1 The Genesis of Fourier Analysis

2 Basic properties of Fourier Series

2.1 In Class

2.1.1 Examples and Basic Definitions

Example 1. difference between pointwise convergence and mean-square convergence:

$$\sum_{k=1}^{\infty} \frac{\sin kx}{\sqrt{k}}$$

Definition 1 (**Convolution**).

$$(f * K_n)(x) \coloneqq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) K_n(y) \,\mathrm{d}x$$

Lemma 2.1 (**Dirichlet Kernel**). $f, T = 2\pi$. *All Fourier coefficients are well defined. Then:*

$$S_n(x) = f * D_n(x) = \frac{1}{2\pi} \int_0^{\pi} [f(x-t) + f(x+t)] D_n(t) \,\mathrm{d}t$$

Lemma 2.2 (**Riemann – Lebesgue lemma**). *If f is integrable on the circle then* $\hat{f} \to 0$ *as* $|n| \to \infty$.

If f is integrable on $[0, 2\pi]$*,then*

$$\int_0^{2\pi} f(\theta) \sin(N\theta) \,\mathrm{d}\theta \to 0 \textit{ as } N \to \infty$$

$$\int_0^{2\pi} f(\theta) \cos(N\theta) \,\mathrm{d}\theta \to 0 \textit{ as } N \to \infty$$

Proof. Given $f(\in L^1(\mathbf{T}^n))$, let P be a trigonometric polynomial such that $||f - P||_{L^1} < \epsilon$. If $|m| > degree(P)$ then $\hat{P}(m) = 0$ and thus ... □

Remark (*F is integrable on the circle*). If F is a function on the circle, then we may define for each real number θ,

$$f(\theta) = F(e^{i\theta}),$$

and observe that with this definition, the function f is periodic on $\mathbb{R}$ of period 2π, that is, $f(\theta + 2\pi) = f(\theta)$ for all θ. The integrability, continuity and other smoothness properties of F are determined by those of f.

Theorem 1 (**Localization Principle**).

Definition 2 (**Dini Condition**, **Hölder Condition**).

Theorem 2 (**the Dini Test of the Pointwise Convergence**).

Example 2. $f(x) = x$

Example 3. $f(x) = \cos(\alpha x), \alpha \in (-1, 1)$

Definition 3 (Fejér Kernel).

$$\sigma_n(x) := \frac{1}{n+1} \sum_{k=0}^{n} S_k(x)$$

Corollary 2.1. *If the fourier series is convergent on $x = x_0$, then must be $f(x_0)$.*

Definition 4 (*fourier series for f with $T(> 0)$ as its period*)**.**

2.1.2 Hilbert Space

2.1.3 (Best approximation: mean-square convergence)

Definition 5. If f is an integrable function on the interval $[-\pi, \pi]$, then the n^{th} Fourier coefficient of f is

$$\hat{f}(n) = a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \, \mathrm{d}\theta, \quad n \in \mathbb{Z},$$

and the Fourier series of f is

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

Here we use θ as a variable since we think of it as an angle ranging from $-\pi$ *to* π.

As it turns out, the smoothness of f is directly related to the decay of the Fourier coefficients, and in general, the smoother the function, the faster this decay. As a result, we can expect that relatively smooth functions equal their Fourier series. Since:

Theorem 3. *Suppose that f is a continuous function on the circle and that the Fourier series of f is absolutely convergent, $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then, the Fourier series converges uniformly to f, that is,*

$$\lim_{N \to \infty} S_N(f)(\theta) = f(\theta) \text{ uniformly in } \theta.$$

[Hints: Recall that if a sequence of continuous functions converges uniformly, then the limit is also continuous. And if $\hat{f}(n) = 0$ *for all* $n \in \mathbf{Z}$, then $f(\theta_0) = 0$ whenever f is continuous at the point θ_0.]

more precisely,

Proposition 2.1 (Fourier inversion). *Suppose that $f \in L^1(\mathbf{T}^n)$ and that*

$$\sum_{m \in \mathbf{Z}^n} |\hat{f}(m)| < \infty$$

Then

$$f(x) = \sum_{m \in \mathbf{Z}^n} \hat{f}(m) e^{2\pi i m \cdot x} \quad a.e.$$

and therefore f is almost everywhere equal to a continuous function.

