LOCAL GAN-GROSS-PRASAD CONJECTURE

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Contents

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1. Introduction

In [GP92],[GP94], Gross and Prasad studied a restriction problem for special orthogonal groups over local fields and gave a precise conjecture. They and Gan ([GGP12]) extended this conjecture to classical groups over local fields, which are called the local Gan-Gross-Prasad conjecture (GGP).

This conjecture consists of four cases: two Bessel cases are of orthogonal and the hermitian, and two Fourier-Jacobi cases are of symplectic-metaplectic and skew-hermitian. This conjecture is now a well-established theorem for classical groups and generic local L-parameters in a lot of cases. More precisely, it is proved for all p-adic cases by J.-L. Waldspurger ([Wal10],[Wal12a],[Wal12b], on the tempered case) and Mœglin and J-L Waldspurger ([MgW12], on the generic case) in the orthogonal case, Beuzart-Plessis ([BP14],[BP16]) on the tempered hermitian case, Gan-Ichino ([GI16]) on the generic skew-hermitian case and H. Atobe ([Ato18]) on the generic symplectic-metaplectic case. In real place, Beuzart-Plessis ([BP20]) and H.Xue ([Xu23] on epsilon dichotomy) proved the conjecture for Hermitian case with tempered L-packet.

The setting of the specific GGP question we will study in this paper (which is called the Bessel model: tempered Hermitian case) is as follows. Let F be a local field of characteristic 0 which is different from $\mathbb{C}$. So, F is either a p-adic field (that is a finite extension of $\mathbb{Q}_p$) or $F = \mathbb{R}$. Let E/F be a quadratic extension of F (if $F = \mathbb{R}$, we have $E = \mathbb{C}$) and let $W \subset V$ be a pair of hermitian spaces over E having the following property: the orthogonal complement $W^\perp$ of W in V is odd-dimensional and its unitary group $U(W^\perp)$ is quasi-split. To such a pair, Gan, Gross and Prasad associate a triple (G, H, ξ). Here, G is equal to the product $U(W) \times U(V)$ of the unitary groups of W and V, H is a certain algebraic subgroup of G and $\xi : H(F) \to \mathbb{C}^\times$ is a continuous character of the F-points of H. For example, in the case where $\dim(W^\perp) = 1$, we just have $H = U(W)$ embedded in G diagonally and the character ξ is trivial. For the definition in codimension greater than 1, we refer the reader to Section 2.2. We call a triple like (G, H, ξ) (constructed from the pair (W, V)) a GGP triple. We will formulate a similar GGP triple for the complete Bessel model, see Section 2.4.

Let π a tempered irreducible representation of $G(F)$. We denote by π^∞ the subspace of smooth vectors in π. This subspace is $G(F)$-invariant and carries a natural topology. Following Gan, Gross and Prasad, we define a multiplicity $m(\pi)$ by

$$m(\pi) = \dim \mathrm{Hom}_H(\pi^\infty, \xi)$$

(where the Fourier-Jacobi cases are of a little difference). Here $\mathrm{Hom}_H(\pi^\infty, \xi)$ denotes the space of continuous linear forms ℓ on π^∞ satisfying the relation $\ell \circ \pi(h) = \xi(h)\ell$ for all $h \in H(F)$. By the main result of [JSZ10] (in the real case) and [AGRS10] (in the p-adic case) together with Theorem 15.1 of [GGP12] (which descent the higher codimension case to codimenstion one), we know that this multiplicity is always less or equal to 1.

Following the initial ideas of Gross-Prasad for orthogonal case in [GP94], Gan-Gross-Prasad extends the multiplicity one theorem to a whole L-packet of tempered representations of $G(F)$, see Conjecture 17.1 of [GGP12]. Actually, the result is better stated if we consider more than one GGP triple at the same time. In any families of GGP triples that we are going to consider, It is natural find out a pair (G^*,H^*) such that both are quasi-split over F, as "base point" in the inner class. So, for convenience, we now make the following additional assumption

$$\underline{G \text{ and } H \text{ are quasi-split}}$$

The other GGP triples that we need to consider are called the pure inner forms of (G, H, ξ)[1]. Those are naturally parametrized by the Galois cohomology set $H^1(F, H)$. A cohomology class $\alpha \in H^1(F, H)$ corresponds to a hermitian space W_α (up to isomorphism) of the same dimension as W. If we set $V_\alpha = W_\alpha \oplus^\perp W^\perp$, then (W_α, V_α) is a pair of hermitian spaces over E which can give rise to a new GGP triple $(G_\alpha, H_\alpha, \xi_\alpha)$. The pure inner forms of (G, H, ξ) are exactly all the GGP triples obtained in this way.

Let φ be a *tempered* Langlands parameter for G. According to the local Langlands correspondence (which is now known in all cases for unitary groups, cf. [KMSW14] and [Mok15]), this parameter determines an L-packet $\Pi^G(\varphi)$ consisting of a *finite* number of tempered representations of $G(F)$. Actually, this parameter also defines L-packets $\Pi^{G_\alpha}(\varphi)$ of tempered representations of $G_\alpha(F)$ for all $\alpha \in H^1(F, H)$. We can now state the main result of this paper as follows.

Theorem 1.1. (i) *There exists exactly one representation π_α in the disjoint union of L-packets*

$$\bigsqcup_{\alpha \in H^1(F,H)} \Pi^{G_\alpha}(\varphi)$$

such that $m(\pi_\alpha) = 1$.

(ii) *Having fixed the Langlands-Vogan parameterization for the unitary group G in hermitian case, and its pure inner forms in the various cases above, the unique representation π_α in the Vogan packet Π_φ which satisfies $\mathrm{Hom}_H(\pi \otimes \overline{\xi}, \mathbb{C}) \neq 0$ enjoys the following characterization:*

$$\pi_\alpha = \pi(\varphi, \chi_\varphi).$$

where χ_φ is defined in Section 3.6.

The goal of this paper is to explain the local Gan-Gross-Prasad conjecture of Bessel model, and give a compelete proof of **Theorem 1.1.**(i) in the hermitian tempered case at both archimedean and non-archimedean places. The proof follows the procedure of Beuzart-Plessis ([BP20]). To save the length of the paper, many technical details inside [BP20], especially the beautiful sorts of local trace formula which is related to the local Gan-Gross-Prasad conjecture, are omitted.

First we will give a brief introduction to our final proof. Fix a tempered representation π of $G(F)$. First consider the L^2-induction of the character ξ from $H(F)$ to $G(F)$ and it consists in the measurable functions $\varphi : G(F) \to \mathbb{C}$ satisfying the relation $\varphi(hg) = \xi(h)\varphi(g)$ ($h \in H(F)$, $g \in G(F)$) almost everywhere and such that

$$\int_{H(F)\backslash G(F)} |\varphi(x)|^2 dx < \infty$$

It is an unitary representation R of $G(F)$ via right translation. Since the triple (G, H, ξ) is of a very particular form, the direct integral decomposition of $L^2(H(F)\backslash G(F), \xi)$ only involves tempered representations and moreover an irreducible tempered representation

[1]Recall that for brd_F the functor from the groupoid of connected reductive group over F to the groupoid of based root data with continuous action of $\Gamma = \mathrm{Gal}(\bar{F}/F)$. For G a connected reductive group over F, the cohomology group $H^1(F, G_{\mathrm{ad}})$ parametrized the set of isomorphism classes of the groupoid fiber of $\mathrm{brd}_F(G)$. All these fibers G' with $\psi : G'_{\bar{F}} \cong G_{\bar{F}}$ are called the inner forms of G. Fix a quasi-split group G^*. Now consider the groupoid of triples (G, ψ, π) where (G, ψ) is an inner form of G^* and π an irreducible smooth representation of $G(F)$. Since for an inner form, its automorphism group in $\mathcal{IT}(G^*)$ (cf. [KT20, Definition 4.1], which is a groupoid isomorphic to $Z^1(F, G^*_{\mathrm{ad}})$) is $G_{\mathrm{ad}}(F)$ (cf. [KT20, Remark 4.2]). The class of pure inner forms classifies those inner forms which the group $G_{\mathrm{ad}}(F)$ acts trivially on the set of isomorphism classes of irreducible smooth representations of $G(F)$.

π of $G(F)$ appears in this decomposition if and only if $m(\pi) = 1$. Thus to study the multiplicity space $\mathrm{Hom}_H(\pi, \xi)$, by Frobenius reciprocity and above discussion:

$$\mathrm{Hom}_H(\pi^\infty|_H, \xi) = \mathrm{Hom}_G(\pi, L^2(H(F)\backslash G(F), \xi))$$

it is equivalent to study the spectral decompositon of $L^2(H\backslash G, \xi)$ as the right translation of $G(F)$.
function $f \in C_c^\infty(G(F))$ naturally acts on this space by

$$(R(f)\varphi)(x) = \int_{G(F)} f(g)\varphi(xg)dg, \quad \varphi \in L^2(H(F)\backslash G(F), \xi),\ x \in G(F)$$

Moreover, this operator $R(f)$ is a kernel operator. More precisely, we have

$$(R(f)\varphi)(x) = \int_{H(F)\backslash G(F)} K_f(x,y)\varphi(y)dy, \quad \varphi \in L^2(H(F)\backslash G(F), \xi),\ x \in G(F)$$

where

$$K_f(x,y) = \int_{H(F)} f(x^{-1}hy)\xi(h)dh, \quad x, y \in G(F)$$

is the kernel function associated to f. In order to study the representation $L^2(H(F)\backslash G(F), \xi)$, we would like to compute the trace of $R(f)$. Unfortunately, the usual definition of trace, which is the integration of the kernel function along the diagonal:

$$J(f) := \int_{H(F)\backslash G(F)} K_f(x,x)dx$$

makes no sense here: the operator $R(f)$ is not generally of trace class[2], and the integral is not usually convergent. The first main step towards the proof is to prove that the integration $J(f)$ is absolutely convergent for a "wide range" of fucntions, $\mathcal{C}_{\mathrm{scusp}}(G(F))$, namely the strongly cuspidal function in $\mathcal{C}(G(F))$ (see the definition in Appendix C). It is proved in [BP20, Chapter 8].

Two different expressions for the distribution $J(\cdot)$ leads us the main information.

Spectral Side. This part is more standard, and works verbatim to other classical groups. Let $\mathcal{X}(G)$ be the set of virtual tempered representations built from Arthur's elliptic representations of G and its Levi subgroups (for details, see Appendix C: on Elliptic Representations.). For all $\pi \in \mathcal{X}(G)$, define a weighted character

$$f \in \mathcal{C}(G(F)) \mapsto J_{M(\pi)}(\pi, f)$$

Here, $M(\pi)$ denotes the Levi subgroup such that π is parabolically induced from an elliptic representation of $M(\pi)$. By definiton, this character depends on the some auxiliary choices (such as the choice of maxmimal compact subgroup K and some normalizations), however, it can be shown that the restriction of $J_{M(\pi)}(\pi, .)$ to $\mathcal{C}_{\mathrm{scusp}}(G(F))$ doesn't depend on any of these choices. For $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$, we define a function $\widehat{\theta}_f$ on $\mathcal{X}(G)$ by

$$\widehat{\theta}_f(\pi) = (-1)^{a_{M(\pi)}} J_{M(\pi)}(\pi, f), \quad \pi \in \mathcal{X}(G)$$

[2] For a separable Hilbert space $\mathcal{H}$, a continuous operator is *of trace class* if for some orthogonal basis $\mathcal{E}$ of $\mathcal{H}$,

$$\sum_{e\in\mathcal{E}} \langle |T|e, e\rangle < \infty.$$

One theorem says for the operator of trace class, the sum

$$\sum_{e\in\mathcal{E}} \langle Te, e\rangle$$

is finite and independent of the choice of orthogonal basis $\mathcal{E}$.

where $a_{M(\pi)}$ is the dimension of $A_{M(\pi)}$ the maximal central split subtorus of $M(\pi)$. Let $D(\pi)$ be the discriminant for elliptic representations, see also Appendix C: on Elliptic Representations. The spectral expansion of the distribution $J(.)$ now reads as follows

Theorem 1.2. *For every strongly cuspidal function $f \in \mathcal{C}_{\text{scusp}}(G(F))$, we have*

$$J(f) = \int_{\mathcal{X}(G)} D(\pi)\widehat{\theta}_f(\pi)m(\pi)d\pi.$$

o The reason for the appearance of $m(\pi)$ here is the explicit description of the Hom space. Let π an irreducible tempered representation of $G(F)$, we may define a certian linear form $\mathcal{L}_\pi \to \mathbb{C}$, by:

$$\mathcal{L}_\pi(e, e') = \int^*_{H(F)} (e, \pi(h)e')\xi(h)dh, \quad e, e' \in \pi^\infty.$$

It is continuous and satisfies the intertwining relation

$$\mathcal{L}_\pi(\pi(h)e, \pi(h')e') = \xi(h)\overline{\xi(h')}\mathcal{L}_\pi(e, e'), \quad e, e' \in \pi^\infty, \ h, h' \in H(F)$$

In particular, we see that for all $e' \in \pi^\infty$ the linear form $e \in \pi^\infty \mapsto \mathcal{L}_\pi(e, e')$ belongs to $\mathrm{Hom}_H(\pi^\infty, \xi)$. Hence, if $\mathcal{L}_\pi$ is not zero so is $m(\pi)$. In fact, the converse is also true (see [BP20, Chapter 7]). This function appears naturally in $J(f)$ (see Lemma 4.4 Proposition 4.5).

Geometric Side. We now come to the geometric expansion of $J(\cdot)$. We first give the statement of the geometric expansion.

Theorem 1.3. *For all strongly cuspidal functions $f \in \mathcal{C}_{\text{scusp}}(G(F))$, we have*

$$J(f) = \lim_{s \to 0^+} \int_{\Gamma(G,H)} c_f(x) D^G(x)^{1/2} \Delta(x)^{s-1/2} dx,$$

where for all $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$, the expression of the right hand side of the equality is absolutely convergent, and the limit exists.

Here $\Gamma(G, H)$ is a subset of semi-simple conjugacy classes of H, which we will construct carefully in Section 4.2. D^G and Δ are some discriminants (defiend in 4.6). The key ingredient here is a function $c_f : G_{\text{ss}}(F) \to \mathbb{C}$. This function coincides with the character of π on regular elements. The definition of c_f involves the weighted orbital integrals of Arthur which is defined in 4.1. Recall that for every Levi subgroup M of G and all $x \in M(F) \cap G_{\text{reg}}(F)$, Arthur has defined a certain distribution

$$f \in \mathcal{C}(G(F)) \mapsto J_M(x, f)$$

called a weighted orbital integral. We define a function θ_f on $G_{\text{reg}}(F)$ by

$$\theta_f(x) = (-1)^{a_{M(x)}} J_{M(x)}(x, f), \quad x \in G_{\text{reg}}(F)$$

where $M(x)$ denotes the minimal Levi subgroup of G containing x. The term $\theta_f(x)$ is of conjugation invariant, and a quasi-character (see Appendix C), in the sense that it has germ expansion as the distribution character of admissible representations. In particular, c_f is its natural extension to G_{ss}.

It is from the equality between the two expansions that we deduce the formula for the multiplicity. To be precise, we first show the spectral expansion, and then we process to show the geometric expansion and the formula for the multiplicity in a common inductive proof, in 4.7. The main reason for processing this way is that we use the spectral expansion together with the multiplicity formula for some "smaller" GGP triples to study $J(\cdot)$.

To continue, the calculation of

$$\sum_{\alpha\in H^1(F,H)}\sum_{\pi\in\Pi^{G_\alpha}(\varphi)} m(\pi)$$

transfers to the expression:

$$\lim_{s\to 0^+}\int_{\Gamma(G_\alpha,H_\alpha)} c_{\varphi,\alpha}(x)D^{G_\alpha}(x)^{1/2}\Delta(x)^{s-1/2}dx$$

where we have set $c_{\varphi,\alpha}=\sum_{\pi\in\Pi^{G_\alpha}(\varphi)}c_\pi$. Then the property of Local Langlands Conjecture leads this to a lot of cancellations. We are left with the contribution of $1\in\Gamma(G,H)$ which is equal to

$$c_{\varphi,1}(1)$$

By a result of Rodier [Ro74] in p-adic case and of Matumoto [Mat92] in real case, the term $c_{\varphi,1}(1)$ has an interpretation in terms of Whittaker models. The property of Local Langlands Correspondence gives $c_{\varphi,1}(1)=1$.

We now give a brief description of the content of each chapter. In the first Chapter, we will introduce the complete setting of the local Gan-Gross-Prasad Conjecture. In Section 2.1, we introduce the Classical Groups attached to an linear space with an bilinear form, and their pure inner forms. In Section 2.2, we introduce the Gan-Gross-Prasad triple (Definition 2.1). As a special application of GGP triple, which is also necessary for our definition of the Local Langlands Correspondence, we will study the generic character and Whittaker datum in Section 2.3. The rough Local Gan-Gross Prasad Conjecture which considering the multiplicity of this representation in a bigger irreducible one, will be stated in Section 2.4. In the second chapter, we introduce the Local Langlands Correspondence. We start from classification of the Weil-Deligne Representation in the Section 3.1. Then we study the Langlands parameter through Section 3.2 and Section 3.3, then introduce the Vogan L-packets of unitary groups in Section 3.4. All these explains the notations appear in Theorem 1.1. A few discussions can be found in Section 3.5 and Section 3.6. In the last Section 3.6, we write down the Local Langlands Classification for discrete series and formulate the precise criterion explicitly. Finally, we will explain the proof of hermitian case in the last chapter. We will start with the integrand, function c_θ, in Section 4.1, then the appropriate region for integration, in Section 4.2. Section 4.5 and Section 4.6 care about two sides of the Local Trace Formula in local GGP setting. Section 4.7 and Section 4.8 gives the final proof of **Theorem 1.1.**(i). Besides, some definitions and results used throughout the text are collected in three appendices, in order to make the paper seems more likely to be self-contained. Appendix A is concerned with group structure theory. Appendix B is concerned with number theoretic background, which concerns mostly about the L-group, whereas Appendix C is about basic representation theory and general study on the characters, and the preliminary for simple Local Trace Formula.

2. Local Gan-Gross-Prasad Conjecture

A motivating example is the following classical branching problem in the theory of compact Lie groups. Let π be an irreducible finite-dimensional representation of the compact unitary group $\mathrm{U}(n)$, and consider its restriction to the naturally embedded subgroup $\mathrm{U}(n-1)$. It is know that this restriction is multiplicity-free. Precisely, recall the classification theorem of the representation of compact Lie group which says that the representation is determined by the highest weight λ. For compact unitary group $\mathrm{U}(n)$, the set of all highest weights are in bijection with the sequence of integers:

$$\lambda=(\lambda_1\geq\cdots\geq\lambda_n)$$

Now for irreducible $\pi = V_\lambda \in \mathrm{Rep}(\mathrm{U}(n))$, $\tau = V_\mu \in \mathrm{Rep}(\mathrm{U}(n-1))$, the classical branching law says

$$\mathrm{Hom}_{\mathrm{U}(n-1)}(\pi|_{\mathrm{U}(n-1)}, \tau) \neq 0$$

if and only if λ and μ satisfies the Cauchy interlacing relation:

$$\lambda_1 \leq \mu_1 \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$

the multiplicity one result follows easily.

2.1. **Classical Groups and Pure Inner Forms.** Let E be a field of characteristic 0. Let σ be an involution of E having F as the fixed field. If $\sigma = 1$, then $E = F$. If $\sigma \neq 1$, E is a quadratic extension of F and σ is the nontrivial element in the Galois group $\mathrm{Gal}(E/F)$.

Let V be a finite dimensional vector space over E. Let

$$\langle , \rangle : V \times V \to E$$

be a non-degenerate, σ-sesquilinear form on V, which is ϵ-symmetric (for $\epsilon = \pm 1$ in $E^\times$):

$$\begin{aligned} \langle \alpha v + \beta w, u \rangle &= \alpha \langle v, u \rangle + \beta \langle w, u \rangle \\ \langle u, v \rangle &= \epsilon \cdot \langle v, u \rangle^\sigma. \end{aligned}$$

Let $G(V) \subset \mathrm{GL}(V)$ be the algebraic subgroup of elements T in $\mathrm{GL}(V)$ which preserve the form $\langle , \rangle$:

$$\langle Tv, Tw \rangle = \langle v, w \rangle.$$

Then $G(V)$ is a classical group, defined over the field F. The different possibilities for $G(V)$ are given in the following table.

(E,ϵ)	$E = F$, $\epsilon = 1$	$E = F$, $\epsilon = -1$	E/F quadratic, $\epsilon = \pm 1$
$G(V)$	orthogonal group $\mathrm{O}(V)$	symplectic group $\mathrm{Sp}(V)$	unitary group $\mathrm{U}(V)$

In our formulation, a classical group will always be associated to a space V, so the hermitian and skew-hermitian cases are distinct. Moreover, the group $G(V)$ is connected except in the orthogonal case. In that case, we let $\mathrm{SO}(V)$ denote the connected component, which consists of elements T of determinant $+1$. We will only work with connected classical groups in this paper.

Now let $W \subset V$ be a subspace, which is non-degenerate for the form $\langle , \rangle$. Then $V = W + W^\perp$. We assume that

$$\begin{array}{ll} (1) & \epsilon \cdot (-1)^{\dim W^\perp} = -1 \\ (2) & W^\perp \text{ is a split space.} \end{array}$$

When $\epsilon = -1$, so $\dim W^\perp = 2n$ is even, condition (2) means that $W^\perp$ contains an isotropic subspace X of dimension n. It follows that $W^\perp$ is a direct sum

$$W^\perp = X + Y,$$

with X and Y isotropic. The pairing $\langle -, - \rangle$ induces a natural map

$$Y \longrightarrow \mathrm{Hom}_E(X, E) = X^\vee$$

which is a F-linear and E-anti-linear isomorphism. When $\epsilon = +1$, so $\dim W^\perp = 2n+1$ is odd, condition (2) means that $W^\perp$ contains an isotropic subspace X of dimension n. It follows that

$$W^\perp = X + Y + \langle e \rangle,$$

where $\langle e \rangle$ is a non-isotropic line orthogonal to $X + Y$, and X and Y are isotropic. As above, one has a F-linear isomorphism $Y \cong X^\vee$.

Let $G(W)$ be the subgroup of $G(V)$ which acts trivially on $W^{\perp}$. This is the classical group, of the same type, associated to the space W. Choose an $X \subset W^{\perp}$ as above, and let P be the parabolic subgroup of $G(V)$ which stabilizes a complete flag of (isotropic) subspaces in X. Then $G(W)$, which acts trivially on both X and $X^{\vee}$, is contained in a Levi subgroup of P, and acts by conjugation on the unipotent radical N of P.
The semi-direct product $H = N \rtimes G(W)$ embeds as a subgroup of the product group $G = G(V) \times G(W)$ as follows. We use the defining inclusion $H \subset P \subset G(V)$ on the first factor, and the projection $H \to H/N = G(W)$ on the second factor.
We now recall the notion of pure inner forms. A *pure inner form* for G is defined formally as a triple (G', ψ, c) where

(i) G' is a connected reductive group defined over F;

(ii) $\psi : G_{\overline{F}} \simeq G'_{\overline{F}}$ is an isomorphism defined over $\overline{F}$;

(iii) $c : \sigma \in \Gamma_F \to c_\sigma \in G(\overline{F})$ is a 1-cocycle such that $\psi^{-1\sigma}\psi = \mathrm{Ad}(c_\sigma)$ for all $\sigma \in \Gamma_F$.

There is a natural notion of isomorphism between pure inner forms the equivalence classes of which are naturally in bijection with $H^1(F, G)$ (the isomorphism class of (G', ψ, c) being parametrized by the image of c in $H^1(F, G)$). Moreover, inside an equivalence class of pure inner forms (G', ψ, c), the group G' is well-defined up to $G'(F)$-conjugacy. We will always assume fixed for all $\alpha \in H^1(F, G)$ a pure inner form in the class of α that we will denote by $(G_\alpha, \psi_\alpha, c_\alpha)$ or simply by G_α if no confusion arises.
For connected, classical groups $G(V) \subset \mathrm{GL}(V)$, the pointed set $H^1(F, G)$ and the pure inner forms G' correspond bijectively to forms V' of the space V with its sesquilinear form $\langle , \rangle$ (cf. Chapt.29D and Chapt.29E in [KMRT98]). Especially, for $U(V)$, elements of the pointed set $H^1(F, G)$ correspond bijectively to the isomorphism classes of hermitian (or skew-hermitian) spaces V' over E with $\dim(V') = \dim(V)$. The corresponding pure inner form G' of G is the unitary group $\mathrm{U}(V')$.

Example 1. *We have the following explicit description of the pure inner forms of $U(V)$. The cohomology set $H^1(F, U(V))$ naturally classifies the isomorphism classes of hermitian spaces of the same dimension as V. Let $\alpha \in H^1(F, U(V))$ and choose a hermitian space V_α in the isomorphism class corresponding to α. Set $V_{\overline{F}} = V \otimes_F \overline{F}$ and $V_{\alpha,\overline{F}} = V_\alpha \otimes_F \overline{F}$. Fix an isomorphism $\phi_\alpha : V_{\overline{F}} \simeq V_{\alpha,\overline{F}}$ of $\overline{E}$-hermitian spaces. Then, the triple $(U(V_\alpha), \psi_\alpha, c_\alpha)$, where ψ_α is the isomorphism $U(V)_{\overline{F}} \simeq U(V_\alpha)_{\overline{F}}$ given by $\psi_\alpha(g) = \phi_\alpha \circ g \circ \phi_\alpha^{-1}$ and c_α is the 1-cocycle given by $\sigma \in \Gamma_F \mapsto \phi_\alpha^{-1\sigma}\phi_\alpha$, is a pure inner form of $U(V)$ in the class of α.*

Remark 2.1. *For the calculation purpose, we classify the hermitian spaces over the field $\mathbb{R}$ and $\mathbb{Q}_p$ separately,as follows:*

Real case: there are exactly $n+1$ hermitian spaces of dimension n, determined by pairs: (a, b), where $a + b = n$, and

$$U(a, b) = \{g \in \mathrm{GL}_n(\mathbb{C}) | \bar{g}^t \cdot 1_{a,b} \cdot g = 1_{a,b}\}, \quad 1_{a,b} = \mathrm{diag}(1_a, -1_b).$$

p-adic case: Let E/F be a quadratic extension of p-adic fields. Then there are exactly two hermitin space $V^{\pm}$ over E/F of dimension n, defined by

$$\epsilon(V^{\pm}) = \epsilon_{E/F}((-1)^{n(n-1)/2}\mathrm{det}V^{\pm}) = \pm 1.$$

where $\epsilon_{E/F} : F^\times \to F^\times/\mathbb{N}E^\times \cong \{\pm 1\}$ is the quadratic character and $\mathrm{det}V^{\pm}$ is the determinant of $((x_i, x_j))_{i,j}$ for any basis $\{x_i\}_i$ of $V^{\pm}$, which is well-defined in $F^\times/\mathbb{N}E^\times$.

2.2. **Gan-Gross-Prasad Triples.** The goal of this section is to describe a distinguished character of H associated to a specific pair (W, V) (Definition 2.1). Finally, we introduce the pure inner form of the given GGP triple.

Orthogonal and Hermitian Cases (Bessel Models)

Assume that $\dim W^{\perp} = 2n+1$ and write

$$W^{\perp} = X + X^{\vee} + \langle e\rangle,$$

where X and $X^{\vee}$ are maximal isotropic subspaces which are in duality using the form $\langle -,-\rangle$ of V, and e is a non-isotropic vector. Let $P(X)$ be the parabolic subgroup in $G(V)$ stabilizing the subspace X, and let $M(X)$ be the Levi subgroup of $P(X)$ which stabilizes both X and $X^{\vee}$, so that

$$M(X) \cong \mathrm{GL}(X) \times G(W \oplus \langle e\rangle).$$

We have

$$P(X) = M(X) \ltimes N(X)$$

where $N(X)$ is the unipotent radical of $P(X)$. The group $N(X)$ sits in an exact sequence of $M(X)$-modules,

$$0 \to Z(X) \to N(X) \to N(X)/Z(X) \to 0,$$

through the form on X, one has natural isomorphisms

$$Z(X) \cong \{\text{skew-hermitian forms on } X^{\vee}\},$$

and

$$N(X)/Z(X) \cong \mathrm{Hom}(W + \langle e\rangle, X) \cong (W + \langle e\rangle) \otimes X.$$

Now let

$$\ell_X : X \to E$$

be a nonzero E-linear homomorphism, and let

$$\ell_W : W \oplus \langle e\rangle \longrightarrow E$$

be a nonzero E-linear homomorphism which is zero on the hyperplane W. Together, these give a map

$$\ell_X \otimes \ell_W : X \otimes (W + \langle e\rangle) \longrightarrow E,$$

and one can consider the composite map

$$\ell_{N(X)} : N(X) \to N(X)/Z(X) \cong X \otimes (W + \langle e\rangle) \overset{\ell_X \otimes \ell_W}{\to} E.$$

Let U_X be any maximal unipotent subgroup of $\mathrm{GL}(X)$ which stabilizes ℓ_X. Then the subgroup

$$U_X \times G(W) \subset M(X)$$

fixes the homomorphism $\ell_{N(X)}$. Now the subgroup $H \subset G = G(V) \times G(W)$ is given by $H = N(X) \rtimes (U_X \cdot G(W)) = N \rtimes G(W)$. We may extend the map $\ell_{N(X)}$ of $N(X)$ to H, by making it trivial on $U_X \times G(W)$. If ψ is a non-trivial additive character of E, and

$$\lambda_X : U_X \to \mathbb{C}$$

is a generic character of U_X, then the representation ξ of H is defined by

$$\xi = (\psi_E \circ \ell_{N(X)}) \boxtimes \lambda_X.$$

The pair (H, ξ) is uniquely determined *up to conjugacy* in the group $G = G(V) \times G(W)$ by the pair $W \subset V$.

