up to a conj. of $W(A:G)$. If a is fixed as $\exp(H)$, H regular, then k_1 is uniquely determined up to a right multiplication by M. $\square$

In $SL_2(\mathbb{R})$, let $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ as A-component.

Let π with character χ be an adm. rep'n, $k_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. If we have $\pi(k_\theta)v_1 = e^{im\theta'}v_1 = \tau_1(k_{\theta'})v_1$, $\pi(k_\theta)v_2 = e^{in\theta}v_2 = \tau_1(k_\theta)v_2$.

$$\Rightarrow \quad F(k_\theta x k_{\theta'}) = e^{i(n\theta + m\theta')}F(x) = \tau_1(k_\theta)F(x)\tau_2(k_{\theta'}).$$

<radical part> of a differential operator on $SL_2(\mathbb{R})$. For any fixed $a \neq 1$,

$$sl_2(\mathbb{R}) = \mathfrak{g}_0 = (Ad(a^{-1}))\mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}_0.$$

By PBW, every member of $\mathcal{U}(\mathfrak{g})$ is a linear combination of terms

$$(Ad(a^{-1})X)HY, \quad X \in \mathcal{U}(\mathfrak{k}),\ Y \in \mathcal{U}(\mathfrak{k}),\ H \in \mathcal{U}(\mathfrak{a}).$$

For F above, $(Ad(a^{-1})X)HYF(a) = \tau_1(X)(HF)(a)\tau_2(Y)$.

Consider this as a differential operator on functions on $A^+ = \{a_t \mid t > 0\}$

Then to each $u \in \mathcal{U}(\mathfrak{g})$, $\rightsquigarrow D_{\tau_1,\tau_2}(u)$ on A^+, (def'n of radical part), s.t.

$$(uF)|_{A^+} = \underline{D_{\tau_1,\tau_2}(u)}(F|_{A^+}).$$

Generally: (τ_1, U_1), (τ_2, U_2) are fin.-dim. rep'ns of K. $\tau := (\tau_1, \tau_2)$.

<τ-spherical function>: $F \in C^\infty(G, \mathrm{Hom}_{\mathbb{C}}(U_2, U_1))$ s.t.

$$F(k_1 x k_2) = \tau_1(k_1)F(x)\tau_2(k_2) \quad \text{for all} \cdots.$$

Thus $(Ad(a^{-1})X)HYF(a) = \tau_1(X)(HF)(a)\tau_2(Y) =: \underline{D_\tau(u)}(F|_{A^+})(a)$

def'n a function: $A^+ \longmapsto \mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(U_2, U_1))$

$$a \longmapsto \Big(F(a) \longmapsto \underline{D_\tau(u)}(F|_{A^+})(a)\Big)$$

$\nwarrow$ radial component

take $M = Z_G(A)$, by

$$\tau_1(m)F(a) = F(ma) = F(am) = F(a)\tau_2(m), \text{ we get}$$

$F(a) \in \mathrm{Hom}_M(U_2, U_1) \subseteq \mathrm{Hom}_{\mathbb{C}}(U_2, U_1) \rightsquigarrow A^+ \longmapsto \mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_M(U_2, U_1))$ actually

Example: $D_{\tau_1,\tau_2}(\Omega)$, $\Omega = \frac{1}{2}h^2 + ef + fe = \frac{1}{2}h^2 - h + 2ef \in Z(sl_2(\mathbb{R}))$

<Lem>: $\alpha = e_1 - e_2$ be the positive restricted root. let $\xi_\alpha = e^{\alpha \log(\cdot)}$.

Let $Y \in \mathfrak{k}$, given by $Y = \frac{1}{2}(e + \theta e) = \frac{1}{2}(e - f)$. Then

$$e = \frac{2\xi_\alpha(a)}{\xi_\alpha^2(a) - 1}\{\xi_\alpha(a)Y - Ad(a^{-1})Y\}.$$

Proof: $Ad(a^{-1})Y = \frac{1}{2}Ad(a^{-1})e - \frac{1}{2}Ad(a^{-1})f$

$$= \frac{1}{2}\xi_\alpha(a)^{-1}e - \frac{1}{2}\xi_{-\alpha}(a)^{-1}f = \frac{1}{2}\xi_\alpha(a)^{-1}e + \xi_\alpha(a)Y - \frac{1}{2}\xi_\alpha(a)e$$

