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Minimal Representations and Applications

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I will define minimal representations of p-adic groups and explain how one can work with them. As an application, using minimal representations, I will decompose certain Weil representations attached to multiplicity-free symplectic representations as defined and classified by Knop. These results are consistent with predictions in the relative Langlands program of Ben-Zvi, Sakellaridis

Lecture 1: Minimal rep'ns

F: p-adic or finite. $G = GL_{2n}(F)$. (π, V): irr. (sm.) rep'n of G

an example: $F = \mathbb{F}_q$. $P = MN = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$.

??

$M = GL_n \times GL_n$. $N = M_n(F)$. Fix $\psi: F \to \mathbb{C}^\times$.

$N \cong \hat{N} \cong M_n(F)$

$A \mapsto [\psi_A : B \mapsto \psi(\mathrm{tr}(AB))]$.

$V^{N, \psi_A} := \{ v \in V \mid \pi(B) v = \psi_A(B) v, \ \forall B \in N \}$

$\rightsquigarrow V = \bigoplus_{A = M_n(F)} V^{N, \psi_A}$ $\qquad M \curvearrowright N, N^\vee$ by conj $\rightsquigarrow$ "$\cong$"

$M_n(F) = \bigcup_{i=0}^{n} W_i$ (the set of rank i matrices)

$\rightsquigarrow \dim V^{N, \psi_A} = \dim V^{N, \psi_B}$ if $A \sim B$

<Def'n> N-rank

$\pi \mapsto i$ if $\begin{cases} ① \ V^{N, \psi_A} \neq 0 \text{ for all } A \in W_i \\ ② \ V^{N, \psi_A} = 0 \text{ for all } A \in W_j, \ j > i \end{cases}$

Ex: N-rank $= 0 \iff V|_{SL_{2n}}$ is trivial

$\dim V \geq |W_1| = (q^n - 1)(q^n - 1)/(q-1) = q^{2n-1} \cdots$

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ex: Heisenberg parabolic $Q = LU$

$\begin{pmatrix} * & X & * \\ 0 & * & Y \\ 0 & 0 & * \end{pmatrix}$ $\quad Z$ $\qquad L = GL_1 \times \underbrace{GL_{2n-2}}_{=: G_1} \times GL_1$, $U \triangleright Z$ $\qquad$ symplectic space $(U/Z = V)$

$\psi: Z \to \mathbb{C}^\times$, $a \mapsto [x \mapsto \psi(ax)]$, $a \in F^\times$

ψ_a

$(\pi, V) \mapsto V = V^Z \oplus \bigoplus_{a \in F^\times} V^{Z, \psi_a}$ $\qquad := \{ v \in V \mid \pi(z) v = \psi(z) \cdot v, \ \forall z \in Z \}$

is a multiple of the Heisenberg repn.,

(denoted by W_{ψ_a}) $\quad W_{\psi_a} \cong \mathbb{C}[x]$ $\quad \dim W_{\psi_a} = q^{2n-2}$

$\dim V \geq (q-1) q^{2n-2} = q^{2n-1} \pm \cdots$

- Fourier-Jacobi Functor.

$(\pi, V) \rightsquigarrow V^{Z, \psi} = V_1 \otimes W_\psi$, $\quad V_1 = \mathrm{Hom}_U(W_\psi, V^{Z, \psi})$

$\circlearrowleft G_1$ action "conjugation"

- G_1 acts on $W_\psi \cong \mathbb{C}[x] \ni f$. by

($(W_\psi(g_1) f)(x) = f(xg)$)

$FJ: (\pi, V) \rightsquigarrow (\pi_1, V_1)$ $\qquad (P_1 = M_1 N_1 = P \cap G_1)$

(1) exact: $N_1\text{-rank}(\pi_1) = N\text{-rank}(\pi) - 1$

(2) π irred, $FJ(\pi) = 0 \iff \pi$ is one-dimensional.

<Def> (minimal rep'n): π minimal if $FJ(\pi) = 1_{G_1}$

(finite field case) (weakly) $\qquad$ (copies of 1_{G_1})

(also p-adic ?) $\qquad$ (existence)

F: p-adic case $\quad P = MN$: Siegel parabolic

$s \in \mathbb{R}$, $\rightsquigarrow$ $I(s) = \mathrm{Ind}_P^G(\chi_s)$. $\quad \chi_s(m) = \left|\frac{\det a}{\det b}\right|^{\frac{1}{2}}$, for $m = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$A_s : I(s) \to I(-s)$ $\quad$ ($I(n)$ has trivial repn as unique quotient)

$A_s(f_s^\circ) = 0$, for $s = 1, 2, \dots n$.

$\hookrightarrow$ spherical vector.

$$\begin{tikzcd} 0 \arrow[r] & \pi' \arrow[r] \arrow[d, "FJ"] & I(n-1) \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi_1' \arrow[r] & I_1(n-1) \arrow[r] & \mathbb{B} \arrow[r] & 0 \end{tikzcd}$$

(proof of existence)

(for GL_n)

$\langle$Thm$\rangle$ $FJ(I(s)) = I_1(s)$. $\quad I_1(s) = \mathrm{Ind}_{P_1}^{G_1}(\chi_s)$. $\rightsquigarrow$

Back to finite fields

$B \subseteq G$, Borel. $\quad \mathrm{End}(\mathrm{Ind}_B^G 1) \cong \mathbb{C}[W]$. $\quad W$ = Weyl group.

