Langlands Classification, Discrete Series (1)

Example: $GL(2,\mathbb{R})$, $\underline{Sp(4,\mathbb{R}), SU(2,2)}$ ← this topic.

and unitary dual question: $\{\text{unitary repn}\}/\sim$

1. $GSp_4(\mathbb{R})$, $Sp_4(\mathbb{R})$ with emphasis on Discrete series

① Prelude: $G = SL_2(\mathbb{R}) = NAK$, $A := \{\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \mid y > 0\}$

$K := \{\gamma_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\}$

Classification:

(1) (limits of) discrete series

As a $\mathbb{C}$-basis of $\mathfrak{sl}_2(\mathbb{C})$, take

$$H = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad X_\pm := \frac{1}{2}\begin{pmatrix} 1 & \pm\sqrt{-1} \\ \pm\sqrt{-1} & 1 \end{pmatrix}.$$

consider Verma-module: $M(k) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} \mathbb{C}_k$,

$\mathbb{C}_k$ is 1-dim $\bar{\mathfrak{b}}$-mod, characterized by $H \cdot v_k = k v_k$, $X_- \cdot v_k = 0$

- $M(k)$ is unitarizable if $k \in \mathbb{Z}_{>0}$.

$\rightsquigarrow \exists D_k^+ \in \mathrm{Repn}(G)$, s.t.

$(\mathfrak{g}, K)$-mod $D_k^+ \cong M(k)$.

$\rightsquigarrow D_k^-$ as its contragredient. (its highest wt $= -k$)

If $k \geq 2$, $D_k^\pm$: discrete series with Blattner parameter $\pm k$.

fact: For a congruence subgp Γ, $k \geq 1$,

$$M_k(\Gamma) \supseteq \mathrm{Hom}_{\mathfrak{g},K}(D_k^+, \mathcal{A}(\Gamma\backslash G))$$

$$S_k(\Gamma) \supseteq \mathrm{Hom}_{\mathfrak{g},K}(D_k^+, \mathcal{A}^{cusp}(\Gamma\backslash G)) \cong \mathrm{Hom}_G(D_k^+, L^2(\Gamma\backslash G)).$$

(2) (non-unitary) principal series

For each $\epsilon : M \to \mathbf{C}^\times$ and $\nu \in \mathbf{C}$, the (non-unitary) principal series representation of G is given by

$$I(\epsilon, \nu) := \{f : G \xrightarrow{C^\infty} \mathbf{C} \mid f(\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} g) = \epsilon(a/|a|)|a|^{\nu+1} f(g),\ \forall \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \in B, \forall g \in G\}.$$

Then we have the following exact sequence:

$$0 \to D_k^+ \oplus D_k^- \to I((-1)^k, k-1) \to F_{k-2} \to 0, \quad k \geq 2. \tag{1.1}$$

Moreover we have

$$I(-, 0) \cong D_1^+ \oplus D_1^-.$$

Here for a non-negative integer $m \geq 0$, $F_m = \mathrm{Sym}^m(\mathbf{C}^2)$ stands for the m-th symmetric tensor representation of the natural (tautological) two dimensional representation of G.

Classification: discrete series/$\sim$, tempered series ($L^{2+\varepsilon}$)/$\sim$

②

$\widehat{G}_{DS} \subset \widehat{G}_{temp} \subset \widehat{G}_{adm}$. ← admissible / ~

Then we have

(i) $\widehat{G}_{adm} \setminus \widehat{G}_{temp} = \{I(+;\nu) \mid \mathrm{Re}(\nu) > 0, \nu \neq 1,3,5,\cdots\} \cup \{I(-;\nu) \mid \mathrm{Re}(\nu) > 0, \nu \neq 2,4,6,\cdots\} \cup \{F_m \mid m \geq 0\}$;

(ii) $\widehat{G}_{temp} \setminus \widehat{G}_{DS} = \{D_1^+, D_1^-\} \cup \{I(+;\nu) \mid \nu \in \sqrt{-1}\mathbf{R}\} \cup \{I(-;\nu) \mid \nu \in \sqrt{-1}\mathbf{R} \setminus \{0\}\}$;

(iii) $\widehat{G}_{DS} = \{D_k^+, D_k^- \mid k \geq 2\}$.

(unitary dual question)

Moreover the set $\widehat{G}_u$ of unitary equivalence classes of irreducible unitary representations of G is given by $\widehat{G}_u = \widehat{G}_{temp} \cup \{I(+;\nu) \mid 0 < \nu < 1\} \cup \{F_0\}$. The family $\{I(+;\nu) \mid 0 < \nu < 1\}$

irr. unitary, called: complementary series.

