the $U(\mathfrak{a})$ part directly?

Lemma 4.5. *The following two statements are equivalent.*
(1) $\gamma_0\left(U(\mathfrak{g})^K\right)$ *is contained in* $U(\mathfrak{a})^W$ $\quad (W = W(\mathfrak{g}, \mathfrak{a}))$.
(2) $\Xi_\mu = \Xi_{s\mu}$ $\quad$ *for all* $s \in W,\ \mu \in (\mathfrak{a}_\mathbb{C})^*$.

When $\Delta_0 = \{\alpha\}$, we can check the following easily:

$$\gamma_0(U(\mathfrak{g})^K) = \gamma_0(\mathrm{symm}(S(\mathfrak{p}_\mathbb{C})^K)) = \{h \in U(\mathfrak{a}) \mid s_\alpha h = h)\}.$$

Then we have finished the proof. Actually, if combines these results with the map property of p_γ (to find the kernel), we will have:

Theorem 31. *The following sequence of algebra homomorphisms is exact:*

$$0 \to U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k} \to U(\mathfrak{g})^K \overset{\gamma_0}{\to} U(\mathfrak{a})^W \to 0.$$

Furthermore, $\gamma_0 \circ \mathrm{symm} : S\left(\mathfrak{p}_{\mathbb{C}}\right)^K \to U(\mathfrak{a})^W$ *is a linear bijection.*

Then we will give crude estimate for Ξ_μ. The technique here is to substitute a "good" μ for the initial μ, thanks to the proposition above. Define Φ^+ as usual. Let $(\mathfrak{a}^*)^+ = \{\mu \in \mathfrak{a}^* \mid (\mu, \alpha) > 0 \text{ for } \alpha \in \Phi^+\}$. Meanwhile, let ${}^+\mathfrak{a} = \left\{H \in \mathfrak{a} : \mu(H) > 0 \text{ for } \mu \in \mathrm{Cl}\left((\mathfrak{a}^*)^+\right) - \{0\}\right\}$. and $\mathfrak{a}^+$ similarly.

Lemma 4.6. *Let G be a real reductive group with Cartan involution θ and Iwasawa decomposition $G = NAK$. Set $\bar{N} = \theta(N)$. Let $a \in A$ be such that* $\log a \in \mathrm{Cl}\left(\mathfrak{a}^+\right)$ *and let $\bar{n} \in \bar{N}$. Then* $\log\left(a\left(a\bar{n}a^{-1}\right)\right) - \log(a(\bar{n})) \in \mathrm{Cl}\left({}^+\mathfrak{a}\right)$.

By the lie alg. theory, we can fix the "good" linear functional in $W\mu \cap \mathrm{Cl}\left((\mathfrak{a}^*)^+\right)$ for every μ uniquely, $|\mu|$. Let $A^+ = \exp(\mathfrak{a}^+)$.

Lemma 4.7. *Let $\mu \in (\mathfrak{a}_{\mathbb{C}})^*$ then $|\Xi_\mu(a)| \le a^{|\operatorname{Re}\mu|}\Xi_0(a)$ for all $a \in \mathrm{Cl}\left(A^+\right)$.*

It follows from Lemma 3.7 and our discussions above:

$$\begin{aligned}
\Xi_{|\mu|}(a) &= \int_K a(ka)^{\mu+\rho} dk \\
&= \int_{\bar{N}} a(\bar{n})^{2\rho} a(k(\bar{n})a)^{\mu+\rho} d\bar{n} \\
&= a^{\mu+\rho} \int_{\bar{N}} a(\bar{n})^{-\mu+\rho} a\left(a^{-1}\bar{n}a\right)^{\mu+\rho} d\bar{n} \\
&\le a^{\mu}\left(a^{\rho}\int_N a(\bar{n})^{\rho} a\left(a^{-1}\bar{n}a\right)^{\rho} dn\right) = a^{\mu}\Xi_0(a)
\end{aligned}$$

In particular, since $\Xi_0 \le 1$,

$$\Xi_\mu(a) \le a^{|\mathrm{Re}\mu|}$$

Remark. We will prove that: There exist positive constants C and d such that

$$a^{-\rho} \le \Xi_0(a) \le Ca^{-\rho}(1 + \log\|a\|)^d$$

for $a \in \mathrm{Cl}\left(A^+\right)$ in next chapter.

