March, IHP, 2019

Theta correspondence of classical groups / Finite Field $\mathbb{F}_q$ (q odd)

Anne-Marie Aubert

The theta correspondence for classical groups over finite fields of odd characteristic is defined via the Weil representation of a finite symplectic group.
It maps a given irreducible representation of a group to a finite collection of irreducible representations of a group , such that is a dual pair.

I will first recall the definition of the correspondence and some its useful properties. Next, I will provide a full and explicit description of the theta correspondence via a conjecture that I have formulated, and proved in the case of linear and unitary dual pairs, with J. Michel and R. Rouquier, and which has been recently established in all cases by Pan and by Ma, Qiu and Zou, independently.

I will later explain several ways to extract a bijective correspondence, inclu~~ding the construction~~ by Gurevich and Howe of the eta correspondence, and give some applications. I will also describe the links between the theta correspondence and the finite analogue of the Gan-Gross-Prasad

(The complete document is on RTNCG page.)

Lecture 1

Notation: $k = \mathbb{F}_q$ q odd. $\bar{k}$: alg. closure $(W/_{\mathbb{F}_q}: \dim = 2N, \langle \cdot, \cdot \rangle)$

non-degenerated alternated form

$\rightsquigarrow Sp(W) \simeq Sp_{2N}(k)$

$M_{2N}(\mathbb{F}_q)$: ring of $2N \times 2N$ matrices, $*$: transport

- $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_{2n}(\mathbb{F}_q) \Leftrightarrow a^*c = c^*a,\ b^*d = d^*b,\ a^*d - c^*b = 1$
  $\Leftrightarrow ab^* = ba^*,\ cd^* = dc^*,\ ad^* - bc^* = 1$
- Heisenberg group: $H(W) := W \times k \simeq k^{2N+1}$, with
  $(w,t)(w',t') := (w+w', t+t' + \frac{1}{2}\langle w, w' \rangle)$

Remark

$H = H(W)$ is a special p-group with exponent p, with derived subgroup $H' = (\mathbb{F}_q, +)$.
If $q = p$, with p a prime number, H is an extraspecial p-group.
[The *exponent* of a group G is the least positive n s.t. $g^n = 1\ \forall g \in G$.
A p-group G is called *special* if either G is elementary abelian or G is of class 2 (i.e., $G' \subset Z_G$), and $G' = Z_G = \Phi(G)$ is elementary abelian.
G is called *extraspecial* if it is special and $|G'| = p$.
Here Z_G is the center of G and $\Phi(G)$ is the *Frattini subgroup* of G, i.e., the intersection proper maximal subgroups of G.]

- Finite analogue of Stone-von-Neumann Thm
- $\forall$ non-trivial ψ of $Z_{H(W)} \simeq k \rightsquigarrow \exists!$ irr. rep'n (ρ_ψ, δ) of $H(W)$. s.t
  $\rho_\psi(z) = \psi(z)\, Id_\delta$, for all $z \in Z_{H(W)}$.
- Weil Rep'n ω_ψ: projective rep'n of $Sp(W)$

$$\rho_\psi(g \cdot w, t) = \omega_\psi(g)\, \rho_\psi(w,t)\, \omega_\psi(g)^{-1}$$

<Rmk>: For $a \in k$, let $\psi_a: Z_{H(W)} \to \mathbb{C}$ defn by $\psi_a(x) := \psi(ax)$
- $\omega_{\psi_a} \simeq \omega_{\psi_{ab^2}}$, $\forall b \in k^\times$
- $[a] \in k^\times / (k^\times)^2$ inequivalent

- reductive dual pair:
  a pair of red. subgp of $Sp(W)$ is called <u>dual pair</u> if
  $$C_{Sp(W)}(G) = G', \quad C_{Sp(W)}(G') = G$$

$\rightsquigarrow$ (irreducible pair):

type I: $(Sp_{2n}(k), O_{N'}(k))$: with $nN' = N$
$(U_n(k), U_{n'}(k))$: $nn' = 2N$

type II: $(GL_n(k), GL_{n'}(k))$: $nn' = 2N$

- Stable range: (with G' smaller), if the defining vector space for G has totally isotropic subspace of dim greater or equal than the dim the defining vector space for G' (e.g. $(Sp_{2n}(k), O_{2n'}(k))$, s.t $n \geq 2n'$)

<Def>: (theta corr.)

$$\omega_\psi|_{G \times G'} =: \omega_{G,G'} = \sum_{(\pi,\pi')} \overset{\geq 0}{m_{\pi,\pi'}}\, \pi \otimes \pi'$$

<Def> $\Theta_{G'}: \mathbb{Z}\mathrm{Irr}(G) \to \mathbb{Z}\mathrm{Irr}(G')$ by $\in \mathrm{Irr}\times\mathrm{Irr}$

$$\Theta_{G'}(\pi) := \{\pi' : m_{\pi,\pi'} \neq 0\}$$

<Goal>: determine $\Theta_{G'}(\pi)$.

<rmk> 1) in general, ≥ 1 elements. (reducible) ($\mathrm{tr} = 0$)

2) hope: preserve "cusp", "temp", "uni", "parabolic induction"...

<Witt tower>

- two quadratic vector space V and V' over k are said to lie in the same Witt tower, if $\simeq$ after $\oplus$ some hyperbolic planes.

Notations

Focus on pairs formed by a symplectic group and an orthogonal group of even dimension. The other cases can be treated in a similar way.

- n and n' be two fixed non-negative integers
- V_n a symplectic space of (even) dimension $2n$ over $\mathbf{k}$. The corresponding group of isometries is the symplectic group $G_n := \mathrm{Sp}(V_n)$.
- $V'_{n'}$ a quadratic space of dimension $2n'$ over $\mathbf{k}$, i.e., a space endowed with a non-degenerate symmetric $\mathbf{k}$-bilinear form. The orthogonal group $G'_{n'} := \mathrm{O}(V_{n'})$ is the corresponding group of isometries.
- $(G_n, G'_{n'})$ is a dual pair in $\mathrm{Sp}(W)$ where $W := V_n \otimes_{\mathbf{k}} V'_{n'}$ with

$$\langle v_1 \otimes v'_1, v_2 \otimes v'_2\rangle := \langle v_1, v_2\rangle_{G_n} \cdot \langle v'_2, v'_1\rangle_{G'_{n'}}$$

for $v_1, v_2 \in V$ and $v'_1, v'_2 \in V'$.

<tower of thetas>:

$V_{n'} = V'_{an} \oplus H^{n'}$, $V_{m'} := V'_{an} \oplus H^{m'}$

$\rightsquigarrow T := \{W_m, m \geq 0\}$. $T' := \{V_{m'} : m' \geq 0\}$ (with tower of vector space)

For fixed $G = G_n \rightsquigarrow (G, G'_{m'})_{m' \geq 0} \rightsquigarrow \Theta_{m'}(\pi) := \Theta_{G'_{m'}}(\pi)$

Remark

There are two Witt towers of even orthogonal groups depending if $G'_{n'} = \mathrm{O}(V_{n'})$ is split or not. We denote them $\mathcal{T}'_+$ and $\mathcal{T}'_-$, respectively.

$\rightsquigarrow$ similarly for $(G_m, G')_{m \geq 0}$

$\theta^0(\pi') := \cdots$

First occurence

Let $\pi \in \mathrm{Irr}(G)$.

- The smallest non-negative integer $n'_{T'}(\pi)$ such that $\theta_{n'_{T'}(\pi)}(\pi) \neq 0$ is called the first occurrence index of π for the Witt tower $\mathcal{T}'$.
- The representation

$$\theta^0(\pi) := \theta_{n'_{T'}(\pi)}(\pi)$$

is called the first occurrence of π for this Witt tower.

<prop>: $n'_{T'}(\pi)$, $n_T(\pi')$ exists, and

$$n'_{T'}(\pi) \leq n', \quad n_T(\pi') \leq n;$$

$\Theta_{m'}(\pi) \neq 0$, for $\forall\, m' \geq n'_{T'}(\pi)$.

