In particular, G acts on itself by conjugation, and so it acts on $\mathfrak{g} = T_e(G)$. This representation is called the *adjoint representation* and is denoted as Ad.

First, The following diagram is commutative.

$$\begin{tikzcd} G \arrow[r, "f"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{d}f"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

Let $H = G$, fix $g \in G$, $f_g(h) = ghg^{-1}$. $\mathrm{d}f : \mathfrak{g} \to \mathfrak{g}$ is denoted $\mathrm{Ad}(g)$.

$$\begin{tikzcd} G \arrow[r, "f_g"] & G \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{Ad}(g)"'] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd}$$

(We have $\exp(\mathrm{Ad}(g)(X)) = g(\exp(X))g^{-1}$) We should confirmed that Ad is a smooth homomorphism from G into $GL(\mathfrak{g})$.

We show next that the differential of Ad is ad ($\mathrm{ad}\, x(y) = [x, y],\ x, y \in \mathfrak{g}$):

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & \mathrm{GL}(\mathfrak{g}) \\ T_e(G) = \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"'] & \mathrm{End}(\mathfrak{g}) \arrow[u, "\exp"'] \end{tikzcd}$$

Proof. It will be most convenient for us to think of elements of the Lie algebra as tangent vectors at the identity or as local derivations of the local ring there. Let $X,\ Y \in \mathfrak{g}$. If $f \in C^\infty(G)$. Then through our definition:

$$(\mathrm{Ad}(g)Y)f = Y(g.f).$$

To compute the differential of Ad, note that the path $t \to \exp(tX)$ in G is tangent to the identity at $t = 0$ with tangent vector X.

$$f \to \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Ad}\left(\mathrm{e}^{tX}\right)Y\right)f|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}u}f\left(\mathrm{e}^{tX}\mathrm{e}^{uY}\mathrm{e}^{-tX}\right)|_{t=u=0}.$$

By the chain rule, our last expression equals$XYf(1) - YXf(1)$. $\square$

Consequently $\mathrm{Ad}(\exp X)$=$e^{\mathrm{ad}X}$ under this identification. In the special case that G is a closed linear map, we can regard X and g as matrics. Note that:

$$\exp(\mathrm{Ad}(g)(tX)) = g(\exp(tX))g^{-1}$$

differentiate and set $t = 0$. Then we see that $\mathrm{Ad}(g)X = gXg^{-1}$ (it has no specific meaning except equal in amount). If we denote $g \in G$ as $\exp(Y),\ Y \in \mathfrak{g}$, then combine the equations before:

$$(e^{\mathrm{ad}(Y)})X = \exp(Y)X\exp(-Y), \quad X, Y \in \mathfrak{g}$$

This equation is essential in Lie algebra theory and can be proved directly. (Humphreys pg.9)