1. Remarks on this note

2. L1-2: Introduction, Tilt Functor

2.1. Introduction. The basic algebraic number theory gives us that if we consider the extension of $\mathbb{Q}_p$, we get $\overline{\mathbb{Q}_p}/\mathbb{Q}_p^t/\mathbb{Q}_p^{nr}/\mathbb{Q}_p$, where $\mathbb{Q}_p^{nr}/\mathbb{Q}_p$ represents the maximal unramified subextension with Galois group $\hat{\mathbb{Z}}$, and $\mathbb{Q}_p^t$ represents the maximal tamely ramified subextension with Galois group $\mathrm{Gal}(\mathbb{Q}_p^t/\mathbb{Q}_p^{nr}) = \prod_{l\neq p}\mathbb{Z}_l$. The mysterious part is its algebraic closure $\overline{\mathbb{Q}_p}$. We want to understand $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

Remark 2.1. $\mathrm{Gal}(\mathbb{Q}_p^t/\mathbb{Q}_p)$ is a semidirect product with $\hat{\mathbb{Z}}$ acts on $\prod_{l\neq p}\mathbb{Z}_l$ through after lifting Frob to $\mathbb{Q}_p^t/\mathbb{Q}_p$, it acts by conjugation on the normal subgroup $\mathrm{Gal}(\mathbb{Q}_p^t/\mathbb{Q}_p^{nr})$. Another thing is, for $K/\mathbb{Q}_p$ finite, we choose an algebraic closure $\overline{K}/K$, and its absolute Galois group is only defined up to inner automorphism.

Recall that a characteristic p ring R is perfect is the Frobenius $\phi : R \to R$ is an isomorphism. Define the field $\mathbb{Q}_p(p^{1/p^\infty})$ after adjoining all p-power roots of p, that is, $\mathbb{Q}_p(p^{1/p^\infty}) := \bigcup \mathbb{Q}_p(p^{1/p^n})$. This is one perfection of $\mathbb{Q}_p$. Similarly, For $\mathbb{F}_p((t))$, denote $\mathbb{F}_p((t^{1/p^\infty})) := \bigcup \mathbb{F}_p((t^{1/p^n}))$. "The basic result which we want to put into a larger context is the following canonical isomorphism of Galois groups, due to Fontaine and Wintenberger. A special case is the following result."

Theorem 2.1. *The absolute Galois group of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((t^{1/p^\infty}))$ are canonically isomorphic.*

For simplicity, we denote K to be the completion of $\mathbb{Q}_p(p^{1/p^\infty})$ and let $K^\flat$ be the completion of $\mathbb{F}_p((t^{1/p^\infty}))$ temporarily. "Let us first explain the relation between K and $K^\flat$, which in vague terms consists in replacing the prime number p by a formal variable t." Let K° and $K^{b\circ}$ be the subrings of integral elements. Then

$$K^\circ/p = \mathbb{Z}_p[p^{1/p^\infty}]/p \cong \mathbb{F}_p[t^{1/p^\infty}]/t = K^{b\circ}/t$$

where the isomorphism sends p^{1/p^n} to t^{1/p^n}. Using this, one can define a continuous multiplicative, but nonadditive, map $K^\flat \to K, x \mapsto x^\sharp$, which sends t to p. On $K^{\flat\circ}$, it is given by sending x to $\lim_{n\to\infty} y_n^{p^n}$, where $y_n \in K^\circ$ is any lift of the image of $x_n = x^{1/p^n}$ in $K^{b\circ}/t = K^\circ/p$,i.e. $y_n \in K^\circ$ such that $y_n \equiv \overline{x_n} (\mathrm{mod}\ p)$. Then one has an identification

$$K^\flat = \varprojlim_{x\mapsto x^p} K, x \mapsto (x^\sharp, (x^{1/p})^\sharp, \ldots).$$

In order to prove the theorem, one has to construct a canonical finite extension $L^\sharp$ of K for any finite extension L of $K^\flat$. Say L is the splitting field of a polynomial $X^d + a_{d-1}X^{d-1} + \ldots + a_0$, which is also the splitting field of $X^d + a_{d-1}^{1/p^n}X^{d-1} + \ldots + a_0^{1/p^n}$ for all $n \geq 0$. Then $L^\sharp$ can be defined as the splitting field of $X^d + (a_{d-1}^{1/p^n})^\sharp X^{d-1} + \ldots + (a_0^{1/p^n})^\sharp$ for n large enough. these fields stabilize as $n \to \infty$. This gives $\mathrm{Gal}(L^\sharp/K) \cong \mathrm{Gal}(L/K^\flat)$.

Here is the generalization.

Definition 2.1. A perfectoid field is a complete topological field K whose topology is induced by a nondiscrete rank 1 valuation, such that the Frobenius Φ is surjective on K°/p.

Here $K^\circ \subset K$ denotes the set of powerbounded elements. Generalizing the example above, a construction of Fontaine associates to any perfectoid field K another perfectoid field $K^\flat$ of characteristic p:

$$K^\flat = \varprojlim_{x \mapsto x^p} K.$$

Remark 2.2. Choose some element $\varpi \in \mathfrak{m} \subset K^\circ$ such that $|p| \leq |\varpi| < 1$ (which eqivalent to $p \in \varpi K^\circ$), then $\Phi : K^\circ/p \to K^\circ/p$ surjective is equivalent to $\Phi : K^\circ/\varpi \to K^\circ/\varpi$ is surjective. Here ϖ is called the pesudouniformizer. (sometimes we will denote it as π or t, anyway.) Note that $\varprojlim_{x \mapsto x^p} K/\varpi$ gives a perfect ring of characteristic p.

Remark 2.3. Leader Lun told me that we can view the image of a rank-1-valuation $\Gamma \cup \{0\}$ to be $\mathbb{R}$. We can use some simple definitions here: The subset K° will coincides with $\{x \in K ||x| \leq 1\}$, called the valuation ring; Its maximal ideal $K^{\circ\circ} := \{x \in K||x| < 1\}$ will coincides with the topologically nilpotent elements of K, i.e., those $t \in K$ such that $t^n \to 0$ as $n \to \infty$. Their quotient k is called the residue field of K. In fact, For a pesudouniformizer t, the t-adic topology of K° coincides with the valuation topology. (One may check that the valuation (holds also for higher rank valuation) is continuous if and only if for one (or equivalently, any) pesudouniformizer t, we have $|t|^n \to 0$ as $n \to \infty$.)

Example 1. (i) $\mathbb{Q}_p$ is not perfectoid. also $\overline{\mathbb{Q}_p}$.

(ii) Let K is a NA field of characteristic p. Then K is perfectoid if and only if K is perfect.

(iii) $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is a perfectoid field with $K^\circ = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$. The value group is $\bigcup_{n=1}^\infty \frac{1}{p^n}\mathbb{Z}$ which is nondiscrete. Here $K^\circ/p \cong \frac{\mathbb{F}_p[t^{1/p^\infty}]}{t}$ satisfies the surjective condition. Then $K^{\flat\circ} = \varprojlim_{x\mapsto x^p} K^\circ/p \cong K^\flat = \varprojlim_{x \mapsto x^p} \frac{\mathbb{F}_p[t^{1/p^\infty}]}{t} \cong \widehat{\mathbb{F}_p[[t^{1/p^\infty}]]}$. That is, $K^\flat = \widehat{\mathbb{F}_p((t^{1/p^\infty}))}$.(We need the following Lemma.)

(iv) Similarly, for $K = \widehat{\mathbb{Q}_p(\mu_p^\infty)}$ which has tilt $K^\flat = \widehat{\mathbb{F}_p((t^{1/p^\infty}))}$. This extension is known as the cyclotomic perfectoid field and denoted by $\mathbb{Q}_p^{cycl}$. By definition this is complete and the value group is $\bigcup_{n=1}^\infty \frac{1}{p^n}\mathbb{Z}$ which is nondiscrete. Considering that $K^\circ/p \cong \frac{\mathbb{F}_p[t^{1/p^\infty}]}{t^{p-1}}$, it is a perfectoid field. describe this!

(v) $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ is a perfectoid field. Its tilt is equal to the completion of the algebraic closure of $\mathbb{Q}_p^{cycl,\flat} = \widehat{\mathbb{F}_p(t^{1/p^\infty})}$, which itself is $\widehat{\overline{\mathbb{F}_p((t))}}$.(I don't know how to observe this without tilt equivalence.)

Remark 2.4. By these examples, we observe that the tilt functor $K \mapsto K^\flat$ is not fully faithful on perfectoid field $K/\mathbb{Q}_p$. We shall see later that this a consequence of working over the non-perfectoid base $\mathbb{Q}_p$: this functor is fully faithful on perfectoid algebras over a perfectoid based field K.

2.2. **Tilt and Untilt.** In this subsetion, we start to study these operations: tilt and untilt.

Lemma 2.1. *Let K be a perfectoid field. Then the value group is p-divisible.*

Proof. It is easy to see that the value group is generated by $|x|$ where $x \in K^\circ$, since it is discrete. Then it follows the "perfectness" and NA property. $\square$

Lemma 2.2. (i) *There is a multiplicative homeomorphism*

$$\varprojlim_{x \mapsto x^p} K^\circ \xrightarrow{\cong} \varprojlim_{\Phi} K^\circ/\varpi$$

given by projection. In particular, the right-hand side is independent of ϖ. Moreover, we get a map

$$\varprojlim_{\Phi} K^\circ/\varpi \to K^\circ : x \mapsto x^\sharp$$

This makes $\varprojlim_\Phi K^\circ/\varpi$ a domain.

(ii) *There is an element $\varpi^\flat \in \varprojlim_\Phi K^\circ/\varpi$ with $|(\varpi^\flat)^\sharp| = |\varpi|$. Define*

$$K^\flat = (\varprojlim_{\Phi} /\varpi)[(\varpi^\flat)^{-1}]$$

(iii) *There is a multiplicative homeomorphism*

$$K^\flat = \varprojlim_{x \mapsto x^p} K$$

In particular, there is a map $K^\flat \to K$, $x \mapsto x^\sharp$. Then $K^\flat$ is a perfectoid field of characteristic p,

$$K^{\flat\circ} = \varprojlim_{x \mapsto x^p} K^\circ \cong \varprojlim_{\Phi} K^\circ/\varpi,$$

and the rank-1-valuation on $K^\flat$ can be defined by $|x|_{K^\flat} = |x^\sharp|_K$. We have $|K^{b\times}| = |K^\times|$. Moreover

$$K^{\flat\circ}/\varpi^\flat \cong K^\circ/\varpi, \quad K^{\flat\circ}/\mathfrak{m}^\flat = K^\circ/\mathfrak{m}$$

where $\mathfrak{m}$, resp. $\mathfrak{m}^\flat$, is the maximal ideal of K°, resp. $K^{\flat\circ}$.