$\Theta_m(\pi') \neq 0$, for $\forall\, m \geq n_T(\pi')$

bijection

<prop>: for cuspidal (irr) π, $\overset{\text{bijection}}{\longleftrightarrow}$ cuspidal irr. π'

<def> G/k conn. red. alg group. $G(k)$ $P = L \ltimes U$

i_P^G, $r_{L,P}^G$, both exact. $r_{L,P}^G \dashv i_P^G$

↳ <def>: cuspidal, $\mathrm{Irr}_c(G)$

<Harish-Chandra>: $(L, \sigma)_G$: cusp supp. $S_c(\pi)$

for $\sigma \in \mathrm{Irr}_c(L) \rightsquigarrow \mathrm{Irr}^\sigma(G)$: section of $\mathrm{Irr}(G) \twoheadrightarrow S_c(\pi)$

<Notation>: P: max'l parabolic subgp of G,

$L := Sp_{2n_0} \times GL_m(k)$. with $n_0 + m = n$.

$\pi \in \mathrm{Irr}(G) \mapsto c(\pi) := \max\{m : \mathrm{Hom}_{GL_m(k)}(r_{L,P}^G(\pi), 1) \neq 0\}$, $4n+2$

<thm> (Ma - Qin - Zou 2024)

(conservation relation): $n'_{T_+}(\pi) + n'_{T_-}(\pi) + c(\pi) = 4n+2$

(local: $4n+4$)

Levi subgroups of G

A Levi subgroup $L = L_n$ of a parabolic subgroup $P = P_n$ of $G = G_n$ is isomorphic to

$$G_{n_0} \times \mathrm{GL}_{n_1}(\mathbf{k}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbf{k})$$

where $n_0 + n_1 + \cdots + n_r = n$.
Every $\pi \in \mathrm{Irr}(G)$ is isomorphic to a subquotient of the parabolic induced representation $i_{L,P}^G(\sigma)$ where $\sigma \in \mathrm{Irr}_c(L)$. We have

$$\sigma \simeq \pi_{n_0'} \otimes \tau_1 \otimes \cdots \otimes \tau_r$$

no prime!

where $\pi_{n_0'} \in \mathrm{Irr}_c(G'_{n_0'})$ and $\tau_i \in \mathrm{Irr}_c(\mathrm{GL}_{m_i}(\mathbf{k}))$ for $1 \le i \le r$.

(and $|\mathbb{F}_q^\times / \mathbb{F}_q^{\times 2}| = 2$)

<rmk>: if $G'_{n'} = O^\varepsilon_{2n'}(k)$. we have $G'_{n_0'} = O^\varepsilon_{2n_0'}(k)$ ($\varepsilon = +1$: split; $\varepsilon = -1$: other)

First occurance of $\pi_{n_0} \in \mathrm{Irr}_c(G_{n_0})$ w.r.t. with T_ε':

$\theta^0(\pi_{n_0}) \in \mathrm{Irr}_c(G'_{n_0'})$ where $n_0' = n'_{T_\varepsilon'}(\pi_{n_0})$.

<notation>: $\bar{n} := n - n_0$. $\bar{n}' := n' - n_0'$, $d := n - n'$, $d_0 := n_0 - n_0'$

<Thm 1> (Compatibility with Harish-Chandra Series)

(first analogue of Kudla's thm)

$\sigma \in \mathrm{Irr}_c(L)$, $\pi \in \mathrm{Irr}^\sigma(G)$, where $G = G_n$, then every $\pi' \in \mathrm{Irr}^{\sigma'}(G')$ ($\Theta_{G'}(\pi)$)

belongs to $\mathrm{Irr}^{\sigma'}(G')$, where

(1) if $\bar{n} \le \bar{n}'$ $\quad \sigma' = \theta^0(\pi_{n_0}) \otimes \tau_1 \otimes \cdots \otimes \tau_r \otimes 1 \cdots \otimes 1$. (for GL_1)

(unique)

(2) otherwise $\exists \{i_j\}_{j=1}^t$ s.t.

$\sigma' = \theta^0(\pi_{n_0}) \otimes \tau_1 \otimes \cdots \hat{\tau}_{i_1} \otimes \cdots \otimes \hat{\tau}_{i_t} \otimes \cdots \otimes \tau_r$.

from «AKT»

We thank Alberto Mínguez for his interest and his suggestion of considering the case of dual pairs of type II.

2. Unipotent representations

Let G be the group of $\mathbb{F}_q$-rational points of a connected algebraic reductive group $\mathbf{G}$ over $\bar{\mathbb{F}}_q$, defined over $\mathbb{F}_q$, and let $F\colon \mathbf{G} \to \mathbf{G}$ be the corresponding Frobenius map so that $\mathrm{G} = \mathbf{G}^F$ (fixed points by F).

To each G-conjugacy class of pairs $(\mathbf{T}, \theta)$ where $\mathbf{T}$ is an F-stable maximal torus in $\mathbf{G}$ and θ is an irreducible character of $T = \mathbf{T}^F$, Deligne and Lusztig attached a virtual character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ of G, [DL76].

Recall that the *uniform class functions* on G are, by definition, the complex linear combinations of Deligne-Lusztig characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. Recall also that an irreducible representation of G is called *unipotent* if its character has non-zero scalar product with $R_{\mathbf{T}}^{\mathbf{G}}(1)$ for some $\mathbf{T}$. If $\mathrm{G} = \mathrm{GL}_n(\mathbb{F}_q)$, then the uniform class functions span the space of all class functions on G. For G arbitrary, it is not the case in general: for instance, the character of the cuspidal unipotent representation θ_{10} of the symplectic group $\mathrm{Sp}_4(\mathbb{F}_q)$ defined by Srinivasan is not uniform.

Because we will need to include the case of orthogonal groups, it is necessary to extend the definition of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ to the case when $\mathbf{G}$ is a disconnected reductive algebraic group. In this case, we put $R_{\mathbf{T}}^{\mathbf{G}}(\theta) := \mathrm{Ind}_{\mathrm{G}^\circ}^{\mathrm{G}}(R_{\mathbf{T}}^{\mathbf{G}^\circ}(\theta))$, where $\mathbf{G}^\circ$ denotes the identity connected component of $\mathbf{G}$ and $\mathrm{G}^\circ := (\mathbf{G}^\circ)^F$. We will call *uniform class functions* all the linear combinations of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. An irreducible representation of G is called *unipotent* if its character has non-zero scalar product with $R_{\mathbf{T}}^{\mathbf{G}}(1)$ for some $\mathbf{T}$.

Since the cyclic group ${\mathbb{F}_q}^*$ is of even order, $|{\mathbb{F}_q}^*/{\mathbb{F}_q}^{*2}| = 2$. Therefore there are exactly two non-equivalent non-degenerate symmetric bilinear forms on the vector space ${\mathbb{F}_q}^{2n}$, see [Jac74, Theorem 6.9], one is split and the other one is not split. Let $\mathrm{O}_{2n}^+(q) = \mathrm{O}_{2n}^+(\mathbb{F}_q)$ (resp. $\mathrm{O}_{2n}^-(q) = \mathrm{O}_{2n}^-(\mathbb{F}_q)$) denotes the corresponding split (resp. non-split) orthogonal group. See [DM91, sec. 15.3] for more details. Also, we shall write $\mathrm{Sp}_{2n}(q) := \mathrm{Sp}_{2n}(\mathbb{F}_q)$.