- triv rep $\longleftrightarrow$ triv rep of G
- sgn rep $\longleftrightarrow$ Steinberg rep of G
- reflection rep E $\longleftrightarrow$ minimal rep of G (defined by reflection)

q°

$q^{|\Phi^+|}$

$q^{d_1 - 1} + \dots + q^{d_\ell - 1}$

G: $\forall$ s.c. simply laced Chevalley group.

$\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ a root system.

where $\{d_i\}_{i=1}^{\ell}$ are composed of G

$d_i = \deg P_i \quad \mathbb{C}[E]^W = \mathbb{C}[P_1, \dots, P_\ell]$

$d_1 \le \dots \le d_\ell$

$$|G| = \prod_{i=1}^{\ell} (q^{d_i} - 1)\, q^{|\Phi^+|}$$

minimal $\quad$ regular

d_ℓ: coexter number h

G	A_n	D_n	E_6	E_7	E_8
h	$n+1$	$2n-2$	12	18	30

It will be convinient for us to think of G as the group of F-points of a Chevalley group $\mathbf{G}$. Let $\mathcal{R}$ be the ring of integers of F. Then K is simply the group of $\mathcal{R}$-points of $\mathbf{G}$. Let ϖ be a uniformizing element of $\mathcal{R}$ and $\mathcal{R}/\varpi\mathcal{R} \cong \mathbb{F}_q$ the residue field of F.

Let K_1 be the first principal congruence subgroup. Then $K/K_1 \cong \mathbb{G}$ is the finite group of $\mathbb{F}_q$-points of $\mathbf{G}$. Let I, $K_1 \subset I \subset K$ be an Iwahori subgroup of G. Then $I/K_1 \cong \mathbb{B}$, a Borel subgroup of $\mathbb{G}$. Let H be the Hecke algebra of I-biinvariant compactly supported functions on G. The space of I-fixed vectors of a smooth representaton of G is naturally an H-module. It is a well known result of Borel [1] that this correspondence defines an equivalence between the category of representations of G generated by its I-fixed vectors and the category of representations of H.

The algebra H can be described as follows. Let $\Delta = \{\alpha_1, \dots \alpha_n\}$ be a set of simple roots. Let $-\alpha_0$ be the maximal root and let $\bar{\Delta} = \Delta \cup \{\alpha_0\}$. Let $<,>$ be a Killing form normalized so that $<\alpha_i, \alpha_i> = 2$ for all i. Then H is generated by T_i, $i = 0, \dots n$ satisfying the following relations:

$$\begin{aligned} T_iT_j &= T_jT_i \quad \text{if} \quad <\alpha_i, \alpha_j> = 0; \\ T_iT_jT_i &= T_jT_iT_j \quad \text{if} \quad <\alpha_i, \alpha_j> = -1; \\ &\text{and} \quad (T_i - q)(T_i + 1) = 0. \end{aligned}$$

Define an irreducible H-module E by (see [7])

$$E = \oplus_{i=0}^{n} \mathbb{C} e_i$$

with the action of H given by

$$T_i e_j = \begin{cases} -e_j & \text{if } \alpha_i = \alpha_j; \\ qe_j + q^{\frac{1}{2}} e_i & \text{if } <\alpha_i, \alpha_j> = -1; \\ qe_j & \text{if } <\alpha_i, \alpha_j> = 0. \end{cases}$$

$\rightsquigarrow$ trivial $T_i \mapsto q \longleftrightarrow$ trivial repn of G

sgn: $T_i \mapsto -1 \longleftrightarrow$ St.

reflection E of $H \longleftrightarrow$ minimal

- Gelfand – Kirillov dimension

F p-adic. $\{K_n\}_{n=1}$ Principal congruence subgroup:

Let $A = \mathcal{O}_F$, $P \subseteq A$: max'l ideal. $\quad 1 \to K_n \to G(A) \to G(A/P^n) \to 1$

$(\pi, V) \in \mathrm{Irr}(G)$, $V^{K_n} \subseteq V^{K_{n+1}}$, $\quad V = \bigcup_{n=1}^{\infty} V^{K_n}$, $\quad \dim V^{K_n} = c \cdot q^{d \cdot n} + \dots$

GK-dim.

triv: $d = 0$

St. $d = |\Phi^+| = \frac{1}{2} \dim \mathcal{O}_{reg}$

minimal: $d = h - 1 = \frac{1}{2} \dim \mathcal{O}_{min}$

Lecture 2 (General discussion)

$\mathfrak{g}$: split simply connected of type A, D, E

β: highest wt. $\alpha_0 = -\beta$

$\varphi: SL_2 \to \mathfrak{g}$ corresponding to β

Let $Q = LU$: max'l parabolic corresponding to the circled ?.

Then U: Heisenberg. Center $Z = U_\beta$ root group

minimal rep'n $\circlearrowleft G$

$\pi_{Z,\psi} = \omega_\psi \Leftrightarrow$ Heisenberg rep.

$G = L^{der}$, $V = U/Z$ is symplectic rep of G.

F: Knop ($F = \mathbb{C}$), (G, V) "multi free" symp. rep.