One can give a more explicit description of (H, ξ), by explicating the choices of ℓ_X, U_X and λ_X above. To do this, choose a basis $\{v_1, \cdots, v_n\}$ of X, with dual basis $\{v_i'\}$ of $X^{\vee}$. Let $P \subset G(V)$ be the parabolic subgroup which stabilizes the flag

$$0 \subset \langle v_1\rangle \subset \langle v_1, v_2\rangle \subset \cdots \subset \langle v_1, \cdots, v_n\rangle = X,$$

and let

$$L = (E^{\times})^n \times G(W + \langle e\rangle)$$

be the Levi subgroup of P which stabilizes the lines $\langle v_i \rangle$ as well as the subspace $W + E$. The torus $T = (E^\times)^n$ scales these lines:

$$t(v_i) = t_i v_i,$$

and $G(W + \langle e \rangle)$ acts trivially.
Let N be the unipotent radical of P, so that

$$N = U_X \ltimes N(X)$$

where U_X is the unipotent radical of the Borel subgroup in $\mathrm{GL}(X)$ stabilizing the chosen flag above. Now define a homomorphism $f : N \to E^n$ by

$$\begin{array}{rcl} f(u) & = & (x_1, \cdots, x_{n-1}, z), \\ x_i & = & \langle uv_{i+1}, v_i' \rangle, \quad i = 1, 2, \cdots, n-1 \\ z & = & \langle ue, v_n' \rangle. \end{array}$$

The subgroup of L which fixes f is $G(W)$, the subgroup of $G(W + \langle e \rangle)$ fixing the vector e. The torus acts on f by

$$f(tut^{-1}) = (t_1^\sigma / t_2 x_1, t_2^\sigma / t_3 x_2, \cdots, t_n^\sigma z).$$

Consider the subgroup $H = N \ltimes G(W)$ of $G = G(V) \times G(W)$. Then, for a non-trivial additive character ψ of E, the representation ξ is given by:

$$\begin{array}{rcl} \xi & : & H \to \mathbb{C}^\times \\ & & (u, g) \mapsto \psi(\sum x_i + z). \end{array}$$

As noted above, up to G-conjugacy, the pair (H, ξ) depends only on the initial data $W \subset V$, and not on the choices of ψ, $\{v_i\}$, or e used to define it.

Definition 2.1. *For the specific pair $W \subset V$, we call (G, H, ξ) the GGP triple associated to (W, V).*

Example 2. *Let me give two particular examples of the situation:*
If $W = 0$, then $G = G(V)$, N is a maximal unipotent subgroup of G and ξ is a generic character of $N(F)$. This is called the Whittaker case because then $m(\pi)$ is the dimension of the space of Whittaker functionals for π with respect to (N, ξ).
If $\dim(W) = \dim(V) - 1$, then $N = 1$ and $\xi = 1$, and the embedding of H is the diagonal one. Then the Hom space is the branching law problem. This is called the Codimensional one case.

Now we will end this section by introducing a class triple based on the given GGP triple. assume that Now assuming that G and H are quasi-split. Let $\alpha \in H^1(F, H)$. We are going to associate to α a *new* GGP triple $(G_\alpha, H_\alpha, \xi_\alpha)$ well-defined up to conjugacy. Let $\alpha \in H^1(F, H)$. Let W_α be the corresponding space with bilinear form in this isomorphism class and set $V_\alpha = W_\alpha \oplus^\perp (V/W)^\perp$. Then, the pair (V_α, W_α) is easily seen to be a pair of hermitian spaces over E and hence there is a GGP triple $(G_\alpha, H_\alpha, \xi_\alpha)$ associated to it. Of course, this GGP triple is well-defined up to conjugacy. We call such a GGP triple a *pure inner form* of (G, H, ξ). By definition, these pure inner forms are parametrized by $H^1(F, H)$. Note that for all $\alpha \in H^1(F, H)$, G_α is a pure inner form of G in the class corresponding to the image of α in $H^1(F, G)$ and that the natural map $H^1(F, H) \to H^1(F, G)$ is injective.

Example 3. *Moreover, we may also fix the other parts of the data $(G_\alpha, \psi_\alpha, c_\alpha)$ of a pure inner form of G in the class of α for hermitian case, as follows. Choose an isomorphism $\phi_\alpha^W : W_{\overline{F}} \simeq W_{\alpha,\overline{F}}$ of $\overline{E}$-hermitian spaces, where we have set as usual $W_{\overline{F}} = W \otimes_F \overline{F}$ and $W_{\alpha,\overline{F}} = W_\alpha \otimes_F \overline{F}$, and extend it to an isomorphism $\phi_\alpha^V : V_{\overline{F}} \simeq V_{\alpha,\overline{F}}$ that is the identity on $Z_{\overline{F}}$. Then, we may take ψ_α to be the isomorphism*

$$G_{\overline{F}} = U(W_{\overline{F}}) \times U(V_{\overline{F}}) \simeq G_{\alpha,\overline{F}} = U(W_{\alpha,\overline{F}}) \times U(V_{\alpha,\overline{F}})$$

given by

$$(g_W, g_V) \mapsto \left(\phi_\alpha^W \circ g_W \circ (\phi_\alpha^W)^{-1}, \phi_\alpha^V \circ g_V \circ (\phi_\alpha^V)^{-1}\right)$$

and we may take the 1*-cocycle* c_α *to be given by*

$$\sigma \in \Gamma_F \mapsto \left((\phi_\alpha^W)^{-1\sigma}\phi_\alpha^W, (\phi_\alpha^V)^{-1\sigma}\phi_\alpha^V\right) \in U(W_{\overline{F}}) \times U(V_{\overline{F}}) = G_{\overline{F}}.$$

2.3. **Generic Characters and Whittaker Datum.** As a direct application of all the constructions in Section 2.2, which is also a preliminary of local Gan-Gross-Prasad conjecture, we introduce the generic characters in this section. These considerations are also general.

Let G be a quasi-split, connected, reductive group over a local field F.Let $B = TN$ be a Borel subgroup of G over F. The quotient torus $T = B/N$ acts on the group $\mathrm{Hom}(N, \mathbb{C}^\times)$. We call a character $\theta : N(F) \to \mathbb{C}^\times$ *generic* if its stabilizer in $T(F)$ is equal to the center of G, $Z(F)$. And we call such pair (N, θ) which is a $G(F)$-conjugacy class a *Whittaker datum* for G. If π is an irreducible representation of $G(F)$ and θ is a generic character, then the complex vector space $\mathrm{Hom}_{N(F)}(\pi, \theta)$ has dimension ≤ 1. When the dimension is 1, we say π is θ-generic. This depends only on the $T(F)$-orbit of θ. In [GGP12, Proposition 12.1], they classify the set D of $T(F)$-orbits on the set of all generic characters θ of $N(F)$. Specifically, for unitary group with odd dimension, D is trivial. If V is hermitian of even dimension, we have a bijection

$$D \longleftrightarrow \mathbb{N}E^\times - \text{orbits on nontrivial additive } \psi : E/F \to \mathbb{C},$$

Then we come back to the explicit construction and classification of Whittaker datum. Actually, there is a bijection between the set of Whittaker datum and the set of regular nilpotent orbits in $\mathfrak{g}(F)$. Specifically, using the bilinear form $B(.,.)$ and the additive character ψ, we can define a bijection $\mathcal{O} \mapsto (U_\mathcal{O}, \xi_\mathcal{O})$ between $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$ and the set of Whittaker data for G as follows. Let $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$. Pick $Y \in \mathcal{O}$ and extend it to an $\mathfrak{sl}_2$-triple (Y, H, X). Then X is a regular nilpotent element and hence belongs to exactly one Borel subalgebra $\mathfrak{b}_\mathcal{O}$ of $\mathfrak{g}$ that is defined over F. Let $B_\mathcal{O}$ be the corresponding Borel subgroup and $U_\mathcal{O}$ be its unipotent radical. The assignment $u \in U_\mathcal{O}(F) \mapsto \xi_\mathcal{O}(u) = \psi\left(B(Y, \log(u))\right)$ defines a generic character on $U_\mathcal{O}(F)$. Moreover, the $G(F)$-conjugacy class of $(U_\mathcal{O}, \xi_\mathcal{O})$ only depends on $\mathcal{O}$ and this defines the desired bijection.

Remark 2.2. *We make a remark on the regular nilpotent orbits* $\mathfrak{g}(F)$*, where* G *is the unitary group attached to a hermitian space* (V, h)*. If* $U(V)$ *is not quasi-split then there are no such orbit. Assume that* $U(V)$ *is quasi-split. If* $\dim(V)$ *is odd or zero, then there is only one regular nilpotent orbits. Assume moreover that* $\dim(V) > 0$ *is even. Then, there are exactly two regular nilpotent orbits. Since* $U(V)$ *is quasi-split, there exists a basis* $(z_i)_{i=\pm1,\ldots,\pm k}$ *such that* $h(z_i, z_j) = \delta_{i,-j}$ *for all* $i, j \in \{\pm1, \ldots, \pm k\}$ *(where* $\delta_{i,-j}$ *denotes the Kronecker symbol). Let* B *be the Borel subgroup in* $U(V)$ *of the flag*

$$\langle z_k \rangle \subset \langle z_k, z_{k-1} \rangle \subset \ldots \subset \langle z_k, \ldots, z_1 \rangle$$

Denote by U *its unipotent radical. For all* $\mu \in E$ *with trace zero, define an element* $X(\mu) \in \mathfrak{g}(F)$ *by the assignments*

$$X(\mu)z_k = 0, \ X(\mu)z_i = z_{i+1} \text{ for } 1 \leqslant i \leqslant k-1 \text{ and}$$

$$X(\mu)z_{-1} = \mu z_1, \ X(\mu)z_{-i} = -z_{1-i} \text{ for } 2 \leqslant i \leqslant k.$$

Then, for all $\mu \in E^\times$ *with* $\mathrm{Tr}_{E/F}(\mu) = 0$*,* $X(\mu)$ *is regular nilpotent. Moreover the orbits of* $X(\mu)$ *and* $X(\mu')$ *coincide if and only if* $\mu N_{E/F}(E^\times) = \mu' N_{E/F}(E^\times)$*. It follows that for all* $\lambda \in F^\times \setminus N_{E/F}(E^\times)$*, the elements* $X(\eta)$ *and* $X(\lambda\eta)$ *are representatives of the two regular nilpotent conjugacy classes. Notice that in particular multiplication by any element of* $F^\times \setminus N_{E/F}(E^\times)$ *permutes the two regular nilpotent orbits in* $\mathfrak{g}(V)(F)$*.*

Remark 2.3. *Recall our classification of the hermitian spaces. When $F = \mathbb{R}$, $U(p,q)$ is quasi-split if and only if $|p-q| \le 1$. When F is p-adic, the group with odd rank is always quasi-split; however, when the group is of even rank, V^+ is quasi-split but V^- isn't. Combine all these, we have*

(i) *A unitary group of odd rank $2k+1$ has two quasi-split pure inner forms and each of these support a unique conjugacy class of Whittaker data.*

(ii) *A unitary group of even rank $2k$ has unique quasi-split pure inner forms but there are two conjugacy classes of Whittaker data on this pure inner form.*

Let π be a tempered irreducible representation of $G(F)$. For $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(F)$, we will say that π has a *Whittaker model of type $\mathcal{O}$* if there exists a nonzero continuous linear form $\ell : \pi^\infty \to \mathbb{C}$ such that $\ell \circ \pi(u) = \xi_{\mathcal{O}}(u)\ell$ for all $u \in U_{\mathcal{O}}(F)$. Recall that the character θ_π of π is a quasi-character on $G(F)$ (See Appendix C). For all $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$, we set $c_{\pi,\mathcal{O}}(1) = c_{\theta_\pi,\mathcal{O}}(1)$. Then we can define such characters for virtual representations by linearity. The following proposition is crucial, by Rodier in [Ro74].

Proposition 2.1. *Let π a irreducible tempered representation of G. Then for all $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$, we have*

$$c_{\pi,\mathcal{O}}(1) = \begin{cases} 1 & \text{if } \pi \text{ has a Whittaker model of type } \mathcal{O}; \\ 0 & \text{otherwise} \end{cases}$$

2.4. **Restriction Problems and Multiplicity One Theorems.** In this section, we will introduce the rough edition of the local Gan-Gross-Prasad Conjecture of the bessel models.
Let $W \subset V$ with GGP triple (G, H, ξ) as before. Let $\pi = \pi_V \boxtimes \pi_W$ be an irreducible representation of G. Then the restriction problem of interest is to determine

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\xi}, \mathbb{C}).$$

More precisely, we have: in the orthogonal or hermitian cases, the representation ξ of H depends only on $W \subset V$ and so we set

$$d(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\xi}, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, \xi).$$

Remark 2.4. *We remark that in the orthogonal and hermitian cases, since ξ is 1-dimensional and unitary, one has:*

$$\mathrm{Hom}_H(\pi \otimes \overline{\xi}, \mathbb{C}) \cong \mathrm{Hom}_H(\pi \otimes \xi^\vee, \mathbb{C}) \cong \mathrm{Hom}_H(\pi, \xi).$$

However, it is not the case for Fourier-Jacobi models.

A basic conjecture is the following:

Theorem 2.1. *Assume that $\dim W^\perp = 0$ or 1. In the Bessel case,*

$$d(\pi) \le 1.$$

Proof. This case is due to Aizenbud-Gourevitch-Rallis-Schiffmann [AGRS10] in the p-adic case and to Sun-Zhu [SZ12] and Aizenbud-Gourevitch [AG08] in the archimedean case. □

In Chapter 4, we will use this multiplicity one property directly.

3. Local Langlands Correspondence

In this section we recall the local Langlands correspondence in a form that will be suitable for us. Let G be a quasi-split connected reductive group over F and denotes by

${}^LG = \widehat{G}(\mathbb{C}) \rtimes W_F$ its Langlands dual, where W_F denotes the Weil group of F. Recall that a *Langlands parameter* for G is a homomorphism from the group

$$WD_F = \begin{cases} W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } F \text{ is } p-\text{adic} \\ W_F & \text{if } F = \mathbb{R} \end{cases}$$

to ${}^LG$ satisfying the usual conditions of continuity, semi-simplicity, algebraicity and compatibility with the projection ${}^LG \to W_F$. A Langlands parameter φ is said to be *tempered* if $\varphi(W_F)$ is bounded. By the hypothetical local Langlands correspondence, a tempered Langlands parameter φ for G should give rise to a finite set $\Pi_\varphi(G)$, called a *L-packet*, of (isomorphism classes of) tempered representations of $G(F)$. Actually, such a parameter φ should also give rise to tempered L-packets $\Pi_\varphi(G_\alpha) \subset \mathrm{Temp}(G_\alpha)$ for all $\alpha \in H^1(F, G)$ (although it may be empty). These families of L-packets should of course satisfy some conditions. We begin with the study of Weil-Deligne representation.

To start our discussion, we still begin with a motivating example. Recall the example we introduced at the beginning of Chapter 2. When $n = 2$, where we have seen that the representation V_λ of $\mathrm{U}(2)$ contains V_μ precisely when $\lambda_1 \leq \mu_1 \lambda_2$. Consider instead the non-compact unitary group $\mathrm{U}(1,1)$. Indeed, one has an isomorphism of real Lie groups:

$$\mathrm{U}(1,1) \cong (\mathrm{SL}_2(\mathbb{R}) \times S^1)/\Delta\mu_2.$$

Now let π be an irreducible discrete representation of $\mathrm{U}(1,1)$; note that π is typically infinite-dimensional but since $\mathrm{U}(1)$ is compact, the restriction of π is a direct sum of irreducible character of $\mathrm{U}(1)$. It is known that this decomposition is multiplicity one, so one is interested in determining precisely which character of $\mathrm{U}(1)$ occur.
Recall the classification theory of the discrete series for real reductive group by Harish Chandra (which we will aslo recall at Section 3.6). For $\mathrm{U}(1,1)$, it turns out that the discrete series representation are classified by a pair of integers $\lambda = (\lambda_1 \leq \lambda_2)$. Each such λ gives rise to two discrete series representations V_λ^+ and V_λ^-. Then one can show that an irreducible representation V_μ occurs in the restriction of V_λ^+ (resp. V_λ^-) if and only if

$$\mu_1 > \lambda_2 \quad (\text{resp., } mu_1 < \lambda_1)$$

i.e. $V_\lambda^\pm$ contains V_μ (with multiplicity one) if and only if λ and μ don't interlace!
An important observation is, it is useful to group certain representations of different groups together. In our example, we see that if one groups together the representations of V_λ of $\mathrm{U}(2)$ and $V_\lambda^\pm$ of $\mathrm{U}(1,1)$, then the branching problem has a uniform answer:

$$\dim \mathrm{Hom}_{\mathrm{U}(1)}(V_\lambda|_{\mathrm{U}(1)}^{\mathrm{U}(2)}, V_\mu) + \dim \mathrm{Hom}_{\mathrm{U}(1)}(V_\lambda|_{\mathrm{U}(1)}^{\mathrm{U}(1,1)}, V_\mu) = 1.$$

3.1. Selfdual and Conjugate-Dual Representations.

Recall that a representation of WD_F is a continuous homomorphism

$$\varphi : WD_F \to \mathrm{GL}(M),$$

where M is a finite dimensional complex vector space, with

(i) φ trivial on an open subgroup of I,

(ii) $\varphi(F)$ semi-simple, here F is the geometric Frobenius,

(iii) $\varphi : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(M)$ algebraic.

The equivalence of this formulation, is a homomorphism $\rho : W_F \to \mathrm{GL}(M)$ and a nilpotent endomorphism N of M which satisfies $Ad\rho(w)(N) = |w| \cdot N$, $\forall w \in W_F$. These details can be checked in [GR10]. We say two representations M and M' of WD_F are isomorphic if there is a linear isomorphism $f : M \to M'$ which commutes with the action of WD_F. If M and M' are two representations of WD_F, we have the direct sum representation $M \oplus M'$ and the tensor product representation $M \otimes M'$. The dual representation $M^\vee$ is defined by the natural action on $\mathrm{Hom}(M, \mathbb{C})$, and the determinant representation $\det(M)$ is defined by the natural action on the top exterior power. Since $\det(M)$ is 1-dimensional, it factors through the quotient $W_F^{ab} \to F^\times$ of WD_F.

We now define certain self-dual representations of WD_F. We say the representation M is orthogonal if there is a non-degenerate bilinear form

$$B : M \times M \to \mathbb{C}$$

which satisfies

$$\begin{cases} B(\tau m, \tau n) = B(m, n) \\ B(n, m) = B(m, n), \end{cases}$$

for all τ in WD_F.
We say M is symplectic if there is a non-degenerate bilinear form B on M which satisfies

$$\begin{cases} B(\tau m, \tau n) = B(m, n) \\ B(n, m) = -B(m, n), \end{cases}$$

for all τ in WD_F.

We note that:

Lemma 3.1. *Given any two non-degenerate forms B and B' on M preserved by WD_F with the same sign $b = \pm 1$, there is an automorphism T of M which commutes with WD_F and such that $B'(m, n) = B(Tm, Tn)$.*

Next, assume that σ is a nontrivial representative element in $\mathrm{Gal}(E/F)$, with fixed field F. Let s be an element of WD_F which generates the quotient group

$$WD_F/WD_E = \mathrm{Gal}(E/F) = \langle 1, \sigma \rangle.$$

If M is a representation of WD_E, let M^s denote the conjugate representation, with the same action of $\mathrm{SL}_2(\mathbb{C})$ and the action $\tau_s(m) = s\tau s^{-1}(m)$ for τ in W_E.

We say the representation M is conjugate-orthogonal if there is a non-degenerate bilinear form $B : M \times M \to \mathbb{C}$ which satisfies

$$\begin{cases} B(\tau m, s\tau s^{-1} n) = B(m, n) \\ B(n, m) = B(m, s^2 n), \end{cases}$$

for all τ in WD_E. We say M is conjugate-symplectic if there is a non-degenerate bilinear form on M which satisfies

$$\begin{cases} B(\tau m, s\tau s^{-1} n) = B(m, n) \\ B(n, m) = -B(m, s^2 n), \end{cases}$$

for all τ in WD_E.
In both cases, the form B gives an isomorphism of representations

$$f : M^s \to M^\vee,$$

whose conjugate-dual $(f^\vee)^s : M^s \to ((M^s)^\vee)^s \overset{\varphi(s)^2}{\to} M$ satisfies

$$(f^\vee)^s = b \cdot f \quad \text{with } b = \text{the sign of } B.$$

We now note:

Lemma 3.2. *Given two such non-degenerate forms B and B' on M with the same sign and preserved by WD_E, there is an automorphism of M which commutes with WD_E and such that $B'(m, n) = B(Tm, Tn)$.*

It is easy to note that the isomorphism class of the representation M^s is independent of the choice of s in $WD_F - WD_E$. If $s' = ts$ is another choice, then the map

$$\begin{aligned} f \quad : \quad & M^s \to M^{s'} \\ & m \mapsto t(m) \end{aligned}$$

is an isomorphism of representations of WD_E. We denote the isomorphism class of M^s and $M^{s'}$ simply by M^σ. If M is conjugate-orthogonal or conjugate-symplectic by the pairing B relative to s, then it is conjugate-orthogonal or conjugate-symplectic by the pairing

$$B'(m, f(n)) = B(m, n)$$

relative to s'. In both cases, M^σ is isomorphic to the dual representation $M^\vee$.
If M is conjugate-dual via a pairing B with sign $b = \pm 1$, then $\det(M)$ is conjugate-dual with sign $= (b)^{\dim(M)}$. Any conjugate-dual representation of WD_E of dimension 1 gives a character $\chi : E^\times \to \mathbb{C}^\times$ which satisfies $\chi^{1+\sigma} = 1$. Hence χ is trivial on the subgroup $\mathbb{N}E^\times$, which has index 2 in $F^\times$, by local class field theory. We denote this 1-dimensional representation by $\mathbb{C}(\chi)$.

Lemma 3.3. *The representation $\mathbb{C}(\chi)$ is conjugate-orthogonal if and only if χ is trivial on $F^\times$, and conjugate-symplectic if and only if χ is nontrivial on $F^\times$ but trivial on $\mathbb{N}E^\times$.*

We use the following equivalence to define the conjugate dual property of the Weil-Deligne representation of WD_F:

Lemma 3.4. *The representation M of WD_E is of conjugate dual with sign b if and only if $N = \mathrm{Ind}_{WD_E}^{WD_F} M$ is selfdual with sign b and has maximal isotropic subspace M.*

3.2. **Component Groups, Local Root Numbers and Characters.** Define $C = C(M, B)$ as the subgroup of $\mathrm{Aut}(M, B) \subset GL(M)$ which centralizes the image of WD_E, here $\mathrm{Aut}(M, B)$ represents the automorphism that preserve the pairing B which introduced before.
If we write M as a direct sum of irreducible representations M_i, with multiplicities m_i, and consider their images in $M^\vee$ under the isomorphism $M^\sigma \to M^\vee$ provided by B, we find that there are three possibilities:

(i) M_i^σ is isomorphic to $M_i^\vee$, via a pairing B_i of the same sign b as B.

(ii) M_i^σ is isomorphic to $M_i^\vee$, via pairing B_i of the opposite sign $-b$ as B. In this case the multiplicity m_i is even.

(iii) M_i^σ is isomorphic to $M_j^\vee$, with $j \neq i$. In this case $m_i = m_j$.

Hence, we have a decomposition

$$M = \bigoplus m_i M_i + \bigoplus 2n_i N_i + \bigoplus p_i(P_i + (P_i^\sigma)^\vee)$$

where the M_i are selfdual or conjugate-dual of the same sign b, the N_i are selfdual or conjugate-dual of the opposite sign $-b$, and P_i^σ is not isomorphic to $P_i^\vee$, so that P_i and $P_j = (P_i^\sigma)^\vee$ are distinct irreducible summands.

In this case, we have

$$C \simeq \prod \mathrm{O}(m_i, \mathbb{C}) \times \prod \mathrm{Sp}(2n_i, \mathbb{C}) \times \prod \mathrm{GL}(p_i, \mathbb{C}).$$

In particular, the component group of C is

$$A = \pi_0(C) \simeq (\mathbb{Z}/2\mathbb{Z})^{\#I},$$

where I is the set of all irreducible summands M_i of the same type as M. Denote the element in A as the following form:

$$(e_i)_{i \in I} \in (\mathbb{Z}/2\mathbb{Z})^{\#I}, \text{ (or simply denoted by } (e_i)), \text{ with } e_i \in \{0, 1\},$$

where for each such M_i, -1 is a representative of the simple nontrivial reflection in the orthogonal group $\mathrm{O}(m_i)$.
For any element $(e_i)_{i\in I}$, we define

$$M^{(e_i)} = \{m \in M : (e_i)_{i\in I} m = -m\}.$$

This is a representation of WD_E, and the restricted pairing $B : M^a \times M^a \to \mathbb{C}$ is non-degenerate, of the same type as M. We can use these representations to define several characters $\chi \in \widehat{A}$. A direct example is the following: since $M^{(e_i)} \pmod 2$ depends only on the coset of $(e_i) \pmod{C^0}$, this gives a character of A:

$$\eta(a) = (-1)^{\dim M^{(e_i)}}.$$

To construct other characters of A, we need more sophisticated signed invariants $d(M)$. We will obtain these from local root numbers.

Given a Weil Deligne represnetation M of WD_E, and let ψ be a nontrivial additive character of E, Let dx be the unique Haar measure on E which is selfdual for Fourier transform with respect to ψ. We define for Weil group representation V:

$$\epsilon(V, \psi) = \epsilon(V, \psi, dx, 1/2) \quad \text{in} \quad \mathbb{C}^\times,$$

in the notation of [De73]. In the non-archimdean case, such a M has a form:

$$M = \bigoplus_{d\geq 0} M_d \boxtimes \mathrm{Sym}^d,$$

where M_d is a semisimple reprsentation of W_E and Sym^d is the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of $\dim = d + 1$.
We define (cf. [GR10]):

$$\epsilon(M, \psi) = \prod_{n\geq 0} \epsilon(M_n, \psi)^{n+1} \cdot \det(-F|M_n^I)^n.$$

This constant depends only on the isomorphism class of M. Since our interest lies in the hermitian case, we will only list the formulation of the desired character of conjugate dual representations. For a uniform statement, check [GGP12, Chapter 6].

First assume M and N are conjugate selfdual representations, with signs $b(M)$ and $b(N)$. Fix ψ of E with $\psi^\sigma = \psi^{-1}$, and for $a \in C_M$, define

$$\chi_N(a) = \epsilon(M^a \otimes N, \psi).$$

Theorem 3.1. (i) *The value $\chi_N(a)$ depends only on the image of a in A, and defines a character $\chi_N : A_M \to \langle \pm 1 \rangle$.*

(ii) *If $b(M) \cdot b(N) = +1$, then $\chi_N = 1$ on A_M.*

(iii) *If $b(M) \cdot b(N) = -1$, let $\psi'(x) = \psi(tx)$ with t the nontrivial class in $F^\times/\mathbb{N}E^\times$, and define*

$$\chi'_N(a) = \epsilon(M^a \otimes N, \psi').$$

Then

$$\chi'_N = \chi_N \cdot \eta^{\dim(N)} \in \mathrm{Hom}(A_M, \pm 1).$$

We use this theorem to define the quadratic character

$$\chi_N(a_M) \cdot \chi_M(a_N)$$

on elements (a_M, a_N) in the component group $A_M \times A_N$. Here M and N are two conjugate dual representations, although by part 2 of Theorem 3.1 the character $\chi_N \times \chi_M$ can only be non-trivial when $b(M) \cdot b(N) = -1$.

3.3. **Langlands parameter of Classical Groups.** If G is a connected reductive group over F, the L-group of G is a semi-direct product

$$^{L}G = \widehat{G} \rtimes \mathrm{Gal}(K/F)$$

where $\widehat{G}$ is the complex dual group and K is a splitting field for the quasi-split inner form of G, with $\mathrm{Gal}(K/F)$ acting on $\widehat{G}$ via pinned automorphisms. Details are in Appendix B. In our case we normalize M as follows, so that the weights for $\widehat{T}$ on M are dual to the weights for the torus $T = U(1)^n$ on V: For $G = \mathrm{GL}(V/F)$, we have $\widehat{G} = \mathrm{GL}(M)$ with $\dim M = \dim V$. If $\langle e_1, \cdots, e_n \rangle$ is the basis of the character group of a maximal torus $T \subset G$ given by the weights of V, then the weights of the dual torus $\widehat{T}$ on M are the dual basis $\langle e_1', \cdots, e_n' \rangle$.