We recall some results from [Lus80]. The group $\mathrm{Sp}_{2n}(q)$ has a unipotent cuspidal irreducible representation if and only if n is a triangular number, that is, $n = k^2 + k$ for some $k \in \mathbb{N}$. The group $\mathrm{Sp}_{2(k^2+k)}(q)$ has a unique unipotent cuspidal representation. Similarly, the group $\mathrm{SO}_{2n}^\varepsilon(q)$, with $\varepsilon \in \{-, +\}$, has a unipotent cuspidal irreducible representation if and only if n is a square, that is, $n = k^2$ for some $k \in \mathbb{N}$. The group $\mathrm{SO}_{2k^2}^\varepsilon(q)$ has a unique unipotent cuspidal representation, say Π_k. It follows that $\mathrm{O}_{2n}^\varepsilon(q)$ admits unipotent cuspidal representations if and only if $n = k^2$ for some $k \in \mathbb{N}$, and that $\mathrm{O}_{2k^2}^\varepsilon(q)$ has exactly two unipotent cuspidal representations, Π_k^{I} and Π_k^{II}. (Indeed, we have $\mathrm{Ind}_{\mathrm{SO}_{2k^2}^\varepsilon(q)}^{\mathrm{O}_{2k^2}^\varepsilon(q)} \Pi_k = \Pi_k^{\mathrm{I}} + \Pi_k^{\mathrm{II}}$. Both Π_k^{I} and Π_k^{II} have the same restriction to $\mathrm{SO}_{2k^2}^\varepsilon(q)$ and thus differ by tensoring with the determinant character of $\mathrm{O}_{2k^2}^\varepsilon(q)$.) See [Lus77, Theorem 8.2] or [AM93, Theorem 5.1] for the details.

Let $\mathbf{M} := \mathrm{Sp}_{2(k^2+k)} \times \mathbf{T}$ (resp. $\mathbf{M} := \mathrm{O}_{2k^2}^\varepsilon \times \mathbf{T}$), where $\mathbf{T}$ is a split torus of $\mathbf{G} = \mathrm{Sp}_{2n}$ (resp. $\mathbf{G} = \mathrm{O}_{2n}^\varepsilon$), and let $\Pi^{\mathbf{M}}$ be a unipotent cuspidal irreducible representation of $\mathbf{M}$. The representation $\Pi^{\mathbf{M}}$ is the tensor product of the unipotent cuspidal representation of $\mathrm{Sp}_{2(k^2+k)}$ (resp. Π_k^{I} or Π_k^{II}) with the trivial representation of $\mathbf{T}$. On the other hand, $\mathbf{M}$ is an $\mathbb{F}_q$-rational Levi subgroup of an $\mathbb{F}_q$-rational

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ and the commuting algebra $\mathrm{End}_{\mathrm{G}}(\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\Pi^{\mathbf{M}}))$ (where the cuspidal representation $\Pi^{\mathbf{M}}$ of $\mathbf{M}$ is trivially extended to the unipotent radical of $\mathbf{P}$, that is, $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\Pi^{\mathbf{M}})$, also denoted by $R_{\mathbf{M}}^{\mathbf{G}}(\Pi^{\mathbf{M}})$, is the usual Harish-Chandra induction) is an Iwahori-Hecke algebra of type $\mathrm{B}_{\tilde{n}}$, with $\tilde{n} := n - (k^2 + k)$ (resp. $\tilde{n} := n - k^2$), see for instance [Lus80] or [AMR96, § 3.A]. Hence the irreducible representations of G which occur in $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\Pi^{\mathbf{M}})$ are in bijection with $\mathrm{Irr}(\mathsf{W}_{\tilde{n}})$, where $\mathsf{W}_{\tilde{n}} = \mathsf{W}(\mathrm{B}_{\tilde{n}}) = (\mathbb{Z}/2\mathbb{Z})^{\tilde{n}} \rtimes \mathfrak{S}_{\tilde{n}}$ (*cf.* [Car93, Chapter 10] or [Lus84, Corollary 8.7]).

We will denote by $\Pi^{\mathrm{G}}_{\Pi^{\mathbf{M}},\rho}$ the irreducible representation of G which corresponds to $\rho \in \mathrm{Irr}(\mathsf{W}_{\tilde{n}})$ by this bijection.

We put

$$\mathrm{Sp} := \{\mathrm{Sp}_{2n}(q) \,:\, n \in \mathbb{N}\} \quad \text{and} \quad \mathrm{O}^{\varepsilon} := \{\mathrm{O}^{\varepsilon}_{2n}(q) \,:\, n \in \mathbb{N}\}.$$

We call Sp (resp. O^{ε}) a *Witt tower of symplectic* (resp. *orthogonal*) *type*. Let $\mathcal{T}$, $\mathcal{T}'$ be two Witt towers, one is of symplectic type and the other one is of orthogonal type.

For a finite group H let $\mathfrak{R}(H)$ denote the free abelian group generated by the irreducible characters of H. Thus the subset of the irreducible characters $\mathrm{Irr}(H) \subseteq \mathfrak{R}(H)$ is a base of $\mathfrak{R}(H)$ over $\mathbb{Z}$. Let $\mathrm{G}'_{m'}$ be an element of $\mathcal{T}'$ and let G_m be an element of $\mathcal{T}$. Denote by $\omega_{m',m}$ the projection onto the space of the uniform class functions on $\mathrm{G}'_{m'} \times \mathrm{G}_m$ of the pullback of the character of the oscillator representation (determined by one fixed character of the field $\mathbb{F}_q$) via the map $\mathrm{G}'_{m'} \times \mathrm{G}_m \ni (g', g) \to g'g \in \mathrm{Sp}_{4m'm}(q)$. By Howe correspondence for the dual pair $(\mathrm{G}'_{m'}, \mathrm{G}_m)$ we shall understand the map

$$\theta^{\mathrm{G}_m} \colon \mathfrak{R}(\mathrm{G}'_{m'}) \to \mathfrak{R}(\mathrm{G}_m) \tag{1}$$

defined by

$$\omega_{m',m} = \sum_{\Pi' \in \mathrm{Irr}(\mathrm{G}'_{m'})} \Pi' \otimes \theta^{\mathrm{G}_m}(\Pi'). \tag{2}$$

(See [AMR96, (1.4)], where $\theta^{\mathrm{G}_m}(\Pi')$ was denoted by $\Theta_{\mathrm{G}_m}(\Pi')$.)

Let Π' be a cuspidal irreducible representation of an element $\mathrm{G}'_{m'}$ of $\mathcal{T}'$. Then there exists $\mathrm{G}_m \in \mathcal{T}$ such that $\theta^{\mathrm{G}_m}(\Pi')$ is a cuspidal irreducible representation of G_m, see [AMR96, Theorem 3.7]. Moreover (see *loc. cit.*), the image by Howe correspondence for the dual pair $(\mathrm{G}'_{m'+l'}, \mathrm{G}_{m+l})$, with $l', l \in \mathbb{N}$, of each component of the Harish-Chandra parabolic induced representation $R^{\mathbf{G}'_{m'+l'}}_{\mathbf{G}'_{m'} \times \mathbf{T}'}(\Pi')$ (where $\mathbf{T}'$ is a split torus in $\mathbf{G}'_{m'+l'}$) belongs to the Harish-Chandra parabolic induced representation $R^{\mathbf{G}_{m+l}}_{\mathbf{G}_m \times \mathbf{T}}(\theta^{\mathrm{G}_m}(\Pi'))$ (where $\mathbf{T}$ is a split torus in $\mathbf{G}_{m+l}$).

Using the description of the uniform part of the restriction of $\mathrm{G}_m \times \mathrm{G}'_{m'}$ of the Weil representation obtained by Srinivasan in [Sri79], Adams and Moy proved that Howe correspondence sends unipotent representations to unipotent representations, [AM93, Theorem 3.5], and that the unique cuspidal unipotent representation of the group $\mathrm{Sp}_{2(k^2+k)}(q)$ corresponds to the representation Π^{II}_k of $\mathrm{O}^{\varepsilon}_{2k^2}(q)$ if ε is the sign of $(-1)^k$ and to the representation Π^{I}_{k+1} of $\mathrm{O}^{\varepsilon}_{2(k+1)^2}(q)$, where ε is the sign of $(-1)^{k+1}$, otherwise, [AM93, Theorem 5.2]. (In fact this defines the representations Π^{I}_k and Π^{II}_k.)

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

· Theta corres between cuspidal unipotent rep's

$\theta : \mathbb{N} \to \mathbb{N} : \theta(c) = \begin{cases} c & \text{if } \varepsilon \text{ is the sign of } (-1)^c \\ c+1 & \text{otherwise} \end{cases}$

· ramification group. $\sigma \in \mathrm{Irr}_c(L)$. The group

$$W_\sigma = \{ n \in N_G(L)/L \ ; \ {}^w\sigma \simeq \sigma \} = W_c^0 \rtimes R_\sigma$$

$R_\sigma := \{ w \in W_\sigma : w\Phi_\sigma^+ \subseteq \Phi_\sigma^+ \}$

finite Weyl gp.

root system: Φ_σ

<Thm> (DL) + (AMR)

$\sigma \in \mathrm{Irr}_{c,u}(L)$, the irr. component of $i_{L,P}^G(\sigma)$ are in bijection with irr. rep's of W_σ.

