NOTES ON PERFECTOID SPACES

ZHIQI ZHU

Contents

Date: December 4, 2023.

1. Remarks on this note

We introduce perfectoid spaces, which naturally appear in number theory and algebraic geometry. The main philosophy of this story is to reduce certain problems about mixed characteristic rings to problems about rings in characteristic p. Tilt functor plays a crucial role here. The study on the structure sheaf allows one to define general perfectoid spaces by gluing affinoid perfectoid spaces. Further, one can define étale morphisms of perfectoid spaces, and then étale topoi. This leads to an improvement on Faltings's almost purity theorem:

Theorem 1.1. *Let R be a perfectoid K-algebra, and let S/R be finite etale. Then S is perfectoid and S° is almost finite étale over R°.*

In the last part of our course, we learned the pro-étale site and established its basic properties. Moreover, going from the étale to the pro-étale site does not change the affinoid perfectoid subsets. These can be treated as the application of the theory of perfectoid spaces. In the last course, we introduced some period sheaves on the pro-étale topology, and gave proofs of de Rham comparison isomorphisms for rigid-analytic varieties with coefficients.

Chapter 6 is an "interlude". Prof.Ding asked me to give him a talk on an application of perfectoid spaces theory, as the further study of this course. The reference is Vector Bundles on Curves and p-adic Hodge Theory, written by Fargues and Fontaine. We give a detailed construction of the "fundamental curve of p-adic Hodge theory" together with sketches of proofs of the main properties of the objects. However, I'm sorry I didn't have enough time to type out the full talk note because I had to get busy with applying to schools. So I've chosen to upload what I've reported to Prof Ding as images and type down in Latex what I didn't finish and what I didn't do well.

Explanations. This is my note on the Spring 2023 course "Topics in Number Theory", which is taught by Professor Yiwen Ding. The order of this note is *almost* strictly based on the schedule of the curriculum. The content is based on Prof's nice and meticulous explanations, Peter Scholze's famous papers: Perfectoid Spaces, and p-Adic Hodge Theory For Rigid-Analytic Varieties, and Bhott's lecture notes: Lecture notes for a class on perfectoid spaces, and Prof Yiwen Ding's Lecture notes on Number Theory, which was my elightened reading material on Number Theory. Knowledge of comparison theories for rigid-analytic varieties remains too difficult for me, so I decided not to "copy" them before understanding them thoroughly (actually I think more motivations and classical theories before reading Scholze will help a lot...). However, I'm still interested in them, and at least, our argument in this note is rather elementary and self-contained.

Acknowledgments. First, I want to express my deep gratitude to Professor Yiwen Ding, who offered such a great opportunity for me to learn something about the "hottest" area in pure mathematics, in particular, enhanced my taste in mathematics and cemented my knowledge of commutative algebra, basic number theory, and algebraic geometry. It was a beautiful and crazy journey. Next, I want to thank my advisor Shanwen Wang, who introduced the "true" mathematics to me when I was a sophomore. Also my friends, Changlun Li and Bichang Lei, offered several crucial remarks on this note and explained some rigid geometry to me. Moreover, the course was taken while I was visiting BICMR, Peking

University, thus the author wants to thank the greatest institutions of China for their hospitality and kindness.

References and links are waiting to be added. This note was written on and off, so there are definitely quite a few small errors. I'll keep trying to perfect it.

2. L1-2: Introduction: Tilt functor

2.1. **Introduction.** The basic algebraic number theory gives us that if we consider the extension of $\mathbb{Q}_p$, we get $\overline{\mathbb{Q}_p}/\mathbb{Q}_p^t/\mathbb{Q}_p^{nr}/\mathbb{Q}_p$, where $\mathbb{Q}_p^{nr}/\mathbb{Q}_p$ represents the maximal unramified subextension with Galois group $\hat{\mathbb{Z}}$, and $\mathbb{Q}_p^t$ represents the maximal tamely ramified subextension with Galois group $\mathrm{Gal}(\mathbb{Q}_p^t/\mathbb{Q}_p^{nr}) = \prod_{l\neq p}\mathbb{Z}_l$. The mysterious part is its algebraic closure $\overline{\mathbb{Q}_p}$. We want to understand $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

Remark 2.1. $\mathrm{Gal}(\mathbb{Q}_p^t/\mathbb{Q}_p)$ is a semidirect product with $\hat{\mathbb{Z}}$ acts on $\prod_{l\neq p}\mathbb{Z}_l$ through after lifting Frobenius to $\mathbb{Q}_p^t/\mathbb{Q}_p$, it acts by conjugation on the normal subgroup $\mathrm{Gal}(\mathbb{Q}_p^t/\mathbb{Q}_p^{nr})$. Another thing is, for $K/\mathbb{Q}_p$ finite, we choose an algebraic closure $\overline{K}/K$, and its absolute Galois group is only defined up to inner automorphism.

Recall that a characteristic p ring R is perfect if the Frobenius $\phi : R \to R$ is an isomorphism. Define the field $\mathbb{Q}_p(p^{1/p^\infty})$ after adjoining all p-power roots of p, that is, $\mathbb{Q}_p(p^{1/p^\infty}) := \bigcup \mathbb{Q}_p(p^{1/p^n})$. This is one perfection of $\mathbb{Q}_p$. Similarly, For $\mathbb{F}_p((t))$, denote $\mathbb{F}_p((t^{1/p^\infty})) := \bigcup \mathbb{F}_p((t^{1/p^n}))$. "The basic result which we want to put into a larger context is the following canonical isomorphism of Galois groups, due to Fontaine and Wintenberger. A special case is the following result."

Theorem 2.1. *The absolute Galois group of* $\mathbb{Q}_p(p^{1/p^\infty})$ *and* $\mathbb{F}_p((t^{1/p^\infty}))$ *are canonically isomorphic.*

For simplicity, we denote K to be the completion of $\mathbb{Q}_p(p^{1/p^\infty})$ and let $K^\flat$ be the completion of $\mathbb{F}_p((t^{1/p^\infty}))$ temporarily. "Let us first explain the relation between K and $K^\flat$, which in vague terms consists in replacing the prime number p by a formal variable t." Let K° and $K^{b\circ}$ be the subrings of integral elements. Then

$$K^\circ/p = \mathbb{Z}_p[p^{1/p^\infty}]/p \cong \mathbb{F}_p[t^{1/p^\infty}]/t = K^{b\circ}/t$$

where the isomorphism sends p^{1/p^n} to t^{1/p^n}. Using this, one can define a continuous multiplicative, but nonadditive, map $K^\flat \to K, x \mapsto x^\sharp$, which sends t to p. On $K^{\flat\circ}$, it is given by sending x to $\lim_{n\to\infty} y_n^{p^n}$, where $y_n \in K^\circ$ is any lift of the image of $x_n = x^{1/p^n}$ in $K^{b\circ}/t = K^\circ/p$,i.e. $y_n \in K^\circ$ such that $y_n \equiv \overline{x_n}(\mathrm{mod}\ p)$. Then one has an identification

$$K^\flat = \varprojlim_{x\mapsto x^p} K, x \mapsto (x^\sharp, (x^{1/p})^\sharp, \ldots).$$

In order to prove the theorem, one has to construct a canonical finite extension $L^\sharp$ of K for any finite extension L of $K^\flat$. Say L is the splitting field of a polynomial $X^d + a_{d-1}X^{d-1} + \ldots + a_0$, which is also the splitting field of $X^d + a_{d-1}^{1/p^n}X^{d-1} + \ldots + a_0^{1/p^n}$ for all $n \geq 0$. Then $L^\sharp$ can be defined as the splitting field of $X^d + (a_{d-1}^{1/p^n})^\sharp X^{d-1} + \ldots + (a_0^{1/p^n})^\sharp$ for n large enough. these fields stabilize as $n \to \infty$. This gives $\mathrm{Gal}(L^\sharp/K) \cong \mathrm{Gal}(L/K^\flat)$.

Here is the generalization.

Definition 2.1. A perfectoid field is a complete topological field K whose topology is induced by a nondiscrete rank 1 valuation, such that the Frobenius Φ is surjective on K°/p.

Here $K^\circ \subset K$ denotes the set of powerbounded elements. Generalizing the example above, a construction of Fontaine associates to any perfectoid field K another perfectoid field $K^\flat$ of characteristic p:

$$K^\flat = \varprojlim_{x\mapsto x^p} K.$$

Remark 2.2. Choose some element $\varpi \in \mathfrak{m} \subset K^\circ$ such that $|p| \leq |\varpi| < 1$ (which eqivalent to $p \in \varpi K^\circ$), then $\Phi : K^\circ/p \to K^\circ/p$ surjective is equivalent to $\Phi : K^\circ/\varpi \to K^\circ/\varpi$ is surjective. Here ϖ is called the pesudouniformizer. (sometimes we will denote it as π or t, anyway.) Note that $\varprojlim_{x\mapsto x^p} K/\varpi$ gives a perfect ring of characteristic p.

Remark 2.3. Leader Lun told me that we can view the image of a rank-1-valuation $\Gamma \cup \{0\}$ as $\mathbb{R}$ (specifically, we can find another valuation isomorphic to this, where the value group lies in $\mathbb{R}$). We can use some simple definitions here: The subset K° will coincides with $\{x \in K||x| \leq 1\}$, called the valuation ring; Its maximal ideal $K^{\circ\circ} := \{x \in K||x| < 1\}$ will coincides with the topologically nilpotent elements of K, i.e., those $t \in K$ such that $t^n \to 0$ as $n \to \infty$. Their quotient k is called the residue field of K. In fact, For a pesudouniformizer t, the t-adic topology of K° coincides with the valuation topology. (One may check that the valuation (holds also for higher rank valuation) is continuous if and only if for one (or equivalently, any) pesudouniformizer t, we have $|t|^n \to 0$ as $n \to \infty$.)

Example 1. (i) $\mathbb{Q}_p$ is not perfectoid. also $\overline{\mathbb{Q}_p}$.

(ii) Let K is a NA field of characteristic p. Then K is perfectoid if and only if K is perfect.

(iii) $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is a perfectoid field with $K^\circ = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$. The value group is $\bigcup_{n=1}^\infty \frac{1}{p^n}\mathbb{Z}$ which is nondiscrete. Here $K^\circ/p \cong \frac{\mathbb{F}_p[t^{1/p^\infty}]}{t}$ satisfies the surjective condition. Then $K^{\flat\circ} = \varprojlim_{x\mapsto x^p} K^\circ/p \cong K^\flat = \varprojlim_{x\mapsto x^p} \frac{\mathbb{F}_p[t^{1/p^\infty}]}{t} \cong \mathbb{F}_p\widehat{[[t^{1/p^\infty}]]}$. That is, $K^\flat = \mathbb{F}_p\widehat{((t^{1/p^\infty}))}$.(We need the following Lemma.)

(iv) Similarly, for $K = \widehat{\mathbb{Q}_p(\mu_p^\infty)}$ which has tilt $K^\flat = \mathbb{F}_p\widehat{((t^{1/p^\infty}))}$. This extension is known as the cyclotomic perfectoid field and denoted by $\mathbb{Q}_p^{cycl}$. By definition this is complete and the value group is $\bigcup_{n=1}^\infty \frac{1}{p^n}\mathbb{Z}$ which is nondiscrete. Since for ζ_{p^k} as a p^k-primary root, element

$$(0, 1-\zeta_p, 1-\zeta_{p^2}, \cdots) \in \varprojlim_{x\mapsto x^p} K^\circ/p,$$

$K^\circ/p \cong \frac{\mathbb{F}_p[t^{1/p^\infty}]}{t^{p-1}}$, it is a perfectoid field. Here $t^\sharp = \lim_{n\to\infty}(1-\zeta_{p^n})^{p^n} \neq 0$.

(v) $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ is a perfectoid field. Its tilt is the completion of the algebraic closure of $\mathbb{Q}_p^{cycl,\flat} = \mathbb{F}_p\widehat{(t^{1/p^\infty})}$, which itself is $\widehat{\overline{\mathbb{F}_p((t))}}$.(I don't know how to observe this without tilt equivalence.)

Remark 2.4. By these examples, we observe that the tilt functor $K \mapsto K^\flat$ is not fully faithful on perfectoid field $K/\mathbb{Q}_p$. We shall see later that this a consequence of working over the non-perfectoid base $\mathbb{Q}_p$: this functor is fully faithful on perfectoid algebras over a perfectoid based field K.

2.2. **Tilt and Untilt.** In this subsetion, we start to study these operations: tilt and untilt.

Lemma 2.1. *Let K be a perfectoid field. Then the value group is p-divisible.*

Proof. It is easy to see that the value group is generated by $|x|$ where $x \in K^\circ$, since it is discrete. Then it follows the "perfectness" and NA property. □

Lemma 2.2. (i) *There is a multiplicative homeomorphism*

$$\varprojlim_{x \mapsto x^p} K^\circ \xrightarrow{\cong} \varprojlim_{\Phi} K^\circ/\varpi$$

given by projection. In particular, the right-hand side is independent of ϖ. Moreover, we get a map

$$\varprojlim_{\Phi} K^\circ/\varpi \to K^\circ : x \mapsto x^\sharp$$

This makes $\varprojlim_\Phi K^\circ/\varpi$ a domain.

(ii) *There is an element $\varpi^\flat \in \varprojlim_\Phi K^\circ/\varpi$ with $|(\varpi^\flat)^\sharp| = |\varpi|$. Define*

$$K^\flat = (\varprojlim_{\Phi} /\varpi)[(\varpi^\flat)^{-1}]$$

(iii) *There is a multiplicative homeomorphism*

$$K^\flat = \varprojlim_{x \mapsto x^p} K$$

In particular, there is a map $K^\flat \to K$, $x \mapsto x^\sharp$. Then $K^\flat$ is a perfectoid field of characteristic p,

$$K^{\flat\circ} = \varprojlim_{x \mapsto x^p} K^\circ \cong \varprojlim_{\Phi} K^\circ/\varpi,$$

and the rank-1-valuation on $K^\flat$ can be defined by $|x|_{K^\flat} = |x^\sharp|_K$. We have $|K^{b\times}| = |K^\times|$. Moreover

$$K^{\flat\circ}/\varpi^\flat \cong K^\circ/\varpi, \quad K^{\flat\circ}/\mathfrak{m}^\flat = K^\circ/\mathfrak{m}$$

where $\mathfrak{m}$, resp. $\mathfrak{m}^\flat$, is the maximal ideal of K°, resp. $K^{\flat\circ}$.

(iv) *If K is of characteristic p, then $K^\flat = K$.*

We call $K^\flat$ the tilt of K, and the functor from $K^\flat$ to K is called untilt.

Proof. It is a constructive proof. for $(\overline{x_0}, \overline{x_1}, \dots) \in \varprojlim_\Phi K^\circ/\varpi$, then we claim that the limit

$$x^\sharp = \lim_{n\to\infty} x_n^{p^n}$$

exists and is independent of all choices. Clearly it is a multiplicative and continuous map, now the map

$$\varprojlim_{\Phi} K^\circ/\varpi \to \varprojlim_{x \mapsto x^p} K^\circ : \ x \mapsto (x^\sharp, (x^{1/p})^\sharp, \dots)$$

gives the inverse of the projection.

The following three points says this map is compatible with the valuation. Furthermore, we can choose $\varpi^\flat$ such that $\varpi = (\varpi^\flat)^\sharp$. □

Remark 2.5. In Scholze's explanation of perfectoid spaces posted on MO (also his paper: Perfectoid Spaces, A Survey), he used Theorem 2.1 to explain this construction as the following:

"At this point, it may be instructive to explain Theorem 2.1 in the example where K is the completion of $\mathbb{Q}_p(p^{1/p^\infty})$; in all examples to follow, we make this choice of K. It says that there is a natural equivalence of categories between the category of finite extensions L of K and the category of finite extensions M of $K^\flat$. Let us give an example: Say M is the extension of $K^\flat$ given by adjoining a root of $X^2 - 7tX + t^5$. Basically, the idea is that one replaces t by p, so that one would like to define L as the field given by adjoining a root of $X^2 - 7pX + p^5$. However, this is obviously not well-defined: If $p = 3$, then $X^2 - 7tX + t^5 = X^2 - tX + t^5$, but $X^2 - 7pX + p^5 \neq X^2 - pX + p^5$, and one will not expect in general that the fields given by adjoining roots of these different polynomials are the same.

However, there is the following way out: M can be defined as the splitting field of $X^2 - 7t^{1/p^n}X + t^{5/p^n}$ for all $n \geq 0$ (using that $K^\flat$ is perfect), and if we choose n very large, then one can see that the fields L_n given as the splitting field of $X^2 - 7p^{1/p^n}X + p^{5/p^n}$ will stabilize as $n \to \infty$; this is the desired field L. Basically, the point is that the discriminant of the polynomials considered becomes very small, and the difference between any two different choices one might make when replacing t by p becomes comparably small."

Since in adic space setting, the continuous valutions are important (although we didn't define what is an adic space and explain why this is important.) Before the next proposition, I have to add the definition of valuation in advance.

Definition 2.2. Let R be some ring. A valuation on R is given by a multiplicative map $|\cdot| : R \to \Gamma \cup \{0\}$, where Γ is some totally ordered abelian group, written multiplicatively, such that $|0| = 0$, $|1| = 1$ and $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in R$. If R is a topological ring, we say a valuation is continuous if for all $\gamma \in \Gamma$, subset $\{x \in R||x| < \gamma\}$ is open.

Proposition 2.1. *Let K be a perfectoid field with tilt $K^\flat$. Then the continuous valuations $|\cdot|$ of K (up to equivalence and of any rank) are mapped bijectively to the continuous valuations $|\cdot|^\flat$ of $K^\flat$ (up to equivalence) via $|x|^\flat = |x^\sharp|$.*

Proof. First, we check that the map $|\cdot| \mapsto |\cdot|^\flat$ maps valuations to valuations. All properties except $|x + y|^\flat \leq \max(|x|^\flat, |y|^\flat)$ are immediate, using that $x \mapsto x^\sharp$ is multiplicative. The NA property follows from:

$$\begin{aligned} |x + y|^\flat &= \lim_{n\to\infty} |(x^{1/p^n})^\sharp + (y^{1/p^n})^\sharp|^{p^n} \\ &\leq \max(\lim_{n\to\infty} |(x^{1/p^n})^\sharp|^{p^n}, \lim_{n\to\infty} |(y^{1/p^n})^\sharp|^{p^n}) = \max(|x^\sharp|, |y^\sharp|). \end{aligned}$$

On the other hand, we need to prove the bijection. First, using the remark 2.3, we can construct the 1-1 correspondence between the valuation with the the "valuation ring" which contains $K^{\circ\circ}$ inside its maximal ideal and lies in K° (here the valuation ring for any rank is defined several sections later, readers may check it after then): The valuation ring $R \subset K$ attached to $|\cdot|$ will satisfies this; Conversely, such valuation subring defines a continuous valuation. This point can be checked using the remark 2.3.

Passing to the quotient, we learn the continuous valuation identify bijectively with valuation rings in $K^\circ/K^{\circ\circ}$, which is isomorphic to $K^{\flat,\circ}/K^{\flat,\circ\circ}$ and thus in bijection with $K^\flat$ continuous valuation. □

Proposition 2.2. *Let K be a perfectoid field with tilt $K^\flat$. If $K^\flat$ is algebraically closed, then K is algebraically closed.*

Proof. Loading...Number Theory (1) □

3. L2-4: Almost mathematics

Our goal in this section is to illustrate the following diagrams (although almost everything in this diagram are not defined yet):

For $K \supseteq K^\circ \supseteq \mathfrak{m} \ni \varpi$ with tilt $K^\flat \supseteq K^{\flat\circ} \supseteq \mathfrak{m}^\flat \ni \varpi^\flat$,

$$\begin{tikzcd}
K\text{−Perf} \arrow[r, leftrightarrow, "\sim"] & K^{\circ a}\text{−Perf} \arrow[r, leftrightarrow] & K^{\circ a}/\varpi\text{−Perf} \arrow[r, equal, "\text{Tilt}"] & K^{\flat\circ a}/\varpi^\flat\text{−Perf} \arrow[d, leftrightarrow, "\wr"] \\
R \arrow[r, mapsto] & R^{\circ a} \arrow[r] & R^{\circ a}/\varpi & K^{\flat\circ}\text{−Perf} \arrow[d, leftrightarrow, "\wr"] \\
A_*[1/\varpi] & A \arrow[l, mapsto] \arrow[r, mapsto, "\text{cot complex}"'] & {} & K^\flat\text{−Perf}
\end{tikzcd}$$

The point is, a perfectoid K-alg which is an object over the generic fibre, has a canonical extension to the almost integral level as a perfectoid $K^{\circ a}$-algebra, and the latter is determined by its reduction modulo ϖ. Or in a more general category, for K-Perf R:

$$\begin{tikzcd}
B_*[1/\varpi] & B \arrow[l, mapsto] \arrow[r, mapsto] & B/\varpi
\end{tikzcd}$$

$$\begin{tikzcd}
R_{\text{fét}} & R^{\circ a}_{\text{fét}} \arrow[l, "\cong"'] \arrow[r, "\cong"] & (R^{\circ a}/\varpi)_{\text{fét}} \arrow[dd, equal, "\text{tilt}"] \\
\text{almost purity} & \text{cotangent complex} & \\
R^\flat_{\text{fét}} & R^{\flat\circ a}_{\text{fét}} \arrow[l, "\cong"] \arrow[r, "\cong"] & (R^{\flat\circ a}/\varpi^\flat)_{\text{fét}}
\end{tikzcd}$$

3.1. Construct the category.

Fix a perfectoid field K with valuation ring K° and maximal ideal $\mathfrak{m} = K^{\circ\circ}$ which is the subset of topologically nilpotent elements. Since the valuation on K is non-discrete, we have $\mathfrak{m}^2 = \mathfrak{m}$. (The general setup: Assume $I \subset R$ is a flat ideal and satisfies $I_2 = I$. This implies $I \otimes_R I \cong I^2 \cong I$.)

Definition 3.1. Let M be a K°-module. We say $x \in M$ is almost zero if $\mathfrak{m}x = 0$. We say M is almost zero if every element of M is almost zero, i.e., $\mathfrak{m}M = 0$.

We would like to "quotient out" all almost zero modules.

Remark 3.1. We recall the machinery to do so: the quotient of an abelian category by a Serre subcategory.

Definition 3.2. Let $\mathcal{A}$ be an abelian category. A Serre subcategory is a full subcategory $\mathcal{B}$ of $\mathcal{A}$ such that for any exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in $\mathcal{A}$, one has $M \in \mathcal{B}$ if and only if $M', M'' \in \mathcal{B}$.

Suppose $\mathcal{B}$ is a Serre subcategory, the one can form the quotient category $\mathcal{A}/\mathcal{B}$, whose objects are the objects of $\mathcal{A}$, and for $M, N \in \mathcal{A}$

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M,N) := \varinjlim_{\substack{\alpha : M' \hookrightarrow M,\ \beta : N \twoheadrightarrow N'' \\ \ker\alpha,\ \mathrm{coker}\beta \in \mathrm{Mor}(\mathcal{B})}} \mathrm{Hom}(M', N'')$$

The quotient category is again an abelian category. By construction, one has a canonical localization functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$. If $M \in \mathcal{B}$, then $Q(M) \cong 0$ in $\mathcal{A}/\mathcal{B}$. The quotient category enjoys the following universal property: suppose $\mathcal{C}$ is another abelian category and $F : \mathcal{A} \to \mathcal{C}$ is an exact functor such that $F(M) = 0$ for any $M \in \mathcal{B}$, then F uniquely factors through $\mathcal{A}/\mathcal{B}$.

