1 Complex Geometry

I can only solve $2, 4, 6$ which is far from full marks (and I learnt 4 in Griffiths Harris for the first time). Particularly, I really want to calculate the Chern class for the first time (No.5), though I have no idea about how to give a explicit representation of Chern connection and the related curvature form. I apologize.

2 Partial Differential Equation

I have no passion in PDEs, as a purely algebra/geometry learner.

3 Riemannian Geometry

1.(I copied them from Peter Petersen, since they're standard.)

Proof. (1)We use the cyclic sum notation and in addition that

$$R(X,Y)Z = [\nabla X, \nabla Y]Z - \nabla[X,Y]Z$$

Denote the cyclic sum for operator T:

$$\mathfrak{S}T(X,Y,Z) = T(X,Y,Z) + T(Z,X,Y) + T(Y,Z,X)$$

Note that,

$$\begin{aligned}\left(\nabla_Z R\right)(X,Y)W =& \nabla_Z(R(X,Y)W) - R\left(\nabla_Z X, Y\right)W \\ & - R\left(X, \nabla_Z Y\right)W - R(X,Y)\nabla_Z W \\ =& \left[\nabla_Z, R(X,Y)\right]W - R\left(\nabla_Z X, Y\right)W - R\left(X, \nabla_Z Y\right)W\end{aligned}$$

Keeping in mind that we only do cyclic sums over X, Y, Z and that we have the Jacobi identity for operators:

$$\mathfrak{S}\left[\nabla_X, \left[\nabla_Y, \nabla_Z\right]\right] = 0$$

We obtain

$$
\begin{aligned}
&\mathfrak{S}\left(\nabla_X R\right)(Y,Z)W \\
=&\mathfrak{S}\left[\nabla_X, R(Y,Z)\right]W - \mathfrak{S}R\left(\nabla_X Y, Z\right)W - \mathfrak{S}R\left(Y, \nabla_X Z\right)W \\
=&\mathfrak{S}\left[\nabla_X, \left[\nabla_Y, \nabla_Z\right]\right]W - \mathfrak{S}\left[\nabla_X, \nabla_{[Y,Z]}\right]W \\
&- \mathfrak{S}R\left(\nabla_X Y, Z\right)W - \mathfrak{S}R\left(Y, \nabla_X Z\right)W \\
=&-\mathfrak{S}\left[\nabla_X, \nabla_{[Y,Z]}\right]W - \mathfrak{S}R\left(\nabla_X Y, Z\right)W + \mathfrak{S}R\left(\nabla_Y X, Z\right)W \\
=&-\mathfrak{S}\left[\nabla_X, \nabla_{[Y,Z]}\right]W - \mathfrak{S}R([X,Y],Z)W \quad \text{(Levi Civita)} \\
=&-\mathfrak{S}\left[\nabla_X, \nabla_{[Y,Z]}\right]W - \mathfrak{S}\left[\nabla_{[X,Y]}, \nabla_Z\right]W + \mathfrak{S}\nabla_{[[X,Y],Z]}W \\
=&\mathfrak{S}\left[\nabla_{[X,Y]}, \nabla_Z\right]W - \mathfrak{S}\left[\nabla_{[X,Y]}, \nabla_Z\right]W \\
=&0
\end{aligned}
$$

(2) Scalar curvature is the trace of the ricci curvature. Choose a normal orthonormal frame E_i at $p \in M$, i.e., $\nabla E_i|_p = 0$, and let W be a vector field such that $\nabla W|_p = 0$. Using the second Bianchi identity:

$$
\begin{aligned}
&(d\,\mathrm{tr}(\mathrm{Ric}))(W)(p) \\
&= D_W \sum g\left(\mathrm{Ric}\left(E_i\right), E_i\right) \\
&= D_W \sum g\left(R\left(E_i, E_j\right)E_j, E_i\right) \\
&= \sum g\left(\nabla_W\left(R\left(E_i, E_j\right)E_j\right), E_i\right) \\
&= \sum g\left(\left(\nabla_W R\right)\left(E_i, E_j\right)E_j, E_i\right) \\
&= -\sum g\left(\left(\nabla_{E_j} R\right)\left(W, E_i\right)E_j, E_i\right) \\
&- \sum g\left(\left(\nabla_{E_i} R\right)\left(E_j, W\right)E_j, E_i\right) \\
&= -\sum\left(\nabla_{E_j} R\right)\left(W, E_i, E_j, E_i\right) - \sum\left(\nabla_{E_i} R\right)\left(E_j, W, E_j, E_i\right) \\
&= \sum\left(\nabla_{E_j} R\right)\left(E_j, E_i, E_i, W\right) + \sum\left(\nabla_{E_i} R\right)\left(E_i, E_j, E_j, W\right) \\
&= 2\sum\left(\nabla_{E_j} R\right)\left(E_j, E_i, E_i, W\right) \\
&= 2\sum \nabla_{E_j}\left(R\left(E_j, E_i, E_i, W\right)\right) \\
&= 2\sum \nabla_{E_j} g\left(\mathrm{Ric}\left(E_j\right), W\right) \\
&= 2\sum \nabla_{E_j} g\left(\mathrm{Ric}(W), E_j\right) \\
&= 2\sum g\left(\nabla_{E_j}(\mathrm{Ric}(W)), E_j\right) \\
&= 2\sum g\left(\left(\nabla_{E_j}\mathrm{Ric}\right)(W), E_j\right) \\
&= 2\,\mathrm{div}(\mathrm{Ric})(W)(p).
\end{aligned}
$$

$\square$

2.[Geodesic ray]:

Proof. We assume $d(x_i, p) \geq i$ and since Hopf-Rinow theorem, we assume that each $\gamma_i(t) = \exp_p(t\gamma_i'(0))$ (which connects p to x_i) is unit speed. Since the unit sphere in T_pM is compact, the sequence $\{\gamma_i'(0)\}$ must have some convergent subsequence, say, with limit v. Use the notation $\gamma_i'(0)$ to refer to this subsequence. Set $\gamma(t) = \exp_p(tv)$.

Because d and $\exp_p$ are continuous,

$$\lim_{i\to\infty} d(\gamma(t), \gamma_i(t)) = d\left(\gamma(t), \exp_p\left(\lim_{i\to\infty} t\gamma_i'(0)\right)\right) = d(\gamma(t), \gamma(t)) = 0.$$

Now, assume for a contradiction that there is some time t for which $\gamma(t)$ is not minimizing. That is, assume there is a $t > 0$ for which $d(p, \gamma(t)) < t$.

For any $i > t$, we know $d(\gamma_i(t), p) = t$. From the lemma, we know that there is an I with the property that for all $i > I, d(\gamma_i(t), \gamma(t)) < t - d(\gamma(t), p)$. Then, for any $i \geq \max\{t, I\}$, the triangle inequality gives

$$t = d(\gamma_i(t), p) \leq d(\gamma_i(t), \gamma(t)) + d(\gamma(t), p) < t - d(\gamma(t), p) + d(\gamma(t), p) = t$$

which gives a contradiction. Thus the limit of the sequence is the desired direction of the geodesic ray. $\square$

3.[Generalized version]:

Proof. Let p and q be any two points in M. Since M is complete, there exists a minimizing geodesic $\gamma : [0,1] \to M$ joining p to q. Let us consider parallel fields $e_1(t), \ldots, e_{n-1}(t)$ along γ which are orthonormal, for each $t \in [0,1]$ (I forgot to take unit speed here (I apologize)), belong to the orthogonal complement of $\gamma'(t)$. Specifically, after choosing the orthogonal basis in T_pM, the parallel translation, $\nabla_{\dot\gamma(t)} e_i(t) = 0$, $\exists!$ ODE solution. Let $e_n(t) = \frac{\gamma'(t)}{l}$ and let Y_j be a vector field along γ given by

$$Y_j(t) = (\sin \pi t) e_j(t), \quad j = 1, \ldots, n-1$$

It is clear that $Y_j(0) = Y_j(1) = 0$, therefore Y_j generates a proper variation of γ, whose energy we denote by E_j. Using the formula above, we obtain

