Topological K-theory of reductive p-adic groups

Maarten Solleveld (Nijmegen)

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The goal of this talk is a simple, concrete description of the topological K-theory of the reduced C*-algebra of a reductive p-adic group , in terms coming from representation theory.

We will discuss the structure of several group algebras of , and how the K-theory can be computed from the space of irreducible tempered

G: p-adic group (locally compact top, norm, totally disconnected) + reductive + connected

▷ first example: $T = GL_1(F) \supseteq \mathcal{O}_F^\times$. $\mathcal{H}(T)^{\mathcal{O}_F^\times} = \mathbb{C}[F^\times/\mathcal{O}_F^\times] = \mathbb{C}[z] = \mathbb{C}[z, z^{-1}] = \mathcal{O}(\mathbb{C}^\times)$

$C_r^*(T)^{\mathcal{O}_F^\times} \cong C(\mathbb{S}^1)$. $\mathcal{S}(T)^{\mathcal{O}_F^\times} \cong \mathcal{S}(\mathbb{Z}) \cong C^\infty(\mathbb{S}^1)$

reduced C^* algebra $\nearrow$ $(C_r^*(\mathbb{Z}))$ $\quad$ $\uparrow$ Schwartz algebra

<Def>: $\mathcal{H}(G)$, $C_r^*(G)$ = completion of $\mathcal{H}(G)$ in $B(L^2(G))$

$\mathcal{H}(G) = \bigcup_K \mathcal{H}(G, K) = C_c(K \backslash G / K)$ $\quad$ $C_r^*(G) = \varinjlim_K C_r^*(G, K)$

also Schwartz version (inverse of p-ly) $\mathcal{S}(G)$. $\mathcal{S}(G, K)$

<Def>: smooth (fixed open) G-rep'n (π, V) | non-degenerate $\mathcal{H}(G)$-mod V: $\mathcal{H}(G)V = V$

<Def>: $\mathrm{Irr}_{temp}(G) := \mathrm{Irr}(\underline{\mathrm{Mod}-\mathcal{S}(G)})$ cat of tempered rep'n

Components of tempered dual (1)

$\mathrm{Mod}-C_r^*(G)$: consists of Hilbert space completions of unitary tempered G-reps

<Def> (discrete series): matrix coefficients square integrable $G/Z(G)$.

<Eg>: $St = \mathrm{ind}_B^G(1) / \sum_{B \subsetneq P \subseteq G} \mathrm{ind}_P^G(1)$.

<Def> (unramified character): $\underline{X_{nr}(G)}$, $\underline{X_{unr}(G)}$, ↙ {trivial on every cpt subgp / complex torus}

(unitary char / max'l cpt real subtorus)

$\Downarrow$ "discrete" conn. component: tensor with X_{nr}

Components of tempered dual (2)

$P = MU$. I_P^G: normalized parabolic induction (temper + unitary) $\rightsquigarrow$ $\mathrm{Irr}_{temp}(G)_{[M,\delta]}$ $\{$summands of $I_P^G(\chi \otimes \delta)$, $\chi \in X_{unr}(M)\}$

δ discrete

$\rightsquigarrow$ $\mathrm{Irr}_{temp}(G) = \bigsqcup_{[M,\delta] \in \Delta(G)} \mathrm{Irr}_{temp}(G)_{[M,\delta]}$

HC

$\Delta(G)$: $\{[M,\delta]\} / X_{unr}(M)$ and G-conjugation

(Structure of $\mathcal{S}(G)$)

(1). <Def> (of f. grp that indexes intertwining operators)

For $[M,\delta] \in \Delta(G)$, $W_{[M,\delta]}$ is generated by

① $\chi \in X_{unr}(M)$, with $\chi \otimes \delta \cong \delta$ $\quad$ this?