Rmk: (1) Translation functor: For $k \geq 2$:

$$D_k^+ \otimes F_1 \cong D_k^+ \oplus D_{k-1}^+$$

$$\rightsquigarrow \quad D_{k-1}^+ = \{v \in D_k^+ \otimes F_1 \mid \Omega v = \{(k-2)^2 - 1\} v\} \quad (\Omega: \text{Casimir})$$

(2) Selberg 1/4 conjecture: Let $f: \Gamma \backslash G/K \to \mathbb{C}$ be a wave cusp form w.r.t. a congruence subgp. Γ. Then, f generates an irr. unitary rep'n $\pi_f \cong I(+,\nu)$ with $\nu \in \sqrt{-1}\mathbb{R} \cup (0,1)$. This gives

$$\Delta f(z) \equiv -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(z) = \frac{1}{4}(1-\nu^2) f(z).$$

Conjecture asserts: $\frac{1}{4}(1-\nu^2) \geq 1/4$. This $\Leftrightarrow$ π_f should be tempered.

<Def-Prop>: temper := Every K-fin. matrix coefficient of π belongs to $\bigcap_{\varepsilon > 0} L^{2+\varepsilon}(G) \Leftrightarrow \Theta_\pi$ can be extended to the Schwartz space $S(G)$.

<Def>: $G \supseteq P = MAN$. For each $\mu \in \mathfrak{a}_{\mathbb{C}}^*$, denote the corresponding quasi-char. of A by a^μ. define $\rho\ (\rho_P) \in \mathfrak{a}_{\mathbb{C}}^*$ by

$$a^\rho = \det\left(\mathrm{Ad}(a)\big|_{\mathrm{Lie}(N)}\right)^{1/2}.$$

<Thm> $(\widehat{G}_{adm} \leftrightarrow \widehat{M}_{temp})$: $\forall \pi \in \widehat{G}_{adm}$, $\exists P = MAN \subseteq G$, $\sigma \in \widehat{M}_{temp}$, $\exists \nu \in \mathfrak{a}_{\mathbb{C}}^*$ with $\mathrm{Re}(\nu)$ belonging to the open Weyl Chamber defined from N, s.t.

$$\pi \cong J(P,\sigma,\nu) := \text{unique irr. quotient of } I(P,\sigma,\nu) := \mathrm{Ind}_P^G(\sigma \otimes a^{\nu+\rho} \otimes 1_N).$$

the triple (P,σ,ν) is unique up to conjugation.

<Thm> $(\widehat{M}_{temp} \leftrightarrow \widehat{M}_{1,DS})$

For $\forall \sigma$, $\exists P_1 = M_1 A_1 N_1 \subseteq M$, $\exists \sigma \in \widehat{M}_{1,DS}$, $\exists \nu \in \sqrt{-1}\mathfrak{a}^*$ s.t.

σ unitarily equivalent to a direct sum of $I(P_1,\sigma_1,\nu)$.

rmk: note that $\exists P' \subseteq G$ parabolic, $P' \subseteq P \subseteq G$, s.t. $P' \cap M = P_1$. by double induction, we can change $\sigma \in \widehat{M}_{temp}$ in <Thm> to $\sigma \in \widehat{M}_{DS}$.

rmk: we also have Casselman's subrep'n theorem.

③ $G = Sp(4,\mathbb{R})$. up to conjugation, it has 3 proper parabolic subgp:

(i) The minimal parabolic subgroup. $P_0 = M_0 A_0 N_0$.

$$M_0 = \{\mathrm{diag}(\epsilon_1,\epsilon_2,\epsilon_1,\epsilon_2) \mid \epsilon_i = \pm 1\}; \quad A_0 = \{\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1}) \mid a_i > 0\};$$

$$N_0 = \left\{ n(x_0,x_1,x_2,x_3) := \left(\begin{array}{cc|cc} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & x_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array}\right) \in G \right\}.$$

(ii) The Siegel parabolic subgroup. $P_S = M_S A_S N_S$.

$$M_S = \left\{ \left(\begin{array}{c|c} m & \\ \hline & {}^t m^{-1} \end{array}\right) \mid m \in SL^{\pm}(2,\mathbf{R}) \right\}; \quad A_S = \{\mathrm{diag}(t,t,t^{-1},t^{-1}) \mid t > 0\};$$

$$N_S = \left\{ \left(\begin{array}{c|c} 1_2 & x \\ \hline & 1_2 \end{array}\right) \mid {}^t x = x \in M_2(\mathbf{R}) \right\}.$$

(iii) The Jacobi (or Klingen) parabolic subgroup. $P_J = M_J A_J N_J$.

