1 Local Properties of Smooth Manifolds

2 Vector Fields, Integral Curves and Flows

3 Sard's Theorem and Embedding Theorem

3.1 Sard's Theorem

We understand the local behavior of smooth maps at regular points by the Local Submersion Theorem (or by **Constant Rank Theorem**, up to diffeomorphism they look locally like the canonical submersion).

Theorem 1 (**Local Submersion Theorem**). *Suppose that $f: X \to Y$ is a submersion at x, and $y = f(x)$. Then there exist local coordinates around x and y such that*

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_m)$$

In other words, f is locally equivalent to the canonical submersion.

But what about the local behavior at critical points, regular values or regular points? We may have a brief observation here (We will discuss about **Morse function** later):

Example 1. Denote f as the height function of a torus, the homotopy type of the fiber can change at critical points.

What properties we appreciate are:

Definition 1 (**Generic Property**). A property P of maps is called *generic* if, for any f_0, there exists a homotopy F for f_0 and an $\epsilon > 0$ such that $f_t = F(-, t)$ has property P for all $t \in (0, \epsilon)$.

Remark. Transversality is generic.

In this section, we are ready to prove: The set of all critical values is of measure zero.

3.1.1 Sets of Measure Zero

The following examples from real analysis will help us understand the definition of *measure zero* in manifolds.

Example 2 (the measure of the measure zero set under a continuous (measurable) function). Let $C \in [0, 1]$ be the Cantor set and $f : [0, 1] \to [0, 1]$ be the Cantor-Lebesgue function.

1° Show that $F : [0, 1] \to [0, 2]$ defined by

$$F(x) := f(x) + x$$

is continuous, strictly increasing and measurable.

2° Show that $F(C) \subset [0, 2]$ is closed with $\mu(F(C)) = 1$.

3° Conclude that the Cantor-Lebesgue function is not Lipschitz continuous.

Example 3. Assume $f : [0,1] \to \mathbb{R}$ is Lipschitz continuous (i.e. there exists $\alpha > 0$ such that $|f(x) - f(y)| \leq \alpha |x - y|$ for any $x, y \in [0,1]$.)

1° Show that f maps measure zero sets to measure zero sets.

2° Show that smooth functions are Lipschitz continuous.

3° However, the Cantor-Lebesgue function is Hölder continuous,(I didn't check though.) i.e.

$$|f(x) - f(y)| \leq |x - y|^s . \ s = \frac{log2}{log3}$$

Definition 2 (**measure zero in** M)**.** We say that a subset $A \subseteq M$ has *measure* zero *in M* if for every smooth chart(U, φ) for M; the subset $A \subseteq M$ has n-dimensional measure zero.

The following lemma shows that we need only check this condition for a single collection of smooth charts whose domains cover A.

Lemma 3.1. *Let M be a smooth n-manifold with or without boundary and $A \subseteq M$. Suppose that for some collection $\{(U_\alpha, \varphi_\alpha)\}$ of smooth charts whose domains cover A, $\varphi_\alpha(A \cap U_\alpha)$ has measure zero in $\mathbb{R}^n$ for each α. Then A has measure zero in M.*

3.1.2 Sard's Theorem

Theorem 2 (**Sard′s Theorem**)**.** *$f : M \to N$. The set of all critical values is of measure zero.*

Remark. Why is the third step necessary? Consider $f(x) = e^{-\frac{1}{x^2}}$.

Remark. What happens if $\dim M < \dim N$?

Remark. On the other hand, the set of critical values could be a dense subset. For example, we can list all rational numbers as $\mathbb{Q} = \{r_1, r_2, \cdots\}$. Then we take a smooth function f_0 defined on $\mathbb{R}$ such that

$$\text{supp}\,(f_0) \subset (-1/3, 1/3) \quad \text{and} \quad f_0 \equiv 1 \text{ on } (-1/4, 1/4)$$

Let

$$f(x) = \sum_{k=1}^{\infty} r_k f_0(x - k)$$

Then each $k \in \mathbb{N}$ is a critical point of f, and thus the set of critical values of f contains $f(\mathbb{N}) = \mathbb{Q}$.

Remark. The alternative form of Sard's Theorem of maps between infinitely dimensional manifolds.

3.2 The Whitney Embedding Theorem

Historically, the word **manifold** first appeared in Riemann's doctoral thesis in 1851. At the early times, manifolds are defined extrinsically: "smooth manifolds" are objects that are (locally) defined by smooth equations and, according to last lecture, are regular submanifolds in Euclidean spaces. In 1912 Weyl gave an intrinsic definition for smooth manifolds. A natural question is: what is the difference between the extrinsic definition and the intrinsic definition? Is there any "abstract" manifold that cannot be embedded into any Euclidian space?

Two cases below share the same thoughts. Before our discussion, we recall the properties of *embedding* and **P.O.U. theorem**.

3.2.1 Embedding

Definition 3 (embedding)**. subspace topology**!

Example 4. The difference between immersed submanifold and embedding.

