$\Rightarrow \forall D \in U(\mathfrak{g}^{\mathbb{C}}),\ D(z(\mathfrak{g})u, v) = z(\mathfrak{g})z(D)u, v)$

$= \chi(z)(z(f)v_i, v_i)$

Summing on $i \Rightarrow (z\Theta)(f) = \chi(z)\Theta(f)$, as required.

When $z\Theta = \chi(z)\Theta$, we say that the global character is an *eigendistribution* (of $Z(\mathfrak{g})$). We now know that global characters of irreducible admissible representations are invariant eigendistributions.

Now let us make use of the differential equations obtained from $Z(\mathfrak{g})$ as in the above proposition.

Theorem 1. *Suppose that Θ is an invariant eigendistribution on G with $z\Theta = \chi(z)\Theta$ for $z \in Z(\mathfrak{g})$. The restriction of Θ to smooth functions compactly supported in the regular set G' is a real analytic function invariant under conjugation. That is, Θ is of the form $\Theta(x)\,dx$ on G' with dx equal to Haar measure and with Θ real analytic and invariant under conjugation.*

To have a formula for this function $\Theta(x)$, it is enough to have a formula on each of the standard Cartan subgroups $H = TA$. On the conjugates of the regular set H' in H, let us write $\Theta(h) = \tau_H(h)/D_H(h)$, where $D_H(h)$ is a standard denominator as in the Weyl character formula.

We want to define $D_H(h)$, modulo technicalities. For this we use the linearity of G: We think of G as embedded in a complex group $G^{\mathbb{C}}$ and let $H^{\mathbb{C}}$ be the subgroup corresponding to $\mathfrak{h}$. For any root α, let ξ_α be the multiplicative character of $H^{\mathbb{C}}$ with differential α, and restrict ξ_α back to H. We do the same thing for δ, the half sum of the positive roots in some order (ignoring that δ may not be integral). Then $D_H(h)$ is given by $D_H(h) = \xi_\delta(h)\prod_{\alpha\in\Delta^+}(1 - \xi_{-\alpha}(h))$.

Theorem 2. *Suppose that $\chi(z) = \chi_\lambda(z)$ relative to H. Then the numerator τ_H of Θ on H satisfies $\gamma(z)\tau_H = \chi_\lambda(z)\tau_H$ for $z \in Z(\mathfrak{g})$, where γ is the Harish-Chandra isomorphism.*

The system $\gamma(z)\tau_H = \chi_\lambda(z)\tau_H$ is a system of partial differential equations with constant coefficients on H. Here H, apart from disconnectedness, is the product of a torus and a Euclidean space. The system can be solved, and here is the result.

Corollary. *Under the assumptions of Theorem 2, let $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$. Fix $h \in H$ and let $\mathfrak{h}_1$ be a connected component of the set of all X in $\mathfrak{h}_0$ such that $D_H(\exp X) \neq 0$. Then there exist uniquely determined polynomial functions p_w on $\mathfrak{h}_0$ for $w \in W$ such that $p_{ws} = p_w$ for $s \in W_\lambda$ and such that the numerator τ_H satisfies*

$$\tau_H(\exp X) = \sum_{w\in W} p_w(X)e^{w\lambda(X)}$$

for all $X \in \mathfrak{h}_1$. Moreover, the degrees of the polynomials p_w are all less than $|W_\lambda|$.

Qualitatively we can think of the polynomials as constant. Then the numerator is a linear combination of exponentials, with the allowable exponents being the Weyl group transforms of λ. It turns out that the assumption that the polynomials are constants is indeed valid if Θ is a global character.