$W_\sigma, W_{\sigma'}$ are finite Weyl gp of type $B_{\bar n}, B_{\bar n'}$ respectively where

$$\bar n = n - c^2 - c, \quad \bar n' = n - \theta(c)^2.$$

Irreducible representations of the Weyl group W of type $B_{\bar n}$

They are parametrized by ordered pairs of partitions (λ, μ) of $\bar n$. We denote by $E_{\lambda,\mu}$ the irrep of W which corresponds to the bipartition (λ, μ) of $\bar n$.

order: $\lambda \preceq \lambda'$ iff $\lambda'_{i+1} \le \lambda_i \le \lambda'_i$ for all $i \in \mathbb{N}$ (in Young diagram)

<rmk>: $|\lambda| = |\lambda'|$. $\lambda \preceq \lambda' \Leftrightarrow \lambda = \lambda'$.

(Big theta)

$\sigma = \pi_{n_0} \otimes \tau_1 \otimes \cdots \otimes \tau_r$

unip. cusp.

$\sigma = \pi_{n_0} \otimes 1 \cdots \otimes 1$.

cusp. unip

with $n_0 = c + c^2$

A correspondence between representations of Weyl groups

We write $\bar n(\nu) := \bar n - |\nu|$ if ν is a partition such that $|\nu| \le \bar n$. Then we define

1. Cases $(\mathrm{Sp}_{2n}, \mathrm{O}^+_{2n'})$ with c even and $(\mathrm{Sp}_{2n}, \mathrm{O}^-_{2n'})$ with c odd:

$$\Omega_{\bar n, \bar n'} := \sum_{r=0}^{\min(\bar n, \bar n')} \sum_{(\lambda,\mu) \in \mathcal{P}_2(r)} \sum_{\nu, \nu'} E_{\lambda,\nu} \otimes E_{\lambda,\nu'},$$

where the third sum is over the partitions $\nu \dashv \bar n(\lambda)$ and $\nu' \dashv \bar n'(\lambda)$ such that $\mu \preceq \nu$ and $\mu \preceq \nu'$.

2. Cases $(\mathrm{Sp}_{2n}, \mathrm{O}^+_{2n'})$ with a odd and $(\mathrm{Sp}_{2n}, \mathrm{O}^+_{2n'})$ with c even:

$$\Omega_{\bar n, \bar n'} = \sum_{r=0}^{\min(\bar n, \bar n')} \sum_{(\lambda,\mu) \in \mathcal{P}_2(r)} \sum_{\nu, \nu'} E_{\nu,\mu} \otimes E_{\lambda,\nu'},$$

where the third sum is over the partitions $\lambda' \dashv \bar n(\nu)$ and $\nu' \dashv \bar n'(\lambda)$ such that $\lambda \preceq \nu$ and $\mu \preceq \nu'$.

(Conj) (AMR. reformed by Ebequin) (proved by Pan, Ma-Liu-Zou)

$$\pi \in \mathrm{Irr}_\sigma(G) \overset{1\leftrightarrow 1}{\longleftrightarrow} E_{\lambda,\mu}, \quad \pi' \in \mathrm{Irr}_{\sigma'}(G) \overset{1\leftrightarrow 1}{\longleftrightarrow} E_{\lambda',\mu'}$$

$$\pi \overset{\Theta}{\longleftrightarrow} \pi' \iff E_{\lambda,\mu} \overset{\Omega}{\longleftrightarrow} E_{\lambda',\mu'}$$

Lecture 2:

· review of the classification of the unipotent repn of classical group

<Def> A (reduced) symbol is an array Λ of the form

$$\Lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{pmatrix} \rightsquigarrow \mathrm{def}(\Lambda) := r - s$$

(defect)

s.t. $\{a_i\}\downarrow$, $\{b_i\}\downarrow$ (for all i, s.t. $0 \notin A \cap B$) ?

(Let $S_{n,d}$:= set of reduced symbol of rk $= n$, def $= d$.

<rank>: $\mathrm{rank}(\Lambda) := \sum_{i=1}^{r} a_i + \sum_{i=1}^{s} b_i - \left\lfloor \left(\frac{r+s-1}{2}\right)^2 \right\rfloor$

$\underline{S_n} := \bigcup S_{n,d}$

<Thm> (Lusztig)

bijection $\Lambda \longmapsto \pi_\Lambda$

$$\mathcal{S}_{n,\equiv 1} := \{\Lambda \in \mathcal{S}_n : \mathrm{def}(\Lambda) \equiv 1 \pmod 4\} \xrightarrow{1-1} \mathrm{Irr_u}(\mathrm{Sp}_{2n}(\mathbf{k}))$$

$$\mathcal{S}_{n,\equiv 0} := \{\Lambda \in \mathcal{S}_n : \mathrm{def}(\Lambda) \equiv 0 \pmod 4\} \xrightarrow{1-1} \mathrm{Irr_u}(\mathrm{O}^+_{2n}(\mathbf{k}))$$

$$\mathcal{S}_{n,\equiv 2} := \{\Lambda \in \mathcal{S}_n : \mathrm{def}(\Lambda) \equiv 2 \pmod 4\} \xrightarrow{1-1} \mathrm{Irr_u}(\mathrm{O}^-_{2n}(\mathbf{k})).$$

Order on partitions

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$ be a partition. We set $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_m$. Let ${}^t\lambda$ denote the transpose of λ. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell)$ be two partitions. We may assume that $\ell = m$ by adding several0's if necessary. Then, we write

$$\lambda \preceq \mu \quad \text{if } \lambda_m \leq \mu_m \leq \lambda_{m-1} \leq \cdots \leq \lambda_2 \leq \mu_2 \leq \lambda_1 \leq \mu_1.$$

Notation

Let $\mathcal{P}_2(n) := \{\binom{\lambda}{\mu}\}$ be the set of bi-partitions of n, where λ and μ are partitions such that $|\lambda| + |\mu| = n$.

(Notation / Rmk)

$\Lambda = \begin{pmatrix} A \\ B \end{pmatrix} \longmapsto \Lambda^t := \begin{pmatrix} B \\ A \end{pmatrix}$

(thus $\mathrm{rk}(\Lambda) = \mathrm{rk}(\Lambda^t)$,

$\det(\Lambda) = -\det(\Lambda^t)$,

for $G = O^\varepsilon_{2n}$, $\pi_{\Lambda^t} = \mathrm{sgn} \cdot \pi_\Lambda$.

· we have

$$c = \begin{cases} \frac{|\mathrm{def}(\Lambda)| - 1}{2} & \text{if } \pi_\Lambda \in \mathrm{Irr}_{\mathrm{cusp}}(\mathrm{Sp}_{2(c^2+c)}(k)) \\ \frac{|\mathrm{def}(\Lambda)|}{2} & \text{if } \pi_\Lambda \in \mathrm{Irr}_{\mathrm{cusp}}(O^\varepsilon_{2c^2}(k)) \end{cases}$$

· for cusp. uni irr. π_{Λ_0} of $\mathrm{Sp}_{2(c^2+c)}(k)$,

$$\Lambda_0 = \begin{cases} \begin{pmatrix} 2c, 2c-1, \cdots, 1, 0 \\ - \end{pmatrix} & \text{if } c \text{ even}, \\ \begin{pmatrix} - \\ 2c, 2c-1, \cdots, 1, 0 \end{pmatrix} & \text{if } c \text{ odd}. \end{cases}$$

· (From symbols to bipartitions)

$$\Lambda = \begin{pmatrix} a_1 a_2 \cdots a_r \\ b_1 b_2 \cdots b_s \end{pmatrix} \xmapsto{\gamma} \begin{pmatrix} a_1 - (r-1) & a_2 - (r-2) & \cdots & a_{r-1} - 1 & a_r \\ b_1 - (s-1) & b_2 - (s-2) & \cdots & b_{r-1} - 1 & b_r \end{pmatrix} = \begin{pmatrix} \lambda \\ \mu \end{pmatrix} =: \begin{pmatrix} \gamma(\Lambda)^* = \lambda \\ \gamma(\Lambda)_* = \mu \end{pmatrix}$$

(bijection): $\gamma : S_{n,d} \to \begin{cases} \mathcal{P}_2(n - \frac{d^2-1}{4}) & \text{if } d \text{ odd} \\ \mathcal{P}_2(n - \frac{d^2}{4}) & \text{if } d \text{ even}. \end{cases}$

(Hence not 1-1/π_Λ).