Now assume that $G \subset \mathrm{GL}(V/F)$ is a connected classical group, defined by a σ-sesquilinear form $\langle , \rangle : V \times V \to E$ of sign ϵ. The group G and its L-group ^{L}G, as well as the splitting field E of its quasi-split inner form, are given by the following table:

(E, ϵ)	G	$\widehat{G}$	K	^{L}G
$E = F$ $\epsilon = 1$	$\mathrm{SO}(V)$, $\dim V = 2n+1$	$\mathrm{Sp}_{2n}(\mathbb{C})$	F	$\mathrm{Sp}_{2n}(\mathbb{C})$
$E = F$ $\epsilon = 1$	$\mathrm{SO}(V)$, $\dim V = 2n$	$\mathrm{SO}_{2n}(\mathbb{C})$	$F(\sqrt{\mathrm{disc}(V)})$	$\mathrm{O}_{2n}(\mathbb{C})$ $(\mathrm{disc}(V) \notin E^{\times 2})$ $\mathrm{SO}_{2n}(\mathbb{C})$ $(\mathrm{disc}(V) \in E^{\times 2})$
$E = F$ $\epsilon = -1$	$\mathrm{Sp}(V)$, $\dim V = 2n$	$\mathrm{SO}_{2n+1}(\mathbb{C})$	F	$\mathrm{SO}_{2n+1}(\mathbb{C})$
$E \neq F$ $\epsilon = \pm 1$	$\mathrm{U}(V)$, $\dim V = n$	$\mathrm{GL}_n(\mathbb{C})$	E	$\mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(K/F)$ $=$ L-group of compact $\mathrm{U}(n)$

Composite the L parameter with the standard representation, it can determine a selfdual or conjugate dual represnetation M of WD_E, with the following structure:

G	$\dim(V)$	M	$\dim M$	
$\mathrm{Sp}(V)$	$2n$	Orthogonal	$2n+1$	$\det M = 1$
$\mathrm{SO}(V)$	$2n+1$	Symplectic	$2n$	
$\mathrm{SO}(V)$	$2n$	Orthogonal	2n	$\det M = \mathrm{disc}\, V$
$\mathrm{U}(V)$	$2n+1$	Conjugate-Orthogonal	$2n+1$	
$\mathrm{U}(V)$	$2n$	Conjugate-Symplectic	$2n$	

Specifically, for unitary group $U(V)$, we have the theorem ([GGP12, Lemma 3.3]):

Theorem 3.2. *The isomorphism class of the representation M determines the equivalence class of the parameter φ. The group $C_\varphi \subset \widehat{G}$ which centralizes the image of φ is isomorphic to the group C of elements in $\mathrm{Aut}(M, B)$ which centralize the image $WD(E) \to \mathrm{GL}(M)$.*

Proof. Give a sketch of proof. □

3.4. **Vogan L-packets.** We are now ready to describe the desiderata for Vogan L-packets, which will be assumed in the rest of this paper. Let G be a quasi-split, connected reductive group over a local field F. Here G' is any of the pure inner forms of G. They have the common dual group $\widehat{G}$.

(i) Every irreducible tempered representation π of G' (up to isomorphism) determines a tempered Langlands parameter

$$\varphi : WD_F \to \widehat{G} \rtimes \mathrm{Gal}(K/F)$$

(up to equivalence).
Each tempered Langlands parameter φ for G corresponds to a finite set Π_φ of irreducible representations of $G(F)$ and its pure inner forms $G'(F)$. Moreover, the cardinality of the finite set Π_φ is equal to the number of irreducible representations χ of the finite group $A_\varphi = \pi_0(C_\varphi)$.

(ii) (WHITT): Each choice of a $T(F)$-orbit θ of generic characters for G gives a bijection of finite sets

$$J(\theta) : \Pi_\varphi \to \mathrm{Irr}(A_\varphi).$$

and the L-packet Π_φ contains at most one θ'-generic representation, for each T-orbit of generic characters θ' of G. Consider the Remark 2.2, it also means that, for every $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$, there exists exactly one representation in the L-packet $\Pi^G(\varphi)$ admitting a Whittaker model of type $\mathcal{O}$. It contains such a representation precisely when the adjoint L-function of φ is regular at the point $s = 1$. In this case, we say the L-packet Π_φ is *generic*.
Assume that the L-packet Π_φ is generic. In the bijection $J(\theta)$, the unique θ-generic representation π in Π_φ corresponds to the trivial representation of A_φ. The θ'-generic representations correspond to the one dimensional representations η_g described as in [GGP12, Chapter 9 (3)].

(iii) In any of the bijections $J(\theta)$, the pure inner form which acts on the representation with parameter (φ, χ) is constrained by the restriction of the irreducible representation χ to the image of the group $\pi_0(Z(\widehat{G}))^{\mathrm{Gal}(K/F)}$ in A_φ.
More precisely, when $F \neq \mathbb{R}$, Kottwitz has identified the pointed set $H^1(F, G)$ with the group of characters of the component group of $Z(\widehat{G})^{\mathrm{Gal}(K/F)}$. The inclusion

$$Z(\widehat{G})^{\mathrm{Gal}(K/F)} \to C_\varphi$$

induces a map on component groups, whose image is central in A_φ (details: see [Ma15]). Hence an irreducible representation χ of A_φ has a central character on $\pi_0(Z(\widehat{G}))^{\mathrm{Gal}(K/F)}$, and determines a class in $H^1(F, G)$. This is the pure inner form G' that acts on the representation $\pi(\varphi, \chi)$.
The recipe for G when $k = \mathbb{R}$ is more complicate.

(iv) Since all the pure inner forms G' of G have the same center Z over F, we want all of the irreducible representations π in Π_φ have the same central character ω_π. This character is determined by φ, using the recipe in [GR10, Chapter 7].

We now make the desiderata of Vogan L-packets completely explicit for the unitary groups.

The Odd Unitary Group $G = \mathrm{U}(V)$, $\dim V = 2n + 1$

(i) A Langlands parameter is a conjugate-orthogonal representation M of WD_F, with $\dim(M) = \dim(V)$. The group $A_\varphi = A_M$ has order $2^{\#I}$, where I is the set of distinct irreducible conjugate-orthogonal summands in M.

(ii) There is a unique T-orbit on the set of generic characters, and hence a single natural isomorphism $J : \Pi_\varphi \to \mathrm{Hom}(A_\varphi, \pm 1)$.

(iii) The pure inner forms of G are the groups $G' = \mathrm{U}(V')$, where V' is a hermitian space over E with $\dim(V') = \dim(V)$.

If F is non-archimedean, there is a unique pure inner form G' such that the discriminant of V' is distinct from the discriminant of V. The representation $\pi(\varphi, \chi)$ is a representation of G if $\chi(-1) = +1$ and a representation of G' if $\chi(-1) = -1$. If $F = \mathbb{R}$ and $G = \mathrm{U}(p, q)$, then the pure inner forms are the groups $G' = \mathrm{U}(p', q')$, and $\pi(\varphi, \chi)$ is a representation of one of the groups G' with $(-1)^{q-q'} = \chi(-1)$.

(iv) The center $Z(F) = E^\times/F^\times = \mathrm{U}(1)$, and the central character of $\pi(\varphi, \chi)$ has parameter $\det(M)$.

The Even Unitary Group $G = \mathrm{U}(V)$, $\dim V = 2n$

(i) A Langlands parameter is a conjugate-symplectic representation M of WD_F, with $\dim(M) = \dim(V)$. The group $A_\varphi = A_M$ has order $2^{\#I}$, where I is the set of distinct irreducible conjugate-symplectic summands in M.

(ii) The choice of a hermitian space V identifies the set D of T-orbits on generic characters is isomorphic with the set of $\mathbb{N}E^\times$-orbits on the nontrivial additive characters ψ_0 of E/F.

(iii) The pure inner forms of G are the groups $G' = \mathrm{U}(V')$, where V' is a hermitian (or skew-hermitian) space over E with $\dim(V') = \dim(V)$.

If F is non-archimedean, there is a unique pure inner form G' such that the discriminant of V' is distinct from the discriminant of V. The representation $\pi(\varphi, \chi)$ is a representation of G if $\chi(-1) = +1$ and a representation of G' if $\chi(-1) = -1$. If $F = \mathbb{R}$ and $G = \mathrm{U}(p, q)$, then the pure inner forms are the groups $G' = \mathrm{U}(p', q')$, and $\pi(\varphi, \chi)$ is a representation of one of the groups G' with $(-1)^{q-q'} = \chi(-1)$.

(iv) The center $Z(F) = E^\times/F^\times = \mathrm{U}(1)$, and the central character of $\pi(\varphi, \chi)$ has parameter $\det(M)$.

Remark 3.1. *The references for the local Langlands Correspondence of Unitary Groups are as follows: When $F = \mathbb{R}$, the local Langlands correspondence has been constructed by Langlands himself* [Lan] *building on previous results of Harish-Chandra. When F is p-adic, the local Langlands correspondence is known in a variety of cases. In particular, for unitary groups, which are our main concern, the existence of the Langlands correspondence is now fully established thanks to Mok* [Mok15] *and Kaletha-Minguez-Shin-White* [KMSW14] *both building up on previous work of Arthur who dealt with orthogonal and symplectic groups* [Art13]*.*

Remark 3.2. *For the unitary group $G(V)$, the proof of the Local Langlands Classification given in* [Mok15] *characterizes the Local Langlands Correspondence via a family of character identities arising in the theory of endoscopy. This elaborate theory requires one to normalize certain "transfer factors". By the work of Kottwitz and Shelstad* [KS99]*, one can fix a normalization of the transfer factors by fixing a Whittaker Datum for unitary group.*

3.5. **Precise Criterion: Local Gan-Gross-Prasad Conjecture.** In this section, we propose a conjecture for the restriction problem formulated in Section 2.4. Recall that we are considering the restriction of irreducible representations $\pi = \pi_V \boxtimes \pi_W$ of $G = G(V) \times G(W)$ to the subgroup $H = N \cdot G(W) \subset G$. Recall also that, we have defined a unitary representation ξ of H of dimension 1 in Bessel case. Then we are interested in

$$d(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\xi}, \mathbb{C}),$$

which is known to be ≤ 1. In this section, we shall give precise criterion for this Hom space to be nonzero, in terms of the Langlands-Vogan parameterization of irreducible representations of G. We first formulate the desired characters.

$G = \mathrm{U}(V) \times \mathrm{U}(W)$, $\dim W^{\perp}$ **odd** Recall that we have fixed a nontrivial character

$$\psi : E/F \to \mathbb{C}$$

up to the action of $\mathbb{N}E^{\times}$ in order to define a generic character of the even unitary group, which determines its Local Langlands Correspondence. If $\delta \in F^{\times}$ is the discriminant of the odd hermitian space in the pair (V, W), consider the character $\psi_{-2\delta}(x) = \psi(-2\delta x)$. Then we set

$$\chi_{\varphi}((a_i),(b_j)) = \epsilon(\varphi_1^{(a_i)} \otimes \varphi_2, \psi_{-2\delta})\epsilon(\varphi_1 \otimes \varphi_2^{(b_j)}, \psi_{-2\delta})$$

as the character for $A_{\varphi_1} \times A_{\varphi_2}$, where A_{φ_1}, A_{φ_2} are the component groups of the Vogan L-packet for $\mathrm{U}(V)$, $\mathrm{U}(W)$, respectively.

Theorem 3.3. (i) *There exists exactly one representation π_α in the disjoint union of tempered L-packets*

$$\bigsqcup_{\alpha \in H^1(F,H)} \Pi^{G_\alpha}(\varphi)$$

such that $m(\pi_\alpha) = 1$.

(ii) *Having fixed the Langlands-Vogan parameterization for the unitary group G in hermitian case, and its pure inner forms in the various cases above, the unique representation π_α in the tempered Vogan packet Π_φ which satisfies $\mathrm{Hom}_H(\pi \otimes \overline{\xi}, \mathbb{C}) \neq 0$ enjoys the following characterization:*

$$\pi_\alpha = \pi(\varphi, \chi_\varphi).$$

where χ_φ is defined in Section 3.6.

Beuzart Plessis proved this precise conjecture for p-adic hermitian case in [BP14], Hang Xue proved this precise one for archimedean hermitian case in [Xu23] for tempered case.

3.6. **Example: Discrete Series.** Before starting our final proof, we want to give an explicit example. In this section, we will consider the precise formulation of the GGP conjecture in codimension one case for the discrete series of real unitary group.
First fix the following notation: the symmetric group on n letters is denoted by S_n. For non-negative integers p and q, we set $n = p + q$ and define the unitary group $\mathrm{U}(p, q)$ of signature (p, q) by

$$\mathrm{U}(p,q) = \left\{ g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^t\overline{g} \begin{pmatrix} frm[o]--_p & 0 \\ 0 & -frm[o]--_q \end{pmatrix} g = \begin{pmatrix} frm[o]--_p & 0 \\ 0 & -frm[o]--_q \end{pmatrix} \right\}.$$

We also denote by $\mathrm{U}_n(\mathbb{R})$ a unitary group of size n, i.e., $\mathrm{U}_n(\mathbb{R})$ isomorphic to $\mathrm{U}(p, q)$ for some non-negative integers p, q such that $n = p + q$.
For a reductive Lie group G, we denote by $\mathrm{Irr}_{\mathrm{disc}}(G)$ (resp $\mathrm{Irr}_{\mathrm{temp}}(G)$) the set of equivalence classes of discrete series representations (resp tempered representations) of G. In this paper, we only identify them with the $(\mathfrak{g}_{\mathbb{C}}, K)$-modules associated to real groups, where $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of Lie algebra of G and K is a fixed maximal compact subgroup of G.

3.6.1. *Harish-Chandra parameters.* The Harish-Chandra parameters classify irreducible discrete series representations. Let $G = \mathrm{U}(p,q)$. We set $K \cong \mathrm{U}(p) \times \mathrm{U}(q)$ to be the maximal compact subgroup of G consisting of the usual block diagonal matrices, and T to be the maximal compact torus of G consisting of diagonal matrices. We denote the Lie algebras of G, K and T by $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{t}$, and its complexifications by $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$, respectively. The set Δ_c of compact roots of $\mathfrak{g}_{\mathbb{C}}$ with respect $\mathfrak{t}_{\mathbb{C}}$ and the set Δ_n of non-compact roots are given by

$$\begin{aligned} \Delta_c &= \{e_i - e_j \mid 1 \leq i,j \leq p\} \cup \{f_i - f_j \mid 1 \leq i,j \leq q\}, \\ \Delta_n &= \{\pm(e_i - f_j) \mid 1 \leq i \leq p,\ 1 \leq j \leq q\}, \end{aligned}$$

respectively. Here, $e_i, f_j \in \mathfrak{t}_{\mathbb{C}}^*$ are defined by

$$e_i \colon \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i, \quad f_j \colon \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_{p+j}.$$

Note that e_i and f_j belong to $\sqrt{-1}\mathfrak{t}^*$, i.e., the images of $e_i(\mathfrak{t})$ and $f_j(\mathfrak{t})$ are in $\sqrt{-1}\mathbb{R}$. The Harish-Chandra parameter $\mathrm{HC}(\pi)$ of a discrete series representation π of $\mathrm{U}(p,q)$ is of the form

$$\mathrm{HC}(\pi) = (\lambda_1, \dots, \lambda_p; \lambda'_1, \dots, \lambda'_q) \in \sqrt{-1}\mathfrak{t}^*,$$

where

(i) $\lambda_i, \lambda'_j \in \mathbb{Z} + \frac{n-1}{2}$;

(ii) $\lambda_i \neq \lambda'_j$ for $1 \leq i \leq p$ and $1 \leq j \leq q$;

(iii) $\lambda_1 > \cdots > \lambda_p$ and $\lambda'_1 > \cdots > \lambda'_q$.

Here, using the basis $\{e_1, \dots, e_p, f_1, \dots, f_q\}$, we identify $\sqrt{-1}\mathfrak{t}^*$ with $\mathbb{R}^p \times \mathbb{R}^q$. Via this identification, we regard $\mathrm{HC}(\pi)$ as an element of $(\mathbb{Z} + \frac{n-1}{2})^p \times (\mathbb{Z} + \frac{n-1}{2})^q$. Hence we obtain an injection

$$\mathrm{HC} \colon \mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p,q)) \hookrightarrow \left(\mathbb{Z} + \frac{n-1}{2}\right)^p \times \left(\mathbb{Z} + \frac{n-1}{2}\right)^q.$$

The infinitesimal character τ_λ of π is the W_G-orbit of $\lambda = \mathrm{HC}(\pi)$, where W_G is the Weyl group of G relative to T. As well as $\lambda = \mathrm{HC}(\pi)$ is regarded as an element of $(\mathbb{Z} + \frac{n-1}{2})^p \times (\mathbb{Z} + \frac{n-1}{2})^q$, we regard τ_λ as an element of $(\mathbb{Z} + \frac{n-1}{2})^n / S_n$. Note that given $\tau \in (\mathbb{Z} + \frac{n-1}{2})^n / S_n$, there are exactly $n!/(p!q!)$ discrete series representations of $\mathrm{U}(p,q)$ whose infinitesimal characters are equal to τ.

3.6.2. *L-parameters.* Local Langlands correspondence is a parametrization of irreducible tempered representations of $\mathrm{U}(p,q)$ in terms of L-parameters.

For $\alpha \in \frac{1}{2}\mathbb{Z}$, we define a unitary character $\chi_{2\alpha}$ of $\mathbb{C}^\times$ by

$$\chi_{2\alpha}(z) = \overline{z}^{-2\alpha}(z\overline{z})^\alpha = (z/\overline{z})^\alpha.$$

Note that $\chi_{2\alpha}(\overline{z}) = \chi_{2\alpha}(z)^{-1} = \chi_{-2\alpha}(z)$. When $a > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we have

$$\chi_{2\alpha}(ae^{\sqrt{-1}\theta}) = e^{2\alpha\sqrt{-1}\theta}.$$

Define $\Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$ by

$$\Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R})) = \left\{ \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \mid \alpha_i \in \frac{1}{2}\mathbb{Z},\ 2\alpha_i \equiv n-1,\ \alpha_1 > \cdots > \alpha_n \right\}.$$

For $\phi = \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$, we define a component group A_ϕ of ϕ by

$$A_\phi = (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_1} \oplus \cdots \oplus (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_n}.$$

Namely, A_ϕ is a free $(\mathbb{Z}/2\mathbb{Z})$-module of rank n equipped with a canonical basis $\{e_{2\alpha_1}, \dots, e_{2\alpha_n}\}$ associated to $\{\chi_{2\alpha_1}, \dots, \chi_{2\alpha_n}\}$. More generally, we define $\Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ by the set of representations ϕ of $\mathbb{C}^\times$ of the form

$$\phi = (m_1\chi_{2\alpha_1} \oplus \cdots \oplus m_u\chi_{2\alpha_u}) \oplus (\xi_1 \oplus \cdots \oplus \xi_v) \oplus ({}^c\xi_1^{-1} \oplus \cdots \oplus {}^c\xi_v^{-1}),$$

where

(i) $\alpha_i \in \frac{1}{2}\mathbb{Z}$ satisfies $2\alpha_i \equiv n-1$ and $\alpha_1 > \cdots > \alpha_u$;

(ii) $m_i \geq 1$ is the multiplicity of $\chi_{2\alpha_i}$ in ϕ;

(iii) $m_1 + \cdots + m_u + 2v = n$;

(iv) ξ_i is a unitary character of $\mathbb{C}^\times$, which is not of the form $\chi_{2\alpha}$ with $2\alpha \equiv n-1$;

(v) ${}^c\xi_i^{-1}$ is the unitary character of $\mathbb{C}^\times$ defined by ${}^c\xi_i^{-1}(z) = \xi_i(\overline{z}^{-1})$.

For such ϕ, we define a component group A_ϕ of ϕ by

$$A_\phi = (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_1} \oplus \cdots \oplus (\mathbb{Z}/2\mathbb{Z})e_{2\alpha_u}.$$

We denote the Pontryagin dual of A_ϕ by $\widehat{A_\phi}$. For $\eta \in \widehat{A_\phi}$, define $-\eta \in \widehat{A_\phi}$ by $(-\eta)(e_{2\alpha_i}) = -\eta(e_{2\alpha_i})$ for $i = 1, \dots, u$.

We define an additive character $\psi^{\mathbb{C}}_{-2}$ of $\mathbb{C}$ by

$$\psi^{\mathbb{C}}_{-2}(z) = \exp(2\pi(\overline{z} - z))$$

for $z \in \mathbb{C}$. For a (continuous, completely reducible) representation ϕ of $\mathbb{C}^\times$, the ϵ-factor of ϕ satisfies the following:

(i) $\epsilon(s, \phi_1 \oplus \phi_2, \psi^{\mathbb{C}}_{-2}) = \epsilon(s, \phi_1, \psi^{\mathbb{C}}_{-2}) \cdot \epsilon(s, \phi_2, \psi^{\mathbb{C}}_{-2})$;

(ii) $\epsilon(1/2, \xi \oplus {}^c\xi^{-1}, \psi^{\mathbb{C}}_{-2}) = 1$ for any character ξ of $\mathbb{C}^\times$;

(iii) $\epsilon(1/2, \chi_{2\alpha}, \psi^{\mathbb{C}}_{-2}) = 1$ for $\alpha \in \mathbb{Z}$;

(iv) When $\alpha \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$,

$$\epsilon\left(\frac{1}{2}, \chi_{2\alpha}, \psi^{\mathbb{C}}_{-2}\right) = \begin{cases} -1 \text{ if } \alpha > 0, \\ +1 \text{ if } \alpha < 0. \end{cases}$$

The local Langlands correspondence for $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q))$ is as follows:

Theorem 3.4. (i) *There is a canonical surjection*

$$\bigsqcup_{p+q=n} \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p,q)) \to \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R})).$$

Meanwhile, there is a bijection

$$J \colon \Pi_\phi \to \widehat{A_\phi}.$$

(ii) *If $\phi = \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$, the L-packet Π_ϕ consists of discrete series representations of various $\mathrm{U}(p,q)$ whose infinitesimal characters are $(\alpha_1, \dots, \alpha_n) \in (\mathbb{Z} + \frac{n-1}{2})^n/S_n$.*

(iii) *If $\phi = \chi_{2\alpha_1} \oplus \cdots \oplus \chi_{2\alpha_n} \in \Phi_{\mathrm{disc}}(\mathrm{U}_n(\mathbb{R}))$ with $\alpha_1 > \cdots > \alpha_n$, the Harish-Chandra parameter*

$$\mathrm{HC}(\pi(\phi, \eta)) = (\lambda_1, \dots, \lambda_p; \lambda'_1, \dots, \lambda'_q)$$

of $\pi(\phi, \eta) \in \Pi_\phi$ is given so that

- $\{\lambda_1, \dots, \lambda_p, \lambda'_1, \dots, \lambda'_q\} = \{\alpha_1, \dots, \alpha_n\}$;
- $\alpha_i \in \{\lambda_1, \dots, \lambda_p\}$ *if and only if* $\eta(e_{2\alpha_i}) = (-1)^{i-1}$.

In particular, $\pi(\phi, \eta) \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{U}(p,q))$ with

$$p = \#\{i \mid \eta(e_{2\alpha_i}) = (-1)^{i-1}\}, \quad q = \#\{i \mid \eta(e_{2\alpha_i}) = (-1)^i\}.$$

(iv) *If* $\phi = \xi \oplus \phi_0 \oplus {}^c\xi^{-1}$ *with a unitary character* ξ *of* $\mathbb{C}^\times$ *and an element* ϕ_0 *in* $\Phi_{\mathrm{temp}}(\mathrm{U}_{n-2}(\mathbb{R}))$*, for any* $\pi(\phi_0, \eta_0) \in \Pi_{\phi_0} \cap \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p-1, q-1))$*, the induced representation* $\mathrm{Ind}_P^{\mathrm{U}(p,q)}(\xi \otimes \pi(\phi_0, \eta_0))$ *decomposes as follows:*

$$\mathrm{Ind}_P^{\mathrm{U}(p,q)}(\xi \otimes \pi(\phi_0, \eta_0)) = \bigoplus_{\substack{\eta \in \widehat{A_\phi}, \\ \eta | A_{\phi_0} = \eta_0}} \pi(\phi, \eta).$$

Here, P *is a parabolic subgroup of* $\mathrm{U}(p, q)$ *with Levi subgroup* $M_P = \mathbb{C}^\times \times \mathrm{U}(p-1, q-1)$*.*

(v) *The contragredient representation of* $\pi(\phi, \eta)$ *is given by* $\pi(\phi^\vee, \eta^\vee)$*, where* $\eta^\vee : A_{\phi^\vee} \to \{\pm 1\}$ *is defined by*

$$\eta^\vee(e_{-2\alpha_i}) = \begin{cases} \eta(e_{2\alpha_i}) \text{ if } n \textit{ is odd,} \\ -\eta(e_{2\alpha_i}) \text{ if } n \textit{ is even} \end{cases}$$

for any $e_{-2\alpha_i} \in A_{\phi^\vee}$*.*

(vi) *If* $\pi = \pi(\phi, \eta) \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(p, q))$*, then* $\pi(\phi, -\eta) \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}(q, p))$ *is the same representation as* π *via the canonical identification* $\mathrm{U}(p, q) = \mathrm{U}(q, p)$ *as subgroups of* $\mathrm{GL}_n(\mathbb{C})$*.*

(vii) *If* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$*, then* $\phi\chi_2 = \phi \otimes \chi_2 \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ *and there is a canonical identification* $A_\phi = A_{\phi\chi_2}$*. If* $\pi = \pi(\phi, \eta)$*, the corresponding representation* $\pi(\phi\chi_2, \eta)$ *is the determinant twist* $\pi \otimes \det$*.*

In this Theorem, we see that

$$(-1)^{i-1} = \epsilon(\phi \otimes \chi_{-2\alpha_i} \otimes \chi_{-1}, \psi^{\mathbb{C}}_{-2}).$$

Hence the Harish-Chandra parameter of π and the unitary group $\mathrm{U}(p, q)$ which acts on π can be determined by the L-parameter $\lambda = (\phi, \eta)$ of π and certain root numbers.

3.6.3. *Local Gan–Gross–Prasad conjecture.* We take the following pair

$$(\mathrm{U}_n(\mathbb{R}), \mathrm{U}_{n+1}(\mathbb{R})) = \begin{cases} (\mathrm{U}(p, q), \mathrm{U}(p+1, q)) \text{ if } n \text{ is even,} \\ (\mathrm{U}(p, q), \mathrm{U}(p, q+1)) \text{ if } n \text{ is odd} \end{cases}$$

The multiplicity one theorem gives: for $\pi_n \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ and $\pi_{n+1} \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{U}_{n+1}(\mathbb{R}))$, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta \mathrm{U}_n(\mathbb{R})}(\pi_n \otimes \pi_{n+1}, \mathbb{C}) \leq 1.$$

The precise theorem 3.3 is:

Theorem 3.5. *Let* $\phi_n \in \Phi_{\mathrm{temp}}(\mathrm{U}_n(\mathbb{R}))$ *and* $\phi_{n+1} \in \Phi_{\mathrm{temp}}(\mathrm{U}_{n+1}(\mathbb{R}))$ *such that*

$$\phi_n = (m_1\chi_{2\alpha_1} \oplus \cdots \oplus m_u\chi_{2\alpha_u}) \oplus (\xi_1 \oplus \cdots \oplus \xi_v) \oplus ({}^c\xi_1^{-1} \oplus \cdots \oplus {}^c\xi_v^{-1}),$$
$$\phi_{n+1} = (m'_1\chi_{2\beta_1} \oplus \cdots \oplus m'_{u'}\chi_{2\beta_{u'}}) \oplus (\xi'_1 \oplus \cdots \oplus \xi'_{v'}) \oplus ({}^c\xi'^{-1}_1 \oplus \cdots \oplus {}^c\xi'^{-1}_{v'}),$$

where

- $\alpha_i, \beta_j \in \frac{1}{2}\mathbb{Z}$ *such that* $2\alpha_i \equiv n-1 \bmod 2$ *and* $2\beta_j \equiv n \bmod 2$*;*
- $m_i \geq 1$ *(resp* $m'_j \geq 1$*) is the multiplicity of* $\chi_{2\alpha_i}$ *(resp* $\chi_{2\beta_j}$*) in* ϕ_n *(resp* ϕ_{n+1}*);*
- $m_1 + \cdots + m_u + 2v = n$ *and* $m'_1 + \cdots + m'_{u'} + 2v' = n+1$*;*
- ξ_i *(resp* ξ'_j*) is a unitary character of* $\mathbb{C}^\times$*, which is not of the form* $\chi_{2\alpha}$ *with* $2\alpha \equiv n-1 \bmod 2$ *(resp* $\chi_{2\beta}$ *with* $2\beta \equiv n \bmod 2$*).*

Then there exists a unique pair of representations $(\pi_n, \pi_{n+1}) \in \Pi_{\phi_n} \times \Pi_{\phi_{n+1}}$ *such that*

- (π_n, π_{n+1}) *is a pair of representations of a relevant pair* $(\mathrm{U}_n(\mathbb{R}), \mathrm{U}_{n+1}(\mathbb{R}))$*;*
- $\mathrm{Hom}_{\Delta \mathrm{U}_n(\mathbb{R})}(\pi_n \otimes \pi_{n+1}, \mathbb{C}) \neq 0$*.*

Moreover,

$$J(\pi_n)(e_{2\alpha_i}) = \epsilon(\chi_{2\alpha_i} \otimes \phi_{n+1}, \psi^{\mathbb{C}}_{-2}) = (-1)^{\#\{j\in\{1,\cdots,u'\}|\beta_j+\alpha_i>0\}},$$
$$J(\pi_{n+1})(e_{2\beta_j}) = \epsilon(\phi_n \otimes \chi_{2\beta_j}, \psi^{\mathbb{C}}_{-2}) = (-1)^{\#\{i\in\{1,\cdots,u\}|\alpha_i+\beta_j>0\}}$$

for $e_{2\alpha_i} \in A_{\phi_n}$ *and* $e_{2\beta_j} \in A_{\phi_{n+1}}$.