Discrete Series and Asymptotics

take $$G = SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \middle| \, |\alpha|^2 - |\beta|^2 = 1 \right\},$$

which is isomorphic to $SL(2,\mathbb{R})$.

take $\alpha = a+bi$, $\beta = c+di$ $\Rightarrow a^2+b^2-(c^2+d^2)=1$
$\Rightarrow (a+c)(a-c)-(d-b)(d+b)=1$
$\Rightarrow \begin{pmatrix} a+c & d-b \\ d+b & a-c \end{pmatrix} \in SL_2(\mathbb{R})$

We write

$$K = T = \left\{ \overset{K_\theta}{\overset{\shortparallel}{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}}} \right\}.$$

The group G acts on $\Omega = \{|z| < 1\}$ by $g(z) = \frac{\alpha z + \beta}{\bar\beta z + \bar\alpha}$.

For f analytic on Ω and $n \geq 2$, put

$$\mathcal{D}_n \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} f(z) = (-\bar\beta z + \alpha)^{-n} f\left(\frac{\bar\alpha z - \beta}{-\bar\beta z + \alpha} \right)$$

$$\|f\|^2 = \int_\Omega |f(z)|^2 (1-|z|^2)^{n-2} \, dx \, dy.$$

Property: ① $G \curvearrowright \{f \in C^\infty(\Omega) \mid \|f\|^2 < +\infty\}$ is complete, G-action is continuous

② unitary: change variable, $(1-|z|^2)^{-2} dx dy$ is G-invariant.

③ z^N are orthogonal in the Hilbert space, and

$$D^n(K_\theta) z^N = e^{-(n+2N)i\theta} z^N,$$

$\Rightarrow$ the only K-type appears are of form $e^{-(n+2N)i\theta}$ with $N \geq 0$, multi $=1$.

④ Irreducibility: Consider $\mathcal{U}(\mathfrak{g})$ invariant subspace in K-fin. elements.

Computing $D_n\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$, then differentiating yields the action of Lie alg. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z^N = (n+N) z^{N+1} - N z^{N-1}.$$

(5) (square integrability: D_n is a discrete series)

Consider: $\int_G |(D_n(g)1,1)|^2 \, dg$

since $(D_n(g)1,1) = \int_\Omega (\bar\beta z + \alpha)^{-n} (1-|z|^2)^{n-2} \, dx \, dy$

$$= \alpha^{-n} \int_\Omega (1-|z|^2)^{n-2} dx \, dy = C_n \alpha^{-n}$$

$$\Rightarrow |(D_n(g)1,1)|^2 = C_n^2 |\alpha|^{-2n}.$$

Now $g(0) = \frac{\beta}{\alpha}$ implies $1 - |g(0)|^2 = 1 - \frac{|\beta|^2}{|\alpha|^2} = \frac{|\alpha|^2 - |\beta|^2}{|\alpha|^2} = |\alpha|^{-2}$

$$\Rightarrow |(D_n(g)1,1)|^2 = C_n^2 (1 - |g(0)|^2)^n$$

For a suitable normalization of dg, whenever a function h of G is of form $h(g) = h_0(g(0))$, then

$$\int_G h(g) \, dg = \int_\Omega h_0(z) (1-|z|^2)^{-2} dx \, dy$$

Apply to $h_0(z) = C_n^2 (1-|z|^2)^n$ gives

$$\int_G |D_n(g)1,1|^2\,dg = C_n^2\int_\Omega (1-|z|)^{n-2}\,dx\,dy = C_n^3 < \infty \qquad \square$$

Theorem (Godement). *For an irreducible unitary representation π of a unimodular Lie group G, the following three conditions are equivalent:*

(a) *Some nonzero matrix coefficient is in $L^2(G)$.*
(b) *All matrix coefficients are in $L^2(G)$.*
(c) *π is equivalent with a direct summand of the right regular representation of G on $L^2(G)$.*

In particular, <prop> shows (a), thus (c), (b).

Such repn is said to be in the discrete series of G.

<Problem>: all discrete series repn when G is linear connected.

Ans for $G = SU(1,1)$: All D_n, as well as $D_n \circ \begin{pmatrix}\text{complex}\\ \text{conjugation}\end{pmatrix}$ for $n \geq 2$.