<Proof>: For $F \in C^\infty(G/\cdots)$ $\square$

$$D_{\tau_1,\tau_2}(\Omega)F(a) = \frac{1}{2}h^2F(a) + \frac{\xi^2+1}{\xi^2-1}hF(a)$$
$$+ \frac{8\xi^2}{(\xi^2-1)^2}(F(a)\tau_2(Y)^2 + \tau_1(Y)^2F(a))$$
$$- \frac{8\xi(\xi^2+1)}{(\xi^2-1)^2}\tau_1(Y)F(a)\tau_2(Y).$$

方法：把Ω中e,f用$Ad(\sigma^{-1})Y$，Y代替.

With $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}\end{pmatrix}$, we have $\xi = \xi_\alpha(a_t) = e^t$, and the differential operator h becomes $2\frac{d}{dt}$. Substitution gives

$$\tfrac{1}{2}D_{\tau_1,\tau_2}(\Omega)F(a_t) = \frac{d^2F}{dt^2} + (\coth t)\frac{dF}{dt}$$
$$+ \frac{1}{(\sinh t)^2}(F(a_t)\tau_2(Y)^2 + \tau_1(Y)^2F(a_t))$$
$$- \frac{2\cosh t}{\sinh^2 t}\tau_1(Y)F(a_t)\tau_2(Y).$$

denote $z = e^{-t}$, $\frac{d}{dt} = -z\,dz$

take $t \to \infty$

$$\tfrac{1}{2}\chi(\Omega)F = \left(z\frac{d}{dz}\right)^2F - \frac{1+z^2}{1-z^2}\left(z\frac{d}{dz}\right)F$$
$$+ \frac{4z^2}{(1-z)^2}(R_{\tau_2(Y))^2} + L_{\tau_1(Y)^2})F$$
$$- \frac{4(z+z^3)}{(1-z^2)^2}(R_{\tau_2(Y)}L_{\tau_1(Y)})F.$$

To study for general G, we use Harish-Chandra homo. study $Z(\mathfrak{g}^{\mathbb{C}})$ ($\mathfrak{g}_0$) ✓

A natural and important result (also an application), is.

Example. If G is a compact connected Lie group and π is an irreducible representation of G with highest weight λ, then the infinitesimal character of π is $\lambda + \delta$, where δ is half the sum of the positive roots.

Proposition 8.22. Let MAN be a parabolic subgroup of G, let $\mathfrak{t}$ be a θ-stable Cartan subalgebra of $\mathfrak{m}$, and let σ be an irreducible unitary representation of M. If σ has infinitesimal character Λ_σ relative to $\mathfrak{t}^{\mathbb{C}}$ and if ν is in $(\mathfrak{a}')^{\mathbb{C}}$, then $U(MAN, \sigma, \nu)$ has infinitesimal character $\Lambda_\sigma + \nu$ relative to $(\mathfrak{a} \oplus \mathfrak{t})^{\mathbb{C}}$.

▷Differential Equations Satisfied by Matrix Coefficients

Assume G has cpt. center.

A choice of positive Weyl Chamber $\mathfrak{a}_p^+$ fixs a positive system $\Sigma^+ \subseteq \Sigma$. Take Σ: set of nonzero restrictions to $\mathfrak{a}_p$. Thus $\mathfrak{b}_p$: Cartan subalg. of $\mathfrak{m}_p$. $\mathfrak{a}_p \oplus \mathfrak{b}_p$: Cartan subalg. of $\mathfrak{g}$. Let Δ = set of roots $\Delta((\mathfrak{a}_p \oplus \mathfrak{b}_p)^{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$, define Δ^+: Σ^+ with nonzero restriction of member of Δ^+. For $\tau = (\tau_1, \tau_2)$. def in $C_\tau^\infty(G^{(0)})$. With $A^+ = \exp(\mathfrak{a}_p^+)$.

$$C_\tau^\infty(A^+) = C^\infty(A^+, \mathrm{Hom}_M(U_2, U_1))$$

<Prop>: $F \mapsto F|_{A^+}$ gives $C_\tau^\infty(G^{(0)}) \xrightarrow{\sim} C_\tau^\infty(A^+)$.

<Lem>: α: restricted root, let $\xi_\alpha(a) = e^{\alpha(\log a)}$ for $a \in A$.