(iv) *If K is of characteristic p, then $K^\flat = K$.*

We call $K^\flat$ the tilt of K, and the functor from $K^\flat$ to K is called untilt.

Proof. It is a constructive proof. for $(\overline{x_0}, \overline{x_1}, \dots) \in \varprojlim_\Phi K^\circ/\varpi$, then we claim that the limit

$$x^\sharp = \lim_{n \to \infty} x_n^{p^n}$$

exists and is independent of all choices. Clearly it is a multiplicative and continuous map, now the map

$$\varprojlim_{\Phi} K^\circ/\varpi \to \varprojlim_{x \mapsto x^p} K^\circ : \ x \mapsto (x^\sharp, (x^{1/p})^\sharp, \dots)$$

gives the inverse of the projection.

The following three points says this map is compatible with the valuation. Furthermore, we can choose $\varpi^\flat$ such that $\varpi = (\varpi^\flat)^\sharp$. $\square$

Remark 2.5. In Scholze's explanation of perfectoid spaces posted on MO (also his paper: Perfectoid Spaces, A Survey), he used Theorem 2.1 to explain this construction as the following:

"At this point, it may be instructive to explain Theorem 2.1 in the example where K is the completion of $\mathbb{Q}_p(p^{1/p^\infty})$; in all examples to follow, we make this choice of K. It says that there is a natural equivalence of categories between the category of finite extensions L of K and the category of finite extensions M of $K^\flat$.

Let us give an example: Say M is the extension of $K^\flat$ given by adjoining a root of $X^2 - 7tX + t^5$. Basically, the idea is that one replaces t by p, so that one would like to define L as the field given by adjoining a root of $X^2 - 7pX + p^5$. However, this is obviously not well-defined: If $p = 3$, then $X^2 - 7tX + t^5 = X^2 - tX + t^5$, but $X^2 - 7pX + p^5 \neq X^2 - pX + p^5$, and one will not expect in general that the fields given by adjoining roots of these different polynomials are the same.

However, there is the following way out: M can be defined as the splitting field of $X^2 - 7t^{1/p^n}X + t^{5/p^n}$ for all $n \geq 0$ (using that $K^\flat$ is perfect), and if we choose n very large, then one can see that the fields L_n given as the splitting field of $X^2 - 7p^{1/p^n}X + p^{5/p^n}$ will stabilize as $n \to \infty$; this is the desired field L. Basically, the point is that the discriminant of the polynomials considered becomes very small, and the difference between any two different choices one might make when replacing t by p becomes comparably small."

Since in adic space setting, the continuous valutions are important (although we didn't define what is an adic space and explain why this is important.) Before the next proposition, I have to add the definition of valuation in advance.

Definition 2.2. Let R be some ring. A valuation on R is given by a multiplicative map $|\cdot| : R \to \Gamma \cup \{0\}$, where Γ is some totally ordered abelian group, written multiplicatively, such that $|0| = 0$, $|1| = 1$ and $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in R$. If R is a topological ring, we say a valuation is continuous if for all $\gamma \in \Gamma$, subset $\{x \in R \mid |x| < \gamma\}$ is open.

Proposition 2.1. *Let K be a perfectoid field with tilt $K^\flat$. Then the continuous valuations $|\cdot|$ of K (up to equivalence and of any rank) are mapped bijectively to the continuous valuations $|\cdot|^\flat$ of $K^\flat$ (up to equivalence) via $|x|^\flat = |x^\sharp|$.*

Proof. First, we check that the map $|\cdot| \mapsto |\cdot|^\flat$ maps valuations to valuations. All properties except $|x + y|^\flat \leq \max(|x|^\flat, |y|^\flat)$ are immediate, using that $x \mapsto x^\sharp$ is multiplicative. The NA property follows from:

$$\begin{aligned}|x+y|^\flat &= \lim_{n\to\infty} |(x^{1/p^n})^\sharp + (y^{1/p^n})^\sharp|^{p^n} \\ &\leq \max(\lim_{n\to\infty} |(x^{1/p^n})^\sharp|^{p^n}, \lim_{n\to\infty} |(y^{1/p^n})^\sharp|^{p^n}) = \max(|x^\sharp|, |y^\sharp|).\end{aligned}$$

On the other hand, we need to prove the bijection. First, using the remark 2.3, we can construct the 1-1 correspondence between the valuation with the the "valuation ring" which contains $K^{\circ\circ}$ inside its maximal ideal and lies in K° (here the valuation ring for any rank is defined several sections later, readers may check it after then): The valuation ring $R \subset K$ attached to $|\cdot|$ will satisfies this; Conversely, such valuation subring defines a continuous valuation. This point can be checked using the remark 2.3.

Passing to the quotient, we learn the continuous valuation identify bijectively with valuation rings in $K^\circ/K^{\circ\circ}$, which is isomorphic to $K^{\flat,\circ}/K^{\flat,\circ\circ}$ and thus in bijection with $K^\flat$ continuous valuation. □

Proposition 2.2. *Let K be a perfectoid field with tilt $K^\flat$. If $K^\flat$ is algebraically closed, then K is algebraically closed.*

3. L2-4: Almost mathematics

Our goal in this section is to illustrate the following diagrams (although almost everything in this diagram are not defined):

For $K \supseteq K^\circ \supseteq \mathfrak{m} \ni \varpi$ with tilt $K^\flat \supseteq K^{\flat\circ} \supseteq \mathfrak{m}^\flat \ni \varpi^\flat$,

$$\begin{tikzcd}
K\text{–Perf} \arrow[r, leftrightarrow, "\sim"] & K^{\circ a}\text{–Perf} \arrow[r, leftrightarrow] & K^{\circ a}/\varpi\text{–Perf} \arrow[r, equal, "\text{Tilt}"] & K^{\flat\circ a}/\varpi^\flat\text{–Perf} \arrow[d, leftrightarrow, "\wr"] \\
R \arrow[r, mapsto] & R^{\circ a} \arrow[r] & R^{\circ a}/\varpi & K^{\flat\circ}\text{–Perf} \arrow[d, leftrightarrow, "\wr"] \\
A_*[1/\varpi] & A \arrow[l, mapsto] \arrow[r, mapsto, "\text{cot complex}"'] & {} & K^\flat\text{–Perf}
\end{tikzcd}$$

The point is, a perfectoid K-alg which is an object over the generic fibre, has a canonical extension to the almost integral level as a perfectoid $K^{\circ a}$-algebra, and the latter is determined by its reduction modulo ϖ. Or in a more general category, for K-Perf R:

$$\begin{tikzcd}
B_*[1/\varpi] & B \arrow[l, mapsto] \arrow[r, mapsto] & B/\varpi
\end{tikzcd}$$

$$\begin{tikzcd}
R_{\text{fét}} \arrow[dr, phantom, "\text{almost purity}"] & R^{\circ a}_{\text{fét}} \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, phantom, "\text{cotangent complex}"] & (R^{\circ a}/\varpi)_{\text{fét}} \arrow[d, equal, "\text{tilt}"] \\
{} & {} & {} \\
R^\flat_{\text{fét}} & R^{\flat\circ a}_{\text{fét}} \arrow[l, "\cong"'] \arrow[r, "\cong"] & (R^{\flat\circ a}/\varpi^\flat)_{\text{fét}}
\end{tikzcd}$$

3.1. **Construct the Category.** Fix a perfectoid field K with valuation ring K° and maximal ideal $\mathfrak{m} = K^{\circ\circ}$ which is the subset of topologically nilpotent elements. Since the valuation on K is non-discrete, we have $\mathfrak{m}^2 = \mathfrak{m}$. (The general setup: Assume $I \subset R$ is a flat ideal and satisfies $I_2 = I$. This implies $I \otimes_R I \cong I^2 \cong I$.)

Definition 3.1. Let M be a K°-module. We say $x \in M$ is almost zero if $\mathfrak{m}x = 0$. We say M is almost zero if every element of M is almost zero, i.e., $\mathfrak{m}M = 0$.

We would like to "quotient out" all almost zero modules.

Remark 3.1. We recall the machinery to do so: the quotient of an abelian category by a Serre subcategory.

Definition 3.2. Let $\mathcal{A}$ be an abelian category. A Serre subcategory is a full subcategory $\mathcal{B}$ of $\mathcal{A}$ such that for any exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in $\mathcal{A}$, one has $M \in \mathcal{B}$ if and only if $M', M'' \in \mathcal{B}$.

Suppose $\mathcal{B}$ is a Serre subcategory, the one can form the quotient category $\mathcal{A}/\mathcal{B}$, whose objects are the objects of $\mathcal{A}$, and for $M, N \in \mathcal{A}$

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M,N) := \varinjlim_{\substack{\alpha : M' \hookrightarrow M,\ \beta : N \twoheadrightarrow N'' \\ \ker\alpha,\ \mathrm{coker}\beta \in \mathrm{Mor}(\mathcal{B})}} \mathrm{Hom}(M', N'')$$

The quotient category is again an abelian category. By construction, one has a canonical localization functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$. If $M \in \mathcal{B}$, then $Q(M) \cong 0$ in $\mathcal{A}/\mathcal{B}$. The quotient category enjoys the following universal property: suppose $\mathcal{C}$ is another abelian category and $F : \mathcal{A} \to \mathcal{C}$ is an exact functor such that $F(M) = 0$ for any $M \in \mathcal{B}$, then F uniquely factors through $\mathcal{A}/\mathcal{B}$.

Proposition 3.1. *The full subcategory of* $K^\circ - \mathrm{mod}$ *consisting of almost zero* K°*-modules is a Serre subcategory of* $K^\circ - \mathrm{mod}$. *Denote the quotient category by* $K^{\circ a} - \mathrm{mod}$.