If you notice that:

Lemma 2.3.
$$\hat{f'}(n) = in\hat{f}(n), \ \ for \ all \ n \in \mathbf{Z}$$
if they exist.

Lemma 2.4. *Suppose that f is periodic and of class C^k. Show that*
$$\hat{f}(n) = o\left(1/|n|^k\right),$$

[Hint: Use the Riemann-Lebesgue lemma or Bessel inequality.]
more precisely, we will attain:

Proposition 2.2. *$m \in \mathbb{N}$, f is periodic and of class C^{m-1} and*
$$f^{(j)}(\pi) = f^{(j)}(-\pi), \quad j = 0, 1, 2, ..., m-1$$
then,
$$S_n \rightrightarrows f, \ n \to \infty$$
more precisely,
$$\sup |S_n(x) - f(x)| \leq \frac{\epsilon_n}{n^{m-1/2}}$$

Remark. We can choose $\epsilon_n := (\frac{2}{2m-1} \sum_{|k|<n} |c_k(f^{(m)})|^2)^{1/2}$

2.1.4 differentiability and integrability

2.1.5 Legendre expansion, Hermite function and Chebyshev function

2.1.6 several applications to Geomotry and Boundary Value Problems

2.2 More Details

2.2.1 The Poisson Summation Formula

2.2.2 Decay of Fourier Coefficients

2.2.3 Absolutely Summable Fourier Coefficients

These two sections' results can be put into use in the pointwise convergence of Fourier series. More details can be found in $GTM249$.

2.2.4 Tauberian Theorem

We have followed the course of history before: Alfred Tauber(1987) → Littlewood(1911) → Karamara(1930) → Wiener-Ikehara Theorem(1931).

2.2.5 A continuous function with diverging fourier series

Our technique is symmetry-breaking. Details can be found in [2], which presented a direct construction. (Abstract theory is in the funtional annlysis part below.)

Btw, the global convergence of the series still connot ensure that it's the fourier series of a (Riemann) integrable function. (Problem 1 Chapter 3,[2]) We left it as an exercise (example).

Example 4. For each $0 < \alpha < 1$ the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}}$$

converges for every x but is not the Fourier series of a Riemann integrable function.
(a) If the conjugate Dirichlet kernel is defined by

$$\tilde{D}_N(x) = \sum_{|n| \leq N} \operatorname{sign}(n) e^{inx}$$

then show that

$$\tilde{D}_N(x) = \frac{\cos(x/2) - \cos((N+1/2)x)}{\sin(x/2)} i$$

and

$$\int_{-\pi}^{\pi} \left| \tilde{D}_N(x) \right| dx \leq c \log N$$

(b) As a result, if f is Riemann integrable, then

$$\left(f * \tilde{D}_N \right)(0) = O(\log N)$$

(c) In the present case, this leads to

$$\sum_{n=1}^{N} \frac{1}{n^{\alpha}} = O(\log N)$$

which is a contradiction.

But for $1 < p < \infty$ and if Cesaro operators of the Fourier series are bounded uniformly,

$$\sup_{n \geq 1} ||\sigma_n||_{L^p} < \infty$$

then there always exist $f \in L^p[0, 2\pi]$, which makes σ_n be its Cesaro means of the Fourier series. ([4])