<realization>

To generalize, we give a group-theoretic formulation of the construction of $\mathcal{D}_n$ for $G = SU(1,1)$. Let $G^{\mathbb{C}} = SL(2,\mathbb{C})$, and let B be the subgroup $\left\{\begin{pmatrix} a & 0 \\ c & a^{-1}\end{pmatrix}\right\}$ of $SL(2,\mathbb{C})$. It is not hard to verify the following facts: By $B = \left\{\begin{pmatrix}\gamma^{-1} & 0\\ 0 & \gamma\end{pmatrix}\times\begin{pmatrix}1 & 0\\ \zeta & 1\end{pmatrix}\right\}$, $\zeta, \gamma \in \mathbb{C}^\times$, $|z|<1$ and

- Elements of GB are in one-one correspondence with all products

$$\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix} = \begin{pmatrix}1 & \beta/\bar\alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}\bar\alpha^{-1} & 0\\ 0 & \bar\alpha\end{pmatrix}\begin{pmatrix}1 & 0\\ \bar\beta/\bar\alpha & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}\gamma^{-1} & 0\\ 0 & \gamma\end{pmatrix}\begin{pmatrix}1 & 0\\ \zeta & 1\end{pmatrix} \quad \text{with} \quad \zeta \in \mathbb{C},\ \gamma \in \mathbb{C}^\times,\ |z| < 1.$$

- GB is open in $G^{\mathbb{C}}$, and its product complex structure from the above formula is the same as its complex structure as an open subset of $G^{\mathbb{C}}$.

In particular, left translation by G is a holomorphic automorphism of GB.

Now we can construct the representation. Put $\xi_n\begin{pmatrix} a & 0 \\ c & a^{-1}\end{pmatrix} = a^{-n}$. Let V_n consist of all $F : GB \to \mathbb{C}$ such that $n \geq 2$

(a) F is holomorphic,
(b) $F(xb) = \xi_n(b)^{-1}F(x)$ for $x \in GB,\ b \in B$,
(c) $\|F\|^2 = \int_G |F(g)|^2\,dg < \infty$,

and let $L(g)F(x) = F(g^{-1}x)$ for $F \in V_n,\ g \in G,\ x \in GB$. Then L is a unitary representation, the only nontrivial part of the verification being that the underlying space is complete.

We can directly exhibit a unitary equivalence of this representation with $\mathcal{D}_n$. In fact, the correspondence is that $F \leftrightarrow f$, where f on $\Omega = \{|z|<1\}$

$$f(z) = F(z,1,0)$$
$$F(z,\gamma,\zeta) = \gamma^{-n} f(z).$$

Proof: (a)(b)(c) is obvious,

for unitary equivalence: start from (the normalization), for $\forall\ h(g) = h_0(g(0))$

$$\int_G h(g)\,dg = \int_\Omega h_0(g)\,(1-|z|^2)^{-2}\,dx\,dy \qquad (*)$$

where $g(z) = \dfrac{\alpha z+\beta}{\bar\beta z+\bar\alpha}$, for $g = \begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}$. For $F \in V_n$, gives

$F(g) = F(\beta/\bar{\alpha}, \bar{\alpha}, \bar{\beta}/\bar{\alpha}) = \bar{\alpha}^{-n} f(\beta/\bar{\alpha}) = \bar{\alpha}^{-n} f(g(0))$

Thus $|F(g)|^2 = |\alpha|^{-2n} |f(g(0))|^2$.

