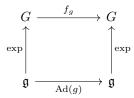
In particular, G acts on itself by conjugation, and so it acts on $\mathfrak{g} = T_e(G)$. This representation is called the *adjoint representation* and is denoted as Ad. First, The following diagram is commutative.

$$\begin{array}{ccc}
G & \xrightarrow{f} & H \\
\exp & & & \uparrow \\
 & & & \downarrow \\
 & \downarrow \\$$

Let H = G, fix $g \in G$, $f_g(h) = ghg^{-1}$. $df : \mathfrak{g} \to \mathfrak{g}$ is denoted Ad(g).



(We have $\exp(\operatorname{Ad}(g)(X)) = g(\exp(X))g^{-1}$) We should confirmed that Ad is a smooth homomorphism from G into $GL(\mathfrak{g})$.

We show next that the differential of Ad is ad (ad $x(y) = [x, y], x, y \in \mathfrak{g}$):

$$G \xrightarrow{\operatorname{Ad}} \operatorname{GL}(\mathfrak{g})$$

$$\stackrel{\exp}{\longrightarrow} \stackrel{\exp}{\longrightarrow} \operatorname{End}(\mathfrak{g})$$

$$T_e(G) = \mathfrak{g} \xrightarrow{\operatorname{ad}} \operatorname{End}(\mathfrak{g})$$

Proof. It will be most convenient for us to think of elements of the Lie algebra as tangent vectors at the identity or as local derivations of the local ring there. Let $X, Y \in \mathfrak{g}$. If $f \in C^{\infty}(G)$. Then through our definition:

$$(\operatorname{Ad}(g)Y)f = Y(g.f).$$

To compute the differential of Ad, note that the path $t \to \exp(tX)$ in G is tangent to the identity at t = 0 with tangent vector X.

$$f \to \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{Ad} \left(\mathrm{e}^{tX} \right) Y \right) f|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathrm{d}}{\mathrm{d}u} f \left(\mathrm{e}^{tX} \mathrm{e}^{uY} \mathrm{e}^{-tX} \right) |_{t=u=0}.$$

By the chain rule, our last expression equals XYf(1) - YXf(1).

Consequently $\operatorname{Ad}(\exp X) = e^{\operatorname{ad}X}$ under this identification. In the special case that G is a closed linear map, we can regard X and g as matrics. Note that:

$$\exp(\operatorname{Ad}(q)(tX)) = q(\exp(tX))q^{-1}$$

differentiate and set t = 0. Then we see that $Ad(g)X = gXg^{-1}$ (it has no specific meaning except equal in amount). If we denote $g \in G$ as $\exp(Y)$, $Y \in \mathfrak{g}$, then combine the equations before:

$$(e^{\operatorname{ad}(Y)})X = \exp(Y)X\exp(-Y), \quad X, Y \in \mathfrak{g}$$

This equation is essential in Lie algebra theory and can be proved directly. (Humphreys pg.9)