Notation

We define

$$\mathcal{A}^{\pm}_{n,n'} := \{(\Lambda, \Lambda') \in \mathcal{S}_{n,\equiv 1} \times \mathcal{S}_{n',\equiv 0} : \mathrm{def}(\Lambda') = -\mathrm{def}(\Lambda) \pm 1\}.$$

$$\mathcal{B}^{+}_{n,n'} := \left\{(\Lambda, \Lambda') \in \mathcal{A}^{+}_{n,n'} : {}^t(\Upsilon(\Lambda')_*) \preceq {}^t(\Upsilon(\Lambda)^*), {}^t(\Upsilon(\Lambda)_*) \preceq {}^t(\Upsilon(\Lambda')^*)\right\}$$

$$\mathcal{B}^{-}_{n,n'} := \left\{(\Lambda, \Lambda') \in \mathcal{A}^{+}_{n,n'} : {}^t(\Upsilon(\Lambda')^*) \preceq {}^t(\Upsilon(\Lambda)_*), {}^t(\Upsilon(\Lambda)^*) \preceq {}^t(\Upsilon(\Lambda')_*)\right\}$$

Theorem [Pan; Ma-Qiu-Zou] (Conjecture A.-Michel-Rouquier)

For the reductive dual pair $(G, G') = (Sp_{2n}(\mathbf{k}), \mathrm{O}^\epsilon_{2n'}(\mathbf{k})$, let $\omega^\epsilon_{n,n'} = \omega_{G,G'}$ be the restriction of the Weil representation to $G \cdot G'$.
Let $\pi_\Lambda \in \mathrm{Irr_u}(\mathrm{Sp}_{2n}(\mathbf{k}))$ and $\pi_{\Lambda'} \in \mathrm{Irr_u}(\mathrm{O}^\epsilon_{2n'}(\mathbf{k}))$.
Then

$$\pi_\Lambda \otimes \pi_{\Lambda'} \text{ occurs in } \omega^\epsilon_{n,n'} \text{ if and only } (\Lambda, \Lambda') \in \mathcal{B}^\epsilon_{n,n'}.$$

Then the following question arises naturally: is there a "preferred" representation among the irreducible representations of G which correspond to Π'? It is the aim of this article to propose a candidate for such a preferred irreducible representation, assuming that Π' is unipotent.

Zelevinsky: stable range (I, II)

Howe: etc correspondence for dual pairs $(Sp_{2n}(k), O_{N'}(k))$ for stable range

$\Downarrow$

Pan: all irr. pairs in type I.

$\rightsquigarrow$ (Geometrization, Ma's talk) Hecke algebra

• <Theta correspondence and unipotent orbits>

· Order on unipotent orbits

$\mathcal{U}_G := \{$set of unipotent classes of $G\}$ $\rightsquigarrow$ partially order

$$\mathcal{O}_1 \leq \mathcal{O}_2 \Leftrightarrow \mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$$

<Parametrization of unipotent orbits>

orbits$/\bar{k}$ $\leftrightarrow$ "partitions" / symplectic : each odd part occur even multiplicity

orthogonal : each even part occur even multiplicity

unipotent orbit (conj-class) $Sp_{2n}(\bar{k})$ $O_{2n}(\bar{k})$ $\lambda \leftrightarrow \mathcal{O}_\lambda$

• Springer correspondence : $G \to W = N_G(T)/T$

$E_{\lambda,\mu} \in \mathrm{Irr}(W) \leftrightarrow \mathcal{U}_e(G)$ = enhancement of the unipotent var. of G

$E \mapsto (\mathcal{O}_{\lambda,\mu}, \rho)$ $\rho \in Z_{G_u}(u)/Z_{G_u}(u)^\circ$, $u \in \mathcal{O}_{\lambda,\mu}$

<Def> $(\lambda', \mu') \in \mathcal{P}_2(\bar{n}')$,

set

$\Theta_{\lambda',\mu'} = \{(\lambda,\mu) \mid (\lambda,\mu) \in \mathcal{P}_2(\bar{n}'),\ E_{\lambda,\mu} \otimes E_{\lambda',\mu'}$ occurs in $\Omega_{\bar{n},\bar{n}'}\}$

<Def> $(\lambda,\mu) \in \mathcal{P}_2(\bar{n})$, say $E_{\lambda_M,\mu_M} \in \Theta_{\lambda',\mu'}$ is <u>maximal</u> if

$$\mathcal{O}_{\lambda,\mu} \leq \mathcal{O}_{\lambda_M,\mu_M} \text{ for all } (\mu,\lambda) \in \Theta_{\lambda',\mu'}.$$

(<u>rmk</u> : exist $\Rightarrow$ unique, but not a priori exist)

similarly : minimal, extremal

<notation> : $\lambda \cup \mu \in \mathcal{P}(n+m)$

<thm> (Ekeqvin)

Cases $(\mathrm{Sp}_{2n}, \mathrm{O}^+_{2n'})$ with c even and $(\mathrm{Sp}_{2n}, \mathrm{O}^-_{2n'})$ with c odd, and $n \geq 2n'$:
Let $(\lambda', \mu') \in \mathcal{P}_2(\bar{n}')$.

1. $(\lambda, /mu) \in \Theta_{\lambda',\mu'}$ if and only if μ and μ' have a common predecessor for $\prec$, and $\lambda = \lambda'$.
2. there exists a unique minimal representation $E_{\lambda_m,\mu_m} \in \Theta_{\lambda',\mu'}$, it is given by
$$\lambda_m := \lambda' \quad \text{and} \quad \mu_m := (\bar{n} - \bar{n}') \cup \mu'.$$
3. there exists a unique maximal representation $E_{\lambda_M,\mu_M} \in \Theta_{\lambda',\mu'}$, it is given by
$$\lambda_M := \lambda' \quad \text{and} \quad \mu_M := (\bar{n} - \bar{n}' + \mu'_1 + \mu'_2, \mu'_3, \ldots, \mu'_m).$$

Cases $(\mathrm{Sp}_{2n}, \mathrm{O}^+_{2n'})$ with c odd and $(\mathrm{Sp}_{2n}, \mathrm{O}^+_{2n'})$ with c even , and $n \geq 2n'$:

1. there exists a unique minimal representation $E_{\lambda_m,\mu_m} \in \Theta_{\lambda',\mu'}$, it is given by
$$\lambda_m := (\bar{n} - \bar{n}') \cup \lambda' \quad \text{and} \quad \mu_m := \mu'.$$
2. there exists a unique maximal representation $E_{\lambda_M,\mu_M} \in \Theta_{\lambda',\mu'}$, it is given by
$$\lambda_M := (\bar{n} - \bar{n}' + \lambda'_1 + \mu'_1, \lambda'_2, \ldots, \lambda'_r) \quad \text{and} \quad \mu_M := (\mu'_2 \ldots, \mu'_s)).$$

<thm> (Lusztig p-large, Achar-A p good)

G/k, $\pi \in \mathrm{Irr}(G)$ $k = \bar{\mathbb{F}}_p$ $\rightsquigarrow \chi_\pi$

conn. red. grp

then $\exists$ rational unipotent class $\mathcal{O}_\pi$ in G

which has property : $\exists\ u \in \mathcal{O}_\pi(k)$. s.t.

① $\chi_\pi(u) \neq 0$.