4. Solution for Hermitian Case

In this section we focus on the proof of main Theorem 3.3. First we will introduce the preliminary materials in the expansion of the geometric side, including the region for integration $\Gamma(G,H)$, and a part of integrand, c_θ. The reason for how $m(\pi)$ appears in the trace formula is described in Section 4.4. Then using the spectral and geometric expansions for the distributions J, we will establish a certain integral formula for the multiplicity $m(\pi)$ when representation π is tempered. These results will be proved together in a common inductive way, in the Section 4.7. Finally, we will combine all these to calculate the multiplicity $m(\pi)$ directly. Appendix C is an important prerequisite for this Chapter where includes sufficient analytic preparation.

4.1. **Function c_θ.** In this section, we may define the main ingredient of the geometric expansion. Recall the local expansion of the quasi-characters in Appendix C.2.1. With these expansions, we can work over $\mathbb{R}$ and $\mathbb{Q}_p$ uniformly. Let θ be a quasi-character on $G(F)$. Then, for all $x \in G_{\mathrm{ss}}(F)$ we have a local expansion

$$D^G(xe^X)^{1/2}\theta(xe^X) = D^G(xe^X)^{1/2} \sum_{\mathcal{O}\in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{\theta,\mathcal{O}}(x)\widehat{j}(\mathcal{O},X) + O(|X|)$$

for all $X \in \mathfrak{g}_{x,\mathrm{reg}}(F)$ sufficiently near 0 (in the p-adic case, this follows from the fact that $D^G(X)^{1/2}\widehat{j}(\mathcal{O},X) = O(|X|)$ near 0 for all $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})\backslash\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$, see Appendix A on the orbital integral). It follows from the homogeneity property of the functions $\widehat{j}(\mathcal{O},.)$ and their linear independence, that the coefficients $c_{\theta,\mathcal{O}}(x)$, $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$, are uniquely defined. We set

$$c_\theta(x) = \frac{1}{|\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)|} \sum_{\mathcal{O}\in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{\theta,\mathcal{O}}(x)$$

for all $x \in G_{\mathrm{ss}}(F)$. This defines a function

$$c_\theta : G_{\mathrm{ss}}(F) \to \mathbb{C}$$

Similarly, to any quasi-character θ on $\mathfrak{g}(F)$ we associate a function

$$c_\theta : \mathfrak{g}_{\mathrm{ss}}(F) \to \mathbb{C}$$

Proposition 4.1. *Let θ be a quasi-character on $G(F)$ and let $x \in G_{\mathrm{ss}}(F)$. Then*

(i) *If G_x is not quasi-split then $c_\theta(x) = 0$.*

(ii) *Assume that G_x is quasi-split. Let $B_x \subset G_x$ be a Borel subgroup and $T_{\mathrm{qd},x} \subset B_x$ be a maximal torus (both defined over F). Then, we have*

$$D^G(x)^{1/2}c_\theta(x) = |W(G_x, T_{\mathrm{qd},x})|^{-1} \lim_{x'\in T_{\mathrm{qd},x}(F)\to x} D^G(x')^{1/2}\theta(x')$$

(in particular, the limit exists).

(iii) *Let $\Omega_x \subseteq G_x(F)$ be a G-good open neighborhood of x. Then, we have*

$$D^G(y)^{1/2}c_\theta(y) = D^{G_x}(y)^{1/2}c_{\theta_{x,\Omega_x}}(y)$$

for all $y \in \Omega_{x,ss}$.

Proposition 4.2. *Let θ be a quasi-character on $\mathfrak{g}(F)$ and let $X \in \mathfrak{g}_{\mathrm{ss}}(F)$. Then*

(i) *If G_X is not quasi-split then $c_\theta(X) = 0$.*

(ii) *Assume that G_X is quasi-split. Let $B_X \subset G_X$ be a Borel subgroup and $T_{\mathrm{qd},X} \subset B_X$ be a maximal torus (both defined over F). Then, we have*

$$D^G(X)^{1/2}c_\theta(X) = |W(G_X, T_{\mathrm{qd},X})|^{-1} \lim_{X' \in \mathfrak{t}_{\mathrm{qd},X}(F) \to X} D^G(X')^{1/2}\theta(X')$$

(in particular, the limit exists).

(iii) *For all $\lambda \in F^\times$ let $M_\lambda\theta$ be the quasi-character defined by $(M_\lambda\theta)(X) = |\lambda|^{-\delta(G)/2}\theta(\lambda^{-1}X)$ for all $X \in \mathfrak{g}_{\mathrm{reg}}(F)$. Then we have*

$$D^G(X)^{1/2}c_{M_\lambda\theta}(X) = D^G(\lambda^{-1}X)^{1/2}c_\theta(\lambda^{-1}X)$$

for all $X \in \mathfrak{g}_{\mathrm{ss}}(F)$ and all $\lambda \in F^\times$.

(iv) *Assume that G is quasi-split. Let $B \subset G$ be a Borel subgroup and $T_{\mathrm{qd}} \subset B$ be a maximal torus (both defined over F). Then, for all $X \in \mathfrak{t}_{\mathrm{qd,reg}}(F)$, we have*

$$c_{\widehat{j}(X,.)}(0) = 1$$

4.2. **More Conjugacy Classes and the Kottwitz Sign.** This section separates to two parts: first is to introduce the objects on the geometric side, which is crucial to define certain continuous linear forms on the spaces of quasi-characters $QC(G(F))$ and $SQC(\mathfrak{g}(F))$ (Definition for the space of functions: Appendix C); second one is to study one specific relation called the "strongly stably conjugate", which establishs the transfer between different pure inner forms.

Fix a Hermitian GGP triple (G, H, ξ). Let $x \in H_{\mathrm{ss}}(F)$. We first give an explicit description of the triple (G_x, H_x, ξ_x) where $\xi_x = \xi_{|H_x(F)}$. Up to conjugation, we may assume that $x \in U(W)_{\mathrm{ss}}(F)$. Denote by W'_x and V'_x the kernel of $1 - x$ in W and V respectively and by W''_x the image of $1-x$. We then have the orthogonal decompositions $W = W'_x \oplus^\perp W''_x$ and $V = V'_x \oplus^\perp W''_x$. Set $H'_x = U(W'_x) \ltimes N_x$ (where N_x is the centralizer of x in N), $G'_x = U(W'_x) \times U(V'_x)$, $H''_x = U(W''_x)_x$ and $G''_x = U(W''_x)_x \times U(W''_x)_x$. We have natural decompositions

$$U(V)_x = U(V'_x) \times U(W''_x)_x,\ \ U(W)_x = U(W'_x) \times U(W''_x)_x \text{ and } H_x = U(W)_x \ltimes N_x$$

Moreover, we easily check that $U(W''_x)_x$ commutes with N_x. Hence, we also have natural decompositions

$$G_x = G'_x \times G''_x \text{ and } H_x = H'_x \times H''_x$$

the inclusions $H_x \subset G_x$ being the product of the two inclusions $H'_x \subset G'_x$ and $H''_x \subset G''_x$. It is clear that ξ_x is trivial on H''_x, so that we get a decomposition

$$(G_x, H_x, \xi_x) = (G'_x, H'_x, \xi'_x) \times (G''_x, H''_x, 1)$$

where $\xi'_x = \xi_{|H'_x}$ and the product of triples is obviously defined. Note that the triple (G'_x, H'_x, ξ'_x) coincides with the GGP triple associated to the pair (V'_x, W'_x). The second triple $(G''_x, H''_x, 1)$ is also of a particular shape: the group G''_x is the product of two copies of H''_x and the inclusion $H''_x \subset G''_x$ is the diagonal one. Finally, note that although we have assumed $x \in U(W)_{\mathrm{ss}}(F)$, there is a decomposition similar to this for any $x \in H_{\mathrm{ss}}(F)$ (just conjugated x inside $H(F)$ to an element in $U(W)_{\mathrm{ss}}(F)$) and that if $x, y \in H_{\mathrm{ss}}(F)$ are $H(F)$-conjugate there are natural isomorphisms of triples

$$(G'_x, H'_x, \xi'_x) \simeq (G'_y, H'_y, \xi'_y) \text{ and } (G''_x, H''_x, 1) \simeq (G''_y, H''_y, 1)$$

well-defined up to inner automorphisms (by $H'_x(F)$ and $H''_x(F)$ respectively).

Following these notations, we denote by $\Gamma(H)$, $\Gamma(H_x)$, $\Gamma(G)$ and $\Gamma(G_x)$ the sets of semi-simple conjugacy classes in $H(F)$, $H_x(F)$, $G(F)$ and $G_x(F)$ respectively and we equip them with topologies. We see easily using the above descriptions of both H_x and

G_x that the two maps $\Gamma(H_x) \to \Gamma(G_x)$ and $\Gamma(H) \to \Gamma(G)$ are injective. Since these maps are continuous and proper as we just said, and $\Gamma(H_x)$, $\Gamma(G_x)$, $\Gamma(H)$, $\Gamma(G)$ are all Hausdorff and locally compact, it follows that $\Gamma(H_x) \to \Gamma(G_x)$ and $\Gamma(H) \to \Gamma(G)$ are closed embeddings.

We now define a subset $\Gamma(G, H)$ of $\Gamma(H)$ as follows: let $x \in H_{\mathrm{ss}}(F)$, $x \in \Gamma(G, H)$ if and only if H''_x is an anisotropic torus (and hence G''_x also). By closed embedding, we may also see $\Gamma(G, H)$ as a subset of $\Gamma(G)$. Notice that $\Gamma(G, H)$ is a subset of $\Gamma_{\mathrm{ell}}(G)$ that contains 1. We now equip $\Gamma(G, H)$ with a topology, which is finer than the one induced from $\Gamma(G)$, and a measure. For this, we need to give a more concrete description of $\Gamma(G, H)$.
We can write $\Gamma(G, H)$ be the following set of conjugacy classes in $H_{\mathrm{ss}}(F) = U(W)_{\mathrm{ss}}(F)$:

$$\Gamma(G, H) = \Big(\coprod_{W' \subseteq W \ nondeg} U(W')(F)_{\mathrm{ell}}\Big)/U(W)(F)$$

where the union is over the set of nondegenerate subspaces $W' \subseteq W$. Here $U(W')(F)_{\mathrm{ell}}$ denots the elliptic regular locus. Note that since $U(W)(F)$-conjugation in $U(W')(F)$ is the same than $U(W')(F)$-conjugation, we may rewrite

$$\Gamma(G, H) = \coprod_{W' \subseteq W \ nondeg/U(W)(F)} U(W')(F)_{\mathrm{ell}}/conj$$

where this time the disjoint union is over the set of $U(W)(F)$-orbits of nondegenerate subspace $W' \subseteq W$. Fix on $U(W')(F)_{\mathrm{ell}}$ the measure as the following:

$$\int_{U(W')(F)_{\mathrm{ell}}/conj} \varphi(x)dx = \sum_{T \in \mathcal{T}_{\mathrm{ell}}(U(W'))} |W(T)|^{-1}\nu(t) \int_{T(F)} \varphi(t)dt,$$

where $\mathcal{T}_{\mathrm{ell}}(U(W'))$ is a set of representatives for the conjugacy classes of maximal elliptic tori in $U(W')$, and $\nu(t)$ is the unique measure gives $T(F)/A_T(F)$ the measure 1.

More generally, for all $x \in H_{\mathrm{ss}}(F)$ we may construct a subset $\Gamma(G_x, H_x)$ of $\Gamma(G_x)$ which is equipped with its own topology as follows. By the natural isomorphisms of triples, we have a decomposition $\Gamma(G_x) = \Gamma(G'_x) \times \Gamma(G''_x)$. Since the triple (G'_x, H'_x, ξ_x) is a GGP triple, the previous construction provides us with a space $\Gamma(G'_x, H'_x)$ of semi-simple conjugacy classes in $G'_x(F)$. On the other hand, we define $\Gamma(G''_x, H''_x)$ to be the image of $\Gamma_{ani}(H''_x)$, by the natural inclusion $\Gamma(H''_x) \subset \Gamma(G''_x)$. In Appendix A, we already equipped $\Gamma(G''_x, H''_x) = \Gamma_{ani}(H''_x)$ with a quotient topology. We now set

$$\Gamma(G_x, H_x) = \Gamma(G'_x, H'_x) \times \Gamma(G''_x, H''_x)$$

and we equip this set with the product of the topologies defined on $\Gamma(G'_x, H'_x)$ and $\Gamma(G''_x, H''_x)$. Note that $\Gamma(G_x, H_x) = \emptyset$ unless $x \in G(F)_{\mathrm{ell}}$ (because otherwise $\Gamma_{ani}(H''_x) = \emptyset$). Similarly, we may define $\Gamma^{\mathrm{Lie}}(G, H)$ as the set of semi-simple conjugacy classes $X \in \Gamma(\mathfrak{h})$ such that $H''_X := U(\mathrm{Im}(X)_{|W})_X$ is an anisotropic torus. It is again a subset of $\Gamma_{\mathrm{ell}}(\mathfrak{g})$ that contains 0, and can be decomposed as $\Gamma^{\mathrm{Lie}}(G, H) = \coprod_{T \in \mathcal{T}} \mathfrak{t}_\natural(F)/W(T)$, where for $T \in \mathcal{T}$, $\mathfrak{t}_\natural$ denotes the Zariski open subset consisting of elements $X \in \mathfrak{t}$ that are regular in $\mathfrak{u}(W''_T)$ and acting without the eigenvalue 0 on W''_T. It again endows with natural topology and measure (details: see [BP20, Subsection 11.1]).

Recall that two regular elements $x, y \in G_{\mathrm{reg}}(F)$ are said to be *stably conjugate* if there exists $g \in G(\overline{F})$ such that $y = gxg^{-1}$ and $g^{-1}\sigma(g) \in G_x$ for all $\sigma \in \Gamma_F = \mathrm{Gal}(\overline{F}/F)$. We will need to extend this definition to more general semi-simple elements. The definition that we will adopt is as follows. We will say that two semi-simple elements $x, y \in G_{\mathrm{ss}}(F)$ are *strongly stably conjugate* and we will write

$$x \sim_{\mathrm{stab}} y$$

if there exists $g \in G(\overline{F})$ such that $y = gxg^{-1}$ and the isomorphism $\mathrm{Ad}(g) : G_x \simeq G_y$ is defined over F ([BP20]). This last condition has the following concrete interpretation: it means that the 1-cocycle $\sigma \in \Gamma_F \mapsto g^{-1}\sigma(g)$ takes its values in $Z(G_x)$ the center of G_x (this is because $Z(G_x)$ coincides with the centralizer of G_x in $Z_G(x)$ since G_x contains a maximal torus of G which is its own centralizer). Moreover, for $x \in G_{\mathrm{ss}}(F)$ the set of $G(F)$-conjugacy classes inside the strong stable conjugacy class of x is easily seen to be in natural bijection with

$$\mathrm{Im}\left(H^1(F, Z(G_x)) \to H^1(F, Z_G(x))\right) \cap \mathrm{Ker}\left(H^1(F, Z_G(x)) \to H^1(F, G)\right).$$

Let (G', ψ, c) be a pure inner form of G. Then, we will say that two semi-simple elements $x \in G_{\mathrm{ss}}(F)$ and $y \in G'_{\mathrm{ss}}(F)$ are *strongly stably conjugate* and we will write

$$x \sim_{\mathrm{stab}} y$$

(this extends the previous notation) if there exists $g \in G(F)$ such that $y = \psi(gxg^{-1})$ and the isomorphism $\psi \circ \mathrm{Ad}(g) : G_x \simeq G_y$ is defined over F. Again, the last condition has an interpretation in terms of cohomological classes: it means that the 1-cocycle $\sigma \in \Gamma_F \mapsto g^{-1}c_\sigma \sigma(g)$ takes its values in $Z(G_x)$. For $x \in G_{\mathrm{ss}}(F)$ the set of semi-simple conjugacy classes in $G'(F)$ that are strongly stably conjugate to x is naturally in bijection with

$$\mathrm{Im}\left(H^1(F, Z(G_x)) \to H^1(F, Z_G(x))\right) \cap p_x^{-1}(\alpha)$$

where $\alpha \in H^1(F, G)$ parametrizes the equivalence class of the pure inner form (G', ψ, c) and p_x denotes the natural map $H^1(F, Z_G(x)) \to H^1(F, G)$. Specifically, the bijection is given as follows: for $y \in \Gamma(G_\alpha)$ such that $x \sim_{\mathrm{stab}} y$ choose $g \in G(\overline{F})$ so that $\psi_\alpha(gxg^{-1}) = y$. Then the cohomological class associated to y is the image of the 1-cocycle $\sigma \in \Gamma_F \mapsto g^{-1}c_{\alpha,\sigma}\sigma(g)$ in $H^1(F, G_x)$ (where $c_{\alpha,\sigma}$ is the cocycle of α, and take value at σ) which is by definition belongs to $\mathrm{Im}\left(H^1(F, Z(G_x)) \to H^1(F, Z_G(x))\right) \cap p_x^{-1}(\alpha)$.
We will need the following lemma:

Lemma 4.1. *Let $y \in G'_{\mathrm{ss}}(F)$ and assume that G and G'_y are both quasi-split. Then, the set*

$$\{x \in G_{\mathrm{ss}}(F);\ x \sim_{stab} y\}$$

is non-empty.

Recall our definition of pure inner forms in Section 2.1. We continue to consider a pure inner form (G', ψ, c) of G. We say of a quasi-character θ on $G(F)$ that it is *stable* if for all regular elements $x, y \in G_{\mathrm{reg}}(F)$ that are stably conjugate we have $\theta(x) = \theta(y)$. Let θ and θ' be stable quasi-characters on $G(F)$ and $G'(F)$ respectively and assume moreover that G is quasi-split. Then, we say that θ' is a *transfer* of θ if for all regular points $x \in G_{\mathrm{reg}}(F)$ and $y \in G'_{\mathrm{reg}}(F)$ that are stably conjugate we have $\theta'(y) = \theta(x)$. Note that if θ' is a transfer of θ the quasi-character θ' is entirely determined by θ (this is because every regular element in $G'(F)$ is stably conjugate to some element of $G(F)$ for example by the lemma above). We will need the following:

Proposition 4.3. *Let θ and θ' be stable quasi-characters on $G(F)$ and $G'(F)$ respectively and assume that θ' is a transfer of θ. Then, for all $x \in G_{\mathrm{ss}}(F)$ and $y \in G'_{\mathrm{ss}}(F)$ that are strongly stably conjugate we have*

$$c_{\theta'}(y) = c_\theta(x).$$

Proof. Let $x \in G_{\mathrm{ss}}(F)$ and $y \in G'_{\mathrm{ss}}(F)$ be two strongly stably conjugate semi-simple elements. Choose $g \in G(\overline{F})$ such that $y = \psi(gxg^{-1})$ and the isomorphism $\psi \circ \mathrm{Ad}\,(g) : G_x \simeq G_y$ is defined over F. We will denote by ι this isomorphism. If G_x and G'_y are not quasi-split there is nothing to prove since by Proposition 4.1(i) both sides of the equality we want to establish are equal to zero. Assume now that the groups G_x, G'_y

are quasi-split. Let B_x be a Borel subgroup of G_x and $T_x \subset B_x$ be a maximal torus, both defined over F. Set $B_y = \iota(B_x)$ and $T_y = \iota(T_x)$. Then, B_y is a Borel subgroup of G'_y and $T_y \subset B_y$ is a maximal torus. By Proposition 4.1(ii), we have

$$D^G(x)^{1/2}c_\theta(x) = |W(G_x, T_x)|^{-1} \lim_{x' \in T_x(F) \to x} D^G(x')^{1/2}\theta(x')$$

and

$$\begin{aligned} D^{G'}(y)^{1/2}c_{\theta'}(y) &= |W(G'_y, T_y)|^{-1} \lim_{y' \in T_y(F) \to y} D^{G'}(y')^{1/2}\theta'(y') \\ &= |W(G'_y, T_y)|^{-1} \lim_{x' \in T_x(F) \to x} D^{G'}(\iota(x'))^{1/2}\theta'(\iota(x')) \end{aligned}$$

For all $x' \in T_x(F) \cap G_{\text{reg}}(F)$, the elements x' and $\iota(x')$ are stably conjugate. Hence, since θ' is a transfer of θ, we have $\theta'(\iota(x')) = \theta(x')$ for all $x' \in T_x(F) \cap G_{\text{reg}}(F)$. On the other hand, we also have $D^G(x) = D^{G'}(y)$, $|W(G_x, T_x)| = |W(G'_y, T_y)|$ and $D^{G'}(\iota(x')) = D^G(x')$ for all $x' \in T_x(F) \cap G_{\text{reg}}(F)$. Consequently, the two formulas above imply the equality $c_{\theta'}(y) = c_\theta(x)$. □

Let us assume henceforth that G is quasi-split. Following Kottwitz [Kot83], we may associate to any class of pure inner forms $\alpha \in H^1(F, G)$ a sign $e(G_\alpha)$, which only depends on the isomorphism class of the group G_α. Recall the natural two-fold cover we constructed in Appendix B. The connection homomorphism induced from the natural exact sequence:

$$H^1(F, G) \to H^2(F, \{\pm 1\}) = Br_2(F) \simeq \{\pm 1\}.$$

Then we define the sign $e(G_\alpha)$ as the image of α.

We will need the following

Lemma 4.2. *Let T be a (not necessarily maximal) subtorus of G. Then, the composition with the natural map $H^1(F, T) \to H^1(F, G)$ is a group morphism $H^1(F, T) \to Br_2(F)$. Moreover, if T is anisotropic this morphism is onto if and only if the inverse image $\widetilde{T}$ of T in $\widetilde{G}$ is a torus (i.e., is connected).*

Then we combine to two parts of this section, in order to characterize the main property of $H^1(F, T_x) \to H^1(F, G) \to Br_2(F)$.

Lemma 4.3. (i) *Let $\alpha \in H^1(F, H)$ and $y \in \Gamma(G_\alpha, H_\alpha)$ be such that $G_{\alpha,y}$ is quasi-split (Denote the set of such y as $\Gamma^{\text{qd}}_{\text{stab}}(G_\alpha, H_\alpha)$). Then, the set*

$$\{x \in \Gamma(G, H);\ x \sim_{\text{stab}} y\}$$

is non-empty. This subset $\Gamma^{\text{qd}}_{\text{stab}}(G_\alpha, H_\alpha)$ is open and closed in $\Gamma_{\text{stab}}(G_\alpha, H_\alpha)$.

(ii) *Let $\alpha \in H^1(F, H)$, $x \in \Gamma(G, H)$ and $y \in \Gamma(G_\alpha, H_\alpha)$ be such that $x \sim_{\text{stab}} y$. Choose $g \in G_\alpha(\overline{F})$ such that $g\psi_\alpha(x)g^{-1} = y$ and $\operatorname{Ad}(g) \circ \psi_\alpha : G_x \simeq G_y$ is defined over F. Then, $\operatorname{Ad}(g) \circ \psi_\alpha$ restricts to an isomorphism*

$$T_x \simeq T_y$$

that is independent of the choice of g.

(iii) *Let $x \in \Gamma(G, H)$. Then, for all $\alpha \in H^1(F, H)$ there exists a natural bijection between the set*

$$\{y \in \Gamma(G_\alpha, H_\alpha);\ x \sim_{\text{stab}} y\}$$

and the set

$$q_x^{-1}(\alpha)$$

where q_x denotes the natural map $H^1(F,T_x) \to H^1(F,G)$.

(iv) *Let $x \in \Gamma(G,H)$, $x \neq 1$. Then, the composition of the map $\alpha \in H^1(F,G) \mapsto e(G_\alpha) \in Br_2(F)$ with the natural map $H^1(F,T_x) \to H^1(F,G)$ gives a surjective morphism of groups $H^1(F,T_x) \to Br_2(F)$.*

4.3. **More on Local Langlands Correspondence.** In this section, we state our main requirements on tempered L-packets, under the form of two hypothesis, (STAB) and (TRANS), which pertain to the aforementioned character identities.

Recall the local Langlands correspondence, a tempered Langlands parameter φ for G should give rise to a finite set $\Pi^G(\varphi)$, called a *L-packet*, of (isomorphism classes of) tempered representations of $G(F)$. Actually, such a parameter φ should also give rise to tempered L-packets $\Pi^{G_\alpha}(\varphi) \subseteq \mathrm{Temp}(G_\alpha)$ for all $\alpha \in H^1(F,G)$ These families of L-packets should of course satisfy some conditions. Among them, we expect the following properties to hold for every tempered Langlands parameter φ of G:

(STAB) For all $\alpha \in H^1(F,G)$, the character

$$\theta_{\alpha,\varphi} = \sum_{\pi \in \Pi^{G_\alpha}(\varphi)} \theta_\pi$$

is stable

Notice that our different notion of "strongly stable conjugate" does not affect this property since it only involves the values of $\theta_{\alpha,\varphi}$ at regular semi-simple elements (for which the two notions of stable conjugacy coincide). For $\alpha = 1 \in H^1(F,G)$, in which case $G_\alpha = G$, we shall simply set $\theta_\varphi = \theta_{1,\varphi}$.

(TRANS) For all $\alpha \in H^1(F,G)$, the stable character $\theta_{\alpha,\varphi}$ is the transfer of $e(G_\alpha)\theta_\varphi$.

Notice that two conditions are far from characterizing the compositions of the L-packets uniquely. However, by the linear independence of characters, conditions (STAB) and (TRANS) uniquely characterize the L-packets $\Pi^{G_\alpha}(\varphi)$, $\alpha \in H^1(F,G)$, in terms of $\Pi^G(\varphi)$.

When $F = \mathbb{R}$, the Local Langlands correspondence indeed satisfies the three conditions stated above. That (STAB) and (TRANS) hold is a consequence of early work of Shelstad ([She] Lemma 5.2 and Theorem 6.3). The property (WHITT) (recall this in Section 3.4) follows from results of Kostant ([Kost] Theorem 6.7.2) and Vogan ([Vo] Theorem 6.2).
When F is p-adic, the tempered L-packets constructed verify the conditions (STAB) and (TRANS) follows from [Mok15] Theorem 3.2.1(a) and [KMSW14] Proposition 1.5.2. Moreover, the L-packets on the quasi-split form G satisfy condition (WHITT) by [Mok15] Corollary 9.2.4.

4.4. **Explicit Tempered Intertwining.** In this section an explicit relation between the non-vanishing of $m(\pi)$ and an associated linear form $\mathcal{L}_\pi$ for any tempered representation $\pi \in \mathrm{Temp}(G)$ is stated. As a corollary, we are able to establish Theorem 3.3 for tempered L-parameters. Moreover, the results will be indispensable for the proof of the spectral expansion of the distribution J. First we fix a hermitian GGP triple (G,H,ξ).

4.4.1. *The ξ-integral.* For any $f \in \mathcal{C}(G)$, the integral

$$\int_{H(F)} f(h)\xi(h)dh$$

is absolutely convergent. Moreover, it defines a continuous linear form on $\mathcal{C}(G)$. Recall that the Harish-Chandra Schzartz space, $\mathcal{C}(G)$ is a dense subspace of the weak Harish-Chandra Schwartz space $\mathcal{C}^w(G)$ (c.f. [BP20, Subsection 1.5.1]).

Proposition 4.4. *The linear form*

$$f \to \mathcal{C}(G) \to \int_{H(F)} f(h)\xi(h)dh$$

extends continuously to $\mathcal{C}^w(G)$.