② $w \in N_G(M)/M$ with $w.\delta \in \delta \otimes X_{unr}(M)$

• (Plancherel isomorphism) (HC, Waldspurger)

Plancherel isomorphism (Harish-Chandra, Waldspurger)

$$\mathcal{S}(G) \cong \bigoplus_{[M,\delta]\in\Delta(G)} \left(C^\infty(X_{\mathrm{unr}}(M)) \otimes \mathrm{End}^{\mathrm{fr}}_{\mathbb{C}}(I_P^G(V_\delta))\right)^{W_{[M,\delta]}}$$

$$\mathcal{S}(G,K) \cong \bigoplus_{[M,\delta]\in\Delta(G)} \left(C^\infty(X_{\mathrm{unr}}(M)) \otimes \mathrm{End}_{\mathbb{C}}(I_P^G(V_\delta)^K)\right)^{W_{[M,\delta]}}$$

for any compact open $K \subset G$ only finitely many nonzero summands

$W_{[M,\delta]}$ acts on $C^\infty(X_{\mathrm{unr}}(M)) \otimes \mathrm{End}^{\mathrm{fr}}_{\mathbb{C}}(I_P^G(V_\delta))$ (fr = finite rank):

$$w \cdot (f \otimes A)(\chi) = f(w^{-1}\chi) \otimes I(w,P,\delta,\chi)\, A\, I(w,P,\delta,\chi)^{-1}$$

for suitable $I(w,P,\delta,\chi) \in \mathrm{Hom}_G(I_P^G(\delta\otimes\chi), I_P^G(\delta\otimes w(\chi)))$

Eg: $G = SL_2(\mathbb{Q}_p)$ (with $p>2$)

$M = T$ or G, $X_{unr}(G) = 1$, $X_{unr}(T) \cong S^1$, $W_{[M,\delta]} = S_2$ or 1

$\mathcal{S}(G)$ is Morita equivalent with

$$(C^\infty(S^1) \otimes M_2(\mathbb{C}))^{S_2} \oplus C^\infty(S^1) \rtimes S_2 \oplus \bigoplus_{j=1}^{\infty} C^\infty(S^1) \oplus \mathbb{C}_{St} \oplus \bigoplus_{i=1}^{\infty} \mathbb{C}$$

· structure of $C_r^*(G)$

$\mathfrak{K}(H)$: compact operators on a Hilbert space H

Theorem (Plymen)

$$C_r^*(G) \cong \bigoplus_{[M,\delta]\in\Delta(G)} \left(C(X_{\mathrm{unr}}(M)) \otimes \mathfrak{K}(I_P^G(V_\delta))\right)^{W_{[M,\delta]}}$$

$$C_r^*(G,K) \cong \bigoplus_{[M,\delta]\in\Delta(G)} \left(C(X_{\mathrm{unr}}(M)) \otimes \mathrm{End}_{\mathbb{C}}(I_P^G(V_\delta)^K)\right)^{W_{[M,\delta]}}$$

for any compact open subgroup $K \subset G$: only finitely many nonzero summands

component of sm. dual (Bernstein): $I_P^G(X \otimes \sigma)$: $X \in X_{nr}(M)$, σ supercuspidal

subquotient of

· structure of $\mathcal{H}(G)$ (1)

$\Omega(G) := \{[M,\sigma] : M \subset G$ Levi subgroup, σ supercuspidal rep of $M\}$, modulo $X_{nr}(M)$ and G-conjugation

Can always take σ unitary, so $\Omega(G) \subset \Delta(G)$

Bernstein decomposition

$$\mathrm{Irr}(G) = \bigsqcup_{[M,\sigma]\in\Omega(G)} \mathrm{Irr}(G)_{[M,\sigma]}$$
$$\mathcal{H}(G) = \bigoplus_{[M,\sigma]\in\Omega(G)} \mathcal{H}(G)_{[M,\sigma]}$$

Theorem (Heiermann)

$$\mathcal{H}(G)_{[M,\sigma]} \cong \mathcal{O}(X_{\mathrm{nr}}(M)) \otimes \mathrm{End}^{\mathrm{fr}}_{\mathbb{C}}(I_P^G(V_\sigma)) \cap \left(\mathbb{C}(X_{\mathrm{nr}}(M)) \otimes \mathrm{End}^{\mathrm{fr}}_{\mathbb{C}}(I_P^G(V_\sigma))\right)^{W_{[M,\sigma]}}$$

$\mathbb{C}(X)$ = quotient field of ring of regular functions $\mathcal{O}(X)$

Warning: the intertwining operators from $W_{[M,\sigma]}$ may have poles on $I_P^G(\sigma\otimes\chi)$ when $\sigma\otimes\chi \in \mathrm{Irr}(M)$ not unitary

eg (SL_2)