② $\mathcal{O}_\pi$ has max'l dim for this property

(unipotent support)

• π is unipotent repn, then $\exists\ E \in \mathrm{Irr}(W)$ such that $(\chi_\pi, R_E)_G \neq 0$

W : Weyl group of G.

R_E : almost char (certain linear combination of DL char)

Moreover, if E' is another irrep of W such that $(\chi_\pi, R_{E'})_G \neq 0$, then E and E' belong to the same "family of characters" of W.

Thus, we can associate with π a unique family of characters of W, or equivalently, a unique "two-sided cell" in W.

Let E_s be the unique special character in this family.

$\pi \mapsto \hat{W} \ni E_s$: special character

Remark

For G of classical type a family of characters of W corresponds to a similarity class of u-symbols, and the symbol corresponding to E_s is the unique distinguished u-symbol in that family.

Then the class $\mathcal{O}_\pi$ coincides with the unipotent class corresponding to E_s by the Springer correspondence for the group W. In particular the unipotent class $\mathcal{O}_\pi$ is always special.

$\mathcal{O}_\pi \longleftrightarrow E_s$ by Springer

Theorem [Achar-A. p good]

Let $\pi \in \mathrm{Irr_u}(G)$. Every rational unipotent class $\mathcal{O}$ on which χ_π does not vanish (i.e., such that there exists $u \in \mathcal{O}(\mathbf{k})$ with $\chi_\pi(u) \neq 0$) satisfies

$$\mathcal{O} \leq \mathcal{O}_\pi.$$

Theorem [A.-Kraśkiewicz-Przebinda]

Let $\pi \in \mathrm{Irr_u}(G)$ with $G = \mathrm{Sp}_4(\mathbf{k})$ (resp. $G = \mathrm{O}_4^+(\mathbf{k})$) which belongs to the principal series of G.
Let $(\lambda, \mu) \in \mathcal{P}_2(n)$ s.t. $\pi = \pi_{\lambda,\mu}$.
Let $n' \geq 3$, and let $(\lambda_0', \mu_0') := (\lambda, (n'-2) \cup \mu)$.

Then every irrep of $G' := \mathrm{O}_{2n'}^+(\mathbf{k})$ (resp. $G' = \mathrm{Sp}_{2n'}(\mathbf{k})$) which occurs in the image of π by the Theta correspondence for the dual pair (G, G') is s.t. the closure of its unipotent support contains the closure of the unipotent support of $\pi'_{\lambda_0', \mu_0'}$.

$\mathrm{Sp}_4 \longleftrightarrow \mathrm{O}^+_{2n'}$

$\mathrm{O}^+_4 \longleftrightarrow \mathrm{Sp}'_{2n}$

$\pi \longmapsto \pi'$

(uni. support)

5. Howe correspondence and wave front set

Recall (see [Lus92], [GM99], [AA07]) that for every irreducible character χ_Π of the $\mathbb{F}_q$-points $G = \mathbf{G}^F$ of a split connected reductive group $\mathbf{G}$ defined over $\mathbb{F}_q$ (assuming that the characteristic p of $\mathbb{F}_q$ is "good for G": for instance, if G a symplectic group or a split special orthogonal group, then p must be odd) there is a unique rational unipotent class $\mathcal{O}_\Pi$ in $\mathbf{G}$ which has the property that there exists $u \in \mathcal{O}_\Pi(q)$ such that $\chi_\Pi(u) \neq 0$ and $\mathcal{O}_\Pi$ has maximal dimension among classes with that property. The class $\mathcal{O}_\Pi$ is called the *unipotent support* of χ_Π. It coincides with the class defined in [Lus84, §13.3].

More precisely, suppose Π is unipotent. Then there exists an irreducible representation ρ of the Weyl group W of $\mathbf{G}$ such that the scalar product between χ_Π and the *almost character* R_ρ (which is defined as a certain linear combination of Deligne-Lusztig generalized characters in [Lus84, page 347 and (4.24.1)], and coincides with the virtual character in Eqn. (3) when $\mathrm{G} = \mathrm{GL}_n(q)$) is non-zero. Moreover, if ρ' is another irreducible representation of W such that χ_Π has non-scalar product with $R_{\rho'}$, then ρ and ρ' belong to the same family of characters of W (see [Lus84, Theorems 5.25 and 6.17]). Thus, we can associate with χ_Π a unique family of characters of W, or equivalently, a unique two-sided cell in W. Let ρ_{spe} be the unique *special character* in this family (for $\mathbf{G}$ of classical type a family of characters of W corresponds to a similarity class of u-symbols, and the symbol corresponding to ρ_{spe} is the unique distinguished u-symbol in that family, [Lus84, (4.5.6)]). Then the class $\mathcal{O}_\Pi$ coincides with the unipotent class corresponding to ρ_{spe} by the Springer correspondence for the group W. In particular the unipotent class $\mathcal{O}_\Pi$ is always special.

Moreover, every rational unipotent class $\mathcal{O}$ on which χ_Π does not vanish (i.e., such that there exists $u \in \mathcal{O}(q)$ with $\chi_\Pi(u) \neq 0$) satisfies

$$\mathcal{O} \leq \mathcal{O}_\Pi, \tag{19}$$

see [AA07, Theorem 6.1].

Suppose Π is an irreducible unipotent representation of a split group G, such as $\mathrm{O}^+_{2n}(q)$ or $\mathrm{Sp}_{2n}(q)$, which belongs to the principal series. The algebra of the endomorphisms of the principal series which commute with the action of G is the Iwahori-Hecke algebra, whose irreducible representations coincide with the irreducible representations of the Weyl group W. Hence, as we remarked previously, there is a one to one correspondence between the irreducible representations of W and the irreducible representations of G which occur in the principal series. Given an irreducible representation ρ of W we denoted by Π_ρ the corresponding representation of G. Furthermore, the almost character R_ρ has a non-zero scalar product with the character of Π_ρ. (This follows from [Lus84, Theorem 4.23]. For an explicit argument see pages 297 and 298 in [Lus84].)

If the group $\mathbf{G}$ is disconnected and Π be an irreducible representation of $\mathrm{G} = \mathbf{G}^F$, we define the unipotent support of Π, denoted $\mathcal{O}_\Pi$, to be the union of the rational unipotent classes $\mathcal{O} \subseteq \mathbf{G}$ of maximal dimension, such that $\mathcal{O} \cap \mathrm{G}$ has a non-empty intersection with the support of the character χ_Π.

Let $\Pi = \Pi_{\rho_{\xi,\eta}}$ be an irreducible unipotent representation of $\mathrm{O}^+_{2n}(q)$ which belongs to the principal series, where (ξ, η) is a pair of partitions of n. Two cases can occur: the restriction Π_{SO} of Π to $\mathrm{SO}^+_{2n}(q)$ is either irreducible, or is the direct sum of two nonequivalent irreducible representations $\Pi^{\mathrm{I}}_{\mathrm{SO}}$ and $\Pi^{\mathrm{II}}_{\mathrm{SO}}$. The latter case

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

arises if and only if the restriction of the representation $\rho_{\xi,\eta}$ to the Weyl group of the special orthogonal group splits into the sum of two inequivalent representations, i.e. the partition (ξ,η) is such that $\xi=\eta$, see [**Car93**, Prop. 11.4.4].

$(*)$ We recall (see for instance [**Spa82**]) that any rational unipotent class $\mathcal{O}$ in O^+_{2n} is either a rational unipotent class in SO^+_{2n} or is the disjoint union of a rational unipotent class $\mathcal{O}(u)$ in SO^+_{2n} and a rational unipotent class $\mathcal{O}(sus^{-1})$ in SO^+_{2n}, with $u\in\mathrm{SO}^+_{2n}(q)$ unipotent and some $s\in\mathrm{O}_{2n}(q)\setminus\mathrm{SO}_{2n}(q)$. Both these classes have the same dimension.