Proof. Suppose M is $\mathfrak{m}$-torsion, then clearly any sub or quotient of M is also $\mathfrak{m}$-torsion. Conversely, if M is an extension of M'' by M', where M' and M'' are $\mathfrak{m}$-torsion, then m^2 kills M. Then the result follows from $\mathfrak{m}^2 = \mathfrak{m}$. $\square$

We denote this localization functor K°-mod to $K^{\circ a}$-mod: $M \mapsto M^a$.

Definition 3.3. Define

$$M^a \otimes_{K^{\circ a}} N^a = (M \otimes_{K^\circ} N)^a.$$

It is a well-defined on $K^{\circ a} -$ mod . For $f : M^a \to N^a$, define $\ker(f) = \ker(f_*)^a$, $\mathrm{coker}(f) = \mathrm{coker}(f_*)^a$, where $f_* : M \to N$.They make $K^{\circ a} - \mathrm{mod}$ an abelian tensor category.

One can show that homomorphisms between two almost modules can be described alternatively by

$$\mathrm{Hom}_{K^{\circ a}}(M^a, N^a) = \mathrm{Hom}_{K^\circ}(\mathfrak{m} \otimes_{K^\circ} M, N)$$

This is a K°-mod with no almost zero elements.

We define

$$\mathrm{alHom}(M^a, N^a) = \mathrm{Hom}_{K^{\circ a}}(M^a, N^a)^a.$$

Then for any three $K^{\circ a}$-modules $L,\ M,\ N$, there is a functorial isomorphism as usual:

$$\mathrm{Hom}(L, \mathrm{alHom}(M,N)) = \mathrm{Hom}_{K^{\circ a}}(L \otimes_{K^{\circ a}} M, N).$$

This means that $K^{\circ a}$-mod has all has all formal properties of the category of modules over a commutative ring. So one can define the notion of almost $K^{\circ a}$-algebras, or $K^{\circ a}$-algebras: these are commutative unitary monoid objects in $K^{\circ a}$-mod. Let A be a $K^{\circ a}$-algebra, one can also define the notation of A-modules and A-algebras...

Remark 3.2. There is a right adjoint functor to the localization functor, called *the functor of almost elements*:

$$M \mapsto M_* = \mathrm{Hom}_{K^{\circ a}}(K^{\circ a}, M)$$

One easily checks (by the previous remark) that $(-^a, -_*)$ is an adjoint pair and

$$(M_*)^a \cong M, \quad (N^a)_* \cong \mathrm{Hom}_{K^\circ}(\mathfrak{m}, N).$$

To show that $(-)_*$, $(-)_a$ are "good" functors from the category of commutative algebra in R-mod to the category of commutative algebras in almost R-mod as well its right adjoint, we need to check there is a canonical map $M_* \otimes_{K^\circ} N_* \cong (M \otimes_{K^{\circ a}} N)_*$.

Then we want to give a realization of such category. In the following of this subsection, I will use (R, I) to repalace $(K^\circ, \mathfrak{m})$, and for simplicity, (sometimes) I will use Mod_R to represent R-mod.

We introduce the category $\mathcal{A} \in \mathrm{Mod}_R$ be the full subcategory spanned by all $M \in \mathrm{Mod}_R$ such that the action map $I \otimes_R M \mapsto M$ is an isomorphism; This functor is exact by the flatness of I. Also we can check this is a abelian subcategory. Then we wil construct a series of functors and eventually to realize $\mathcal{A}$ as a quotient of Mod_R.

(i) Write $i_* : \mathrm{Mod}_{R/I} \to \mathrm{Mod}_R$ as functor given by the restriction of scalars along $R \to R/I$;

(ii) it has a left adjoint i^* given by $M \mapsto M \otimes_R R/I$ and a right adjoint $i^!$ given by $M \mapsto \mathrm{Hom}_R(R/I, M) = M[I]$.

(iii) Write $j_! : \mathcal{A} \to \mathrm{Mod}_R$ for the exact inclusion. It has a exact right adjoint j^* given by $M \mapsto I \otimes_R M$ and This right adjoint j^* has a further right adjoint j_* given by the formula

$$j_*(M) = \mathrm{Hom}_R(I, M)$$

Proposition 3.2. *The unit map $N \to j^* j_! N$ is an isomorphism for any $N \in \mathcal{A}$. The counit map $j^* j_* M \to M$ is an isomorphism for any $M \in \mathcal{A}$.*

(i) We have $i^* j_!$, $i^! j_*$, $j^* i_*$ are 0.

(ii) The kernel of j^* is exactly $\mathrm{Mod}_{R/I}$: $\supseteq$ is obvious; For $\forall M \in \ker j^*$, Tensoring M with the standard exact sequence $0 \to R \to R/I \to 0$ shows that $M \cong M/IM$.

(iii) The quotient functor $q : \mathrm{Mod}_R \to \mathrm{Mod}_R^a$ (the quotient to its almost part) is realized by j^*.

Remark 3.3. This also proved that the quotient functor admits fully faithful left and right adjoints, thus commutes with all limits and colimits.

Proof. To prove the last point, we need to check the following:

$j^*(\mathrm{Mod}_{R/I}) = 0$ and exact: This follows from $I = I^2$ and I is flat.

j^* is universal with the previous two properties: let $q' : \mathrm{Mod}_R \to \mathcal{B}$ be such a functor of abelian categories, then it factors through j^*: Fix some $M \in \mathrm{Mod}_R$, then we have the action map $I \otimes_R M \to M$. The kernel and cokernel of this map are $\mathrm{Tor}_i^R(R/I, I)$ for $i = 1, 0$ which are killed by I, thus also q'. As q' is exact, we have $q'(j_! j^* M) = q'(I \otimes_R M) \cong q'(M)$. Thus $q' \cong q' j_! j^*$ as wanted. □

Remark 3.4. We can draw a diagram to describe these adjoint pairs:

$$\begin{tikzcd} \mathrm{Mod}_{R/I} \arrow[r, "i_*"] & \mathrm{Mod}_R \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathrm{Mod}_R^a = \mathcal{A} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Bhott said in his note:"We may view Mod_R as quasi-coherent sheaves on $X = \mathrm{Spec}(R)$, and $\mathrm{Mod}_{R/I}$ as quasi-coherent sheaves on $Z = \mathrm{Spec}(R/I)$. And we can think the category $\{\mathrm{Mod}_R^a = \mathcal{A}\}$ as quasi-coherent sheaves on some non-exist open $\overline{U} \subset X$ that contains $X - Z$ (but not U)."

In Bhott note, He defines

$$M_* := j_* j^* M = \mathrm{Hom}_{K^\circ}(\mathfrak{m}, \mathfrak{m} \otimes_{K^\circ} M)$$

Thus we have $M_* = \mathrm{Hom}_{K^\circ}(\mathfrak{m}, M_*) \cong \mathrm{Hom}_{K^{\circ a}}(K^{\circ a}, M)$, which is scholze's definition.

3.2. **Commutative Algebra in Almost World.** We now extend some basic notions of commutative algebra to the almost world.

Definition 3.4. Let $M \in \mathrm{Mod}_R$ with image $M^a \in \mathrm{Mod}_R^a$.

We say that M or M^a is almost flat if $M^a \otimes (-)$ is exact on Mod_R^a; equivalently, $\mathrm{Tor}_{>0}^R(M, N)$ is almost zero for any R-module N.

We say that M or M^a is almost projective if $\mathrm{alHom}(M, -)$ is exact; equivalently, $\mathrm{Ext}_R^{>0}(M, N)$ is almost zero for any R-module N.

We say that M or M^a is almost finitely generated (resp. almost finitely presented) if for each $\epsilon \in I$, there exists a finitely generated (resp. finitely presented) R-module N_ϵ and a map $N_\epsilon \to N$ with kernel and cokernel killed by ϵ. If the number of generators of N_ϵ can be bounded independently of ϵ, we say that M is uniformly almost finitely generated.

Remark 3.5. (Why needs almost-projective) The notion of almost projectivity is distinct from the categorical notion of projectivity in the abelian category Mod_R^a: the latter is far more restrictive. Indeed, the ring R is almost projective with the above definition. However, R^a need not be a projective object of Mod_R^a. In fact, consider $R = K^\circ$ and $I = K^{\circ\circ}$ for a perfectoid field K with residue field k, note that if we apply $(-)^a$ to

$$0 \to \mathrm{Hom}_{K^\circ}(K^\circ, K^\circ/\mathfrak{m}) \to K^\circ \to \mathrm{Hom}_{K^\circ}(K^\circ, \mathfrak{m}) \to \mathrm{Ext}^1_{K^\circ}(K^\circ, K^\circ/\mathfrak{m})$$

the group $\mathrm{Ext}^1_{K^{\circ a}}(K^{\circ a}, K^{\circ a})$ identifies with $\mathrm{Ext}^2_{K^\circ}(k, K^\circ)$ is the obstacle for $K^{\circ a}$ being projective, and is nonzero if K is not spherically complete.

Remark 3.6. As an example of an almost finitely presented module, consider the case that K is the p-adic completion of $\mathbb{Q}_p(p^{1/p^\infty})$, $p \neq 2$. Consider the extension $L = K(p^{1/2})$. Then $L^{\circ a}$ is an almost finitely presented $K^{\circ a}$-mod. Indeed, for any $n \geq 1$, we have injective maps

$$K^\circ \oplus p^{1/2p^n} K^\circ \to L^\circ$$

since L° is the completion of $\varinjlim(K^\circ \oplus p^{1/2p^n} K^\circ)$, whose cokernel is killed by $p^{1/2p^n}$. That is, L° is a uniformly almost finitely presented projective K°-mod.

Proposition 3.3. *Let A be a R^a-algebra. Then $A-\mathrm{mod}$ M is flat and almost finitely presented if and only if it is almost projective and almost finitelt generated.*

Definition 3.5. Let A be a R^a-algebra, and let B be an A-algebra. Let $\mu : B \otimes_A B \to B$ denote the multiplication morphism.

(i) The morphism $A \to B$ is said to be *unramified* if there is some element $e \in (B \otimes_A B)_*$ such that $e^2 = e,\ \mu_*(e) = 1$ and $xe = 0$ for all $x \in \ker(\mu)_*$.

(ii) The morphism $A \to B$ is said to be étale if it is unramified and B is a flat A-module.