Proposition 3.1. *The full subcategory of* $K^\circ - \mathrm{mod}$ *consisting of almost zero* K°*-modules is a Serre subcategory of* $K^\circ - \mathrm{mod}$. *Denote the quotient category by* $K^{\circ a} - \mathrm{mod}$.

Proof. Suppose M is $\mathfrak{m}$-torsion, then clearly any sub or quotient of M is also $\mathfrak{m}$-torsion. Conversely, if M is an extension of M'' by M', where M' and M'' are $\mathfrak{m}$-torsion, then m^2 kills M. Then the result follows from $\mathfrak{m}^2 = \mathfrak{m}$. ☐

We denote this localization functor K°-mod to $K^{\circ a}$-mod: $M \mapsto M^a$.

Definition 3.3. Define

$$M^a \otimes_{K^{\circ a}} N^a = (M \otimes_{K^\circ} N)^a.$$

It is a well-defined on $K^{\circ a} - \mathrm{mod}$. For $f : M^a \to N^a$, define $\ker(f) = \ker(f_*)^a$, $\mathrm{coker}(f) = \mathrm{coker}(f_*)^a$, where $f_* : M \to N$.They make $K^{\circ a} - \mathrm{mod}$ an abelian tensor category.

One can show that homomorphisms between two almost modules can be described alternatively by

$$\mathrm{Hom}_{K^{\circ a}}(M^a, N^a) = \mathrm{Hom}_{K^\circ}(\mathfrak{m} \otimes_{K^\circ} M, N)$$

This is a K°-mod with no almost zero elements.

We define

$$\mathrm{alHom}(M^a, N^a) = \mathrm{Hom}_{K^{\circ a}}(M^a, N^a)^a.$$

Then for any three $K^{\circ a}$-modules $L,\ M,\ N$, there is a functorial isomorphism as usual:

$$\mathrm{Hom}(L, \mathrm{alHom}(M,N)) = \mathrm{Hom}_{K^{\circ a}}(L \otimes_{K^{\circ a}} M, N).$$

This means that $K^{\circ a}$-mod has all has all formal properties of the category of modules over a commutative ring. So one can define the notion of almost $K^{\circ a}$-algebras, or $K^{\circ a}$-algebras: these are commutative unitary monoid objects in $K^{\circ a}$-mod. Let A be a $K^{\circ a}$-algebra, one can also define the notation of A-modules and A-algebras...

Remark 3.2. There is a right adjoint functor to the localization functor, called *the functor of almost elements*:

$$M \mapsto M_* = \mathrm{Hom}_{K^{\circ a}}(K^{\circ a}, M)$$

One easily checks (by the previous remark) that $(-^a, -_*)$ is an adjoint pair and

$$(M_*)^a \cong M, \quad (N^a)_* \cong \mathrm{Hom}_{K^\circ}(\mathfrak{m}, N).$$

To show that $(-)_*$, $(-)_a$ are "good" functors from the category of commutative algebra in R-mod to the category of commutative algebras in almost R-mod as well its right adjoint, we need to check there is a canonical map $M_* \otimes_{K^\circ} N_* \cong (M \otimes_{K^{\circ a}} N)_*$.

Then we want to give a realization of such category. In the following of this subsection, I will use (R, I) to repalace $(K^\circ, \mathfrak{m})$, and for simplicity, (sometimes) I will use Mod_R to represent R-mod.

We introduce the category $\mathcal{A} \in \mathrm{Mod}_R$ be the full subcategory spanned by all $M \in \mathrm{Mod}_R$ such that the action map $I \otimes_R M \mapsto M$ is an isomorphism; This functor is exact by the flatness of I. Also we can check this is a abelian subcategory. Then we wil construct a series of functors and eventually to realize $\mathcal{A}$ as a quotient of Mod_R.

(i) Write $i_* : \mathrm{Mod}_{R/I} \to \mathrm{Mod}_R$ as functor given by the restriction of scalars along $R \to R/I$;

(ii) it has a left adjoint i^* given by $M \mapsto M \otimes_R R/I$ and a right adjoint $i^!$ given by $M \mapsto \mathrm{Hom}_R(R/I, M) = M[I]$.

(iii) Write $j_! : \mathcal{A} \to \mathrm{Mod}_R$ for the exact inclusion. It has a exact right adjoint j^* given by $M \mapsto I \otimes_R M$ and This right adjoint j^* has a further right adjoint j_* given by the formula

$$j_*(M) = \mathrm{Hom}_R(I, M)$$

Proposition 3.2. *The unit map $N \to j^* j_! N$ is an isomorphism for any $N \in \mathcal{A}$. The counit map $j^* j_* M \to M$ is an isomorphism for any $M \in \mathcal{A}$.*

(i) We have $i^* j_!$, $i^! j_*$, $j^* i_*$ are 0.

(ii) The kernel of j^* is exactly $\mathrm{Mod}_{R/I}$: $\supseteq$ is obvious; For $\forall M \in \ker j^*$, Tensoring M with the standard exact sequence $0 \to R \to R/I \to 0$ shows that $M \cong M/IM$.

(iii) The quotient functor $q : \mathrm{Mod}_R \to \mathrm{Mod}_R^a$ (the quotient to its almost part) is realized by j^*.

Remark 3.3. This also proved that the quotient functor admits fully faithful left and right adjoints, thus commutes with all limits and colimits.

Proof. To prove the last point, we need to check the following:

$j^*(\mathrm{Mod}_{R/I}) = 0$ and exact: This follows from $I = I^2$ and I is flat.

j^* is universal with the previous two properties: let $q' : \mathrm{Mod}_R \to \mathcal{B}$ be such a functor of abelian categories, then it factors through j^*: Fix some $M \in \mathrm{Mod}_R$, then we have the action map $I \otimes_R M \to M$. The kernel and cokernel of this map are $\mathrm{Tor}_i^R(R/I, I)$ for $i = 1, 0$ which are killed by I, thus also q'. As q' is exact, we have $q'(j_! j^* M) = q'(I \otimes_R M) \cong q'(M)$. Thus $q' \cong q' j_! j^*$ as wanted. $\square$

Remark 3.4. We can draw a diagram to describe these adjoint pairs:

$$\begin{tikzcd} \mathrm{Mod}_{R/I} \arrow[r, "i_*"] & \mathrm{Mod}_R \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathrm{Mod}_R^a = \mathcal{A} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Bhott said in his note:"We may view Mod_R as quasi-coherent sheaves on $X = \mathrm{Spec}(R)$, and $\mathrm{Mod}_{R/I}$ as quasi-coherent sheaves on $Z = \mathrm{Spec}(R/I)$. And we can think the category $\{\mathrm{Mod}_R^a = \mathcal{A}\}$ as quasi-coherent sheaves on some non-exist open $\overline{U} \subset X$ that contains $X - Z$ (but not U)."

In Bhott note, He defines

$$M_* := j_* j^* M = \mathrm{Hom}_{K^\circ}(\mathfrak{m}, \mathfrak{m} \otimes_{K^\circ} M)$$

Thus we have $M_* = \mathrm{Hom}_{K^\circ}(\mathfrak{m}, M_*) \cong \mathrm{Hom}_{K^{\circ a}}(K^{\circ a}, M)$, which is scholze's definition.

After reading a few articles about six-functors, I started to know why Bhott wrote this.

3.2. **Commutative algebra in almost world.** We now extend some basic notions of commutative algebra to the almost world.

Definition 3.4. Let $M \in \mathrm{Mod}_R$ with image $M^a \in \mathrm{Mod}_R^a$.

We say that M or M^a is almost flat if $M^a \otimes (-)$ is exact on Mod_R^a; equivalently, $\mathrm{Tor}^R_{>0}(M, N)$ is almost zero for any R-module N.

We say that M or M^a is almost projective if $\mathrm{alHom}(M, -)$ is exact; equivalently, $\mathrm{Ext}_R^{>0}(M, N)$ is almost zero for any R-module N.

We say that M or M^a is almost finitely generated (resp. almost finitely presented) if for each $\epsilon \in I$, there exists a finitely generated (resp. finitely presented) R-module N_ϵ and a map $N_\epsilon \to N$ with kernel and cokernel killed by ϵ. If the number of generators of N_ϵ can be bounded independently of ϵ, we say that M is uniformly almost finitely generated.

Remark 3.5. (Why needs almost-projective) The notion of almost projectivity is distinct from the categorical notion of projectivity in the abelian category Mod_R^a: the latter is far more restrictive. Indeed, the ring R is almost projective with the above definition. However, R^a need not be a projective object of Mod_R^a. In fact, consider $R = K^\circ$ and $I = K^{\circ\circ}$ for a perfectoid field K with residue field k, note that if we apply $(-)^a$ to

$$0 \to \mathrm{Hom}_{K^\circ}(K^\circ, K^\circ/\mathfrak{m}) \to K^\circ \to \mathrm{Hom}_{K^\circ}(K^\circ, \mathfrak{m}) \to \mathrm{Ext}^1_{K^\circ}(K^\circ, K^\circ/\mathfrak{m})$$

the group $\mathrm{Ext}^1_{K^{\circ a}}(K^{\circ a}, K^{\circ a})$ identifies with $\mathrm{Ext}^2_{K^\circ}(k, K^\circ)$ is the obstacle for $K^{\circ a}$ being projective, and is nonzero if K is not spherically complete.

Remark 3.6. As an example of an almost finitely presented module, consider the case that K is the p-adic completion of $\mathbb{Q}_p(p^{1/p^\infty})$, $p \neq 2$. Consider the extension $L = K(p^{1/2})$. Then $L^{\circ a}$ is an almost finitely presented $K^{\circ a}$-mod. Indeed, for any $n \geq 1$, we have injective maps

$$K^\circ \oplus p^{1/2p^n} K^\circ \to L^\circ$$

since L° is the completion of $\varinjlim(K^\circ \oplus p^{1/2p^n} K^\circ)$, whose cokernel is killed by $p^{1/2p^n}$. That is, L° is a uniformly almost finitely presented projective K°-mod.

The following conclusions explain how these properties relate to the "classical" algebraic geometry, and their benefits to the structure.

Proposition 3.3. *Let A be a R^a-algebra. Then $A-\mathrm{mod}$ M is flat and almost finitely presented if and only if it is almost projective and almost finitelt generated.*

Definition 3.5. Let A be a R^a-algebra, and let B be an A-algebra. Let $\mu : B \otimes_A B \to B$ denote the multiplication morphism.

(i) The morphism $A \to B$ is said to be *unramified* if there is some element $e \in (B \otimes_A B)_*$ such that $e^2 = e,\ \mu_*(e) = 1$ and $xe = 0$ for all $x \in \ker(\mu)_*$.

(ii) The morphism $A \to B$ is said to be étale if it is unramified and B is a flat A-module.

A morphism $A \to B$ of R^a-algebras is said to be almost *finite étale* if it is étale and B is an almost finitely presented A-module. Write $A_{\text{afét}}$ for the category of almost finite étale maps $A \to B$.

We will see how to characterize this property. Here is an example.

Example 2. Recall the setting in Lemma 3.6. We shall show L°/K° is almost finite étale. We have shown that L° is a uniformly almost finitely presented projective K°-mod, clearly L° is flat. It remains the unramifiedness. Note that L/K is a Galois extension of degree 2, hence we have a canonical isomorphism (after fixing $\sigma : L \to L$ the nontrivial element):

$$\mathrm{can} : L \otimes_K L \cong L \times L, \quad a \otimes b \mapsto (ab, a\sigma(b))$$

where sends $e = \frac{1}{2\sqrt{p}\otimes 1}(\sqrt{p}\otimes 1 + 1 \otimes \sqrt{p})$ to $(1,0)$. Using the isomorphism above, it is easy to vertify that e is the diagonal idempotent we want, by deducing $p^{1/p^n} \cdot e \in L^\circ \otimes_{K^\circ} L^\circ$.

In fact, we have the following theorem by Tate and Gabber-Ramero:

Theorem 3.1. *If L/K is a finite extension, then $\mathcal{O}_L/\mathcal{O}_K$ is almost finite étale. Similarly, if $M/K^\flat$ is finite, then $\mathcal{O}_M/\mathcal{O}_{K^\flat}$ is almost finite étale.*

Lemma 3.1. *For a perfectoid field K, set $K \supseteq K^\circ \supseteq \mathfrak{m} \ni \varpi$. Let M be a $K^{\circ a}$-module.*

(i) M is almost flat if and only if M_* is flat over K° if and only if M_* has no ϖ-torsion.

(ii) If N is a flat K°-module and $M = N^a$, then M is flat over $K^{\circ a}$ and we have $M_* = \{x \in N[\frac{1}{\varpi}] \mid \forall \epsilon \in \mathfrak{m} : \epsilon x \in N\}$.

(iii) If M is flat over $K^{\circ a}$, then for all $x \in K^\circ$, we have $(xM)_* = xM_*$. Moreover, $M_*/xM_* \subset (M/xM)_*$, and for all $\epsilon \in \mathfrak{m}$ the image of $(M/x\epsilon M)_* in (M/xM)_*$ is equal to the image of $M_*/xM_* \subset (M/xM)_*$.

(iv) If M is almost flat, then M is ϖ-adically complete if and only if M_* is ϖ-adically complete.

Proof. For (ii), We have

$$M_* = \mathrm{Hom}_{K^{\circ a}}(K^{\circ a}, M) = \mathrm{Hom}_{K^\circ}(\mathfrak{m}, N).$$

As N is flat over K°, we can apply fraction to the last term, as the subset of those $x \in \mathrm{Hom}_K(K, N[\frac{1}{\varpi}]) = N[\frac{1}{\varpi}]$ satisfying the condition that for all $\epsilon \in \mathfrak{m},\ \epsilon x \in N$.

For (iii), $(xM)_* = xM_*$ is clear. For the rest, consider the canonical map of the short exact sequence of $K^{\circ a}$-mod:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "x"] \arrow[d] & M \arrow[r] \arrow[d] & M/xM \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M/\epsilon M \arrow[r] & M/x\epsilon M \arrow[r] & M/xM \arrow[r] & 0
\end{tikzcd}$$

Then apply $(-)_*$ functor, we get a diagram:

$$\begin{tikzcd}
0 \arrow[r] & M_*/xM_* \arrow[r, "a"] \arrow[d] & (M/xM)_* \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{K^{\circ a}}(K^{\circ a}, M)[x] \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & (M/x\epsilon M)_* \arrow[r, "b"] & (M/xM)_* \arrow[r] & \mathrm{Ext}^1_{K^{\circ a}}(K^{\circ a}, M/\epsilon M) &
\end{tikzcd}$$

We wish to show that a and b have the same image, thus it suffices to show c is injective.

For (iv), it follows from (iii) clearly. □

3.3. **The dictionary.**

Definition 3.6 (K-Banach algebra)**.** Let K be a complete NA field. A Banach K-algebra R is a K algebra R equipped with a map $|\cdot| : R \to \mathbb{R}_{\geq 0}$ extending the norm on K such that

1. (Norm) $|f| = 0$ only if $f = 0$.
2. (Submultiplicativity) $|fg| \leq |f||g|$, with equality if $f \in K$.
3. (NA property) $|f + g| \leq \max(|f|, |g|)$.
4. R is complete in the metric d given by $d(f, g) = |f - g|$.

For such a K-Banach algebra R, define the set $R^\circ \subset R$ of powerbounded elements as

$$R^\circ := \{f \in R | \{|f^n|\} \text{ is bounded.}\} = \{f \in R | \{f^n\} \subset R \text{ is bounded.}\}$$

Since $R_{<1} \subset R_{\leq 1} \subset R^\circ$, thus R° is an open subring.

Example 3. For $K = \mathbb{Q}_p \subset \mathbb{Z}_p = K^\circ$, A is a flat p-adically complete $\mathbb{Z}_p$-algebra, $R = A[\frac{1}{p}]$, then $R_{\leq 1} = A$. Thus is seminorm on R is $|f| = \min\{|p|^n | f \in p^n A\}$. But R° could be larger then $R_{\leq 1}$: if $A = K^\circ[x]/(x^2)$, then $\frac{1}{p^n} xA \subset R^\circ$ for each n, but these not lies in $R_{\leq 1} = A$.

Definition 3.7. For a perfectoid field K, set $K \supseteq K^\circ \supseteq \mathfrak{m} \ni \varpi$ with tilt $K^\flat$. Define:

(i) A Banach K-algebra R is *perfectoid* if $R^\circ \subset R$ is bounded, and the Frobenius map $R^\circ/\varpi \to R^\circ/\varpi$ is surjective. With continuous morphisms as morphisms of K-alg, this gives the category K- Perf of perfectoid K-algebras.

(ii) A $K^{\circ a}$-algebra A is *perfectoid* if:

(1) A is t-adically complete and flat over K°.

(2) The map $K^\circ/\varpi \to A/\varpi$ is relatively perfect, i.e., the Frobenius induces an isomorphism $A/\varpi^{\frac{1}{p}} \simeq A/\varpi$.

With continuous morphisms as morphisms of $K^{\circ a}$-alg, this gives the category $K^{\circ a} -$ Perf of perfectoid $K^{\circ a}$ algebras.

(iii) A $K^{\circ a}/\varpi$-algebra A is perfectoid if:
(1) A is flat over K°/ϖ.
(2) The map $K^{\circ}/\varpi \to A$ is relatively perfect, i.e., the Frobenius induces an isomorphism $A/\varpi^{\frac{1}{p}} \simeq A$.
With continuous morphisms as morphisms of $K^{\circ a}/\varpi$-alg, this gives the category $K^{\circ a}/\varpi - \mathrm{Perf}$ of perfectoid $K^{\circ a}/\varpi$ algebras.

Then we start to explain the first diagram of equivalences of categories at the beginning of this section.

4. L4-7: Perfectoid algebras and almost purity (1)

Our main theorem is:

Theorem 4.1 (Tilting from characteristic 0 to characteristic p)**.** *We have the chain of equivalences:*

$$K - \mathrm{Perf} \cong K^{\circ a} - \mathrm{Perf} \cong K^{\circ a}/\varpi - \mathrm{Perf} \cong K^{\flat\circ a}/\varpi^{\flat} - \mathrm{Perf} \cong K^{\flat} - \mathrm{Perf}$$

4.1. $K-\mathrm{Perf} \cong K^{\circ a}-\mathrm{Perf}$**.** First we are going to prove the canonical equivalence: $K-\mathrm{Perf} \cong K^{\circ a}-\mathrm{Perf}$. "In other words, a perfectoid K-algebra, which is an object over the generic fibre, has a canonical extension to the almost integral level as a perfectoid $K^{\circ a}$-algebra". This proof is quite explicit, so I want to copy it directly from scholze's thesis. This helps me to understand what is "almost".

Proposition 4.1. *Let $R \in K-\mathrm{Perf}$. Then Φ induces an isomorphism: $R^{\circ}/\varpi^{1/p} \cong R^{\circ}/\varpi$ and $R^{\circ a} \in K^{\circ a} - \mathrm{Perf}$.*

Proof. It is clear that Φ is a surjection. By Lemma 3.1 we have R° is complete and flat over K°. Moreover, the perfectoidness of R ensures that the Frobenius is surjective. Thus it suffices to show that Frobenius has kernel $(\varpi^{1/p})$. ☐

Lemma 4.1. *Let A be a perfectoid $K^{\circ a}$-algebra, and let $R = A_*[\varpi^{-1}]$. Equip R with the Banach K-algebra structure making A_* open and bounded. Then $A_* = R^{\circ}$ is the set of power-bounded elements, which is ϖ-adically complete, ϖ torsionfree, p-root closed in R and has a surjective Frobenius modulo ϖ.*

Proof. The completeness results and torsion free is clear by Lemma 3.1.
p-root closed: Since Φ is an isomorphism $A/\varpi^{1/p} \cong A/\varpi$, hence Φ is an almost isomorphism $A_*/\varpi^{1/p} \to A_*/\varpi$. It is injective: If $x \in A_*$ and $x^p \in \varpi A_*$, then for all $\epsilon \in \mathfrak{m}$, $\epsilon x \in \varpi^{1/p}A_*$ by almost injectivity, hence $x \in (\varpi^{1/p}A)_* = \varpi^{1/p}A_*$.

Thus if $y \in A_*$ satisfies $y^p \in \varpi A_*$, then $y \in \varpi^{\frac{1}{p}}A_*$. Now fix $x \in R$ with $x^p \in A_*$. There is some positive integer k such that $y = \varpi^{\frac{k}{p}}x \in A_*$, and as long as $k \geq 1$, we have $y^p \in \varpi, \varpi^k A_* \subset \varpi A_*$, so that $y \in \varpi^{\frac{1}{p}}A_*$ by injectivity. Thus $\varpi^{\frac{k-1}{p}}x = \frac{y}{\varpi^{1/p}} \in A_*$. By induction, $x \in A_*$.

$A_* = R^{\circ}$: Obviously, A_* consists of power-bounded elements. Now assume that $x \in R$ is powerbounded. Then ϵx is topologically nilpotent for all $\epsilon \in \mathfrak{m}$. In particular, $(\epsilon x)^{p^N} \in A_*$ for N sufficiently large. By the p-root closedness, this implies $\epsilon x \in A_*$. Since this is true for all $\epsilon \in \mathfrak{m}$, we have $x \in A_*$ by the key lemma.

For surjectivity of Frobenius: It is almost surjective, hence it suffices to show that the composition $A_*/\varpi^{1/p} \to A_*/\varpi \to A_*/\mathfrak{m}$ is surjective. Let $x \in A_*$. By almost surjectivity, $\varpi^c x \equiv y^p$ modulo ϖA_*, for some $y \in A_*$ and $c < 1$. Let $z = \frac{y}{\varpi^{c/p}} \in R$. This implies $z^p \in A_*$ also $z \in A_*$, by p-root closedness. Thus $y \in \varpi^{c/p} A_*$ which implies $x \equiv z^p$ modulo $\varpi^{1-c} A_* \supseteq \mathfrak{m} A_*$. This gives the desired surjectivity. $\square$

Remark 4.1. In fact, we have the following proposition:

Proposition 4.2. *Fix a pseudouniformizer $t \in K$. The following categories are equivalent:*

** The category $\mathcal{C}$ of uniform Banach K-algebras R with continuous K-algebra maps.*

** The category $\mathcal{D}_{tic}$ of t-adically complete and t-torsionfree K°-algebras A with A totally integrally closed (i.e. the given $f \in A[\frac{1}{t}]$ with $f^{\mathbb{N}}$ lying in a finite generated A-submodule of $A[\frac{1}{t}]$ will lie in A) in $A[\frac{1}{t}]$.*

The functors are given by $F : R \mapsto R^\circ$ and $G : A \mapsto A[\frac{1}{t}]$, and $G \circ F \cong \mathrm{id}$.