$$M_J = \left\{ \left(\begin{array}{cc|cc} \epsilon & & & \\ & a & & b \\ \hline & & \epsilon & \\ & c & & d \end{array}\right) \middle| \epsilon = \pm 1, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R}) \right\};$$

$$A_J = \{\mathrm{diag}(a_1,1,a_1^{-1},1) \mid a_1 > 0\}; \quad N_J = \{n(x_0,x_1,x_2,0) \mid x_i \in \mathbf{R}\}.$$

degenerate principle series rep'n

$I(P_S; \mathrm{triv}, \nu)$

$:= \{ f: G \xrightarrow{C^\infty} \mathbb{C} \mid f(\begin{pmatrix} m & \\ & {}^t m^{-1} \end{pmatrix} g) = |\det(m)|^{\nu+3} f(g), \ \forall (m,g) \in GL_2(\mathbb{R}) \times G \}$

where $g \in G$ acts as right translation.

$\forall (m,a,n,g) \in M_0 \times A_0 \times N_0 \times G$

<rmk>: $I(P_0; \mathrm{triv}, \frac{\nu+1}{2}, \frac{\nu-1}{2}) := \{ f: G \xrightarrow{C^\infty} \mathbb{C} \mid f(mang) = a_1^{(\nu+1)/2+2} a_2^{(\nu+1)/2+1} f(g) \}$.

For ν in general position $I(P_S; \mathrm{triv}, \nu)$ is irreducible. $\rightsquigarrow$ if such ν satisfies $\mathrm{Re}(\nu) > 0$, then $I(P_S; \mathrm{triv}, \nu) \cong J(P_0; \mathrm{triv}, \frac{\nu+1}{2}, \frac{\nu-1}{2})$

<rmk>: $I(P_S; \mathrm{triv}, \nu)$ and its p-adic counterparts are the local components of the Siegel Eisenstein Series.

④ Harish Chandra's parametrization of discrete series.

G, $G := G_{\mathbb{R}}$, assume G: simply connected $\rightsquigarrow G$ is a connected gp.

$K \subseteq G$: cpt.

<Thm>: set $\hat{G}_{DS} \neq \phi$ iff $\mathrm{rk}(G) = \mathrm{rk}(K)$.

(Construction): fix notation: Δ: root system $\Delta(G,T)$

set $\mathrm{rk}(G) = \mathrm{rk}(K)$, then the Cartan subgp. T of K is also a Cartan subgp of G.

W_G: Weyl gp of Δ.

Δ_c: set of compact roots $= \Delta(K,T) = \Delta_c^+ \sqcup \Delta_c^-$

$\uparrow$ simple roots

another 刻画 is: for $H \in \mathfrak{t}_0, \alpha \in \Delta, X \in \mathfrak{g}_\alpha$ ($\dim = 1$), then $[H, \theta X] = \alpha(H)\theta X$, then $\theta X \in \mathfrak{g}_\alpha \rightsquigarrow \theta^2 = 1$ $\rightsquigarrow \mathfrak{g}_\alpha \in \mathfrak{t}$ compact or $\mathfrak{g}_\alpha \in \mathfrak{p}$ non-compact

W_K: Weyl gp. of Δ_c.

$\tau_\mu \in \hat{K}$: irr. rep'n of K with highest wt. $\mu \in \sqrt{-1}\mathfrak{t}^*$ w.r.t. Δ_c^+

$P(\Delta) := \{ \lambda \in \mathfrak{t}_{\mathbb{C}}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \}$: the wt. lattice of Δ.

<Thm>: $\hat{G}_{DS}$ parametrized by set

$\lambda + \delta$ is integral

$$\Xi := \{ \lambda \in P(\Delta) \mid \langle \lambda, \alpha^\vee \rangle \neq 0 \ (\forall \alpha \in \Delta), \ \langle \lambda, \alpha^\vee \rangle > 0 \ (\forall \alpha \in \Delta_c^+) \}$$

Then the discrete series π_λ is characterized by:

(i) *The infinitesimal character of* π *is given by* $\lambda \in \mathfrak{t}_{\mathbf{C}}^*/\sim_{W_G}$.
(ii) *Let* Δ_λ^+ *be the (unique) positive system of* Δ *such that* $\langle \lambda, \alpha^\vee \rangle > 0$ $(\forall \in \Delta_\lambda^+)$. *Put*

$$\Lambda = \lambda + \frac{1}{2}\sum_{\alpha \in \Delta_\lambda^+} \alpha - \sum_{\alpha \in \Delta_c^+} \alpha.$$

Then the irreducible representation τ_Λ *of* K *with highest weight* Λ *occurs in* π_λ *with multiplicity one. The specific* K*-type* τ_Λ *of* π_λ *is called* **the minimal** K**-type of** π_λ.
(iii) *If an irreducible representation* τ_μ *of* K *occurs in* π_λ, *then we have*

$$\mu = \Lambda + \sum_{\alpha \in \Delta_\lambda^+ \setminus \Delta_c^+} m_\alpha \alpha, \qquad \text{for some } m_\alpha \in \mathbf{Z}_{\geq 0}.$$