Since $|\alpha|^2 - |\beta|^2 = 1 \Rightarrow |F(g)|^2 = (1 - |g(0)|^2)^n |f(g(0))|^2$

$\Rightarrow$ putting $h_0(z) = (1-|z|^2)^n |f(z)|^2$, and $h(g) = |F(g)|^2$ in $(*)$, we get

$$\|F\|^2 = \int_G |F(g)|^2 dg = \int_\Omega |f(z)|^2 (1-|z|^2)^{n-2} dx\, dy = \|f\|^2.$$

- For $F \Leftrightarrow f$ commutes with group actions, compute

怎么分解的?

$$L(g) F(z, \gamma, \delta) = F\left(\frac{\bar{\alpha} z - \beta}{-\bar{\beta} z + \alpha}, \gamma(-\bar{\beta} z + \alpha), \delta'\right)$$

$$= (-\bar{\beta} z + \alpha)^{-n} f\left(\frac{\bar{\alpha} z - \beta}{-\bar{\beta} z + \alpha}\right) = \mathcal{D}_n(g) f(z) \qquad \square$$

This realization of $\mathcal{D}_n$ has a vector-bundle interpretation. We have a holomorphic bundle $G^{\mathbb{C}} \to G^{\mathbb{C}}/B$ and an open subbundle $GB \to GB/B$. We can identify the base with G/T, where $T = K$ is a compact Cartan subgroup. Form the associated holomorphic line bundle relative to $\xi_n : B \to \mathbb{C}^\times$ and restrict to the part of the bundle over G/T, obtaining

$$G \times_T \mathbb{C}^\times \to G/T.$$

Then V_n consists of the square-integrable holomorphic sections.

Construction: G: reductive gp. Assume that G/K has a G invariant complex structure. If $\mathfrak{c}_0$ is the center of $\mathfrak{k}_0$, this condition can be shown to be equivalent with the assumption $Z_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$. Under this hypothesis, $\mathfrak{g}_0$ always has a compact Cartan subalgebra $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$. Let $T = \exp \mathfrak{t}_0$. (this assumption is satisfied in the case G is compact.)

The roots relative to the complexification $\mathfrak{t}$ of $\mathfrak{t}_0$ are of two kinds, called "compact" and "noncompact". To define these terms, form $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$. For $H \in \mathfrak{t}_0$, $\alpha \in \Delta$, and $X \in \mathfrak{g}_\alpha$, we have $[H, \theta X] = \theta[\theta H, X] = \theta[H, X] = \theta(\alpha(H)X) = \alpha(H)\theta X$. Thus θX is in $\mathfrak{g}_\alpha$, and θ is a linear transformation of $\mathfrak{g}_\alpha$ to itself with $\theta^2 = 1$. Since $\dim \mathfrak{g}_\alpha = 1$, $\mathfrak{g}_\alpha \subseteq \mathfrak{k}$ or $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$. Call α *compact* or *noncompact* accordingly. Let $\Delta_K = \Delta(\mathfrak{k}, \mathfrak{t})$ be the subset of compact roots. $\Rightarrow$ \mathfrak{k}$(\mathfrak{k}) = \mathfrak{t} \oplus \sum_{\alpha \in \Delta_K} \mathfrak{g}_\alpha$

Δ_n for non-cpt. roots $\qquad \mathfrak{p} = \sum_{\alpha \in \Delta_n} \mathfrak{g}_\alpha$

Proposition. *The compact roots are exactly those vanishing on* $\mathfrak{c}_0$.

$\alpha \in \Delta$, then $\alpha(C) = 0 \Leftrightarrow [C, \mathfrak{g}_\alpha] = 0 \Leftrightarrow \mathfrak{g}_\alpha \in Z_{\mathfrak{g}}(C) \Leftrightarrow \mathfrak{g}_\alpha \subseteq \mathfrak{k} \Leftrightarrow \alpha$ is compact

We use this proposition to introduce a "good ordering". We have $i\mathfrak{t}_0 = i\mathfrak{c}_0 \oplus i(\mathfrak{t}_0 \cap [\mathfrak{k}_0, \mathfrak{k}_0])$. Introduce an ordering that takes $i\mathfrak{c}_0$ before the rest. Observe by the proposition that every noncompact positive root is greater than every compact root. Hence $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ is 0 if α and β are positive noncompact.