The continuous linear form on $\mathcal{C}^w(G)$ proved above is called the ξ-*integral* on $H(F)$ and will be denoted by

$$f \in \mathcal{C}^w(G) \to \int_{H(F)}^{*} f(h)\xi(h)dh$$

4.4.2. *Definition of $\mathcal{L}_\pi$.* Let $\pi \in \mathrm{Temp}(G)$. First we denote by $\mathrm{End}(\pi)$ the space of continuous endomorphisms of the space of π, equipped with the natural operator norm. Thus it has a Banach space structure. Denote $\mathrm{End}(\pi)^\infty$ the subspace of smooth vectors with own locally convex topology. For any $T \in \mathrm{End}(\pi)^\infty$, the function

$$g \in G(F) \to \mathrm{Trace}(\pi(g^{-1})T)$$

belongs to $\mathcal{C}^w(G)$ by [BP20, Subsection 2.2.4]. Hence one may define a (continuous) linear form $\mathcal{L}_\pi : \mathrm{End}(\pi)^\infty \to \mathbb{C}$ by

$$\mathcal{L}_\pi(T) = \int_{H(F)}^{*} \mathrm{Trace}(\pi(h^{-1})T)\xi(h)dh, \quad T \in \mathrm{End}(\pi)^\infty.$$

Recall that there exists a canonical continuous $G(F) \times G(F)$-equivariant embedding with dense image $\pi^\infty \otimes \overline{\pi^\infty} \hookrightarrow \mathrm{End}(\pi)^\infty$, $e \otimes e' \to T_{e,e'} : e_0 \in \pi \mapsto (e_0, e')e$ (which is an isomorphism in the p-adic case). In any case, $\mathrm{End}^\infty(\pi)^\infty$ is naturally isomorphic to the completed projective topological tensor product $\pi^\infty \widehat{\otimes}_p \overline{\pi^\infty}$, see [BP20, Appendix and Section 2.7]. Thus $\mathcal{L}_\pi$ can be identified with the continuous sesquilinear form on π^∞ given by

$$\mathcal{L}_\pi(e, e') := \mathcal{L}_\pi(T_{e,e'})$$

for any $e, e' \in \pi^\infty$. Expanding the definitions,

$$\mathcal{L}_\pi(e, e') = \int_{H(F)}^{*} (e, \pi(h)e')\xi(h)dh$$

for any $e, e' \in \pi^\infty$. Fixing $e' \in \pi^\infty$, the map $e \in \pi^\infty \to \mathcal{L}(e, e')$ then lies in $\mathrm{Hom}_H(\pi^\infty, \xi)$. By the density, it follows that

$$\mathcal{L}_\pi \neq 0 \Rightarrow m(\pi) \neq 0.$$

The main theorem is the following:

Theorem 4.1.

$$\mathcal{L}_\pi \neq 0 \Leftrightarrow m(\pi) \neq 0$$

for any $\pi \in \mathrm{Temp}(G)$.

The complete proof is given in [BP20, Chapter 7], which highly relies on the asymptotic results and the geometry of hermitian GGP triple (see [BP20, Chapter 6]). Here we will skip the proof, however, explain the quantitative identities which will appear in the proof and play an important role in next section.

Lemma 4.4. (i) *Let $f \in \mathcal{C}(G)$ and assume that its Plancherel transform $\pi \in \mathcal{X}_{\mathrm{temp}}(G) \to \pi(f)$ is compactly supported (which is automatic when F is p-adic). Then the following identity holds,*

$$\int_{H(F)} f(h)\xi(h)dh = \int_{\mathcal{X}_{\mathrm{temp}}(G)} \mathcal{L}_\pi(\pi(f))\mu(\pi)d\pi$$

where both integrals are absolutely convergent.

(ii) *Let $\mathcal{K} \subset \mathcal{X}_{\text{temp}}(G)$ be a compact subset. There exists a section $T \in \mathcal{C}(\mathcal{X}_{\text{temp}}(G), \mathcal{E}(G))$ such that*

$$\mathcal{L}_\pi(T_\pi) = m(\pi)$$

for any $\pi \in \mathcal{K}$. Moreover, the same equality is satisfied for every sub-representation π of some $\pi' \in \mathcal{K}$.

Remark 4.1. *Here elements of $\mathcal{C}(\mathcal{X}_{\text{temp}}(G), \mathcal{E}(G))$ are certain assignments $T : \pi \in \text{Temp}(G) \mapsto T_\pi \in \text{End}(\pi)^\infty$ with certain smooth properties and convergent properties. The following matricial Paley-Weiner theorem gives one characterization:*

Theorem 4.2. (i) *The map $f \in \mathcal{C}(G) \mapsto (\pi \in \text{Temp}(G) \mapsto \pi(f) \in \text{End}(\pi)^\infty)$ induces a topological isomorphism $\mathcal{C}(G) \cong \mathcal{C}(\mathcal{X}_{\text{temp}}(G), \mathcal{E}(G))$.*

(ii) *The inverse of that isomorphism is given by sending $T \in \mathcal{C}(\mathcal{X}_{\text{temp}}(G), \mathcal{E}(G))$ to the function f_T defined by*

$$f_T(g) = \int_{\mathcal{X}_{\text{temp}}(G)} \text{Trace}(\pi(g^{-1})T_\pi)\mu(\pi)d\pi.$$

4.5. **Local Trace Formula: Spectral Side.** In this section the spectral expansion of the distribution J is established. First we give the statement of the theorem.

Theorem 4.3. *For any $f \in \mathcal{C}_{\text{scusp}}(G)$, set*

$$J_{\text{spec}}(f) = \int_{\mathcal{X}(G)} D(\pi)\widehat{\theta}_f(\pi)m(\bar{\pi})d\pi.$$

Then the integral is absolutely convergent and

$$J(f) = J_{\text{spec}}(f)$$

for any $f \in \mathcal{C}_{\text{scusp}}(G)$.

Sketch of Proof: First, From [BP20, Lemma 5.4.2] and [BP20, Theorem 8.1.1], both sides of the equality are continuous on $\mathcal{C}_{\text{scusp}}(G)$. Hence by [BP20, Lemma 5.3.1 (ii)] it is sufficient to establish the equality for functions $f \in \mathcal{C}_{\text{scusp}}(G)$ which have compactly supported Fourier transforms (where the Fourier transform is understood as the spectral transform appearing in Matricial Paley-Wiener theorem). Throughout the section a function $f \in \mathcal{C}_{\text{scusp}}(G)$ with compactly supported Fourier transform is fixed.

4.5.1. *Study of an auxiliary distribution.* For any $f' \in \mathcal{C}(G)$, the following integrals are introduced,

$$K^A_{f,f'}(g_1, g_2) = \int_{G(F)} f(g_1^{-1} g g_2) f'(g) dg, \quad g_1, g_2 \in G(F),$$

$$K^1_{f,f'}(g, x) = \int_{H(F)} K^A_{f,f'}(g, hx)\xi(h)dh, \quad g, x \in G(F),$$

$$K^2_{f,f'}(x, y) = \int_{H(F)} K^1_{f,f'}(h^{-1}x, y)\xi(h)dh, \quad x, y \in G(F),$$

$$J_{\text{aux}}(f, f') = \int_{H(F)\backslash G(F)} K^2_{f,f'}(x, x)dx.$$

The following proposition gives estimations for the auxiliary distributions.

Proposition 4.5. (i) *The integral defining $K^A_{f,f'}(g_1, g_2)$ is absolutely convergent. For any $g_1 \in G(F)$ the map*

$$g_2 \in G(F) \to K^A_{f,f'}(g_1, g_2)$$

belongs to $\mathcal{C}(G)$. More precisely, for any $d>0$ there exists $d'>0$ such that for every continuous semi-norm ν on $\mathcal{C}^w_{d'}(G(F))$, there exists a continuous semi-norm μ on $\mathcal{C}(G)$ satisfying

$$\nu(K^A_{f,f'}(g,\cdot)) \leq \nu(f')\Xi^G(g)\sigma(g)^{-d}$$

for any $f' \in \mathcal{C}(G)$ and $g \in G(F)$.

(ii) *The integral defining $K^1_{f,f'}(g,x)$ is absolutely convergent. More precisely, for any $d>0$, there exists $d'>0$ and a continuous semi-norm $\nu_{d,d'}$ on $\mathcal{C}(G)$ such that*

$$|K^1_{f,f'}(g,x)| \leq \nu_{d,d'}(f')\Xi^G(g)\sigma(g)^{-d}\Xi^{H\backslash G}(x)\sigma_{H\backslash G}(x)^{d'}$$

for any $f' \in \mathcal{C}(G)$ and $g,x \in G(F)$.

(iii) *The integral defining $K^2_{f,f'}(x,y)$ is absolutely convergent convergent. Moreover*

$$K^2_{f,f'}(x,y) = \int_{\mathcal{X}_{\text{temp}}(G)} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(y^{-1}))\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}\mu(\pi)d\pi$$

for any $f' \in \mathcal{C}(G)$ and $x,y \in G(F)$. The integral is absolutely convergent.

(iv) *The integral defining $J_{\text{aux}}(f,f')$ is absolutely convergent. More precisely, for every $d>0$ there exists a continuous semi-norm ν_d on $\mathcal{C}(G)$ such that*

$$|K^2_{f,f'}(x,x)| \leq \nu_d(f')\Xi^{H\backslash G}(x)^2\sigma_{H\backslash G}(x)^{-d}$$

for any $f' \in \mathcal{C}(G)$ and $x \in H(F)\backslash G(F)$. In particular, the linear form

$$f' \in \mathcal{C}(G) \to J_{\text{aux}}(f,f')$$

is continuous.

(v) ([BP20, Proposition 9.2.2]) *The following equality holds*

$$J_{\text{aux}}(f,f') = \int_{\mathcal{X}(G)} D(\pi)\widehat{\theta}_f(\pi)\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}d\pi$$

for any $f' \in \mathcal{C}(G)$.

Remark 4.2. *The function $\Xi^{H\backslash G}$ is defined in* [BP20, Section 6.7]. *σ is the natural norm equipped on algebraic varieties, which is described in* [BP20, Section 1.2].

Now we end of proof of the theorem 4.3.
Sketch of Proof: Recall that a function $f \in \mathcal{C}_{\text{scusp}}(G)$ with compactly supported Fourier transform has been fixed. By Lemma 4.4 (i),

$$K_f(x,x) = \int_{\mathcal{X}_{\text{temp}}(G)} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(x^{-1}))\mu(\pi)d\pi$$

for any $x \in H(F)\backslash G(F)$. By Lemma 4.4(ii), there exists a function $f' \in \mathcal{C}(G)$ such that

$$\mathcal{L}_\pi(\pi(\overline{f'})) = m(\pi)$$

for any $\pi \in \mathcal{X}_{\text{temp}}(G)$ such that $\pi(f) \neq 0$. Also, since any $\pi \in \mathcal{X}_{\text{temp}}(G)$,

$$\mathcal{L}_\pi \neq 0 \Leftrightarrow m(\pi) = 1.$$

Hence the further equality holds

$$K_f(x,x) = \int_{\mathcal{X}_{\text{temp}}(G)} \mathcal{L}_\pi(\pi(x)\pi(f)\pi(x^{-1}))\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}\mu(\pi)d\pi$$

and by Proposition 4.5(iii), it follows that

$$K_f(x,x) = K^2_{f,f'}(x,x)$$

for any $x \in H(F)\backslash G(F)$. Consequently,

$$J(f) = J_{\text{aux}}(f,f').$$

After applying Proposition 4.5(v),

$$J(f)=\int_{\mathcal{X}(G)} D(\pi)\widehat{\theta}_f(\pi)\overline{\mathcal{L}_\pi(\pi(\overline{f'}))}d\pi.$$

Let $\pi \in \mathcal{X}(G)$ be such that $\widehat{\theta}_f(\pi) \neq 0$ and let π' be the unique representation in $\mathcal{X}_{\text{temp}}(G)$ such that π is a linear combination of sub-representations of π'. Then $\pi'(f) \neq 0$. Hence, by the definition of f' and Lemma 4.4 (ii), $\overline{\mathcal{L}_\pi(\pi(\overline{f'}))} = \overline{m(\pi)} = m(\overline{\pi})$. It follows that

$$J(f)=\int_{\mathcal{X}(G)} D(\pi)\widehat{\theta}_f(\pi)m(\overline{\pi})d\pi$$

and this ends the proof of Theorem 4.3. ■

4.5.2. *The distribution* J^{Lie}. As a bridge of the proof of geometric expansion, Beuzart-Plessis also introduced the "spectral" expansion in Lie algebra level: express $J^{\text{Lie}}(.)$ in terms of (weighted) orbital integrals of the Fourier trasnform of the test function. To save the length and focus more on our main story, we only state the result here.

For all $f \in \mathcal{S}(\mathfrak{g}(F))$, define a function $K^{\text{Lie}}(f,.)$ on $H(F)\backslash G(F)$ by

$$K^{\text{Lie}}(f,x)=\int_{\mathfrak{h}(F)} f(x^{-1}Xx)\xi(X)dX, \quad x \in H(F)\backslash G(F)$$

the above integral being absolutely convergent. Similar to the theory above, the integral

$$J^{\text{Lie}}(f)=\int_{H(F)\backslash G(F)} K^{\text{Lie}}(f,x)dx$$

is absolutely convergent for all $f \in \mathcal{S}_{\text{scusp}}(\mathfrak{g}(F))$ and define a continuous linear form

$$\mathcal{S}_{\text{scusp}}(\mathfrak{g}(F)) \to \mathbb{C}$$
$$f \mapsto J^{\text{Lie}}(f).$$

The key step is to rewrite the kernel fucntion as an integral over a certain affine subspace $\Sigma(F)$ of $\mathfrak{g}(F)$. First we will define the space Σ.

Fix a Hermitian GGP triple (G,H,ξ). Recall our definition of $P = U(W) \times P_V \subseteq W = U(W) \times U(V)$ in Section 2.1. Let $\overline{P} = M\overline{N}$ be the parabolic subgroup opposite to P with respect to Levi component M. Then the unipotent radicals N and $\overline{N}$ can also be seen as subgroups of $U(V)$ and we will identify them as such in what follows.
Using the G-invariant bilinear pairing B on $\mathfrak{g}$ we fixed in Remark 5.1, there exist a unique element $\Xi \in \overline{\mathfrak{n}}(F)$ such that

$$\xi(X)=\psi(B(\Xi,X))$$

for all $X \in \mathfrak{n}(F)$.
In fact, we have the following explicit description of Ξ (seen as an element of $\mathfrak{u}(V)$):

$$\Xi z_i = z_{i-1}, \text{ for } 1 \leqslant i \leqslant r, \ \Xi z_{-i} = -z_{-i-1}, \text{ for } 0 \leqslant i \leqslant r-1, \ \Xi z_{-r}=0 \text{ and } \Xi(W)=0.$$

Set $\Sigma = \Xi + \mathfrak{h}^\perp$ where $\mathfrak{h}^\perp$ is the orthogonal of $\mathfrak{h}$ in $\mathfrak{g}$ for $B(.,.)$. The study of this space can be found in [BP20, Section 10.1-10.7].
Let us define $\Gamma(\Sigma)$ to be the subset of $\Gamma(\mathfrak{g})$ consisting of the conjugacy classes of the semi-simple parts of elements in $\Sigma(F)$. We equip this subset with the restriction of the measure defined on $\Gamma(\mathfrak{g})$. Thus, if $\mathcal{T}(G)$ is a set of representatives for the $G(F)$-conjugacy classes of maximal tori in G and if for all $T \in \mathcal{T}(G)$ we denote by $\mathfrak{t}(F)_\Sigma$ the subset of elements $X \in \mathfrak{t}(F)$ whose conjugacy class belongs to $\Gamma(\Sigma)$, then we have

$$\int_{\Gamma(\Sigma)} \varphi(X)dX = \sum_{T\in\mathcal{T}(G)} |W(G,T)|^{-1}\int_{\mathfrak{t}(F)_\Sigma} \varphi(X)dX$$

for all $\varphi \in C_c(\Gamma(\Sigma))$.

Theorem 4.4. *We have*

$$J^{\mathrm{Lie}}(f) = \int_{\Gamma(\Sigma)} D^G(X)^{1/2}\widehat{\theta}_f(X)dX$$

for all $f \in \mathcal{S}_{\mathrm{scusp}}(\mathfrak{g}(F))$.

4.6. **Local Trace Formula: Geometric Side.** In this section, three types of Geometric expansion will be introduced: one is prepared for $J(f)$, one is prepared for semi-simple descent, one is for Lie-algebra descent. First we set some determinant functions:

$$\Delta(x) = D^G(x)D^H(x)^{-2}$$

for all $x \in H_{\mathrm{ss}}(F)$ where we recall that $D^G(x) = |\det(1 - \mathrm{Ad}\,(x))_{|\mathfrak{g}/\mathfrak{g}_x}|$ and $D^H(x) = |\det(1 - \mathrm{Ad}\,(x))_{|\mathfrak{h}/\mathfrak{h}_x}|$. Then, we easily check that

$$\Delta(x) = \left|N_{E/F}(\det(1-x)_{|W''_x})\right|$$

for all $x \in H_{\mathrm{ss}}(F)$. Similarly, we define

$$\Delta(X) = D^G(X)D^H(X)^{-2}$$

for all $X \in \mathfrak{h}_{\mathrm{ss}}(F)$ and we have the equality

$$\Delta(X) = \left|N_{E/F}(\det(X_{|W''_X}))\right|$$

for all $X \in \mathfrak{h}_{\mathrm{ss}}(F)$.

Let $\omega \subseteq \mathfrak{g}(F)$ be a G-excellent open neighborhood of 0. Then, we easily check that $\omega \cap \mathfrak{h}(F)$ is a H-excellent open neighborhood of 0. We may thus set

$$j_G^H(X) = j^H(X)^2 j^G(X)^{-1}$$

for all $X \in \omega \cap \mathfrak{h}(F)$. Thus $j_G^H(X) = \Delta(X)\Delta(e^X)^{-1}$ for all $X \in \omega \cap \mathfrak{h}_{\mathrm{ss}}(F)$. Note that j_G^H is a smooth, positive and $H(F)$-invariant function on $\omega \cap \mathfrak{h}(F)$. It actually extends (not uniquely although) to a smooth, positive and $G(F)$-invariant function on ω. We fix the entension as follows: we can embed the groups $H_1 = U(W)$ and $H_2 = U(V)$ into GGP triples (G_1, H_1, ξ_1) and (G_2, H_2, ξ_2). Then the function $X = (X_W, X_V) \in \omega \mapsto j_{G_1}^{H_1}(X_W)^{1/2} j_{G_2}^{H_2}(X_V)^{1/2}$ is easily seen to be such an extension.

Let $x \in H_{\mathrm{ss}}(F)$. Then, we define

$$\Delta_x(y) = D^{G_x}(y)D^{H_x}(y)^{-2}$$

for all $y \in H_{x,\mathrm{ss}}(F)$. On the other hand, since the triple (G'_x, H'_x, ξ'_x) is a GGP triple, the previous construction yields a function $\Delta^{G'_x}$ on $H'_{x,\mathrm{ss}}(F)$. We easily check that

$$\Delta_x(y) = \Delta^{G'_x}(y')$$

for all $y = (y', y'') \in H_{x,\mathrm{ss}}(F) = H'_{x,\mathrm{ss}}(F) \times H''_{x,\mathrm{ss}}(F)$.

Let $\Omega_x \subseteq G_x(F)$ be a G-good open neighborhood of x. Then, it is easy to see that $\Omega_x \cap H(F) \subseteq H_x(F)$ is a H-good open neighborhood of x. This allow us to set

$$\eta = \eta_x^H(y)^2\eta_x^G(y)^{-1}$$

for all $y \in \Omega_x \cap H(F)$. Thus we have

$$\eta_{G,x}^H(y) = \Delta_x(y)\Delta(y)^{-1}$$

for all $y \in \Omega_x \cap H_{\mathrm{ss}}(F)$. Note that $\eta_{G,x}^H$ is a smooth, positive and $H_x(F)$-invariant function on $\Omega_x \cap H(F)$. It actually extends (not uniquely although) to a smooth, positive and $G_x(F)$-invariant function on Ω_x. We will always still denote by $\eta_{G,x}^H$ such an extension.

The definitions of the distributions $m_{\rm geom}$ and $m_{\rm geom}^{\rm Lie}$ are contained in the following proposition:

Proposition 4.6. (i) *Let $\theta \in QC(G(F))$. Then, for all $s \in \mathbb{C}$ such that $Re(s) > 0$ the integral*

$$\int_{\Gamma(G,H)} D^G(x)^{1/2} c_\theta(x) \Delta(x)^{s-1/2} dx$$

is absolutely convergent and the limit

$$m_{\rm geom}(\theta) := \lim_{s\to 0^+} \int_{\Gamma(G,H)} D^G(x)^{1/2} c_\theta(x) \Delta(x)^{s-1/2} dx$$

exists. Similarly, for all $x \in H_{\rm ss}(F)$ and for all $\theta_x \in QC(G_x(F))$, the integral

$$\int_{\Gamma(G_x,H_x)} D^{G_x}(y)^{1/2} c_{\theta_x}(y) \Delta_x(y)^{s-1/2} dy$$

is absolutely convergent for all $s \in \mathbb{C}$ such that $Re(s) > 0$ and the limit

$$m_{x,\rm geom}(\theta_x) := \lim_{s\to 0^+} \int_{\Gamma(G_x,H_x)} D^{G_x}(y)^{1/2} c_{\theta_x}(y) \Delta_x(y)^{s-1/2} dy$$

exists. Moreover, $m_{\rm geom}$ is a continuous linear form on $QC(G(F))$ and for all $x \in H_{\rm ss}(F)$, $m_{x,\rm geom}$ is a continuous linear form on $QC(G_x(F))$.

(ii) *Let $x \in H_{\rm ss}(F)$ and let $\Omega_x \subseteq G_x(F)$ be a G-good open neighborhood of x and set $\Omega = \Omega_x^G$. Then, if Ω_x is sufficiently small, we have*

$$m_{\rm geom}(\theta) = m_{x,\rm geom}((\eta_{G,x}^H)^{1/2} \theta_{x,\Omega_x})$$

for all $\theta \in QC_c(\Omega)$.

(iii) *Let $\theta \in QC_c(\mathfrak{g}(F))$. Then, for all $s \in \mathbb{C}$ such that $Re(s) > 0$ the integral*

$$\int_{\Gamma^{\rm Lie}(G,H)} D^G(X)^{1/2} c_\theta(X) \Delta(X)^{s-1/2} dX$$

is absolutely convergent and the limit

$$m_{\rm geom}^{\rm Lie}(\theta) := \lim_{s\to 0^+} \int_{\Gamma^{\rm Lie}(G,H)} D^G(X)^{1/2} c_\theta(X) \Delta(X)^{s-1/2} dX$$

exists. Moreover, $m_{\rm geom}^{\rm Lie}$ is a continuous linear form on $QC_c(\mathfrak{g}(F))$ that extends continuously to $SQC(\mathfrak{g}(F))$ and we have

$$m_{\rm geom}^{\rm Lie}(\theta_\lambda) = |\lambda|^{\delta(G)/2} m_{\rm geom}^{\rm Lie}(\theta)$$

for all $\theta \in SQC(\mathfrak{g}(F))$ and all $\lambda \in F^\times$ (recall that $\theta_\lambda(X) = \theta(\lambda^{-1}X)$ for all $X \in \mathfrak{g}_{\rm reg}(F)$).

(iv) *Let $\omega \subseteq \mathfrak{g}(F)$ be a G-excellent open neighborhood of 0 and set $\Omega = \exp(\omega)$. Then, we have*

$$m_{\rm geom}(\theta) = m_{\rm geom}^{\rm Lie}((j_G^H)^{1/2} \theta_\omega)$$

for all $\theta \in QC_c(\Omega)$.

Remark 4.3. *By Proposition 4.1(i), in the integral defining $m_{\rm geom}(\theta)$ above only the conjugacy classes $x \in \Gamma(G,H)$ such that G_x is quasi-split contribute. This means that we could have replaced $\Gamma(G,H)$ by the, usually smaller, set $\Gamma_{\rm qd}(G,H)$ consisting of conjugacy classes $x \in \Gamma(G,H)$ such that G_x is quasi-split. Of course, a similar remark applies to $m_{x,\rm geom}(\theta_x)$ and $m_{\rm geom}^{\rm Lie}(\theta)$. This observation will be applied in the end of the proof.*

4.7. **Geometric Expansion for Multiplicity.** With these preparation, we set

$$J_{\text{geom}}(f) = m_{\text{geom}}(\theta_f), \quad \text{for all } f \in \mathcal{C}_{\text{scusp}}(G(F))$$
$$m_{\text{geom}}(\pi) = m_{\text{geom}}(\theta_\pi), \quad \text{for all } \pi \in \text{Temp}(G)$$
$$J^{\text{Lie}}_{\text{geom}}(f) = m^{\text{Lie}}_{\text{geom}}(\theta_f), \quad \text{for all } f \in \mathcal{S}_{\text{scusp}}(G(F))$$

Recall the definition of $J(.)$ on $\mathcal{C}_{\text{scusp}}(G(F))$, $J^{\text{Lie}}(.)$ on $\mathcal{S}_{\text{scusp}}(G(F))$, and multiplicity $m(\pi)$ for all $\pi \in \text{Temp}(G)$ which arised initially. Goal of this section is to explain the proof of the following three theorems.

Theorem 4.5. *We have*

$$J(f) = J_{\text{geom}}(f)$$

for all $f \in \mathcal{C}_{\text{scusp}}(G(F))$.

Theorem 4.6. *We have*

$$m(\pi) = m_{\text{geom}}(\pi)$$

for all $\pi \in \text{Temp}(G)$.

Theorem 4.7. *We have*

$$J^{\text{Lie}}(f) = J^{\text{Lie}}_{\text{geom}}(f)$$

for all $f \in \mathcal{S}_{\text{scusp}}(\mathfrak{g}(F))$.

The proof is by induction on $\dim(G)$ (the case $\dim(G) = 1$ being obvious). Hence, until the end of this section, we make the following induction hypothesis

> **(HYP)** Theorem 4.5, Theorem 4.6 and Theorem 4.7 hold for all GGP triples (G', H', ξ') such that $\dim(G') < \dim(G)$.

4.7.1. *Equivalence of theorem 4.5 and theorem 4.6.* Recall the sets of elliptic representation we defined in Appendix C and the decomposition: $R_{\text{temp}}(G) = R_{ind}(G) \oplus R_{\text{ell}}(G)$.

Proposition 4.7. (i) *Let* $\pi \in R_{ind}(G)$. *Then, we have*

$$m(\pi) = m_{\text{geom}}(\pi)$$

(ii) *For all* $f \in \mathcal{C}_{\text{scusp}}(G(F))$, *we have the equality*

$$J(f) = J_{\text{geom}}(f) + \sum_{\pi \in \mathcal{X}_{\text{ell}}(G)} D(\pi)\widehat{\theta}_f(\pi)\left(m(\overline{\pi}) - m_{\text{geom}}(\overline{\pi})\right)$$

the sum in the right hand side being absolutely convergent.

(iii) *Theorem 4.5 and Theorem 4.6 are equivalent.*

<u>Sketch of Proof</u>:
(i) is the direct corollary from the induction hypothesis and the relation between geometric multiplicity and parabolic induction, which is discussed can be found in [BP20, Section 3.4, 4.7, 11.3].

(ii): Let $f \in \mathcal{C}_{\text{scusp}}(G(F))$. The absolutely convergence on the right hand side of the identity follows from Lemma 5.4.2, Proposition 4.8.1(ii) and 2.7.2 in [BP20]. By Theorem 4.3, we have the equality

$$\begin{aligned} J(f) &= \int_{\mathcal{X}(G)} D(\pi)\widehat{\theta}_f(\pi)m(\overline{\pi})d\pi \\ &= \int_{\mathcal{X}_{ind}(G)} D(\pi)\widehat{\theta}_f(\pi)m(\overline{\pi})d\pi + \sum_{\pi \in \mathcal{X}_{\text{ell}}(G)} D(\pi)\widehat{\theta}_f(\pi)m(\overline{\pi}) \end{aligned}$$

Since the (virtual) representations in $\mathcal{X}_{ind}(G)$ are properly induced, by (i), we have

$$m(\overline{\pi}) = m_{\text{geom}}(\overline{\pi})$$

for all $\pi \in \mathcal{X}_{ind}(G)$.
Meanwhile, we have the equality follows from Theorem 7.1:

$$J_{\text{geom}}(f) = \int_{\mathcal{X}(G)} D(\pi)\widehat{\theta}_f(\pi)m_{\text{geom}}(\overline{\pi})d\pi$$

Combine all these, we prove (ii).

(iii): It is obvious from (ii) to prove one direction. For the converse, it suffices to use (i) and one existence result:

Lemma 4.5. *Assume that G admits an elliptic maximal torus and that the center of $G(F)$ is compact. Then for all $\pi \in \mathcal{X}_{\text{ell}}(G)$ there exists $f \in \mathcal{C}_{\text{scusp}}(G(F))$ such that for all $\pi' \in \mathcal{X}_{\text{ell}}(G)$ we have*

$$\widehat{\theta}_f(\pi') \neq 0 \Leftrightarrow \pi' = \pi$$

4.7.2. *Semisimple descent and the support of $J_{\text{qc}} - m_{\text{geom}}$.*

Proposition 4.8. *There exists a unique continuous linear form J_{qc} on $QC(G(F))$ such that*

- *$J(f) = J_{\text{qc}}(\theta_f)$ for all $f \in \mathcal{C}_{\text{scusp}}(G(F))$;*
- $\text{Supp}(J_{\text{qc}}) \subseteq G(F)_{\text{ell}}$.

