Tempered representations with unipotent parahoric restriction: A noncommutative geometry viewpoint

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(RTCNG, Feb 27th Paris.)
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The category of smooth representations of a -adic group admits a decomposition into Bernstein blocks. With Paul Baum, Roger Plymen and Maarten Solleveld, we have formulated a conjecture which relates these blocks to (possibly twisted) extended quotients of complex algebraic varieties by finite groups.

In this talk, I will first introduce p-adic groups and some of their representations. Next, I will describe the conjecture in the case of tempered representations with unipotent parahoric restriction, focusing on its links with the generalized Springer correspondence for complex disconnected groups and its K-theoretical aspects. As an application, I will show how it

Setup: parabolic induction functor $I_{L,P}^{G} : \mathcal{C}(L) \to \mathcal{C}(G)$,
satisfies (1) subquotient (construction)
(2) $S_c^G : \pi \mapsto G\cdot(L,\sigma)$: cuspidal support (uniqueness)
supercuspidal / simple object

e.g. G/F with F finite, p-adic (smooth repn), complex (connected reductive)

$\rightsquigarrow \mathcal{C}(G) = \prod_{s\in\mathcal{B}} \underbrace{\mathcal{C}_s(G)}_{\text{fibre of } s \text{ under } S_c^G}$

= "G-equivariant preserved sheaves of a unipotent variety of G"

$\rightsquigarrow$ rough [ABPS]:
for non-archi local F, the structure of $\mathrm{Irr}^s(G)$ (subset) of $\mathcal{C}^s(G)$ is given by a (possibly twisted by a 2-cocycle) extended quotient $(T^s /\!/ W^s)$

$W^s := \{w \in N_G(L)/L : {}^w s = s\}$

complex torus

Note: If G is a quasi-split classical group [A.-Moussaoui-Solleveld] or the exceptional group G_2 [A.-Xu], no twisting is needed. However, for $G = \mathrm{SL}_n(D)$, with D/F a division algebra, there are cases which require a twisting [A.-Baum-Plymen-Solleveld].

Geometric extended quotient

Let X be a space and Γ a (finite) group acting on X. For $x \in X$, let $\Gamma_x \subset \Gamma$ be the fixator of x: $\Gamma_x := \{\gamma \in \Gamma : \gamma \cdot x = x\}$.
The quotient X/Γ of X by (the action of) Γ is the set Γ-orbits in X.
We replace X by a bigger space $\widetilde{X}$ on which Γ is still acting, and take the quotient of $\widetilde{X}$ by Γ:
the geometric extended quotient of X by Γ is the quotient

$$(X//\Gamma)_{\mathrm{geo}} := \{(x,\tau) : x \in X, \gamma \in \Gamma_x\}/\Gamma.$$

Spectral extended quotient

Instead of elements of Γ_x, we can consider irred. repres. of Γ_x.
The (spectral) extended quotient of X by Γ is the quotient

$$X//\Gamma := \{(x,\tau) : x \in X, \tau \in \mathrm{Irr}(\Gamma_x)\}/\Gamma.$$

The extended quotients $(X//\Gamma)_{\mathrm{geo}}$ and $X//\Gamma$ are in bijection but not in a canonical way in general.

Notation: A: k-alg. Prim: primitive ideal; <u>k-ideal</u>: ideal + $\lambda a \in I$ for all $\lambda \in k$, $a \in I$.

Morita equivalent $\underset{\text{weaker}}{\rightsquigarrow}$ stratified equiv.

<Def>: A, B f.t k-alg. a morphism is spectrum preserving if:
(1) $\mathrm{Prim}(B) \to \mathrm{Prim}(A)$ is bijection.
(2) ideal of $B \leftrightarrow$ ideal of A

$\Rightarrow$ f is spectrum preserving w.r.t filtration if there are k-ideal:

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_r = A$$
$$0 = J \quad - \quad - \quad = B, \text{ s.t.}$$

> stratification

$f(I_j) \subseteq J_j$ + $I_j/I_{j-1} \to J_j/J_{j-1}$ is spectrum preserving.
Prim ideal of I_j/I_{j-1} and J_j/J_{j-1} are called strata.