If K is a perfectoid field, $\mathcal{D}_{\mathrm{tic}}$ is equivalent to the category $\mathcal{D}_{prc}$ of t-adically complete and t-torsion free K° algebras A with A p-root closed on $A[\frac{1}{t}]$ and $A \cong A_*$. Thus for $R \in K$-Perf, $A \in K^{\circ a}$-Perf, we have $(R^{\circ a})_*[\frac{1}{\varpi}] \cong R$ and $A \cong (A_*[\frac{1}{\varpi}])^{\circ a}$.

4.2. **Deformation and tilt equivalence.**

In order to finish the proof of the main theorem, it suffices to prove:

Theorem 4.2. *The functor $A \mapsto \bar{A} = A/\varpi$ induces an equivalence of categories $K^{\circ a} - \mathrm{Perf} \cong (K^{\circ a}/\varpi) - \mathrm{Perf}$.*

To construct the inverse, we need the deformation theory. Since I am not familiar with this language before (now as well), I don't want to talk about the construction here. You can check the reference note by Bhott for the definition of $P^\bullet_{B/A}$ (which is the canonical simplicial A-algebra resolution of B) and the cotangent complex $L_{B/A} := \Omega^1_{P^\bullet/A} \otimes_{P^\bullet} B$ (where $P^\bullet \to B$ is a simplicial resolution of B by polynomial A-algebras. Here we view the simplicial B-module $\Omega^1_{P^\bullet/A} \otimes_{P^\bullet} B$ as a B-complex by taking an alternating sum of the face maps as a differential). You can also check the basic properties there. "The main reason to introduce the cotangent complex is that it controls deformation theory in complete generality, analogous to how the tangent bundle controls deformations of smooth varieties":

Theorem 4.3 (Infinitesimal invariance of formally étale rings)**.** *For any ring A, write $\mathcal{C}_A$ for the category of flat A-algebras B such that $L_{B/A} \simeq 0$. Then for any surjective map $\tilde{A} \to A$ with nilpotent kernel, base change induces an equivalence $\mathcal{C}_{\tilde{A}} \simeq \mathcal{C}_A$.*

Any étale A-algebra B is an object of $\mathcal{C}_A$. For our purposes, the following class of examples is crucial:

Proposition 4.3 (Perfect rings have a trivial cotangent complex)**.** *Assume A has characteristic p. Let $A \to B$ be a flat that is relatively perfect, i.e., the relative Frobenius $F_{B/A} : B^{(1)} := B \otimes_{A,F_A} A \to B$ is an isomorphism. Then $L_{B/A} \cong 0$.*

We can draw the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "Frob_A"] \arrow[dd] & & A \arrow[d] \arrow[ddr, bend left] & \\
& & B \otimes_{A,Frob_A} B \arrow[dr, dashed, "\sim"] & \\
B \arrow[urr] \arrow[rrr, "Frob_B"'] & & & B
\end{tikzcd}$$

Corollary 4.1. *Let R be a ring equipped with a nonzero divisor $f \in R$. Then reduction modulo f gives and equivalence between the category of f-adically complete and f-torsionfree R-algebras S with $R/f \to S/f$ flat and $L_{S/R} \otimes_R R/f \cong 0$, and the category $\mathcal{C}_{R/f}$.*

Then we will prove the equivalence $K^{\circ a} - \mathrm{Perf} \cong K^{\circ a}/\varpi - \mathrm{Perf}$.

Remark 4.2. Before the proof, we need a further functor. Recall that $M \mapsto M_*$ which is a right adjoint to the almostification functor $(\ )^a$, takes commutative algebra to commutative algebra (notice that there is a canonical map $M_* \otimes_R N_* \to (M \otimes N)_*$). However, the left adjoint $M \mapsto M_!$ does not, since $R_!^a = I$ does not coincide with R, so one does not have a unit for $A_!$. This leads us to define the functor $(\ )_{!!}$ as the pushout R°-alg of $R^\circ \leftarrow I \cong R_!^a \to A_!$. Here are some properties:

(i) There is a unique way to make $A_{!!}$ into a commutative ring such that the defining map $R \to A_{!!}$ is a unit and the defining map $A_! \to A_{!!}$ is compatible with the multiplication.

(ii) $(\ )_{!!}$ commutes with colimits.

(iii) $(\ )_{!!}$ preserves faithful flatness, i.e. for A a perfectoid faithful flat $R^{\circ a}/\varpi$ algebra, $A_{!!}$ is a perfectoid faithful flat R°/ϖ algebra.

Proof of the equivalence $\mathrm{Perf}_{K^{\circ a}} \simeq \mathrm{Perf}_{K^{\circ a}/\varpi}$. There is an obvious functor $\mathrm{Perf}_{K^{\circ a}} \to \mathrm{Perf}_{K^{\circ a}/\varpi}$ given by reduction moduloϖ. To construct the inverse, we need some deformation theory.

Write $\mathcal{C}_n$ for the category of flat K°/ϖ^n-algebras B_n with such that the relative Frobenius $K^\circ/\varpi \to B_n/\varpi$ is an isomorphism. By deformation theory Thm 4.3 and Prop 4.3, we have $\mathcal{C}_{n+1} \simeq \mathcal{C}_n$ via the reduction map. Moreover, by taking inverse limits, these categories are also identified with $\mathcal{C}$, the category of t-adically complete and flat K°-algebras B such that the relative Frobenius for $K^\circ/\varpi \to B$ is an isomorphism. Write $B \mapsto \widetilde{B}$ for the inverse equivalence $\mathrm{C}_1 \simeq \mathrm{C}$.

Now say $A \in \mathrm{Perf}_{K^{\circ a}/\varpi}$. We shall show that A deforms uniquely to a perfectoid $K^{\circ a}$-algebra, then it will automatically become faithfully flat over $K^{\circ a}/\varpi$ (this is an/a exercise/lemma). Using the functor $(-)_{!!}$ from above remark, we have $A_{!!} \in \mathcal{C}_1$: the functor $(\ )_{!!}$ preserves faithful flatness, pushout diagrams (or all colimits), and carries Frobenius to Frobenius (in characteristic p). We have the corresponding lift $\widetilde{A_{!!}} \in \mathcal{C}$ to K°. Write $\widetilde{A} := \widetilde{A_{!!}}^a$ for the corresponding almost algebra. Then $\widetilde{A_{!!}}$ is K°-flat and t-adically complete, so the same holds true for $\widetilde{A}$ (as almostification preserves limits, colimits and flatness). Moreover, we have $\tilde{A}/\varpi = \widetilde{A_{!!}}^a/\varpi \simeq (\widetilde{A_{!!}}/\varpi)^a \simeq A_{!!}^a \simeq A$. Thus, the construction $A \mapsto \widetilde{A}$ sending A

to the almostification of the unique lift to $\mathcal{C}$ of $A_{!!}$ provides a right-inverse to the canonical projection $\mathrm{Perf}_{K^{\circ a}} \to \mathrm{Perf}_{K^{\circ a}/\varpi}$.

It remains to check that the functor in the previous paragraph also gives a left-inverse. Fix some $A \in \mathrm{Perf}_{K^{\circ a}}$. We want to show that $A \simeq \widetilde{(A/\varpi)}$. We may assume $A \neq 0$, and thus A is faithfully flat. By the preservation of colimits and faithful flatness under $(-)_{!!}$, the ring $A_{!!}$ is a faithfully flat K°-algebra with $K^\circ/\varpi \to A_{!!}/\varpi \simeq (A/\varpi)_{!!}$ relatively perfect. By Lemma 3.1, the ring A_* is complete, and hence so is $A_{!!}$: the canonical map $A_{!!} \to A_*$ is injective with almost zero cokernel, so we can apply the following lemma to conclude that $A_{!!}$ is complete. Thus $A_{!!} \in \mathcal{C}$. The corresponding object of $\mathcal{C}_1$ is given by $A_{!!}/\varpi \simeq (A/\varpi)_{!!}$ as $(-)_{!!}$ commutes with colimits. The construction in the previous paragraph then shows that $\widetilde{A/\varpi}$ is the almostification of the unique lift to $\mathcal{C}of(A/\varpi)_{!!}$, and thus $\widetilde{A/\varpi} \simeq A_{!!}^a \simeq A$, as wanted. □

This lemma is used in the last paragraph of this proof, also a part in the proof of Lemma 3.1.

Lemma 4.2. *Let A be a ring equipped with a nonzerodivisor t. Let $\alpha : M \to N$ be a map of t-torsionfree A-modules. Assume that α is injective with t-torsion cokernel Q. Then M is t-adically complete if and only if N is so.*

In particular, we also arrive at the tilting equivalence, $K\text{-}\mathrm{Perf} \cong K^\flat\text{-}\mathrm{Perf}$. We want to compare this with Fontaine's explicit construction. Let R be a perfectoid K-algebra, with $A = R^{\circ a}$. Define

$$A^\flat = \varprojlim_{\Phi} A/\varpi \ ,$$

which we regard as a $K^{\flat\circ a}$-algebra via

$$K^{\flat\circ a} = (\varprojlim_{\Phi} K^\circ/\varpi)^a = \varprojlim_{\Phi}(K^\circ/\varpi)^a = \varprojlim_{\Phi} K^{\circ a}/\varpi \ ,$$

and set $R^\flat = A_*^\flat[(\varpi^\flat)^{-1}]$ where $A*$. Recall that $(\ )_*$ represents the functor of almost elements.

Proposition 4.4. *This defines a perfectoid $K^\flat$-algebra $R^\flat$ with corresponding perfectoid $K^{\flat\circ a}$-algebra $A^\flat$, and $R^\flat$ is the tilt of R. Moreover,*

$$R^\flat = \varprojlim_{x\mapsto x^p} R \ , \ A_*^\flat = \varprojlim_{x\mapsto x^p} A_* \ , \ A_*^\flat/\varpi^\flat \cong A_*/\varpi \ .$$

In particular, we have a continuous multiplicative map $R^\flat \to R$, $x \mapsto x^\sharp$.

Proof. First, we have

$$A_*^\flat = (\varprojlim_{\Phi} A/\varpi)_* = \varprojlim_{\Phi}(A/\varpi)_* = \varprojlim_{\Phi} A_*/\varpi \ ,$$

because $_*$ commutes with inverse limits and using Lemma 3.1. Note that the image of $\Phi : (A/\varpi)_* \to (A/\varpi)_*$ is A_*/ϖ, because it factors over $(A/\varpi^{1/p})_*$, and the image of the projection $(A/\varpi)_* \to (A/\varpi^{1/p})_*$ is $A_*/\varpi^{1/p}$. But

$$\varprojlim_{\Phi} A_*/\varpi = \varprojlim_{x\mapsto x^p} A_* \ ,$$

as in the proof of 2.2.

This shows that $A_*^\flat$ is a $\varpi^\flat$-adically complete flat $K^{\flat\circ}$-algebra. Moreover, the projection $x \mapsto x^\sharp$ of $A_*^\flat$ onto the first component $x^\sharp \in A_*$ induces an isomorphism

$$A_*^\flat/\varpi^\flat \cong A_*/\varpi \ ,$$

Therefore $A^\flat$ is a perfectoid $K^{\flat\circ a}$-algebra.

Then we can easily go through all equivalences to check that $R^\flat$ is the tilt of R. □

Remark 4.3. It follows that the tilting functor is independent of the choice of ϖ and $\varpi^\flat$. We note that this explicit description comes from the fact that the deformation from perfectoid $K^{\flat\circ a}/\varpi^\flat$-algebras to perfectoid $K^{\flat\circ a}$-algebras can be made explicit by means of the inverse limit over the Frobenius.

4.3. **Recall: Witt vector.** In fact, we can write down the functors in both directions "explicitly", without using almost mathematics in the proof of the tilting equivalence. I have learnt the basic story about Witt lifting from Wenwei Li's beautiful book. Witt vector can be used to give an alternate perspective on the tilting corresondence, and explicit description of the untilting functor.

Let R be a perfect ring of characteristic p. Then R is relatively perfect over $\mathbb{F}_p$, then we can construct $W(R)$ as its Witt vectors, which can also be seen as the unique p-adically complete p-torsionfree $\mathbb{Z}_p$-algebra lifting R (combining to the deformation theory, this comes from $L_{R/\mathbb{F}_p} \simeq 0$). Explicitly, one simply sends $r \in R$ to $\tilde{r_n}^{p^n}$, where $\tilde{r_n} \in W_n(R)$ denotes some lift of $r_n = r^{1/p^n}$. The resulting multiplicative maps $R \to W(R)$ are called the Teichmuller lifts and denotes as $r \mapsto [r]$. From the universal property describing $W(R)$, it is clear that if R is f-adically complete for some element $f \in R$, then $W(R)$ is $(p, [f])$-adically complete.

(Fontaine's map θ and A_{inf}) Fix a map $A \to B$ in $\mathcal{C}_A$. One fact is, if $C' \to C$ is surjective with nilpotent kernel, then every A-algebra map $B \to C$ lifts unique to an A-algebra map $B \to C'$ (proved before). In particular, we can use this to get a unique map $W(R) \to C$. In perfectoid theory, this observation shows:

Proposition 4.5 (The kernel of θ)**.** *Given a perfectoid field K, the canonical map $\bar{\theta} : K^{\circ\flat} \to K^\circ/p$ lifts to a unique map $\theta : A_{inf}(K^\circ) := W(K^{\circ\flat}) \to K^\circ$. The kernel of θ is a principal ideal generated by a nonzerodivisor. In fact, for K having characteristic 0, one may choose $\xi \in$ ker(θ) to be any element such that ξ generates the kernel of $K^{\circ\flat} \to K^\circ/p$; when K has characteristic p, we have* ker$(\theta) = (p)$.

Thus we obtain a pushout square:

$$\begin{tikzcd}
A_{inf}(K^\circ) \arrow[r, "\theta"] \arrow[d] & K^\circ \arrow[d] \\
K^{\circ\flat} \arrow[r] & K^{\circ\flat}/t \cong K^\circ/\varpi
\end{tikzcd}$$

In characteristic p, the right vertical map and the bottom horizonal map are isomorphisms, while the remaining two maps coincide with reduction modulo p. In characteristic 0, all maps are quotients by nonzerodivisors along which the source ring is complete. "In particular, in both cases, all rings involved can be viewed as pro-infinitesimal thickenings[1] of K°/p."

In particular, since any relatively perfect (or, equivalentlly, perfect) $K^{\circ\flat}$-algebra A has unique lift $W(A)$ along $A_{inf}(K^\circ) \to K^{\circ\flat}$, and the base change $W(A) \otimes_{A_{inf}(K^\circ)} K^\circ$ provides a K°-algebra that lifts $A \otimes_{K^\circ} K^{\circ\flat}/\varpi$ by the above diagram. From this, for $S \in \mathrm{Perf}_{K^\flat}$, then its untilt $S^\sharp \in \mathrm{Perf}_K$ is given by

$$S^\sharp := (W(S^\circ) \otimes_{A_{inf}(K^\circ)} K^\circ)[\frac{1}{\varpi}]$$

Along our notation, we can prove the following Tilting correspondence for integral perfectoid rings (actually the same thing, maybe the notations are easier):

Theorem 4.4. *Fix a integral perfectoid ring A. Then the tilting induces an equivalence of categories $A-\mathrm{Perf}$ and $A^\flat-\mathrm{Perf}$, with inverse given above.*

Proof. Let $\varpi \in A$ be a perfectoid pseudo-uniformiser admitting p-power-roots, and $\varpi^\flat = (\varpi, \varpi^{1/p}, \dots) \in A^\flat$ the associated perfectoid pseudo-uniformiser of $A^\flat$; also let $\xi = p + \left[\varpi^b\right]^p z \in W(A^\flat)$ be the generator of the ideal $\mathrm{Ker}(\theta : W(A^\flat) \to A)$ which we constructed.

Step 1: Letting B be a perfectoid A-algebra, we show $(B^\flat)^\sharp = B$. We obviously have a commutative diagram (with surjective horizontal arrows)

$$\begin{tikzcd} W(B^\flat) \arrow[r, "\theta_B"] & B \\ W(A^\flat) \arrow[u] \arrow[r, "\theta_A"'] & A \arrow[u] \end{tikzcd}$$

and so the image of ξ in $W(B^\flat)$ lands in $\mathrm{Ker}\,\theta_B$ (this denotes the θ-map for the integral perfectoid ring B). But the first coordinate in the Witt vector expansion of ξ is a unit of $A^\flat$, and so its image in $B^\flat$ is also a unit; therefore Theorem for the ring B implies that $\mathrm{Ker}\,\theta_B = \xi W(A^\flat)$. In other words, the above diagram is a pushout and so the induced map $(B^\flat)^\sharp = W(B^\flat) \otimes_{W(A^\flat),\theta} A \to B$ is an isomorphism, as required.

Step 2: Letting C be a perfectoid A-algebra, we show that $C^\sharp$ is a perfectoid $A^\flat$-algebra and that $(C^\sharp)^\flat = C$. Since θ is surjective with kernel $\xi W(A^\flat)$, we can write $C^\sharp = W(C)/\xi W(C)$ viewed as an A-algebra via the identification $\theta : W(A^\flat)/\xi W(A^\flat) \xrightarrow{\sim} A$. Then shows that $C^\sharp$ is complete for the ϖ-adic topology and that ϖ is a non-zero-divisor of C. It remains to show that $\Phi : C^\sharp/\varpi C^\flat \to C^\sharp/\varpi^p C^\flat$ is an isomorphism. But again writing $C^\sharp = W(C)/\xi W(C)$ and recalling that $\xi \equiv p \bmod [\varpi^b]^p$, this map may be rewritten as $\Phi : C/\varpi^\flat C \to C/\varpi^{\flat p} C$, which is indeed an isomorphism since $\varpi^\flat$ is a perfectoid pseudo-uniformiser of C. This completes the proof that $C^\sharp$ is a perfectoid A-algebra.

Finally, as we already used in the previous paragraph, we have $C^\sharp/\varpi C^\sharp = C/\varpi^\flat C$. Tilting obtains

$$(C^\sharp)^\flat = \varprojlim_{x \mapsto x^p} C^\sharp/\varpi C^\sharp = \varprojlim_{x \mapsto x^p} C/\varpi^\flat C = C^\flat = C,$$

[1]We explain the definition. Let R be a ϖ-complete $\mathbb{Z}_p$-algebra. A surjection $D \to R$ of algebra with kernel I, such that D is $I + (\varpi)$-adically complete is called a ϖ adic pro-infinitesimal thickening of R. A_{inf} is the universal ϖ adic pro-infinitesimal thickening of R in our case.

where the final equality is the fact that tilting an integral perfectoid ring of characteristic p has no effect. □

4.4. **Almost purity (1).** Finally, let us discuss finite étale covers of perfectoid algebras, and finish the proof of Theorem 2.1.

Proposition 4.6. *Let $\overline{A}$ be a perfectoid $K^{\circ a}/\varpi$-algebra, and let $\overline{B}$ be a finite étale $\overline{A}$-algebra. Then $\overline{B}$ is a perfectoid $K^{\circ a}/\varpi$-algebra.*

Notice that (I don't actually know how it comes out, but we will come back later.)

Theorem 4.5 (finite étale covers lift uniquely over nilpotents)**.** *Let A be a $K^{\circ a}$-algebra. Assume that A is flat over $K^{\circ a}$ and ϖ-adically complete, i.e.*

$$A \cong \varprojlim A/\varpi^n \ .$$

Then the functor $B \mapsto B \otimes_A A/\varpi$ induces an equivalence of categories $A_{\text{fét}} \cong (A/\varpi)_{\text{fét}}$. Any $B \in A_{\text{fét}}$ is again flat over $K^{\circ a}$ and ϖ-adically complete. Moreover, B is a uniformly finite projective A-module if and only if $B \otimes_A A/\varpi$ is a uniformly finite projective A/ϖ-module.

This provides the following commutative diagram: for $R, A = R^{\circ a}, \overline{A}, A^\flat$ and $R^\flat$ form a sequence of rings under the tilting procedure.

$$\begin{tikzcd}
R_{\text{fét}} & A_{\text{fét}} \arrow[l] \arrow[r, "\cong"] \arrow[d] & \overline{A}_{\text{fét}} \arrow[d] & A^\flat_{\text{fét}} \arrow[l, "\cong"'] \arrow[r] \arrow[d] & R^\flat_{\text{fét}} \\
K\text{-Perf} & K^{\circ a}\text{-Perf} \arrow[l] \arrow[r] & K^{\circ a}/\varpi - \text{Perf} \arrow[r] & K^{\flat\circ a} - \text{Perf} \arrow[r] & K^\flat - \text{Perf}
\end{tikzcd}$$

It follows from this diagram that the functors $A_{\text{fét}} \to R_{\text{fét}}$ and $A^\flat_{\text{fét}} \to R^\flat_{\text{fét}}$ are fully faithful. Then we want to show that both of them are equivalences, this involves Falting's almost purity theorem. At first we want to introduce the trace map, this helps us to prove the characteristic p case.

Define the trace map (the idea is: trace morphism is an equivalent characterization of finite étale morphisms): let R be a ring, and let $R \to S$ be a finite étale extension. Morphism $s \mapsto [s' \mapsto ss']$ induces $S \hookrightarrow \text{End}_R(S)$, since S is finite flat and R is noetherian, the R module S is locally free R-module and multiplication os therefore locally given by multiplication by a matrix. We define the trace as the trace of the matrix. Let $e \in S \otimes_R S$ be the diagonal idempotent cutting out the multiplication $\mu : S \otimes_R S \to S$, i.e., $e^2 = e$ and $\ker(\mu) = 0$. Write $e = \sum_{i=1}^n a_i \otimes b_i$ for $a_i, b_i \in S$. Then we can explicitly realize S is a direct summand of R^n via the maps

$$S \xrightarrow{\alpha} R^n \xrightarrow{\beta} S$$

given by

$$\alpha(f) = \left(\text{Tr}_{S/R}(fa_i)\right) \quad \text{and} \quad \beta((g_i)) = \sum_{i=1}^n g_i b_i.$$

First we must show that $\beta \circ \alpha = id$. In other words, we want to check that

$$\sum_{i=1}^n \text{Tr}_{S/R}(fa_i)\, b_i = f$$

for any $f \in S$. To prove this, note that

$$\mathrm{Tr}_{i_2}(e) = \mathrm{Tr}_{S/S}(1) = 1$$

where $i_2 : S \to S \otimes_R S$ is the second inclusion $s \mapsto 1 \otimes s$. Plugging in $e = \sum_i a_i \otimes b_i$ above and using the compatibility of trace maps with base change, we get

$$\sum_{i=1}^{n} \mathrm{Tr}_{S/R}(a_i)\, b_i = 1$$

In particular, this verifies the formula for $f = 1$. In general, one repeats the same argument by replacing e with $(f \otimes 1) \cdot e$ (which equals $(1 \otimes f) \cdot e$ as $\ker(\mu) \cdot e = 0$):

$$\sum_{i=1}^{n} \mathrm{Tr}_{S \otimes_S / R}(f a_i) b_i = \mathrm{Tr}_{S \otimes_R S/S}((f \otimes 1)e) = \mathrm{Tr}_{S \otimes_R S/S}((1 \otimes f)e) = f\, \mathrm{Tr}_{S \otimes_R S/S}(e) = f$$

Using this construction (Finite étale algebras explicitly as finite projective modules), we arrive at the almost purity theorem in characteristic p:

Theorem 4.6 (Almost purity in char p: primitive version.)**.** *Let $\eta : R \to S$ be an integral map of perfect rings. Assume that $\eta[\frac{1}{t}]$ is finite étale for some $t \in R$. Then η is almost finite étale with respect to the ideal $I = (t^{\frac{1}{p^\infty}})$.*

"In other words, the assumption that $\eta[\frac{1}{t}]$ on the "generic fibre" spreads out to the conclusion that η is almost finite etale on the "almost integral fibre"."