We can now define the subgroup B of $G^{\mathbb{C}}$. Let $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha$. This is a Borel subalgebra built from the *negative* roots. Let $B \subseteq G^{\mathbb{C}}$ be the analytic subgroup corresponding to $\mathfrak{b}$. Then GB may be shown to be open in $G^{\mathbb{C}}$.

Examples:

A **bounded symmetric domain** Ω is a bounded open subset of $\mathbb{C}^n$ of the form G/K, with G linear connected semisimple, such that G operates holomorphically on Ω. There are four infinite classes of classical examples, and we give three of them here. In each case it should be noted that the center of K has positive dimension.

conjugate transpose

Example 1. Let $m \leq n$ and $\Omega = \{Z \in M_{nm}(\mathbb{C}) \mid I_m - Z^*Z > 0\}$. Here M_{nm} refers to all n-by-m matrices, I_m is the identity of size m, and >0 means "is positive definite." Let

$$G = \mathrm{SU}(n, m) \subseteq \mathrm{SL}(n + m, \mathbb{C})$$

$$K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(m))$$

$$= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A \in \mathrm{U}(n), D \in \mathrm{U}(m), \det A \det D = 1 \right\}$$

The group G operates holomorphically on Ω by

$$g(Z) = (AZ + B)(CZ + D)^{-1} \qquad \text{if} \quad g = \begin{array}{cc} n & m \\ \end{array} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{array}{c} n \\ m \end{array}. \tag{6.4}$$

One can show that the action of G is transitive, and it is easy to see that the isotropy subgroup at $Z = 0$ is K.

Let us verify that $(CZ + D)^{-1}$ is defined in (6.4) and that $g(\Omega) \subseteq \Omega$. We write

$$\begin{aligned} &(AZ + B)^*(AZ + B) - (CZ + D)^*(CZ + D) \\ &\quad = (Z^* \quad I_m) g^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} g \begin{pmatrix} Z \\ I_m \end{pmatrix} \\ &\quad = (Z^* \quad I_m) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} Z \\ I_m \end{pmatrix} \\ &\quad = Z^*Z - I_m. \end{aligned} \tag{6.5}$$

Let $(CZ + D)v = 0$. Unless $v = 0$, we have

$$0 \leq v^*(AZ + B)^*(AZ + B)v = v^*(Z^*Z - I_m)v < 0$$

by (6.5), a contradiction. So $(CZ + D)^{-1}$ exists, and then (6.5) gives

$$g(Z)^*g(Z) - I = (CZ + D)^{-1*}(Z^*Z - I_m)(CZ + D)^{-1} < 0.$$

Thus $g(\Omega) \subseteq \Omega$.

Example 2. Let $\Omega = \{Z \in M_{nn}(\mathbb{C}) \mid I_n - Z^*Z > 0, Z = -Z^{tr}\}$. Let

$$G = SO^*(2n) = \left\{ g \in SU(n, n) \,\middle|\, g^{tr}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right\}$$

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \,\middle|\, A \in U(n) \right\}.$$

Then G acts in the same way as in Example 1, preserving Ω. The action is transitive with K the isotropy subgroup at $Z = 0$.

Example 3. Let $\Omega = \{Z \in M_{nn}(\mathbb{C}) \mid I_n - Z^*Z > 0, Z = Z^{tr}\}$. For G we use the alternate realization of $Sp(n, \mathbb{R})$ in Chapter I:

$$G = \left\{ g \in SU(n, n) \,\middle|\, g^{tr}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\}$$

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \,\middle|\, A \in U(n) \right\}.$$

Then G acts in the same way as in Example 1, preserving Ω. The action is transitive with K the isotropy subgroup at $Z = 0$.

<rmk>: Example 2, 3 are all subdomains of Example I.
Obviously they all satisfies the condition $Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{t}$. Specifically, we may choose $\mathfrak{h}$: diagonal, $\mathfrak{t}$: block diagonal. $\mathfrak{p}^+$: has the block form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, $\mathfrak{p}^-$ has the form $\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$.