To $X \in$ restricted root space $\mathfrak{g}_\alpha$, $\rightsquigarrow Y \in \mathfrak{k}$, by $Y = \frac{1}{2}(X + \theta X)$, then

$$X = \frac{2\xi_\alpha(a)}{\xi_\alpha^2(a) - 1}\{\xi_\alpha(a)Y - Ad(a)^{-1}Y\}.$$

Introduce coordinate: $\{\alpha_1, \cdots \alpha_\ell\}$: simple restricted roots, defn

$$\iota : A \mapsto (\mathbb{R}^+)^\ell, \text{ by } \iota(\exp(H)) = (e^{-\alpha_1(H)}, \cdots, e^{-\alpha_\ell(H)})$$

Then $\iota(A^+) = (0,1)^\ell \subseteq D^\ell$, where D: unit disc in $\mathbb{C}$. Under ι, $e^{-\alpha_j(\log(a))}$ corresponds to z_j: If $\beta = \sum n_j \alpha_j \in \Sigma^+$,

$$e^{-\beta(\log(a))} \longleftrightarrow z_1^{n_1} \cdots z_\ell^{n_\ell}.$$

Let $H^{(1)}, \cdots, H^{(\ell)}$ be the dual basis of $\mathfrak{a}_p$ with $\alpha_i(H^{(j)}) = \delta_j^i$.

We shall show that differential operator $H^{(j)}$ on A corresponds to $-z_j \frac{\partial}{\partial z_j}$. In fact, let $F \in C^\infty(A)$, put $f = F \circ \iota^{-1}$. The

$$H^{(j)}F(a) = H^{(j)}(f \circ \iota)(a) = \frac{d}{dt} f \circ \iota(a \exp(tH^{(j)}))\Big|_{t=0}$$

$$= \frac{d}{dt} f(e^{-\alpha_1 \log a}, \cdots, e^{-\alpha_j \log a} e^{-t\alpha_j(H^{(j)})}, \cdots, e^{-\alpha_\ell \log(a)})\Big|_{t=0}$$

$$= -e^{-\alpha_j(\log(a))} f_j(e^{-\alpha_1 \log a}, \cdots, e^{-\alpha_\ell \log a})$$

$$= -z_j \frac{\partial f}{\partial z_j}(z).$$

Thus $U(\mathfrak{a}_p^{\mathbb{C}})$ corresponds to $\mathbb{C}[-z_1\frac{\partial}{\partial z_1}, \cdots, -z_\ell \frac{\partial}{\partial z_\ell}]$

Denote: $U := \mathrm{Hom}_M(U_2, U_1)$

$\mathscr{H}_U = \{\text{holomorphic functions from } D^l \text{ to } U\}$

$\mathscr{H}_{\mathrm{End}\, U} = \{\text{holomorphic functions from } D^l \text{ to End } U\}$

$$\partial^k = \left(\frac{\partial}{\partial z_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial z_l}\right)^{k_l} \qquad \text{for } k \text{ in } (\mathbb{Z}^+)^l$$

$$(z\partial)^k = \left(z_1 \frac{\partial}{\partial z_1}\right)^{k_1} \cdots \left(z_l \frac{\partial}{\partial z_l}\right)^{k_l} \qquad \text{for } k \text{ in } (\mathbb{Z}^+)^l$$

$$\mathscr{D} = \left\{ \sum_{\substack{k \in (\mathbb{Z}^+)^l \\ \text{finite sum}}} A_k \partial^k \,\middle|\, A_k \in \mathscr{H}_{\mathrm{End}\, U} \text{ for all } k \right\}$$

$$\mathscr{D}^* = \left\{ \sum_{\substack{k \in (\mathbb{Z}^+)^l \\ \text{finite sum}}} A_k (z\partial)^k \,\middle|\, A_k \in \mathscr{H}_{\mathrm{End}\, U} \text{ for all } k \right\}.$$

$\mathfrak{l} := \mathfrak{a}_p \oplus \mathfrak{m}_p \rightsquigarrow \mathfrak{g} = \bar{\mathfrak{n}}_p \oplus \mathfrak{l} \oplus \mathfrak{n}_p$

$\rightsquigarrow Z(\mathfrak{g}^{\mathbb{C}}) \subseteq \bar{\mathfrak{n}}_p^{\mathbb{C}} U(\mathfrak{g}^{\mathbb{C}}) \oplus U(\mathfrak{l}^{\mathbb{C}})$.