Proof. Sketch:

(i) We first reducing to t-torsionfree case by observating that the t-power torsion ideals in R and S are almost zero.

(ii) Then we reduce to the case where R is integrally closed in $R[\frac{1}{t}]$, also for S respectively. This follows from $R \to R_{int}$, where R_{int} is the integral closure of R in $R[\frac{1}{t}]$, is an almost isomorphism.

(iii) Next we check almost unramifiedness.

(iv) It remains to show that S is almost finite projective over R. This follows from the realization of "finite étale" into "trace map": Since there is an integer $m \geq 0$ such that $t^{\frac{1}{p^m}} \cdot e \in (S \otimes_R S)[\frac{1}{t}]$ as an element $\sum_{i=1}^{n} a_i \otimes b_i \in S \otimes_R S$, we have an equality of map $\beta \circ \alpha = t^{\frac{1}{p^m}}$. Here we need all the reductions/assumptions. In particular,multiplication by $t^{\frac{1}{p^m}}$ on S factors through a finite free R-module. This shows almost finite projective property.

□

Then we can upgrade this to an equivalence of categories.

Theorem 4.7. *Let R be a perfect ring of characteristic p, and consider almost mathematics with respect to $I = (t^{\frac{1}{p^\infty}})$ for a fixed element $t \in R$. Then $S \mapsto S_*[\frac{1}{t}]$ gives an equivalence of categories $R^a_{\text{fét}} \cong R[\frac{1}{t}]_{\text{fét}}$.*

Proof. (i) As the proof before, we may assume R has no t-torsion (t is a nonzero-divisor on A).

(ii) Since S_* is finite étale, $(S \otimes_R S)_*[\frac{1}{t}] \cong (S_* \otimes_R S_*)[\frac{1}{t}]$ gives the idempotent. Almost projective also can be checked easily.

(iii) perfectness induces essential suerjectivity, by the above theorem.

(iv) For faithfulness, fix some $S \in R^a_{\text{fét}}$. We claim:

$$S \cong T^a \text{ for the integral closure } T \text{ of } R \text{ in } S_*[\frac{1}{t}].$$

To claim this, we need the following lemma.

Lemma 4.3. *Let $A \to B$ be weakly étale map of $\mathbb{F}_p$-algebra, i.e., both $A \to B$ and $\mu : B \otimes_A B \to B$ are flat. Then the diagram*

$$\begin{tikzcd}
A \arrow[r, "Frob_A"] \arrow[d] & A \arrow[d] \\
B \arrow[r, "Frob_B"] & B
\end{tikzcd}$$

is a pushout square of rings, i.e., the relative Frobenius $F_{B/A} : B \otimes_{A, Frob_A} B \to B$ is an isomorphism. In particular, if A is perfect, so is B.

Proof. remaining to explain.

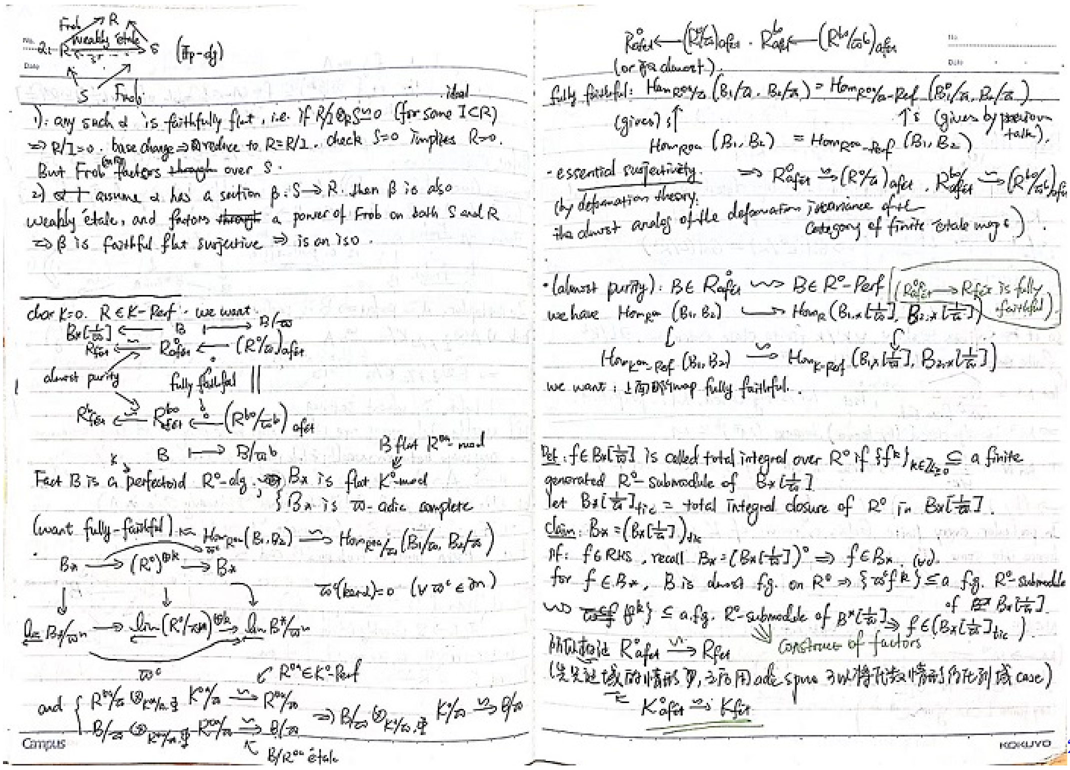
$$\begin{tikzcd}
A \arrow[rr, "Frob_A"] \arrow[dd] & & A \arrow[rr, "Frob_A"] \arrow[d] & & A \arrow[dr] & \\
& & B \otimes_{A, Frob_A} A \arrow[rrr, "Frob"] \arrow[drr, dashed, "F_{B/A}"] & & & B \otimes_{A, Frob_A} A \\
B \arrow[urr] \arrow[rrrr, "Frob_B"] & & & & B \arrow[ur] &
\end{tikzcd}$$

So it suffices to show that any weakly étale map $\alpha : R \to S$ of $\mathbb{F}_p$-algebra that factors a power of Frobenius on R and S is an isomorphism. □

Since almost étale maps are easily seen to be weekly étale maps, S is perfect, so as S_*. Moreover, as S is almost flat over R, the element $t \in S_*$ is a nonzerodivisor, so $S_* \subset S_*[\frac{1}{t}]$. Since T is perfect with a nonzerodivisor t, and $R \to T$ is an integral extension of perfect rings that is identified with $R[\frac{1}{t}] \to S_*[\frac{1}{t}]$ on inverting t. Then we shall check $T_* = S_*$. This is standard by checking all the definitions. The lemma will reovers S functorially from the map $R \to S_*[\frac{1}{t}]$. □

This provides us to upgrade our diagram: $A^\flat_{\text{fét}} \cong R^\flat_{\text{fét}}$ (where $A = R^{\circ a}$ as usual). To summarize, we (want to) have the following theorem:

Theorem 4.8. *Let R be a perfectoid K-algebra with tilt $R^\flat$. There is a fully faithful functor from $R^\flat_{\text{fét}}$ to $R_{\text{fét}}$ inverse to the tilting functor. The essential image of this functor consists of the finite étale covers S of R, for which S (with its natural topology) is perfectoid and $S^{\circ a}$ is finite étale over $R^{\circ a}$. In this case, $S^{\circ a}$ is a uniformly finite projective $R^{\circ a}$-module.*


(almost purity): $B \in R^{\circ}_{afet} \rightsquigarrow B \in R^{\circ}\text{-Perf}$

Def: $f \in B_*[\frac{1}{\varpi}]$ is called total integral over R° if $\{f^k\}_{k \in \mathbb{Z}_{\geq 0}} \subseteq$ a finite generated R°-submodule of $B_*[\frac{1}{\varpi}]$.

Let $B_*[\frac{1}{\varpi}]_{tic}$ = total integral closure of R° in $B_*[\frac{1}{\varpi}]$.

Claim: $B_* = (B_*[\frac{1}{\varpi}])_{tic}$

[2]

We will later prove that this is an equivalence in general (characteristic case/algebra). For now, we prove that it is an equivalence for perfectoid fields, i.e. we finish the proof of Theorem 2.1.

Proof. Let K be a perfectoid field, characteristic 0, with tilt $K^\flat$. It is enough to show that the fully faithful functor $\sharp : K^\flat_{\text{fét}} \to K_{\text{fét}}$ is an equivalence. This functor is fully faithful (as this true always), perserve degrees (by constructions). Let $M = \widehat{\overline{K^\flat}}$ be the completion of an algebraic closure of $K^\flat$. M is complete and perfect, i.e. M is perfectoid. Let $M^\sharp$ be the untilt of M. Then by Lemma 2.2, $M^\sharp$ is an algebraically closed perfectoid field containing K (Actually for all $R \in K$-Perf with tilt $R^\flat$, R is a perfectoid field iff $R^\flat$ is a perfectoid field). Any finite extension $L \subset M$ of $K^\flat$ gives the untilt $L^\sharp \subset M^\sharp$, a finite extension of K. It is easy to see that the union $N = \bigcup_L L^\sharp \subset M^\sharp$ is a dense subfield. Now Krasner's lemma implies that N is algebraically closed. Hence any finite extension F of K is contained in N; this means that there is some Galois extension L of $K^\flat$ such that F is contained in $L^\sharp$. Note that $L^\sharp$ is still Galois, as the functor $L \mapsto L^\sharp$ preserves degrees and automorphisms. In particular, F is given by some subgroup H of $\mathrm{Gal}(L^\sharp/K) = \mathrm{Gal}(L/K^\flat)$, which gives the desired finite extension $F^\flat = L^H$ of $K^\flat$ that untilts to F: The equivalence of categories shows that $(F^\flat)^\sharp \subset (L^\sharp)^H = F$, and as they have the same degree, they are equal. $\square$

Remark 4.4. Recall the Krasner's lemma: Let $(K, |\cdot|)$ be a complete non-archimedean valuation field. Let $\alpha, \beta \in \overline{K}$ such that $|\alpha - \beta| < |\alpha - \alpha'|$ for all Galois conjugate α' of α different from α. Then $\alpha \in K(\beta)$.

In the end of this section, we offer some "basic examples" for tilting process.

[2]This is what a equivalence of category actually needs... Too strict.

Example 4. Let

$$R = K\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle = K^\circ[\widehat{T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}}][\varpi^{-1}] .$$

Then R is a perfectoid K-algebra, and its tilt $R^\flat$ is given by $K^\flat\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle$.

Perfectoid fields are identified while tilting.

Lemma 4.4. *Let R be a perfectoid K-algebra with tilt $R^\flat$. Then R is a perfectoid field if and only if $R^\flat$ is a perfectoid field.*

Proof. Note that R is a perfectoid field if and only if it is a nonarchimedean field, i.e. its topology is induced by a rank-1-valuation. This valuation is necessarily given by the spectral norm

$$||x||_R = \inf\{|t|^{-1} \mid t \in K^\times, tx \in R^\circ\}$$

on R. It is easy to check that for $x \in R^\flat$, we have $||x||_{R^\flat} = ||x^\sharp||_R$. In particular, if $||\cdot||_R$ is multiplicative, then so is $||\cdot||_{R^\flat}$, i.e. if R is a perfectoid field, then so is $R^\flat$.

Conversely, assume that $R^\flat$ is a perfectoid field. We have to check that the spectral norm $||\cdot||_R$ on R is multiplicative.

To see that R is a field, choose x such that $x \in R^\circ$, but not in ϖR°, and take $x^\flat$ as before. Then by multiplicativity of $||\cdot||_R$, $||1 - \frac{x}{(x^\flat)^\sharp}||_R < 1$, and hence $\frac{x}{(x^\flat)^\sharp}$ is invertible, and then also x. □

5. L8-17: Perfectoid spaces

Recall in algebraic geometry, a scheme is glued by affine schemes the locally ringed spaces which are isomrophic to spectrums of some rings A. We call such local space model space. In rigid geometry, the model space we study is $B^n(\bar{K}) := \operatorname{Sp} K\langle T_1, \cdots, T_n\rangle = \{(x_1, \cdots, x_n) \in \bar{K}^n, |x_i| \leq 1\}$ where K is a non-archimedean field with valuation $|\cdot| : K \to \mathbb{R}_{\geq 0}$. We call restricted series like

$$K\langle T_1, \cdots, T_n\rangle = \left\{\sum_v c_v T^v \in k[[T_1, \cdots, T_n]]\,; \lim_{v\to\infty} |c_v| = 0\right\}$$

Tate algebra and call the maximal spectrum $\operatorname{Sp} K\langle T_1, \cdots, T_n\rangle$ of some Tate algebra *affinoid space*. A motivation to construct rigid space by glueing affinoid space is finding a suitable geometrical object associated with the sheaf of power series. However, affinoid space is totally disconnected, which means we cannot give any restrictions on the global sections. The solution of Tate is to replace Zariski topology by Grothendieck topology $(\mathscr{C}, \operatorname{Cov}\mathscr{C})$, where $\mathscr{C}$ is the category of admissible open subsets of $X = \operatorname{Sp} K\langle T_1, \cdots, T_n\rangle$ and $\operatorname{Cov}\mathscr{C}$ contains of the families of morphisms $\{U_i \to U\}_{i\in I}$ in $\mathscr{C}$ with some finiteness condition.

As a topological space, X is still not good enough. For example, when $n = 1$, we can cover X by open subsets $U_1 = \bigcup_{\epsilon<1}\{x \in X; |T|_x \leq 1\}$ and $U_2 = \{x \in X; |T|_x = 1\}$, i.e., X is not connected. To solve this, Huber adds more points which are out of U_i and gives a new structure named *adic space*. Just as mentioned before, we focus on the model space of adic space first. It also has a corrospending algebraic object. One sees immediately that the explicit description of the tilting corresondence involves p-adic limits, so a formalization will necessarily need to use some framework of rigid geometry.

5.1. **Adic spaces.** Recall the definition 2.2. If $|\cdot| : R \to \Gamma \cup \{0\}$ is a valuation on R, let $\Gamma_{|\cdot|} \subset \Gamma$ denote the subgroup generated by all $|x|$, $x \in R$, which are nonzero. The set $\operatorname{supp}(|\cdot|) = \{x \in R \mid |x| = 0\}$ is a prime ideal of R called the support of $|\cdot|$. Let K be the quotient field of $R/\operatorname{supp}(|\cdot|)$. Then the valuation factors as a composite $R \to K \to \Gamma \cup \{0\}$. Let $R(|\cdot|) \subset K$ be the valuation subring, i.e. $R(|\cdot|) = \{x \in K \mid |x| \le 1\}$.

Definition 5.1. Two valuations $|\cdot|$, $|\cdot|'$ are called equivalent if the following equivalent conditions are satisfied.

(i) There is an isomorphism of totally ordered groups $\alpha : \Gamma_{|\cdot|} \cong \Gamma_{|\cdot|'}$ such that $|\cdot|' = \alpha \circ |\cdot|$.

(ii) The supports $\operatorname{supp}(|\cdot|) = \operatorname{supp}(|\cdot|')$ and valuation rings $R(|\cdot|) = R(|\cdot|')$ agree.

(iii) For all $a, b \in R$, $|a| \ge |b|$ if and only if $|a|' \ge |b|'$.

For K a complete non-archimedean field, $\varpi \in K^\circ \subset K$,

Definition 5.2. (i) A Tate k-algebra is a topological k-algebra R for which there exists a subring $R_0 \subset R$ such that aR_0, $a \in k^\times$, forms a basis of open neighborhoods of 0. A subset $M \subset R$ is called bounded if $M \subset aR_0$ for some $a \in k^\times$. An element $x \in R$ is called power-bounded if $\{x^n \mid n \ge 0\} \subset R$ is bounded. Let $R^\circ \subset R$ denote the subring of powerbounded elements.

(ii) An affinoid k-algebra is a pair (R, R^+) consisting of a Tate k-algebra R and an open and integrally closed subring $R^+ \subset R^\circ$.

(iii) An affinoid k-algebra (R, R^+) is said to be of topologically finite type (tft for short) if R is a quotient of $k\langle T_1, \dots, T_n\rangle$ for some n, and $R^+ = R^\circ$.

Here,

$$k\langle T_1, \dots, T_n\rangle = \{ \sum_{i_1,\dots,i_n \ge 0} x_{i_1,\dots,i_n} T_1^{i_1} \cdots T_n^{i_n} \mid x_{i_1,\dots,i_n} \in k, x_{i_1,\dots,i_n} \to 0\}$$

is the ring of convergent power series on the ball given by $|T_1|, \dots, |T_n| \le 1$.

Remark 5.1. One fact is R° is an open integrally closed subring of R. Another fact is any Tate k-algebra R, resp. affinoid k-algebra (R, R^+), admits the completion $\hat{R}$, resp. $(\hat{R}, \hat{R}^+)$, which is again a Tate, resp. affinoid, k-algebra. Everything "depends" only on the completion, so one may simply assume that (R, R^+) is complete in the following.

Definition 5.3. Let (R, R^+) be an affinoid k-algebra. Let

$$X = \operatorname{Spa}(R, R^+) = \{|\cdot| : R \to \Gamma \cup \{0\} \text{ continuous valuation} \mid \forall f \in R^+ : |f| \le 1\}/\cong .$$

For any $x \in X$, write $f \mapsto |f(x)|$ for the corresponding valuation on R. We equip X with the topology which has the open subsets

$$U(\frac{f_1, \dots, f_n}{g}) = \{x \in X \mid \forall i : |f_i(x)| \le |g(x)|\} ,$$

called rational subsets, as basis for the topology (check it: closed under finite intersection), where $f_1, \dots, f_n \in R$ generate R as an ideal and $g \in R$.

The following remark is quite useful for calculation.

Remark 5.2. Let $\varpi \in k$ be topologically nilpotent, i.e. $|\varpi| < 1$. Then to $f_1, \ldots, f_n$ one **can add** $f_{n+1} = \varpi^N$ for some big integer N without changing the rational subspace. Indeed, there are elements $h_1, \ldots, h_n \in R$ such that $\sum h_i f_i = 1$. Multiplying by ϖ^N for N sufficiently large, we have $\varpi^N h_i \in R^+$, as $R^+ \subset R$ is open. Now for any $x \in U(\frac{f_1,\ldots,f_n}{g})$, we have

$$|\varpi^N(x)| = |\sum(\varpi^N h_i)(x) f_i(x)| \leq \max |(\varpi^N h_i)(x)||f_i(x)| \leq |g(x)| \ ,$$

as desired. In particular, we see that on rational subsets, $|g(x)|$ is nonzero, and bounded from below.

Remark 5.3. Let $|f| = 0$, then $U(\frac{f}{g}) = \{| \cdot | \mid |g| \neq 0\}$. Hence, the topology on $\mathrm{Spa}(R, R^+)$ is a refinement of Zariski topology. In fact, we can write

$$U(\frac{f}{g}) = \cup_n U(\frac{f, \varpi^n}{g}),$$

as $|g| \neq 0$ implies that $|g(x)| > |t^n(x)|$ for $n \gg 0$. This expresses the open set on the left as a (typically infinite) union of the opens on the right.

Recall that a topological space X is quasicompact if every open covering of X has a finite cover. X is called quasiseparated if the intersection of any two quasicompact open subsets is again quasicompact. In the following we will often abbreviate quasicompact, resp. quasiseparated, as qc, resp. qs.

Definition 5.4. A topological space X is called spectral if it satisfies the following equivalent properties.

(i) There is some ring A such that $X \cong \mathrm{Spec}(A)$.

(ii) One can write X as an inverse limit of finite T_0 spaces.

(iii) The space X is quasicompact, has a basis of quasicompact open subsets stable under finite intersections, and is sober (i.e. every irreducible closed subset admits a unique generic point).

In particular, spectral spaces are quasicompact, quasiseparated and T_0.

Huber proved the following basic properties:

Proposition 5.1. *For any affinoid k-algebra (R, R^+), the space $\mathrm{Spa}(R, R^+)$ is spectral. The rational subsets form a basis of quasicompact open subsets stable under finite intersections. Thus the construction $(A, A^+) \to \mathrm{Spa}(A, A^+)$ naturally defines a functor from affinoid Tate rings to spectral spaces, i.e., pulling back valuations along a map of affinoid Tate rings gives rise to a spectral map on adic spectra.*

Proposition 5.2. *Let (R, R^+) be an affinoid k-algebra with completion $(\hat{R}, \hat{R}^+)$. Then $\mathrm{Spa}(R, R^+) \cong \mathrm{Spa}(\hat{R}, \hat{R}^+)$, identifying rational subsets.*

Proposition 5.3. *Let (R, R^+) be an affinoid k-algebra, $X = \mathrm{Spa}(R, R^+)$.*

(i) *If $X = \emptyset$, then $\hat{R} = 0$.*

(ii) *Let $f \in R$ be such that $|f(x)| \neq 0$ for all $x \in X$. If R is complete, then this holds if and only if f is invertible.*

(iii) *Let $f \in R$. Then $f \in R^+$ if and only if $|f(x)| \leq 1$ for all $x \in X$.*

(iv) *Let $f \in R$. Then f is nilpotent if and only if $|f(x)|^n \to 0$ for all $x \in X$.*

Proof. It suffices to show that if $\hat{R} \neq 0$, then there exist a continuous valuation of $\hat{R}$. Take an element ϖ which is not invertible, then there exists an prime ideal $\mathfrak{m} \subset R^+$, such that $\varpi \in \mathfrak{m}$, $(R^+)_{\mathfrak{m}}[\frac{1}{\varpi}] \neq 0$.

Take an nonzero prime ideal in $(R^+)_{\mathfrak{m}}[\frac{1}{\varpi}]$, then find its pullback $\mathfrak{p} \subset \mathfrak{m} \subset R^+$. Note that ϖ will not contained in $\mathfrak{p}$.

Recall that the valuation rings of a field are the maximal elements of the set of the local subrings in the field partially ordered by dominance or refinement, where

$$(A, \mathfrak{m}_A) \text{ dominates } (B, \mathfrak{m}_B) \text{ if } B \subset A \text{ and } \mathfrak{m}_A \cap B = \mathfrak{m}_B.$$

By Zorn lemma, we have the following **fact**: there exist a valuation ring, write as $(R_{\mathfrak{p}}, \mathfrak{m}')$, we have

$$R^+/\mathfrak{p} \subset R_{\mathfrak{p}} \subset \mathrm{Frac}(R^+/\mathfrak{p}), \text{ such that } \exists \mathfrak{m}' \subset R_{\mathfrak{p}},\ \mathfrak{m}' \cap R^+/\mathfrak{p} = \mathfrak{m}$$

We can choose the related valuation. It will be a continuous valuation.

For (ii), use the fact for $(\hat{R}/f, \hat{R^+}/f)$. ☐

We want to endow $X = \mathrm{Spa}(R, R^+)$ with a structure sheaf $\mathcal{O}_X$. The construction is as follows.

