If $\mu_1 = \nu_1 \Rightarrow a' \in \mathbb{C}a$, $\mathbb{C}a$ invariant (this is). Contradiction! $\Rightarrow a' \in \mathbb{C}b$, means $\nu := \nu_1 = \mu_2$, $\mu := \mu_1 = \nu_2$ then take $\alpha = a$, $\beta = a'$, $l = \mu - \nu$, $t = \frac{1}{2}(\mu + \nu)$. $\square$

⟨Prop⟩: Every fin. s.s. repn of $W_{\mathbb{R}}$ is fully reducible to dim = 1 or 2.

▶ L-function and ε-factor

here $G_0 = GL_n(\mathbb{R})$, ρ an irr. adm. repn. $\psi: \mathbb{R} \mapsto \mathbb{C}^\times$, $x \mapsto e^{2\pi i x}$

$I_0 := \{M_n(K) \to \mathbb{R}$ of the form $P(x_{ij}) \exp(-\pi \sum x_{ij}^2)$, P a poly$\}$

$\forall$ K_0-finite matrix coefficient c of ρ, $\forall f \in I_0$, $s \in \mathbb{C}$, defn

$$\mathcal{Z}(f, c, s) = \int_{M_n(\mathbb{R})} f(x) c(x) |\det(x)|^s \underbrace{d^\times x}_{= |\det x|^{-n} dx}$$

fixed invariant measure for $M_n(\mathbb{R})$

⟨prop⟩: $\mathcal{Z}(f, c, s)$ converges in right half plane, and extend to a meromorphic function (in terms of s) on $\mathbb{C}$. Moreover, $\exists$ a finite collection $\{(c_i, f_i)\}$, s.t

$$L(s, \rho) := \sum_i \mathcal{Z}(f_i, c_i, s)$$

unique up to scalar

s.t. $\forall (c, f)$, $\mathcal{Z}(f, c, s + \frac{1}{2}(n-1)) = P(f, c, s) L(s, \rho)$ with a Poly P in s.

Now for $\forall f \in I_0$, defn $\hat{f} = \int_{M_n(\mathbb{R})} f(y) \psi(\mathrm{Tr}(xy)) dy$ ($\in I_0$)

dual Haar measure; still

⟨Prop⟩: $\exists$ meromorphic $\gamma(s, \rho, \psi)$, s.t.

$$\mathcal{Z}(\hat{f}, \check{c}, 1 - s + \tfrac{1}{2}(n-1)) = \gamma(s, \rho, \psi) \mathcal{Z}(f, c, s + \tfrac{1}{2}(n-1))$$

for $\forall f \in I_0$, $\forall c$ of ρ. Here $\check{c}(x) = c(x^{-1})$.

⟨Def⟩ (ε-factor): $\varepsilon(s, \rho, \psi) = \dfrac{\gamma(s, \rho, \psi) L(s, \rho)}{L(1-s, \tilde{\rho})}$ $\tilde{\rho}$: the admissible dual.

Then we attach L-function and ε-factor to representation of $W_{\mathbb{R}}$:

$$L(s, \varphi) = \begin{cases} \pi^{-\frac{s+t}{2}} \Gamma\left(\frac{s+t}{2}\right) & \text{for } \varphi: \langle +, t \rangle \\ \pi^{-\frac{s+t+1}{2}} \Gamma\left(\frac{s+t+1}{2}\right) & \text{for } \varphi: \langle -, t \rangle \\ 2(2\pi)^{-(s+t+\frac{1}{2})} \Gamma\left(\frac{2s+2t+l}{2}\right) & \text{for } \varphi: \langle l, t \rangle \end{cases} \qquad \varepsilon(s, \varphi, \psi) = \begin{cases} 1 \\ i \\ i^{l+1} \end{cases}$$

⟨Thm⟩ $\exists$ bijection between

$$\{\text{s.s. repn } W_{\mathbb{R}} \to G_0^L\} \longleftrightarrow \{\text{L-packet of irr. adm. repn of } G_0\}$$

For $G_0 = GL_n(\mathbb{R})$, each L-packet has <u>size one</u>.