let μ'_Σ: projection $Z(\mathfrak{g}^{\mathbb{C}}) \mapsto U(\mathfrak{l}^{\mathbb{C}})$.

then $z \in Z(\mathfrak{g}^{\mathbb{C}}) \Leftrightarrow \mu'_{\Sigma^+}(z) \in \underline{Z(\mathfrak{l}^{\mathbb{C}})}$
$= U(\mathfrak{a}_p^{\mathbb{C}}) Z(\mathfrak{m}_p^{\mathbb{C}})$

Since $Z(\mathfrak{l}^{\mathbb{C}}) = U(\mathfrak{a}_p^{\mathbb{C}}) Z(\mathfrak{m}_p^{\mathbb{C}})$,

$\mu'_{\Sigma^+}(z)$ decompose as sum of "H". $\rightsquigarrow$

$D_\tau(\mu'_{\Sigma^+}(z)) \xleftrightarrow{\iota} (z\partial)^k$ without const. term.

$\cong U(\mathfrak{a}_p^{\mathbb{C}})$, by ι.

extends $\mu'_{\Sigma^+}: Z(\mathfrak{g}^{\mathbb{C}}) \to Z(\mathfrak{l}^{\mathbb{C}})$ to $\mu'_{\Sigma^+}: U(\mathfrak{g}^{\mathbb{C}}) \to U(\mathfrak{a}_p^{\mathbb{C}}) U(\mathfrak{t}^{\mathbb{C}})$:

write $\mathfrak{g} = \bar{\mathfrak{n}}_p \oplus \mathfrak{a}_p \oplus \mathfrak{t}$, apply PBW $\rightsquigarrow U(\mathfrak{g}^{\mathbb{C}}) = (U(\mathfrak{a}_p^{\mathbb{C}}) U(\mathfrak{t}^{\mathbb{C}})) \oplus \bar{\mathfrak{n}}_p^{\mathbb{C}} U(\mathfrak{g}^{\mathbb{C}})$

projection gives μ'_{Σ^+}

<Thm> For each $z \in Z(\mathfrak{g}^{\mathbb{C}})$, τ-radical component $D_\tau(z)$ corresponds under "ι" to $\mathscr{D}^*$. Moreover, $D_\tau(z) - D_\tau(\mu'_{\Sigma^+}(z)) \xrightarrow{\iota} \mathscr{D}^*$ whose coefficient function vanish at $z=0$.

$\Rightarrow$

(这在讲什么？算不清楚)

Application: for $\lambda \in \mathrm{Rep}(G)$ admissible. $\rightsquigarrow \exists \Lambda$, $\lambda(z) = \chi_\Lambda(z) I$ for $z \in Z(\mathfrak{g}^{\mathbb{C}})$

Let F be a τ-spherical function built from matrix coefficients of π as at the start of §4. Formula (8.10) says that

$$ZF = \chi_\Lambda(Z)F \qquad \text{for } Z \in Z(\mathfrak{g}^{\mathbb{C}}),$$

hence that $\boxed{D_\tau(Z)(F|_{A^+}) = \chi_\Lambda(Z)(F|_{A^+})}$ for $Z \in Z(\mathfrak{g}^{\mathbb{C}})$.

Thus $F|_{A^+}$ is annihilated by $D_\tau(Z)$ for all Z in the kernel I of χ_Λ. Since χ_Λ is a linear functional, its kernel has codimension one. Thus Theorems 8.26 and 8.28 apply, and these theorems say that we can apply the theory of Appendix B.

E_1, E_2: projection on U_1, U_2 of $\mathrm{Rep}(G)$. Then $E_1 \pi(x) E_2 \in C^\infty_{(\tau_1,\tau_2)}(G)$.

§8: Asymptotic Expansions and Leading Coefficients

$F(x) := E_1 \pi(x) E_2 \in U := \mathrm{Hom}_{\mathbb{C}}(U_2, U_1)$ is an eigenfunction of $Z(\mathfrak{g}^{\mathbb{C}})$ say with eigenvalue χ_Λ.