4.6 The subquotient theorem

Our main results is

Theorem 32. *every irreducible* $(\mathfrak{g}, K)$*-module is equivalent with a subrepresentation of some* $H^{\sigma,\mu}$*.*

all the tools prepared for the proof are really deep also useful in later developments in our discussions. Thus we will prove it later.

Proposition 4.6 (Osborne lemma)**.** *Let* V *be a finitely generated, admissible* $(\mathfrak{g}, K)$*-module. Then* V *is finitely generated as a* $U(\mathfrak{g}_{\mathbb{C}})$*-module.*

This is discussed in more detail in the remark part. Let $H_{\sigma,\mu}$ denote the $(\mathfrak{p}, {}^0M)$-module H_σ with $\mathfrak{a}$ acting by $(\mu+\rho)I$ and $\mathfrak{n}$ acting by 0. Let V be a $(\mathfrak{g}, K)$-module. If $T \in \mathrm{Hom}_{\mathfrak{g},K}(V, H^{\sigma,\mu})$ then set $T^\wedge(v) = T(v)(1)$ for $v \in V$. It is easily seen that $T^\wedge \in \mathrm{Hom}_{\mathfrak{p},{}^0M}(V/\mathfrak{n}V, H_{\sigma,\mu})$, since $\left(\mathfrak{n}\left(H^{\sigma,\mu}\right)_K\right)(1) = 0$.

Proposition 4.7. *The map* $T \mapsto T^\wedge$ *induces an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g},K}(V, H^{\sigma,\mu}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{p},{}^0M}(V/\mathfrak{n}V, H_{\sigma,\mu})$$

where the inverse is defined by: Let $S \in \mathrm{Hom}_{\mathfrak{p},{}^0M}(V/\mathfrak{n}V, H_{\sigma,\mu})$*,*

$$S^\sim(v)(k) = S(kv), \quad \forall k \in K,\ v \in V.$$

Finally if we can prove the following theorem, then the subrepresentation theorem is clear.

Theorem 33. *Let* V *a fin. generated* $(\mathfrak{g}, K)$*-mod. If* $V = \mathfrak{n}V$ *then* $V = 0$*.*

We prove the theorem by induction on $\dim \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]$. We may assume that G is connected and semi-simple (which is actually not a restriction).

We note that there exists $k \in K$ such that $\theta\mathfrak{n} = \mathrm{Ad}(k)\mathfrak{n}$. Thus $\bar{\mathfrak{n}}V = V$. Let $\alpha \in \Delta_0$ and let $F = \Delta_0 - \{\alpha\}$. Let (P_F, A_F) be the corresponding p-pair with $P_F = M_F N_F$, as usual. Osborne lemma implies that $V/\bar{\mathfrak{n}}_F V$ is finitely generated as a $U(\bar{\mathfrak{n}} \cap \mathfrak{m}_F)$-module. Thus if $V/\bar{\mathfrak{n}}_F V$ is non-zero then $V/\bar{\mathfrak{n}}V = \left(V/\bar{\mathfrak{n}}_F V\right)/{}^*\bar{\mathfrak{n}}_F\left(V/\bar{\mathfrak{n}}_F V\right)$ is non-zero by the *inductive hypothesis*. We therefore conclude that

(1) $V = \bar{\mathfrak{n}}_F V$ for all $F = \Delta_0 - \{\alpha\},\ \alpha \in \Delta_0$.

Then we can derive the contradiction following these steps:

(2) Use subquotient theorem to find a realization for V, i.e. an admissible representation, (π, H), with H_K equivalent with V as a $(\mathfrak{g}, K)$-mod.

(3) There exists a K-invariant semi-norm, q, on H_K and $\lambda \in a^*$ s.t. for all $v, w \in H_K$ and $a \in \mathrm{Cl}\left(A^+\right)$,

$$|c_{v,w}(a)| \leq q(v)q(w)a^\lambda$$

(4) Let Using $\bar{\mathfrak{n}}_F V = V$ to describe V through the basis for $\bar{\mathfrak{n}}_F$, we can refine our estimate: set D equal to the maximum of the $q\left(v_k\right) q\left(\pi^*\left(Y_k\right) w\right)$ (where $v = \Sigma Y_k v_k$). Then

$$|\langle \pi(a)v, w\rangle| \leq \dim(\mathfrak{n}) D a^{\lambda-\alpha} \text{ for } a \in \mathrm{Cl}\left(A^+\right).$$