A morphism $A \to B$ of R^a-algebras is said to be almost *finite étale* if it is étale and B is an almost finitely presented A-module. Write $A_{\text{afét}}$ for the category of almost finite étale maps $A \to B$.

We will see how to characterize this property. Here is an example.

Example 2. Recall the setting in 3.6. We shall show L°/K° is almost finite étale. We have shown that L° is a uniformly almost finitely presented projective K°-mod, clearly L° is flat. It remains the unramifiedness. Note that L/K is a Galois extension of degree 2, hence we have a canonical isomorphism (after fixing $\sigma : L \to L$ the nontrivial element):

$$\mathrm{can} : L \otimes_K L \cong L \times L, \quad a \otimes b \mapsto (ab, a\sigma(b))$$

where sends $e = \frac{1}{2\sqrt{p}\otimes 1}(\sqrt{p}\otimes 1 + 1 \otimes \sqrt{p})$ to $(1,0)$. Using the isomorphism above, it is easy to vertify that e is the diagonal idempotent we want, by deducing $p^{1/p^n} \cdot e \in L^\circ \otimes_{K^\circ} L^\circ$.

In fact, we have the following theorem by Tate and Gabber-Ramero:

Theorem 3.1. *If L/K is a finite extension, then $\mathcal{O}_L/\mathcal{O}_K$ is almost finite étale. Similarly, if $M/K^\flat$ is finite, then $\mathcal{O}_M/\mathcal{O}_{K^\flat}$ is almost finite étale.*

Lemma 3.1. *For a perfectoid field K, set $K \supseteq K^\circ \supseteq \mathfrak{m} \ni \varpi$. Let M be a $K^{\circ a}$-module.*

(i) M is almost flat if and only if M_* is flat over K° if and only if M_* has no ϖ-torsion.

(ii) If N is a flat K°-module and $M = N^a$, then M is flat over $K^{\circ a}$ and we have $M_* = \{x \in N[\frac{1}{\varpi}] \mid \forall \epsilon \in \mathfrak{m} : \epsilon x \in N\}$.

(iii) If M is flat over $K^{\circ a}$, then for all $x \in K^\circ$, we have $(xM)_* = xM_*$. Moreover, $M_*/xM_* \subset (M/xM)_*$, and for all $\epsilon \in \mathfrak{m}$ the image of $(M/x\epsilon M)_* in (M/xM)_*$ is equal to the image of $M_*/xM_* \subset (M/xM)_*$.

(iv) If M is almost flat, then M is ϖ-adically complete if and only if M_* is ϖ-adically complete.

Proof. For (ii), We have

$$M_* = \mathrm{Hom}_{K^{\circ a}}(K^{\circ a}, M) = \mathrm{Hom}_{K^\circ}(\mathfrak{m}, N).$$

As N is flat over K°, we can apply fraction to the last term, as the subset of those $x \in \mathrm{Hom}_K(K, N[\frac{1}{\varpi}]) = N[\frac{1}{\varpi}]$ satisfying the condition that for all $\epsilon \in \mathfrak{m}$, $\epsilon x \in N$.

For (iii), $(xM)_* = xM_*$ is clear. For the rest, consider the canonical map of the short exact sequence of $K^{\circ a}$-mod:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "x"] \arrow[d] & M \arrow[r] \arrow[d] & M/xM \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M/\epsilon M \arrow[r] & M/x\epsilon M \arrow[r] & M/xM \arrow[r] & 0
\end{tikzcd}$$

Then apply $(-)_*$ functor, we get a diagram:

$$\begin{tikzcd}
0 \arrow[r] & M_*/xM_* \arrow[r, "a"] \arrow[d] & (M/xM)_* \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{K^{\circ a}}(K^{\circ a}, M)[x] \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & (M/x\epsilon M)_* \arrow[r, "b"] & (M/xM)_* \arrow[r] & \mathrm{Ext}^1_{K^{\circ a}}(K^{\circ a}, M/\epsilon M) &
\end{tikzcd}$$

We wish to show that a and b have the same image, thus it suffices to show c is injective.

For (iv), it follows from (iii) clearly. $\square$

3.3. **The dictionary.**

Definition 3.6 (K-Banach algebra)**.** Let K be a complete NA field. A Banach K-algebra R is a K algebra R equipped with a map $|\cdot|: R \to \mathbb{R}_{\geq 0}$ extending the norm on K such that

1. (Norm) $|f| = 0$ only if $f = 0$.
2. (Submultiplicativity) $|fg| \leq |f||g|$, with equality if $f \in K$.
3. (NA property) $|f + g| \leq \max(|f|, |g|)$.
4. R is complete in the metric d given by $d(f, g) = |f - g|$.

For such a K-Banach algebra R, define the set $R^\circ \subset R$ of powerbounded elements as

$$R^\circ := \{f \in R | \{|f^n|\} \text{ is bounded.}\} = \{f \in R | \{f^n\} \subset R \text{ is bounded.}\}$$

Since $R_{<1} \subset R_{\leq 1} \subset R^\circ$, thus R° is an open subring.

Example 3. For $K = \mathbb{Q}_p \subset \mathbb{Z}_p = K^\circ$, A is a flat p-adically complete $\mathbb{Z}_p$-algebra, $R = A[\frac{1}{p}]$, then $R_{\leq 1} = A$. Thus is seminorm on R is $|f| = \min\{|p|^n | f \in p^n A\}$. But R° could be larger then $R_{\leq 1}$: if $A = K^\circ[x]/(x^2)$, then $\frac{1}{p^n} xA \subset R^\circ$ for each n, but these not lies in $R_{\leq 1} = A$.

Definition 3.7. For a perfectoid field K, set $K \supseteq K^\circ \supseteq \mathfrak{m} \ni \varpi$ with tilt $K^\flat$. Define:

(i) A Banach K-algebra R is $perfectoid$ if $R^\circ \subset R$ is bounded, and the Frobenius map $R^\circ/\varpi \to R^\circ/\varpi$ is surjective. With continuous morphisms as morphisms of K-alg, this gives the category $K - \text{Perf}$ of perfectoid K-algebras.

(ii) A $K^{\circ a}$-algebra A is $perfectoid$ if:

(1) A is t-adically complete and flat over K°.

(2) The map $K^\circ/\varpi \to A/\varpi$ is relatively perfect, i.e., the Frobenius induces an isomorphism $A/\varpi^{\frac{1}{p}} \simeq A/\varpi$.

With continuous morphisms as morphisms of $K^{\circ a}$-alg, this gives the category $K^{\circ a} - \text{Perf}$ of perfectoid $K^{\circ a}$ algebras.

(iii) A $K^{\circ a}/\varpi$-algebra A is perfectoid if:

(1) A is flat over K°/ϖ.

(2) The map $K^\circ/\varpi \to A$ is relatively perfect, i.e., the Frobenius induces an isomorphism $A/\varpi^{\frac{1}{p}} \simeq A$.

With continuous morphisms as morphisms of $K^{\circ a}/\varpi$-alg, this gives the category $K^{\circ a}/\varpi - \text{Perf}$ of perfectoid $K^{\circ a}/\varpi$ algebras.

Then we start to explain the first diagram of equivalences of categories at the beginning of this section.

4. L4-7: Perfectoid Algebras and Almost Purity (1)

Our main theorem is:

Theorem 4.1 (Tilting from characteristic 0 to characteristic p)**.** *We have the chain of equivalences:*

$$K-\mathrm{Perf} \cong K^{\circ a}-\mathrm{Perf} \cong K^{\circ a}/\varpi-\mathrm{Perf} \cong K^{\flat\circ a}/\varpi^{\flat}-\mathrm{Perf} \cong K^{\flat}-\mathrm{Perf}$$

4.1. $K-\mathrm{Perf} \cong K^{\circ a}-\mathrm{Perf}$**.** First we are going to prove the canoncal equivalence: $K-\mathrm{Perf} \cong K^{\circ a}-\mathrm{Perf}$. "In other words, a perfectoid K-algebra, which is an object over the generic fibre, has a canonical extension to the almost integral level as a perfectoid $K^{\circ a}$-algebra". This proof is quite explicit, so I want to copy it directly from scholze's thesis. This helps me to understand what is "almost".

Proposition 4.1. *Let $R \in K-\mathrm{Perf}$. Then Φ induces an isomorphism: $R^{\circ}/\varpi^{1/p} \cong R^{\circ}/\varpi$ and $R^{\circ a} \in K^{\circ a}-\mathrm{Perf}$.*

Proof. It is clear that Φ is a surjection. By Lemma 3.1 we have R° is complete and flat over K°. Moreover, the perfectoidness of R ensures that the Frobenius is surjective. Thus it suffices to show that Frobenius has kernel $(\varpi^{1/p})$. ☐

Lemma 4.1. *Let A be a perfectoid $K^{\circ a}$-algebra, and let $R = A_*[\varpi^{-1}]$. Equip R with the Banach K-algebra structure making A_* open and bounded. Then $A_* = R^{\circ}$ is the set of power-bounded elements, which is ϖ-adically complete, ϖ torsionfree, p-root closed in R and has a surjective Frobenius modulo ϖ.*

Proof. The completeness results and torsion free is clear by Lemma 3.1.

p-root closed: Since Φ is an isomorphism $A/\varpi^{1/p} \cong A/\varpi$, hence Φ is an almost isomorphism $A_*/\varpi^{1/p} \to A_*/\varpi$. It is injective: If $x \in A_*$ and $x^p \in \varpi A_*$, then for all $\epsilon \in \mathfrak{m}$, $\epsilon x \in \varpi^{1/p}A_*$ by almost injectivity, hence $x \in (\varpi^{1/p}A)_* = \varpi^{1/p}A_*$.

Thus if $y \in A_*$ satisfies $y^p \in \varpi A_*$, then $y \in \varpi^{\frac{1}{p}}A_*$. Now fix $x \in R$ with $x^p \in A_*$. There is some positive integer k such that $y = \varpi^{\frac{k}{p}}x \in A_*$, and as long as $k \geq 1$, we have $y^p \in \varpi, \varpi^k A_* \subset \varpi A_*$, so that $y \in \varpi^{\frac{1}{p}}A_*$ by injectivity. Thus $\varpi^{\frac{k-1}{p}}x = \frac{y}{\varpi^{1/p}} \in A_*$. By induction, $x \in A_*$.