<Lem>: $[\mathfrak{t}, \mathfrak{p}^+] \subseteq \mathfrak{p}^+$, $[\mathfrak{t}, \mathfrak{p}^-] \subseteq \mathfrak{p}^-$, $\mathfrak{p}^+, \mathfrak{p}^-$ are abelian

Theorem 6.3 (Harish-Chandra decomposition). Let G be linear connected reductive, and suppose the center $\mathfrak{c}$ of $\mathfrak{k}$ has $Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$. Then multiplication from $P^+ \times K^{\mathbb{C}} \times P^-$ into $G^{\mathbb{C}}$ is one-one, holomorphic, and regular (with image open in $G^{\mathbb{C}}$), GB is open in $G^{\mathbb{C}}$, and there exists a bounded open subset $\Omega \subseteq P^+$ such that

$$GB = GK^{\mathbb{C}}P^- = \Omega K^{\mathbb{C}}P^-.$$

Example (Remark): For $g \in G$, $w \in \Omega$, the thm. allows us to define
$g(w) := p^+$ component of the product gw.
In this way, $G \curvearrowright \Omega$ transitively with isotropy subgp K at $w = \exp(0)$.

So G/K is realized as the complex mfd. Ω, and G acts holo.

In the case of $SU(n,m)$, the Harish-Chandra decomp. is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A-BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}$$

and P^+-component of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & Z \\ 0 & 0 \end{pmatrix}$ is

$$\begin{pmatrix} I & (AZ+B)(CZ+D)^{-1} \\ 0 & I \end{pmatrix}$$

thus this action reproduces the classical action of symmetric domain.

To construct a representation, let $\lambda \in \mathfrak{t}'$ be integral and satisfy $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta_K^+$. Define V_λ to consist of all $F : GB \to \mathbb{C}$ such that

take an analytic integral in $(\mathfrak{h}^{\mathbb{C}})'$ — dominant

ξ_λ: repn of $T^{\mathbb{C}}$, extends to 1-dim repn B. (by defining ξ_λ to be 1 on N^-)

(a) F is holomorphic,
(b) $F(xb) = \xi_\lambda(b)^{-1} F(x)$ for $x \in GB$ and $b \in B$, $\}\, T(\lambda)$
(c) $\|F\|^2 = \int_G |F(g)|^2 \, dg < \infty$,

and let $L(g)F(x) = F(g^{-1}x)$ for $F \in V_\lambda$, $g \in G$, and $x \in GB$.

$L(g)$ preserves both $T(\lambda)$ and V_λ and acts in unitary fashion on V_λ.

Theorem (Harish-Chandra, 1956). *If $\langle \lambda + \delta, \alpha \rangle < 0$ for all positive noncompact roots α, then (L, V_λ) is a (nonzero) irreducible unitary representation in the discrete series of G.* (and the irr. repn of K with highest wt. λ occurs in $L|_K$.

Δ_K^+

Since these representations and their twists by complex conjugation exhaust the discrete series for $SL(2,\mathbb{R})$, one might hope that a similar thing happens for all linear connected reductive groups. The following example indicates that this is not the case.

Example (Dixmier). The group $G = SO_0(4,1)$, in which G/K does not have an invariant complex structure, has discrete series.

Proof:

Example of $SU(1,1)$:

$$\mathfrak{h} = \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix}, \quad \mathfrak{h}^{\mathbb{C}} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad e_1\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = a, \quad b = \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix}$$

and also, $\lambda = -ne_1$, $\alpha = 2e_1$, $\delta = \frac{1}{2}\alpha = e_1$. The form λ is analytically integral if $n \in \mathbb{Z}$, and since $\Delta_K^+ = \phi \Rightarrow$ it is automatically dominant. The form $\lambda + \delta$ is $-(n-1)e_1$, thus $\langle \lambda+\delta, \alpha \rangle < 0 \Leftrightarrow n > 1$. Therefore, $\xi_\lambda\begin{pmatrix} \gamma^{-1} & 0 \\ 0 & \gamma \end{pmatrix} = \gamma^n$ with $n \geq 2$.