Lemma 11. *If Π_{SO} is irreducible then*

$$\mathcal{O}_{\Pi}=\mathcal{O}_{\Pi_{\mathrm{SO}}}. \tag{20}$$

If $\Pi_{\mathrm{SO}}=\Pi^{\mathrm{I}}_{\mathrm{SO}}\oplus\Pi^{\mathrm{II}}_{\mathrm{SO}}$, then

$$\mathcal{O}_{\Pi}=\mathcal{O}_{\Pi^{\mathrm{I}}_{\mathrm{SO}}}\cup\mathcal{O}_{\Pi^{\mathrm{II}}_{\mathrm{SO}}}. \tag{21}$$

In both cases, $\mathcal{O}_{\Pi}$ is a single unipotent class in O^+_{2n}.

Proof. Let us assume first that Π_{SO} is irreducible. Then the restrictions of the two characters χ_{Π} and $\chi_{\Pi_{\mathrm{SO}}}$ to $\mathrm{SO}_{2n}(q)$ are equal. In particular, $\chi_{\Pi_{\mathrm{SO}}}(g)=\chi_{\Pi}(g)=\chi_{\Pi}(sgs^{-1})=\chi_{\Pi_{\mathrm{SO}}}(sgs^{-1})$ for any $g\in\mathrm{SO}_{2n}(q)$ and s as in $(*)$. Let $\mathcal{O}(u)$ be a rational unipotent class in O^+_{2n}, as in $(*)$. We see that, with the notation of $(*)$, the restriction of χ_{Π} to $\mathcal{O}(u)\cap\mathrm{SO}^+_{2n}(q)$ is non-zero if and only if the restriction of χ_{Π} to $\mathcal{O}(sus^{-1})\cap\mathrm{SO}^+_{2n}(q)$ is non-zero. But the classes $\mathcal{O}(sus^{-1})$ and $\mathcal{O}(u)$ have the same dimension. Therefore, if that dimension is maximal among the unipotent classes which have a non-empty intersection with the support of $\chi_{\Pi_{\mathrm{SO}}}$, we get a contradiction. Thus the unipotent support of Π_{SO} is the unipotent class in O^+_{2n} which is also a single unipotent class in SO^+_{2n}. Hence, (20) follows.

Assume now that $\Pi_{\mathrm{SO}}=\Pi^{\mathrm{I}}_{\mathrm{SO}}\oplus\Pi^{\mathrm{II}}_{\mathrm{SO}}$. In this case the representations $\Pi^{\mathrm{I}}_{\mathrm{SO}}$ and $\Pi^{\mathrm{II}}_{\mathrm{SO}}$ are permuted via the action of the group element s, as in $(*)$, and so are their unipotent supports. More precisely, $\chi_{\Pi^{\mathrm{II}}_{\mathrm{SO}}}(u)=\chi_{\Pi^{\mathrm{I}}_{\mathrm{SO}}}(sus^{-1})$, $\mathcal{O}_{\Pi^{\mathrm{I}}_{\mathrm{SO}}}=\mathcal{O}(u)$, $\mathcal{O}_{\Pi^{\mathrm{II}}_{\mathrm{SO}}}=\mathcal{O}(sus^{-1})$ and the right hand side of (21) is a single unipotent rational class in O_{2n}. (Since, as we noticed before, $\xi=\eta$, these classes have the same set of elementary divisors and hence the same dimension, see [**Car93**, page 399]. They are described explicitly in [**Car93**, § 13.3, Type D_l]).

Let $u\in\mathrm{SO}^+_{2n}(q)\cap\mathcal{O}_{\Pi}$ be such that $\chi_{\Pi}(u)\neq 0$. Since $\chi_{\Pi}(u)=\chi_{\Pi^{\mathrm{I}}_{\mathrm{SO}}}(u)+\chi_{\Pi^{\mathrm{II}}_{\mathrm{SO}}}(u)$, we see that that $\chi_{\Pi^{\mathrm{I}}_{\mathrm{SO}}}(u)\neq 0$ or $\chi_{\Pi^{\mathrm{II}}_{\mathrm{SO}}}(u)\neq 0$. Hence, $\mathcal{O}_{\Pi}=\mathcal{O}_{\Pi^{\mathrm{I}}_{\mathrm{SO}}}\cup\mathcal{O}_{\Pi^{\mathrm{II}}_{\mathrm{SO}}}$. □

Since (19) holds for the representations of $\mathrm{SO}_{2n}(q)$, we see from Lemma 11 that it also holds for the representations of $\mathrm{O}^+_{2n}(q)$.

Corollary 12. *Let $\Pi'_{\rho_{\xi',\eta'}}$ be an irreducible unipotent representation of $\mathrm{G}'=\mathrm{Sp}_4(q)$ (resp. $\mathrm{G}'=\mathrm{O}^+_4(q)$) which belongs to the principal series of G'. Let $n\geq 3$, and let $(\xi_0,\eta_0):=(\xi',(n-2)\cup\eta')$.*

Assume that Conjecture 1 holds. Then every representation of $\mathrm{G}=\mathrm{O}^+_{2n}(q)$ (resp. $\mathrm{G}'=\mathrm{Sp}_{2n}(q)$) which occurs in the image of Π' by Howe correspondence for the dual pair (G',G) is such that the closure of its unipotent support contains the closure of the unipotent support of Π_{ξ_0,η_0}.

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

Proof. This follows directly from (12) and from the fact that, for the map $\vartheta_1^{2,n}$ with $n \geq 3$, the underlined orbit in the output set is contained in the closure of each orbit in that set. □

Recall Alvis-Curtis Duality $D_{\mathrm{G}} \colon \mathfrak{R}(\mathrm{G}) \to \mathfrak{R}(\mathrm{G})$ ([Alv79], [Cur80], [Aub92]), which is defined for representations of $\mathrm{G} = \mathbf{G}^F$, when $\mathbf{G}$ is connected.

Let Π be an irreducible unipotent representation of $\mathrm{O}_{2n} := \mathrm{O}^+_{2n}(q)$, as in Lemma 11. If Π_{SO} is irreducible, define $D_{\mathrm{O}_{2n}}(\Pi)$ to be the unique irreducible representation $\tilde{\Pi}$ of O_{2n} such that $\tilde{\Pi}_{\mathrm{SO}} = D_{\mathrm{SO}_{2n}}(\tilde{\Pi}_{\mathrm{SO}})$. If $\Pi_{\mathrm{SO}} = \Pi^I_{\mathrm{SO}} \oplus \Pi^{II}_{\mathrm{SO}}$, let $D_{\mathrm{O}_{2n}}(\Pi)$ to be the only irreducible representation $\tilde{\Pi}$ of $\mathrm{O}^+_{2n}(q)$ such that $\tilde{\Pi}_{\mathrm{SO}} = D_{\mathrm{SO}_{2n}}(\Pi^I_{\mathrm{SO}}) \oplus D_{\mathrm{SO}_{2n}}(\Pi^{II}_{\mathrm{SO}})$. Then in both cases, $D_{\mathrm{O}_{2n}}(\Pi_{\rho_{\xi,\eta}}) = \Pi_{\rho_{t_\eta,t_\xi}}$. In other words, $D_{\mathrm{O}_{2n}}(\Pi_\rho) = \Pi_{\rho\otimes\mathrm{sgn}}$, see [Lus84, (6.8.6)]. Hence, $\mathcal{O}_\Pi = \mathcal{O}_{\xi,\eta}$ if and only if $\mathcal{O}_{D_{\mathrm{O}_{2n}}(\Pi)} = \mathcal{O}_{t_\xi,t_\eta}$. Also, tensoring with the sign representation of the Weyl group translates via Springer correspondence to an order reversing involution on the special unipotent orbits, see [Car93, pages 389, 390]. By combining this with Corollary 12, (18), (17) and [AKP13, Proposition 5], we deduce the following theorem.

Theorem 13. *Consider the dual pair* $(\mathrm{G}' = \mathrm{Sp}_4(q), \mathrm{G} = \mathrm{O}^+_{2n}(q))$ *with* $n \geq 4$. *(This is a dual pair in the stable range with* G' *the smaller member.) Assume that Conjecture 1 holds.*

Let π' *be an irreducible representation of* G' *such that* $D_{\mathrm{G}'}(\pi')$ *is unipotent and belongs to the principal series of* G'. *Then there is a unique irreducible representation* π_{pref} *of* G *such that* $D_{\mathrm{G}}(\pi_{\mathrm{pref}})$ *corresponds to* $D_{\mathrm{G}'}(\pi')$ *via Howe Correspondence for the pair* $(\mathrm{G}', \mathrm{G})$ *and the unipotent support* $\mathcal{O}_{\pi_{\mathrm{pref}}}$ *of* π_{pref} *contains in its closure the unipotent support of any irreducible representation* π *of* G *such that* $D_{\mathrm{G}}(\pi)$ *corresponds to* $D_{\mathrm{G}'}(\pi')$.