$A_* = R^{\circ}$: Obviously, A_* consists of power-bounded elements. Now assume that $x \in R$ is powerbounded. Then ϵx is topologically nilpotent for all $\epsilon \in \mathfrak{m}$. In particular, $(\epsilon x)^{p^N} \in A_*$ for N sufficiently large. By the p-root closedness, this implies $\epsilon x \in A_*$. Since this is true for all $\epsilon \in \mathfrak{m}$, we have $x \in A_*$ by the key lemma.

For surjectivity of Frobenius: It is almost surjective, hence it suffices to show that the composition $A_*/\varpi^{1/p} \to A_*/\varpi \to A_*/\mathfrak{m}$ is surjective. Let $x \in A_*$. By almost surjectivity, $\varpi^c x \equiv y^p$ modulo ϖA_*, for some $y \in A_*$ and $c < 1$. Let $z = \frac{y}{\varpi^{c/p}} \in R$. This implies $z^p \in A_*$ also $z \in A_*$, by p-root closedness. Thus $y \in \varpi^{c/p}A_*$ which implies $x \equiv z^p$ modulo $\varpi^{1-c}A_* \supseteq \mathfrak{m}A_*$. This gives the desired surjectivity. ☐

Remark 4.1. In fact, we have the following proposition:

Proposition 4.2. *Fix a pseudouniformizer $t \in K$. The following categories are equivalent:*

** The category $\mathcal{C}$ of uniform Banach K-algebras R with continuous K-algebra maps.*

** The category $\mathcal{D}_{tic}$ of t-adically complete and t-torsionfree K°-algebras A with A totally integrally closed (i.e. the given $f \in A[\frac{1}{t}]$ with $f^{\mathbb{N}}$ lying in a f.g. A-submodule of $A[\frac{1}{t}]$ will lie in A) in $A[\frac{1}{t}]$.*

The functors are given by $F : R \mapsto R^\circ$ and $G : A \mapsto A[\frac{1}{t}]$, and $G \circ F \cong \mathrm{id}$.

If K is a perfectoid field, $\mathcal{D}_{\mathrm{tic}}$ is equivalent to the category $\mathcal{D}_{prc}$ of t-adically complete and t-torsion free K° algebras A with A p-root closed on $A[\frac{1}{t}]$ and $A \cong A_*$. Thus for $R \in K$-Perf, $A \in K^{\circ a}$-Perf, we have $(R^{\circ a})_*[\frac{1}{\varpi}] \cong R$ and $A \cong (A_*[\frac{1}{\varpi}])^{\circ a}$.

4.2. **Deformation and Tilt equivalence.** In order to finish the proof of the main theorem, it suffices to prove:

Theorem 4.2. *The functor $A \mapsto \bar{A} = A/\varpi$ induces an equivalence of categories $K^{\circ a} - \mathrm{Perf} \cong (K^{\circ a}/\varpi) - \mathrm{Perf}$.*

To construct the inverse, we need the deformation theory. Since I am not familiar with this language before (now as well), I don't want to talk about the construction here. You can check the reference note by Bhott for the definition of $P^\bullet_{B/A}$ (which is the canonical simplicial A-algebra resolution of B) and the cotangent complex $L_{B/A} := \Omega^1_{P^\bullet/A} \otimes_{P^\bullet} B$ (where $P^\bullet \to B$ is a simplicial resolution of B by polynomial A-algebras. Here we view the simplicial B-module $\Omega^1_{P^\bullet/A} \otimes_{P^\bullet} B$ as a B-complex by taking an alternating sum of the face maps as a differential). You can also check the basic properties there. "The main reason to introduce the cotangent complex is that it controls deformation theory in complete generality, analogous to how the tangent bundle controls deformations of smooth varieties":

Theorem 4.3 (Infinitesimal invariance of formally étale rings)**.** *For any ring A, write $\mathcal{C}_A$ for the category of flat A-algebras B such that $L_{B/A} \simeq 0$. Then for any surjective map $\tilde{A} \to A$ with nilpotent kernel, base change induces an equivalence $\mathcal{C}_{\tilde{A}} \simeq \mathcal{C}_A$.*

Any étale A-algebra B is an object of $\mathcal{C}_A$. For our purposes, the following class of examples is crucial:

Proposition 4.3 (Perfect rings have a trivial cotangent complex)**.** *Assume A has characteristic p. Let $A \to B$ be a flat that is relatively perfect, i.e., the relative Frobenius $F_{B/A} : B^{(1)} := B \otimes_{A,F_A} A \to B$ is an isomorphism. Then $L_{B/A} \cong 0$.*

We can draw the following diagram:

$$\begin{tikzcd}
A \arrow[r, "Frob_A"] \arrow[dd] & A \arrow[d] \arrow[ddr, bend left] & \\
& B \otimes_{A,Frob_A} B \arrow[dr, dashed, "\sim"] & \\
B \arrow[ur] \arrow[rr, "Frob_B"'] & & B
\end{tikzcd}$$

Corollary 4.1. *Let R be a ring equipped with a nonzero divisor $f \in R$. Then reduction modulo f gives and equivalence between the category of f-adically complete and f-torsionfree R-algebras S with $R/f \to S/f$ flat and $L_{S/R} \otimes_R R/f \cong 0$, and the category $\mathcal{C}_{R/f}$.*

Example 4. Let R be a perfect ring of characteristic p. Then R is relatively perfect over $\mathbb{F}_p$, then we can construct $W(R)$ as its Witt vectors, which can also be seen as the unique p-adically complete p-torsionfree $\mathbb{Z}_p$-algebra lifting R by the latest corollary. Explicitly, one simply sends $r \in R$ to $\tilde{r_n}^{p^n}$, where $\tilde{r_n} \in W_n(R)$ denotes some lift of $r_n = r^{1/p^n}$. The resulting multiplicative maps $R \to W(R)$ are called the Teichmuller lifts and denotes as $r \mapsto [r]$. From the universal property describing $W(R)$, it is clear that if R is f-adically complete for some element $f \in R$, then $W(R)$ is $(p, [f])$-adically complete.

(Fontaine's map θ and A_{inf}) Fix a map $A \to B$ in $\mathcal{C}_A$. One fact is, if $C' \to C$ is surjective with nilpotent kernel, then every A-algebra map $B \to C$ lifts unique to an A-algebra map $B \to C'$. In particular, we can use this to get a unique map $W(R) \to C$. In perfectoid theory, this observation shows:

Proposition 4.4 (The kernel of θ)**.** *Given a perfectoid field K, the canonical map $\bar{\theta} : K^{\circ\flat} \to K^\circ/p$ lifts to a unique map $\theta : A_{inf}(K^\circ) := W(K^{\circ\flat}) \to K^\circ$. The kernel of θ is a principal ideal generated by a nonzerodivisor. In fact, for K having characteristic 0, one may choose $\xi \in \ker(\theta)$ to be any element such that ξ generates the kernel of $K^{\circ\flat} \to K^\circ/p$; when K has characteristic p, we have* $\ker(\theta) = (p)$.

Thus we obtain a pushout square:

$$\begin{array}{ccc} A_{inf}(K^\circ) & \xrightarrow{\theta} & K^\circ \\ \downarrow & & \downarrow \\ K^{\circ\flat} & \longrightarrow & K^\circ/p \end{array}$$

In characteristic p, the right vertical map and the bottom horizonal map are isomorphisms, while the remaining two maps coincide with reduction modulo p. In characteristic 0, all maps are quotients by nonzerodivisors along which the source ring is complete. "In particular, in both cases, all rings involved can be viewed as pro-infinitesimal thickenings of K°/p." We will come back later.

Then we will prove the equivalence $K^{\circ a} - \mathrm{Perf} \cong K^{\circ a}/\varpi - \mathrm{Perf}$.

Proof. □

In particular, we also arrive at the tilting equivalence, $K - \mathrm{Perf} \cong K^\flat - \mathrm{Perf}$. We want to compare this with Fontaine's explicit construction. Let R be a perfectoid K-algebra, with $A = R^{\circ a}$. Define

$$A^\flat = \varprojlim_{\Phi} A/\varpi \ ,$$

which we regard as a $K^{\flat\circ a}$-algebra via

$$K^{\flat\circ a} = (\varprojlim_{\Phi} K^\circ/\varpi)^a = \varprojlim_{\Phi} (K^\circ/\varpi)^a = \varprojlim_{\Phi} K^{\circ a}/\varpi \ ,$$

and set $R^\flat = A^\flat_*[(\varpi^\flat)^{-1}]$ where $A*$. Recall that $(\ )_*$ represents the functor of almost elements.

Proposition 4.5. *This defines a perfectoid $K^\flat$-algebra $R^\flat$ with corresponding perfectoid $K^{\flat\circ a}$-algebra $A^\flat$, and $R^\flat$ is the tilt of R. Moreover,*

$$R^\flat = \varprojlim_{x \mapsto x^p} R \ , \ A^\flat_* = \varprojlim_{x \mapsto x^p} A_* \ , \ A^\flat_*/\varpi^\flat \cong A_*/\varpi \ .$$

In particular, we have a continuous multiplicative map $R^\flat \to R$, $x \mapsto x^\sharp$.

Remark 4.2. It follows that the tilting functor is independent of the choice of ϖ and $\varpi^\flat$. We note that this explicit description comes from the fact that the deformation from perfectoid $K^{\flat\circ a}/\varpi^\flat$-algebras to perfectoid $K^{\flat\circ a}$-algebras can be made explicit by means of the inverse limit over the Frobenius.

Proof. First, we have

$$A_*^\flat = (\varprojlim_\Phi A/\varpi)_* = \varprojlim_\Phi (A/\varpi)_* = \varprojlim_\Phi A_*/\varpi \ ,$$

because $_*$ commutes with inverse limits and using Lemma 3.1. Note that the image of $\Phi : (A/\varpi)_* \to (A/\varpi)_*$ is A_*/ϖ, because it factors over $(A/\varpi^{1/p})_*$, and the image of the projection $(A/\varpi)_* \to (A/\varpi^{1/p})_*$ is $A_*/\varpi^{1/p}$. But

$$\varprojlim_\Phi A_*/\varpi = \varprojlim_{x\mapsto x^p} A_* \ ,$$

as in the proof of 2.2.