$\mathfrak{g}$	D_n	E_6	E_7	E_8
G	$SL_2 \times Spin_{2n-4}$	SL_6	$Spin_{12}$	E_7
V	$V_2 \otimes V_{2n-4}$	$\Lambda^3 V_6$	V_{32}	V_{56}
M^{der}		$SL_2 \times SL_3$	SL_6	E_6
N		$V_3 \otimes V_3^*$	$\Lambda^2 V_6$	V_{27} & cubic

$\mathbb{C}[V]^G \curvearrowright \mathrm{End}_G(\omega_\psi)$

$SL_2 = \varphi(SL_2) \subseteq \mathfrak{g}$

$SL_2 \times G \subseteq \mathfrak{g}$

$SL_2 \supseteq B = TU$, $U = U_\beta = Z$

(U, ψ): a fixed Whittaker data

$\pi|_{SL_2 \times G} \rightsquigarrow$ decomposition of ω_ψ as G-mod

If σ: irr. rep'n of SL_2, $\Theta(\sigma) = (\pi \otimes \sigma^\vee)_{SL_2}$

π — — — G, $\Theta(\pi) = (\pi \otimes \pi^\vee)_G$

$$(\Theta(\pi))_{U,\psi} = ((\pi \otimes \pi^\vee)_G)_{U,\psi}$$
$$= (\pi_{U,\psi} \otimes \pi^\vee)_G = (\omega_\psi \otimes \pi^\vee)_G$$
$$\mathrm{Hom}_G(\omega_\psi, \pi) \cong (\omega_\psi \otimes \pi^\vee)^*_G$$

- Determine $\Theta(\sigma)$ for all σ.
- $\Theta(\pi)$ is of finite length.
- $\Theta(\pi)$ is irreducible

<thm>: Let π: an irrep of G, then $\mathrm{Hom}_G(\omega_\psi, \pi) \neq 0$ iff $\pi = \Theta(\sigma)$ for some σ, irrep of SL_2. Such that $\sigma_{U,\psi} \neq 0$

moreover, $\mathrm{Hom}_G(\omega_\psi, \pi) = \mathbb{C}$

· Tools to compute $\Theta(\sigma)$.

(1) Jacquet functor

(2) Fourier Jacobi functor

$$\mathrm{Hom}_{SL_2}(\pi, \sigma) = \mathrm{Hom}_{SL_2}(\pi, \mathrm{Ind}_B^{SL_2} \chi)$$
$$\cong \mathrm{Hom}_T(r_B(\pi), \chi)$$

understand

$r_B(\pi) := \delta_B^{-1/2} \pi_U$ as $T \times G$ mod

Assume $\sigma = \mathrm{Ind}_B^{SL_2}(\chi)$. $\mathfrak{g} \supseteq SL_2 \times G$

$Q = LU$, $U \triangleright U$

$\Omega \subseteq V \subseteq V^\vee$: minimal G-orbit in $\hat{V} \cong V$

$$0 \to C_c(\Omega) \to \pi_U \to \pi_U \to 0$$

$U/U = V$

Let P: stabilizer of a highest wt line in V, $P = MN \subseteq G$

<Thm> (example of Kudla filtration) $r_B(\Pi)$ as $G\times T$ module has a filtration with two subquotients:

- $\delta_B^{-1/2}\,\Pi_{\mathcal{U}} = \underline{\Pi[G]} \otimes |a|^{2+\delta} \oplus \mathbb{C} \otimes |a|^{2+2\delta}$ (minimal)
- $\mathrm{Ind}_{P\times T}^{G\times T}(C_c(GL_1))$ $\qquad \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} N \mapsto |a|$

($GL_1 \simeq P/P^{der}$, $P \subset GL_4$ via fundamental cut)

$$\mathrm{Hom}_{SL_2}(\Pi, \mathrm{Ind}_B^{SL_2}\chi) \simeq \mathrm{Hom}_T(r_B(\Pi), \chi) \simeq \mathrm{Hom}_T(\mathrm{Ind}_{P\times T}^{G\times T}(C_c(GL_1)), \chi)$$

assuming χ is unitary

$$\simeq \mathrm{Hom}_G(\mathrm{Ind}_P^G(\chi), \Pi)$$

$$\Rightarrow \underline{\Theta(\mathrm{Ind}_B^{SL_2}(\chi)) = \mathrm{Ind}_P^G(\chi)}$$

(example of Jacquet functor) ($\Theta \leftrightarrow$ algebraic tensor product)

Irreducibility of $\Theta(\sigma)$ for (tempered) σ.

Use FJ functor w.r.t G. Let $Q = LH$ be a Heisenberg parabolic of G

$FJ_G(\pi) \leftarrow$ rep of L^{der}

$\uparrow$ rep of G

$\mathfrak{g}$	$\mathfrak{so}_6$	$\mathfrak{so}_7$	$\mathfrak{e}_8$
G	SL_6	$Spin_{12}$	E_7
L^{der}	$SL_4 \simeq Spin_6$	$SL_2 \times Spin_8$	$Spin_{12}$

<Prop>: $FJ_G(\pi) = \Omega_\psi$ (minimal) Weil Rep for $SL_2 \times Spin_{2n}$

$2n = 6, 8, 12$

For $\mathfrak{g} = \mathfrak{e}_7$ case, the factor SL_2^1 acts trivially.

$$FJ_G(\Theta(\sigma)) = FJ_G((\Pi\otimes\sigma^\vee)_{SL_2})$$
$$= (FJ_G(\Pi)\otimes\sigma^\vee)_{SL_2}$$

$\rightsquigarrow FJ_G(\Theta(\sigma)) = \Theta_\psi(\sigma)$ $\Leftarrow$ irreducible

<Prop>: For σ tempered, $\Theta(\sigma)$ is irreducible and $\neq 0$, and

$$[\Theta(\sigma) = \Theta(\sigma')] \Rightarrow [\sigma = \sigma'].$$

<Prop>: $\Theta(\pi)$ is of finite length.