Some terminologies and remarks:
(1) λ (resp. Λ) is called the **Harish-Chandra parameter** (resp. the **Blattner parameter**) of π_λ. Some authors interchange the letters λ and Λ.
(2) **Infinitesimal character.** By Schur's lemma, the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ acts on each irreducible admissible representation (π, H_π) as scalar multiplication. That is, we have

$$\pi(z)v = \chi_\pi(z)v, \quad (\forall v \in H_\pi, \forall z \in Z(\mathfrak{g}))$$

for some $\mathbf{C}$-algebra homomorphism $\chi_\pi : Z(\mathfrak{g}) \to \mathbf{C}$. On the other hand, there is a $\mathbf{C}$-algebra isomorphism (the Harish-Chandra isomorphism):

$$q : Z(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{t}_{\mathbf{C}})^{W_G}.$$

Hence there exists $\nu \in \mathfrak{t}_{\mathbf{C}}^*/\sim_{W_G}$ such that

$$\chi_\pi(z) = \langle \nu, q(z) \rangle, \qquad \forall z \in Z(\mathfrak{g}).$$

We call χ_π or $\nu \in \mathfrak{t}_{\mathbf{C}}^*$ the infinitesimal character of π. The theorem implies that there are $\sharp(W_G/W_K)$ numbers of discrete series representations having the same infinitesimal character.

Ex. 1. $G = SL_2(\mathbb{R})$, Blattner parameter of $D_k^\pm$ $(k \geq 2)$ is $\pm k$, HC para. is $\pm(k-1)$.

Ex. 2. $G = Sp(4, \mathbf{R})$ As a maximal compact subgroup, we take

$$K := G \cap O(4) = \left\{ k_{A,B} := \left(\begin{array}{c|c} A & B \\ \hline -B & A \end{array} \right) \in G \right\}.$$

Fix an isomorphism $\kappa : U(2) \cong K$ by $A + \sqrt{-1}B \mapsto k_{A,B}$. As a Cartan subgroup T of K, we can take

$$T = \left\{ t = \kappa\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \right) \mid t_i \in \mathbf{C}^{(1)} \right\}.$$

The character group $\widehat{T}$ of T is given by

$$\widehat{T} = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2, \qquad e_i(\kappa(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix})) = t_i \ (i = 1, 2).$$

We frequently write (a_1, a_2) in place of $a_1e_1 + a_2e_2 \in \widehat{T}$ $(a_i \in \mathbf{Z})$. Then we have

$$\Delta = \Delta(G, T) = \{\pm(2,0), \pm(1,1), \pm(0,2), \pm(1,-1)\}, \quad \Delta_c = \{\pm(1,-1)\}$$

$$W_G \cong (\mathbf{Z}/2\mathbf{Z})^2 \rtimes S_2, \quad W_K \cong S_2.$$

We fix a positive system Δ_c^+ of Δ_c as $\Delta_c^+ = \{(1,-1)\}$. The parameter set Ξ is a disjoint union of $\Xi^{p,q}$ $(p, q \geq 0, p + q = 3)$ below:

$$\Xi^{3,0} := \{\lambda = (\lambda_1, \lambda_2) \in \widehat{T} \mid \lambda_1 > \lambda_2 > 0\};$$
$$\Xi^{2,1} := \{\lambda = (\lambda_1, \lambda_2) \in \widehat{T} \mid \lambda_1 > -\lambda_2 > 0\};$$
$$\Xi^{1,2} := \{\lambda = (\lambda_1, \lambda_2) \in \widehat{T} \mid -\lambda_2 > \lambda_1 > 0\};$$
$$\Xi^{0,3} := \{\lambda = (\lambda_1, \lambda_2) \in \widehat{T} \mid 0 > \lambda_1 > \lambda_2\}.$$

The Blattner parameter Λ is given by

$$\lambda \in \Xi^{3,0} \Rightarrow \Lambda = \lambda + (1,2); \qquad \lambda \in \Xi^{2,1} \Rightarrow \Lambda = \lambda + (1,0);$$
$$\lambda \in \Xi^{1,2} \Rightarrow \Lambda = \lambda + (0,-1); \qquad \lambda \in \Xi^{0,3} \Rightarrow \Lambda = \lambda + (-2,-1).$$

For $\lambda \in \Xi$, we set $\lambda^\vee := (-\lambda_2, -\lambda_1)$. Then the contragredient representation $\pi_\lambda^\vee$ of π_λ is given by $\pi_{\lambda^\vee}$. Note that if $\lambda \in \Xi^{3,0}$ (resp. $\Xi^{2,1}$) then $\lambda^\vee \in \Xi^{0,3}$ (resp. $\lambda^\vee \in \Xi^{1,2}$). The discrete series representations of $G = Sp(4, \mathbf{R})$ are divided into two classes:

- $\lambda \in \Xi^{3,0} \cup \Xi^{0,3} \Rightarrow \pi_\lambda$ is called an **(anti-)holomorphic** discrete series representation.
- $\lambda \in \Xi^{2,1} \cup \Xi^{1,2} \Rightarrow \pi_\lambda$ is called a **large** discrete series representation.