This shows that $A_*^\flat$ is a $\varpi^\flat$-adically complete flat $K^{\flat\circ}$-algebra. Moreover, the projection $x \mapsto x^\sharp$ of $A_*^\flat$ onto the first component $x^\sharp \in A_*$ induces an isomorphism

$$A_*^\flat/\varpi^\flat \cong A_*/\varpi \ ,$$

Therefore $A^\flat$ is a perfectoid $K^{\flat\circ a}$-algebra.

Then we can easilt go through all equivalences to check that $R^\flat$ is the tilt of R. ∎

Remark 4.3. (Untilting via A_{inf}).

Witt vector can be used to give an alternate perspective on the tilting correson-dence, and explicit description of the untilting functor. Recall that we have the pushout:

$$\begin{array}{ccc} A_{inf}(K^\circ) & \xrightarrow{\theta} & K^\circ \\ \downarrow & & \downarrow \\ K^{\circ\flat} & \longrightarrow & K^{\circ\flat}/t \cong K^\circ/\pi \end{array}$$

“where all rings could be viewed as pro-infinitesimal thickenings of K°/π”. In particular, since any relatively perfect (or, equivalentlly, perfect) $K^{\circ\flat}$-algebra A has unique lift $W(A)$ along $A_{inf}(K^\circ) \to K^{\circ\flat}$, and the base change $W(A) \otimes_{A_{inf}(K^\circ)} K^\circ$ provides a K°-algebra that lifts $A \otimes_{K^\circ} K^{\circ\flat}/\pi$ by the above diagram. From this, for $S \in \mathrm{Perf}_{K^\flat}$, then its untilt $S^\sharp \in \mathrm{Perf}_K$ is given by

$$S^\sharp := (W(S^\circ) \otimes_{A_{inf}(K^\circ)} K^\circ)[\frac{1}{\pi}]$$

4.3. **almost purity (1).** Finally, let us discuss finite étale covers of perfectoid algebras, and finish the proof of Theorem 2.1.

Proposition 4.6. *Let $\overline{A}$ be a perfectoid $K^{\circ a}/\varpi$-algebra, and let $\overline{B}$ be a finite étale $\overline{A}$-algebra. Then $\overline{B}$ is a perfectoid $K^{\circ a}/\varpi$-algebra.*

Notice that (I don't actually know how it comes out, but we will come back later.)

Theorem 4.4. *Let A be a $K^{\circ a}$-algebra. Assume that A is flat over $K^{\circ a}$ and ϖ-adically complete, i.e.*

$$A \cong \varprojlim A/\varpi^n \ .$$

Then the functor $B \mapsto B \otimes_A A/\varpi$ induces an equivalence of categories $A_{\text{fét}} \cong (A/\varpi)_{\text{fét}}$. Any $B \in A_{\text{fét}}$ is again flat over $K^{\circ a}$ and ϖ-adically complete. Moreover, B is a uniformly finite projective A-module if and only if $B \otimes_A A/\varpi$ is a uniformly finite projective A/ϖ-module.

This provides the following commutative diagram: for $R, A = R^{\circ a}, \overline{A}, A^\flat$ and $R^\flat$ form a sequence of rings under the tilting procedure.

$$\begin{array}{ccccccccc}
R_{\text{fét}} & \longleftarrow & A_{\text{fét}} & \xrightarrow{\cong} & \overline{A}_{\text{fét}} & \xleftarrow{\cong} & A^\flat_{\text{fét}} & \longrightarrow & R^\flat_{\text{fét}} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
K-\text{Perf} & \longleftarrow & K^{\circ a}-\text{Perf} & \longrightarrow & K^{\circ a}/\varpi-\text{Perf} & \longrightarrow & K^{\flat \circ a}-\text{Perf} & \longrightarrow & K^\flat-\text{Perf}
\end{array}$$

It follows from this diagram that the functors $A_{\text{fét}} \to R_{\text{fét}}$ and $A^\flat_{\text{fét}} \to R^\flat_{\text{fét}}$ are fully faithful. Then we want to show that both of them are equivalences, this involves Falting's almost purity theorem. At first we want to introduce the trace map, this helps us to prove the characteristic p case.

Define the trace map: let R be a ring, and let $R \to S$ be a finite étale extension. Morphism $s \mapsto [s' \mapsto ss']$ induces $S \hookrightarrow \text{End}_R(S)$, since S is finite flat and R is noetherian, the R module S is locally free R-module and multiplication os therefore locally given by multiplication by a matrix. We define the trace as the trace of the matrix. Let $e \in S \otimes_R S$ be the diagonal idempotent cutting out the multiplication $\mu : S \otimes_R S \to S$, i.e., $e^2 = e$ and $\ker(\mu) = 0$. Write $e = \sum_{i=1}^n a_i \otimes b_i$ for $a_i, b_i \in S$. Then we can explicitly realize S is a direct summand of R^n via the maps

$$S \xrightarrow{\alpha} R^n \xrightarrow{\beta} S$$

given by

$$\alpha(f) = \left(\text{Tr}_{S/R}(fa_i)\right) \qquad \text{and} \qquad \beta\left((g_i)\right) = \sum_{i=1}^n g_i b_i.$$

First we must show that $\beta \circ \alpha = id$. In other words, we want to check that

$$\sum_{i=1}^n \text{Tr}_{S/R}(fa_i)\, b_i = f$$

for any $f \in S$. To prove this, note that

$$\text{Tr}_{i_2}(e) = \text{Tr}_{S/S}(1) = 1$$

where $i_2 : S \to S \otimes_R S$ is the second inclusion $s \mapsto 1 \otimes s$. Plugging in $e = \sum_i a_i \otimes b_i$ above and using the compatibility of trace maps with base change, we get

$$\sum_{i=1}^n \text{Tr}_{S/R}(a_i)\, b_i = 1$$

In particular, this verifies the formula for $f=1$. In general, one repeats the same argument by replacing e with $(f\otimes 1)\cdot e$ (which equals $(1\otimes f)\cdot e$ as $\ker(\mu)\cdot e=0$):

$$\sum_{i=1}^{n} \mathrm{Tr}_{S\otimes_S/R}(fa_i)b_i = \mathrm{Tr}_{S\otimes_R S/S}((f\otimes 1)e) = \mathrm{Tr}_{S\otimes_R S/S}((1\otimes f)e) = f\,\mathrm{Tr}_{S\otimes_R S/S}(e) = f$$

Using this construction (Finite étale algebras explicitly as finite projective modules), we arrive at the almost purity theorem in characteristic p:

Theorem 4.5 (Almost purity in char p: primitive version.). *Let $\eta R\to S$ be an integral map of perfect rings. Assume that $\eta[\frac{1}{t}]$ is finite étale for some $t\in R$. Then η is almost finite étale with respect to the ideal $I=(t^{\frac{1}{p^\infty}})$.*

"In other words, the assumption that $\eta[\frac{1}{t}]$ on the "generic fibre" spreads out to the conclusion that η is almost finite etale on the "almost integral fibre"."

Proof. (i) We first reducing to t-torsionfree case by observating that the t-power torsion ideals ...

(ii) Then we reduce to the case where R is integrally closed in $R[\frac{1}{t}]$, also for S respectively.

(iii) Next we check almost unramifiedness.

(iv) It remains to show that S is almost finite projective over R.

□

Then we can upgrade this to an equivalence of categories.

Theorem 4.6. *Let R be a perfect ring of characteristic p, and consider almost mathematics with respect to $I=(t^{\frac{1}{p^\infty}})$ for a fixed element $t\in R$. Then $S\mapsto S_*[\frac{1}{t}]$ gives an equivalence of categories $R^a_{\text{fét}}\cong R[\frac{1}{t}]_{\text{fét}}$.*

Proof. (i) As the proof before, we may assume R has no t-torsion (t is a nonzero-divisor on A).

(ii) Since S_* is finite étale, $(S\otimes_R S)_*[\frac{1}{t}]\cong(S_*\otimes_R S_*)[\frac{1}{t}]$ gives the idempotent. Almost projective also can be checked easily.

(iii) perfectness induces essential suerjectivity, by the above theorem.

(iv) For faithfulness, fix some $S\in R^a_{\text{fét}}$. We claim:

$$S\cong T^a \text{ for the integral closure } T \text{ of } R \text{ in } S_*[\frac{1}{t}].$$

□

We prove the lemma now.

Lemma 4.2. *Let $A\to B$ be weakly étale map of $\mathbb{F}_p$-algebra, i.e., both $A\to B$ and $\mu: B\otimes_A B\to B$ are flat. Then the diagram*

$$\begin{array}{ccc} A & \xrightarrow{Frob_A} & A \\ \downarrow & & \downarrow \\ B & \xrightarrow{Frob_B} & B \end{array}$$

is a pushout square of rings, i.e., the relative Frobenius $F_{B/A}: B\otimes_{A,Frob_A} B\to B$ is an isomorphism. In particular, if A is perfect, so is B.

Proof. remains to explain.

$$\begin{tikzcd}
A \arrow[r, "Frob_A"] \arrow[dd] & A \arrow[r, "Frob_A"] \arrow[d] & A \arrow[dr] & \\
& B \otimes_{A,Frob_A} A \arrow[rr, "Frob"] \arrow[dr, dashed, "F_{B/A}"] & & B \otimes_{A,Frob_A} A \\
B \arrow[ur] \arrow[rr, "Frob_B"] & & B \arrow[ur] &
\end{tikzcd}$$

So it suffices to show that any weakly etale map $\alpha : R \to S$ of $\mathbb{F}_p$-algebra that factors a power of Frobenius on R and S is an isomorphism.

remains to explain. □

This provides us to upgrade our diagram: $A^\flat_{\text{fét}} \cong R^\flat_{\text{fét}}$ (where $A = R^{\circ a}$ as usual). To summarize, we have the following theorem:

Theorem 4.7. *Let R be a perfectoid K-algebra with tilt $R^\flat$. There is a fully faithful functor from $R^\flat_{\text{fét}}$ to $R_{\text{fét}}$ inverse to the tilting functor. The essential image of this functor consists of the finite étale covers S of R, for which S (with its natural topology) is perfectoid and $S^{\circ a}$ is finite étale over $R^{\circ a}$. In this case, $S^{\circ a}$ is a uniformly finite projective $R^{\circ a}$-module.*

We will later prove that this is an equivalence in general. For now, we prove that it is an equivalence for perfectoid fields, i.e. we finish the proof of Theorem 2.1.