<Thm> (expansion for $F|_{A^+}$)

Theorem 8.32. Let F be the τ-spherical eigenfunction of $Z(\mathfrak{g}^{\mathbb{C}})$ in (8.50). Then $F \circ \iota^{-1}$, initially defined on $(0, 1)^l$, extends to a global multiple-valued holomorphic solution on $(D^\times)^l$ of the system corresponding to

$$D_\tau(Z)(F|_{A^+}) = \chi_\Lambda(Z)(F|_{A^+}), \qquad Z \in Z(\mathfrak{g}^{\mathbb{C}}).$$

Moreover, there exist an integer q_0 and a finite subset $\mathscr{F} \subseteq \mathbb{C}^l$ such that $F \circ \iota^{-1}$ is given on $(D^\times)^l$ by

$$F \circ \iota^{-1}(z) = \sum_{s \in \mathscr{F}} \sum_{0 \le |q| \le q_0} z^s (\log z)^q F_{s,q}(z), \tag{8.51}$$

with each $F_{s,q}(z)$ in $\mathscr{H}_U$.

Since $F_{s,q}(z)$ is holo. on D^l, $\rightsquigarrow$ have a convergent multiple power series expansion. (absolute + uniform on $\forall$ cpt $\subseteq D^l$) $\rightsquigarrow$

$a \in A^+$, $\quad F_{s,q}(\iota(a)) = \sum_{k \in (\mathbb{Z}^+)^l} c'_{s,q,k} e^{-(k_1\alpha_1 + \dots + k_l\alpha_l)(\log a)}, \qquad c'_{s,q,k} \in U,$

with the convergence absolute and uniform as long as $\alpha_j(\log a) \ge \varepsilon_j > 0$

since (8.51) is a finite sum, $F(a)$ itself has an analogous asym. expan., by $\quad z^s(\log(z))^q \longleftrightarrow a_1(\log a)^{q_1} \cdots a_l(\log a)^{q_l} e^{-(s_1\alpha_1 + \dots s_l\alpha_l)(\log a)}$.

let ρ: half sum of Σ^+. write

$$F(\exp(H)) = \sum_\nu F_{\nu-\rho}(\exp H), \quad F_{\nu-\rho}(\exp H) = e^{-\rho(H)} \sum_{|q| \le q_0} C_{\nu,q}\, \alpha(H)\, e^{\nu(H)}$$

(8.52a) (8.52b)

$\rightsquigarrow$ $\nu - \rho$: exponent of F if $F_{\nu-\rho} \not\equiv 0$

2 exponents equivalent if difference $\in \mathrm{Span}_{\mathbb{Z}}\{\alpha_1, \dots, \alpha_l\}$

(or integrally equivalent)

leading exponent: $\nu-\rho$ is highest exponent. i.e. $\nu-\nu' \in \mathrm{Span}_{\mathbb{Z}^+}\{\alpha_1, \dots, \alpha_k\}$ for all {integrally equivalent} classes of $\nu-\rho$.

leading term: $F_{\nu-\rho}$

Theorem 8.33. Let F be the τ-spherical eigenfunction of $Z(\mathfrak{g}^{\mathbb{C}})$ in (8.50) with eigenvalue χ_Λ under $Z(\mathfrak{g}^{\mathbb{C}})$. Then any leading exponent $\nu-\rho$ of F has the property that $\nu = w\Lambda|_{\mathfrak{a}_{\mathfrak{p}}}$ for some w in the Weyl group $W = W((\mathfrak{a}\oplus\mathfrak{b})^{\mathbb{C}}:\mathfrak{g}^{\mathbb{C}})$.

Corollary 8.34. Let π be an admissible representation of G with an infinitesimal character. Then the set of leading exponents of π is finite and nonempty. Moreover, if $\nu'-\rho$ is any exponent of π, then there is some leading exponent $\nu-\rho$ of π with $\nu' \le \nu$.

Proposition 8.35. Let π be an admissible representation of G with an infinitesimal character. As τ varies, the powers q of $\alpha(H)$ that can appear in the expansion (8.52b) of leading terms have $|q|$ uniformly bounded.