Remark 5.4 (on the completion progress).

Definition 5.5. We say that a Huber ring R is *complete* if and only if R is complete and Hausdorff in the usual topological sense.

In general, Let $\hat{R}$ be the Hausdorff completion of R. (Recall: if X is a $C2$ topological space, then we may define $\hat{R} = R^{\mathbb{N}}/\{\text{Cauchy sequences}\}$ to be its Hausdorff completion.)

In practice, it is more convenient to define $\hat{R}$ algebraically as $\hat{R} = \hat{R}_0 \otimes_{R_0} R$, where $\hat{R}_0 := \varprojlim_r R_0/I^r$ is the I-adic completion of a chosen subring of definition R_0 with respect to an ideal of definition I. The next proposition shows that this process really gives the same result:

Proposition 5.4. *$\hat{R}$ is a Huber ring. Moreover, letting $I \subset R_0 \subset R$ be any ideal and subring of definition:*

(i) *$I\hat{R}_0 \subset \hat{R}_0$ are an ideal and subring of definition of $\hat{R}_0$, where $\hat{R}_0 = \varprojlim_n R_0/I^n$ is the I-adic completion of R_0;*

(ii) *the canonical map $\hat{R}_0 \otimes_{R_0} R \to \hat{R}$ is an isomorphism.*

Finally, for the subring of integral elements in R, its Hausdorff completion is a subring of integral elements in $\hat{R}$, where $(\hat{R}, \hat{R^+})$ is a Huber pair (which is called the completion of the former huber pair).

Another description is, the completion of a Huber pair obviously satisfies the following universal property:

Corollary 5.1. *Let (R, R^+) be a Huber pair. Then $(R, R^+) \to (\hat{R}, \hat{R^+})$ is an adic morphism with the following universal property: given any complete Huber pair (S, S^+) and morphism $\phi : (R, R^+) \to (S, S^+)$, there exists a unique morphism $\hat{\phi} : (\hat{R}, \hat{R^+}) \to (S, S^+)$ such that*

$$\begin{tikzcd} (R, R^+) \arrow[r] \arrow[rr, bend right, "\phi"'] & (\hat{R}, \hat{R^+}) \arrow[r, "\hat{\phi}"] & (S, S^+) \end{tikzcd}$$

commutes.

Definition 5.6. Let (R, R^+) be an affinoid k-algebra, and let $U = U(\frac{f_1,\ldots,f_n}{g}) \subset X = \mathrm{Spa}(R, R^+)$ be a rational subset. Choose some $R_0 \subset R$ such that aR_0, $a \in k^\times$, is a basis of open neighborhoods of 0 in R. Consider the subalgebra $R[\frac{f_1}{g},\ldots,\frac{f_n}{g}]$ of $R[g^{-1}]$, and equip it with the topology making $aR_0[\frac{f_1}{g},\ldots,\frac{f_n}{g}]$, $a \in k^\times$, a basis of open neighborhoods of 0. Let $B \subset R[\frac{f_1}{g},\ldots,\frac{f_n}{g}]$ be the integral closure of $R^+[\frac{f_1}{g},\ldots,\frac{f_n}{g}]$ in $R[\frac{f_1}{g},\ldots,\frac{f_n}{g}]$. Then $(R[\frac{f_1}{g},\ldots,\frac{f_n}{g}], B)$ is an affinoid k-algebra. Let $(R\langle\frac{f_1}{g},\ldots,\frac{f_n}{g}\rangle, \hat{B})$ be its completion.

Obviously,

$$\mathrm{Spa}(R\langle\frac{f_1}{g},\ldots,\frac{f_n}{g}\rangle, \hat{B}) \to \mathrm{Spa}(R, R^+)$$

factors over the open subset $U \subset X$.

Proposition 5.5. *In the situation of the definition, the following universal property is satisfied. For every complete affinoid k-algebra (S, S^+) with a map $\iota : (R, R^+) \to (S, S^+)$ such that the induced map $\mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ factors over U, there is a unique map*

$$(R\langle\frac{f_1}{g},\ldots,\frac{f_n}{g}\rangle, \hat{B}) \to (S, S^+)$$

making the obvious diagram commute.

In particular, $(R\langle\frac{f_1}{g},\ldots,\frac{f_n}{g}\rangle, \hat{B})$ depends only on U. Define

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (R\langle\frac{f_1}{g},\ldots,\frac{f_n}{g}\rangle, \hat{B}) \ .$$

For example, $(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ is the completion of (R, R^+).

Proof. Since there exists N. s.t. $|g(x)| \leq |\varpi^N(x)|, \forall x \in U$, thus for any $x \in \mathrm{Spa}(S, S^+)$, $|\iota(g)(x)| \neq 0$, i.e. $\iota(g)$ is invertible. Thuen $\iota(\frac{f_i}{g}) \in S^+$. ☐

We define presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on X as above on rational subsets, and for general open $U \subset X$ by requiring

$$\mathcal{O}_X(W) = \varprojlim_{U \subset W \text{ rational}} \mathcal{O}_X(U) \ ,$$

and similarly for $\mathcal{O}_X^+$.

Proposition 5.6. *For any $x \in X$, the valuation $f \mapsto |f(x)|$ extends to the stalk $\mathcal{O}_{X,x}$, and*

$$\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x} \mid |f(x)| \leq 1\} \ .$$

The ring $\mathcal{O}_{X,x}$ is a local ring with maximal ideal given by $\{f \mid |f(x)| = 0\}$. The ring $\mathcal{O}_{X,x}^+$ is a local ring with maximal ideal given by $\{f \mid |f(x)| < 1\}$. Moreover, for any open subset $U \subset X$,

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \mid \forall x \in U : |f(x)| \leq 1\} \ .$$

If $U \subset X$ is rational, then $U \cong \mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ compatible with rational subsets, the presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$, and the valuations at all $x \in U$ (use the universal proerty 5.5 to check).

It is not in general that $\mathcal{O}_X$ is a sheaf. The proposition ensures that $\mathcal{O}_X^+$ is a sheaf if $\mathcal{O}_X$ is. “The basic problem is that completion behaves in general badly for nonnoetherian rings”.

Let’s give two example: for k a complete and algebraically closed field, we are going to describe $\mathrm{Spa}(k, k^\circ)$ and $\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$, which is an affinoid k-algrebra of tft. I’ll highly recommend this note for details (thanks leader Li again).

It is easy to see that the space $\mathrm{Spa}(k, k^\circ)$ consist of a single point. For $\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$, here is the classification:

(1) The classical points: Let $x \in k^\circ$, i.e. $x \in k$ with $|x| \leq 1$. Then for any $f \in k\langle T\rangle$, we can evaluate f at x to get a map $R \to k$, $f = \sum a_n T^n \mapsto \sum a_n x^n$. Composing with the norm on k, one gets a valuation $f \mapsto |f(x)|$ on R, which is obviously continuous and ≤ 1 for all $f \in R^+$.

(2), (3) The rays of the tree: Let $0 \leq r \leq 1$ be some real number, and $x \in k^\circ$. Then

$$f = \sum a_n (T-x)^n \mapsto \sup |a_n| r^n = \sup_{y \in k^\circ : |y-x| \leq r} |f(y)|$$

defines another continuous valuation on R which is ≤ 1 for all $f \in R^+$. It depends only on the disk $D(x, r) = \{y \in k^\circ \mid |y - x| \leq r\}$. If $r = 0$, then it agrees with the classical point corresponding to x. For $r = 1$, the disk $D(x, 1)$ is independent of $x \in k^\circ$, and the corresponding valuation is called the Gaußpoint.

If $r \in |k^\times|$, then the point is said to be of type (2), otherwise of type (3). Note that a branching occurs at a point corresponding to the disk $D(x, r)$ if and only if $r \in |k^\times|$, i.e. a branching occurs precisely at the points of type (2).

(4) Dead ends of the tree: Let $D_1 \supset D_2 \supset \dots$ be a sequence of disks with $\bigcap D_i = \emptyset$. Such families exist if k is not spherically complete, e.g. if $k = \mathbb{C}_p$. Then

$$f \mapsto \inf_i \sup_{x \in D_i} |f(x)|$$

defines a valuation on R, which again is ≤ 1 for all $f \in R^+$.

(5) Finally, there are some valuations of rank 2 which are only seen in the adic space. Let us first give an example, before giving the general classification. Consider the totally ordered abelian group $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, where we require that $r < \gamma < 1$ for all real numbers $r < 1$. It is easily seen that there is a unique such ordering. Then

$$f = \sum a_n (T-x)^n \mapsto \max |a_n| \gamma^n$$

defines a rank-2-valuation on R. This is similar to cases (2), (3), but with the variable r infinitesimally close to 1. One may check that this point only depends on the disc $D(x, <1) = \{y \in k^\circ \mid |y - x| < 1\}$.

Similarly, take any $x \in k^\circ$, some real number $0 < r < 1$ and choose a sign $? \in \{<, >\}$. Consider the totally ordered abelian group $\Gamma_{?r} = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, where $r' < \gamma < r$ for all real numbers $r' < r$ if $? =<$, and $r' > \gamma > r$ for all real numbers $r' > r$ if $? =>$. Then

$$f = \sum a_n (T-x)^n \mapsto \max |a_n| \gamma^n$$

defines a rank 2-valuation on R. If $? =<$, then it depends only on $D(x, < r) = \{y \in k^\circ \mid |y - x| < r\}$. If $? =>$, then it depends only on $D(x, r)$.

Remark 5.5. In fact, for any v a rank 1 valuation with generic support, the valuation ring R_v satisfies $R_v \cap k = k^\circ$. In particular, the residue field $\kappa(v)$ of R_v is natually an extension of the residue field κ of k, and the value group contains the value group of $(\kappa, |\cdot|)$ as an ordered group. In our case (k) is algebraic closed, $\kappa(v) = \kappa$ when v is a type (3) point, and $\kappa(v)$ is the transcendental extension of κ when v is a type (2) point.

Remark 5.6. All points except those of type (2) are closed. Let κ be the residue field of k. Then the closure of the Gaußpoint is exactly the Gaußpoint together with the points of type (5) around it, and is homeomorphic to $\mathbb{A}^1_\kappa$, with the Gaußpoint as the generic point. At the other points of type (2), one gets $\mathbb{P}^1_\kappa$.

Let (R, R^+) be an affinoid k-algebra, and let $X = \mathrm{Spa}(R, R^+)$. We need not assume that $\mathcal{O}_X$ is a sheaf in the following. For any $x \in X$, we let $k(x)$ be the residue field of $\mathcal{O}_{X,x}$, and $k(x)^+ \subset k(x)$ be the image of $\mathcal{O}^+_{X,x}$. We have the following crucial property:

Proposition 5.7. *Let $\varpi \in k$ be topologically nilpotent. Then the ϖ-adic completion of $\mathcal{O}^+_{X,x}$ is equal to the ϖ-adic completion $\widehat{k(x)}^+$ of $k(x)^+$.*

Proof. It is enough to note that kernel of the map $\mathcal{O}^+_{X,x} \to k(x)^+$, which is also the kernel of the map $\mathcal{O}_{X,x} \to k(x)$, is ϖ-divisible. □

Definition 5.7. An affinoid field is pair (K, K^+) consisting of a nonarchimedean field K and an open valuation subring $K^+ \subset K^\circ$.

Remark 5.7. The completion of an affinoid field is again an affinoid field. Also note that the affinoid fields for which $k \subset K$ are affinoid k-algebra.

Here are some basic geometric properties of affinoid k-algebra (also for affinoid Tate ring).

Proposition 5.8. *Let (R, R^+) be an affinoid k-algebra. The points of* $\mathrm{Spa}(R, R^+)$ *are in bijection with maps $(R, R^+) \to (K, K^+)$ to complete affinoid fields (K, K^+) such that the quotient field of the image of R in K is dense.*

Definition 5.8. For two points x, y in some topological space X, we say that x specializes to y (or y generalizes to x), written $x \succ y$ (or $y \prec x$), if y lies in the closure $\overline{\{x\}}$ of x.

Proposition 5.9. *Let (R, R^+) be an affinoid k-algebra, and let $x, y \in X =$* $\mathrm{Spa}(R, R^+)$ *correspond to maps $(R, R^+) \to (K, K^+)$, resp. $(R, R^+) \to (L, L^+)$. Then $x \succ y$ if and only if $K \cong L$ as topological R-algebras and $L^+ \subset K^+$.*

Proof. If part is obvious. Conversely, If $|g(y)| \neq 0$, then we have $|g(y)| \geq |\varpi^N(y)|$ for some N. Since y lies in the closure $\overline{\{x\}}$, $x \in U(\frac{\varpi^N}{g})$ as well. If $|f(x)| \neq 0$ and $f(y) = 0$, Take N such that $|g(x)| \geq |\varpi^N(x)|$. However, $y \in U(\frac{f, \varpi^N}{\varpi^N})$ but x does not. This shows that $\mathrm{supp}(x) = \mathrm{supp}(y)$. Thus $K \cong L$, $L^+ \subset K^+$. □

Lemma 5.1. *For a constructible closed set $T \subset X$, the interior is exactly those $t \in T$ such that all generalizations of t in X are also obtained in T. Moreover, this set is closed under specializations in X. Furthermore, for a generic point $y \in X$, any quasi-compact open neighborhood W of the closure $\overline{\{y\}}$, we can find a smaller open neighborhood W' of $\overline{\{y\}}$ such that W' is closed under specialization in X.*

Remark 5.8. For any point $y \in X$, the set $\{x \mid x \succ y\}$ of generalizations of y is a totally ordered chain of length exactly the rank of the valuation corresponding to y.

Remark 5.9. Why we consider the specialization here? Bhott gives a remark: Let (A, A^+) be an affinoid Tate ring, we have seen that the kernel map ker : $\mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A)$ is continuous, but not spectral, i.e. the preimage of a quasi-compact open subsets of $\mathrm{Spec}(A)$ under ker need not be quasicompact. Then we may define another natural continuous map

$$\mathrm{sp} : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A^+/A^{\circ\circ}),$$

called the *specialization* map, is sepctral.

To explain this, note first that $\mathrm{Spec}(A^+) - \mathrm{Spec}(A) = \mathrm{Spec}(A^+/A^{\circ\circ})$ as subsets of $\mathrm{Spec}(A^+)$. Now given a point $x \in \mathrm{Spa}(A, A^+)$ corresponding to the map ϕ_x : $A^+ \to R_x$ (where R_x is the induced valuation ring lies inside $\kappa(\mathrm{supp}\, x)$), we set $\mathrm{sp}(x)$ to be the image in $\mathrm{Spec}(A^+)$ of the closed point of $\mathrm{Spec}(R_x)$; as ϕ carries pseudouniformizers to pseudouniformizers, it is clear that $\mathrm{sp}(x)$ lies in $\mathrm{Spec}(A^+) - \mathrm{Spec}(A) = \mathrm{Spec}(A^+/A^{\circ\circ})$. For continuity and spectrality, fix some $\bar{f} \in A^+/A^{\circ\circ}$ defining an open set $D(\bar{f}) \subset \mathrm{Spec}(A^+/A^{\circ\circ})$. Representing $\bar{f}$ by some $f \in A^+$, the preimage can be written explicitly:

$$\mathrm{sp}^{-1}(D(\bar{f})) = \mathrm{Spa}(A, A^+)(\frac{1}{f}) = \{x \in \mathrm{Spa}(A, A^+) | |f(x)| = 1\}.$$

where the right side is rational. This proves both continuity and spectrality.

It offers us a comment on the Hausdorffness of the space.

Proposition 5.10. *Let $X := \mathrm{Spa}(A, A^+)$ for an affinoid Tate ring (A, A^+), and let X be the quotient of X by the equivalence relation generated by specializations (check: equivalence of the definitions), endowed with the quotient topology. Then X is Hausdorff, and is the maximal Hausdorff quotient of X.*

(This is a "boring" remark...) We come back to the question: when $\mathcal{O}_X$ is a sheaf?

Definition 5.9. A Tate k-algebra R is called strongly noetherian if

$$R\langle T_1, \dots, T_n\rangle = \{ \sum_{i_1,\dots,i_n \geq 0} x_{i_1,\dots,i_n} T_1^{i_1} \cdots T_n^{i_n} \mid x_{i_1,\dots,i_n} \in \hat{R}, x_{i_1,\dots,i_n} \to 0\}$$

is noetherian for all $n \geq 0$.

For example, if R is of tft, then R is strongly noetherian.

Theorem 5.1. *If (R, R^+) is an affinoid k-algebra such that R is strongly noetherian, then $\mathcal{O}_X$ is a sheaf.*

Later we will prove that this theorem is true under the assumption that R is a perfectoid k-algebra.

Finally, we introduce the related category: consider the category (V) of triples $(X, \mathcal{O}_X, (|\cdot(x)| \mid x \in X))$ consisting of a locally ringed topological space $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is a sheaf of complete topological k-algebras, and a continuous valuation $f \mapsto |f(x)|$ on $\mathcal{O}_{X,x}$ for every $x \in X$. Morphisms are given by morphisms of locally ringed topological spaces which are continuous k-algebra morphisms on $\mathcal{O}_X$, which are compatible with the valuations.

Definition 5.10. Any affinoid k-algebra (R, R^+) for which $\mathcal{O}_X$ is a sheaf gives rise to such a triple $(X, \mathcal{O}_X, (|\cdot(x)| \mid x \in X))$. Call an object of (V) isomorphic to such a triple an affinoid adic space.

An adic space over k is an object $(X, \mathcal{O}_X, (|\cdot(x)| \mid x \in X))$ of (V) that is locally on X an affinoid adic space. An adic space over k is called locally of finite type if it is locally of the form $\mathrm{Spa}(R, R^+)$, where (R, R^+) is of tft.

Proposition 5.11. *For any affinoid k-algebra (R, R^+) with $X = \mathrm{Spa}(R, R^+)$ such that $\mathcal{O}_X$ is a sheaf, and any adic space Y over k, we have*

$$\mathrm{Hom}(Y, X) = \mathrm{Hom}((\hat{R}, \hat{R}^+), (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))) \ .$$

Here, the latter set denotes the set of continuous k-algebra morphisms $\hat{R} \to \mathcal{O}_Y(Y)$ such that $\hat{R}^+$ is mapped into $\mathcal{O}_Y^+(Y)$.

Example 5. rigid analytic space: We have a fully faithful functor:

$$r : \{\text{rigid} - \text{analytic varieties}/k\} \to \{\text{adic spaces}/k\} \ : \ X \mapsto X^{\mathrm{ad}}$$

sending $\mathrm{Sp}(R) := \{X, \text{Grothendieck topology}, \mathcal{O}_X\} \to \mathrm{Spa}(R, R^+)$ for any affinoid k-algebra of tft. It induces an equivalence:

$$\{\text{qs rigid} - \text{analytic varieties}/k\} \cong \{\text{qs adic spaces locally of finite type}/k\} \ ,$$

Let X be a rigid-analytic variety over k with corresponding adic space X^{ad}. As any classical point defines an adic point, we have $X \subset X^{\mathrm{ad}}$. If X is quasiseparated, then mapping a quasicompact open subset $U \subset X^{\mathrm{ad}}$ to $U \cap X$ defines a bijection

$$\{\text{qc admissible opens in } X\} \cong \{\text{qc opens in } X^{\mathrm{ad}}\} \ ,$$

the inverse of which is denoted $U \mapsto \tilde{U}$. Under this bijection a family of quasi-compact admissible opens $U_i \subset X$ forms an admissible cover if and only if the corresponding subsets $\tilde{U}_i \subset X^{\mathrm{ad}}$ cover X^{ad}.

In particular, for any rigid-analytic variety X, the topos of sheaves on the Grothendieck site associated to X is equivalent to the category of sheaves on the sober topological space X^{ad}.

5.2. **Perfectoid spaces: analytic topology.** In the following, we are interested in the adic spaces associated to perfectoid algebras.

Definition 5.11. A perfectoid affinoid K-algebra is an affinoid K-algebra (R, R^+) such that R is a perfectoid K-algebra.

Lemma 5.2. *The categories of perfectoid affinoid K-algebras and perfectoid affinoid $K^\flat$-algebras are equivalent. If (R, R^+) maps to $(R^\flat, R^{\flat+})$ under this equivalence, then $x \mapsto x^\sharp$ induces an isomorphism $R^{\flat+}/\varpi^\flat \cong R^+/\varpi$. Also $R^{\flat+} = \varprojlim_{x \mapsto x^p} R^+$.*

Recall the proposition 2.1, we have the following "algebraic version":

Theorem 5.2. *Let (R, R^+) be a perfectoid affinoid K-algebra, and let $X = \mathrm{Spa}(R, R^+)$ with associated presheaves $\mathcal{O}_X$, $\mathcal{O}_X^+$. Also, let $(R^\flat, R^{\flat+})$ be the tilt given before, and let $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$ etc..*

Then we have a homeomorphism $X \cong X^\flat$, given by mapping $x \in X$ to the valuation $x^\flat \in X^\flat$ defined by $|f(x^\flat)| = |f^\sharp(x)|$. This homeomorphism identifies rational subsets.

In fact, the map $X \to X^\flat$ is continuous, because the preimage of the rational subset $U(\frac{f_1,\ldots,f_n}{g})$ is given by $U(\frac{f_1^\sharp,\ldots,f_n^\sharp}{g^\sharp})$, assuming as in Remark 5.2 that f_n is a power of $\varpi^\flat$ to ensure that $f_1^\sharp,\ldots,f_n^\sharp$ still generate R. Base on this homeomorphism, we have the following main theorem:

Theorem 5.3. *Let (R,R^+) be a perfectoid affinoid K-algebra, and let $X = \mathrm{Spa}(R,R^+)$ with associated presheaves $\mathcal{O}_X$, $\mathcal{O}_X^+$. Also, let $(R^\flat,R^{\flat+})$ be the tilt, and let $X^\flat = \mathrm{Spa}(R^\flat,R^{\flat+})$ etc. .*

(i) *For any rational subset $U \subset X$ with tilt $U^\flat \subset X^\flat$, the complete affinoid K-algebra $(\mathcal{O}_X(U),\mathcal{O}_X^+(U))$ is perfectoid, with tilt $(\mathcal{O}_{X^\flat}(U^\flat),\mathcal{O}_{X^\flat}^+(U^\flat))$.*

(ii) *The presheaves $\mathcal{O}_X$, $\mathcal{O}_{X^\flat}$ are sheaves.*

(iii) *The cohomology group $H^i(X,\mathcal{O}_X^+)$ is $\mathfrak{m}$-torsion for $i>0$.*

Scholze gave a nice outline of the following beautiful proof in his thesis. Here let's prove this theorem step by step directly.

Lemma 5.3. *Let $U = U(\frac{f_1,\ldots,f_n}{g}) \subset \mathrm{Spa}(R^\flat,R^{\flat+})$ be rational, with preimage $U^\sharp \subset \mathrm{Spa}(R,R^+)$. Assume that all $f_i, g \in R^{\flat\circ}$ and that $f_n = \varpi^{\flat N}$ for some N; this is always possible without changing the rational subspace.*

(i) *Consider the ϖ-adic completion*

$$R^\circ\langle\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\ldots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\rangle$$

of the subring

$$R^\circ[\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\ldots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}] \subset R[\frac{1}{g^\sharp}]\ .$$

Then $R^\circ\langle\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\ldots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\rangle^a$ is a perfectoid $K^{\circ a}$-algebra.