Proof. Let K be a perfectoid field, characteristic 0, with tilt $K^\flat$. It is enough to show that the fully faithful functor $\sharp : K^\flat_{\text{fét}} \to K_{\text{fét}}$ is an equivalence. This functor is fully faithful (as this true always), perserve degrees (by constructions). Let $M = \widehat{\overline{K^\flat}}$ be the completion of an algebraic closure of $K^\flat$. M is complete and perfect, i.e. M is perfectoid. Let $M^\sharp$ be the untilt of M. Then by Lemma 2.2, $M^\sharp$ is an algebraically closed perfectoid field containing K (Actually for all $R \in K-\text{Perf}$ with tilt $R^\flat$, R is a perfectoid field iff $R^\flat$ is a perfectoid field). Any finite extension $L \subset M$ of $K^\flat$ gives the untilt $L^\sharp \subset M^\sharp$, a finite extension of K. It is easy to see that the union $N = \bigcup_L L^\sharp \subset M^\sharp$ is a dense subfield. Now Krasner's lemma implies that N is algebraically closed. Hence any finite extension F of K is contained in N; this means that there is some Galois extension L of $K^\flat$ such that F is contained in $L^\sharp$. Note that $L^\sharp$ is still Galois, as the functor $L \mapsto L^\sharp$ preserves degrees and automorphisms. In particular, F is given by some subgroup H of $\text{Gal}(L^\sharp/K) = \text{Gal}(L/K^\flat)$, which gives the desired finite extension $F^\flat = L^H$ of $K^\flat$ that untilts to F: The equivalence of categories shows that $(F^\flat)^\sharp \subset (L^\sharp)^H = F$, and as they have the same degree, they are equal. □

Remark 4.4. Recall the Krasner's lemma: Let $(K, |\cdot|)$ be a complete non-archi. valuation field. Let $\alpha, \beta \in \overline{K}$ such that $|\alpha - \beta| < |\alpha - \alpha'|$ for all Galois conjugate α' of α different from α. Then $\alpha \in K(\beta)$.

5. L8-10: Adic Spaces

Recall the definition 2.2. If $|\cdot| : R \to \Gamma \cup \{0\}$ is a valuation on R, let $\Gamma_{|\cdot|} \subset \Gamma$ denote the subgroup generated by all $|x|$, $x \in R$, which are nonzero. The set $\operatorname{supp}(|\cdot|) = \{x \in R \mid |x| = 0\}$ is a prime ideal of R called the support of $|\cdot|$. Let K be the quotient field of $R/\operatorname{supp}(|\cdot|)$. Then the valuation factors as a composite $R \to K \to \Gamma \cup \{0\}$. Let $R(|\cdot|) \subset K$ be the valuation subring, i.e. $R(|\cdot|) = \{x \in K \mid |x| \leq 1\}$.

Definition 5.1. Two valuations $|\cdot|$, $|\cdot|'$ are called equivalent if the following equivalent conditions are satisfied.

(i) There is an isomorphism of totally ordered groups $\alpha : \Gamma_{|\cdot|} \cong \Gamma_{|\cdot|'}$ such that $|\cdot|' = \alpha \circ |\cdot|$.

(ii) The supports $\operatorname{supp}(|\cdot|) = \operatorname{supp}(|\cdot|')$ and valuation rings $R(|\cdot|) = R(|\cdot|')$ agree.

(iii) For all $a, b \in R$, $|a| \geq |b|$ if and only if $|a|' \geq |b|'$.

For K a complete non-archimedean field, $\varpi \in K^\circ \subset K$,

Definition 5.2. (i) A Tate k-algebra is a topological k-algebra R for which there exists a subring $R_0 \subset R$ such that aR_0, $a \in k^\times$, forms a basis of open neighborhoods of 0. A subset $M \subset R$ is called bounded if $M \subset aR_0$ for some $a \in k^\times$. An element $x \in R$ is called power-bounded if $\{x^n \mid n \geq 0\} \subset R$ is bounded. Let $R^\circ \subset R$ denote the subring of powerbounded elements.

(ii) An affinoid k-algebra is a pair (R, R^+) consisting of a Tate k-algebra R and an open and integrally closed subring $R^+ \subset R^\circ$.

(iii) An affinoid k-algebra (R, R^+) is said to be of topologically finite type (tft for short) if R is a quotient of $k\langle T_1, \ldots, T_n\rangle$ for some n, and $R^+ = R^\circ$.

Here,

$$k\langle T_1, \ldots, T_n\rangle = \{\sum_{i_1,\ldots,i_n \geq 0} x_{i_1,\ldots,i_n} T_1^{i_1} \cdots T_n^{i_n} \mid x_{i_1,\ldots,i_n} \in k, x_{i_1,\ldots,i_n} \to 0\}$$

is the ring of convergent power series on the ball given by $|T_1|, \ldots, |T_n| \leq 1$.

Remark 5.1. One fact is R° is an open integrally closed subring of R. Another fact is any Tate k-algebra R, resp. affinoid k-algebra (R, R^+), admits the completion $\hat{R}$, resp. $(\hat{R}, \hat{R}^+)$, which is again a Tate, resp. affinoid, k-algebra. Everything "depends" only on the completion, so one may assume that (R, R^+) is complete in the following.

Definition 5.3. Let (R, R^+) be an affinoid k-algebra. Let

$$X = \operatorname{Spa}(R, R^+) = \{|\cdot| : R \to \Gamma \cup \{0\} \text{ continuous valuation} \mid \forall f \in R^+ : |f| \leq 1\}/\cong .$$

For any $x \in X$, write $f \mapsto |f(x)|$ for the corresponding valuation on R. We equip X with the topology which has the open subsets

$$U(\frac{f_1, \ldots, f_n}{g}) = \{x \in X \mid \forall i : |f_i(x)| \leq |g(x)|\} ,$$

called rational subsets, as basis for the topology, where $f_1, \ldots, f_n \in R$ generate R as an ideal and $g \in R$.

The following remark is quite useful for calculation.

Remark 5.2. Let $\varpi \in k$ be topologically nilpotent, i.e. $|\varpi| < 1$. Then to $f_1, \dots, f_n$ one **can add** $f_{n+1} = \varpi^N$ for some big integer N without changing the rational subspace. Indeed, there are elements $h_1, \dots, h_n \in R$ such that $\sum h_i f_i = 1$. Multiplying by ϖ^N for N sufficiently large, we have $\varpi^N h_i \in R^+$, as $R^+ \subset R$ is open. Now for any $x \in U(\frac{f_1,\dots,f_n}{g})$, we have

$$|\varpi^N(x)| = |\sum(\varpi^N h_i)(x) f_i(x)| \leq \max|(\varpi^N h_i)(x)||f_i(x)| \leq |g(x)| \ ,$$

as desired. In particular, we see that on rational subsets, $|g(x)|$ is nonzero, and bounded from below.

Remark 5.3. Recall that a topological space X is quasicompact if every open covering of X has a finite cover. X is called quasiseparated if the intersection of any two quasicompact open subsets is again quasicompact. In the following we will often abbreviate quasicompact, resp. quasiseparated, as qc, resp. qs.

Definition 5.4. A topological space X is called spectral if it satisfies the following equivalent properties.

(i) There is some ring A such that $X \cong \mathrm{Spec}(A)$.

(ii) One can write X as an inverse limit of finite T_0 spaces.

(iii) The space X is quasicompact, has a basis of quasicompact open subsets stable under finite intersections, and every irreducible closed subset has a unique generic point.

In particular, spectral spaces are quasicompact, quasiseparated and T_0.

Huber proved the following basic properties:

Proposition 5.1. *For any affinoid k-algebra (R, R^+), the space* $\mathrm{Spa}(R, R^+)$ *is spectral. The rational subsets form a basis of quasicompact open subsets stable under finite intersections.*

Proposition 5.2. *Let (R, R^+) be an affinoid k-algebra with completion $(\hat{R}, \hat{R}^+)$. Then* $\mathrm{Spa}(R, R^+) \cong \mathrm{Spa}(\hat{R}, \hat{R}^+)$, *identifying rational subsets.*

Proposition 5.3. *Let (R, R^+) be an affinoid k-algebra, $X = \mathrm{Spa}(R, R^+)$.*

(i) *If $X = \emptyset$, then $\hat{R} = 0$.*

(ii) *Let $f \in R$ be such that $|f(x)| \neq 0$ for all $x \in X$. If R is complete, then f is invertible.*

(iii) *Let $f \in R$ be such that $|f(x)| \leq 1$ for all $x \in X$. Then $f \in R^+$.*

Proof. It suffices to show that if $\hat{R} \neq 0$, then there exist a continuous valuation of $\hat{R}$. Take an element ϖ which is not invertible, then there exists an prime ideal $\mathfrak{m} \subset R^+$, such that $\varpi \in \mathfrak{m}$, $(R^+)_{\mathfrak{m}}[\frac{1}{\varpi}] \neq 0$.

Take an nonzero prime ideal in $(R^+)_{\mathfrak{m}}[\frac{1}{\varpi}]$, then find its pullback $\mathfrak{p} \subset \mathfrak{m} \subset R^+$. Note that ϖ will not contained in $\mathfrak{p}$.

Recall that the valuation rings of a field are the maximal elements of the set of the local subrings in the field partially ordered by dominance or refinement, where

$$(A, \mathfrak{m}_A) \text{ dominates } (B, \mathfrak{m}_B) \text{ if } B \subset A \text{ and } \mathfrak{m}_A \cap B = \mathfrak{m}_B.$$

By Zorn lemma, we have the following **fact**: there exist a valuation ring, write as $(R_{\mathfrak{p}}, \mathfrak{m}')$, we have

$$R^+/\mathfrak{p} \subset R_{\mathfrak{p}} \subset \mathrm{Frac}(R^+/\mathfrak{p}),\ \text{such that } \exists \mathfrak{m}' \subset R_{\mathfrak{p}},\ \mathfrak{m}' \cap R^+/\mathfrak{p} = \mathfrak{m}$$

We can choose the related valuation. It will be a continuous valuation.