☆ Application: Subrepresentation Theorem

Theorem 8.37. Let π be an irreducible admissible representation of G. Then π is infinitesimally equivalent with a subrepresentation of some nonunitary principal series representation. More specifically, if $\nu-\rho$ is a leading exponent of π, then π is infinitesimally equivalent with a subrepresentation of $U(\bar{S}_{\mathfrak{p}}, \sigma, \nu)$ for some irreducible unitary representation σ

Proof: Choose two K-finite vect. v_1, v_2, st. the asymptotic expansion of $(\pi(a)v_1, v_2)$ along $A_{\mathfrak{p}}^+$ contains the term $\alpha(H)^q e^{\nu-\rho}$ with $\neq 0$ coefficient.

Define linear functional ℓ, on the space of K-finite vectors of π: $\ell(v)$: coefficient of $\alpha(H)^q e^{\nu-\rho}$ in asymptotic expansion of $(\pi(a)v_1, v_2)$.

v_1, v_2 construction ensure $\ell \neq 0$.

Since $\nu-\rho$ is a leading exponent $\rightsquigarrow$

$$\ell(Xv)=0, \text{ for all } X\in\bar{\mathfrak{n}}_{\mathfrak{p}}, \text{ all K-fin. } v.$$

Since q is as large as possible, gives

$$\ell(Hv) = (\nu-\rho)(H)\,\ell(v), \text{ all } H\in\mathfrak{a}_{\mathfrak{p}}, \text{ all K-fin. } v.$$

Defn

$$L: \{\text{K-fin. vect.}\} \longrightarrow (\text{K-fin.}) \text{ func}: K\to\mathbb{C}$$
$$v \longmapsto (k\mapsto \ell(\pi(k)^{-1}v))$$

extends $K\to\mathbb{C}$ to G: $(\mathrm{ind}_{A_{\mathfrak{p}}\bar{N}_{\mathfrak{p}}}^{G}(e^{\nu}\otimes 1))$:

$$f(ka\bar{n}) = e^{(-\nu+\rho)(\log(a))} f(k).$$

Then L is a equivariant, and $L(Xv) = X(L(v))$ for all $X\in\mathfrak{g}$

Then check L is $\mathfrak{g}$-equivariant, i.e. $L(xv) = x(Lv)$ for all $x \in \mathfrak{g}$.

need to verify: $$l(\pi(k)^{-1}Xv) = X(e^{(-\nu+\rho)H(x)}\pi(\kappa(x))^{-1}v)_{x=k}, \tag{8.55}$$

where H is the log of the $A_{\mathfrak{p}}$ component when $G = KA_{\mathfrak{p}}\bar{N}_{\mathfrak{p}}$. We do so for one value of k at a time. Put $X = \mathrm{Ad}(k)Y$. Then the left side of (8.55) is $l(Y\pi(k)^{-1}v)$. The right side is easily checked to be

$$\begin{array}{ll} 0 & \text{if } Y \in \bar{\mathfrak{n}}_{\mathfrak{p}} \\ (\nu-\rho)(Y)l(\pi(k)^{-1}v) & \text{if } Y \in \mathfrak{a}_{\mathfrak{p}} \\ l(Y\pi(k)^{-1}v) & \text{if } Y \in \mathfrak{k}. \end{array}$$

Hence the right side is always $l(Y\pi(k)^{-1}v)$, and thus (8.55) holds. Consequently L is $\mathfrak{g}$-equivariant.

[remark: common way to study such linear functional:

suppose $(\pi, V^\pi) \in \mathrm{Rep}(G)$, $\exists\, l \neq 0 \in (V^\pi)'$, s.t.

$$\begin{cases} l(\pi(n)v) = l(v) & \text{for } n \in N_{\mathfrak{p}},\ v \in V^\pi \\ l(\pi(a)v) = e^{(\nu+\rho_{\mathfrak{p}})\log(a)}\, l(v), & \text{for } a \in A_{\mathfrak{p}},\ v \in V^\pi \end{cases}$$

then $v \mapsto f_v := [x \mapsto l(\pi(x)^{-1}v)]$ gives

$$\begin{cases} f_v(xan) = e^{-(\nu+\rho_{\mathfrak{p}})\log(a)} f_v(x) \\ f_{\pi(g)v}(x) = f_v(g^{-1}x). \end{cases}$$

$\rightsquigarrow$ we get a non-zero G-map from $V^\pi \to \mathrm{Ind}_{A_{\mathfrak{p}}N_{\mathfrak{p}}}^{G}(\exp(\nu)\otimes 1)$.