Let λ', λ' *be the partitions describing the rational unipotent class* $\mathcal{O}_{\pi'}$ *and* $\mathcal{O}_{\pi_{\mathrm{pref}}}$, *respectively. Then* λ *is obtained from* λ' *by adding a column of length* $2N-4$ *to* λ', *as in* [AKP13, Theorem 1].

Lusztig has proved in [Lus92, Theorem 11.2], under the assumption that p is large enough, that the closure of the unipotent support of Π coincides with the wave front set (as defined by Kawanaka in [Kaw87]) of its dual. Recall that $D_{\mathrm{GL}_n(q)}$ maps the unipotent character R_{ρ_μ} to the unipotent character $R_{\rho_{t_\mu}}$. Hence Theorem 3 and Theorem 13 imply the following result.

Corollary 14. *Let* $(\mathrm{G}', \mathrm{G})$ *be one of the dual pairs* $(\mathrm{GL}_{n'}(q), \mathrm{GL}_n(q))$ *or* $(\mathrm{Sp}_4(q), \mathrm{O}^+_{2n}(q))$ *with* $n \geq 4$. *In the latter case, we assume that Conjecture 1 holds.*

Let Π' *be a unipotent irreducible representation of* G' *that belongs to the principal series of* G'. *Then there is a unique irreducible representation* Π_{pref} *of* G *such that* Π_{pref} *corresponds to* Π' *via Howe Correspondence for the pair* $(\mathrm{G}', \mathrm{G})$ *and the wave front set of* Π_{pref} *contains the wave front set of any irreducible representation* Π *of* G *such that* Π *corresponds to* Π'.

Proof. Let $(\mathrm{G}, \mathrm{G}') = (\mathrm{Sp}_4(q), \mathrm{O}^+_{2n}(q))$. We put $\pi' := D_{\mathrm{G}'}(\Pi')$. Since $D_{\mathrm{G}'}$ is an involution, we have $\Pi' = D_{\mathrm{G}'}(\pi')$. Then we apply Theorem 13 to the representation π', and we set

$$\Pi_{\mathrm{pref}} := D_G(\pi_{\mathrm{pref}}).$$

From Theorem 13, it follows that Π_{pref} corresponds to Π' by Howe correspondence, and that the unipotent support $\mathcal{O}_{\pi_{\mathrm{pref}}}$ of π_{pref} contains in its closure the unipotent

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

Dual pairs of type II

Definition/Remark

Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\mathbf{k}$, and let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$. Let F be the Frobenius endomorphism and W the Weyl group of $\mathbf{G}$.

- F-conjugation in $\mathbf{G}$ means the action of $\mathbf{G}$ on itself defined for any $g \in \mathbf{G}$ by $x \mapsto gx({}^F g)^{-1}$.
- The F-conjugacy class of the image of $g^{-1}{}^F g$ in W is called the type of the torus ${}^g\mathbf{T}$ with respect to $\mathbf{T}$; a representative of this F-conjugacy class is called a type of ${}^g\mathbf{T}$ with respect to $\mathbf{T}$.
- By the conjugation action of g^{-1}, the torus ${}^g\mathbf{T}$, endowed with the action of F, is identified with the torus $\mathbf{T}$ endowed with the action of wF, if w is the image of $g^{-1}{}^F g$ in W.
  We will denoted by $\mathbf{T}_w$.

- unipotent character

$\{$char. of unipotent irrep of $G\}$

$\updownarrow$ 1:1

$\{$irrep of symmetric gp$\}$ $\quad E_\lambda$

$\updownarrow$ char $\quad \uparrow$

$\{$partition of $n\}$ $\quad \lambda$

<def> almost character

$$R_\lambda := \frac{1}{n!} \sum_{w \in S_n} \mathrm{Tr}(E_\lambda)\, R_{T_w}^{G}(1)$$

where T_w : max'l torus of type w.

<prop>

R_λ is a char. of unipotent irrep of G.

and all such char. is of this form.

<Thm>: $(G, G') = (GL_n(k), GL_{n'}(k))$ with $n \le n'$

and rep of G, G'

$\updownarrow \quad \updownarrow$

$R_\lambda \quad R_{\lambda \cup (n'-n)}$, where $\lambda \vdash n$, $\lambda' \vdash n'$

corresponds under theta.

moreover, $\forall$ rep G' belongs to image of R_λ under theta, is of form (not have to be irreducible),

$R_{\mu'}$, with $\mu' \supseteq \lambda \cup (n'-n)$.

The generalized Springer correspondence

Let $\mathcal{G}$ be a connected reductive group over $\mathbb{C}$ and let $\mathrm{Unip}_{\mathcal{G}}$ denote the unipotent variety of $\mathcal{G}$.
The category of $\mathcal{G}$-equivariant sheaves on $\mathrm{Unip}_{\mathcal{G}}$ has been the subject of many studies in geometric representation theory:

- Springer (1969, 1973) constructed the "Springer resolution" of $\mathrm{Unip}_{\mathcal{G}}$, and used it to give a geometric construction of the Weyl group representations from their actions on the cohomology of Springer fibers.
- Lusztig (1984) generalized the method and gave a description of all perverse sheaves on $\mathrm{Unip}_{\mathcal{G}}$ in terms representations of various relative Weyl groups.

(Generalized Springer Variety)

$\mathcal{P} = \mathcal{L}\mathcal{U}$ parabolic subgp $\subseteq \mathcal{G}$

$u \in \mathcal{G}$, $v \in \mathcal{L}$: unipotent elements

$Z_{\mathcal{G}}(u) \times Z_{\mathcal{L}}(u)\, \mathcal{U} \circlearrowright Y_{u,v} := \{y \in \mathcal{G} : y^{-1}uy \in v\mathcal{U}\}$

by

$(g, p) \cdot y := gyp^{-1}$.

$A_{\mathcal{G}}(u) := C_{\mathcal{G}}(u)/C_{\mathcal{G}}(u)^\circ$.

Definition [Lusztig, Invent. math. 1984]

Let $\rho \in \mathrm{Irr}(A_{\mathcal{G}}(u))$. Then ρ is called cuspidal if

$$\langle \rho, \sigma_{u,v} \rangle_{A_{\mathcal{G}}(u)} \neq 0 \text{ for any unipotent } v \in \mathcal{L} \quad \Rightarrow \quad \mathcal{P} = \mathcal{G},$$

where $\langle\ ,\ \rangle_{A_{\mathcal{G}}(u)}$ is the usual scalar product on the space of class functions on $A_{\mathcal{G}}(u)$ with values in $\overline{\mathbb{Q}}_\ell$.

- $A_{\mathcal{G}}(u) \times A_{\mathcal{L}}(v) \circlearrowright$ irr. components of $Y_{u,v}$ of max'l dim:

$\dim \mathcal{U} + \frac{1}{2}(\dim Z_{\mathcal{G}}(u) + \dim Z_{\mathcal{L}}(v))$.

$\sigma_{u,v}$: corresponding permutation rep'n

Disconnected complex reductive groups [A.-Moussaoui-Solleveld, 2018]

Let $\mathcal{G}$ be a possibly disconnected reductive group over $\mathbb{C}$, with identity component $\mathcal{G}^\circ$. Let $u \in \mathrm{U}(\mathcal{G})$ and $\rho \in \mathrm{Irr}(A_{\mathcal{G}}(u))$.

- We have $A_{\mathcal{G}^\circ}(u) \subset A_{\mathcal{G}}(u)$.
- The pair (u, ρ) is called cuspidal if the restriction of ρ to $A_{\mathcal{G}^\circ}(u)$ is a direct sum of irreducible representations ρ° such that one (or equivalently any) of the pairs (u, ρ°) is cuspidal.