Moreover,Let $\theta \in QC(G(F))$ and assume that $1 \notin \text{Supp}(\theta)$. Then, we have the homogeneity

$$J_{\text{qc}}(\theta) = m_{\text{geom}}(\theta)$$

4.7.3. *Lie-algebra descent and equivalence of theorem 4.5 and theorem 4.7.* Let $\omega \subseteq \mathfrak{g}(F)$ be a $G(F)$-excellent open neighborhood of 0 and set $\Omega = \exp(\omega)$. Recall that for any quasi-character $\theta \in QC(\mathfrak{g}(F))$ and all $\lambda \in F^\times$, θ_λ denotes the quasi-character given by $\theta_\lambda(X) = \theta(\lambda^{-1}X)$ for all $X \in \mathfrak{g}_{\text{reg}}(F)$.

Proposition 4.9. (i) *For all $f \in \mathcal{S}_{\text{scusp}}(\Omega)$, we have*

$$J(f) = J^{\text{Lie}}((j_G^H)^{1/2} f_\omega)$$

(ii) *There exists a unique continuous linear form $J_{\text{qc}}^{\text{Lie}}$ on $SQC(\mathfrak{g}(F))$ such that*

$$J^{\text{Lie}}(f) = J_{\text{qc}}^{\text{Lie}}(\theta_f)$$

for all $f \in \mathcal{S}_{\text{scusp}}(\mathfrak{g}(F))$. Moreover, we have

$$J_{\text{qc}}^{\text{Lie}}(\theta_\lambda) = |\lambda|^{\delta(G)/2} J_{\text{qc}}^{\text{Lie}}(\theta)$$

for all $\theta \in SQC(\mathfrak{g}(F))$ and all $\lambda \in F^\times$.

(iii) *Theorem 4.5 and Theorem 4.7 are equivalent.*

(iv) *Let $\theta \in SQC(\mathfrak{g}(F))$ and assume that $0 \notin \text{Supp}(\theta)$. Then, we have*

$$J_{\text{qc}}^{\text{Lie}}(\theta) = m_{\text{geom}}^{\text{Lie}}(\theta)$$

Sketch of Proof:
(i) follows from the direct calculation:

$$\begin{aligned}\int_{H(F)} f(g^{-1}hg)\xi(h)dh &= \int_{\omega_{\mathfrak{h}}} j^H(X) f(g^{-1}e^X g)\xi(X)dX \\ &= \int_{\mathfrak{h}(F)} j_G^H(X)^{1/2} f_\omega(g^{-1}Xg)\xi(X)dX\end{aligned}$$

for all $f \in \mathcal{S}(\Omega)$ and all $g \in G(F)$, where the first equality follows from the Lie-algebra descent.

For (ii), the uniqueness follows from Proposition 7.5. For the existence, set

$$J_{\mathrm{qc}}^{\mathrm{Lie}}(\theta) = \int_{\Gamma(\Sigma)} D^G(X)^{1/2}\widehat{\theta}(X)dX$$

for all $\theta \in SQC(\mathfrak{g}(F))$. We can prove that the integral above is absolutely convergent and defines a continuous linear form. Moreover, the Lie version of spectral expansion, Theorem 4.4, implies

$$J^{\mathrm{Lie}}(f) = J_{\mathrm{qc}}^{\mathrm{Lie}}(\theta_f)$$

for all $f \in \mathcal{S}_{\mathrm{scusp}}(\mathfrak{g}(F))$.

(iii): By (i) and Proposition 4.6(iv), it is clear that Theorem 4.7 implies Theorem 4.5. The converse follows from the density in Proposition 7.5 and the continuity of the functionals in both sides.
First we need to show that we may assume $\theta \in QC_c(\mathfrak{g}(F))$. By homogeneity of $J_{\mathrm{qc}}^{\mathrm{Lie}}$ and $m_{\mathrm{geom}}^{\mathrm{Lie}}$, we may even assume that $\mathrm{Supp}(\theta) \subseteq \omega$. Finally, it follows from (i), Proposition 7.5 (i), Proposition 4.8, and Proposition 4.6(iv).

4.7.4. *End of the proof.* By all the reduction above, it only remains to show that $J_{\mathrm{qc}}^{\mathrm{Lie}}$ equals to $m_{\mathrm{geom}}^{\mathrm{Lie}}$ always. Follows from the homogeneity of these two functionals and $\widehat{j}(\mathcal{O}, .)$ (details: see [BP20, Subsection 11.8]), there exists a constant $c \in \mathbb{C}$, such that

$$J_{\mathrm{qc}}^{\mathrm{Lie}}(\theta) - m_{\mathrm{geom}}^{\mathrm{Lie}}(\theta) = c.c_\theta(0),$$

for all $\theta \in SQC(\mathfrak{g}(F))$.
To prove $c = 0$, we only need to consider G is quasi-split (see Proposition 4.1). Fix a Borel subgroup $B \subset G$ and a maximal torus $T_{\mathrm{qd}} \subset B$ (both defined over F). Denote by $\Gamma_{\mathrm{qd}}(\mathfrak{g})$ the subset of $\Gamma(\mathfrak{g})$ consisting of the conjugacy classes that meet $\mathfrak{t}_{\mathrm{qd}}(F)$. Recall the inclusion $\Gamma(\Sigma) \subset \Gamma(\mathfrak{g})$. It consists in the conjugacy classes of the semi-simple parts of elements in the affine subspace $\Sigma(F) \subset \mathfrak{g}(F)$ defined in Section 4.5. We have

$$\Gamma_{\mathrm{qd}}(\mathfrak{g}) \subseteq \Gamma(\Sigma)$$

Let $\theta_0 \in C_c^\infty(\mathfrak{t}_{\mathrm{qd,reg}}(F))$ be a $W(G, T_{\mathrm{qd}})$-invariant and such that

$$\int_{\mathfrak{t}_{\mathrm{qd}}(F)} D^G(X)^{1/2}\theta_0(X)dX \neq 0$$

Then extend θ_0 to a smooth invariant function on $\mathfrak{g}_{\mathrm{reg}}(F)$, which is zero outside $\mathfrak{t}_{\mathrm{qd,reg}}(F)^G$. Thus is becomes a compactly supported quasi-character. Consider its Fourier transform $\theta = \widehat{\theta}$. By the integral formula (7.1(ii), 7.2), and the direct calculation (function $D^G(Y)^{1/2}\widehat{j}^G(X_{\mathrm{qd}}, Y)$ is supported at $\mathfrak{t}_{\mathrm{qd,reg}}^G(F)$), θ is supported in $\Gamma_{\mathrm{qd}}(\mathfrak{g})$. Since $\Gamma_{\mathrm{qd}}(\mathfrak{g}) \cap \Gamma(G, H) = \{1\}$, by definition of $m_{\mathrm{geom}}^{\mathrm{Lie}}$, we have

$$m_{\mathrm{geom}}^{\mathrm{Lie}}(\theta) = c_\theta(0)$$

On the other hand, by the integral formula (7.1(ii), 7.2) again, and Proposition 4.1(iv), we have

$$c_\theta(0) = \int_{\Gamma(\mathfrak{g})} D^G(X)^{1/2}\theta_0(X)c_{\widehat{j}(X,.)}(0)dX = \int_{\Gamma_{\mathrm{qd}}(\mathfrak{g})} D^G(X)^{1/2}\theta_0(X)dX = J_{\mathrm{qc}}^{\mathrm{Lie}}(\theta).$$

Hence $m_{\mathrm{geom}} = J_{\mathrm{qc}}^{\mathrm{Lie}}$. Since our choice of θ assures $c_\theta(0) \neq 0$, we end the proof. ■

4.8. **Proof of Theorem 3.3.** Let φ be a tempered Langlands parameter for G. We want to show that the sum

$$\sum_{\alpha\in H^1(F,H)}\sum_{\pi\in\Pi^{G_\alpha}(\varphi)} m(\pi)$$

is equal to 1. Let $\pi\in\Pi^{G_\alpha}(\varphi)$, Using the geometric expansion of $m(\pi)$, and summing the integral over the L-packet, we deduce that

$$\sum_{\pi\in\Pi^{G_\alpha}(\varphi)} m(\pi)=\lim_{s\to0^+}\int_{\Gamma(G_\alpha,H_\alpha)} c_{\varphi,\alpha}(y)D^{G_\alpha}(y)^{1/2}\Delta(y)^{s-1/2}dy,$$

where we set

$$c_{\varphi,\alpha}=\sum_{\pi\in\Pi^{G_\alpha}(\varphi)} c_\pi$$

Denote by $\Gamma_{\mathrm{stab}}(G_\alpha,H_\alpha)$ the set of strongly stable conjugacy classes in $\Gamma(G_\alpha,H_\alpha)$. We endow this set with the quotient topology and with the unique measure such that the projection map $\Gamma(G_\alpha,H_\alpha)\to\Gamma_{\mathrm{stab}}(G_\alpha,H_\alpha)$ is locally measure-preserving. By the condition (STAB) of Section 4.3, the character $\theta_{\varphi,\alpha}=\sum_{\pi\in\Pi^{G_\alpha}(\varphi)}\theta_\pi$ is stable and it follows from Proposition 4.3 that the character is constant on the fibers of the map $\Gamma(G_\alpha,H_\alpha)\to\Gamma_{\mathrm{stab}}(G_\alpha,H_\alpha)$. Since it is also trivially true for the functions D^{G_α} and Δ, we may rewrite the right hand side as an integral over $\Gamma_{\mathrm{stab}}(G_\alpha,H_\alpha)$ to obtain

$$\sum_{\pi\in\Pi^{G_\alpha}(\varphi)} m(\pi)=\lim_{s\to0^+}\int_{\Gamma_{\mathrm{stab}}(G_\alpha,H_\alpha)} |p^{-1}_{\alpha,\mathrm{stab}}(y)|c_{\varphi,\alpha}(y)D^{G_\alpha}(y)^{1/2}\Delta(y)^{s-1/2}dy,$$

where $p_{\alpha,\mathrm{stab}}:\Gamma(G_\alpha,H_\alpha)\to\Gamma_{\mathrm{stab}}(G_\alpha,H_\alpha)$ is the natural projection.
By Proposition 4.1(i), the function $c_{\varphi,\alpha}$ vanishes on the complement $\Gamma_{\mathrm{stab}}(G_\alpha,H_\alpha)\smallsetminus\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G_\alpha,H_\alpha)$, the sum becomes

$$\sum_{\pi\in\Pi^{G_\alpha}(\varphi)} m(\pi)=\lim_{s\to0^+}\int_{\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G_\alpha,H_\alpha)} |p^{-1}_{\alpha,\mathrm{stab}}(y)|c_{\varphi,\alpha}(y)D^{G_\alpha}(y)^{1/2}\Delta(y)^{s-1/2}dy$$

By Proposition 4.3(i), we have an injection $\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G_\alpha,H_\alpha)\hookrightarrow\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G,H)$ such that if $y\mapsto x$ then y and x are strongly stably conjugate. For all $y\in\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G_\alpha,H_\alpha)$ (see also Remark 4.3), denoting by x its image in $\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G,H)$, we have the following commutative diagram

$$\begin{array}{ccc} T_y(F) & \xrightarrow{\sim} & T_x(F)\\ \downarrow & & \downarrow\\ \Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G_\alpha,H_\alpha) & \hookrightarrow & \Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G,H)\end{array}$$

where the two vertical arrows are only defined in some neighborhood of 1, given by $t\mapsto ty$, $t\mapsto tx$ and are both locally preserving measures (when $T_y(F)$ and $T_x(F)$ are both equipped with their unique Haar measure of total mass one) and the top vertical arrow is the restriction to the F-points of the isomorphism provided by Proposition 4.3(ii). The commutative implies tht the embedding $\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G_\alpha,H_\alpha)\hookrightarrow\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G,H)$ preserves measures. Moreover, by Proposition 4.3 and the condition (TRANS), if $y\in\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G_\alpha,H_\alpha)$ maps to $x\in\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G,H)$, we have the equality (where we set $c_\varphi=c_{\varphi,1}$)

$$c_{\varphi,\alpha}(y)=e(G_\alpha)c_\varphi(x).$$

Using these two facts, we map now express the integral over $\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G,H)$:

$$\sum_{\alpha\in H^1(F,H)}\sum_{\pi\in\Pi^{G_\alpha}(\varphi)} m(\pi)=\lim_{s\to 0^+}\int_{\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G,H)}\sum_{\alpha\in H^1(F,H)} e(G_\alpha)n_\alpha(x)c_\varphi(x)D^G(x)^{1/2}\Delta(x)^{s-1/2}dx$$

where we have the set

$$n_\alpha(x)=|\{y\in\Gamma(G_\alpha,H_\alpha);y\sim_{\mathrm{stab}}x\}|$$

for all $x\in\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G,H)$. Summing the above equality over $\alpha\in H^1(F,H)$, we get

$$\sum_{\alpha\in H^1(F,H)}\sum_{\pi\in\Pi^{G_\alpha}(\varphi)} m(\pi)=\lim_{s\to 0^+}\int_{\Gamma^{\mathrm{qd}}_{\mathrm{stab}}(G,H)}\sum_{\alpha\in H^1(F,H)} e(G_\alpha)n_\alpha(x)c_\varphi(x)D^G(x)^{1/2}\Delta(x)^{s-1/2}dx.$$

Then consider the inner sum, by main Lemma 4.3, we have

$$\sum_{\alpha\in H^1(F,H)} e(G_\alpha)n_\alpha(x)=\sum_{\beta\in H^1(F,T_x)} e(G_\beta).$$

Moreover, by main Lemma 4.3(iv), the map $\beta\in H^1(F,T_x)\mapsto e(G_\beta)\in Br_2(F)\simeq\{\pm 1\}$ is a group homomorphism which is non-trivial for $x\neq 1$. Thus this sum is zero unless $x=1$. Taking into account these cancellations, we see that the right hand side of expansion reduces to the contribution of $1\in\Gamma(G,H)$ and thus we get

$$\sum_{\alpha\in H^1(F,H)}\sum_{\pi\in\Pi^{G_\alpha}(\varphi)} m(\pi)=c_\varphi(1).$$

By Proposition 2.1, the term $c_\varphi(1)$ has the following representation-theoretic interpretation: it equals the number of representations in $\Pi^G(\varphi)$ admitting a Whittaker model, a representation being counted as many times as the number of different types of Whittaker model it has, divided by the number of different type of Whittaker models for G. By the condition (WHITT), this number is 1. It follows that the left hand side of equation is also equal to 1 and we are done. ■

5. Appendix A: Structure Theory Background

These considerations are general, let G be any connected reductive group defined over F. First we study some special subgroups or quotients of G, then choose appropriate topology and measure on them. Next, we introduce orbital integrals and their Fourier transforms. In the last, we explain the descent methods briefly.

5.0.1. *Conjugacy Classes.* First we fix some notations. We shall denote by Z_G the center of G and by A_G its split component. If H is an algebraic group, we shall denote by H^0 its neutral connected component. We define the real vector space:

$$\mathcal{A}_G=\mathrm{Hom}(X^*(G),\mathbb{R})$$

and its dual

$$\mathcal{A}_G^*=X^*(G)\otimes\mathbb{R}$$

where $X^*(G)$ stands for the module of F-rational characters of G. We have a natural homomorphism

$$H_G:G(F)\to\mathcal{A}_G$$

given by

$$\langle\chi,H_G(g)\rangle=\log\left(|\chi(g)|\right),\quad g\in G(F),\ \chi\in X^*(G)$$

Let $P=MU$ be a parabolic subgroup of G. Of course, the previous constructions apply to M. We will denote by $R(A_M,P)$ the set of roots of A_M in the unipotent radical of P. If K is a maximal compact subgroup of $G(F)$ which is special in the p-adic case, we have the Iwasawa decomposition $G(F)=M(F)U(F)K$. We may then choose maps $m_P\colon G(F)\to M(F)$, $u_P\colon G(F)\to U(F)$ and $k_P\colon G(F)\to K$ such that

$g = m_P(g)u_P(g)k_P(g)$ for all $g \in G(F)$. We then extend the homomorphism H_M to a map $H_P : G(F) \to \mathcal{A}_M$ by setting $H_P(g) = H_M(m_P(g))$. This extension depends of course on the maximal compact K but its restriction to $P(F)$ doesn't and is given by $H_P(mu) = H_M(m)$ for all $m \in M(F)$ and all $u \in U(F)$. By a Levi subgroup of G we mean a subgroup of G which is the Levi component of some parabolic subgroup of G. We will also use Arthur's notation: if M is a Levi subgroup of G, then we denote by $\mathcal{P}(M)$, $\mathcal{L}(M)$ and $\mathcal{F}(M)$ the finite sets of parabolic subgroups admitting M as a Levi component, of Levi subgroups containing M and of parabolic subgroups containing M respectively. If $M \subset L$ are two Levi subgroups, we set $\mathcal{A}_M^L = \mathcal{A}_M/\mathcal{A}_L$.

For $x \in G$ (resp. $X \in \mathfrak{g}$), we denote by $Z_G(x)$ (resp. $Z_G(X)$) the centralizer of x (resp. X) in G and by $G_x = Z_G(x)^0$ (resp. $G_X = Z_G(X)^0$) the neutral component of the centralizer. Recall that if $X \in \mathfrak{g}$ is semi-simple then $Z_G(X) = G_X$. We will denote by G_{ss} and G_{reg} (resp. $\mathfrak{g}_{\mathrm{ss}}$ and $\mathfrak{g}_{\mathrm{reg}}$) the subsets of semi-simple and regular semi-simple elements in G (resp. in $\mathfrak{g}$). For any subset A of $G(F)$ (resp. of $\mathfrak{g}(F)$), we will denote by A_{reg} the intersection $A \cap G_{\mathrm{reg}}(F)$ (resp. $A \cap \mathfrak{g}_{\mathrm{reg}}(F)$) and by A_{ss} the intersection $A \cap G_{\mathrm{ss}}(F)$ (resp. $A \cap \mathfrak{g}_{\mathrm{ss}}(F)$). We will usually denote by $\mathcal{T}(G)$ a set of representatives for the conjugacy classes of maximal tori in G. Recall that a maximal torus T of G is said to be elliptic if $A_T = A_G$. Elliptic maximal tori always exist in the p-adic case but not necessarily in the real case. An element $x \in G(F)$ will be said to be *elliptic* if it belongs to some elliptic maximal torus (in particular it is semi-simple). Similarly, an element $X \in \mathfrak{g}(F)$ will be said to be *elliptic* if it belongs to the Lie algebra of some elliptic maximal torus. We will denote by $G(F)_{\mathrm{ell}}$ and $\mathfrak{g}(F)_{\mathrm{ell}}$ the subsets of elliptic elements in $G(F)$ and $\mathfrak{g}(F)$ respectively. For all $x \in G_{\mathrm{ss}}(F)$ (resp. all $X \in \mathfrak{g}_{\mathrm{ss}}(F)$), we set

$$D^G(x) = \left|\det(1 - \mathrm{Ad}\,(x))_{|\mathfrak{g}/\mathfrak{g}_x}\right| \quad \left(\text{resp. } D^G(X) = \left|\det\ \mathrm{ad}\,(X)_{|\mathfrak{g}/\mathfrak{g}_X}\right|\right).$$

If a group H acts on a set X and A is a subset of X, we shall denote by $\mathrm{Norm}_H(A)$ the normalizer of A in H. For every Levi subgroup M and every maximal torus T of G, we will denote by $W(G, M)$ and $W(G, T)$ the Weyl groups of $M(F)$ and $T(F)$ respectively, that is

$$W(G, M) = \mathrm{Norm}_{G(F)}(M)/M(F) \quad \text{and} \quad W(G, T) = \mathrm{Norm}_{G(F)}(T)/T(F).$$

If H is a connected linear algebraic group defined over F, we will denote by $\Gamma(H)$ the set of semi-simple conjugacy classes in $H(F)$. Thus, we have a natural projection

$$H_{\mathrm{ss}}(F) \twoheadrightarrow \Gamma(H)$$

and we endow $\Gamma(H)$ with the quotient topology. Then, $\Gamma(H)$ is Hausdorff and locally compact. Moreover for every connected linear algebraic group H' over F and every embedding $H' \hookrightarrow H$ the induced map $\Gamma(H') \to \Gamma(H)$ is continuous and proper. We define similarly the space $\Gamma(\mathfrak{h})$ of semi-simple conjugacy classes in $\mathfrak{h}(F)$. This space satisfies similar properties.

About measure: we will usually denote by d_Lg (resp. d_Rg) a left (resp. a right) Haar measure on G. If the group is unimodular then we will usually denote both by dg. Finally δ_G will stand for the modular character of G that is defined by $d_L(gg'^{-1}) = \delta_G(g')d_Lg$ for all $g \in G$. To further discussion, first we need to fix a (unitary) continuous non-trivial additive character $\psi\colon F \to \mathbb{S}^1$ and we equip F with the autodual Haar measure with respect to ψ. We also fix a Haar measure $d^\times t$ on $F^\times$ to be $|t|^{-1}dt$ where dt is the Haar measure on F that we just fixed.
Then fix a $G(F)$-invariant nondegenerate bilinear form B on $\mathfrak{g}(F)$. If $F = \mathbb{R}$, we choose B so that for every maximal compact subgroup K of $G(F)$ the restriction of B to $\mathfrak{k}(F)$ is negative definite and the restriction to $\mathfrak{k}(F)^\perp$ (the orthogonal of $\mathfrak{k}(F)$ with respect to

B) is positive definite. We endow $\mathfrak{g}(F)$ with the autodual measure with respect to B, it is the only Haar measure dX on $\mathfrak{g}(F)$ such that the Fourier transform

$$\widehat{f}(Y) = \int_{\mathfrak{g}(F)} f(X)\psi\left(B(X,Y)\right) dX, \quad f \in \mathcal{S}(\mathfrak{g}(F))$$

satisfies $\widehat{\widehat{f}}(X) = f(-X)$. We equip $G(F)$ with the unique Haar measure such that the exponential map has a Jacobian equal to 1 at the origin. Similarly, for every F-algebraic subgroup H of G such that the restriction of $B(.,.)$ to $\mathfrak{h}(F)$ is non-degenerate, we equip $\mathfrak{h}(F)$ with the autodual measure with respect to B and we lift this measure to $H(F)$ by means of the exponential map. This fixes for example the Haar measures on the Levi subgroups of G as well as on the maximal subtori of G. For other subgroups of $G(F)$, for example unipotent radicals of parabolic subgroups of G, we fix an arbitrary Haar measure on the Lie algebra and we lift it to the group, again using the exponential map.

We will say of a subset $A \subseteq \mathfrak{h}(F)$ (resp. $A \subseteq H(F)$) that it is *completely $H(F)$-invariant* if it is $H(F)$-invariant and if moreover for all $X \in A$ (resp. $g \in A$) its semi-simple part X_s (resp. g_s) also belongs to A. Closed invariant subsets are automatically completely $H(F)$-invariant. We easily check that the completely $H(F)$-invariant open subsets of $\mathfrak{h}(F)$ (resp. of $H(F)$) define a topology. We will call it the *invariant topology*. This topology coincides with the pull-back of the topology on $\Gamma(\mathfrak{h})$ (resp. on $\Gamma(H)$) just defined by the natural map

$$\mathfrak{h}(F) \to \Gamma(\mathfrak{h}) \quad (\text{resp. } H(F) \to \Gamma(H))$$

which associates to $X \in \mathfrak{h}(F)$ (resp. $g \in H(F)$) the conjugacy class of the semi-simple part of X (resp. of g). We will say of an invariant subset $L \subseteq \mathfrak{h}(F)$ (resp. $L \subseteq H(F)$) that it is *compact modulo conjugation* if it is closed and if there exists a compact subset $\mathcal{K} \subseteq \mathfrak{h}(F)$ (resp. $\mathcal{K} \subseteq H(F)$) such that $L = \mathcal{K}^G$, it is equivalent to ask that L is completely $H(F)$-invariant and that for every maximal torus $T \subset H$ the intersection $L \cap \mathfrak{t}(F)$ (resp. $L \cap T(F)$) is compact, it is also equivalent to the fact that L is compact for the invariant topology.

We will denote by $\Gamma_{\mathrm{ell}}(G)$ and $\Gamma_{\mathrm{reg}}(G)$ (resp. $\Gamma_{\mathrm{ell}}(\mathfrak{g})$ and $\Gamma_{\mathrm{reg}}(\mathfrak{g})$) the subsets of elliptic and regular conjugacy classes in $\Gamma(G)$ (resp. in $\Gamma(\mathfrak{g})$) respectively. The subset $\Gamma_{\mathrm{ell}}(G)$ (resp. $\Gamma_{\mathrm{ell}}(\mathfrak{g})$) is closed in $\Gamma(G)$ (resp. in $\Gamma(\mathfrak{g})$) whereas $\Gamma_{\mathrm{reg}}(G)$ (resp. $\Gamma_{\mathrm{reg}}(\mathfrak{g})$) is an open subset of $\Gamma(G)$ (resp. of $\Gamma(\mathfrak{g})$). Let $\mathcal{T}(G)$ be a set of representatives for the conjugacy classes of maximal tori in G. We equip $\Gamma(G)$ and $\Gamma(\mathfrak{g})$ with the unique regular Borel measures dx and dX such that

$$\int_{\Gamma(G)} \varphi_1(x)dx = \sum_{T \in \mathcal{T}(G)} |W(G,T)|^{-1} \int_{T(F)} \varphi_1(t)dt$$

$$\int_{\Gamma(\mathfrak{g})} \varphi_2(X)dX = \sum_{T \in \mathcal{T}(G)} |W(G,T)|^{-1} \int_{\mathfrak{t}(F)} \varphi_2(X)dX$$

for all $\varphi_1 \in C_c(\Gamma(G))$ and all $\varphi_2 \in C_c(\Gamma(\mathfrak{g}))$.

We say that an element $x \in G(F)$ is *anisotropic* if it is regular semi-simple and $G_x(F)$ is compact. We will denote by $G(F)_{ani}$ the subset of anisotropic elements and by $\Gamma_{ani}(G)$ the set of anisotropic conjugacy classes in $G(F)$. We equip $\Gamma_{ani}(G)$ with the quotient topology relative to the natural projection $G(F)_{ani} \twoheadrightarrow \Gamma_{ani}(G)$. Let $\mathcal{T}_{ani}(G)$ be a set of representatives for the $G(F)$-conjugacy classes of maximal anisotropic tori of G (a torus T is *anisotropic* if $T(F)$ is compact). We equip $\Gamma_{ani}(G)$ with the quotient topology and we endow it with the unique regular Borel measure such that

$$\int_{\Gamma_{ani}(G)} \varphi(x)dx = \sum_{T \in \mathcal{T}_{ani}(G)} |W(G,T)|^{-1}\nu(T) \int_{T(F)} \varphi(t)dt$$

for all $\varphi \in C_c(\Gamma_{ani}(G))$, where the factor $\nu(T)$ is defined as follows: consider T a maximal subtorus of G. Besides the Haar measure dt that hase been fixed above on $T(F)$, define $d_c t$ is the unique Haar measure on $T(F)$ such that the quotient measure $d_c t/d_c a$ gives $T(F)/A_T(F)$ the measure 1. To avoid confusions, we shall only use the Haar measure dt but we need to introduce the only factor $\nu(T) > 0$ such that $d_c t = \nu(T)dt$.

Remark 5.1. *In the paper, for GGP triple (G, H, ξ), we use the the following $B(.,.)$*

$$B\left((X_W, X_V), (X'_W, X'_V)\right) = \frac{1}{2}\left(\mathrm{Tr}_{E/F}\left(\mathrm{Trace}(X_W X'_W)\right) + \mathrm{Tr}_{E/F}\left(\mathrm{Trace}(X_V X'_V)\right)\right)$$

to normalize the Haar measures on both $\mathfrak{g}(F)$ and $G(F)$.