Explizitly: $GL(j, \mathbb{R})$, $j = 1, 2$.

$\forall X \in GL(j, \mathbb{R})$ admits $X = YZ$, where $Y \in SL^{\pm}(j, \mathbb{R})$, Z a positive scalar.

Denote two character of $SL^{\pm}(1, \mathbb{R})$ as $\underline{1}$ and $\underline{\mathrm{sgn}}$.

For $l \in \mathbb{Z}^+$. D_l^+ : discrete series

$SL_2(\mathbb{R}) \curvearrowright \{$ analytic f on $\mathcal{H}^+$: $\|f\|^2 = \iint |f(z)|^2 y^{l-1}\, dx\, dy < \infty \}$

$$\left(D_l^+\begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right)(z) = (bz+d)^{-l-1} f\left(\frac{az+c}{bz+d}\right)$$

Consider $D_l = \mathrm{ind}_{SL_2(\mathbb{R})}^{GL_2^+(\mathbb{R})}(D_l^+)$. For $GL_1(\mathbb{R})$, $\forall t \in \mathbb{C}$, we have

$$1 \otimes |\cdot|^t : x \mapsto |z|^t$$

$$\mathrm{sgn} \otimes |\cdot|^t : x \mapsto \mathrm{sgn}|Y|\, |z|^t$$

For $GL_2(\mathbb{R})$:

$$D_l \otimes |\cdot|^t : x \mapsto D_l(Y)\, |\det(z)|^t$$

Given φ : a repn of $W_{\mathbb{R}}$, suppose $\varphi = \bigoplus_{i=1}^{j} \varphi_j$: φ_j is n_j-dim irr. s.s. repn

then

$$\boxed{\begin{aligned} \langle +, t\rangle &\mapsto 1 \otimes |\cdot|^t \\ \langle -, t\rangle &\mapsto \mathrm{sgn} \otimes |\cdot|^t \\ \langle l, t\rangle &\mapsto D_l \otimes |\det(\cdot)|^t \\ \varphi_j &\mapsto \text{a repn } \psi_j \text{ of } \underbrace{GL(n_j, \mathbb{R})} \end{aligned}}$$

$D = \prod_{i=1}^{j} GL_{n_j}(\mathbb{R})$ unipotent

extends trivially to DU

induction

$\mathrm{Ind}_{DU}^{GL_n}$ acts trivially

<u>Prop</u>: $\mathrm{Ind}_{DU}^{GL_n}$ has a unique irr. quotient ψ which is admissible.

Then $\varphi \mapsto \psi$ is desired.

Interlude: Beilinson-Bernstein Localization

Reference: https://people.math.harvard.edu/~campbell/sl2r.pdf

Recall: choose a positive root system Δ^+ for $\mathfrak{g}$ $\rightsquigarrow$

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \oplus \underbrace{\mathfrak{h}}_{\text{Cartan}} \oplus \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$$

$\mathfrak{n} \nearrow$ $\quad$ $\nwarrow \mathfrak{n}^-$ $\quad$ $\mathfrak{b} := \mathfrak{n} \oplus \mathfrak{h}$

$\Pi = \{\alpha_j\}_{1 \le j \le n}$: simple roots

<Prop>: each $\alpha_i \in \Pi$, $\exists\, \alpha_i^\vee \in \mathfrak{h}$ s.t. $\alpha_i^\vee(\alpha_i) = 2$ and $s_{\alpha_i^\vee, \alpha_i}(\Delta) = \Delta$, where the map $s_{\alpha_i^\vee, \alpha_i} : \mathfrak{h}^* \to \mathfrak{h}^*$ is given by $\underline{v \mapsto v - \alpha^\vee(v)\alpha}$.

<Def> <u>Weyl group</u>: subgp of $GL(\mathfrak{h}^*)$ generated by $\{s_{\alpha_i^\vee, \alpha_i}\}_{1 \le i \le n}$.