One important property of large discrete series representations π_λ is that they have Whittaker models. That is, for a maximal unipotent subgroup N_0 of G and its non-degenerate character ψ, the intertwining space $\mathrm{Hom}_{\mathfrak{g},K}(\pi_\lambda, C_\psi^\infty(N_0\backslash G))$ is non-zero, where we set

$$C_\psi^\infty(N_0\backslash G) := \{W : G \xrightarrow{C^\infty} \mathbf{C} \mid W(ng) = \psi(n)W(g), \quad \forall (n,g) \in N_0 \times G\}.$$

A realization of π_λ in $C_\psi^\infty(N_0\backslash G)$ is called a Whittaker model of π_λ. On the other hand, (anti-)holomorphic discrete series representations do not have Whittaker models.

(+) Langlands parameter, $W_{\mathbb{R}} = \mathbb{C}^\times \sqcup \mathbb{C}^\times \cdot j$ $(jz = \bar{z}j, j^2 = -1)$

Examples. Define two dimensional representations $\phi_{\mu,N} : \underline{W_{\mathbf{R}} \to GL(2, \mathbf{C})}$ $(\mu \in \mathbf{C}, N \in \mathbf{Z}_{\geq 0})$ as follows:

$$\phi_{\mu,N}(re^{\sqrt{-1}\theta}) = \begin{pmatrix} r^{2\mu-N}e^{-\sqrt{-1}N\theta} & 0 \\ 0 & r^{2\mu-N}e^{+\sqrt{-1}N\theta} \end{pmatrix}, \qquad \phi_{\mu,N}(j) = \begin{pmatrix} 0 & (-1)^N \\ 1 & 0 \end{pmatrix}.$$

The representation $\phi_{\mu,N}$ is irreducible if $N > 0$. The L- and ϵ-factors corresponding to the representation $\phi_{\mu,N}$ $(N > 0)$ are defined by

$$L(s, \phi_{\mu,N}) := \Gamma_{\mathbf{C}}(s + \mu), \quad \epsilon(s, \phi_{\mu,N}, \psi_\infty) := (\sqrt{-1})^{N+1}.$$

chose (f)

choose (2)

this normalization is the same as we did before: $\ell \leftrightarrow N$, $t = \frac{2\mu + N}{2}$.

$\boxed{G = GL(2,\mathbf{R})}$ In this case, $\mathsf{G}^\vee \cong GL(2,\mathbf{C})$. We denote by $\pi = D_k[c]$ $(k \geq 1, c \in \mathbf{C})$ the irreducible admissible representation of $GL(2,\mathbf{R})$ characterized by $\pi|_{SL(2,\mathbf{R})} \cong D_k^+ \oplus D_k^-$ and $\pi(zI_2) = z^c \times \mathrm{id}$ $(z > 0)$. Then we have

L-packet: fibre of $\widehat{G}_{adm} \to \Phi(G)$.

$$\Pi_{\phi_{\mu,N}}(GL(2,\mathbf{R})) = \{D_k[c]\}, \quad \text{with } \mu = (c+k-1)/2,\ N = k-1.$$

$\boxed{G = GSp(4,\mathbf{R})}$ In this case, $\mathsf{G}^\vee \cong GSp(4,\mathbf{C})$. Let $\pi_\lambda[c]$ be the representation of $GSp(4,\mathbf{R})$ with $\pi_\lambda[c]|_{Sp(4,\mathbf{R})} \cong \pi_\lambda \oplus \pi_\lambda^\vee$ and $\pi(zI_4) = z^c \times \mathrm{id}.$ $(z > 0)$. For $\lambda_1 > \lambda_2 > 0$, we consider the Langlands parameter $\phi : W_\mathbf{R} \to \mathsf{G}^\vee = GSp(4,\mathbf{C})$ defined by

$$\phi(w) = \left(\begin{array}{cc|cc} a_1 & & b_1 & \\ & a_2 & & b_2 \\ \hline c_1 & & d_1 & \\ & c_2 & & d_2 \end{array}\right) \quad \text{with} \quad \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = \phi_{\mu_i, N_i}(w) \quad (i = 1, 2),$$

where we set

$$\mu_1 = (c + \lambda_1 - \lambda_2)/2, \quad N_1 = \lambda_1 - \lambda_2; \quad \mu_2 = (c + \lambda_1 + \lambda_2)/2, \quad N_2 = \lambda_1 + \lambda_2.$$