5.0.2. *Orbits, Orbital Integrals and their Fourier Transforms.* For $x \in G_{\mathrm{reg}}(F)$ (resp. $X \in \mathfrak{g}_{\mathrm{reg}}(F)$), we define the *normalized orbital integral* at x (resp. at X) by

$$J_G(x, f) = D^G(x)^{1/2} \int_{G_x(F)\backslash G(F)} f(g^{-1}xg)dg, \quad f \in \mathcal{C}(G(F))$$

$$(\text{resp. } J_G(X, f) = D^G(X)^{1/2} \int_{G_X(F)\backslash G(F)} f(g^{-1}Xg)dX, \quad f \in \mathcal{S}(\mathfrak{g}(F)))$$

the integral being absolutely convergent for all $f \in \mathcal{C}(G(F))$ (resp. for all $f \in \mathcal{S}(\mathfrak{g}(F))$). This defines a tempered distribution $J_G(x,.)$ (resp. $J_G(X,.)$) on $G(F)$ (resp. on $\mathfrak{g}(F)$). For all $f \in \mathcal{C}(G(F))$ (resp. $f \in \mathcal{S}(\mathfrak{g}(F))$), the function $x \in G_{\mathrm{reg}}(F) \mapsto J_G(x, f)$ (resp. $X \in \mathfrak{g}_{\mathrm{reg}}(F) \mapsto J_G(X, f)$) is locally bounded on $G(F)$ (resp. on $\mathfrak{g}(F)$).
Similarly, for $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})$, we define the orbital integral on $\mathcal{O}$ by

$$J_{\mathcal{O}}(f) = \int_{\mathcal{O}} f(X)dX, \quad f \in \mathcal{S}(\mathfrak{g}(F))$$

We have

$$J_{\mathcal{O}}(f_\lambda) = |\lambda|^{\dim(\mathcal{O})/2} J_{\mathcal{O}}(f)$$

for all $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})$ and all $\lambda \in F^{\times 2}$ (recall that $f_\lambda(X) = f(\lambda^{-1}X)$). Denote by $\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$ the subset of regular nilpotent orbits in $\mathfrak{g}(F)$. This set is empty unless G is quasi-split in which case we have $\dim(\mathcal{O}) = \delta(G)$ for all $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$. By the above equality, the distributions $J_{\mathcal{O}}$ for $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$ are all homogeneous of degree $\delta(G)/2 - \dim(\mathfrak{g})$. This characterizes the distributions $J_{\mathcal{O}}$, $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$, among the invariant distributions supported in the nilpotent cone. More precisely, we have:

The invariant distributions on $\mathfrak{g}(F)$ supported in the nilpotent cone and homogeneous of degree $\delta(G)/2 - \dim(\mathfrak{g})$ are precisely linear combinations of the distributions $J_{\mathcal{O}}$ for $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g})$. This follows from Lemma 3.3 of [HCDS] in the p-adic case and from Corollary 3.9 of [BV80] in the real case.

According to Harish-Chandra, there exists a unique smooth function $\widehat{j}(.,.)$ on $\mathfrak{g}_{\mathrm{reg}}(F) \times \mathfrak{g}_{\mathrm{reg}}(F)$ which is locally integrable on $\mathfrak{g}(F) \times \mathfrak{g}(F)$ such that

$$J_G(X, \widehat{f}) = \int_{\mathfrak{g}(F)} \widehat{j}(X, Y) f(Y)dY$$

for all $X \in \mathfrak{g}_{\mathrm{reg}}(F)$ and all $f \in \mathcal{S}(\mathfrak{g}(F))$. We have the following control on the size of $\widehat{j}$:

The function $(X, Y) \in \mathfrak{g}_{\mathrm{reg}}(F) \times \mathfrak{g}_{\mathrm{reg}}(F) \mapsto D^G(Y)^{1/2}\widehat{j}(X, Y)$ is globally bounded. This follows from Theorem 7.7 and Lemma 7.9 of [HCDS] in the p-adic case and Proposition 9 of [Va77] in the real case. We will need the following property regarding to the non-vanishing of the function $\widehat{j}$

Assume that G admits elliptic maximal tori. Then, for all $Y \in \mathfrak{g}_{\mathrm{reg}}(F)$ there exists $X \in \mathfrak{g}_{\mathrm{reg}}(F)_{\mathrm{ell}}$ such that $\widehat{j}(X, Y) \neq 0$. In the p-adic case, this follows from Theorem 9.1

and Lemma 9.6 of [HCDS] whereas in the real case, it is a consequence of Theorem 4 and Theorem 11 of [Va77].

Similarly, for any nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})$, there exists a smooth function $\widehat{j}(\mathcal{O}, .)$ on $\mathfrak{g}_{\mathrm{reg}}(F)$ which is locally integrable on $\mathfrak{g}(F)$ such that

$$J_{\mathcal{O}}(\widehat{f}) = \int_{\mathfrak{g}(F)} \widehat{j}(\mathcal{O}, X) f(X) dX$$

for all $f \in \mathcal{S}(\mathfrak{g}(F))$. We know that the function $(D^G)^{1/2} \widehat{j}(\mathcal{O}, .)$ is locally bounded on $\mathfrak{g}(F)$ (Theorem 6.1 of [HCDS] in the p-adic case and Theorem 17 of [Va77] in the real case). The functions $\widehat{j}(\mathcal{O}, .)$ satisfy the following homogeneity property

$$\widehat{j}(\mathcal{O}, \lambda X) = |\lambda|^{-\dim(\mathcal{O})/2} \widehat{j}(\lambda \mathcal{O}, X)$$

for all $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})$, all $X \in \mathfrak{g}_{\mathrm{reg}}(F)$ and all $\lambda \in F^{\times}$. Recall also that for every nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})$, we have $\lambda \mathcal{O} = \mathcal{O}$ for all $\lambda \in F^{\times 2}$.

5.0.3. *Descent.* Recall our definition of $G(F)$-completely invariant. The reason why we mention them now is to explain the definitions of some specific spaces arised in Section 4.2. A more careful and complete discussion can be found in [BP20, Chapter 3]. there collect some well-known facts concerning Harish-Chandra's technique of descents: semisimple descent, lie algebra descent, and parabolic descent.

(Semi-simple Descent) Let $X \in \mathfrak{g}_{\mathrm{ss}}(F)$. An open subset $\omega_X \subseteq \mathfrak{g}_X(F)$ will be called *G-good* if it is completely $G_X(F)$-invariant and if moreover the map

$$\omega_X \times^{G_X(F)} G(F) \to \mathfrak{g}(F) \tag{1}$$
$$(Y, g) \mapsto g^{-1} Y g$$

induces an F-analytic isomorphism between $\omega_X \times^{G_X(F)} G(F)$ and ω_X^G, where $\omega_X \times^{G_X(F)} G(F)$ (the *contracted product*) denotes the quotient of $\omega_X \times G(F)$ by the free $G_X(F)$-action given by

$$g_X \cdot (Y, g) = (g_X Y g_X^{-1}, g_X g), \quad g_X \in G_X(F), (Y, g) \in \omega_X \times G(F)$$

The Jacobian of the map at $(Y, g) \in \omega_X \times^{G_X(F)} G(F)$ is equal to

$$\eta_X^G(Y) = \left| \det \mathrm{ad}\,(Y)_{|\mathfrak{g}/\mathfrak{g}_X} \right|$$

It follows that an open subset $\omega_X \subseteq \mathfrak{g}_X(F)$ is G-good if and only if the following conditions are satisfied

(i) ω_X is completely $G_X(F)$-invariant;

(ii) For all $Y \in \omega_X$, we have $\eta_X^G(Y) \neq 0$

(iii) For all $g \in G(F)$, the intersection $g^{-1} \omega_X g \cap \omega_X$ is nonempty if and only if $g \in G_X(F)$.

Let $\omega_X \subseteq \mathfrak{g}_X(F)$ be a G-good open neighborhood of X and set $\omega = \omega_X^G$. Then ω is completely $G(F)$-invariant (since ω_X is completely $G_X(F)$-invariant). Moreover, the completely $G(F)$-invariant open subsets obtained in this way form a basis of neighborhood for X in the invariant topology. We have the integration formula

$$\int_{\omega} f(Y) dY = \int_{G_X(F) \backslash G(F)} \int_{\omega_X} f(g^{-1} Y g) \eta_X^G(Y) dY dg$$

for all $f \in L^1(\omega)$. For every function f defined on ω, we will denote by f_{X,ω_X} the function on ω_X given by $f_{X,\omega_X}(Y) = \eta_X^G(Y)^{1/2} f(Y)$. The map $f \mapsto f_{X,\omega_X}$ induces topological isomorphisms

$$C^\infty(\omega)^G \simeq C^\infty(\omega_X)^{G_X} \quad C^\infty(\omega_{\mathrm{reg}})^G \simeq C^\infty(\omega_{X,\mathrm{reg}})^{G_X}$$

(Note that $\omega_X \cap \mathfrak{g}_{X,\mathrm{reg}}(F) = \omega_X \cap \mathfrak{g}_{\mathrm{reg}}(F)$ so that the notation $\omega_{X,\mathrm{reg}}$ is unambiguous).

(Lie algebra Descent) We will say of an open subset $\omega \subseteq \mathfrak{g}(F)$ that it is G-*excellent*, if it satisfies the following conditions

(i) ω is completely $G(F)$-invariant and relatively compact modulo conjugation;

(ii) The exponential map is defined on ω and induces an F-analytic isomorphism between ω and $\Omega = \exp(\omega)$.

For all $X \in \mathfrak{z}_G(F)$ (in particular $X = 0$), the lie algebra of Z_G, the G-excellent open subsets containing X form a basis of neighborhoods of X for the invariant topology.
Let $\omega \subseteq \mathfrak{g}(F)$ be a G-excellent open subset and set $\Omega = \exp(\omega)$. The Jacobian of the exponential map

$$\exp : \omega \to \Omega$$
$$X \mapsto e^X$$

at $X \in \omega_{\mathrm{ss}}$ is given by

$$j^G(X) = D^G(e^X)D^G(X)^{-1}$$

Hence, we have the integration formula

$$\int_\Omega f(g)dg = \int_\omega f(e^X)j^G(X)dX$$

for all $f \in L^1(\Omega)$. For every function f on Ω, we will denote by f_ω the function on ω defined by $f_\omega(X) = j^G(X)^{1/2}f(e^X)$. The map $f \mapsto f_\omega$ induces topological isomorphisms

$$C^\infty(\Omega) \simeq C^\infty(\omega) \qquad C^\infty(\Omega_{\mathrm{reg}}) \simeq C^\infty(\omega_{\mathrm{reg}}).$$

6. Appendix B: Number Theoretic Background

Classification of quasi split group over F; Galois group action on the root data!
Recall that a quasi-split connected group over F is classified up to conjugation by its (canonical) based root datum $\Psi_0(G) = (X_G, \Delta_G, X_G^\vee, \Delta_G^\vee)$ together with the natural action of Γ_F on $\Psi_0(G)$. For any Borel pair (B,T) of G that is defined over F, we have a canonical Γ_F-equivariant isomorphism $\Psi_0(G) \simeq (X^*(T), \Delta(T,B), X_*(T), \Delta(T,B)^\vee)$ where $\Delta(T,B) \subseteq X^*(T)$ denotes the set of simple roots of T in B and $\Delta(T,B)^\vee \subseteq X_*(T)$ denotes the corresponding sets of simple coroots. Fix such a Borel pair and set

$$\rho = \frac{1}{2}\sum_{\beta \in R(G,T)} \beta \in X^*(T) \otimes \mathbb{Q}$$

for the half sum of the roots of T in B. The image of ρ in $X_G \otimes \mathbb{Q}$ doesn't depend on the particular Borel pair (B,T) chosen and we shall still denote by ρ this image. Consider now the following based root datum

$$(\widetilde{X}_G, \Delta_G, \widetilde{X}_G^\vee, \Delta_G^\vee)$$

where $\widetilde{X}_G = X_G + \mathbb{Z}\rho \subseteq X_G \otimes \mathbb{Q}$ and $\widetilde{X}_G = \{\lambda^\vee \in X_G^\vee;\ \langle \lambda^\vee, \rho\rangle \in \mathbb{Z}\}$. Note that we have $\Delta_G^\vee \subseteq \widetilde{X}_G^\vee$ since $\langle \alpha^\vee, \rho\rangle = 1$ for all $\alpha^\vee \in \Delta_G^\vee$. Such based root datum, with its natural Γ_F-action, is the based root datum of a unique quasi-split group $\widetilde{G}_0$ over F well-defined up to conjugacy. Moreover, we have a natural central isogeny $\widetilde{G}_0 \to G$, well-defined up to $G(F)$-conjugacy, whose kernel is either trivial or $\{\pm 1\}$ (depending on whether ρ belongs to X_G or not). If the kernel is $\{\pm 1\}$, we set $\widetilde{G} = \widetilde{G}_0$ otherwise we simply set $\widetilde{G} = G \times \{\pm 1\}$. In any case, we obtain a short exact sequence

$$1 \to \{\pm 1\} \to \tilde{G} \to G \to 1$$

well-defined up to $G(F)$-conjugacy, which induces

Example 4. *For $U(V)$ where V is a hermitian space, we have the following explicit description:*

(i) *If* $\dim(V)$ *is odd, then* $\widetilde{U(V)} = U(V) \times \{\pm 1\}$*;*

(ii) *If* $\dim(V)$ *is even then* $\widetilde{U(V)} = \{(g,z) \in U(V) \times \mathrm{Ker} N_{E/F};\ \det(g) = z^2\}$.

7. Appendix C: Representation Theoretic Background

We will use freely the knowledge of the first two chapters in [BP20], except where we need to make the definition more explicitly.

7.1. **Representation Preliminaries.**

7.1.1. *Space $\mathcal{X}_{\mathrm{temp}}(G)$, Elliptic Representations and the Space $\mathcal{X}(G)$.* The second chapter of [BP20], contains some background on representations of $G(F)$. In order to state the spectral expansion, we will introduce the appropriate space for us to do integration. Let us define $\mathcal{X}_{\mathrm{temp}}(G)$ to be the set of isomorphism classes of tempered representations of $G(F)$ which are of the form $i_M^G(\sigma)$ where M is a Levi subgroup of G, $\sigma \in \Pi_2(M)$ is a square-integrable representation, and i represents the unitary induction. According to Harish-Chandra two such representations $i_M^G(\sigma)$ and $i_{M'}^G(\sigma')$ are isomorphic if and only if the pairs (M,σ) and (M',σ') are conjugate under $G(F)$. Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of Levi subgroups of G. Then, $\mathcal{X}_{\mathrm{temp}}(G)$ is naturally a quotient of

$$\widetilde{\mathcal{X}}_{\mathrm{temp}}(G) = \bigsqcup_{M\in\mathcal{M}} \bigsqcup_{\mathcal{O}\in \Pi_2(M)/i\mathcal{A}^*_{M,F}} \mathcal{O}$$

which has a natural structure of real smooth manifold since each orbit $\mathcal{O} \in \{\Pi_2(M)\}$ is a quotient of $i\mathcal{A}^*_{M,F}$ by a finite subgroup hence is naturally a real smooth manifold. We equip $\mathcal{X}_{\mathrm{temp}}(G)$ with the quotient topology. Note that the connected components of $\mathcal{X}_{\mathrm{temp}}(G)$ are the image of unramified classes $\mathcal{O} \in \{\Pi_2(M)\}$, $M \in \mathcal{M}$. We define a regular Borel measure $d\pi$ on $\mathcal{X}_{\mathrm{temp}}(G)$ by requesting that

$$\int_{\mathcal{X}_{\mathrm{temp}}(G)} \varphi(\pi)d\pi = \sum_{M\in\mathcal{M}} |W(G,M)|^{-1} \sum_{\mathcal{O}\in\Pi_2(M)/i\mathcal{A}^*_{M,F}} [i\mathcal{A}^\vee_{M,\sigma} : i\mathcal{A}^\vee_{M,F}]^{-1} \int_{i\mathcal{A}^*_{M,F}} \varphi(i_P^G(\sigma_\lambda))d\lambda$$

for all $\varphi \in C_c(\mathcal{X}_{\mathrm{temp}}(G))$, where for all $M \in \mathcal{M}$ we have fixed $P \in \mathcal{P}(M)$ and for all $\mathcal{O} \in \{\Pi_2(M)\}$ we have fixed a base-point $\sigma \in \mathcal{O}$.

For all $\pi = i_M^G(\sigma) \in \mathcal{X}_{\mathrm{temp}}(G)$, we set $\mu(\pi) = d(\sigma)j(\sigma)^{-1}$. This quantity only depends on π since another pair (M',σ') yielding π, where M' is a Levi subgroup and $\sigma' \in \Pi_2(M')$, is $G(F)$-conjugate to (M,σ).

Then we introduce the space $\mathcal{X}(G)$. First denote by $R_{\mathrm{temp}}(G)$ the space of complex virtual tempered representations of $G(F)$, that is $R_{\mathrm{temp}}(G)$ is the complex vector space with basis $\mathrm{Temp}(G)$.
In [Art93], Arthur defines a set $T_{\mathrm{ell}}(G)$ of virtual tempered representations of $G(F)$, that we will denote by $\mathcal{X}_{\mathrm{ell}}(G)$ in this paper. In fact, we can simply define the elliptic representation as

$$R_{\mathrm{ell}}(G) = \{\pi \in R_{\mathrm{temp}}(G), \forall P \subsetneq G, \pi_P^w = 0\}$$

, and

$$R_{\mathrm{ind}}(G) = \sum_{P=MU\subsetneq G} i_M^G(R_{\mathrm{temp}(M)}),$$

where π_P^w is the left adjoint of the normalized parabolic induction functor which is called the *weak Jacquet module*. The set $T_{\mathrm{ell}}(G)$ is a basis of $R_{\mathrm{ell}}(G)$.

The elements of $\mathcal{X}_{\mathrm{ell}}(G)$ are actually well-defined only up to a scalar of module 1. That is, we have $\mathcal{X}_{\mathrm{ell}}(G) \subset R_{\mathrm{temp}}(G)/\mathbb{S}^1$. These are the so-called *elliptic representations*. Let us recall their definition. Let $P = MU$ be a parabolic subgroup of G and $\sigma \in \Pi_2(M)$. For all $g \in G(F)$, we define $g\sigma$ to be the representation of $gM(F)g^{-1}$ given by $(g\sigma)(m') = \sigma(g^{-1}m'g)$ for all $m' \in gM(F)g^{-1}$. Denote by $\mathrm{Norm}_{G(F)}(\sigma)$ the subgroup of elements $g \in \mathrm{Norm}_{G(F)}(M)$ such that $g\sigma \simeq \sigma$ and set $W(\sigma) = \mathrm{Norm}_{G(F)}(\sigma)/M(F)$. Fix $P \in \mathcal{P}(M)$. Then, we may associate to every $w \in W(\sigma)$ an unitary endomorphism $R_P(w)$ of the representation $i_P^G(\sigma)$ that is well-defined up to a scalar of module 1 as follows. Choose a lift $\widetilde{w} \in \mathrm{Norm}_{G(F)}(\sigma)$ of w and an unitary endomorphism $A(\widetilde{w})$ of σ such that $\sigma(\widetilde{w}^{-1}m\widetilde{w}) = A(\widetilde{w})^{-1}\sigma(m)A(\widetilde{w})$ for all $m \in M(F)$. We define the operator $R_P(w) : i_P^G(\sigma)^\infty \to i_P^G(\sigma)^\infty$ as the composition $R_{P|wPw^{-1}}(\sigma) \circ \mathcal{A}(\widetilde{w}) \circ \lambda(\widetilde{w})$, where

(i) $\lambda(\widetilde{w})$ is the isomorphism $i_P^G(\sigma) \simeq i_{wPw^{-1}}^G(\widetilde{w}\sigma)$ given by $(\lambda(\widetilde{w})e)(g) = e(\widetilde{w}^{-1}g)$;

(ii) $\mathcal{A}(\widetilde{w})$ is the isomorphism $i_{wPw^{-1}}^G(\widetilde{w}\sigma) \simeq i_{wPw^{-1}}^G(\sigma)$ given by $(\mathcal{A}(\widetilde{w})e)(g) = A(\widetilde{w})e(g)$;

(iii) $R_{P|wPw^{-1}}(\sigma) : i_{wPw^{-1}}^G(\sigma)^\infty \to i_P^G(\sigma)^\infty$ is a chosen normalized intertwining operator.

We can check that $R_P(w)$ is $G(F)$-equivariant and that it depends on all the choices ($\widetilde{w}$, $A(\widetilde{w})$ and the normalization of the intertwining operator $R_{P|wPw^{-1}}(\sigma)$) only up to a scalar of module 1. We associate to any $w \in W(\sigma)$ a virtual tempered representation $i_M^G(\sigma, w)$, well-defined up to a scalar of module 1, by setting

$$i_M^G(\sigma, w) = \sum_{\lambda \in \mathbb{C}} \lambda \, i_P^G(\sigma, w, \lambda)$$

where for all $\lambda \in \mathbb{C}$, $i_P^G(\sigma, w, \lambda)$ denotes the subrepresentation of $i_P^G(\sigma)$ where $R_P(w)$ acts by multiplication by λ (as is indicated in the notation this definition doesn't depend on the choice of P). Let $W_0(\sigma)$ be the subgroup of elements $w \in W(\sigma)$ such that $R_P(w)$ is a scalar multiple of the identity and let $W(\sigma)_{\mathrm{reg}}$ be the subgroup of elements $w \in W(\sigma)$ such that $\mathcal{A}_M^w = \mathcal{A}_G$. We will say that the virtual representation $i_M^G(\sigma, w)$, $w \in W(\sigma)$, is *elliptic* if $W_0(\sigma) = \{1\}$ and $w \in W(\sigma)_{\mathrm{reg}}$. The set $\mathcal{X}_{\mathrm{ell}}(G)$ is the set of all virtual elliptic representations (well-defined up to multiplication by a scalar of module 1) that are obtained in this way. Let $\pi \in \mathcal{X}_{\mathrm{ell}}(G)$ and write $\pi = i_M^G(\sigma, w)$ with M, σ and $w \in W(\sigma)_{\mathrm{reg}}$ as before. Then we set

$$D(\pi) = |\det(1-w)_{\mathcal{A}_M^G}|^{-1}|W(\sigma)_w|^{-1}$$

where $W(\sigma)_w$ denotes the centralizer of w in $W(\sigma)$. This number doesn't depend on the particular choice of M, σ and w representing π because any other choice yielding π will be $G(F)$-conjugate to (M, σ, w).

The set $\mathcal{X}_{\mathrm{ell}}(G)$ satisfies the following important property. Denote by $R_{\mathrm{ell}}(G)$ the subspace of $R_{\mathrm{temp}}(G)$ generated by $\mathcal{X}_{\mathrm{ell}}(G)$ and denote by $R_{ind}(G)$ the subspace of $R_{\mathrm{temp}}(G)$ generated by the image of all the linear maps $i_M^G : R_{\mathrm{temp}}(M) \to R_{\mathrm{temp}}(G)$ for M a proper Levi subgroup of G. Then we have the decomposition

$$R_{\mathrm{temp}}(G) = R_{ind}(G) \oplus R_{\mathrm{ell}}(G)$$

The set $\mathcal{X}_{\mathrm{ell}}(G)$ is invariant under unramified twists. We will denote by $\mathcal{X}_{\mathrm{ell}}(G)/i\mathcal{A}_{G,F}^*$ the set of unramified orbits in $\mathcal{X}_{\mathrm{ell}}(G)$. Also, we will denote by $\underline{\mathcal{X}}_{\mathrm{ell}}(G)$ the inverse image of $\mathcal{X}_{\mathrm{ell}}(G)$ in $R_{\mathrm{temp}}(G)$. This set is invariant under multiplication by $\mathbb{S}^1$.

We define $\mathcal{X}(G)$ to be the subset of $R_{\mathrm{temp}}(G)/\mathbb{S}^1$ consisting of virtual representations of the form $i_M^G(\sigma)$ where M is a Levi subgroup of G and $\sigma \in \mathcal{X}_{\mathrm{ell}}(M)$. Also, we will denote by $\underline{\mathcal{X}}(G)$ the inverse image of $\mathcal{X}(G)$ in $R_{\mathrm{temp}}(G)$. Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of Levi subgroups of G. Then, $\mathcal{X}(G)$ is naturally a quotient of

$$\bigsqcup_{M \in \mathcal{M}} \ \bigsqcup_{\mathcal{O} \in \mathcal{X}_{\mathrm{ell}}(M)/i\mathcal{A}_{M,F}^*} \mathcal{O}$$

This defines, as for $\mathcal{X}_{\text{temp}}(G)$, a structure of topological space on $\mathcal{X}(G)$. We also define a regular Borel measure $d\pi$ on $\mathcal{X}(G)$ by requesting that

$$\int_{\mathcal{X}(G)} \varphi(\pi)d\pi = \sum_{M\in\mathcal{M}} |W(G,M)|^{-1} \sum_{\mathcal{O}\in\mathcal{X}_{\text{ell}}(M)/i\mathcal{A}^*_{M,F}} [i\mathcal{A}^\vee_{M,\sigma} : i\mathcal{A}^\vee_{M,F}]^{-1} \int_{i\mathcal{A}^*_{M,F}} \varphi(i_M^G(\sigma_\lambda))d\lambda$$

for all $\varphi \in C_c(\mathcal{X}_{\text{ell}}(G))$, where we have fixed a base point $\sigma \in \mathcal{O}$ for every orbit $\mathcal{O} \in \mathcal{X}_{\text{ell}}(M)/i\mathcal{A}^*_{M,F}$.
Finally, we extend the function $\pi \mapsto D(\pi)$ to $\mathcal{X}(G)$ by setting $D(\pi) = D(\sigma)$ for $\pi = i_M^G(\sigma)$, where M is a Levi subgroup and $\sigma \in \mathcal{X}_{\text{ell}}(M)$.

Remark 7.1. $\mathcal{X}_{\text{ell}}(G)$ *contains* $\Pi_2(G)$*, the isomorphism classes of square-integral representations of* $G(F)$*, but it usually bigger (unless* $G = \mathrm{GL}_n$*). For example, if* $G = \mathrm{SL}_2$*,* $\xi : F^\times \to \{\pm 1\}$ *is nontrivial, and* $i_B^G(\chi) = \pi^+ \oplus \pi^-$*, then* $\pi^+ - \pi^- \in \mathcal{X}_{\text{ell}}(G)$*.*

7.2. **Analytic Preliminaries.**

7.2.1. *Quasi Characters.* The goal of this section is to define and establish some crucial properties of quasi-characters on the group $G(F)$ and its Lie algebra.
First assume that F is p-adic. The definition and basic properties of quasi-characters in this case have been established in [Wal10]. We recall them now: let $\omega \subseteq \mathfrak{g}(F)$ be a completely $G(F)$-invariant open subset. A *quasi-character* on ω is a $G(F)$-invariant smooth function $\theta : \omega_{\text{reg}} \to \mathbb{C}$ satisfying the following condition: for all $X \in \omega_{\text{ss}}$, there exists $\omega_X \subseteq \mathfrak{g}_X(F)$ a G-good open neighborhood of X (See the definiton in Appendix A), such that $\omega_X^G \subseteq \omega$ and coefficients $c_{\theta,\mathcal{O}}(X)$ for all $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_X)$ such that we have

$$\theta(Y) = \sum_{\mathcal{O}\in\mathrm{Nil}(\mathfrak{g}_X)} c_{\theta,\mathcal{O}}(X)\widehat{j}(\mathcal{O}, Y)$$

for all $Y \in \omega_{X,\text{reg}}$. Denote $QC(\omega)$ the space of all quasi-characters on ω, Denote $QC_c(\omega)$ the subspace of all quasi-characters on ω whose support is compact modulo conjugation. To unify notation with the real case, we will also set $SQC(\mathfrak{g}(F)) = QC_c(\mathfrak{g}(F))$ and we will call elements of that space Schwartz quasi-characters on $\mathfrak{g}(F)$.

Parallelly, Let $\Omega \subseteq G(F)$ be a completely $G(F)$-invariant open subset. A *quasi-character* on Ω is a $G(F)$-invariant smooth function $\theta : \Omega_{\text{reg}} \to \mathbb{C}$ satisfying the following condition: for all $x \in \Omega_{\text{ss}}$, there exists $\omega_x \subseteq \mathfrak{g}_x(F)$ a G_x-excellent open neighborhood of 0 such that $(x\exp(\omega_x))^G \subseteq \Omega$ and coefficients $c_{\theta,\mathcal{O}}(x)$ for all $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_x)$ such that we have the equality

$$\theta(xe^Y) = \sum_{\mathcal{O}\in\mathrm{Nil}(\mathfrak{g}_x)} c_{\theta,\mathcal{O}}(x)\widehat{j}(\mathcal{O}, Y)$$

for all $Y \in \omega_{x,\text{reg}}$. Denote $QC(\Omega)$ the space of all quasi-characters on Ω, denote $QC_c(\Omega)$ the subspace of all quasi-characters on Ω whose support is compact modulo conjugation.