The fundamental weights π_i : $\alpha_i(\pi_j) = \delta_{ij}$.

$\rho := \sum \pi_i = \frac{1}{2} \sum \alpha_i$, Q^+ : non-negative integral span of $\{\alpha_i\}$

$P^+ := \bigoplus_{i=1}^{n} \mathbb{Z}_{\ge 0} \pi_i$ (dominant weight),

$\{\lambda \in \mathfrak{h}^* : \alpha_i^\vee(\lambda) < 0\} \subseteq -P^+$: regular weights

given a repn V of $\mathfrak{g}$, $V = \bigoplus V_\lambda$ w.r.t. $\mathfrak{h}^*$: $\{\lambda\}$: weight of V.

(Thm) (Verma module, highest weight repn)

Any irreducible repn V of $\mathfrak{g}$ has a unique maximal weight λ, and a unique minimal μ, w.r.t. the partial ordering given by Q^+.

with $\lambda \in P^+$, $\mu \in -P^+$. Conversely, $\forall \lambda \in P^+$, $\exists!$ rep in of $\mathfrak{g}$, denoted as $L^+(\lambda)$, with maximal weight λ.

<Def> $Z(\mathfrak{g}) \subseteq U(\mathfrak{g}) \curvearrowright L^+(\lambda)$ by scalar $\Rightarrow \underline{\chi_\lambda}: Z(\mathfrak{g}) \to \mathbb{C}$. (central character)

- Consider $f: U(\mathfrak{h}) \to U(\mathfrak{h})$ induced by $h \mapsto h + \rho(h)1$. By

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n} U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^-),$$

$\forall z \in Z(\mathfrak{g})$ can be written as $z = h + n \rightsquigarrow$

<Def> Harish-Chandra homo is the map $\lambda: Z(\mathfrak{g}) \to U(\mathfrak{h})$, $z \mapsto f(h)$.

Moreover, for the central char., $\chi_\lambda(z) = \lambda(\gamma(z))$

(this is because: $(\lambda+\delta)(h_i - 1) = (\lambda+\delta)(h_i) - \underline{(\lambda+\delta)(1)} = \lambda(h_i) + \overset{\delta(h_i)}{\overset{\shortparallel}{1}} - 1 = \lambda(h_i)$)

<u>(Thm)</u> λ induces $Z(\mathfrak{g}) \cong U(\mathfrak{h})^W$, with W acts on $\mathfrak{h}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$, then identify $\mathfrak{h}$ with $\mathfrak{h}^*$.

$(w \cdot h = w(h+1) - 1)$

▸ Algebraic $\mathcal{D}$-module

X: sm. alg. var./$\mathbb{C}$, structure sheaf $\mathcal{O}_X$, tangent sheaf $\mathcal{T}_X$.

Definition 3.2.1 *A linear map $D : \mathcal{O}_X \to \mathcal{O}_X$ is a differential operator of order 0 if it is multiplication by a function. For $k \in \mathbb{Z}^+$, D is a differential operator of order k if for any function $f \in \mathcal{O}_X$, the commutator $[D, f]$ is a differential operator of order $k-1$. Let $D_{X,k}$ be the collection of differential operators of order k and $D_X = \bigcup_{k=0}^{\infty} D_{X,k}$.*

Concretely, we have the following description of D_X.

Proposition 3.2.2 *The graded algebra $\mathrm{gr}(D_X)$ is isomorphic to $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X)$.*

Definition 3.2.3 *A (left) D-module is an $\mathcal{O}_X$-quasicoherent sheaf of (left) modules over D_X.*

Now consider any line bundle $\mathcal{L}$ on X with dual sheaf $\mathcal{L}^\vee$.