$\mathcal{H}(G)$ is Morita equivalent with

$$\mathrm{Hafl}(G) \oplus \mathcal{O}(\mathbb{C}^\times) \rtimes S_2 \oplus \bigoplus_{j=1}^{\infty} \mathcal{O}(\mathbb{C}^\times) \oplus \bigoplus_{i=1}^{\infty} \mathbb{C}$$

$\mathrm{Irr}(G) \rightsquigarrow \mathrm{Irr}_{\mathrm{temp}}(G)$

"compactification"

For any reductive p-adic group G

Group algebras

$\mathcal{H}(G)$ | $\mathcal{S}(G)$ | $C_r^*(G)$

Locally constant functions on group

compactly supported | rapidly decreasing | "C^*-norm bounded"

Representations

smooth | tempered | unitary tempered

Fourier transform

algebraic sections | differentiable sections | continuous sections

- Structure of $\mathcal{S}(G)$ and $C_r^*(G)$ easier than for $\mathcal{H}(G)$
- Advantage of $\mathcal{S}(G)$ over $C_r^*(G)$: can do homological algebra
- Next goal: determine topological K-theory of $C_r^*(G)$

Computation of K_*(summand of $C_r^*(G)$) in examples

Example from $SL_2(F)$

$A = (C(S^1) \otimes M_2(\mathbb{C}))^{S_2}$ $\qquad \mathrm{Irr}(A) =$

$S_2 = \{1, w\}$, $I(w,z) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right)$ for $z \in S^1$

$(w \cdot f)(z) = I(w,z) f(z^{-1}) I(w,z)^{-1}$

Evaluation at -1 gives $\quad 0 \to C_0((-1,1]) \otimes M_2(\mathbb{C}) \to A \to \mathbb{C}^2 \to 0$

$$\begin{array}{ccccc} K^0((-1,1]) = 0 & \to & K_0(A) & \to & K_0(\mathbb{C}^2) = \mathbb{Z}^2 \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{C}^2) = 0 & \leftarrow & K_1(A) & \leftarrow & K^1((-1,1]) = 0 \end{array}$$

$K_*(A) \cong K^*([-1,1] \sqcup \mathrm{pt}) \cong K_*(\mathbb{C}^2)$

$\rightsquigarrow$ generally:

Structure of $\mathrm{Irr}(G)$

$\mathcal{H}(G) = \bigoplus_{[M,\sigma] \in \Omega(G)} \mathcal{H}(G)_{[M,\sigma]}$

$\mathcal{H}(G)_{[M,\sigma]} = \text{"}(\mathcal{O}(X_{\mathrm{nr}}(M)) \otimes \mathrm{End}_{\mathbb{C}}^{\mathrm{fr}}(I_P^G(V_\sigma)))^{W_{[M,\sigma]}}\text{"}$

Twisted group algebra $\mathbb{C}[W, \natural]$ of finite group W

For a 2-cocycle $\natural : W^2 \to \mathbb{C}^\times$, multiplication $T_w \cdot T_{w'} = \natural(w,w') T_{ww'}$

$A \rtimes \mathbb{C}[W, \natural] = \text{"}A \rtimes W$ with $\mathbb{C}[W, \natural]$ instead of $\mathbb{C}[W]\text{"}$

Theorem (ABPS conjecture)

For every reductive p-adic group G and $[M,\sigma] \in \Omega(G)$, there is a bijection

$$\mathrm{Irr}(G)_{[M,\sigma]} = \mathrm{Irr}(\mathcal{H}(G)_{[M,\sigma]}) \longrightarrow \mathrm{Irr}(\mathcal{O}(X_{\mathrm{nr}}(M)) \rtimes \mathbb{C}[W_{[M,\sigma]}, \natural])$$

which is canonical once a suitable 2-cocycle $\natural$ of $W_{[M,\sigma]}$ has been chosen.