In realization of D_n^-, the const. function 1 (of z) plays a role in

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta} z)\, d\theta = f(0)$$

$$\left(\text{i.e. } \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} D_n^-\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} f(z)\, d\theta = f(0)\, 1(z).\right)$$

When turns to F side: by $F(z,\gamma,\xi) = \gamma^{-n} f(z)$, denote ψ_n: $\psi_n \Leftrightarrow 1$,

$\psi_n(z,\gamma,\xi) = \gamma^{-n} \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} F\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} x \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}\right) d\theta = F(1)\psi_n(x)$, for $x \in GB$.

general case: (Φ_λ, V): irr. uni. rep'n of K, highest wt $= \lambda$. generalization of 1

v_λ: highest wt vect. of norm 1. Extend to hol. rep'ns of $K^{\mathbb{C}}$. In the decomp: $P^+ K^{\mathbb{C}} P^+ \subseteq G^{\mathbb{C}}$, let $\mu(\cdot)$: middle component.

<Lem> For $v \in V$, ψ_v is (well) defined on GB by ← by Harish-Chandra decomposition

$$\psi_v(x) = (\Phi_\lambda(\mu(x))^{-1} v, v_\lambda)$$

is in $T(\lambda)$, and satisfies $L(k)\psi_v = \psi_{\Phi_\lambda(k)v}$ for $k \in K$. □

Define $\psi_\lambda(x) = \psi_{v_\lambda}(x) = (\Phi_\lambda(\mu(x))^{-1} v_\lambda, v_\lambda)$.

Lemma 6.8. For F in $\Gamma(\lambda)$ and x in GB,

$$\int_T F(txt^{-1})\, dt = F(1)\psi_\lambda(x).$$

Proof. The same as for Lemma 5.32, but with M replaced by T.

We can deduce the following consequences of Lemma 6.8, as in §5.7.

(1) $|F(1)| \le \|\psi_\lambda\|^{-1}\|F\|$ for all F in $\Gamma(\lambda)$, by the same proof as for (1) in §5.7. (use Schwartz inequality).

☆(2) $V_\lambda \ne 0$ if and only if $\|\psi_\lambda\| < \infty$. [In fact, ψ_λ is in V_λ if $\|\psi_\lambda\| < \infty$. If $\|\psi_\lambda\| = \infty$, then (1) shows F and all functions $L(g)F$ vanish at 1. Hence $F = 0$.]

(3) To each compact set $E \subseteq GB$ corresponds a constant $C_E < \infty$ such that

$$|F(x)| \le C_E\|F\|$$

for all F in V_λ and x in E, by the same proof as for (2) in §5.7.

(4) V_λ is complete, by the same proof as for (3) in §5.7.

☆(5) If $\|\psi_\lambda\| < \infty$, then V_λ is irreducible and the map $v \to \psi_v$ of V into $\Gamma(\lambda)$ carries V into V_λ and exhibits the K representation Φ_λ as occurring in $L|_K$. [In fact, the irreducibility is proved as in (4) of §5.7. Then (6.15) says that $\psi_{\Phi_\lambda(k)v_\lambda}$ is in V_λ for all k; since these elements are ψ_v for a set of v in V that spans V, all ψ_v are in V_λ. Formula (6.15) shows Φ_λ occurs in $L|_K$.