Definition 3.2.4 *A map $D : \mathcal{L} \to \mathcal{L}$ is a differential operator of order 0 on $\mathcal{L}$ if it is multiplication by a function. For $k \in \mathbb{Z}^+$, D is a differential operator of order k on $\mathcal{L}$ if for any function $f \in \mathcal{O}_X$, the commutator $[D, f]$ is a differential operator of order $k-1$. Denote the collection of all such differential operators by $D_{\mathcal{L}}$.*

Proposition 3.2.5 *Concretely, we have $D_{\mathcal{L}} = \mathcal{L} \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee$.*

<Def> quotient G/B is called the flag variety of G.
<Prop> $\{G$-equivalent vect. bundle on $G/B\} \xleftrightarrow{1:1} \{$repn of $B\}$.
<Prop> $\{1$-dim repn of $B\} \xleftrightarrow{1:1} \{$char of $T\} \xleftrightarrow{1:1} \{$integral char of $\mathfrak{h}^*\}$
Hence given an integral wt $\lambda \in \mathfrak{h}^* \mapsto \mathbb{C}_\lambda$: 1-dim repn $\mapsto \mathcal{L}(\lambda)$: the G-equiv line bundle
<Def>: D_λ : sheaf $D_{\mathcal{L}(\lambda+\rho)}$.

We denote by $D_\lambda(G/B)$ the category of D_λ-modules that are quasi-coherent as $\mathcal{O}_{G/B}$-modules, and by $\mathfrak{g}$-mod$_\lambda$ the category of $U(\mathfrak{g})$-modules on which $Z(\mathfrak{g})$ acts by the character χ_λ. Then we have the following crucial result:

Theorem 3.3.5 (Beilinson–Bernstein Localization) *If λ is regular, then the global section functor $\boxed{\Gamma : D_\lambda(G/B) \to \mathfrak{g}\text{-mod}_\lambda}$ is an equivalence of categories with quasi-inverse given by $D_\lambda \otimes_{U(\mathfrak{g})} (\cdot)$*

This is still not exactly what's needed for our purpose, as we are intersted in classifying $(\mathfrak{g}, K)$-modules. As it turns out, the appropriate notion corresponding to $(\mathfrak{g}, K)$-modules with central character χ_λ on the geometric side is the category $D_\lambda(G/B)^K$ of "K-equivariant" D_λ-modules on G/B. In other words, we have an equivalence of categories $D_\lambda(G/B)^K \cong (\mathfrak{g}, K)$-mod$_\lambda$. Thus to classify irreducible $(\mathfrak{g}, K)$-modules, it suffices to classify simple objects in $D_\lambda(G/B)^K$, for which the following two results are necessary.

Proposition 3.3.6 *For every simple $\mathcal{M} \in D_\lambda(G/B)^K$, there is a K-orbit Q of G/B and a simple $\mathcal{N} \in D_\lambda(Q)^K$ such that $\mathcal{M} \cong j_{!*}(\mathcal{N})$. (For a detailed discussion of the functor $j_{!*}$, see [3, Section 5].)*

Proposition 3.3.7 *Fix a point x in a K-orbit Q. Let K_x be the stabilizer of x in K. Then $D_\lambda(Q)^K = 0$ unless λ integrates to a character of the identity component K_x^0 of K_x, in which case $D_\lambda(Q)^K \cong \mathrm{Rep}(K_x/K_x^0)$, the category of representations.*

Remark 3.3.8 *The integrality condition means we can first conjugate K_x so that it is contained in H and then find a character of K_x^0 whose derivative coincides with the restriction of λ.*

Induction in Geometric Context.

<Def>: Jacquet functor r_H^G is given by
$(\mathfrak{g},K)$-mod $\to (\mathfrak{h}, K_H := K\cap B)$-mod, $M \mapsto U(\mathfrak{h}) \otimes_{U(\mathfrak{b})} M$,
i.e. taking $\mathfrak{n}$-coinvariant. It admits a right adjoint: i_B^G.
<rmk>: on the level of K-mod, $i_B^G = \mathrm{ind}_{K_H}^K$.