(ii) *The algebra $\mathcal{O}_X(U^\sharp)$ is a perfectoid K-algebra, with associated perfectoid $K^{\circ a}$-algebra*

$$\mathcal{O}_X(U^\sharp)^{\circ a} = R^\circ\langle\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\ldots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\rangle^a\ .$$

(iii) *The tilt of $\mathcal{O}_X(U^\sharp)$ is given by $\mathcal{O}_{X^\flat}(U)$.*

Proof. We only need to show that modulo ϖ, Frobenius is almost surjective with kernel almost generated by $\varpi^{1/p}$, since flatness and ϖ-adically completeness are obvious. Consider the following surjection:

$$R^\circ[T_1^{1/p^\infty},\ldots,T_n^{1/p^\infty}] \to R^\circ[\left(\frac{f_1}{g}\right)^{1/p^\infty},\ldots,\left(\frac{f_n}{g}\right)^{1/p^\infty}]\ .$$

Its kernel contains the ideal I generated by all $T_i^{1/p^m}g^{1/p^m} - f_i^{1/p^m}$.

First for characteristic p case, we claim that the induced morphism is an almost isomorphism. If f lies in the kernel of this map, there is some k with $\varpi^k f \in I$. But then $(\varpi^{k/p^m}f)^{p^m} \in I$, and because I is perfect, also $\varpi^{k/p^m}f \in I$. This gives the desired statement.

Thus reducing modulo ϖ, we have the following almost isomorphism:

$$R^\circ[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}]/(I,\varpi) \to R^\circ\langle \left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\rangle/\varpi \ .$$

From the definition of I, the Frobenius gives an isomorphism:

$$R^\circ[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}]/(I,\varpi^{1/p}) \cong R^\circ[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}]/(I,\varpi) \ .$$

This finally shows that $R^\circ\langle \left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\rangle^a$ is a perfectoid $K^{\circ a}$-algebra.

For general K, (we need to distinguish $f^\sharp$ and f here, and the philosophy is the tilt equivalence), again we have the map:

$$R^\circ[T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}]/I \to R^\circ[\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\dots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}] \ ,$$

where I is the ideal generated by all $T_i^{1/p^m}(g^{1/p^m})^\sharp - (f_i^{1/p^m})^\sharp$. We may apply our results for the tilt situation. We konw that $(\mathcal{O}_{X^\flat}(U),\mathcal{O}_{X^\flat}^+(U))$ is a perfectoid affinoid $K^\flat$-algebra. Let (S,S^+) be its tilt. Then $\mathrm{Spa}(S,S^+)\to X$ factors over $U^\sharp$, and hence we get a map $(\mathcal{O}_X(U^\sharp),\mathcal{O}_X^+(U^\sharp))\to(S,S^+)$.

Compose these maps in the almost degree, we have a map of perfectoid $K^{\circ a}$-algebra,which is the tilt of the composite map:

$$R^{\flat\circ}\langle T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}\rangle^a \to R^{\flat\circ}\langle T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}\rangle^a/I^\flat \to \mathcal{O}_{X^\flat}(U)^{\circ a} \ ,$$

where $I^\flat$ is the corresponding ideal which occurs in the tilted situation. Note that

$$R^{\flat\circ}\langle T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}\rangle/(I^\flat,\varpi^\flat) = R^\circ\langle T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}\rangle/(I,\varpi)$$

which comes from the explicit map. Since

$$R^{\flat\circ}\langle T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}\rangle^a/(I^\flat,\varpi^\flat) \to \mathcal{O}_{X^\flat}(U)^{\circ a}/\varpi^\flat$$

is an isomorphism, we can untilt it and show the following easily:

$$R^\circ\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty},\dots,\left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\rangle^a/\varpi \cong \mathcal{O}_X(U^\sharp)^{\circ a}/\varpi.$$

Then this isomorphism completes the proof. ∎

We need an approximation lemma (we made a lot of effort to prove it in class. However, I want to skip them):

Lemma 5.4. *Let $R = K\langle T_0^{1/p^\infty},\dots,T_n^{1/p^\infty}\rangle$. Let $f \in R^\circ$ be a homogeneous element of degree $d \in \mathbb{Z}[\frac{1}{p}]$. Then for any rational number $c \geq 0$ and any $\epsilon > 0$, there exists an element*

$$g_{c,\epsilon} \in R^{\flat\circ} = K^{\flat\circ}\langle T_0^{1/p^\infty},\dots,T_n^{1/p^\infty}\rangle$$

homogeneous of degree d such that for all $x \in X = \mathrm{Spa}(R,R^\circ)$, we have

$$|f(x) - g_{c,\epsilon}^\sharp(x)| \leq |\varpi|^{1-\epsilon}\max(|f(x)|,|\varpi|^c) \ .$$

The desired estimate leads to the desired homeomorphism:

Corollary 5.2. *Let (R,R^+) be a perfectoid affinoid K-algebra, with tilt $(R^\flat,R^{\flat+})$, and let $X = \mathrm{Spa}(R,R^+)$, $X^\flat = \mathrm{Spa}(R^\flat,R^{\flat+})$.*

(i) *For any* $f \in R$ *and any* $c \geq 0$, $\epsilon > 0$, *there exists* $g_{c,\epsilon} \in R^\flat$ *such that for all* $x \in X$, *we have*

$$|f(x) - g^\sharp_{c,\epsilon}(x)| \leq |\varpi|^{1-\epsilon} \max(|f(x)|, |\varpi|^c) .$$

(ii) *Given* $f, g \in R$ *and an integer* $c \geq 0$, *there exist* $a, b \in R^\flat$ *such that*

$$X(\frac{f, \varpi^c}{g}) = X(\frac{a^\sharp, \varpi^c}{b^\sharp})$$

as subsets of X.

Using this, we have: every rational subset of X *is the preimage of a rational subset of* $X^\flat$.

(iii) *For each* $U \subset X$ *rational,* $(\mathcal{O}_X(U), \mathcal{O}^+_X(U))$ *is a perfectoid affinoid* (R, R^+)*-algebra.*

(iv) *For any* $x \in X$, *the completed residue field* $\widehat{k(x)}$ *is a perfectoid field.*

(v) *The morphism* $X \to X^\flat$ *induces a homeomorphism, identifying rational subsets.*

Proof. (i) is the direct corollary of the lemma. For (ii), Using (i), we can choose

$$|(g - b^\sharp)(x)| < \max(|g(x)|, |\pi|^c)$$

and

$$\max(|f(x)|, |\pi|^c) = \max(|a^\sharp(x)|, |\pi|^c).$$

Now say $x \in X(\frac{f,\pi^c}{g})$. We shall check that $x \in X(\frac{a^\sharp,\pi^c}{b^\sharp})$. As $|\pi|^c \leq |g(x)|$, estimate gives $|g(x) - b^\sharp(x)| < |g(x)|$. By the strict non-archimedean inequality, this can only happen if $|b^\sharp(x)| = |g(x)|$, so we get $|\pi|^c \leq |b^\sharp(x)|$. Also, the second immediately shows that for such x, we have either $|a^\sharp(x)| \leq |\pi|^c$ or $|a^\sharp(x)| = |f(x)|$; the former implies $|a^\sharp(x)| \leq |\pi|^c \leq |g(x)| = |b^\sharp(x)|$ by the assumption on x and the previous deduction, while the latter implies $|a^\sharp(x)| = |f(x)| \leq |g(x)| = |b^\sharp(x)|$ by the assumption on x and the previous deduction. In either case, we get the desired equality $|a^\sharp(x)| \leq |b^\sharp(x)|$, proving $x \in X(\frac{a^\sharp,\pi^c}{b^\sharp})$.

Conversely, say $x \in X(\frac{a^\sharp,\pi^c}{b^\sharp})$. We shall check that $x \in X(\frac{f,\pi^c}{g})$. First, we check $|\pi^c| \leq |g(x)|$. If this failed, then we would have $|g(x)| > |\pi|^c$; by the inequality and the strict NA inequality, this would mean $|g(x)| = |b^\sharp(x)|$, but the latter implies $|g(x)| = |b^\sharp(x)| \leq |\pi|^c$ by assumption on x, which is a contradiction. Thus, we must have $|\pi|^c \leq |g(x)|$. Next, as in the previous paragraph, this implies that $|g(x)| = |b^\sharp(x)|$ for such x. It remains to check that $|f(x)| \leq |g(x)|$. If not, we must have $|f(x)| > |g(x)| \geq |\pi|^c$. By $\max(|f(x)|, |\pi|^c) = \max(|a^\sharp(x)|, |\pi|^c)$, this shows that $|f(x)| = |a^\sharp(x)|$, and thus $|a^\sharp(x)| > |g(x)|$ as well. But we just checked that $|g(x)| = |b^\sharp(x)|$, so we get $|a^\sharp(x)| > |b^\sharp(x)|$, which contradicts the assumption on x.

For the last result, let $U = X(\frac{f_1,\ldots,f_n}{g})$ be a rational subset of X. We may assume after scaling that $f_i \in R^+$ and $f_n = \pi^c$ for some integer $c \geq 1$. We can then write $U = \cap_{i=1}^{n-1} X(\frac{f_i,\pi^c}{g})$. The claim now follows by applying (ii) $(n-1)$-times.

(iii) is proved. For (iv), note that the open valuation ring $\widehat{k(x)^+} \subset \widehat{k(x)}$ is the π-adic completion of the direct limit of the K°-algebras $\mathcal{O}^+_X(U)$ as U ranges through rational open subsets of X. when K is of characteristic p, we know that $\mathcal{O}_X(U)^{\circ a}$ is perfectoid for any rational subset U. It follows that the ϖ-adic completion of $\mathcal{O}^{\circ a}_{X,x}$ is a perfectoid $K^{\circ a}$-algebra, hence $\widehat{k(x)}$ is a perfectoid K-algebra. As it is also a nonarchimedean field, the result follows. Then the characteristic 0 case follows from (v) and tilting equivalence.

(v): First, we have already shown that $\psi : X \to X^\flat$ is a continuous map such that each rational subset of X is a pullback of a rational subset of $X^\flat$; as X is T_0, this formally gives injectivity of ψ.

Then Using (iii), this will imply that ψ carries rational subsets to rational subsets, proving continuity for the inverse. For surjecitivity, pick $x \in X^\flat$. This point defines a map $(R^\flat, R^{\flat+}) \to (\widehat{k(x)}, \widehat{k(x)^+})$ to the corresponding perfectoid affinoid field by easier part of (v). By tilting and Lemma 4.4, this untilts to a map $(R, R^+) \to (L, L^+)$ where (L, L^+) is a perfectoid affinoid field. This corresponds to a point $y \in \mathrm{Spa}(R, R^+)$. It is then easy to see that $\psi(y) = x$, by definition. Then the surjecitivity follows. $\square$

For any subset $M \subset X$, we write $M^\flat \subset X^\flat$ for the corresponding subset of $X^\flat$. Follows from the universal property of the structure sheaf among the perfectoid affinoid K-algebra, we have the tilt corresondence of the rational subset:

Corollary 5.3. *Let (R, R^+) be a perfectoid affinoid K-algebra with tilt $(R^\flat, R^{\flat+})$. Let $X = \mathrm{Spa}(R, R^+)$, $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$. Then for all rational $U \subset X$, the pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is a perfectoid affinoid K-algebra with tilt $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.*

Our main goal is to prove the following Tate acyclicity and the sheaf-theoretic properties, stating roughly that perfectoid affinoid algebras behave like affine schemes in algebraic geometry. At this point, let us recall some facts about reduced affinoid K-algebras of topologically finite type.

Proposition 5.12. *Let (S, S^+) be a reduced affinoid K-algebra of topologically finite type, and let $X = \mathrm{Spa}(S, S^+)$.*

(i) *The subset $S^+ = S^\circ \subset S$ is open and bounded.*

(ii) *For any rational subset $U \subset X$, the affinoid K-algebra $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is reduced and of topologically finite type.*

(iii) *For any covering $X = \bigcup U_i$ by finitely many rational subsets $U_i \subset X$, each cohomology group of the complex*

$$0 \to \mathcal{O}_X(X)^\circ \to \prod_i \mathcal{O}_X(U_i)^\circ \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^\circ \to \dots$$

is annihilated by some power of ϖ.

Proof. We only prove the third point. The Tate's acyclicity theorem says that

$$0 \to \mathcal{O}_X(X) \xrightarrow{d_0} \prod_i \mathcal{O}_X(U_i) \xrightarrow{d_1} \prod_{i,j} \mathcal{O}_X(U_i \cap U_j) \xrightarrow{d_2} \dots$$

is exact. Then $\ker d_i$ is a closed subspace of a K-Banach space, hence itself a K-Banach space, and d_{i-1} is a surjection onto $\ker d_i$. By Banach's open mapping theorem, d_{i-1} is an open map to $\ker d_i$. This shows that the subspace and quotient topologies on $\ker d_i = \mathrm{Im} d_{i-1}$ coincide. Now consider the sequence

$$0 \to \mathcal{O}_X(X)^\circ \xrightarrow{d_0^\circ} \prod_i \mathcal{O}_X(U_i)^\circ \xrightarrow{d_1^\circ} \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^\circ \xrightarrow{d_2^\circ} \dots .$$

Since parts (i) and (ii) and the coincidence of the topology, the cohomology group is annihilated by some power of ϖ. $\square$

I guess the following idea appears in the next paper somehow (the limit?)

Definition 5.12. Assume K is of characteristic p. Then a perfectoid affinoid K-algebra (R, R^+) is said to be **p-finite** if there exists a reduced affinoid K-algebra (S, S^+) of topologically finite type such that (R, R^+) is the completed perfection of (S, S^+), i.e. R^+ is the ϖ-adic completion of $\varinjlim_{\Phi} S^+$ and $R = R^+[\varpi^{-1}]$.

Proposition 5.13. *Assume that K is of characteristic p, and that (R, R^+) is p-finite, given as the completed perfection of a reduced affinoid K-algebra (S, S^+) of topologically finite type.*

(i) *The map $X = \mathrm{Spa}(R, R^+) \cong Y = \mathrm{Spa}(S, S^+)$ is a homeomorphism identifying rational subspaces.*

(ii) *For any $U \subset X$ rational, corresponding to $V \subset Y$, the perfectoid affinoid K-algebra $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is equal to the completed perfection of $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$.*

(iii) *For any covering $X = \bigcup_i U_i$ by rational subsets, the sequence*

$$0 \to \mathcal{O}_X(X)^{\circ a} \to \prod_i \mathcal{O}_X(U_i)^{\circ a} \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^{\circ a} \to \dots$$

is exact. In particular, $\mathcal{O}_X$ is a sheaf, and $H^i(X, \mathcal{O}_X^{\circ a}) = 0$ for $i > 0$. This is equivalent to the assertion that $H^i(X, \mathcal{O}_X^+)$ is annihilated by $\mathfrak{m}$.

Lemma 5.5. *Assume K is of characteristic p.*

(i) *Any perfectoid affinoid K-algebra (R, R^+) for which R^+ is a K°-algebra is the completion of a filtered direct limit of p-finite perfectoid affinoid K-algebras (R_i, R_i^+).*

(ii) *This induces a homeomorphism $\mathrm{Spa}(R, R^+) \cong \varprojlim \mathrm{Spa}(R_i, R_i^+)$, and each rational $U \subset X = \mathrm{Spa}(R, R^+)$ comes as the preimage of some rational $U_i \subset X_i = \mathrm{Spa}(R_i, R_i^+)$.*

(iii) *In this case $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is equal to the completion of the filtered direct limit of the $(\mathcal{O}_{X_j}(U_j), \mathcal{O}_{X_j}^+(U_j))$, where U_j is the preimage of U_i in X_j for $j \geq i$.*

(iv) *If $U_i \subset X_i$ is some quasicompact open subset containing the image of X, then there is some j such that the image of X_j is contained in U_i.*

Proof. We can "descent" the geometric object to its algebraic object, then the lemma is some exercises on algebra and the apllication of the universal property. ∎

Proposition 5.14. *Let K be of any characteristic, and let (R, R^+) be a perfectoid affinoid K-algebra, $X = \mathrm{Spa}(R, R^+)$. For any covering $X = \bigcup_i U_i$ by finitely many rational subsets, the sequence*

$$0 \to \mathcal{O}_X(X)^{\circ a} \to \prod_i \mathcal{O}_X(U_i)^{\circ a} \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^{\circ a} \to \dots$$

is exact. In particular, $\mathcal{O}_X$ is a sheaf, and $H^i(X, \mathcal{O}_X^{\circ a}) = 0$ for $i > 0$.

Proof. Assume first that K has characteristic p. We may replace K by a perfectoid subfield, such as the ϖ-adic completion of $\mathbb{F}_p((\varpi))(\varpi^{1/p^\infty})$; this ensures that for any perfectoid affinoid K-algebra (R, R^+), the ring R^+ is a K°-algebra. Then use Lemma 5.5 to write $X = \mathrm{Spa}(R, R^+) \cong \varprojlim X_i = \mathrm{Spa}(R_i, R_i^+)$ as an inverse limit, with (R_i, R_i^+) p-finite.

Thus any rational subspace comes from a finite level, and a cover by finitely many rational subspaces is the pullback of a cover by finitely many rational subspaces on a finite level (we can choose one family of algebras here explicitly). Hence the almost exactness of the sequence follows by taking the completion of the direct limit of the corresponding statement for X_i, which is given by Proposition 5.13.

In characteristic 0, we want to use the tilt functor. First we need the lemma:

Lemma 5.6. *Let A, B, C be hte ϖ torsion free and ϖ-adic complete algebra, then*

$$0 \to A \to B \to C \to 0 \text{ exact} \Leftrightarrow 0 \to A/\varpi \to B/\varpi \to C/\varpi \to 0 \text{ exact}.$$

Then use the exactness of the tilted sequence and reduce modulo $\varpi^\flat$, which identify with the original sequence reduced modulo ϖ. □

This proves Theorem 5.3.

Remark 5.10. The vanish degree of the cohomology group, N, cannot be choosen independently of the cover of the rational subset. If this were true, then it would follow that $H^i(X, \mathcal{O}_X^+)$ has bounded ϖ-torsion for any i. However, one example is: $H^1(X, \mathcal{O}_X^+)$ has unbounded ϖ-torsion for $X = \mathrm{Spa}(A, A^+)$ where $A^+ = \mathbb{F}_p[t, x, y]/(y^2 - z^3)$.

We see that in particular, the adic spectrum attached to any perfectoid affinoid K-algebra (R, R^+) can be associated to an affinoid adic space $X = \mathrm{Spa}(R, R^+)$. We call these spaces affinoid perfectoid spaces.

Definition 5.13. A *perfectoid space* is an adic space over K that is locally isomorphic to an affinoid perfectoid space. Morphisms between perfectoid spaces are the morphisms of adic spaces.

Then the progress of tilting glues.

Definition 5.14. We say that a perfectoid space $X^\flat$ over $K^\flat$ is the tilt of a perfectoid space X over K if there is a functorial isomorphism

$$\mathrm{Hom}(\mathrm{Spa}(R^\flat, R^{\flat+}), X^\flat) = \mathrm{Hom}(\mathrm{Spa}(R, R^+), X)$$

for all perfectoid affinoid K-algebras (R, R^+) with tilt $(R^\flat, R^{\flat+})$.

Example 6. $\mathrm{Spa}(R, R^+)^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$.

It is reaonable to expect the following category equivalence:

Proposition 5.15. *Any perfectoid space X over K admits a tilt $X^\flat$, unique up to unique isomorphism. This induces an equivalence between the category of perfectoid spaces over K and the category of perfectoid spaces over $K^\flat$. The underlying topological spaces of X and $X^\flat$ are naturally identified. A perfectoid space X is affinoid perfectoid if and only if its tilt $X^\flat$ is affinoid perfectoid (subcategory equivalence). Finally, for any affinoid perfectoid subspace $U \subset X$, the pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is a perfectoid affinoid K-algebra with tilt $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.*

Remark 5.11. It is not clear if a perfectoid space that is affinoid as an adic space is an affinoid perfectoid space, i.e., if $X = \mathrm{Spa}(R, R^+)$ for a sheafy complete affinoid Tate ring (R, R^+), and X ahs a rational cover $X = \cup_i U_i$ with each U_i being affinoid perfectoid, it is not clear that X is affinoid perfectoid. However, let A of characteristic p, the assertion will hold (Mihara).

We conclude this subsection by discussing one pleasant feature of perfectoid spaces, with respect to the adic space:

Proposition 5.16. *If $X \to Y \leftarrow Z$ are perfectoid spaces over K, then the fibre product $X \times_Y Z$ exists in the category of adic spaces over K, and is a perfectoid space.*

Proof. It is enough to prove this when all objects in sight are of affinoid perfectoid: write $X = \mathrm{Spa}(A, A^+)$, $Y = \mathrm{Spa}(B, B^+)$ and $Z = \mathrm{Spa}(C, C^+)$, and we want to construct $W = X \times_Y Z$. This is given by $W = \mathrm{Spa}(D, D^+)$, where D is the completion of $A \otimes_B C$, and D^+ is the completion of the integral closure of the image of $A^+ \otimes_{B^+} C^+$ in D. Then we need to show that $A^{\circ a} \widehat{\otimes_{B^{\circ a}}} C^{\circ a}$ is a perfectoid $K^{\circ a}$ algebra. The result follows from checking the universal property. The characteristic p case is direct, since then $A^+ \otimes_{B^+} C^+$ is perfect and ϖ-torsion free. The general case needs tilt equivalence. $\square$

Our next goal is the complete almost purity theorem.

5.3. **Perfectoid spaces: étale topology.** Recall the almost purity theorem:

Theorem 5.4. *Let R be a perfectoid K-algebra. Let S/R be finite étale. Then S is a perfectoid K-algebra, and S° is almost finite étale over R°.*

We are left to prove the characteristic 0 case. It is easy to construct a fully faithful functor from the category of finite étale $R^\flat$ algebras to finite étale R algebras as we just see, and the problem can be reformulated as the assertion that this functor is essentially surjective. Finally we observe that the adic spectra X and $X^\flat$ are locally given by perfectoid fields, so we use $X \simeq X^\flat$ to glue the results in the case of fields.

In this section, we consider the locally noetherian adic space over k, i.e. they are locally of the form $\mathrm{Spa}(A, A^+)$, where A is a strongly noetherian Tate k-algebra. If additionally, they are quasicompact and quasiseparated, we call them *noetherian adic spaces.*

Definition 5.15. (i) A morphism $(R, R^+) \to (S, S^+)$ of affinoid k-algebras is called finite étale if S is a finite étale R-algebra with the induced topology, and S^+ is the integral closure of R^+ in S. Write $(R, R^+)_{\text{fét}}$ for the category of all such maps.

(ii) A morphism $f : X \to Y$ of adic spaces over k is called finite étale if there is a cover of Y by open affinoids $V \subset Y$ such that the preimage $U = f^{-1}(V)$ is affinoid, and the associated morphism of affinoid k-algebras

$$(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$

is finite étale. Write $Y_{\text{fét}}$ for the category of all such maps.