Proposition 3.1. *$F: M \to N$ is an embedding, then the image $F(M)$ is a regular submanifold of N.*

Proposition 3.2. *Suppose M and N are smooth manifolds with or without boundary, and $F: M \to N$ is an* **injective smoothness immersion***. If any of the following holds, then F is a smooth embedding:*

1° *open or closed map.*

2° *proper map.*

3° *M is compact.*

Lemma 3.2 (Proper Continuous Maps Are Closed)**.** *Suppose X is a topological space and Y is a locally compact Hausdorff space. Then every proper continuous map $F: X \to Y$ is closed.*

Proof. Let $K \subseteq X$ be a closed subset. To show that $F(K)$ is closed in Y, we show that it contains all of its limit points. Let y be a limit point of $F(K)$, and let U be a precompact neighborhood of y. Then y is also a limit point of $F(K) \cap \bar{U}$. Because F is proper, $F^{-1}(\bar{U})$ is compact, which implies that $K \cap F^{-1}(\bar{U})$ is compact. Because F is continuous, $F\left(K \cap F^{-1}(\bar{U})\right) = F(K) \cap \bar{U}$ is compact and therefore closed in Y. In particular, $y \in F(K) \cap \bar{U} \subseteq F(K)$, so $F(K)$ is closed. $\square$

3.2.2 Existence of P.O.U.

Theorem 3. *Let M be a smooth manifold, and $\{U_\alpha\}$ an open cover of M. Then there exists a P.O.U. subordinate to $\{U_\alpha\}$.*

In other words, we need to find a family $\{\rho_\alpha\}$ of smooth functions so that

1° $0 \leq \rho_\alpha \leq 1$ for all α.

2° $\operatorname{supp}(\rho_\alpha) \subset U_\alpha$ for all α.

3° each point $p \in M$ has a neighborhood which intersects $\operatorname{supp}(\rho_\alpha)$ for only finitely many α.

4° $\sum_\alpha \rho_\alpha(p) = 1$ for all $p \in M$.

3.2.3 The Whitney Embedding Theorem: Compact case

Theorem 4. *Any compact smooth manifold M admits an injective immersion into $\mathbb{R}^K$ for sufficiently large K.*

Theorem 5. *If a smooth manifold M of dimension m admits an injective immersion into $\mathbb{R}^K$ for some $K > 2m+1$, then it admits an injective immersion into $\mathbb{R}^{K-1}$.*

Remark. We don't assume compactness here, also the next proposition.

Proposition 3.3. *If a smooth manifold M of dimension m can be embedded into $\mathbb{R}^{2m+1}$, then it can be immersed into $\mathbb{R}^{2m}$.*

3.2.4 The Whitney Embedding Theorem: Non-Compact case

We have to modify two things:

1° How to construct a *injective immersion* into sufficient (large K) $\mathbb{R}^K$ without **finitely** many coordinate charts?

2° For non-compact manifolds, injective immersion need not to be an embedding. (Properness is a substitution of compactness)

Definition 4 (exhaustion function for manifold)**.** If M is a topological space, an exhaustion function for M is a continuous function $f : M \to \mathbb{R}$ with the property that the set $f^{-1}((-\infty, c])$ is compact for each $c \in \mathbb{R}$.

Proposition 3.4 (Existence of Smooth Exhaustion Functions)**.** *Every smooth manifold admits a smooth positive (proper) exhaustion function.*

Proof. Let M be a smooth manifold with or without boundary, let $\{V_j\}_{j=1}^{\infty}$ be any countable open cover of M by precompact open subsets, and let $\{\psi_j\}$ be a smooth partition of unity subordinate to this cover. Define $f \in C^\infty(M)$ by

$$f(p) = \sum_{j=1}^{\infty} j\psi_j(p)$$

Then f is smooth because only finitely many terms are nonzero in a neighborhood of any point, and positive because $f(p) \geq \sum_j \psi_j(p) = 1$.

To see that f is an exhaustion function, let $c \in \mathbb{R}$ be arbitrary, and choose a positive integer $N > c$. If $p \notin \bigcup_{j=1}^N \bar{V}_j$, then $\psi_j(p) = 0$ for $1 \leq j \leq N$, so

$$f(p) = \sum_{j=N+1}^{\infty} j\psi_j(p) \geq \sum_{j=N+1}^{\infty} N\psi_j(p) = N\sum_{j=1}^{\infty} \psi_j(p) = N > c$$

Equivalently, if $f(p) \leq c$, then $p \in \bigcup_{j=1}^N \bar{V}_j$. Thus $f^{-1}((-\infty, c])$ is a closed subset of the compact set $\bigcup_{j=1}^N \bar{V}_j$ and is therefore compact. □

Theorem 6 (The Whitney Embedding Theorem)**.** *Any smooth non-compact manifold M of dimension m can be embedded into $\mathbb{R}^{2m+1}$.*

Theorem 7 (The Whitney Immersion Theorem)**.** *Any smooth non-compact manifold M of dimension m can be immersed into $\mathbb{R}^{2m}$.*

3.2.5 *try* $S^1 \times S^1 = \mathbb{T}^2$ into $\mathbb{R}^3, \mathbb{RP}^n$, *Klein Bottle* into $\mathbb{R}^4$