For (ii), use the fact for $(\hat{R}/f, \hat{R}^+/f)$. $\square$

We want to endow $X = \mathrm{Spa}(R, R^+)$ with a structure sheaf $\mathcal{O}_X$. The construction is as follows.

Definition 5.5. Let (R, R^+) be an affinoid k-algebra, and let $U = U(\frac{f_1,\dots,f_n}{g}) \subset X = \mathrm{Spa}(R, R^+)$ be a rational subset. Choose some $R_0 \subset R$ such that aR_0, $a \in k^\times$, is a basis of open neighborhoods of 0 in R. Consider the subalgebra $R[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$ of $R[g^{-1}]$, and equip it with the topology making $aR_0[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$, $a \in k^\times$, a basis of open neighborhoods of 0. Let $B \subset R[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$ be the integral closure of $R^+[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$ in $R[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$. Then $(R[\frac{f_1}{g}, \dots, \frac{f_n}{g}], B)$ is an affinoid k-algebra. Let $(R\langle\frac{f_1}{g}, \dots, \frac{f_n}{g}\rangle, \hat{B})$ be its completion.

Obviously,

$$\mathrm{Spa}(R\langle\frac{f_1}{g}, \dots, \frac{f_n}{g}\rangle, \hat{B}) \to \mathrm{Spa}(R, R^+)$$

factors over the open subset $U \subset X$.

Proposition 5.4. *In the situation of the definition, the following universal property is satisfied. For every complete affinoid k-algebra (S, S^+) with a map ι : $(R, R^+) \to (S, S^+)$ such that the induced map $\mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ factors over U, there is a unique map*

$$(R\langle\frac{f_1}{g}, \dots, \frac{f_n}{g}\rangle, \hat{B}) \to (S, S^+)$$

making the obvious diagram commute.

In particular, $(R\langle\frac{f_1}{g}, \dots, \frac{f_n}{g}\rangle, \hat{B})$ depends only on U. Define

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (R\langle\frac{f_1}{g}, \dots, \frac{f_n}{g}\rangle, \hat{B}) \ .$$

For example, $(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ is the completion of (R, R^+).

Proof. Since there exists N. s.t. $|g(x)| \leq |\varpi^N(x)|, \forall x \in U$, thus for any $x \in \mathrm{Spa}(S, S^+)$, $|\iota(g)(x)| \neq 0$, i.e. $\iota(g)$ is invertible. Thuen $\iota(\frac{f_i}{g}) \in S^+$. $\square$

We define presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on X as above on rational subsets, and for general open $U \subset X$ by requiring

$$\mathcal{O}_X(W) = \varprojlim_{U \subset W \text{ rational}} \mathcal{O}_X(U) \ ,$$

and similarly for $\mathcal{O}_X^+$.

Proposition 5.5. *For any $x \in X$, the valuation $f \mapsto |f(x)|$ extends to the stalk $\mathcal{O}_{X,x}$, and*

$$\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x} \mid |f(x)| \leq 1\} \ .$$

The ring $\mathcal{O}_{X,x}$ is a local ring with maximal ideal given by $\{f \mid |f(x)| = 0\}$. The ring $\mathcal{O}^+_{X,x}$ is a local ring with maximal ideal given by $\{f \mid |f(x)| < 1\}$. Moreover, for any open subset $U \subset X$,

$$\mathcal{O}^+_X(U) = \{f \in \mathcal{O}_X(U) \mid \forall x \in U : |f(x)| \leq 1\} .$$

If $U \subset X$ is rational, then $U \cong \mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}^+_X(U))$ compatible with rational subsets, the presheaves $\mathcal{O}_X$ and $\mathcal{O}^+_X$, and the valuations at all $x \in U$ (use the universal proerty 5.4 to check).

It is not in general that $\mathcal{O}_X$ is a sheaf. The proposition ensures that $\mathcal{O}^+_X$ is a sheaf if $\mathcal{O}_X$ is. “The basic problem is that completion behaves in general badly for nonnoetherian rings”.

Let’s give two example: for k a complete and algebraically closed field, we are going to describe $\mathrm{Spa}(k, k^\circ)$ and $\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$, which is an affinoid k-algebra of tft. I’ll highly recommend this note for details (thanks leader Li again).

It is easy to see that the space $\mathrm{Spa}(k, k^\circ)$ consist of a single point. For $\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$, here is the classification:

(1) The classical points: Let $x \in k^\circ$, i.e. $x \in k$ with $|x| \leq 1$. Then for any $f \in k\langle T\rangle$, we can evaluate f at x to get a map $R \to k$, $f = \sum a_n T^n \mapsto \sum a_n x^n$. Composing with the norm on k, one gets a valuation $f \mapsto |f(x)|$ on R, which is obviously continuous and ≤ 1 for all $f \in R^+$.

(2), (3) The rays of the tree: Let $0 \leq r \leq 1$ be some real number, and $x \in k^\circ$. Then

$$f = \sum a_n (T - x)^n \mapsto \sup |a_n| r^n = \sup_{y \in k^\circ : |y - x| \leq r} |f(y)|$$

defines another continuous valuation on R which is ≤ 1 for all $f \in R^+$. It depends only on the disk $D(x, r) = \{y \in k^\circ \mid |y - x| \leq r\}$. If $r = 0$, then it agrees with the classical point corresponding to x. For $r = 1$, the disk $D(x, 1)$ is independent of $x \in k^\circ$, and the corresponding valuation is called the Gaußpoint.

If $r \in |k^\times|$, then the point is said to be of type (2), otherwise of type (3). Note that a branching occurs at a point corresponding to the disk $D(x, r)$ if and only if $r \in |k^\times|$, i.e. a branching occurs precisely at the points of type (2).

(4) Dead ends of the tree: Let $D_1 \supset D_2 \supset \dots$ be a sequence of disks with $\bigcap D_i = \emptyset$. Such families exist if k is not spherically complete, e.g. if $k = \mathbb{C}_p$. Then

$$f \mapsto \inf_i \sup_{x \in D_i} |f(x)|$$

defines a valuation on R, which again is ≤ 1 for all $f \in R^+$.

(5) Finally, there are some valuations of rank 2 which are only seen in the adic space. Let us first give an example, before giving the general classification. Consider the totally ordered abelian group $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, where we require that $r < \gamma < 1$ for all real numbers $r < 1$. It is easily seen that there is a unique such ordering. Then

$$f = \sum a_n (T - x)^n \mapsto \max |a_n| \gamma^n$$

defines a rank-2-valuation on R. This is similar to cases (2), (3), but with the variable r infinitesimally close to 1. One may check that this point only depends on the disc $D(x, < 1) = \{y \in k^\circ \mid |y - x| < 1\}$.

Similarly, take any $x \in k^\circ$, some real number $0 < r < 1$ and choose a sign $? \in \{<, >\}$. Consider the totally ordered abelian group $\Gamma_{?r} = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, where $r' < \gamma < r$ for all real numbers $r' < r$ if $? =<$, and $r' > \gamma > r$ for all real numbers $r' > r$ if $? =>$. Then

$$f = \sum a_n (T - x)^n \mapsto \max |a_n| \gamma^n$$

defines a rank 2-valuation on R. If $? =<$, then it depends only on $D(x, < r) = \{y \in k^\circ \mid |y - x| < r\}$. If $? =>$, then it depends only on $D(x, r)$.

Remark 5.4. In fact, for any v a rank 1 valuation with generic support, the valuation ring R_v satisfies $R_v \cap k = k^\circ$. In particular, the residue field $\kappa(v)$ of R_v is natually an extension of the residue field κ of k, and the value group contains the value group of $(\kappa, |\cdot|)$ as an ordered group. In our case (k) is algebraic closed, $\kappa(v) = \kappa$ when v is a type (3) point, and $\kappa(v)$ is the transcendental extension of κ when v is a type (2) point.

Remark 5.5. All points except those of type (2) are closed. Let κ be the residue field of k. Then the closure of the Gaußpoint is exactly the Gaußpoint together with the points of type (5) around it, and is homeomorphic to $\mathbb{A}^1_\kappa$, with the Gaußpoint as the generic point. At the other points of type (2), one gets $\mathbb{P}^1_\kappa$.

Let (R, R^+) be an affinoid k-algebra, and let $X = \mathrm{Spa}(R, R^+)$. We need not assume that $\mathcal{O}_X$ is a sheaf in the following. For any $x \in X$, we let $k(x)$ be the residue field of $\mathcal{O}_{X,x}$, and $k(x)^+ \subset k(x)$ be the image of $\mathcal{O}^+_{X,x}$. We have the following crucial property:

Proposition 5.6. *Let $\varpi \in k$ be topologically nilpotent. Then the ϖ-adic completion of $\mathcal{O}^+_{X,x}$ is equal to the ϖ-adic completion $\widehat{k(x)}^+$ of $k(x)^+$.*

Proof. It is enough to note that kernel of the map $\mathcal{O}^+_{X,x} \to k(x)^+$, which is also the kernel of the map $\mathcal{O}_{X,x} \to k(x)$, is ϖ-divisible. $\square$

Definition 5.6. An affinoid field is pair (K, K^+) consisting of a nonarchimedean field K and an open valuation subring $K^+ \subset K^\circ$.

Remark 5.6. The completion of an affinoid field is again an affinoid field. Also note that the affinoid fields for which $k \subset K$ are affinoid k-algebra.

Proposition 5.7. *Let (R, R^+) be an affinoid k-algebra. The points of* $\mathrm{Spa}(R, R^+)$ *are in bijection with maps $(R, R^+) \to (K, K^+)$ to complete affinoid fields (K, K^+) such that the quotient field of the image of R in K is dense.*

Definition 5.7. For two points x, y in some topological space X, we say that x specializes to y (or y generalizes to x), written $x \succ y$ (or $y \prec x$), if y lies in the closure $\overline{\{x\}}$ of x.

Proposition 5.8. *Let (R, R^+) be an affinoid k-algebra, and let $x, y \in X =$* $\mathrm{Spa}(R, R^+)$ *correspond to maps $(R, R^+) \to (K, K^+)$, resp. $(R, R^+) \to (L, L^+)$. Then $x \succ y$ if and only if $K \cong L$ as topological R-algebras and $L^+ \subset K^+$.*