$\mathrm{Irr}(G)_{[M,\sigma]}$ can be expressed with only $X_{\mathrm{nr}}(M)$, $W_{[M,\sigma]}$ and $\natural$

Irreducible representations of twisted crossed products

Twisted group algebra $\mathbb{C}[W, \natural]$ of finite group W

For a 2-cocycle $\natural : W^2 \to \mathbb{C}^\times$, multiplication $T_w \cdot T_{w'} = \natural(w,w') T_{ww'}$

$A \rtimes \mathbb{C}[W, \natural] = A \rtimes W$ with $\mathbb{C}[W, \natural]$ instead of $\mathbb{C}[W]$

Take $A = \mathcal{C}(X)$ for a W-space X, where $\mathcal{C}$ means $\mathcal{O}$ or C^∞ or C

$\rightsquigarrow$ Study

Construction and parametrization of $\mathrm{Irr}(\mathcal{C}(X) \rtimes \mathbb{C}[W, \natural])$

- For $x \in X$: stabilizer group W_x
- For $\rho \in \mathrm{Irr}(\mathbb{C}[W_x, \natural])$: $x \otimes \rho \in \mathrm{Irr}(\mathcal{C}(X) \rtimes \mathbb{C}[W_x, \natural])$
- $\pi(x, \rho) := \mathrm{ind}_{\mathcal{C}(X) \rtimes \mathbb{C}[W_x, \natural]}^{\mathcal{C}(X) \rtimes \mathbb{C}[W, \natural]}(x \otimes \rho) \in \mathrm{Irr}(\mathcal{C}(X) \rtimes \mathbb{C}[W, \natural])$
- $\pi(x, \rho) \cong \pi(x', \rho')$ if and only if $(x', \rho') = (w(x), \rho \circ \mathrm{Ad}(T_w)^{-1})$ for some $w \in W$

$\Downarrow$ tempered repn

Structure of $\mathrm{Irr}_{\mathrm{temp}}(G)$ (1)

$\mathcal{S}(G) = \bigoplus_{[L,\delta] \in \Omega(G)} \mathcal{S}(G)_{[L,\delta]}$

$\mathcal{S}(G)_{[L,\delta]} = (C^\infty(X_{\mathrm{unr}}(L)) \otimes \mathrm{End}_{\mathbb{C}}^{\mathrm{fr}}(I_P^G(V_\delta)))^{W_{[L,\delta]}}$

Definition: $[L,\delta] \prec [M,\sigma]$ if $\mathrm{Irr}_{\mathrm{temp}}(G)_{[L,\delta]} \subset \mathrm{Irr}(G)_{[M,\sigma]}$

The completion of $\mathcal{H}(G)_{[M,\sigma]}$ in $\mathcal{S}(G)$ is $\bigoplus_{[L,\delta] \prec [M,\sigma]} \mathcal{S}(G)_{[L,\delta]}$

$$\mathrm{Irr}_{\mathrm{temp}}(G) \cap \mathrm{Irr}(G)_{[M,\sigma]} = \bigsqcup_{\Delta(G) \ni [L,\delta] \prec [M,\sigma]} \mathrm{Irr}_{\mathrm{temp}}(G)_{[L,\delta]}$$

Theorem (ABPS conjecture for tempered representations)

For every reductive p-adic group G and $[M,\sigma] \in \Omega(G)$, there is a bijection

$$\mathrm{Irr}\Big(\bigoplus_{\Delta(G) \ni [L,\delta] \prec [M,\sigma]} \mathcal{S}(G)_{[L,\delta]}\Big) \longrightarrow \mathrm{Irr}(C^\infty(X_{\mathrm{unr}}(M)) \rtimes \mathbb{C}[W_{[M,\sigma]}, \natural])$$

which is canonical once a suitable 2-cocycle $\natural$ of $W_{[M,\sigma]}$ has been chosen

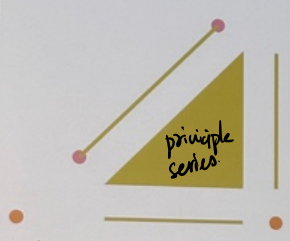
Structure of $\mathrm{Irr}_{\mathrm{temp}}(G)$ (2)

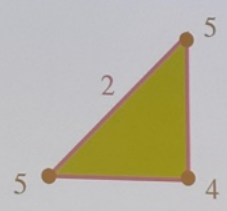
Example: Iwahori-spherical tempered representations of $SO_5(F)$