Algebraic variation of k-structure

Let A be a unital $\mathbb{C}$-algebra, and $\psi\colon k \to Z(A[t,t^{-1}])$ a unital morphism of $\mathbb{C}$-algebras. For $\zeta \in \mathbb{C}^\times$, consider the composition

$$k \xrightarrow{\psi} Z(A[t,t^{-1}]) \xrightarrow{\mathrm{ev}(\zeta)} Z(A).$$

and denote by A_ζ the unital k-algebra so obtained. We call such a family $\{A_\zeta\}_{\zeta\in\mathbb{C}^\times}$ an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$.

Definition

With k fixed, we consider the collection of all finite type k-algebras. On this collection, a stratified equivalence is the equivalence relation generated by the two elementary steps:

ES1 If there is a morphism of k-algebras $f\colon A \to B$ which is spectrum preserving with respect to filtrations, then $A \sim B$.

ES2 If there is $\{A_\eta\}_{\zeta\in\mathbb{C}^\times}$, an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$, such that each A_ζ is a unital finite type k-algebra, then for any $\zeta, \zeta' \in \mathbb{C}^\times$, $A_\zeta \sim A_{\zeta'}$.

(equivalent description)

Equivalent description

Two finite type k-algebras A, B are stratified equivalent if and only if there is a finite sequence $A_0, A_1, A_2, \ldots, A_r$ of finite type k-algebras with $A_0 = A$, $A_r = B$, and for each $j \in \{0, 1, \ldots, r-1\}$ one of the following three possibilities holds:

- a morphism of k-algebras $A_j \to A_{j+1}$ is given which is spectrum preserving with respect to filtrations.
- a morphism of k-algebras $A_{j+1} \to A_j$ is given which is spectrum preserving with respect to filtrations.
- $\{A_\eta\}_{\zeta\in\mathbb{C}^\times}$, an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$, is given such that each A_ζ is a unital finite type k-algebra, and ζ', ζ'' in $\mathbb{C}^\times$ have been chosen such that $A_j = A_{\zeta'}$ and $A_{j+1} = A_{\zeta''}$.

Note

To define a stratified equivalence relating A and B, the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spaces and an isomorphism of periodic cyclic homology are determined:

$$\mathrm{Prim}(A) \leftrightarrow \mathrm{Prim}(B) \quad \text{and} \quad \mathrm{HP}_*(A) \simeq \mathrm{HP}_*(B).$$

Proposition [A-Baum-Plymen-Solleveld]

If two unital finite type k-algebras A, B are Morita equivalent (as k-algebras) then they are stratified equivalent:

$$A \underset{\text{Morita}}{\sim} B \Longrightarrow A \sim B.$$

In contrast, there exist k-algebras that are stratified equivalent but not Morita equivalent, e.g., the affine Hecke algebra $\mathcal{H}_q(W^{\mathrm{aff}})$ (with $q \neq 1$) associated to an affine Weyl group W^{aff} and the group algebra $\mathbb{C}[W^{\mathrm{aff}}]$ are stratified equivalent for almost all q, but they are not Morita equivalent en general.

$\rightsquigarrow$ back to $\mathcal{H}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}$

Conjecture [A.-Baum-Plymen-Solleveld]

Assume that G is quasi-split (to simplify the exposition). Let $\mathfrak{s}$ be a point in the Bernstein spectrum of G. There is a canonical bijection

$$\nu^{\mathfrak{s}} \colon T^{\mathfrak{s}} // W^{\mathfrak{s}} \to \mathrm{Irr}^{\mathfrak{s}}(G),$$

which

1. comes from a canonical stratified equivalence of the two unital finite-type $\mathcal{O}(T^{\mathfrak{s}}/W^{\mathfrak{s}})$–algebras $\mathcal{O}(T^{\mathfrak{s}}) \rtimes W^{\mathfrak{s}}$ and $\mathcal{H}(G)^{\mathfrak{s}}$,
2. maps $T^{\mathfrak{s}}_{\mathrm{cpt}} // W^{\mathfrak{s}}$ onto $\mathrm{Irr}^{\mathfrak{s}}(G) \cap \mathrm{Irr}^{\mathrm{temp}}(G)$, where $T^{\mathfrak{s}}_{\mathrm{cpt}}$ is the maximal compact subgroup of $T^{\mathfrak{s}}$,