<pf>: Assume σ is a cuspidal rep, appearing in $\Theta(\pi)$. then $\pi = \Theta(\sigma)$.

For other Bernstein block, it suffice to prove, $r_B(\Theta(\pi))$ is finite.:

$$r_B(\Theta(\pi)) = r_B((\Pi\otimes\pi^\vee)_{G}) = (r_B(\Pi)\otimes\pi^\vee)_G$$

Thus now we can analyze the Weil representation $\Omega_\psi \cong \Pi_{U,\psi}$ via the theta correspondence.

Theorem 6.5. *Let π be an irreducible representation of G. Then* $\dim \mathrm{Hom}_G(\Omega_\psi, \pi) \leq 1$ *and it is 1 precisely when π is a lift of an irreducible (U, ψ)-generic representation of* SL_2.

Proof. We have

$$(\Omega_\psi \otimes \pi^\vee)_G \cong (\Pi_{U,\psi} \otimes \pi^\vee)_G \cong \Theta(\pi)_{U,\psi}.$$

Since $\Theta(\pi) \neq 0$ if and only if π is a lift of an irreducible representation σ of SL_2, and then $\Theta(\pi) = \sigma$, and the theorem follows. □

7 Mini theta

Let V be the standard (symplectic) 2-dimensional representation of $\mathrm{SL}_2(F)$, realized as the set of 2×1 column vectors. Now we consider the n-fold tensor product $V \otimes \cdots \otimes V$. This a faithful representation of the group

$$S_n = \mathrm{SL}_2^n / \mu_2^{n-1}$$

where μ_2^{n-1} is the subgroup of the center of SL_2^n, consisting of $(x_1, \ldots, x_n)$ such that $x_1 \cdots x_n = 1$. In order to understand the corresponding group of F-points, observe that on $V \otimes \cdots \otimes V$ we have a faithful action of

$$G_n = \mathrm{GL}_2^n / \mathbb{G}_m^{n-1}$$

where $\mathbb{G}_m^{n-1}$ is the subgroup of the center of GL_2^n consisting of n-tuples $(z_1, \ldots, z_n)$ of scalar matrices such that $z_1 \cdots z_n = 1$. The group $G_n(F)$ is given by the same quotient, by Hilbert 90. If π is an irreducible representation of $\mathrm{GL}_2(F)$ then $\pi \otimes \ldots \otimes \pi$ (tensor n-times) is naturally a representation of $G_n(F)$.

Let det be the character of G_n that restricts to the usual determinant on each GL_2. Then S_n is the kernel of the determinant, this is an easy check over algebraic closure of F. This description gives as the following inductive construction of S_n, which will play an important role. Let

$$(G_{n-1} \times \mathrm{GL}_2)^{\det} = \{(g, h) \mid \det g \cdot \det h = 1\}.$$

Let $\Delta\mathbb{G}_m$ be embedding of $\mathbb{G}_m$ in the center of the above group given by $z \mapsto ((z, 1, \ldots, 1), z^{-1}) = (1, z, \ldots, 1), z^{-1}) \ldots$. Then

$$S_n \cong (G_{n-1} \times \mathrm{GL}_2)^{\det} / \Delta\mathbb{G}_m$$

and this is true on the level of F-points. Thus any smooth S_n-module Ω can be viewed as an $(G_{n-1} \times \mathrm{GL}_2)^{\det}$-module. Let π be an irreducible representation of $\mathrm{GL}_2(F)$. Let $\mathrm{SL}_2 \subset (G_{n-1} \times \mathrm{GL}_2)^{\det}$, sitting in the second factor. Observe that $(G_{n-1} \times \mathrm{GL}_2)^{\det}/\mathrm{SL}_2 \cong G_{n-1}$. Thus

$$\Theta(\pi) = (\Omega \otimes \pi^\vee)_{\mathrm{SL}_2}$$

is naturally a smooth G_{n-1}-module. In the following three subsections, for $n = 2, 3, 4$ we shall exhibit Ω such that $\Theta(\pi) = \pi \otimes \ldots \otimes \pi$ for all irreducible Whittaker generic π.

7.1 Two SL_2

In this case $S_2 = \mathrm{SL}_2 \times \mathrm{SL}_2/\mu_2 \cong \mathrm{SO}(4)$. Indeed, $V \otimes V$ is an orthogonal representation of $\mathrm{SL}_2 \times \mathrm{SL}_2/\mu_2$. It can be identified with the space M_2 of 2×2 matrices by the map

$$v \otimes w \mapsto v \cdot w^\top$$

where $w^\top$ is the transpose of w. The determinant is a quadratic form that turns M_2 into a quadratic space. The action of $(g, h) \in (\mathrm{GL}_2 \times \mathrm{GL}_2)^{\det}$ on $x \in M_2$ is given by $gxh^\top$. This gives as isomorphism $(\mathrm{GL}_2 \times \mathrm{GL}_2)^{\det}/\Delta\mathbb{G}_m \cong \mathrm{SO}(4)$. Since the action preserves the determinant, it follows that $(\mathrm{GL}_2 \times \mathrm{GL}_2)^{\det}$ acts naturally on the regular representation $C_c(\mathrm{SL}_2(F))$. Now we have

$\cong \mathrm{SL}_2^2/\mu_2(F)$ $\quad$ $(g,h) \in \mathrm{SL}_2 \times \mathrm{SL}_2$. $((g,h)f)(x) = f(g^\top x h)$.