By double induction:

$$\mathrm{Ind}_{A_{\mathfrak{p}}N_{\mathfrak{p}}}^{G}(\exp(\nu)\otimes 1) = \mathrm{Ind}_{M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}}^{G}\Big[\underbrace{\mathrm{Ind}_{1}^{M_{\mathfrak{p}}}(1)}_{\text{Peter-Weyl decompose discretely.}} \otimes \exp(\nu)\otimes 1\Big].$$

$\rightsquigarrow$ $v \mapsto f_v$ gives us a non-zero G-map, into $U(Sp, 6, \nu)$ for some 6. It is a 1-1 map.

One source of such linear functionals l is the asymptotics of matric coefficients. Simplest case: $\lim_{a\to\infty} e^{-(\nu-\rho_{\mathfrak{p}})\log(a)}\langle \pi(a)v, v'\rangle = C_{v,v'} \neq 0$ exists for all v, some v'.

Then $$l(\pi(a_0)v) = e^{(\nu-\rho_{\mathfrak{p}})\log a_0}\, l(v)$$

trivially, and $$l(\pi(\bar{n})v) = \lim e^{-(\nu-\rho_{\mathfrak{p}})\log(a)}\langle \pi(a\bar{n}a^{-1})\pi(a)v, v'\rangle$$
$$= \lim e^{-(\nu-\rho_{\mathfrak{p}})\log a}\langle \pi(a)v, \pi^{t_\pi}(a\bar{n}a^{-1})v'\rangle$$

since $a\bar{n}a^{-1} \to 1$, for $\bar{n} \in \bar{N}_{\mathfrak{p}} = \Theta N_{\mathfrak{p}}$.

thus we expect $\ell(\pi(\bar{n})v) = \ell(v)$ for $\bar{n} \in \bar{N}_{\mathfrak{p}}$, $v \in V^{\pi}$. $\square$

Example for $SL_2(\mathbb{R})$: expansion in series about $z=0$, gives:

Expansion in series about $z = 0$ gives

$$z^2 \frac{d^2F}{dz^2} + O(z)z\frac{dF}{dz} + (-\tfrac{1}{2}\chi(\Omega) + O(z))F = 0$$

with each instance of $O(z)$ representing an analytic function for $|z| < 1$ with no constant term. This is an ordinary differential equation with a regular singular point at $z = 0$, and one looks for solutions z^s(power series). The power s must satisfy the "indicial equation" $s(s-1) - \frac{1}{2}\chi(\Omega) = 0$. For $\chi = \chi_\lambda$, we have seen that $\chi(\Omega) = \frac{1}{2}\lambda(h)^2 - \frac{1}{2}$. Then $s = \frac{1}{2}(1 \pm \lambda(h))$. The leading terms of the asymptotic expansion are thus $e^{-\frac{1}{2}(1\pm\lambda(h))t}$, possibly with a factor of t. ← how it comes?

From this analysis we can see a connection between the infinitesimal character and the possible square integrability of the matrix coefficient on G. In the KAK decomposition, one can check that Haar measure is given by $(\sinh t)\, d\theta'\, dt\, d\theta$. The product of the leading terms, namely of $e^{-\frac{1}{2}(1+\lambda(h))t}$ and $e^{-\frac{1}{2}(1-\lambda(h))t}$ is e^{-t}, and so one of these leading terms fails to be square integrable. Thus one leading term must have coefficient 0 for a discrete series representation. For the unitary principal series $\mathrm{Ind}_{MAN}^{G}(\sigma \otimes \nu \otimes 1)$, the infinitesimal character is ν. Then $\nu(h)$ is imaginary, and the magnitude of both leading terms is $e^{-t/2}$, which just misses square integrability.

Langlands Classification Thm.

introduce $\{\alpha_i\}$: simple restricted roots, def'n $\{\omega_i\}_{i=1}^{\ell}$, s.t. $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$, $\{\omega_j\}$ also forms a basis for Σ^+.