(iii) A morphism $f : X \to Y$ of adic spaces over k is called étale if for any point $x \in X$ there are open neighborhoods U and V of x and $f(x)$ and a commutative diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "f|_U"'] & W \arrow[d, "p"] \\ & V \end{tikzcd}$$

where j is an open embedding and p is finite étale. Write $Y_{\text{ét}}$ for the category of all such maps.

For locally noetherian adic spaces over k, this recovers the usual notions. We will see that these notions are useful in the case of perfectoid spaces, and will not use them otherwise. However, we will temporarily need a stronger notion of étale morphisms for perfectoid spaces (We shall eventually show that if $(A, A^+) \to (B, B^+)$ is a finite étale map with A perfectoid, then B is also perfgectoid, and the map $A^+ \to B^+$ is almost finite étale). After proving the almost purity theorem, we will see that there is no difference. In the following let K be a perfectoid field again.

Definition 5.16. (i) A morphism $(R, R^+) \to (S, S^+)$ of perfectoid affinoid K-algebras is called strongly finite étale if it is finite étale and additionally $S^{\circ a}$ is a finite étale $R^{\circ a}$-algebra. Write $(R, R^+)_{s\text{fét}}$ for the category of all such maps.
(ii) A morphism $f : X \to Y$ of perfectoid spaces over K is called strongly finite étale if there is a cover of Y by open affinoid perfectoids $V \subset Y$ such that the preimage $U = f^{-1}(V)$ is affinoid perfectoid, and the associated morphism of perfectoid affinoid K-algebras

$$(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$

is strongly finite étale. Write $Y_{s\text{fét}}$ for the category of all such maps.
(iii) A morphism $f : X \to Y$ of perfectoid spaces over K is called strongly étale if for any point $x \in X$ there are open neighborhoods U and V of x and $f(x)$ and a commutative diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "f|_U"'] & W \arrow[d, "p"] \\ & V \end{tikzcd}$$

where j is an open embedding and p is strongly finite étale. Write $Y_{s\text{ét}}$ for the category of all such maps.

From the definitions and the "tilt equivalence" we have proved, we see that $f : X \to Y$ is strongly finite étale, resp. strongly étale, if and only if the tilt $f^\flat : X^\flat \to Y^\flat$ is strongly finite étale, resp. strongly étale. Moreover, in characteristic p, anything (finite) étale is naturally strongly (finite) étale (This is beacuse of the almost purity theorem we have proved). For the single point case, i.e. the almost purity theorem for field, we have also proved it before.

Let us recall the following statement about henselian rings.

Proposition 5.17. *Let A be a flat K°-algebra such that A is henselian along (ϖ). Then $A[\varpi^{-1}]_{\text{fét}} \cong \hat{A}[\varpi^{-1}]_{\text{fét}}$, where $\hat{A}$ is the ϖ-adic completion of A.*

Remark 5.12 (Henselian). For pair (R, I) with a ring R and an ideal $I \subset R$ is *henselian* if for any finite R-algebra S, the map $S \to S/IS$ induces a bijection on idempotents; we sometimes also say R is I-adically henselian in this case. A *henselian local ring* is a local ring $(R, \mathfrak{m})$ which is henselian. Examples can be constructed using the following stability properties: (a) If R is I-adically complete then R is I-adically henselian, and (b) if (R, I) is henselian and $J \subset I$ is an ideal, then $(R/J, I/J)$ and (R, J) are both henselian. If we restrict to the case for affinoid

Tate ring (A, A^+), the henselian property is equivalent to the following: an étale map $A^+ \to B$ admits a section provided it does so modulo $A^{\circ\circ}$.

Our main component of the proof is the following proposition, which states that strongly finite étale maps can be described in terms of algebra if one works over affinoids. This proposition helps us to patch the local ones together.

Proposition 5.18. *If $f : X \to Y$ is a strongly finite étale morphism of perfectoid spaces, then for any open affinoid perfectoid $V \subset Y$, its preimage U is affinoid perfectoid, and*

$$(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$

is strongly finite étale.

The proof relies on "noetherian approximation" techniques and one classical result from noetherian adic space.

Proof. First we can assume $Y = V = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid, then $X = U$. Since strongly finite étale maps hold under tilting, we immediately reduce to the case that K is of characteristic p. Again, we replace K by a perfectoid subfield to ensure that R^+ is a K°-algebra in all cases.

We may assume $Y = V = \mathrm{Spa}(R, R^+)$ is affinoid. Writing (R, R^+) as the completion of the direct limit of p-finite perfectoid affinoid K-algebras (R_i, R_i^+) as in Lemma 5.5. As both rational subsets and finite étale cover pass through such filter colimits, we may assume that Y is already defined as a finite étale cover (via base change) of some $Y_i = \mathrm{Spa}(R_i, R_i^+)$. The fiber product in the category of perfectoid space allows us to assume (R, R^+) is itself p-finite, given as the completed perfection $\varprojlim_\varphi (S, S^+)$.

As the Frobenius map ϕ does not affect the notion of étale morphisms of rings or rational subsets, it follows that the finite étale map $X \to \mathrm{Spa}(R, R^+)$ arises as the base change of a finite étale map $Z \to \mathrm{Spa}(S, S^+)$. The classical theorem declares that all finite étale maps $Z \to \mathrm{Spa}(S, S^+)$ are of form $Z = \mathrm{Spa}(A, A^+)$ with $S \to A$ finite étale and S^+ being the integral closure of S^+ in A. By the description of pushouts in the category of complete uniform affinoid ring, it follows that $X = \mathrm{Spa}(D, D^+)$ where D^+ is the t-adic completion of the integral closure of the image of $A^+ \otimes_{S^+} R^+$ in $A \otimes_S R$.

Now we shall check that D^+ is almost finite étale over R^+. We shall check that the t-adic completion of the integral closure of $\mathrm{Im}(A^+_{\mathrm{perf}} \otimes_{S^+_{\mathrm{perf}}} R^+) \to A \otimes_S R$, is isomorphic to D. $\square$

As a consequence, the notion of strongly finite étale maps has good geometric properties.

Corollary 5.4. *For an affinoid perfectoid space $Y = \mathrm{Spa}(R, R^+)$, the functor $X \mapsto \mathcal{O}_X^+(X)$ induces an equivalence $Y_{\text{sfét}} \cong R^{\circ a}_{\text{afét}}$, and the functor gives a fully faithful functor $Y_{\text{sfét}} \cong R_{\text{fét}}$.*

Furthermore, the following theorem gives a strong form of Falting's almost purity theorem:

Theorem 5.5. *For an affinoid perfectoid space $Y = \mathrm{Spa}(R, R^+)$, with tilt $Y^\flat$.*

(i) *For any open affinoid perfectoid subspace $U \subset Y$, we have a fully faithful functor from the category of strongly finite étale covers of U to the category of finite étale covers of $\mathcal{O}_Y(U)$, given by taking global sections.*

(ii) *For any U, this functor is an equivalence of categories.*

(iii) *For any finite étale cover S/R, S is perfectoid and $S^{\circ a}$ is finite étale over $R^{\circ a}$. Moreover, $S^{\circ a}$ is a uniformly almost finitely generated $R^{\circ a}$-module.*

Proof. (i) By Theorem 4.8 and Proposition 5.18, the perfectoid spaces strongly finite étale over U are the same as the finite étale $\mathcal{O}_X(U)^{\circ a}$-algebras, which are a full subcategory of the finite étale $\mathcal{O}_X(U)$-algebras.

(ii) We may assume that $U = X$. We first check that there exists a cover $\{V_i\}$ of X by rational subsets such that the finite étale $\mathcal{O}_X(V_i)$-algebra $S \otimes_R \mathcal{O}_X(V_i)$ lifts uniquely to an almost finite étale $\mathcal{O}_X^+(V_i)$-algebra S_i^+.

Note that we have an equivalence of categories between the direct limit of the category of finite étale $\mathcal{O}_X(U)$-algebras over all affinoid perfectoid neighborhoods U of x and the category of finite étale covers of the completion $\widehat{k(x)}$ of the residue field at x, where the latter is a perfectoid field, by Proposition 5.18 as

$$\varinjlim_{x\in U}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \cong (\widehat{k(x)}, \widehat{k(x)^+})$$

in the category of complete uniform affinoid Tate rings.

Recall that we have already proved the almost purity theorem for the single point case (field). Applying this latest isomorphism in the tilted setting, and the almost purity, then untilting (by $\varinjlim_{x\in U}(\mathcal{O}_X(U))_{\text{fét}} \cong \varinjlim_{x\in U}(\mathcal{O}_{X^\flat}(U^\flat))_{\text{fét}}$, $\widehat{k(x)}_{\text{fét}} \cong \widehat{k(x^\flat)}_{\text{fét}}$), we get

$$\varinjlim_{x\in U} \mathcal{O}_X^+(U)_{\text{afét}} \cong \widehat{k(x)^+}_{\text{afét}} \cong \widehat{k(x)}_{\text{fét}}$$

is an equivalence. It follows that the canonical map

$$\varinjlim_{x\in U} \mathcal{O}_X^+(U)_{\text{afét}} \cong \varinjlim_{x\in U} \mathcal{O}_X(U)_{\text{fét}}.$$

In particular, there exists some rational subset $V \subset X$ containing x such that the finite étale $\mathcal{O}_X(V)$-algebra $S \otimes_R \mathcal{O}_X(V)$ lifts uniquely to an almost finite étale $\mathcal{O}_X^+(V)$-algebra. As x varies, this gives the desired cover $\{V_i\}$ of X. Then we shall write $V_{ij} = V_i \cap V_j$.

Thus for each i, we have a strongly finite étale map $U_i = \mathrm{Spa}(S_i, S_i^+) \to V_i$ of affinoid perfectoid spaces, where $S_i = S_i^+[\frac{1}{\pi}]$. whose underlying finite étale map corresponds to the finite étale $\mathcal{O}_X(V)$-algebra $S \otimes_R \mathcal{O}_X(V)$. By fully faithfulness property, there is a canonical isomorphism $U_i \times_{V_i} V_{ij} \simeq U_j \times_{V_j} V_{ij}$ in $(V_{ij})_{s\text{fét}}$. Since these isomorphisms satisfy the cocycle condition, so the U_i's can be glued together to give a perfectoid space Y, with a map $Y \to X \in (V_i)_{s\text{fét}}$, for each i. By definition, $Y \to X \in X_{s\text{fét}}$, so $Y = \mathrm{Spa}(T, T^+)$ for the perfectoid affinoid algebra $(T, T^+) \in (R, R^+)_{s\text{fét}}$, by 5.18!

It remains to check that $T \cong S$. This follows directly from the construction and the Lemma:

Lemma 5.7 (Realizing finite projective modules as sheaves)**.** *Let $X = \mathrm{Spa}(A, A^+)$ for a perfectoid affinoid K-algebra (A, A^+). For a finite projective A-mod M, write $\tilde{M} := M \otimes_A \mathcal{O}_X$ for the associated sheaf of $\mathcal{O}_X$-modules.*

1. *For any rational subset $U \subset X$, we have $H^0(U, \tilde{M}) = M \otimes_A \mathcal{O}_X(U)$.*
2. *The functor $M \mapsto \tilde{M}$ is fully faithful.*

Here the word "by construction" means that, we can check $T = S$ by the following sheaf property of $\mathcal{O}_Y$:

$$0 \to T \to \prod_i \mathcal{O}_X(U_i) \otimes_R S \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j) \otimes_R S \text{ is eaxct.}$$

and the sheaf property for $\mathcal{O}_X$:

$$0 \to R \to \prod_i \mathcal{O}_X(U_i) \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j) \text{ is eaxct.}$$

Because S is flat over R, tensoring is exact, and the first sequence is identified with the second sequence after $\otimes_R S$. Therefore $T = S$, as desired.

(iii) is a formal consequence of part (ii), Proposition 5.18 and Lemma 4.8. □

As the direct conclusion, we can define the étale site of a perfectoid space.

Definition 5.17. Let X be a perfectoid space. Then the étale site of X is the category $X_{\text{ét}}$ of perfectoid spaces which are étale over X, and coverings are given by topological coverings. The associated topos is denoted $X^{\sim}_{\text{ét}}$.

The previous results show that all conditions on a site are satisfied, and that a morphism $f : X \to Y$ of perfectoid spaces induces a morphism of sites $X_{\text{ét}} \to Y_{\text{ét}}$. Also, a morphism $f : X \to Y$ from a perfectoid space X to a locally noetherian adic space Y induces a morphism of sites $X_{\text{ét}} \to Y_{\text{ét}}$. With these preparations, we get the main result:

Theorem 5.6. *Let X be a perfectoid space over K with tilt $X^\flat$ over $K^\flat$. Then the tilting operation induces an isomorphism of sites $X_{\text{ét}} \cong X^\flat_{\text{ét}}$. This isomorphism is functorial in X.*

The almost vanishing of the cohomology we have proved may extends to the étale cohomology (for finite dimensional property... they are in the second paper).

Proposition 5.19. *For any perfectoid space X over K, the sheaf $U \mapsto \mathcal{O}_U(U)$ is a sheaf $\mathcal{O}_X$ on $X_{\text{ét}}$, and $H^i(X_{\text{ét}}, \mathcal{O}_X^{\circ a}) = 0$ for $i > 0$ if X is affinoid perfectoid.*

The proof relies on some reductions: Tilting equivalence reduces the assertion to characteristic p case, then use finite approximation lemma to p-finite (R, R^+). Then the result follows from the analogous sequence for noetherian adic spaces, which is "almost" exact, and the process of taking the limit and perfection, which is also almost.

6. Interlude: Fargues Fontaine Curve

Since the Tilt Functor is not fully faithful (for example, If we choose the base field as a perfectoid field, Tilt is fullyfaithful; But for characteristic 0 fields $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\mu_{p^\infty})$, they both have $\mathbb{F}_p((t^{1/p^\infty}))$ as their tilt), It is natural to ask how to parametrize of the untilt of a perfectoid space. Here we use a complete curve to parametrize the untilt of a algebraic closed perfectoid field of characteristic p.

parametrization of the untilt of a perfectoid space: FF curve and motivation of the diamonds ($\mathrm{Spd}\,\mathbb{Q}_p$).

Our goal is: For F a characteristic p perfectoid field, denote (since tilt is not a fully faithful functor.)

$|Y_F|$: the set of untilts (C, ι) ($\iota: C^\flat \cong F$). of char $=0$.

Define the frobenius equiv of untilts : view $(C, \iota\circ\varphi^{\mathbb{Z}})$ as a single pt.

($\exists\, \eta: C_1 \cong C_2$, s.t. for $\eta: C_1^\flat \to C_2^\flat$, we have $C_1^\flat \xrightarrow{\eta} C_2^\flat$, $\iota_1 \downarrow \wr \quad \wr\downarrow \iota_2$, $F \xrightarrow{\varphi^n} F$)

Thm : The set $|Y_F|/\varphi^{\mathbb{Z}}$ is the underlying set of a complete curve X_F^{FF}.

Review : Witt Vector:

Let (R, R^+) a perfectoid Huber pair $/\mathbb{F}_p$, take $(R^\sharp, R^{\sharp +})$ as an untilt,

(Tate $\Leftarrow R^{\circ\circ} \cap R^\times \neq \phi$.) ($R^\circ/\varpi \xrightarrow{\varphi} R^\circ/\varpi^p$ is iso.)

for topo ring R, $\exists$ open subring R_0^+, s.t. $\exists$ f.g. ideal $I \subset R^\circ$, s.t. $\{I^n\}$ is a basis of $\{0\}$.

i.e. $\exists$ iso $\iota: R^{\sharp\flat} \xrightarrow{\sim} R$.

$r_n \xmapsto{\iota^{-1}} R^{\sharp\flat} \xrightarrow{\theta^\sharp} r_n^\sharp$.

identification : $R^{\sharp\flat +} \cong \varprojlim_{\varphi} R^{\sharp +}$, $R^{\sharp\flat +}/\varpi^\flat \cong R^{\sharp +}/\varpi$

Thm. ① $\exists$ surj. ring homo $\theta: W(R^+) \twoheadrightarrow R^{\sharp +}$

$$\sum_{n\geq 0}[r_n]p^n \xrightarrow{\sharp\circ\iota^{-1}} \sum_{n\geq 0} r_n^\sharp p^n.$$

② $\ker(\theta)$ is generated by a non-zero divisor ξ of the form $\xi = p + [\varpi]\alpha$ ($\varpi \in R^+$, $\alpha \in W(R^+)$)

i.e. $\ker(\theta)$ is an ideal of $\deg = 1$. (primitive)

rmk this gives the equivalence of categories :

$W(R^+)/pW(R^+) = R^+$

$R^\flat = \varprojlim R$, $R^{\flat\circ} = \varprojlim_{\varphi} R^\circ \cong \varprojlim R^\circ/p \cong \varprojlim_{\varphi} R^\circ/\varpi$

(for $\exists\, \eta: C_1 \cong C_2$, s.t. $\eta: C_1^\flat \cong C_2^\flat$, we have $\iota_1 \downarrow \wr\, (\iota_2 C^{(i)}) = \varprojlim_{n\geq 0} x_{n+i}^{p^n}$ commute)

①: $\{$ Perfectoid Tate Huber pairs (S, S^+) $\}/\sim$

②: $\{$ Triples (R, R^+, I) where (R, R^+) perf. Tate-Huber pairs of char $=p$. and $I \subseteq W(R^+)$ is primitive of deg 1 $\}$ by.

$(S, S^+) \mapsto (S^\flat, S^{\flat +}, \ker(\theta))$. ; $(R, R^+, I) \mapsto (W(R^+)[[\varpi]]^{-1}/I, W(R^+)/I)$

proof : $R^\circ \xrightarrow{\sim} R^{\sharp\flat\circ} \xrightarrow{\sharp} R^{\sharp\circ}$ $\leftarrow$ "p-adically complete ring $R^{\sharp\circ}$, is a ring hom.

$\exists!$ cont. ring hom.

$$\begin{array}{ccc} R^\circ & \longrightarrow & R^{\sharp\circ} \\ \downarrow & \nearrow & \\ W(R^\circ) & & \end{array}$$

①

pf: 自然会是 non-zero-divisor:

if $\xi \sum_{n\geq 0}[c_n]p^n = 0$, mod $[\varpi] \rightsquigarrow \sum_{n\geq 0}[c_n]p^{n+1} \equiv 0 \pmod{[\varpi]}$. $\rightsquigarrow$ all $c_n \equiv 0 \pmod{\varpi}$

then we can divide all c_n by ϖ. then induce. $\square$

Figure 1. Witt Vector (1)

Pf: 1. Fix $\varpi \in R^+$ a pseudo-uniformizer. st $\varpi^\# \in R^{\#+}$ st $(\varpi^\#)^p | p$.

check θ is surj ring ~~mod.~~ map:

mod $(\varpi^\#)^m$ for all $m \geq 1$. use m-th ghost map.

$$W(R^{\#+}) \to R^\#/(\varpi^\#)^m: \quad (x_0, x_1, \dots) \mapsto \sum_{n=0}^m x_n^{p^{m-n}} p^n$$

factors uniquely over $W(R^{\#+}/\varpi^\#)$, by obvious conjugation. (ring)

$\leadsto$ induce $W(R^{\#+}/\varpi^\#) \to R^{\#+}/(\varpi^\#)^m$, now composite

$$\underline{W(R^+) \to W(R^+/\varpi)} = W(R^{\#+}/\varpi^\#) \to R^{\#+}/(\varpi^\#)^m.$$

$R^+ = \varprojlim R^{\#+}/\varpi^\# \to m$-th component

thus θ mod $(\varpi^\#)^m$ is this map.

surj: $R^+ \to R^{\#+}/\varpi^\#$ is surj. which shows that θ mod $[\varpi]$ is surj. As everything is $[\varpi]$-adically complete, $\leadsto \theta$ is surj.

(2): we claim that $\exists\, f \in \varpi R^+$, st. $f^\# \equiv p \pmod{p\varpi^\# R^{\#+}}$. $\leadsto \alpha = p/\varpi^\# \in R^{\#+}$, $\exists \beta \in R^+$, st $\beta^\# \equiv \alpha \pmod{pR^{\#+}}$ $\leadsto (\varpi\beta)^\# = (\varpi^\#\alpha) \equiv p \pmod{p\varpi^\# R^{\#+}}$, take $f = \varpi\beta$

then we can pick $p = f^\# + p\varpi^\# \sum_{n \geq 0} r_n^\# p^n$ with $r_n \in R^+$.

define $\xi = p - [f] - [\varpi] \sum_{n \geq 0} [r_n] p^{n+1}$, has the desired form lies in $\ker(\theta)$.

For this, note that θ induces surj: $f: W(R^+)/\xi \to R^{\#+}$

It is ~~another~~ enough to show that f is an iso mod $[\varpi]$:

$$W(R^+)/(\xi, [\varpi]) = W(R^+)/(p, [\varpi]) = R^+/\varpi \underbrace{=}_{\text{by tilt}} R^{\#+}/\varpi^\# \qquad \square$$

(Thm). equivalence: - - - .

~~Proof~~ Frobenius equivalence of perfectoid tilt.

$(C_1, \iota_1) \sim (C_2, \iota_2)$ if $C_2 = C_1$, $\iota_2 = \iota_1 \circ \varphi$. $\varphi(\xi)$ is a primitive element iff ξ is primitive.

对应 W(F) 里的 Frobenius:

$$\begin{array}{ccccccccc} F & & W(\mathcal{O}_F) & \sum [a_n] p^n & \xrightarrow{\text{mod } \xi} & C & \to & C^\flat & \overset{\iota}{\simeq} F \\ \varphi \downarrow & \overset{\text{lift}}{\leadsto} & \varphi \downarrow & \downarrow & & & & & \wr\!\mid \\ F & & ~~W(\mathcal{O}_R)~~ \; W(\mathcal{O}_F) & \sum [a_n^p] p^n & \xrightarrow{\text{mod } \varphi(\xi)} & C & \to & C^\flat & \overset{\iota\varphi^{-1}}{\simeq} F \end{array}$$

(2)

Figure 2. Witt Vector (2)

Def: An element $x = \sum_{i=0}^{\infty} [x_i] p^i \in A_{inf}$ is called primitive if $x_0 \neq 0$, and $\exists\, d \geq 0$ s.t. $x_d \in \mathcal{O}_F^\times$. $\deg(x) = \min\{d \mid x_d \in \mathcal{O}_F^\times\}$.

primitive ideal: if it is generated by a primitive element.

eg. For $\{1, \zeta_p, \zeta_{p^2}, \cdots\} \in F^\#$ (assume $\mathbb{C}_p$, $\mathbb{Q}_p(\mu_{p^\infty})$). we can define

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}) \in \varprojlim F^\#/p \cong F, \text{ then } (\xi) = \left(\frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1}\right) \text{ is the ker.}$$

For $\mathbb{Q}_p(p^{1/p^\infty}) \xrightarrow{\text{tilt}} \mathbb{F}_p((t^{1/p^\infty}))$ take $\xi = [t] - p$.

$$p \longmapsto t$$

---

denote $|Y|$: all untilt of alg. closed. perfectoid $F/\mathbb{F}_p$: (C, v)

$\forall y \in |Y|$. denote by (C_y, ι_y), define $\theta_y : A_{inf} = W(\mathcal{O}_F) \xrightarrow{\theta_{C_y}} \mathcal{O}_{C_y}$

$\ker =: (\xi_y) =: \mathfrak{p}_{y}$. $\forall f \in A_{inf}$. define $f(y) = \theta_y(f) \in \mathcal{O}_{C_y}$

$v_y = v_F \circ \iota \circ b : C_y \to \mathbb{R}_{\geq 0}$

Def (want to give $|Y|$ a matrix).

$$y_1, y_2 \in |Y|. \quad d(y_1, y_2) = v_{y_1}(\theta_{y_1}(\xi_{y_2})).$$

Prop: it is an ultrametric. that is:

$d(x,y) = 0 \Leftrightarrow x = y$, $\quad d(x,y) = d(y,x)$, $\quad d(x,z) \leq \max\{d(x,y), d(y,z)\}$.

proof. $x: (C_x, \iota_x)$. $y: (C_y, \iota_y)$. since ~~$C_x C_y$~~ F is alg. closed $\Rightarrow \xi_x(y) = c^\#$ for some $c \in F$.