Proposition 7.1. *Let π be an irreducible representation of $\mathrm{GL}_2(F)$. Then*

$$(C_c(\mathrm{SL}_2(F)) \otimes \pi^\vee)_{\mathrm{SL}_2} \cong \pi$$

as $(\mathrm{GL}_2 \times \mathrm{GL}_2)^{\det}/\mathrm{SL}_2 \cong \mathrm{GL}_2$-modules.

Proof. We have a natural GL_2-intertwining map

$$(C_c(\mathrm{SL}_2) \otimes \pi^\vee)_{\mathrm{SL}_2} \to \pi$$

given by $(f, v) \mapsto \pi(f)v$. To prove that this is bijective, restrict $\pi = \pi_1 + \cdots + \pi_s$ to SL_2, and use

$$(C_c(\mathrm{SL}_2) \otimes \pi_i^\vee)_{\mathrm{SL}_2} \cong \pi_i$$

for any irreducible representation π_i of SL_2.

□

7.2 Three SL_2

Observe that $V \otimes V \otimes V$ is naturally a symplectic space. Hence

$$\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2/\mu_2^2 \subset \mathrm{Sp}_8$$

Fix an additive character ψ of F. It gives us a Whittaker datum (U, ψ) for SL_2, as well as a Weil representation ω_ψ of Sp_8. By restriction, ω_ψ is an S_3-module. Decomposing ω_ψ under the action of S_3 is closely related to the classical theta correspondence, where (any) two SL_2 are the factors of $\mathrm{SO}_4 \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)^{\det}/\Delta(\mathbb{G}_m)$ and the remaining SL_2 is a symplectic group.

We realize ω_ψ, as a Schroedinger model, on the space of compactly supported smooth functions $C_c(M_2(F))$, with the geometric action of $(\mathrm{GL}_2 \times \mathrm{GL}_2)^{\det}$, while the action of the third SL_2 is generated by

$$u \cdot f(x) = \psi(u \det(x)) f(x),$$

for $u \in U$, and a Fourier transform. Namely, if w is the non-trivial Weyl group element of the third SL_2 and $\phi \in C_c(M_2(F))$, then, by [Kud96], we have

$$\omega_\psi(w)(\phi)(x) = \int_{M_2(F)} \psi(\langle x, y \rangle)\phi(-y)\ dy = -\mathcal{F}(\phi)(-x).$$

From [MS97, Lemma 2.2] and the description of the action of U for the third SL_2, it follows at once that

$$(\omega_\psi)_{U,\psi} \cong C_c(\mathrm{SL}_2)$$

as $(\mathrm{GL}_2 \times \mathrm{GL}_2)^{\det}$-modules.

Proposition 7.2. *Let π be an irreducible, Whittaker generic, representation of* $\mathrm{GL}_2(F)$*. Then*

$$\Theta(\pi) := (\omega_\psi \otimes \pi^\vee)_{\mathrm{SL}_2} \cong \pi \otimes \pi$$

as $(G_2 \times \mathrm{GL}_2)^{\det}/\mathrm{SL}_2 \cong G_2$*-modules.*

Proof. Let SL_2 be one of two in GL_2. Without loss of generality we can take the Schroedinger model of ω_ψ such that this SL_2 plays the role of the symplectic group. Since

$$\Theta(\pi)_{U,\psi} \cong ((\omega_\psi)_{U,\psi} \otimes \pi^\vee)_{\mathrm{SL}_2} \cong (C_c(\mathrm{SL}_2(F)) \otimes \pi^\vee)_{\mathrm{SL}_2} \cong \pi$$

where the last isomorphism is Proposition 7.1. Now we can repeat the argument with the other SL_2 in G_2. It follows that $\pi \otimes \pi$ is a subquotient of $\Theta(\pi)$, and all other subquotients are non-generic on both factors, that is, a trivial representation when restricted to $\mathrm{SL}_2 \times \mathrm{SL}_2$. Working with three SL_2, the trivial representation will split off of $\Theta(\pi)$ using the action of the Bernstein center, unless π is Steinberg. In this case, it is a priori possible that $\Theta(\mathrm{St})$ has $\mathrm{St} \otimes \mathrm{St}$ as a quotient and $1 \otimes 1$ as the next subquotient. However, by Küneth formula,

$$\mathrm{Ext}^1_{\mathrm{SL}_2 \times \mathrm{SL}_2}(1 \otimes 1, \mathrm{St} \otimes \mathrm{St}) = 0$$

and $1 \otimes 1$ must be an SL_2^2-invariant quotient of $\Theta(\mathrm{St})$. Thus in all cases, if $\Theta(\pi)$ is larger than $\pi \otimes \pi$, then $1 \otimes 1 \otimes \sigma$ is a quotient of ω_ψ, where σ is an SL_2-summand of π, a contradiction to the fact that the small theta lift of the trivial representation of SL_2 is the trivial representation of SL_2^2 ([Kud96], [BH21]). □