$$\begin{aligned}
\frac{1}{2} E_j''(0) &= -\int_0^1 \left\langle Y_j, Y_j'' + R(\gamma', Y_j)\gamma' \right\rangle \mathrm{d}t \\
&= \int_0^1 \sin^2 \pi t \left(\pi^2 - l^2 K(e_n(t), e_j(t))\right) \mathrm{d}t
\end{aligned}$$

where $K(e_n(t), e_j(t))$ is the sectional curvature. Summing on j and using the definition of the Ricci curvature and the given estimate, we get

$$
\begin{aligned}
\frac{1}{2}\sum_{j=1}^{n-1} E_j''(0) &= \int_0^1 \left\{\sin^2 \pi t \left((n-1)\pi^2 - (n-1)l^2 \operatorname{Ric}_{\gamma(t)}\left(e_n(t)\right)\right)\right\} \mathrm{d}t \\
&\leq (n-1)\left[\frac{\pi^2 - l^2 a}{2} - l\int_0^1 \sin^2(\pi s)\frac{\mathrm{d}f}{\mathrm{d}s}\mathrm{d}s\right] \\
&= (n-1)\left[\frac{\pi^2 - l^2 a}{2} + \pi l\int_0^1 \sin(2\pi s) f(s)\mathrm{d}s\right] \\
&\leq (n-1)\left[\frac{\pi^2 - l^2 a}{2} + c\pi l\int_0^1 |\sin(2\pi s)|\mathrm{d}s\right] \\
&= (n-1)\left[\frac{\pi^2 - l^2 a}{2} + 2cl\right].
\end{aligned}
$$

We apply integration by parts here (the ommitance of ^{2} appears in re-scalar). Then solve the one-variable quadratic inequality, we get the final estimate:

$$
\operatorname{Diam}(M, g) \leq \frac{\pi^2}{\sqrt{c^2 + \pi^2 a} - c}
$$

Since M is complete and bounded, then compact. □

4.[Homogenerous space]:

Proof. Obviously we can use the Hopf-Rinow theorem. Homogeneity implies that all metric balls of the same radius are isometric. Then we can easily do the geodesic extension. Another way (I will explain explicitly) here is to use the definition:

Let $\{x_n\} \subset M$ be a Cauchy sequence and fix $p \in M$. For each $m \in \mathbb{N}$ there is a $g_m \in \operatorname{Iso}(M)$ s.t. $g_m(x_m) = p$. Let $B_\epsilon(p)$ a normal ball around p. There is a $N \in \mathbb{N}$ s.t. for every $n \geq N$,

$$
\epsilon > d(x_N, x_n) = d(g_N p, g_n p) = d\left(p, g_N^{-1} g_n p\right)
$$

So $d(p, x_n) = d(p, g_n p) \in g_N(B_\epsilon(p))$. Then $\{x_n\}$ admits a convergente subsequence. As x_n is a cauchy sequence, x_n converges to the limit of the subsequence. □

5.[conjugate point]

Proof. Let $\gamma : [0, \epsilon] \to M$ be a geodesic curve and (assume) J (to) be a Jacobi vector field along γ with $J(0) = J(\epsilon) = 0$. Now look at the map $\phi : t \mapsto \|J(t)\|^2$. Its second derivative is easy to compute and using the definitions:

$$\begin{aligned}\phi''(t) &= \left\langle \frac{D^2}{dt^2} J, J \right\rangle + \left\langle \frac{D}{dt} J, \frac{D}{dt} J \right\rangle \\ &= - \langle R(\gamma', J)\gamma', J\rangle + \left\| \frac{D}{dt} J \right\|^2 \\ &= - \left[\|X\|^2 \|Y\|^2 - \langle X, Y\rangle^2 \right] \kappa(\gamma', J) + \left\| \frac{D}{dt} J \right\|^2 .\end{aligned}$$

In particular, if all sectional curvatures are non-positive then $\phi'' \geq 0$. So ϕ is a convex map and since $\phi(0) = \phi(\epsilon) = 0, \phi(t) = 0$ for any $t \in [0, \epsilon]$. It follows that J has to be trivial. So there is no conjugate point of p on M. $\square$