Then c belongs to the maximal ideal $\mathfrak{m}_F$.

so. ξ_x and $\xi_x - [c]$ have the same image under

$$A_{inf} \to W(\mathcal{O}_F/\mathfrak{m}_F) = W(k_F).$$

$\rightsquigarrow \xi_x - [c]$ is also a prim. element, vanish at y. assume $\xi_y = \xi_x - [c]$

so $d(y,x) = |\xi_y(x)|_{C_x} = |\xi_x(x) - c^\#|_{C_x} = |c|_F = |c^\#|_{C_y} = |\xi_x(y)|_{C_y} = d(x,y)$

For inequality.

$$d(x,z) = |\xi_x(z)|_{C_z} \leq \max(|\xi_y(z)|_{C_z}, |c^\#|_{C_z}) = \max(d(x,y), d(y,z))$$

$$|\xi_x(z)|_{C_z} = |\xi_y(z) + c^\#|_{C_z}$$

□

③

Figure 3. Metric on $|Y|$

define $|Y_r| := \{y \in |Y| \mid d(y,0) = r\} \stackrel{\sim}{=} |\mathcal{P}|_{C_y}$

property. $\forall r \in (0,\infty)$,
$(|Y_r|, d)$ is complete.
In particular. $|Y|$ is complete.

introduce: $B_I = \mathcal{O}(|Y_I|)$; $B = \mathcal{O}(|Y|)$

$\widetilde{B}^b := \{\sum_{n \gg -\infty} [x_n]p^n : x_n \in \mathcal{O}_F\} = A_{inf}[1/p]$

$\widetilde{B} = A_{inf}[1/p, 1/[\varpi]] = \{\sum_{n \gg -\infty} [x_n]p^n : x_n \in F, |x_n|_F$ is bounded. as $n \to \infty\}$

define topology on $\widetilde{B}$:

for $x = \sum [x_n]p^n \in \widetilde{B}$, and $r \geq 0$, set

(valuation).

$V_r(x) = \inf_{n \in \mathbb{Z}} \{V_F(x_n) + n\underbrace{r}_{V_y(p)}\}$ $(r \geq 0)$. $r = V_y(p) \in (0,1)$

$\sup\{|C_n|_F \cdot \rho^n\}$ norm.

since: if $r = d(y,0)$,

$V_y(\theta_y(x)) = \inf_{i \in \mathbb{Z}} \{V(x_i) + iV_y(p)\} = \inf_{n \in \mathbb{Z}} \{V_F(x_i) + ir\} = V_r(x)$

a sum $\sum_{n \in \mathbb{Z}} [C_n]p^n$ converges w.r.t. $|\cdot|_\rho$ iff $(\rho \in (0,1))$ $\begin{cases} \lim_{n\to\infty} |C_n|_F \rho^n = 0 \\ \lim_{n\to\infty} |C_{-n}| \rho^{-n} = 0 \end{cases}$ hold.

We can extend $\theta_y : \widetilde{B} \to C_y$ to $\theta_y : B_I \to C_y$,
where B_I is the completion of B, w.r.t. to V_r.

~~For F~~ Actually for $\forall$ cpt $I \subseteq (0, +\infty)$

$\mathcal{F} = \{\bigcap_{i=1}^{n} V_{r_i}^{-1}([m, +\infty)) \mid n, m \in \mathbb{N}, r_i \in I\}$ is a topo. basis

$B_I = \varprojlim_{U \subseteq \mathcal{F}} \widetilde{B}^b/U$ is the completion.

then construct $B_{(0,+\infty)} = \varprojlim_{I \subseteq (0,+\infty)} B_I$ (as $\mathcal{O}(|Y|)$.).

review: $\varphi \curvearrowright (\sum_{n \geq 0} [a_n]p^n) = \sum_{n \geq 0} [a_n^p]p^n$ on A_{inf}.

this can be extended to $\widetilde{B}$, B_I, B.

and we have $\varphi : B_I \xrightarrow{\sim} B_{pI}$

$\boxed{4}$

Figure 4. Completion of $A_{\inf}$

crucial properties: $\forall f \in B_I$, can be written as $f = u\,\xi_{y_1}\cdots\xi_{y_n}$, $u \in B_I^\times$; use Newton polygon.

- B_I is a PID. $\mathfrak{p}_y B_I$ is a maximal ideal of B_I, for each $y \in |Y|$, s.t. $d(y,0) \in I$. $\ker(\theta_I : B_I \to C_y)$ generated by ξ_y

$\rightsquigarrow \mathfrak{p}_y B$ is ------. $= \{f \in B \mid f(y) = 0\}$

moreover, $\mathrm{Spm}\,B \longleftrightarrow |Y|$. $\mathfrak{p}_y B \longmapsfrom y$

pf: to prove $\ker(\theta_y : B_I \to C_y) = \xi_y B_I$, we need to ~~show that~~ view that $\ker(\theta_y : B_I \to C_y)$ is the closure of $\xi_y \widehat{B}$ in B_I.

Let $f \in B_I$, be in this closure, write $f = \lim_{n\to\infty} f_n$.

write $f_n = g_n \xi_y \in \xi_y B^b$. let $r \in I$,

$$V_r(g_m - g_n) = V_r(f_m - f_n) - \underbrace{\cancel{V_r(\xi_y)}}_{\neq 0}$$

thus $(g_n)_n$ is again Cauchy $\rightsquigarrow$ converge in B_I. $\square$

B_I is PID, also gives its non-zero ideals, as a monoid, under $\times$,

$\cong \mathrm{Div}^+(Y_I) :=$ monoid of formal finite sum $\sum_{y \in |Y_I|} n_y [y]$ with $n_y \in \mathbb{N}$.

Taking limit over all compact intervals $\rightsquigarrow B$:

$\{$all non-zero closed ideals of $B\} \xrightarrow{\sim} \mathrm{Div}^+(Y) := \{\sum_{y \in |Y|} n_y [y],\ n_y \in \mathbb{N},$ s.t. $\forall I \subseteq (0,\infty)$, $\{y \in |Y_I|,\ n_y \neq 0\}$ is finite$\}$

as monoids.

explicitly, let $\mathrm{ord}_y : B \to \mathbb{N} \cup \{\infty\}$, is the discrete valuation, associated to $y \in |Y|$. since $\mathfrak{p}_y B \in \mathrm{Spm}\,B$, is a PID in B, $\rightsquigarrow B_{\mathfrak{p}_y B}$ is ~~a~~ discrete valuation ring, so ord_y is well defined.

maximal. $\rightarrow$ in $B^+_{dR,y} \cong \varprojlim_n A_{inf}[1/p]/(\xi_y)^n$.

$$\sum_{y \in |Y|} n_y [y] \longmapsto \{f \in B : \mathrm{ord}_y(f) \geq n_y,\ \text{for all } y \in |Y|\}$$

$$[y] \longmapsfrom y$$

for $|Y|/\varphi^{\mathbb{Z}}$, consider

prove: here we can define "$\deg$"$(D) := \sum_{y \in |Y|/\varphi^{\mathbb{Z}}} n_y$!

1:1. this is because we can "move" I together (use φ),

$\{D \in \mathrm{Div}^+(Y) : \varphi^* D = D\} \longleftrightarrow$ monoid of formal sum of pts in $|Y|/\varphi^{\mathbb{Z}}$,

$$\sum_{y \in |Y|} n_{\pi(y)} [y] \longmapsfrom \sum_{y \in |Y|/\varphi^{\mathbb{Z}}} n_y [y]$$

$\pi : |Y| \twoheadrightarrow |Y|/\varphi^{\mathbb{Z}}$.

the "finite supp" for $\forall I$ will convert to $\sum n_y < +\infty$.

FIGURE 5. Divisors on $|Y|$ and $|Y|/\varphi^{\mathbb{Z}}$

Theorem 6.1. *If F is algebraically closed, the morphism of monoids:*

$$\mathrm{div} : (\bigcup_{d\geq 0} B^{\varphi=p^d} \setminus \{0\})/\mathbb{Q}_p^\times \to \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$$

is an isomorphism.

Proof. For injectivity, let $x \in B^{\varphi=p^k}$, $y \in B^{\varphi=k'}$ are non-zero elements such that $\mathrm{div}(x) = y$. Without loss of generality we may assume $k' \geq k$. Then by projecting to $\mathrm{Div}^+(Y_I)$ (which we have seen is isomorphic to the monoid of non-zero ideals of the principal ideal domain B_I) shows that the image of x, y in the projection differ by a unique unit; this being true for every compact interval I, taking the limit implies that there is $u \in B^\times$ such that $f = gu$. Moreover, $\varphi(u) = p^{k-k'}$, i.e.

$$u \in B^{\varphi=p^{k-k'}} = \begin{cases} 0, & \text{if } d \neq d' \\ \mathbb{Q}_p, & \text{if } d = d' \end{cases}$$

by Newton polygon arguments.

Next we explain the proof of surjectivity. We see that $\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$ is generated by its elements of the form $\sum_{n\in\mathbb{Z}}[\varphi^n(y)]$, for $y \in |Y|$. Therefore it is enough to find $t_y \in B$ for such $y \in |Y|$, satisfying $\mathrm{div}(t_y) = \sum_{n\in\mathbb{Z}}[\varphi^n(y)]$. Let $\xi_y = p - [x] \in W(\mathcal{O}_F)$ be a primitive element of degree one corresponding to y, where $x \in \mathfrak{m}_F$.

Considering the infinite product

$$\Pi^+(\xi_y) := \prod_{n\geq 0} \varphi^n(\frac{\xi_y}{p}) = \prod_{n\geq 0}(1 - \frac{[x^{p^n}]}{p}).$$

This product converges in B, and satisfies

$$\mathrm{div}(x) = \sum_{n\geq 0} \varphi^{-n}(y).$$

To find a non-zero element $\Pi^-(\xi_y) \in B$, such that

$$\varphi(\Pi^-(\xi_y)) = \xi_y \Pi^-(\xi_y), \quad \mathrm{div}(\Pi^-(\xi_y)) = \sum_{n<0}[\varphi^n(y)],$$

we need the "solve equation" lemma:

Lemma 6.1. *Let $b \in \tilde{B} \cap W(F)^+$ be any element. Then the $\mathbb{Q}_p$-vector space*

$$\tilde{B}^{\varphi=b} := \{x \in \tilde{B} | \varphi(x) = bx\}$$

is one dimensional.

Then we can take $t_y = \Pi^+(\xi_y)\Pi^-(\xi_y)$ as desired. □

In the example: $y = \mathbb{Q}_p(\mu_{p^\infty})$ with its tilt $\mathbb{F}_p((t^{1/p^\infty}))$, we can choose

$$\xi_y = \frac{[\epsilon] - 1}{[\epsilon^{1/p}] - 1}$$

and we can take $\Pi^-(\xi_y) = [\epsilon^{1/p}] - 1$. Then

$$t_y = ([\epsilon^{1/p}] - 1) \prod_{n\geq 0} \varphi^n(\frac{\xi_y}{p}) = ([\epsilon^{1/p}] - 1) \prod_{n\geq 0} \frac{([\epsilon^{1/p}] - 1)^{n+1}}{p \cdot ([\epsilon^{1/p}] - 1)^n}.$$

Remark 6.1. We can find "another" t_y for every y, using "log" function.

Lemma 6.2. *Let A be an algebra over $\mathbb{Q}_p$, equipped with a norm $|\cdot|_A$ satisfying the condition*

$$|x \cdot y|_A \leq |x|_A \cdot |y|_A.$$

Let $x \in A$ be an element satisfying $|x-1|_A < 1$. Then infinite sum

$$\log(x) = \sum_{k>0} \frac{(-1)^{k+1}}{k}(x-1)^k$$

is a well-defined element in the completion of A with respect to this norm. This is because the individual terms $\frac{(-1)^{k+1}}{k}(x-1)^k$ converge to zero as $k \to \infty$, as the exponent decreases faster than the linear function.

Let x be an element of F satisfying $|x-1|_F < 1$. Note that $[x]-1$ is an element of the ring $W(\mathcal{O}_F)$, and therefore admits a Teichmüller expansion

$$[x] - 1 = \sum_{n \geq 0} [c_n] \, p^n$$

where the coefficients $c_n \in \mathcal{O}_F$. Moreover, we have $c_0 = x - 1$, so that $|c_0|_F < 1$. For each real number $\rho \in (0,1)$, we have

$$|[x]-1|_\rho = \sup\{|c_n|_F \rho^n\} \leq \max(|c_0|_F, 1 \cdot \rho) < 1.$$

Applying the remark, we conclude that the series

$$\log([x]) = \sum_{k>0} \frac{(-1)^{k+1}}{k}([x]-1)^k$$

converges with respect to the Gauss norm $|\cdot|_\rho$. Since ρ is arbitrary, it follows that $\log([x])$ is a well-defined element of the ring B.

For each $x \in 1 + \mathfrak{m}_F$, we have

$$\varphi(\log([x])) = \log(\varphi([x])) = \log\left([x^p]\right) = p \log([x]).$$

That is, $\log([x])$ actually belongs to the eigenspace $B^{\varphi=p} \subseteq B$. Then we want to show that in the example: $C_y = \mathbb{Q}_p(\mu_{p^\infty})$ with its tilt $F = \mathbb{F}_p((t^{1/p^\infty}))$, with $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in 1 + \mathfrak{m}_F$,

$$\operatorname{div}(\log([\epsilon])) = \sum_{y \in |Y|/\varphi^{\mathbb{Z}}} \operatorname{ord}_y(\log([\epsilon]))[y] = \sum_{n \in \mathbb{Z}} \varphi^n(y).$$

We can choose

$$\xi_y = \frac{[\epsilon]-1}{[\epsilon^{1/p}]-1} \in W(\mathcal{O}_F)$$

as the generator of $\ker(\theta_y)$. Note that the image of $[\epsilon^{1/p}]$ in C_y is a primitive pth root of unity ζ_p, so that $\zeta_p - 1$ in invertible in C_y and therefore $[\epsilon^{1/p}]-1$ is invertible in $B_{\mathfrak{p}_y B}$, since $B_{\mathfrak{p}_y B}/(\mathfrak{p}_y) = C_y$. It follows that $[\epsilon]-1$ is a unit multiple ξ_y in $B_{\mathfrak{p}_y B}$, and is therefore a uniformizer. The congrence

$$\log([\epsilon]) = \sum_{k>0} \frac{(-1)^{k+1}}{k}([\epsilon]-1)^k \equiv [\epsilon]-1 \pmod{([\epsilon]-1)^2}$$

shows that $\operatorname{ord}_y(\log([\epsilon])) = 1$. Since $\log([\epsilon]) \in B^{\varphi=p}$, we have the desired equation.

Different Untilt of F will differ from a isomorphism $\iota : C_y^\flat \cong F$ when choosing ϵ, however the whole procedures are the same.

Remark 6.2. This monoid isomorphism is also an isomorphism with respect to the grade. Degree of the left side is given by $\varphi = p^k$; Degree of the right side is given by $\sum_{y\in|Y|/\varphi^{\mathbb{Z}}} n_y$. This is a well defined function since we need the divisor of $|Y|$ is finite support for every compact interval I, and the Frobenius action can gather the support in the sum $\sum_{y\in|Y|/\varphi^{\mathbb{Z}}} n_y[y]$ in a certain compact interval, thus it is finite.

A clear corollary of this theorem is

Corollary 6.1. *The graded ring $P = \oplus_{k\leq 0} B^{\varphi=k}$ is graded factorial with irreducible elements of degree* 1*.*

We start the further study by proving the fundamental exact sequence of p-adic Hodge theory.

First choose $y \in |Y|$ with the related maximal ideal, denoted as (ξ). Define The ring

$$B^+_{\mathrm{dR}} = \varprojlim_n \mathbb{A}_{\inf}[1/\varpi]/(\xi)^n$$

as the ξ-completion of $\mathbb{A}_{\inf}[1/\varpi]$. Directly, we can check that B^+_{dR} is a ϖ-adci completed and completed as a valuation ring. It is a discrete valuation ring, and the residue field is C_y, which is the perfectoid field corresponding to y.

Theorem 6.2. *Let $y \in |Y|$ and set $t \in B^{\varphi=p}$ is one preimage of $\sum_{|Y|/\varphi^{\mathbb{Z}}}[y]$. Then for $d \leq 0$ the natural sequence*

$$0 \to \mathbb{Q}_p t^d \to B^{\varphi=p^d} \to B^+_{\mathrm{dR},y}/\xi_y^d B^+_{\mathrm{dR},y} \to 0$$

is exact.

Proof. If $x \in B^{\varphi=p^d}$ maps to 0 in $B^+_{\mathrm{dR},y}/\xi_y^d B^+_{\mathrm{dR},y}$, then

$$\mathrm{div}(x) \geq d \cdot y$$

which implies that

$$\mathrm{div}(x) \geq d \cdot \mathrm{div}(t)$$

because $\mathrm{div}(x)$ is φ-invariant. By Theorem 6.1 this implies that

$$x \in \mathbb{Q}_p t^d.$$

By induction, the surjectivity of $B^{\varphi=p^d} \to B^+_{\mathrm{dR},y}/\xi_y^d B^+_{\mathrm{dR},y}$ can be reduced to the case that $d = 1$. Then it suffices to see that the map

$$\theta_y : B^{\varphi=p} \to C_y$$

is surjective. But then we can apply the more precise Lemma.

Lemma 6.3. *We have a commutative with exact row*

$$\begin{tikzcd}
1 \arrow[r] & \epsilon^{\mathbb{Q}_p} \arrow[r] \arrow[d] & 1+\mathfrak{m}_F \arrow[r, "\log"] \arrow[d, "\log([-])"] & C_y \arrow[r] \arrow[d, "="] & 1 \\
0 \arrow[r] & \mathbb{Q}_p t \arrow[r] & B^{\varphi=p} \arrow[r, "\theta_y"] & C_y \arrow[r, "0"] & 0
\end{tikzcd}$$

where

$$\epsilon = [1, \zeta_p, \zeta_{p^2}, \ldots] \in \mathcal{O}^\flat_{C_y} \cong \mathcal{O}_F$$

is the compatible system of primitive p^n-roots of unity and $t = \log([\epsilon]) = \Pi(\xi_y)$.

□

Corollary 6.2. *There is a canonical isomorphism*

$$P/tP \cong S := \{f \in C_y[T] \mid f(0) \in \mathbb{Q}_p\}.$$

of graded $\mathbb{Q}_p$-algebras. In particular, $\mathrm{Proj}(P/tP) = (0)$*.*

Proof. Let $\theta_y : B \to C_y$ be the canonical quotient associated to y. Then we claim that the morphism

$$\alpha : P/tP \to S, \sum_{d\geq 0} x_d \mapsto \sum_{d\geq 0} \theta_y(x_d)\, T^d$$

is an isomorphism of graded $\mathbb{Q}_p$-algebras. It is trivially an isomorphism in degree 0 and in degrees ≥ 1 by Theorem 6.2. Indeed, surjectivity is clear as each element in C_y has arbitrary roots. To prove injectivity let $x \in P_d, d \geq 1$, with $\theta_y(t) = 0$. Then $x \equiv t \cdot t' \mod \xi_y^d B_{\mathrm{dR},y}^+$ for some $t' \in B^{\varphi = p^{d-1}}$ by Theorem 6.2. This implies that $x - t\cdot t' \in \mathbb{Q}_p \cdot t^d$ as desired. Let us prove that $\mathrm{Proj}(P/tP) = \{(0)\}$. For this pick a graded prime ideal $\mathfrak{p} \subseteq S$. If $cT^d \in \mathfrak{p}$ for some $d \geq 1$ and $c \in C_y^\times$ multiplying with $c^{-1}T$ yields $T^{d+1} \in \mathfrak{p}$, and then $\mathfrak{p} = (T)$, i.e., $\mathfrak{p}$ does not appear in $\mathrm{Proj}(S)$. □

Now we can prove one main result of this Interlude:

Theorem 6.3. *(Fargues-Fontaine). Let $t \in P_1 = B^{\varphi=p}$ be non-zero. Then*

$$B_t := P[1/t]_0 = B[1/t]^{\varphi=1}$$

is a principal ideal domain[3]*, and*

$$\mathrm{Proj}(P) \cong \mathrm{Spec}\,(B_t) \cup \{\infty_t\}\,.$$

where ∞_t is the unique closed point in the vanishing locus of t.

In particular, X is noetherian and regular of Krull dimension 1*.*

Proof. If $x \in B_t$ is non-zero, then for some $d \geq 0$

$$x = \frac{t'}{t^d}$$

with $t' \in B^{\varphi=p^d}$. Applying 6.1 we can factor t' into elements $t_1, \ldots, t_d \in B^{\varphi=p}$, i.e.,

$$x = \frac{t_1}{t} \cdots \frac{t_d}{t}.$$

By 6.2 and the above Corollary each t_i/t is either a unit or generates a maximal ideal in B_t. This finishes the proof that B_t is a principal ideal domain. To see that X is noetherian and regular of Krull dimension 1, pick two non-$\mathbb{Q}_p$-linear $t, t' \in B^{\varphi=p}$. Then

$$\mathrm{Proj}(P) = \mathrm{Spec}\,(B_t) \cup \mathrm{Spec}\,(B_{t'})$$

which finishes the proof. □

Define $X = \mathrm{Proj}(P)$ the Fargues-Fontaine Curve. Let $|X|$ denote the set of closed points of X.

[3] It coincides with the fundamental open set related to t

Theorem 6.4. *There are canonical bijections*

$$|X| = |Y|/\varphi^{\mathbb{Z}} = (P_1\backslash\{0\})/\mathbb{Q}_p^{\times}.$$

Moreover, for $y \in |Y|$ with image $x \in |X|$ there is a canonical isomorphism

$$\mathcal{O}_{X,x}^{\wedge} \cong B_{\mathrm{dR},y}^{+}$$

Similar to $\mathbb{P}_1(\mathbb{C})$, as the B_t is a principal ideal domain, X is a proper curve:

Proposition 6.1. *Denote $k(X)$ as the function field of Fargues-Fontaine Curve. Then for $f \in k(X)^{\times}$.*

$$\deg(\mathrm{div}(f)) = 0,$$

and the resulting morphism

$$\deg : \mathrm{Pic}(X) \to \mathbb{Z}$$

is an isomorphism (with inverse $n \to \mathcal{O}_{\mathcal{X}}(n)$).

7. L18-22: Pro-étale site (loading)