2.2.6 Lacunary Fourier Series and Delayed Means Operator

2.2.7 Solution to Example 4

Proof. Note that by trigonometric identity,

$$\begin{aligned}
& |\tilde{D}_N(x)| \\
= \quad & \left|\frac{\cos(x/2) - \cos((N+1/2)x)}{\sin(x/2)}\right| \\
\leq \quad & 4 \frac{\sin(\frac{N}{2}x)}{x}, \\
& \int_{-\pi}^{\pi} \left| \tilde{D}_N(\theta) \right| \mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&\leq && 8\int_0^\pi \frac{|\sin(N/2)\theta|}{\theta}\,\mathrm{d}\theta \\
&= && 8\int_0^{(N/2)\pi} \frac{|\sin\theta|}{\theta}\,\mathrm{d}\theta \\
&= && 8\sum_{k=0}^{N-1}\int_{k\pi/2}^{(k+1)\pi/2} \frac{|\sin\theta|}{\theta}\,\mathrm{d}\theta \\
&= && 8\sum_{k=1}^{N-1}\int_{k\pi/2}^{(k+1)\pi/2} \frac{|\sin\theta|}{\theta}\,\mathrm{d}\theta + \int_0^{\pi/2}\frac{\sin\theta}{\theta}\,\mathrm{d}\theta \\
&\leq && 8\frac{2}{\pi}\sum_{k=1}^{N-1}\frac{1}{k}\int_{k\pi/2}^{(k+1)\pi/2} |\sin\theta|\,\mathrm{d}\theta + \int_0^{\pi/2}\frac{\sin\theta}{\theta}\,\mathrm{d}\theta \\
&= && \frac{16}{\pi}\sum_{k=1}^{N-1}\frac{1}{k} + \int_0^{\pi/2}\frac{\sin\theta}{\theta}\,\mathrm{d}\theta \\
&\lesssim && \log N + 1
\end{aligned}
$$

If f an integrable function, then on the one hand,

$$\int_{-\pi}^{\pi} f(t)\tilde{D}_N(t)\,\mathrm{d}t \lesssim \log N + 1$$

on the other hand,

$$
\begin{aligned}
& && |(f * \tilde{D}_N)(0))| \\
&= && |\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\tilde{D}_N(t)\,\mathrm{d}t| \\
&= && |\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{|n|\leq N}\mathrm{sign}(n)e^{int}\,\mathrm{d}t| \\
&= && |\sum_{|n|\leq N}\mathrm{sign}(n)\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{int}\,\mathrm{d}t| \\
&\sim && \sum_{n=1}^{N}\frac{1}{n^\alpha} \\
&\sim && N^{1-\alpha}
\end{aligned}
$$

which is a contradiction. $\square$

2.3 Functional Analysis Point of View

Theorem 4 (**Korovkin's theorem**). *Let (K,d) be a compact metric space (equipped with sup-norm), let $\phi \in \mathcal{C}[0,\infty[$ be a function that satisfies $\phi(t) > 0$ for all $t > 0$, and let the function $\psi_x \in \mathcal{C}(K)$ be defined for each $x \in K$ by*

$$\psi_x(y) \coloneqq \phi(d(x,y)) \quad \text{for all } y \in K$$

Let $(A_n)_{n=0}^{\infty}$ be a sequence of linear operators $A_n : \mathcal{C}(K) \to \mathcal{C}(K)$ that possess the following three properties.

First, each A_n, $n \geq 0$, is **nonnegativity − preserving**, *in the sense that*

$$f \in \mathcal{C}(K) \text{ and } f(x) \geq 0 \text{ for all } x \in K \text{ implies } A_n f(x) \geq 0 \text{ for all } x \in K.$$

Second,

$$\lim_{n\to\infty} \|f_0 - A_n f_0\| = 0$$

where the function $f_0 \in \mathcal{C}(K)$ is defined by $f_0(x) = 1$ for all $x \in K$. Third,

$$\lim_{n\to\infty} \left(\sup_{x\in K} |(A_n \psi_x)(x)| \right) = 0.$$

Then,

$$\text{for each } f \in \mathcal{C}(K),\ \lim_{n\to\infty} \|f - A_n f\| = 0$$

The proof can be found in [3].

With the particular function ϕ, lots of remarkable convergent properties of linear opeartors can be unified (which also be found in [3], for instance, **Bohman′s theorem**, **Bernstein′s theorem**, etc.), including *Fejér* operator. Denote the space $\mathcal{C}_{\text{per}}[0, 2\pi]$ formed by all $2\pi -$ *periodic* continuous real-value function equipped with the sup-norm.