and, there is an algebraic family $\vartheta_z \colon T^{\mathfrak{s}}_{\mathrm{cpt}} // W^{\mathfrak{s}} \to T^{\mathfrak{s}}/W^{\mathfrak{s}}$ of finite morphisms of algebraic varieties, with $z \in \mathbb{C}^\times$, such that ϑ_1 is the natural projection and $\vartheta_{\sqrt{q}} = \mathrm{Sc}^{\mathfrak{s}} \circ \nu^{\mathfrak{s}}$, where q is the order of the residue field of F.
Moreover, $\nu^{\mathfrak{s}}$ is compatible the local Langlands correspondence.

note (1) proved for $G = GL_n(F)$, $G = G_2$; $C_r^*(G) = \bigsqcup_{\mathfrak{s}} C_r^*(G)^{\mathfrak{s}}$.
(2) Galois version

K-theoretical version of the ABPS conjecture

Let $\mathfrak{s} \in \mathfrak{B}(G)$. There exists a canonical isomorphism

$$K^*_{W_{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{cpt}}) \longrightarrow K_*(C_r^*(G)^{\mathfrak{s}}).$$

Decomposition of $C_r^*(G)$ and the theta correspondence [A.]

Let $(G_n, G'_{n'})$ be an irreducible reductive dual pair, formed of a p-adic symplectic group $\mathrm{Sp}_{2n}(F)$ and a p-adic orthogonal group $\mathrm{O}_{2n'}(F)$, such that $n' = n$ or $n' = n+1$.
Then the theta correspondence for the reductive dual pair $(G_n, G'_{n'})$ (which preserves temperedness) induces a correspondence between subsets of simple modules of $C_r^*(G)^{\mathfrak{s}}$ and $C_r^*(G)^{\theta(\mathfrak{s})}$.
The map $\mathfrak{s} \mapsto \theta(\mathfrak{s})$ can be explicitly described.

(preserve Bernstein (by Kudla). temperness)

Notation

- $\beta(G) :=$ Bruhat-Tits building of G, and $\beta_{\mathrm{red}}(G) :=$ reduced BT building. $\beta(G) = \beta_{\mathrm{red}}(G) \times (X_*(Z_G) \otimes_{\mathbb{Z}} \mathbb{R})$, where $X_*(Z_G)$ is the set of F-algebraic cocharacters of the center Z_G of G.
- For $x \in \beta(G)$, let $G_{x,0}$ be the associate parahoric subgroup, and $G_{x,0+}$ the pro-p unipotent radical of $G_{x,0}$.
- Let $\mathbb{G}_{x,0} := G_{x,0}/G_{x,0+}$ a connected reductive group over $\mathbb{F}_q$.

Definition

An irreducible smooth representation (π, V) of G is said to have unipotent parahoric restriction (or, for short, to be unipotent) if there is $x \in \beta(G)$ such the $G_{x,0+}$-invariants in V contain an irreducible cuspidal unipotent representation π_x in the sense of Deligne-Lusztig theory, i.e., such that

$$\langle \pi_x, R_{\mathbb{T}}^{\mathbb{G}_{x,0}}(1) \rangle \neq 0,$$

for a maximal torus $\mathbb{T}$ of $\mathbb{G}_{x,0}$, with $R_{\mathbb{T}}^{\mathbb{G}_{x,0}}(1)$ a Deligne-Lusztig character.

rep's $\mapsto$ rep's
block $\mapsto$ block

Example

Every Iwahori-spherical representation of G is unipotent. Iwahori-spherical representations belongs to $\mathcal{C}^{\mathfrak{s}}(G)$, where $\mathfrak{s} = [T, 1]_G$, with T a maximal torus of G.