Theorem (Langlands). *For an irreducible admissible representation π of G, the following are equivalent:*

1) *All K finite matrix coefficients are in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$.*
2) *Every leading exponential $e^{(\nu-\rho)\log a}$ of π satisfies $\mathrm{Re}\langle \nu, \omega_j \rangle \leq 0$ for $1 \leq j \leq l$.*
3) *π is infinitesimally equivalent with a subrepresentation of a standard induced representation $\mathrm{Ind}_{MAN}^{G}(\sigma_0 \otimes \nu_0 \otimes 1)$ for some parabolic subgroup $MAN \supseteq M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$, some discrete series representation σ_0 of M, and some imaginary parameter ν_0.* (this gives a parametrization)

Proof of (2) ⇒ (3):

Take a leading exponential $e^{(\nu-\rho)\log a}$. There may be more than one leading exponential, and we specify which one to use in a moment. Let

$$\mathcal{F} = \{j \mid 1 \leq j \leq l \text{ and } \mathrm{Re}\,\langle \nu, \omega_j \rangle < 0\}.$$

The set $\mathcal{F}$ corresponds to indices where $e^{(\nu-\rho)\log a}$ decreases at an L^2 rate. Write $\mathrm{Re}\,\nu = \sum_{i=1}^{l} c_i\alpha_i$. Take the inner product with ω_j to see that $c_j < 0$ for $j \in \mathcal{F}$ and

$c_j = 0$ (by (2)) for $j \notin \mathcal{F}$. Now

$$\{\alpha_j \mid j \in \mathcal{F}\} \cup \{\omega_j \mid j \notin \mathcal{F}\}$$

is a basis of $\mathfrak{a}'_{0,\mathfrak{p}}$. The reason is that each set is independent, the two sets are orthogonal to each other, and the total number of elements in the set is l.

Expand ν in terms of this basis to get

$$\nu = \sum_{j \notin \mathcal{F}} b_j \omega_j - \sum_{j \in \mathcal{F}} a_j \alpha_j$$

with $\operatorname{Re} b_j = 0$ for $j \notin \mathcal{F}$ and $\operatorname{Re} a_j > 0$ for $j \in \mathcal{F}$. To do so, first expand $\operatorname{Re} \nu$, using only the second term. Then expand $\operatorname{Im} \nu$, and combine the two expansions.

We define $\mathfrak{a}_0 = \sum_{j \notin \mathcal{F}} \mathbb{R} H_{\omega_j}$, and we use the indices $j \in \mathcal{F}$ to generate the roots defining M. The proof of the Subrepresentation Theorem gives an infinitesimal embedding

$$\pi \subseteq \operatorname{Ind}_{M_\mathfrak{p} A_\mathfrak{p} \overline{N}_\mathfrak{p}}^G (\sigma \otimes e^\nu \otimes 1) \quad \text{for some } \sigma \in \widehat{M_\mathfrak{p}}.$$

▷ rewrite right side by double induction: inside term being induced up to $\underline{M}$, the character ν_0 induced from A being $e^{\sum_{j \notin F} b_j \omega_j}$.
Denote by $\tilde{\sigma}_0$ the rep'n of M, that is obtained.
The desired discrete series σ_0 is a certain subrep'n of $\tilde{\sigma}_0$. ∎

Theorem (Langlands). *The equivalence classes (under infinitesimal equivalence) of irreducible admissible representations of G stand in one-one correspondence with all triples $(MAN, [\sigma_0], \nu_0)$ such that*

MAN is a parabolic subgroup of G containing $M_\mathfrak{p} A_\mathfrak{p} N_\mathfrak{p}$

σ_0 is an irreducible tempered (unitary) representation of M and $[\sigma_0]$ is its equivalence class

ν_0 is a member of $\mathfrak{a}'$ such that $\operatorname{Re}\langle \nu_0, \alpha \rangle > 0$ for every positive restricted root that does not vanish on $\mathfrak{a}_0$.

The correspondence is that $(MAN, [\sigma_0], \nu_0)$ corresponds to the class of the unique irreducible quotient of $\operatorname{Ind}_{MAN}^G (\sigma_0 \otimes \nu_0 \otimes 1)$.

(far from accurate)

REMARK: It does not tell us which parameters of the nonunitary principal series correspond to irreducibility, and it does not tell us much about what irreducible admissible representations are infinitesimally unitary.

There is a supplementary statement to the theorem: The irreducible quotient is explicitly obtained as the image of a specific <u>integral operator</u> sending

$$\operatorname{Ind}_{MAN}^G (\sigma_0 \otimes \nu_0 \otimes 1) \quad \text{equivariantly to} \quad \operatorname{Ind}_{MA\overline{N}}^G (\sigma_0 \otimes \nu_0 \otimes 1).$$

More Accurate parametrization: see another note.