Then the corresponding L-packet consists of two elements:

$$\Pi_\phi(GSp(4,\mathbf{R})) = \{\pi_{(\lambda_1,\lambda_2)}[c], \pi_{(\lambda_1,-\lambda_2)}[c]\}.$$

• spinor L-function: Let $r : \mathsf{G}^\vee \hookrightarrow GL(4,\mathbf{C})$ be a natural inclusion. For $\pi \in \Pi_\phi(GSp(4,\mathbf{R}))$, we have

$$L(s, r \circ \phi) = \Gamma_\mathbf{C}(s + \frac{c + \lambda_1 + \lambda_2}{2})\Gamma_\mathbf{C}(s + \frac{c + \lambda_1 - \lambda_2}{2});$$
$$\epsilon(s, r \circ \phi, \psi_\infty) = (-1)^{\lambda_1 + 1}.$$

Remark. (i) $\phi(W_\mathbf{R})$ is not contained in any proper Levi subgroups of $\mathsf{G}^\vee$.
$\Leftrightarrow \forall \pi \in \Pi_\phi(G)$ belongs to the discrete series (modulo center).
(ii) $\phi(W_\mathbf{R})$ is bounded $\Leftrightarrow \Pi_\phi(G) \subset \widehat{G}_{temp}$.

(5) Automorphic forms on $G = Sp(4,\mathbb{R})$ of discrete series type.

(i) space of automorphic forms: Γ: congruence subgp. of

def'n $\mathcal{A}(\Gamma\backslash G) := \left\{ f : G \xrightarrow{C^\infty} \mathbb{C} \;\middle|\; \begin{array}{l} (i)\ f(\gamma g) = f(g),\ (\gamma, g) \in \Gamma \times G \\ (ii)\ f:\ \text{right } K\text{-fin. } Z(\mathfrak{g})\text{-fin.} \\ (iii)\ f \text{ is of moderate growth} \end{array} \right\}.$

If $f \in \mathcal{A}(\Gamma\backslash G)$, satisfies

$$\int_{\Gamma \cap N_S \backslash N_S} f(ng)\, dn \equiv 0 \quad \text{and} \quad \int_{\Gamma \cap N_J \backslash N_J} f(ng)\, dn \equiv 0$$

we say f is a <u>cusp form</u>. $\rightsquigarrow \mathcal{A}^{cusp}(\Gamma\backslash G)$.

Let $\pi = \pi_\lambda$, discrete series rep'n of G, with minimal K-type $\tau_\lambda = (\tau, V_\tau)$.

$\tau \hookrightarrow \pi$ (unique) gives

dual

$H^\bullet(\mathfrak{g}, \mathcal{A}(\Gamma\backslash G)) \to H^\bullet(\mathfrak{g}, \mathcal{A}(\Gamma\backslash G)) \cong \mathcal{A}(\Gamma\backslash G)^{\vee} \otimes V_\tau^{K}$

$\mathrm{Hom}_{\mathfrak{g},K}(\pi_\lambda, \mathcal{A}(\Gamma\backslash G)) \cong \mathrm{Hom}_K(\tau, \mathcal{A}(\Gamma\backslash G)) \cong \mathcal{A}(\Gamma\backslash G)\otimes V_\tau\}$

$\Phi \longmapsto f : [g \mapsto \sum_i \Phi(v_i)(g)\otimes v_i^\vee]$

where $\{v_i\}$: basis of τ, $\{v_i^\vee\}$ dual basis of $\tau^\vee$.

- $\mathbf{H}_2 := \{z \in M_2(\mathbf{C}) \mid {}^t z = z, \mathrm{Im}(z) \gg 0\}$: the Siegel upper half space of degree 2.
- The action of $G = Sp(4, \mathbf{R})$ on $\mathbf{H}_2$ is given by

$$g\langle z\rangle := (az+b)(cz+d)^{-1} \in \mathbf{H}_2, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, z \in \mathbf{H}_2.$$

(complex structure)

- Identify $X = G/K$ with $\mathbf{H}_2$ through $X \ni gK \mapsto g\langle\sqrt{-1}\rangle \in \mathbf{H}_2$. The holomorphic (resp. anti-holomorphic) part $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) of the complexified tangent space $T_{\bar{e}}(X)\otimes\mathbf{C}$ at $\bar{e} = eK \in X$ is given by

$$\mathfrak{p}_\pm = \left\{ \begin{pmatrix} A & \pm\sqrt{-1}A \\ \pm\sqrt{-1}A & A \end{pmatrix} \mid A = {}^tA \in M_2(\mathbf{C}) \right\}.$$

why this condition?