Supercuspidal unipotent representations

These are the representations π of G such that there exist a vertex $x \in \beta_{\mathrm{red}}(G)$ and an irreducible unipotent cuspidal representation π_x of $\mathbb{G}_{x,0}$, such π is compactly induced from $\widetilde{\pi}_x$, an extension to $\mathrm{N}_G(G_{x,0})$ of the inflation of π_x to $G_{x,0}$:

$$\pi = \mathrm{c}\text{-}\mathrm{Ind}_{\mathrm{N}_G(G_{x,0})}^{G}(\pi_x),$$

where $\mathrm{N}_G(G_{x,0})$ is the normalizer of $G_{x,0}$ in G (a totally disconnected group that is compact mod center). It coincides with the fixator under the action of G on $\beta_{\mathrm{red}}(G)$ of the image of x in $\beta_{\mathrm{red}}(G)$.

From parahorics to parabolics

1. For a facet $\mathcal{F}$ of $\beta(G)$, let $\Phi_{\mathcal{F}}^{\mathrm{aff}}$ be the set of affine roots vanishing on $\mathcal{F}$, and $\Phi_{\mathcal{F}}$ the associate set of roots.
2. $\Phi_{\mathcal{F}}$ is a not necessarily closed root system, and we denote its closure by ${}^{\mathrm{c}}\Phi_{\mathcal{F}}$.
3. There is a connected reductive F-subgroup $\mathbf{H}$ of $\mathbf{G}$ containing $\mathbf{T}$, which has the relative root system ${}^{\mathrm{c}}\Phi_{\mathcal{F}}$ with respect to $\mathbf{T}$.
4. Let $\mathbf{L}$ be the centralizer of the F-split component of the center of $\mathbf{H}$. Then $\mathbf{L}$ is the Levi factor of a parabolic F-subgroup of $\mathbf{G}$.
5. Moreover, $G_{x,0} \cap L = L_{y,0}$ for some vertex $y \in \beta(L)$, i.e., it is a maximal parahoric subgroup of L, and $\mathbb{L}_{y,0} = \mathbb{G}_{x,0}$.

The category $\mathcal{C}^{\mathrm{u}}(G)$ of unipotent representations of G

We have

$$\mathcal{C}^{\mathrm{u}}(G) := \prod_{\substack{\mathfrak{s}=[L,\sigma] \\ \sigma \text{ unipotent}}} \mathcal{C}^{\mathfrak{s}}(G).$$

Parametrization of $\mathrm{Irr}^{\mathrm{u}}(G)$ [Lusztig, Feng-Opdam-Solleveld]

$$\mathrm{Irr}^{\mathrm{u}}(G) \overset{1-1}{\longleftrightarrow} (s, u, \rho)_{G^\vee}$$

where $G^\vee$ is the complex reductive group dual to G, $s, u \in G^\vee$, with s semisimple, u unipotent, such that $sus^{-1} = u^q$, and $\rho \in \mathrm{Irr}(A(s,u))$, with $A(s,u) := \mathrm{Z}_{G^\vee}(s,u)/\mathrm{Z}_{G^\vee} \cdot \mathrm{Z}_{G^\vee}(s,u)^\circ$. We decompose s into its compact and hyperbolic parts: $s = s_{\mathrm{c}} s_{\mathrm{h}}$.The tempered unipotent representations are those such that $s_{\mathrm{h}} = 1$, and the tempiric ones are those such that $s = 1$.

How is it related to the Bernstein decomposition of $\mathrm{Irr}^{\mathrm{u}}(G)$?

Given a pair (s, u), we may choose a homomorphism $\varphi_u \colon \mathrm{SL}_2(\mathbb{C}) \to G^\vee$ such that

$$\varphi_u \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = u \quad \text{and such that } s_u := \varphi_u \left(\begin{smallmatrix} q^{1/2} & 0 \\ q^{-1/2} & 0 \end{smallmatrix}\right) \text{ commutes with } s.$$

Then $t := s_u^{-1} s \in \mathrm{Z}_{G^\vee}(u)$. We set $\mathcal{G}_t := \mathrm{Z}_{G^\vee}(t)$ and decompose the category of $\mathcal{G}_t$-equivariant perverse sheaves on the unipotent variety of $\mathcal{G}_t$ into blocks via the generalized Springer correspondence [Lusztig, A.-Moussaoui-Solleveld]. We have a notion of cuspidality, and the corresponding blocks match.