$L(k)\psi_v = \psi_{\Phi_\lambda(k)v}$ for $k \in K$

(6) If $\|\psi_\lambda\| < \infty$, then $(L(x)\psi_\lambda, \psi_\lambda) = \psi_\lambda(x^{-1})\|\psi_\lambda\|^2$, so that $(L(x)\psi_\lambda, \psi_\lambda)$ is in $L^2(G)$ and every matrix coefficient of L on V_λ is square integrable. [In fact, application of Lemma 6.8 to $L(x)\psi_\lambda$ and use of the identity $\psi_\lambda(txt^{-1}) = \psi_\lambda(x)$ for $t \in T$ gives

$$\begin{aligned}
(L(x)\psi_\lambda, \psi_\lambda) &= \int_G \psi_\lambda(x^{-1}y)\overline{\psi_\lambda(y)}\, dy \\
&= \int_G \psi_\lambda(x^{-1}tyt^{-1})\overline{\psi_\lambda(y)}\, dy \qquad \text{for all } t \in T \\
&= \int_{G\times T} \psi_\lambda(x^{-1}tyt^{-1})\overline{\psi_\lambda(y)}\, dt\, dy \\
&= \int_G \psi_\lambda(x^{-1})\psi_\lambda(y)\overline{\psi_\lambda(y)}\, dy \\
&= \psi_\lambda(x^{-1})\|\psi_\lambda\|^2.
\end{aligned}$$

⇒ discrete series (square integrability)

Thus, the theorem reduces to the following:

<Thm>: If $\langle \lambda+\delta, \alpha \rangle < 0$, for all $\alpha \in \Delta_n^+$, then $\int_G |\psi_\lambda(x)|^2 dx < +\infty$.

The proof: see Knapp Chap. 6, §5: finiteness of integral.

Theorem (Harish-Chandra, 1966). *The linear connected reductive group G has no discrete series unless G has a compact Cartan subgroup. If G has a compact Cartan subgroup $T \subseteq K$ and if $\lambda \in \mathfrak{t}'$ has the property that $\lambda + \delta$ is integral and $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, then there exists a unique invariant eigendistribution Θ_λ on G such that*

(a) $z\Theta_\lambda = \chi_\lambda(z)\Theta_\lambda$,

(b) *the numerator $\tau_{T,\lambda}$ of Θ_λ on T is given by*

$$\tau_{T,\lambda}(\exp X) = \sum_{w \in W(\Delta_K)} (\operatorname{sgn} w) e^{w\lambda(X)}$$

for $X \in \mathfrak{t}_0$,

(c) *the numerator $\tau_{H,\lambda}$ of Θ_λ on each Cartan subgroup H is bounded.*

Up to a sign, Θ_λ is the global character of a discrete series representation π_λ of G. The representations π_λ exhaust the discrete series, and two such π_λ and $\pi_{\lambda'}$ are equivalent if and only if λ' is in $W(\Delta_K)\lambda$.

called the discrete series with H-C parameter λ. The infinitesimal character is λ.

Problem: Realization of π_λ.

The four solutions to this problem that have continued to be useful are the following, in chronological order of completion.

1) (Langlands, Schmid) By L^2 cohomology. This realization was conjectured by Langlands and proved by Schmid. (Schmid. Bolton, Discrete Series in [BK])

2) (Zuckerman, Vogan) By cohomological induction. Zuckerman gave a construction and an outline of a proof, and Vogan gave a full proof. [KV] [Wa1]

3) (Flensted-Jensen) By techniques of analysis on semisimple symmetric spaces.

4) (Kostant, Schmid, Aguilar-Rodriguez) By sheaf cohomology. This realization was roughly conjectured by Kostant and partly proved by Schmid. Aguilar-Rodriguez completed Schmid's proof. [BK]

[K1], Chapter IX.

▸ Asymptotic Expansion.

As we calculated before. $X(\pi(g)v_1, v_2) = (\pi(g)\pi(X)v_1, v_2)$

If π has infinitesimal character χ, then $\pi(z)v_1 = \chi(z)v_1$, for $z \in Z(\mathfrak{g})$. Thus for $z \in Z(\mathfrak{g})$, $\underline{Z(\pi(g)v_1, v_2) = \chi(z)(\pi(g)v_1, v_2)}$.

Want to study: $(\pi(g)v_1, v_2)$ as $g \to \infty$.

Eg: $G = SL_2(\mathbb{R})$.

<Thm> (KAK decomposition): $G = KAK$, A is uniquely determined