7.3 Four SL_2

We have

$$\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2/\mu_2^3$$

in adjoint D_4. Let Π be the minimal representation of D_4. Let (U, ψ) be the Whittaker datum for any one SL_2. Then

$$\Pi_{U,\psi} = \omega_\psi$$

Proposition 7.3. *Let π be an irreducible, Whittaker generic, representation of* $\mathrm{GL}_2(F)$*. Then*

$$\Theta(\pi) := (\Pi \otimes \pi^\vee)_{\mathrm{SL}_2} \cong \pi \otimes \pi \otimes \pi$$

as $(G_3 \times \mathrm{GL}_2)^{\det}/\mathrm{SL}_2 \cong G_3$*-modules.*

Proof. Let (U, ψ) be the Whittaker datum for any of the three SL_2 in G_3. Then

$$\Theta(\pi)_{U,\psi} \cong (\Pi_{U,\psi} \otimes \pi^\vee)_{\mathrm{SL}_2} \cong (\omega_\psi \otimes \pi^\vee)_{\mathrm{SL}_2} \cong \pi \otimes \pi$$

where the last isomorphism is Proposition 7.2. It follows that $\pi \otimes \pi \otimes \pi$ is a subquotient of $\Theta(\pi)$, and all other subquotients are non-generic on all factors, that is, a trivial representation when restricted to SL_2^3. The trivial representation will split off of $\Theta(\pi)$ using the action of the Bernstein center, unless π is Steinberg. In this case, it is a priori possible that $\Theta(\mathrm{St})$ has $\mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St}$ as a quotient and $1 \otimes 1 \otimes 1$ as the next subquotient. However, by Küneth formula,

$$\mathrm{Ext}^1_{\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2}(1 \otimes 1 \otimes 1, \mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St}) = 0$$

and $1 \otimes 1 \otimes 1$ is a quotient of $\Theta(\mathrm{St})$. Thus in all cases, if $\Theta(\pi)$ is larger than $\pi \otimes \pi \otimes \pi$, then $1 \otimes 1 \otimes 1 \otimes \sigma$ is a quotient of ω_ψ, where σ is an SL_2-summand of π, a contradiction to Proposition 2.4 which implies that the small theta lift of the trivial representation of one SL_2 is the trivial representation of SL_2^3. □

Now we state a consequence of Proposition 7.3 that we shall need later. In the adjoint D_4 we have a dual pair

$$\mathrm{SL}_2 \times (\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2/\mu_2^2).$$

If σ is an irreducible representation of SL_2 let

$$\mathrm{mini}(\sigma) = (\Pi \otimes \sigma^\vee)_{\mathrm{SL}_2}$$

be its theta lift to $(\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2/\mu_2^2)$. The correspondence described in Proposition 7.3 is a version of this lift for groups of similitudes, as explained in [BGS24]. We are also using a simple observation that if the big theta lift for similitudes correspondence is irreducible, then the big theta lift for the original correspondence is certainly semi-simple, so also irreducible, by Theorem 5.1 of [BGS24]; see Lemma 5.3. of [BGS24]. Thus we have the following:

Proposition 7.4. *Let π be an irreducible, Whittaker generic, representation of $\mathrm{GL}_2(F)$. Let σ be an irreducible SL_2-summand of π. Then $\mathrm{mini}(\sigma)$ is an $(\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2/\mu_2^2)$-irreducible summand of $\pi \otimes \pi \otimes \pi$. As σ runs over all summands of π, $\mathrm{mini}(\sigma)$ runs over all irreducible summands of $\pi \otimes \pi \otimes \pi$.*

8 Computing parameters of lifts of tempered representations

Here we determine $\Theta(\sigma)$ for supercuspidal σ using a "tower" of theta correspondences, starting with the correspondence in D_4, Proposition 7.4 as the base case. To that end, let $\mathscr{G}_i$, $i = 4, \ldots, 8$ denote one of the following simply connected Chevalley groups in the sequence

$$D_4 < D_5 < E_6 < E_7 < E_8$$

where i denotes the rank of the group. Let $\mathscr{P}_i = \mathscr{M}_i \mathscr{N}_i$ be the (standard) maximal parabolic in $\mathscr{G}_i$ such that $[\mathscr{M}_i, \mathscr{M}_i] = \mathscr{G}_{i-1}$. We can arrange the dual pair $\mathrm{SL}_2 \times G_i$ in $\mathscr{G}_i$ such that SL_2 is contained in the Levi $\mathscr{M}_i$. Let $P_i = M_i N_i$ be the centralizer of SL_2 in $\mathscr{P}_i$. It is a maximal parabolic subgroup in G_i. Let $\bar{\mathscr{N}_i}$ and $\bar{N}_i$ denote the unipotent radical of respective opposite parabolic subgroups. Then $V_i = \bar{\mathscr{N}_i}/\bar{N}_i$ is abelian, and thus a module for $\mathrm{SL}_2 \times M_i$. One checks that $V_i = V_2 \otimes V_{M_i}$ for an M_i-module V_{M_i}. The cases are given by the following table, where the factor SL_2' acts trivially on V_4. The meaning of r_i and t_i will be explained below.