Example:
$G_0 = GL_2(\mathbb{R})$, $G = GL_2(\mathbb{C})$, $K = O(2,\mathbb{C})$, B, H
Langlands Approach: ✓
Geometric Approach: two K-orbit on G/B: $\{p := \begin{pmatrix}1&0\\0&1\end{pmatrix}\}$ and $\{q : \begin{pmatrix}1&0\\i&1\end{pmatrix}\}$
the <u>stabilizer</u>: $K_p = \{\begin{pmatrix}x&0\\0&y\end{pmatrix} : x^2 = y^2 = 1\}$, $K_q = \{\begin{pmatrix}x&y\\-y&x\end{pmatrix} : x^2+y^2 = 1\}$ $\cong SO(\mathbb{R})$
note that $|K_p| = 4 \rightsquigarrow \mathrm{Rep}(K_p/K_p^0)$ has 4 simple objects, and the integrability of λ in <prop. 3.3.7> (to K_p^0) satisfies $\rightsquigarrow$ These $\mathcal{D}$-mod corresponds to the irr. rep'n $\langle \mathrm{sgn}^{\varepsilon_1}, t_1, \mathrm{sgn}^{\varepsilon_2}, t_2 \rangle$,
where $\langle t_1, t_2 \rangle$ parametrize the wt λ, $\langle \varepsilon_1, \varepsilon_2 \rangle \in \mathrm{Rep}(K_p/K_p^0)$
K_q is connected $\Rightarrow \mathrm{Rep}(K_q/K_q^0)$ has exactly one simple object.
note that characters of K_q are parametrized by integer $n \in \mathbb{Z}$
$$\begin{pmatrix}x&y\\-y&x\end{pmatrix} \mapsto (x+yi)^n$$
These $\mathcal{D}$-mod corresponds to $D_\ell \otimes |\det(\cdot)|^t$, with $\ell \in \mathbb{Z}^+$, $t \in \mathbb{C}^\times$.

4.4 Parabolic Induction

For a dominant regular weight λ, let $\Delta_\lambda : (\mathfrak{g}, K)\text{-mod}_\lambda \to D_\lambda(G/B)^K$ be the localization functor from Theorem 3.3.5. Let Q be the orbit of p in G/B and j the inclusion $Q \hookrightarrow G/B$. Finally, let $T_p(-)$ be the functor of taking fiber at p.

Proposition 4.4.1 *We have a commutative diagram:*

$$\begin{tikzcd}
(\mathfrak{g},K)\text{-}mod_\lambda \arrow[r, "\Delta_\lambda"] \arrow[d, hook] & D^\lambda(G/B)^K \arrow[d, "j^*"] \\
(\mathfrak{g},K)\text{-}mod \arrow[d, "r_H^G"'] & D^\lambda(Q)^K = D(Q)^K \arrow[d, "T_p(-)"] \\
(\mathfrak{h},K_H)\text{-}mod \arrow[r, "\mathbb{C}_\lambda \otimes V_\theta(-)"'] & \mathrm{Rep}(K_p/K_p^0)
\end{tikzcd}$$

what is "W"?

Here $\boxed{\theta = W \cdot \lambda}$ *and* $V_\theta = U(\mathfrak{h})/J_\theta U(\mathfrak{h})$ *where* J_θ *is the kernel of* $U(\mathfrak{h}) \to \mathbb{C}$ *determined by* $\lambda + \rho$. *Notably, on the* K*-orbit* Q*, any* λ*-twisting is trivialized.*

Proof: By [11, Theorem 2.5], we have an isomorphism between the left derived functors $LT_p \circ L\Delta_\lambda$ and $\mathbb{C}_\lambda \otimes^L_{V_\theta} (U(\mathfrak{h}) \otimes^L_{U(\mathfrak{b})} -)$. As j^* doesn't change the fiber and r^G_H is exactly $U(\mathfrak{h}) \otimes_{U(\mathfrak{b})} -$, the diagram follows.