(5) Let $\Delta_0 = \{\alpha_1, \ldots, \alpha_r\}$, $\zeta = \Sigma_j \alpha_j$. If we apply (4) to all α_j and iterate, then we conclude:

$$|\langle \pi(a)v, w\rangle| \leq D(p, v, w)a^{\lambda-p\zeta}, \quad \text{for } a \in \mathrm{Cl}\left(A^+\right), \forall p \in \mathbb{N}^+.$$

(6) Fix $f_1, f_2 \in H_K$. Set $f(\mu, g) = \langle \pi_{\sigma,\mu}(g) f_1, f_2\rangle$ for $g \in G$ and $\mu \in \left(\mathfrak{a}_\mathbb{C}\right)^*$. Let v, w be non-zero elements of V. Let F be a finite subset of $K^\wedge$ such that $v, w \in \Sigma_{\gamma\in F} V(\gamma) = W$. Let u_j be an orthonormal basis of W. Let C_p be $(\dim W)^2 |v||w|$ times the maximum of $D(p, u_j, u_k)$. Then (5) implies that

$$|\langle \pi\left(k_1 a k_2\right) v, w\rangle| \leq C_p a^{\lambda-p\zeta}, \quad \forall a \in \mathrm{Cl}\left(A^+\right),\ k_1, k_2 \in K$$

Then we can prove that (basic calculus)

$$\delta(\mu) = \int_G f(\mu, g) \operatorname{conj}\left(c_{v,w}(g)\right) dg$$

is *holomorphic* on $\left(\mathfrak{a}_\mathbb{C}\right)^*$. But $\pi_{\sigma,\mu}$ is unitary for $\mu \in i\mathfrak{a}^*$. If $\delta(\mu)$ is non-zero with $\mu \in i\mathfrak{a}^*$ then V would be equivalent to a subrepresentation of $\pi_{\sigma,\mu}$ (orthogonality relation), which is impossible since $V = \mathfrak{n}V$. Hence δ is identically 0. Then $c_{v,w} = 0$ which is also a contradiction.

He **finally** wrote a clear proof! Congrats!

Remark. To give the initial estimate, we have the following lemma:

Lemma 4.8. Let $u, v \in H^\sigma$, then the function $\mu, g \to \langle \pi_{\sigma,\mu}(g)u, v\rangle$ is a smooth function on $\left(\mathfrak{a}_\mathbb{C}\right)^* \times G$ that is holomorphic in μ. Furthermore, if Ω is a compact subset of $\left(\mathfrak{a}_\mathbb{C}\right)^*$ then there exists a K-invariant semi-norm q on $\left(H^\sigma\right)^\infty$ and $\lambda \in \mathfrak{a}^*$ depending only on Ω such that

$$|\langle \pi_{\sigma,\mu}\left(k_1 a k_2\right) u, v\rangle| \leq q(u)q(v)a^\lambda$$

for all $k_1, k_2 \in K, a \in \mathrm{Cl}\left(A^+\right)$ and $\mu \in \Omega$.

It follows from $|\Xi_\mu(a)| \leq a^{|\operatorname{Re}\mu|}\Xi_0(a)$ and the integration formula:

$$\langle \pi_{\sigma,\mu}(g)u, v\rangle = \int_K a(kg)^{\mu+\rho}\langle u(k(kg)), v(k)\rangle dk.$$

Remark. The Osborne lemma follows from:

Proposition 4.8. *There is a finite dimensional subspace, E, of $U(\mathfrak{g}_{\mathbb{C}})$ s.t.*

$$U(\mathfrak{n}_{\mathbb{C}})EZ_G(\mathfrak{g}_{\mathbb{C}})U(\mathfrak{k}_{\mathbb{C}}) = U(\mathfrak{g}_{\mathbb{C}})$$

In the next chapter, we will prove that

$$(\mathfrak{g}, K)\text{-module} + \text{finitely generated as } U(\mathfrak{n})\text{-module} \Rightarrow \text{admissible}$$

as its inverse proposition.

Remark. We will prove that for any $V \in \mathscr{H}$ there always exists a Hilbert representation as its realization in the next chapter. It will simplify oue discussion to a great extent.