$M = T$, $\sigma = \mathrm{triv}$, $X_{\mathrm{unr}}(M) \cong S^1 \times S^1$, $W_{[M,\sigma]} = W(G,T) \cong D_4$

e.g. SO_5 $\rightsquigarrow$

$\mathrm{Irr}_{\mathrm{temp}}(G) \cap \mathrm{Irr}(G)_{[M,\sigma]}$ $\qquad \mathrm{Irr}(C^\infty(X_{\mathrm{unr}}(M)) \rtimes W_{[M,\sigma]})$





red = discrete series

purple = two representations

black = several representations

More twisted group algebras

ABPS will induce an isomorphism

$$K_*(C^\infty(X_{\mathrm{unr}}(M)) \rtimes \mathbb{C}[W_{[M,\sigma]}, \natural]) \to K_*\Big(\bigoplus_{\Delta(G)\ni[L,\delta]\prec[M,\sigma]} \mathcal{S}(G)_{[L,\delta]}\Big)$$

Theorem (Schur extension)

For a twisted group algebra $\mathbb{C}[W,\natural]$ there exists a finite central extension

$$1 \to Z \to W^* \to W \to 1$$

and a minimal idempotent $e_\natural \in \mathbb{C}[Z]$, from a character $\chi_\natural$ of Z,

such that $\mathbb{C}[W,\natural] \cong e_\natural \mathbb{C}[W^*]$

For any W-C^*-algebra A:

$K_*(A \rtimes \mathbb{C}[W,\natural]) = K_*(e_\natural(A \rtimes W^*)) = e_\natural K_*(A \rtimes W^*)$

Twisted crossed products and equivariant K-theory

$\natural$ 2-cocycle of W, X a locally compact W-space

Consequence of equivariant Serre–Swan theorem

$K_*(C_0(X) \rtimes \mathbb{C}[W,\natural]) = e_\natural K_*(C_0(X) \rtimes W^*) \cong e_\natural K^*_{W^*}(X)$

Define $K^*_{W,\natural}(X) = e_\natural K^*_{W^*}(X)$, this is the K-theory of W^*-equivariant vector bundles on X on which $Z = \ker(W^* \to W)$ acts as $\chi_\natural$

Theorem

For every reductive p-adic group G and $[M,\sigma] \in \Omega(G)$:

$$K_*\Big(\bigoplus_{\Delta(G)\ni[L,\delta]\prec[M,\sigma]} C^*_r(G)_{[L,\delta]}\Big) \otimes_{\mathbb{Z}} \mathbb{C} \cong$$
$$K_*(C(X_{\mathrm{unr}}(M)) \rtimes \mathbb{C}[W_{[M,\sigma]}, \natural]) \otimes_{\mathbb{Z}} \mathbb{C} \cong K^*_{W_{[M,\sigma]},\natural}(X_{\mathrm{unr}}(M)) \otimes_{\mathbb{Z}} \mathbb{C}$$

canonical once a suitable 2-cocycle $\natural$ has been chosen

Topological K-theory of reductive p-adic groups

Theorem (ABPS for K-theory, torsion part in progress)

(a) For every reductive p-adic group G and $[M,\sigma] \in \Omega(G)$:

$$K_*\Big(\bigoplus_{\Delta(G)\ni[L,\delta]\prec[M,\sigma]} C^*_r(G)_{[L,\delta]}\Big) \cong K_*(C(X_{\mathrm{unr}}(M)) \rtimes \mathbb{C}[W_{[M,\sigma]}, \natural_{[M,\sigma]}])$$

canonical once a suitable 2-cocycle $\natural_{[M,\sigma]}$ has been chosen

(b) $\text{"}K_*(\mathrm{Irr}_{\mathrm{temp}}(G))\text{"} = K_*(C^*_r(G)) \cong \bigoplus_{[M,\sigma]\in\Omega(G)} K^*_{W_{[M,\sigma]},\natural_{[M,\sigma]}}(X_{\mathrm{unr}}(M))$

Can be compared with Baum–Connes for reductive p-adic groups

Theorem (V. Lafforgue)

$\mathcal{B}(G)$: Bruhat–Tits building of G

$K^G_*(\mathcal{B}(G)) \cong K_*(C^*_r(G))$

What is the Bernstein decomposition of $K^G_*(\mathcal{B}(G))$?