Remark 4.4.2 *This gives two interpretations of the localization of principal series representations. Now take right adjoints.*

$$\begin{tikzcd}
(\mathfrak{g}, K)\text{-}mod_\lambda & D^\lambda(G/B)^K \arrow[l, "\Gamma_\lambda(-)"'] \\
(\mathfrak{g}, K)\text{-}mod \arrow[u] & D^\lambda(Q)^K = D(Q)^K \arrow[u, "j_*"'] \\
(\mathfrak{h}, K_H)\text{-}mod \arrow[u, "i^G_B"] & \mathrm{Rep}(K_p/K_p^0) \arrow[l] \arrow[u]
\end{tikzcd}$$

We obtain two parametrizations of the principal series representations by the four simple objects in $\mathrm{Rep}(K_p/K_p^0)$. *On the right hand side, this is exactly the geometric approach, while on the left hand side, note that* i^G_B *is the same as* $\mathrm{ind}_{DU}^{\mathrm{GL}(n,\mathbb{R})}$ *from Proposition 2.4.5, so it is essentially the Langlands approach.*

Verma module:

Now let us consider the idea behind the Theorem of the Highest Weight. We continue with the notation G, $\mathfrak{g}_0$, $\mathfrak{g}$, $\mathfrak{t}$, Δ, Δ^+, Φ on V, and φ on V.

We make the following construction: Let $\beta_1, \ldots, \beta_k$ be an enumeration of Δ^+, and let $H_1, \ldots, H_l$ be a basis of $\mathfrak{t}$. The Poincaré-Birkhoff-Witt Theorem implies that all elements of the form

$$(*) \qquad E_{-\beta_1}^{q_1} \cdots E_{-\beta_k}^{q_k} H_1^{m_1} \cdots H_l^{m_l} E_{\beta_1}^{p_1} \cdots E_{\beta_k}^{p_k}$$

form a basis of $U(\mathfrak{g})$. Using φ, we make V into a unital left $U(\mathfrak{g})$ module.

Now let λ be the highest weight of φ, and let v be a nonzero highest weight vector. Let us see the main ideas behind the proof of the Theorem of the Highest Weight.

We begin with the easy parts. Assume that φ irreducible, and apply $(*)$ to v. Any $E_{\beta_j}^{p_j}$ gives 0 if $p_j > 0$. If all $p_j = 0$, H_j gives a factor $\lambda(H_j)$. Then the $E_{-\beta_j}$ push weights down. So the only vectors of weight λ in $U(\mathfrak{g})v$ are $\mathbb{C}v$, and the only weights are λ minus combinations of positive roots. But $U(\mathfrak{g})v = V$ by irreducibility. So our conclusions about $U(\mathfrak{g})v$ are valid for V.

Next we consider the dominance of λ. If α is in Δ^+, then E_α, $E_{-\alpha}$, and $[E_\alpha, E_{-\alpha}]$ generate a copy of $\mathfrak{sl}(2,\mathbb{C})$, and $\frac{2\langle\lambda,\alpha\rangle}{|\alpha|^2}$ is the weight of v under the element h of $\mathfrak{sl}(2,\mathbb{C})$. One uses that the weights of any finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ are closed under negatives. Then $\langle\lambda,\alpha\rangle \geq 0$.

Now let us consider uniqueness. Suppose (φ_1, V_1) and (φ_2, V_2) are given irreducible representations, λ is the common highest weight, and v_1 and v_2 are nonzero highest weight vectors. Form $S = (\varphi_1 \oplus \varphi_2)(U(\mathfrak{g}))(v_1 \oplus v_2)$. One uses $U(\mathfrak{g})$ to check that this is irreducible. Apply Schur's Lemma to the projection of S to V_1 to see that S is equivalent with V_1. In similar fashion S is equivalent with V_2. Therefore V_1 and V_2 are equivalent with each other.

The main step in the proof of the Theorem of the Highest Weight is to prove existence. We give the idea of the proof when G is simply connected, omitting the general case.