Theorem 5. *Let $(A_n)_{n=0}^{\infty}$ be a sequence of linear operators $A_n : \mathcal{C}_{per}[0, 2\pi] \to \mathcal{C}_{per}[0, 2\pi]$ that possess the following two properties:*

First, each $A_n, n \geq 0$, is nonnegativity-preserving:

Second,

$$\lim_{n\to\infty} \|g_p - A_n g_p\| = 0 \qquad \text{for } p = 0, 1, 2$$

where the functions $g_p \in \mathcal{C}_{per}[0, 2\pi]$, $p = 0, 1, 2$, are defined by

$$g_0(\theta) = 1,\ g_1(\theta) = \cos\theta,\ g_2(\theta) = \sin\theta,\ 0 \leq \theta \leq 2\pi.$$

Then,

$$\text{for each } g \in \mathcal{C}_{per}[0, 2\pi],\ \lim_{n\to\infty} \|g - A_n g\| = 0$$

Proof. Let the set:

$$K \coloneqq \{x = (x_1, x_2) \in \mathbb{R}^2; x_1^2 + x_2^2 = 1\}$$

d represents the Euclidean norm. Define $g^\star : K \to \mathbb{R}$, $A_n^\star : \mathcal{C}(K) \to \mathcal{C}(K)$:

$$g^\star \coloneqq g(\theta),\ A_n^\star g^\star \coloneqq (A_n g)^\star$$

with the particular function $\phi(t) = t^2$. Then the conclusion follows from Korovkin's theorem. □

Theorem 6 (**Fejér theorem**). *The family of* **Fejér operators** *is an approximate identity.*

Remark. Fejér operators are continuous and $||F_n|| = 1$.

Remark. An averaging procedure often improves convergence properties.

Denote $\mathcal{Q}[0,2\pi]$ as the space formed by all real $2\pi-$periodic trigonometric polynomials of degree $\leq n$, S_n as the nth *Fourier partial sum*. The methods used here can also be put into use in the same manner as the polynomial situations (divergence phenomenon for polynomial interpolation). At first, we have to admit that the "ideal sequence of operators" is unattainable:

Theorem 7 (**Kharshiladze – Lozinski approximation theorem**). *Any sequence $(B_n)_{n=0}^{\infty}$ of mappings $B_n : \mathcal{C}_{per}\,[0,2\pi] \to \mathcal{Q}_n[0,2\pi] \subset \mathcal{C}_{per}\,[0,2\pi]$ that are linear, continuous, and preserve all trigonometric polynomials of degree $\leq n$, i.e., $B_n q = q$ for all $q \in Q_n[0,2\pi]$ and all $n \geq 0$, is such that*

$$||B_n|| \geq ||S_n|| \quad \textit{for all } n \geq 0$$

.

[Hint: Banach-Steinhaus Theorem]

Note that (more carful: $\frac{4}{\pi^2}\log N$):

$$\begin{aligned}
& \int_{-\pi}^{\pi} |D_N(\theta)|\,\mathrm{d}\theta \\
\geq \quad & 4\int_0^{\pi} \frac{|\sin(N+1/2)\theta|}{\theta}\,\mathrm{d}\theta \\
= \quad & 4\int_0^{(N+1/2)\pi} \frac{|\sin\theta|}{\theta}\,\mathrm{d}\theta \\
\geq \quad & 4\sum_{k=0}^{N-1}\int_{k\pi}^{(k+1)\pi} \frac{|\sin\theta|}{\theta}\,\mathrm{d}\theta \\
\geq \quad & 4\sum_{k=0}^{N-1}\frac{1}{(k+1)\pi}\int_{k\pi}^{(k+1)\pi} |\sin\theta|\,\mathrm{d}\theta \\
\geq \quad & \frac{8}{\pi}\log N
\end{aligned}$$

Since the Banach-Steinhaus theorem again implies that

$$\sup_{n\geq 0} \|B_n g\| = \infty \quad \text{for some } g \in \mathcal{C}_{\text{per}}\,[0,2\pi]$$

Remark. Fejér operators do not preserve all trigonometric polynomials.

3 Some Applications of Fourier Series