$\mathscr{G}_i$	D_5	E_6	E_7	E_8
$\mathscr{M}_i$	D_4	D_5	E_6	E_7
M_i^{der}	$\mathrm{SL}_2' \times \mathrm{SL}_2'' \times \mathrm{SL}_2'''$	$\mathrm{SL}_2' \times \mathrm{SL}_4$	SL_6	Spin_{12}
V_{M_i}	$2V_2'$	V_4	$\wedge^2 V_6$	V_{12}
t_i	1	$3/2$	2	3
r_i	-1	$-3/2$	-3	$-11/2$

Let ϖ_i be the fundamental weight for $\mathcal{G}_i$ with respect to the maximal parabolic $\mathscr{P}_i$. Let

$$|\cdot|_{\mathscr{M}_i} : \mathscr{M}_i \to \mathbb{R}^+$$

be the positive character such that for any $\mathscr{G}_i$ co-root $\alpha^\vee : \mathbb{G}_m \to \mathscr{M}_i$ we have

$$|\alpha^\vee(t)|_{\mathscr{M}_i} = |t|^{\varpi_i(\alpha^\vee)}$$

Let $\Omega_i \subset V_i$ be the $\mathrm{SL}_2 \times M_i$-orbit consisting of highest weight vectors. By Theorem 1.1 and Theorem 6.1 in [MS97], we have an exact sequence of $\mathrm{SL}_2 \times M_i$-modules

$$0 \to C_c(\Omega_i) \to \Pi_{N_i} \to \Pi_{\mathscr{N}_i} \to 0,$$

where

$$\Pi_{\mathscr{N}_i} = \Pi[\mathscr{M}_i] \otimes |\cdot|_{\mathscr{M}_i}^{t_i} \oplus \mathbb{C} \otimes |\cdot|_{\mathscr{M}_i}^{s_i}$$

a direct sum of the minimal representation and the trivial representation of $\mathscr{M}_i$, twisted by the powers of $|\cdot|_{\mathscr{M}_i}$ where t_i is given in the table above. We note that our s_i and t_i are obtained form corresponding s and t in [MS97] by dividing by lengths of fundamental characters ϖ_i. The action of $\mathrm{SL}_2 \times M_i$ on $C_c(\Omega_i)$ is geometric twisted by the character $|\cdot|_{\mathscr{M}_i}^{s_i}$.

Proposition 8.1. *Let σ be an irreducible, supercuspidal representation of SL_2. Let $\Theta_i(\sigma)$ be its lift to G_i. Then, as M_i-modules,*

$$r_{P_i}(\Theta_i(\sigma)) \cong (\Pi[\mathscr{M}_i] \otimes \sigma^\vee)_{\mathrm{SL}_2} \otimes |\cdot|_{M_i}^{r_i}$$

where $|\cdot|_{M_i}$ is defined analogously to $|\cdot|_{\mathscr{M}_i}$ but with respect to G_i and r_i is given in the table above.

Proof. The proposition follows from the filtration of Π_{N_i}, if we can show that σ does not appear as a quotient of the bottom piece $C_c(\Omega_i)$. But this is done exactly as in Proposition 3.2, (the discussion of the sub quotient Π_1/Π_2 there). As far as figuring out r_i goes, we note that the restriction of $|\cdot|_{\mathscr{M}_i}$ to M_i is precisely $|\cdot|_{M_i}$ and then it remains to divide with the modular character $\delta_{P_i}^{1/2}$ to determine r_i.

□

We realize the root system E_7 siting in $\mathbb{R}^8$, in the standard fashion, where the simple roots are

$$\alpha_1 = \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + e_4 + e_5 + e_6 + e_7),\ \alpha_2 = e_1 + e_2,$$
$$\alpha_3 = e_2 - e_1,\ \alpha_4 = e_3 - e_2,\ \alpha_5 = e_4 - e_3,\ \alpha_6 = e_5 - e_4,\ \alpha_7 = e_6 - e_5.$$

The following is the Dynkin diagram of E_7.

We consider G_i, for $i = 6, 7, 8$ to be in the sequence $A_5 < D_6 < E_7$ where A_5 is picked to contain the three black dots. Let $Q_i = L_i U_i$ be the standard parabolic in G_i corresponding to the three simple roots, that is, $[L_i, L_i] = \mathrm{SL}_2^3$. We shall prove that $\Theta_i(\sigma)$, where σ is a tempered representation of SL_2, are Langlands quotients of standard modules supported on

Q_i. To that end, let T_i be the connected component of the center of L_i. It is a split torus of the dimension $2, 3$ and 4, respectively. One checks that $[L_i, L_i] \cap T_i = \mu_2^2$. It follows that

$$L_i/T_i \cong \mathrm{SL}_2^3/\mu_2^2$$

and $\mathrm{mini}(\sigma)$ is thus a representation of L_i. We shall prove that $\Theta_i(\sigma)$ is the Langlands quotient of $\mathrm{Ind}_{Q_i}^{G_i}(\mathrm{mini}(\sigma) \otimes \chi_i)$ where χ_i is a positive character of L_i. We shall describe χ_i by giving the weight λ_i such that

$$\chi_i(\alpha^\vee(t)) = |t|^{\lambda_i(\alpha^\vee)}$$

for any co-root $\alpha^\vee$. At the same time we shall verify that the Langlands parameter of $\Theta_i(\sigma)$ is

$$\varphi_{G_i} : WD_F \to G_i^\vee,$$

described in the introduction in terms of the principal SL_2 in $\mathrm{Aut}(J_i)$. However, this SL_2 is also principal in $S_i^\vee$, the dual group of the Levi factor S_i of the Siegel parabolic from Section 2 and denoted M there. Hence λ_i should be a W_{G_i}-conjugate of ρ_{S_i}. We give details on case by case basis, using reverse engineering.