Verma module: $V(\lambda+\delta) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$

$U(\mathfrak{g})$ acts by left multiplication, $\rightsquigarrow$ it is a infinite-dim $U(\mathfrak{g})$-mod with basis (by PBW): $E_{-\beta_1}^{q_1} \cdots E_{-\beta_k}^{q_k} \otimes 1$, and $\underline{1 \otimes 1}$ the wt. vector with highest wt. λ. Take the sum: all $U(\mathfrak{g})$ submodule not meeting $\mathbb{C}(1\otimes 1)$ is a maximal proper $U(\mathfrak{g})$-submodule $M(\lambda+\delta)$.

For λ: integral dominant, $L(\lambda+\delta) = V(\lambda+\delta)/M(\lambda+\delta)$ is fin. dim'l. irr.

Interlude: Borel-Weil Theorem.

explicit realization of irr. rep'n of a compact Lie group K.

$G: K^{\mathbb{C}}$, $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k}$ ($\mathfrak{k}$: \mathfrak{k}) (Cartan decomp: $\mathfrak{p} = i\mathfrak{k}$)

$= \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (Iwasawa decomposition) $(\mathfrak{n} = Z_{\mathfrak{k}}(\mathfrak{a}))$

$\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a}) = i\mathfrak{a}$. $\mathfrak{m}^{\mathbb{C}} = \mathfrak{a} \oplus \mathfrak{n}$, $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{a} \oplus \theta\mathfrak{n} = \mathfrak{m}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{k}_{-\alpha}$, $B = MA\bar{N}$

For λ: analytically dominant integral, ξ_λ: corresponding character on M, then extend it to holo. one-dim rep'n $\xi_\lambda: B \to \mathbb{C}^\times$, by defining

$$\xi_\lambda(x) = \begin{cases} \exp\{\lambda(\log(x))\} & \text{for } x \in A \\ 1 & \text{for } x \in \bar{N} \end{cases}$$

<Thm> (Borel-Weil thm) $(K, G = K^{\mathbb{C}}, B)$, (λ, ξ_λ), then realize an irr. rep'n of K with highest weight λ is as follows:

$\underset{\text{left regular rep'n}}{K \circlearrowright}$ $\Gamma(\lambda) = \{F : G \to \mathbb{C} \mid$ (i) F is holomorphic, (ii) $F(xb) = \xi_\lambda(b)^{-1} F(x)$ for $x \in G, b \in B\}$

① make $\Gamma(\lambda)$ an inner-product space (not obviously complete)

$$(F_1, F_2) = \int_K F_1(k)\overline{F_2(k)}\, dk$$

where K acts (by left) isometrically. $\|F\| = 0 \Leftrightarrow F|_K = 0 \overset{\text{by } G = KB}{\Rightarrow} F = 0$

(Lem) Let τ: finite-dim. rep'n of K, on complex vect. space V.

Then it <u>extends to a holomorphic rep'n</u> of G on V.

<Proof> Start from (Φ_λ, V): an irr. uni. rep'n of K with highest wt λ.

v_λ: highest wt. vector of norm 1. then extends to Rep'n(G).

For $v \in V$, define ψ_v on G by $\boxed{\psi_v(x) = (\Phi_\lambda(x)^{-1} v, v_\lambda)}$

<Lem>: each $v \in V$, ψ_v is in $\Gamma(\lambda)$. Moreover, $L(k)\psi_v = \psi_{\Phi_\lambda(k)v}$.

so that $\{\psi_v \mid v \in V\}$ is an irr. subspace of $\Gamma(\lambda)$ under K equivalent with Φ_λ.