Remark 7.2. *For both cases, since if* θ *a quasi-character on* ω *(resp.* Ω*) and* $f \in C^\infty(\omega)^G$ *(resp.* $f \in C^\infty(\Omega)^G$*), then* $f\theta$ *is also a quasi-character. Thus we will endow the space of quasi-characters with the projective limit topology where the limit runs through the* QC_c *function supported on the completely* $G(F)$ *invariant open subsets of* ω *(resp.* Ω*), where the transition map is give by natural projection, and the* QC_c *spaces are endowed with its finest locally convex topology.*

Proposition 7.1. (i) *For all* $X \in \mathfrak{g}_{\text{reg}}(F)$*,* $\widehat{j}(X, .)$ *is a quasi-character on* $\mathfrak{g}(F)$*. For all* $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})$*,* $\widehat{j}(\mathcal{O}, .)$ *is a quasi-character on* $\mathfrak{g}(F)$*. For every irreducible admissible representation* π *of* $G(F)$*, the character* θ_π *is a quasi-character on* $G(F)$*.*

(ii) *For all* $\theta \in QC(G(F))$ *(resp.* $\theta \in QC(\mathfrak{g}(F))$*) the function* $(D^G)^{1/2}\theta$ *is locally bounded.*

(iii) *The Fourier transform preserves* $SQC(\mathfrak{g}(F))$ *in the following sense: for all* $\theta \in SQC(\mathfrak{g}(F))$*, there exists* $\widehat{\theta} \in SQC(\mathfrak{g}(F))$ *such that* $\widehat{T_\theta} = T_{\widehat{\theta}}$*. Moreover, for all* $\theta \in SQC(\mathfrak{g}(F))$*, we have the equality*

$$\widehat{\theta} = \int_{\Gamma(\mathfrak{g})} D^G(X)^{1/2}\theta(X)\widehat{j}(X,.)dX$$

the integral being absolutely convergent in $QC(\mathfrak{g}(F))$*.*

(iv) *Let* $\omega \subset \mathfrak{g}(F)$ *be a* G*-excellent open subset. Set* $\Omega = \exp(\omega)$*. Then, the linear map*

$$\theta \mapsto \theta_\omega$$

Example 5. *This remarkable theorem, which was first proved by Howe* [Ho74] *for general linear group, is a qualitative result that leaves many unresolved quantitative questions. Howe* [Ho74] *showed that* $c_{\mathcal{O},\theta_\pi}$ *is an integer for every irreducible supercuspidal representation and every nilpotent orbit. Harish Chandra derived a formula for the leading term* $c_{\{0\},\theta_\pi}$ *in the local character expansion of an irreducible supercuspidal represnetation* π*, Rogawski proved this formula still holds for discrete series in* [Rog80]*. Assem* [Ass94] *determined the function* $\widehat{j}(\mathcal{O},.)$ *for* $\mathrm{SL}_r(F)$ *with* r *a prime. DeBacker and Sally* [DS00] *and Murnaghan* [Mu91] *evaluated an integral formula to obtain values for the* $\widehat{j}(\mathcal{O},.)$*s in the case* $\mathrm{SL}_2(F)$ *and* $\mathrm{GSp}_4(F)$*.*

Then we come to the quasi-characters at archimdean place. Let $\omega \subseteq \mathfrak{g}(\mathbb{R})$ be a completely $G(\mathbb{R})$-invariant open subset. A *quasi-character* on ω is a function $\theta \in C^\infty(\omega_{\mathrm{reg}})^G$ which satisfies the two following conditions

(i) For all $u \in I(\mathfrak{g})$, the function $(D^G)^{1/2}\partial(u)\theta$ is locally bounded on ω;

(ii) For all $u \in I(\mathfrak{g})$, we have the following equality of distributions on ω

$$\partial(u)T_\theta = T_{\partial(u)\theta},$$

where $I(\mathfrak{g})$ by definition is the subalgebra of G-invariant elements in $S(\mathfrak{g})$, $\partial(\cdot)$ is the identification of the symmetric algebra of $\mathfrak{g}$ with the algebra of differential operators on $\mathfrak{g}$, $D^G(\cdot)$ is the discriminant $|\det \mathrm{ad}(X)_{|\mathfrak{g}/\mathfrak{g}_X}|$, and T_θ is the natural identification of the locally integrable function with the distribution.

Remark 7.3. *Here we use the lemma impliedly: the function* $X \mapsto D^G(X)^{-1/2}$ *(resp.* $x \mapsto D^G(x)^{-1/2}$*) is locally integrable on* $\mathfrak{g}(F)$ *(resp. on* $G(F)$*).*

Notice that the notion of quasi-character is local for the invariant topology: if $\theta \in C^\infty(\omega_{\mathrm{reg}})^G$ then θ is a quasi-character on ω if and only if for all $X \in \omega_{\mathrm{ss}}$ there exists $\omega' \subseteq \omega$ a completely G-invariant open neighborhood of X such that $\theta_{|\omega'}$ is a quasi-character on ω'. We will say that a quasi-character θ on ω is *compactly supported* if its support (in ω) is compact modulo conjugation. Finally, a *Schwartz quasi-character* is a quasi-character θ on $\mathfrak{g}(\mathbb{R})$ such that for all $u \in I(\mathfrak{g})$ and for any integer $N \geqslant 1$, we have an inequality

$$D^G(X)^{1/2}|\partial(u)\theta(X)| \ll \|X\|_{\Gamma(\mathfrak{g})}^{-N}$$

for all $X \in \mathfrak{g}_{\mathrm{reg}}(\mathbb{R})$. Note that a compactly supported quasi-character is automatically a Schwartz quasi-character.

Any invariant distribution T on some completely $G(\mathbb{R})$-invariant open subset $\omega \subseteq \mathfrak{g}(\mathbb{R})$ such that $\dim(I(\mathfrak{g})T) < \infty$ is the distribution associated to a quasi-character on ω. This follows from the representation theorem of Harish-Chandra on the Lie algebra, cf. Theorem 28 p.95 of [Va77]. In particular, the functions $\widehat{j}(X,.)$, $X \in \mathfrak{g}_{\mathrm{reg}}(\mathbb{R})$, and the functions $\widehat{j}(\mathcal{O},.)$, $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})$, are quasi-characters on $\mathfrak{g}(\mathbb{R})$.

Proposition 7.2. *Let $\theta \in C^\infty(\mathfrak{g}_{\mathrm{reg}}(\mathbb{R}))^G$ and assume that there exists $k \geq 0$ such that for all $N \geq 1$ we have an inequality*

$$D^G(X)^{1/2}|\theta(X)| \ll \log\left(2 + D^G(X)^{-1}\right)^k \|X\|_{\Gamma(\mathfrak{g})}^{-N}$$

for all $X \in \mathfrak{g}_{\mathrm{reg}}(F)$. Then the function θ is locally integrable, the distribution T_θ is tempered and there exists a quasi-character $\widehat{\theta}$ on $\mathfrak{g}(\mathbb{R})$ such that

$$\widehat{T_\theta} = T_{\widehat{\theta}}$$

Moreover, we have

$$\widehat{\theta}(Y) = \int_{\Gamma(\mathfrak{g})} D^G(X)^{1/2}\theta(X)\widehat{j}(X,Y)dX$$

for all $Y \in \mathfrak{g}_{\mathrm{reg}}(\mathbb{R})$, the integral being absolutely convergent, and the function $X \in \mathfrak{g}_{\mathrm{reg}}(\mathbb{R}) \mapsto D^G(X)^{1/2}\widehat{\theta}(X)$ is (globally) bounded.

Proposition 7.3. *Let $\omega \subseteq \mathfrak{g}(\mathbb{R})$ be a completely $G(\mathbb{R})$-invariant open subset and let $\theta \in C^\infty(\omega_{\mathrm{reg}})^G$. Then for $X \in \mathfrak{g}_{\mathrm{ss}}(\mathbb{R})$ and $\omega_X \subseteq \mathfrak{g}_X(\mathbb{R})$ be a G-good open neighborhood of X, assume that $\omega = \omega_X^G$. Then θ is a quasi-character on ω if and only if θ_{X,ω_X} is a quasi-character on ω_X.*

Let $\omega \subseteq \mathfrak{g}(\mathbb{R})$ be a completely $G(\mathbb{R})$-invariant open subset. We will denote by $QC_c(\omega)$ the space of quasi-characters on ω and by $QC_c(\omega)$ the subspace of compactly supported quasi-characters on ω. If $L \subset \omega$ is invariant and compact modulo conjugation, we will also denote by $QC_L(\omega) \subset QC_c(\omega)$ the subspace of quasi-characters with support in L. Finally, we will denote by $SQC(\mathfrak{g}(\mathbb{R}))$ the space of Schwartz quasi-characters on $\mathfrak{g}(\mathbb{R})$. The topology endows on $QC(\omega)$ is similar to the p-adic case, with a little modification on the locally convex topology and the subsets ran through (See [BP20, Section 4.2]). What we interested in now is the local expansion property for $\theta \in QC(\omega)$.

Lemma 7.1. *Let $X \in \omega_{\mathrm{ss}}$. Then, there exist constants $c_{\theta,\mathcal{O}}(X)$ for $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_X)$ such that*

$$D^G(X+Y)^{1/2}\theta(X+Y) = D^G(X+Y)^{1/2} \sum_{\mathcal{O}\in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_X)} c_{\theta,\mathcal{O}}(X)\widehat{j}(\mathcal{O},Y) + O(|Y|)$$

for all $Y \in \mathfrak{g}_{X,\mathrm{reg}}(\mathbb{R})$ sufficiently near 0.

Then "take the exponent" of these results to the group level: let $\Omega \subseteq G(\mathbb{R})$ be a completely $G(\mathbb{R})$-invariant open subset. A *quasi-character* on Ω is a function $\theta \in C^\infty(\Omega_{\mathrm{reg}})^G$ that satisfies the two following conditions

(i) For all $z \in \mathcal{Z}(\mathfrak{g})$, the function $(D^G)^{1/2}z\theta$ is locally bounded on Ω;

(ii) For all $z \in \mathcal{Z}(\mathfrak{g})$, we have the following equality of distributions on Ω

$$zT_\theta = T_{z\theta},$$

where $\mathcal{Z}(\mathfrak{g})$ by definition is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, where the action is given by differential operators. Similarly, we may endow this an appropriate topology, see [BP20, Section 4.4].

As in the case of the Lie algebra, by the representation theorem of Harish-Chandra, any invariant distribution T defined on some completely $G(\mathbb{R})$-invariant open subset $\Omega \subseteq G(\mathbb{R})$ such that $\dim(\mathcal{Z}(\mathfrak{g})T) < \infty$ is the distribution associated to a quasi-character on Ω. In particular, for every admissible irreducible representation π of $G(\mathbb{R})$ the character θ_π of π is a quasi-character on $G(\mathbb{R})$. Meanwhile, we have the local expansion as follows:

Proposition 7.4. *Let $\Omega \subseteq G(\mathbb{R})$ be a completely $G(\mathbb{R})$-invariant open subset. For all $\theta \in QC(\Omega)$ and all $x \in G_{\mathrm{ss}}(\mathbb{R})$, there exist constants $c_{\theta,\mathcal{O}} \in \mathbb{C}$, for $\mathcal{O} \in \mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)$, such that*

$$D^G(xe^Y)^{1/2}\theta(xe^Y) = D^G(xe^Y)^{1/2} \sum_{\mathcal{O}\in\mathrm{Nil}_{\mathrm{reg}}(\mathfrak{g}_x)} c_{\theta,\mathcal{O}}(x)\widehat{j}(\mathcal{O}, Y) + O(|Y|)$$

for all $Y \in \mathfrak{g}_{x,\mathrm{reg}}(\mathbb{R})$ sufficiently near 0.

7.2.2. *Weighted Orbital Integrals and Weighted Characters.* The goal of this section is to introduce "weighted" edition of the classical objects arised in the local trace formula. These are studied by Arthur initially in [Art05]. We use these factors to study strongly cuspidal fucntions in next subsection.

Geometric Sides. Let M be a Levi subgroup of G. Choose a maximal compact subgroup K of $G(F)$ that is special in the p-adic case. Recall that using K (that's the reason why all distributions constructed here are depend on the choice of K), we may construct for every $P \in \mathcal{P}(M)$ a map

$$H_P : G(F) \to \mathcal{A}_M$$

(cf. Appendix A). For every $g \in G(F)$, the family $(H_P(g))_{P\in\mathcal{P}(M)}$ is a positive (G, M)-orthogonal set, i.e., for all adjacent parabolic subgroups $P, P' \in \mathcal{P}(M)$, there exists a positive number $r_{P,P'} \leq 0$ such that $H_P(g) - H_{P'}(g) = r_{P,P'}\alpha^\vee$, where α is the unique root of A_M that is positive for P and negative for P'. Hence, it defines $(v_P(g, .))_{P\in\mathcal{P}(M)}$ as a family of functions on $i\mathcal{A}_M^*$, by sending λ to $e^{\lambda(H_P(g))}$. Here the number $v_M(g)$ associated to this family is the volume in $\mathcal{A}_M^G$ of the convex hull of the $H_P(g)$, $P \in \mathcal{P}(M)$. Check the complete knowledge of (G, M)-family (with beautiful illustrations and careful calculations) in [Art05, Part II, (G, M)-family]. The function $g \mapsto v_M(g)$ is obviously invariant on the left by $M(F)$ and on the right by K.
Let $x \in M(F) \cap G_{\mathrm{reg}}(F)$. Then, for $f \in \mathcal{C}(G(F))$, we define the *weighted orbital integral* of f at x to be

$$J_M(x, f) = D^G(x)^{1/2} \int_{G_x(F)\backslash G(F)} f(g^{-1}xg)v_M(g)dg$$

(note that the above expression is well-defined since $G_x \subset M$). The integral above is absolutely convergent and this defines a *tempered* distribution $J_M(x, .)$ on $G(F)$ (for tempered distribution, see [BP20, Section 1.5]). More generally, we can associate to the (G, M)-family $(v_P(g, .))_{P\in\mathcal{P}(M)}$ complex numbers $v_L^Q(g)$ for all $L \in \mathcal{L}(M)$ and all $Q \in \mathcal{F}(L)$, which allows us to define tempered distributions $J_L^G(x, .)$ on $G(F)$ for all $L \in \mathcal{L}(M)$ and all $Q \in \mathcal{F}(L)$ by setting

$$J_L^Q(x, f) = D^G(x)^{1/2} \int_{G_x(F)\backslash G(F)} f(g^{-1}xg)v_L^Q(g)dg, \quad f \in \mathcal{C}(G(F))$$

where the space $\mathcal{C}(G(F))$ is the Harish-Chandra Schwartz space defined in [BP20, Section 1.5]. The functions $x \in M(F) \cap G_{\mathrm{reg}}(F) \mapsto J_L^Q(x, f)$ are easily seen to be $M(F)$-invariant.
Let $X \in \mathfrak{m}(F) \cap \mathfrak{g}_{\mathrm{reg}}(F)$. We define similarly *weighted orbital integrals* $J_L^Q(X, .)$, $L \in \mathcal{L}(M)$, $Q \in \mathcal{F}(L)$. These are tempered distributions on $\mathfrak{g}(F)$ given by

$$J_L^Q(X, f) = D^G(X)^{1/2} \int_{G_X(F)\backslash G(F)} f(g^{-1}Xg)v_L^Q(g)dX, \quad f \in \mathcal{S}(\mathfrak{g}(F))$$

Where $\mathcal{S}(G(F))$ is the Schwartz sapce. When $Q = G$, we will simply set $J_L^G(X, f) = J_L(X, f)$.

Example 6. *When $M = G$, weight $v_M = 1$, the weighted integral is the usual orbital integral.*

Spectral Sides. Through [BP20, Section 2.4, 2.5], we associate a family of tempered distributions

$$(J_L^Q(\sigma, .))_{L,Q} = \mathrm{Trace}(\mathcal{R}_L^Q(\sigma, P) i_P^G(\sigma)), \quad L \in \mathcal{L}(M),\ Q \in \mathcal{F}(L)$$

on $G(F)$ with the inputs: M a Levi subgroup of G, σ a tempered representation of $M(F)$, and K a maximal compact subgroup of $G(F)$ that is special in p-adic case. Here $\mathcal{R}_L^Q(\sigma, P)$ is Arthur's Weight, defined using standard normalized interwining operators, taking values in $\mathrm{End}(i_{K_P}^K(\sigma_{K_P})^\infty)$. It will depends on the choice of K and the way we normalized the interwining operators. If $L = Q = G$, this reduces to the usual character of the unitary induction:

$$J_G^G(\sigma, f) = \mathrm{Trace}(i_M^G(\sigma, f))$$

for all $f \in \mathcal{C}(G(F))$. Details can be checked in [Art94].
Considering that the spectral side is more standard, and can be applied verbatim to other classical groups, in order save the length of the paper, we skip the explicit construction and analytic properties.

7.2.3. *Strong Cuspidal Functions.* This section is devoted to the study of *strongly cuspidal functions*. This introduces us enough convergent properties to discuss the local trace formulas, and establishes the bridge between the quasi-characters with orbital integrals. For every parabolic subgroup $P = MU$ of G, we define continuous linear maps

$$f \in \mathcal{C}(G(F)) \mapsto f^P \in \mathcal{C}(M(F))$$
$$\varphi \in \mathcal{S}(\mathfrak{g}(F)) \mapsto \varphi^P \in \mathcal{S}(\mathfrak{m}(F))$$

by setting

$$f^P(m) = \delta_P(m)^{1/2} \int_{U(F)} f(mu)du \text{ and } \varphi^P(X) = \int_{\mathfrak{u}(F)} \varphi(X + N)dN$$

We will say that a function $f \in \mathcal{C}(G(F))$ (resp. $\varphi \in \mathcal{S}(\mathfrak{g}(F))$) is *strongly cuspidal* if $f^P = 0$ (resp. $\varphi^P = 0$) for every proper parabolic subgroup P of G. We will denote by $\mathcal{C}_{\mathrm{scusp}}(G(F))$, $\mathcal{S}_{\mathrm{scusp}}(G(F))$ and $\mathcal{C}_{\mathrm{scusp}}(\mathfrak{g}(F))$ the subspaces of strongly cuspidal functions in $\mathcal{C}(G(F))$, $\mathcal{S}(G(F))$ and $\mathcal{S}(\mathfrak{g}(F))$ respectively. More generally, if $\Omega \subseteq G(F)$ (resp. $\omega \subseteq \mathfrak{g}(F)$) is a completely $G(F)$-invariant open subset, we will set $\mathcal{S}_{\mathrm{scusp}}(\Omega) = \mathcal{S}(\Omega) \cap \mathcal{S}_{\mathrm{scusp}}(G(F))$ (resp. $\mathcal{S}_{\mathrm{scusp}}(\omega) = \mathcal{S}(\omega) \cap \mathcal{S}_{\mathrm{scusp}}(\mathfrak{g}(F))$). Furthermore, the action of $\mathcal{Z}(\mathfrak{g})$ preserves the spaces $\mathcal{C}_{\mathrm{scusp}}(G(F))$, $\mathcal{S}_{\mathrm{scusp}}(G(F))$ and $\mathcal{S}_{\mathrm{scusp}}(\Omega)$ and the action of $I(\mathfrak{g})$ preserves the spaces $\mathcal{S}_{\mathrm{scusp}}(\mathfrak{g}(F))$ and $\mathcal{S}_{\mathrm{scusp}}(\omega)$, and the multiplication with functions in $C_c^\infty(G(F))^G$ (resp. $C_c^\infty(\mathfrak{g}(F))^G$) will also preserve these spaces, respectively. Our main lemma on the geometric side is the following:

Lemma 7.2. *Let $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$ (resp. $f \in \mathcal{S}_{\mathrm{scusp}}(\mathfrak{g}(F))$) be a strongly cuspidal function and fix $x \in M(F) \cap G_{\mathrm{reg}}(F)$ (resp. $X \in \mathfrak{m}(F) \cap \mathfrak{g}_{\mathrm{reg}}(F)$). Then*

(i) *For all $L \in \mathcal{L}(M)$ and all $Q \in \mathcal{F}(L)$, if $L \neq M$ or $Q \neq G$, we have*

$$J_L^Q(x, f) = 0 \quad (\text{resp. } J_L^Q(X, f) = 0)$$

(ii) *The weighted orbital integral $J_M^G(x, f)$ (resp. $J_M^G(X, f)$) doesn't depend on the choice of K;*

(iii) *If $x \notin M(F)_{\mathrm{ell}}$ (resp. $X \notin \mathfrak{m}(F)_{\mathrm{ell}}$), we have*

$$J_M^G(x, f) = 0 \quad (\text{resp. } J_M^G(X, f) = 0)$$

(iv) *For all $y \in G(F)$, we have*

$$J_{yMy^{-1}}^G(yxy^{-1}, f) = J_M^G(x, f) \quad (\text{resp. } J_{yMy^{-1}}^G(yXy^{-1}, f) = J_M^G(X, f)).$$

For all $x \in G_{\text{reg}}(F)$, let us denote by $M(x)$ the centralizer of A_{G_x} in G. It is the minimal Levi subgroup of G containing x. Let $f \in \mathcal{C}_{\text{scusp}}(G(F))$. Then, we set

$$\theta_f(x) = (-1)^{a_G - a_{M(x)}} \nu(G_x)^{-1} D^G(x)^{-1/2} J^G_{M(x)}(x, f)$$

for all $x \in G_{\text{reg}}(F)$. By the point (iv) of the above lemma, the function θ_f is invariant. We define similarly an invariant function θ_f on $\mathfrak{g}_{\text{reg}}(F)$ for all $f \in \mathcal{S}_{\text{scusp}}(\mathfrak{g}(F))$, by setting

$$\theta_f(X) = (-1)^{a_G - a_{M(X)}} \nu(G_X)^{-1} D^G(X)^{-1/2} J^G_{M(X)}(X, f)$$

for all $X \in \mathfrak{g}_{\text{reg}}(F)$, where $M(X)$ denotes the centralizer of A_{G_X} in G.

Parallelly, we have such results for weighted characters of strongly cuspidal functions (spectral side): Let M be a Levi subgroup of G and σ a tempered representation of $M(F)$. Recall that we have defined tempered distributions $J^Q_L(\sigma, .)$ on $G(F)$ for all $L \in \mathcal{L}(M)$ and all $Q \in \mathcal{F}(L)$. These distributions depended on the choice of a maximal compact subgroup K which is special in the p-adic case and also on the way we normalize intertwining operators.

Lemma 7.3. *Let $f \in \mathcal{C}(G(F))$ be a strongly cuspidal function.*

(i) *For all $L \in \mathcal{L}(M)$ and all $Q \in \mathcal{F}(L)$, if $L \neq M$ or $Q \neq G$, then we have*

$$J^Q_L(\sigma, f) = 0$$

(ii) *The weighted character $J^G_M(\sigma, f)$ doesn't depend on the choice of K or on the way we normalized the intertwining operators;*

(iii) *If σ is induced from a proper parabolic subgroup of M then*

$$J^G_M(\sigma, f) = 0$$

(iv) *For all $x \in G(F)$, we have*

$$J^G_{xMx^{-1}}(x\sigma x^{-1}, f) = J^G_M(\sigma, f)$$

In the beginning of this appendix, we have defined a set $\underline{\mathcal{X}}(G)$ of virtual tempered representations of $G(F)$. Let $\pi \in \underline{\mathcal{X}}(G)$. Then, there exists a pair (M, σ) where M is a Levi subgroup of G and $\sigma \in \underline{\mathcal{X}}_{\text{ell}}(M)$ such that $\pi = i^G_M(\sigma)$. We set

$$\widehat{\theta}_f(\pi) = (-1)^{a_G - a_M} J^G_M(\sigma, f)$$

for all $f \in \mathcal{C}_{\text{scusp}}(G(F))$ (recall that the weighted character $J^G_M(\sigma, .)$ is extended by linearity to all virtual tempered representations). This definition makes sense by the point (iv) of the lemma above since the pair (M, σ) is well-defined up to conjugacy.

7.2.4. *Simple Local Trace Formulas for Strongly Cuspidal Functions.* Let us set

$$K^A_{f,f'}(g_1, g_2) = \int_{G(F)} f(g_1^{-1} g g_2) f'(g) dg$$

for all $f, f' \in \mathcal{C}(G(F))$ and all $g_1, g_2 \in G(F)$. Easy to see that the integral above is absolutely convergent. We also define

$$K^A_{f,f'}(x, x) = \int_{\mathfrak{g}(F)} f(x^{-1} X x) f'(X) dX$$

for all $f, f' \in \mathcal{S}(\mathfrak{g}(F))$ and all $x \in A_G(F)\backslash G(F)$. The two theorems below are slight variations around the local trace formula of Arthur ([Art91]) and its version for Lie algebras due to Waldspurger ([Wal95]).

Theorem 7.1. (i) *For all $d \geqslant 0$, there exists $d' \geqslant 0$ and a continuous semi-norm $\nu_{d,d'}$ on $\mathcal{C}(G(F))$ such that*

$$|K^A_{f,f'}(g_1, g_2)| \leqslant \nu_{d,d'}(f)\nu_{d,d'}(f')\Xi^G(g_1)\sigma(g_1)^{-d}\Xi^G(g_2)\sigma(g_2)^{d'}$$

and

$$|K^A_{f,f'}(g_1, g_2)| \leqslant \nu_{d,d'}(f)\nu_{d,d'}(f')\Xi^G(g_1)\sigma(g_1)^{d'}\Xi^G(g_2)\sigma(g_2)^{-d}$$

for all $f, f' \in \mathcal{C}(G(F))$ and for all $g_1, g_2 \in G(F)$.

(ii) *For all $d \geqslant 0$, there exists a continuous semi-norm ν_d on $\mathcal{C}(G(F))$ such that*

$$|K^A_{f,f'}(x, x)| \leqslant \nu_d(f)\nu_d(f')\Xi^G(x)^2\sigma_{A_G\backslash G}(x)^{-d}$$

for all $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$, all $f' \in \mathcal{C}(G(F))$ and all $x \in A_G(F)\backslash G(F)$.

(iii) *Let $f, f' \in \mathcal{C}(G(F))$ with f strongly cuspidal. Then, there exists $c > 0$ such that for all $d \geqslant 0$ there exists $d' \geqslant 0$ such that*

$$|K^A_{f,f'}(g, hg)| \ll \Xi^G(g)^2\sigma_{A_G\backslash G}(g)^{-d}e^{c\sigma(h)}\sigma(h)^{d'}$$

for all $h, g \in G(F)$.
By the point (ii), the function $x \in A_G(F)\backslash G(F) \mapsto K^A_{f,f'}(x, x)$ is integrable as soon as f is strongly cuspidal. We set

$$J^A(f, f') = \int_{A_G(F)\backslash G(F)} K^A_{f,f'}(x, x)dx$$

for all $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$ and all $f' \in \mathcal{C}(G(F))$.

(iv) *We have the geometric expansion*

$$J^A(f, f') = \int_{\Gamma(G)} D^G(x)^{1/2}\theta_f(x)J_G(x, f')dx$$

for all $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$ and all $f' \in \mathcal{C}(G(F))$, the integral above being absolutely convergent.

(v) *We have the spectral expansion*

$$J^A(f, f') = \int_{\mathcal{X}(G)} D(\pi)\widehat{\theta}_f(\pi)\theta_{\overline{\pi}}(f')d\pi$$

for all $f \in \mathcal{C}_{\mathrm{scusp}}(G(F))$ and all $f' \in \mathcal{C}(G(F))$, the integral above being absolutely convergent.

Theorem 7.2. (i) *For all $N \geqslant 0$, there exists a continuous semi-norm ν_N on $\mathcal{S}(\mathfrak{g}(F))$ such that*

$$|K^A_{f,f'}(x, x)| \leqslant \nu_N(f)\nu_N(f')\|x\|^{-N}_{A_G\backslash G}$$

for all $f \in \mathcal{S}_{\mathrm{scusp}}(\mathfrak{g}(F))$, all $f' \in \mathcal{S}(\mathfrak{g}(F))$ and all $x \in A_G(F)\backslash G(F)$.
In particular the function $x \in A_G(F)\backslash G(F) \mapsto K^A_{f,f'}(x, x)$ is integrable as soon as f is strongly cuspidal. We set

$$J^A(f, f') = \int_{A_G(F)\backslash G(F)} K^A_{f,f'}(x, x)dx$$

for all $f \in \mathcal{S}_{\mathrm{scusp}}(\mathfrak{g}(F))$ and all $f' \in \mathcal{S}(\mathfrak{g}(F))$.

(ii) *We have the "geometric" expansion*

$$J^A(f,f') = \int_{\Gamma(\mathfrak{g})} D^G(X)^{1/2}\theta_f(X)J_G(X,f')dX$$

for all $f \in \mathcal{S}_{\text{scusp}}(\mathfrak{g}(F))$ and all $f' \in \mathcal{S}(\mathfrak{g}(F))$, the integral above being absolutely convergent.

(iii) *We have the "spectral" expansion*

$$J^A(f,f') = \int_{\Gamma(\mathfrak{g})} D^G(X)^{1/2}\theta_{\widehat{f}}(X)J_G(-X,\widehat{f'})dX$$

for all $f \in \mathcal{S}_{\text{scusp}}(\mathfrak{g}(F))$ and all $f' \in \mathcal{S}(\mathfrak{g}(F))$, the integral above being absolutely convergent.

The reason we can apply the simple local trace formula of strongly cuspidal functions to study the quasi-characters which arised naturally in the representation theory is the following proposition:

Proposition 7.5. (i) *For all $f \in \mathcal{S}_{\text{scusp}}(\mathfrak{g}(F))$, the function θ_f is a Schwartz quasi-character and we have $\widehat{\theta}_f = \theta_{\widehat{f}}$. Moreover, if G admits an elliptic maximal torus, then the linear map*

$$\mathcal{S}_{\text{scusp}}(\mathfrak{g}(F)) \to SQC(\mathfrak{g}(F))$$
$$f \mapsto \theta_f$$

has dense image and for every completely $G(F)$-invariant open subset $\omega \subseteq \mathfrak{g}(F)$ which is relatively compact modulo conjugation, the linear map

$$\mathcal{S}_{\text{scusp}}(\omega) \to QC_c(\omega)$$
$$f \mapsto \theta_f$$

also has dense image.

(ii) *Let $f \in \mathcal{C}_{\text{scusp}}(G(F))$. Then, the function θ_f is a quasi-character on $G(F)$ and we have an equality of quasi-characters*

$$\theta_f = \int_{\mathcal{X}(G)} D(\pi)\widehat{\theta}_f(\pi)\overline{\theta_\pi}d\pi$$

where the integral above is absolutely convergent in $QC(G(F))$.

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