Case $G_6 = A_5$. Here $S_6 = A_2 \times A_2$ and $\rho_{S_6} = (1, 0, 1, -1, 0, 1)$ which, after using the Weyl group W_{G_6}, can be put in the positive position

$$\lambda_6 = (1, -1, 0, 0, 1, 1).$$

Observe that λ_6 is perpendicular to the three black simple roots. Thus it defines a positive character χ_6 of L_6 such that $\mathrm{Ind}_{Q_6}^{G_6}(\mathrm{mini}(\sigma) \otimes \chi_6)$ is a standard module. The parameter of its Langlands quotient is φ_{G_6}.

Case $G_7 = D_6$. Here $S_7 = A_5$, but this A_5 is not the one containing the three black roots. Hence $\rho_{S_7} = (-5, -3, -1, 1, 3, 5)/2$ which, after using the Weyl group W_{G_7}, can be put in the positive position

$$\lambda_7 = (-1, 1, 3, 3, 5, 5)/2.$$

Then λ_7 is perpendicular to the three black simple roots. Thus it defines a positive character χ_7 of L_7 such that $\mathrm{Ind}_{Q_7}^{G_7}(\mathrm{mini}(\sigma) \otimes \chi_7)$ is a standard module. The parameter of its Langlands quotient is φ_{G_7}.

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Case $G_8 = E_7$. Here $S_8 = D_6$ and $\rho_{S_8} = (0, 1, 2, 3, 4, -4, -4, 4)$ which, after using the Weyl group W_{G_8} can be put in the positive position

$$\lambda_8 = (-1, 1, 3, 3, 5, 5, -11, 11)/2.$$

Again, λ_8 is perpendicular to the three black simple roots. Thus it defines a positive character χ_8 of L_8 such that $\mathrm{Ind}_{Q_8}^{G_8}(\mathrm{mini}(\sigma) \otimes \chi_8)$ is a standard module. The parameter of its Langlands quotient is φ_{G_8}.

Proposition 8.2. *Assume that σ is tempered. In each of the three cases above, $\Theta_i(\sigma)$ is the Langlands quotient of* $\mathrm{Ind}_{Q_i}^{G_i}(\mathrm{mini}(\sigma) \otimes \chi_i)$.

Proof. To keep the exposition simple we shall assume that σ is supercuspidal, otherwise we already determined $\Theta_i(\sigma)$. In this case we shall prove that $r_{Q_i}(\Theta_i(\sigma)) \cong \mathrm{mini}(\sigma) \otimes \chi_i^{-1}$. This implies that $\Theta_i(\sigma)$ is the unique sub of $\mathrm{Ind}_{Q_i}^{G_i}(\mathrm{mini}(\sigma) \otimes \chi_i^{-1})$, an equivalent statement.

Now observe that the normalized Jacquet functor r_{Q_i} is simply the composite

$$r_{Q_i} = r_{P_4} \circ \ldots \circ r_{P_i}$$

hence $r_{Q_i}(\Theta_i(\sigma)) \cong \mathrm{mini}(\sigma) \otimes \chi_i^{-1}$ will be established by induction on i.

We leave as an exercise to the reader to check this for $G_6 = A_5$. We move on to $G_7 = D_6$. Proposition 8.1 says that $r_{P_7}(\Theta_7(\sigma))$ is $\Theta(\sigma_6)$ twisted by a positive character of the Levi $M_7 = A_5$ corresponding to -3 times the fundamental character for the pair (D_6, A_5). Since the difference

$$\lambda_6 - \lambda_7 = (1, -1, 0, 0, 1, 1) - (-1, 1, 3, 3, 5, 5)/2 = -(-3, 3, 3, 3, 3, 3)/2$$

is precisely that character, the proposition follows for $G_7 = D_6$. We move on to $G_8 = E_7$. Proposition 8.1 says that $r_{P_8}(\Theta_8(\sigma))$ is $\Theta(\sigma_7)$ twisted by a positive character of the Levi $M_8 = D_6$ corresponding to $-11/2$ times the fundamental character for the pair (E_7, D_6). Since the difference

$$\lambda_7 - \lambda_8 = (-1, 1, 3, 3, 5, 5, 0, 0)/2 - (-1, 1, 3, 3, 5, 5, -11, 11)/2 = -(0, 0, 0, 0, 0, 0, -11, 11)/2$$

is precisely that character, the proposition follows for $G_8 = E_7$. □

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References

[AG17] Hiraku Atobe and Wee Teck Gan, *Local theta correspondence of tempered representations and Langlands parameters*, Invent. Math. **210** (2017), no. 2, 341–415 (English).

[BGS24] Petar Bakić, Wee Teck Gan, and Gordan Savin, *Similitude exceptional theta correspondences*, Rad Hrvatske Akademije Znanosti i Umjetnosti **558** (2024), no. 28, 193–222.

[BH21] Petar Bakić and Marcela Hanzer, *Theta correspondence for p-adic dual pairs of type I*, J. Reine Angew. Math. **776** (2021), 63–117 (English).

[BS12] Rolf Berndt and Ralf Schmidt, *Elements of the representation theory of the Jacobi group*, reprint of the 1998 original ed., Mod. Birkhäuser Class., Basel: Birkhäuser, 2012 (English).

[BZ76] I. N. Bernshteĭn and A. V. Zelevinskiĭ, *Representations of the group* $\mathrm{GL}(n, F)$ *where* F *is a non-Archimedean local field*, Russ. Math. Surv. **31** (1976), no. 3, 1–68 (English).