Proof: φ_λ: differential of Φ_λ. Since Φ_λ is unitary on K, φ_λ is skew-Hermitian on $\mathfrak{k}$. Since φ_λ is complex-linear on $\mathfrak{g}$,

$$\varphi_\lambda(\theta X) = -\varphi_\lambda(X)^* \quad , \forall X \in \mathfrak{g}.$$

Thus $\Phi_\lambda(\Theta x) = \Phi_\lambda(x^{-1})^*$ for $x \in G$. Hence $b = ma\bar{n} \in MA\bar{N}$ implies

$$\begin{aligned}
\psi_v(xman) &= (\Phi_\lambda(ma\bar{n})^{-1}\Phi_\lambda(x)^{-1} v, v_\lambda) \\
&= (\Phi_\lambda(\Theta x)^{-1} v, \Phi_\lambda(ma^{-1}n) v_\lambda) \quad (\text{for } n = \Theta\bar{n}) \\
&= (\Phi_\lambda(\Theta x)^{-1} v, \Phi_\lambda(ma^{-1}) v_\lambda) \quad \text{since } v_\lambda \text{ highest wt} \\
&= (\Phi_\lambda(x)^{-1} v, \xi_\lambda(m)\xi_\lambda(a)^{-1} v_\lambda) \quad \text{with wt} = \lambda \\
&= \xi_\lambda(m)\xi_\lambda(a)^{-1}(\Phi_\lambda(x)^{-1} v, v_\lambda) \\
&= \xi_\lambda(b)^{-1}\psi_v(x) \quad \Rightarrow \psi_v \in \Gamma(\lambda)
\end{aligned}$$

Finally:

$$\begin{aligned}
\psi_{\Phi_\lambda(k)v}(x) &= (\Phi_\lambda(x)^{-1}\Phi_\lambda(k)v, v_\lambda) \\
&= (\Phi_\lambda(k^{-1}x)^{-1} v, v_\lambda) \\
&= \psi_v(k^{-1}x) = L(k)\psi_v(x)
\end{aligned}$$

Def: $\psi_\lambda := \psi_{v_\lambda}$.

<Lem>: If $F \in \Gamma(\lambda)$, then $\int_M F(mxm^{-1})\, dm = F(1)\psi_\lambda(x)$, $\forall x \in G$.

This allows us to complete the proof:

(1) $|F(1)| \le \|\psi_\lambda\|^{-1}\|F\|$, for all F in $\Gamma(\lambda)$.

$\|F\|^2 = \int_K |F(k)|^2 dk = \int_K |F(mkm^{-1})|^2 dk$ for all $m \in M$

$$= \int_K \int_M |F(mkm^{-1})|^2 \, dm \, dk$$
$$\geq \int_K \left| \int_M F(mkm^{-1}) \, dm \right|^2 dk \quad \text{by Schwartz}$$
$$= |F(1)|^2 \int_K |\psi_\lambda(k)|^2 \, dk \quad \text{(by Lemma)}$$
$$= |F(1)|^2 \, \|\psi_\lambda\|^2$$

(2) To each compact set $E \subseteq G$ corresponds a constant $C_E < \infty$ such that

$$|F(x)| \leq C_E \|F\|$$

for all F in $\Gamma(\lambda)$ and x in E. [In fact, this is a simple consequence of (1) since $\|L(k)F\| = \|F\|$ and since $|F(xb)| = |\xi_\lambda(b)|^{-1}|F(x)|$ for b in B.]

(3) $\Gamma(\lambda)$ is complete. [In fact, Cauchy sequences in $\Gamma(\lambda)$ converge uniformly on compact sets in G, by (2), and the limit function is holomorphic and satisfies the correct transformation law under B.]

(4) $\Gamma(\lambda)$ is irreducible, and hence the map $v \to \psi_v$ of Lemma 5.31 is onto $\Gamma(\lambda)$. [In fact, let $U \subseteq \Gamma(\lambda)$ be a nonzero closed invariant subspace. Let $F \neq 0$ be in U. After applying some $L(k)$, we may assume that $F(1) \neq 0$. Then

$$\int_M \overline{\xi_\lambda(m)} L(m) F \, dm \in U \text{ by (3).}$$

By lemma, this $= F(1)\psi_\lambda$. $\Rightarrow \psi_\lambda \in U$

If $U^\perp \neq 0$, also $\psi_\lambda \in U^\perp$ $\Rightarrow$ contradiction.