For $\Lambda = (\Lambda_1, \Lambda_2) \in \mathbf{Z}^2$ satisfying $\boxed{\Lambda_1 \geq \Lambda_2 \geq 0}$, we set

$$M_\Lambda^{holo}(\Gamma\backslash G) := \{f : \Gamma\backslash G \to V_\tau^\vee \mid C^\infty,\ f(g;x) = 0,\ (\forall x \in \mathfrak{p}_-),$$
$$f(gk) = \tau^\vee(k)^{-1} f(g)\ (\forall (g,k) \in G\times K)\};$$

$$M_\Lambda(\Gamma\backslash X) := \{\phi : X \to V_\tau^\vee \mid \text{holomorphic}, \quad \phi|_\gamma(z) = \phi(z),\ (\forall\gamma\in\Gamma)\},$$

where $\tau = \tau_\Lambda \in \widehat{K}$. Here we use the usual notation:

- $\phi|_\gamma(z) := \tau_\Lambda^\vee(\kappa({}^t j(\gamma,z)^{-1}))^{-1}\phi(\gamma\langle z\rangle)$;

$K : U(2) \cong K := G \cap O_4 = \{k_{A,B} := \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G\}$

$A + \sqrt{-1}B \mapsto k_{A,B}$

$(A^2+B^2 = 1,\ BA = AB)$

- $j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) := cz + d, \qquad (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \in G \times \mathbf{H}_2,$

$cz+d \in U(2)$?

- $K_\mathbf{C}$: the analytic subgroup of $Sp(\cancel{2}4, \mathbf{R})$ corresponding to $\mathfrak{k}_\mathbf{C}$;
- $\kappa : GL(2,\mathbf{C}) \cong K_\mathbf{C}$ is the holomorphic extension of $\kappa : U(2) \cong K$;
- $\tau^\vee : K_\mathbf{C} \to GL(V_\tau^\vee)$ is the holomorphic extension of $\tau^\vee : K \to GL(V_\tau^\vee)$;

rmk: set $C_G := \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1}/2 & 1/2 \end{pmatrix} \in G_\mathbf{C}$, then set $K_\mathbf{C}' = C_G K_\mathbf{C} C_G^{-1}$,
we have $K_\mathbf{C}' = \{\begin{pmatrix} m & 0_2 \\ 0_2 & {}^t m^{-1} \end{pmatrix} \mid m \in GL_2(\mathbf{C})\}$

- $f(g; x' + \sqrt{-1}x'') := \frac{d}{dt}\Big|_{t=0} [f(g\exp(tx')) + \sqrt{-1} f(g\exp(tx''))]$ $\quad x', x'' \in \mathfrak{g}$

<Thm>: suppose $\Lambda_1 \geq \Lambda_2 \geq 3$. If we set $\lambda = (\Lambda_1 - 1, \Lambda_2 - 2)$, then

$$\mathrm{Hom}_{\mathfrak{g},K}(\pi_\lambda, \mathcal{A}(\Gamma\backslash G)) \cong M_\Lambda^{holo}(\Gamma\backslash G) \cong M_\Lambda(\Gamma\backslash X)$$
$$\Phi \longmapsto f_\Phi$$

pf: the second translation ("isomorphism") is classical. For C^∞ function $f : G \to V_\tau^\vee$ satisfying $f(gk) = \tau^\vee(k)^{-1}f(g)$ define a C^∞-section Φ of the vector bundle $G\times V_\tau^\vee \to X$

section s_f of the homogeneous vect. bundle $G\times_K V_\tau \to X$:

$$s_f: X\ni gK \mapsto [g, f(g)] \in G\times_K V_\tau^\vee$$

Set $G_{\mathbb{C}} := Sp_4(\mathbb{C})$. $P_\pm = \exp(\mathfrak{p}_\pm)$, then $P_+K_{\mathbb{C}}P_-$ is an open of $G_{\mathbb{C}}$, and

$$P_+\times K_{\mathbb{C}}\times P_- \ni (P_+, u, P_-) \mapsto P_+uP_- \in P_+K_{\mathbb{C}}P_- \text{ is}$$

bi-holo. Since we have $c_G^{-1}G \subseteq P_+K_{\mathbb{C}}P_-$, through the map

$$G\times_K V_\tau^\vee \ni [g,\xi] \mapsto [c_G^{-1}g, \xi] \in P_+K_{\mathbf{C}}P_- \times_{K_{\mathbf{C}}P_-} V_\tau^\vee,$$

the vector bundle $G\times_K V_\tau^\vee$ on X can be regarded as a pull-back of the *holomorphic* vector bundle $P_+K_{\mathbf{C}}P_- \times_{K_{\mathbf{C}}P_-} V_\tau^\vee$ on P_+. Here P_- acts on $V_\tau^\vee$ trivially. Since $P_+K_{\mathbf{C}}P_- \times_{K_{\mathbf{C}}P_-} V_\tau^\vee \cong \underline{P_+\times V_\tau^\vee}$, the C^∞-section s_f defined above gives rise to a C^∞-function $\phi_f : X \to V_\tau^\vee$:

$k \mapsto \tau^\vee(k)$

action

$$\boxed{\phi_f(g\langle\sqrt{-1}\rangle) := \tau^\vee((c_G^{-1}g)_0)f(g).}$$

$V_\tau^\vee$

Here $(c_G^{-1}g)_0$ is the image of $c_G^{-1}g$ through the projection $P_+K_{\mathbf{C}}P_- \to K_{\mathbf{C}}$. To compute $(c_G^{-1}g)_0$, it is convenient to introduce $\underline{P'_\pm = c_G P_\pm c_G^{-1}}$ and $\underline{K'_{\mathbf{C}} = c_G K_{\mathbf{C}} c_G^{-1}}$. It is easy to check that

$$P'_+ = \{\begin{pmatrix} 1_2 & x \\ 0_2 & 1_2\end{pmatrix} \mid x = {}^t x \in M_2(\mathbf{C})\}, \quad P'_- = \{\begin{pmatrix} 1_2 & 0_2 \\ x & 1_2\end{pmatrix} \mid x = {}^t x \in M_2(\mathbf{C})\}$$

$$K'_{\mathbf{C}} = \{\begin{pmatrix} m & 0_2 \\ 0_2 & {}^t m^{-1}\end{pmatrix} \mid m \in GL(2,\mathbf{C})\}.$$

By a simple computation, we have

$$gc_G^{-1} = \underbrace{\begin{pmatrix} 1_2 & g\langle\sqrt{-1}\rangle \\ 0_2 & 1_2\end{pmatrix}}_{P'_+} \underbrace{\begin{pmatrix} {}^t(c\sqrt{-1}+d)^{-1} & 0_2 \\ 0_2 & c\sqrt{-1}+d\end{pmatrix}}_{K'_{\mathbf{C}}} \underbrace{\begin{pmatrix} 1_2 & 0_2 \\ * & 1_2\end{pmatrix}}_{P'_-}, \quad \forall g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

By $c_G K(m) c_G^{-1} = \begin{pmatrix} m & 0_2 \\ 0_2 & {}^t m^{-1}\end{pmatrix} \rightsquigarrow (c_G^{-1}g)_0 = K({}^t j(g,\sqrt{-1})^{-1})$, therefore

$$\phi_f(g\langle\sqrt{-1}\rangle) = \tau^\vee(K({}^t j(g,\sqrt{-1})^{-1}))f(g)$$

Now it is easy to see that $\phi_f|_\gamma(z) = \phi_f(z)$ for all $\gamma \in \Gamma$. To complete the proof, we have to show that the holomorphy of $\phi_f(z)$ is equivalent to the condition $f(g;x) = 0$ $(\forall x \in \mathfrak{p}_-)$. Since the anti-holomorphic part of $T_{gK}(X)\otimes\mathbf{C}$ is given by $L_g(\mathfrak{p}_-)$, the function $\phi_f(z)$ is holomorphic at $gK \in X$ if and only if

$$\frac{d}{dt}_{|t=0}[\phi_f(g\exp(tx')K) + \sqrt{-1}\phi_f(g\exp(tx'')K)] = 0 \qquad \forall x = x' + \sqrt{-1}x'' \in \mathfrak{p}_-.$$

Pf:

$$\frac{d}{dt}_{|t=0}[\phi_f(g\exp(tx')K) + \sqrt{-1}\phi_f(g\exp(tx'')K)]$$

$$=\tau^\vee((c_G^{-1}g)_0)f(g;x'+\sqrt{-1}x'')$$

(defn of ϕ_f: $g\langle\cdot\rangle \mapsto \tau^\vee((c_G^{-1}g)_0)f(g)$)

$$+\frac{d}{dt}_{|t=0}[\tau^\vee((c_G^{-1}g\exp(tx'))_0)f(g) + \sqrt{-1}\tau^\vee((c_G^{-1}g\exp(tx''))_0)f(g)]$$

$$=\tau^\vee((c_G^{-1}g)_0)f(g;x) + \underline{\frac{d}{dt}_{|t=0}[\tau^\vee((c_G^{-1}g\exp(tx))_0)]f(g)} = 0, \text{ since } \tau^\vee \text{ is holomorphic}$$

$$=\tau^\vee((c_G^{-1}g)_0)f(g;x).$$

$\square$