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It's a quick review of riemannian geometry. 中英混杂方便四处誊抄 (主要是誊抄梅老师的流形与几何初步, 我爱梅老师的书). 八月份上了 BICMR 的 2022 Summer School on Differential Geometry, 简单记录一下黎曼几何的部分因为感觉只是听懂了但是没学很懂, 不记录一下以后都不记得学过这个课 (考试的时候记得就够了, 因为真的不好玩). 只上了复几何和黎曼几何, 二阶... 方程选讲没听. 因为复几何发了讲义以及本身也熟悉一些就只敲黎曼几何的 LaTeX. 总共十节课只 follow 了八节课, 专题课程 (讲了 Ricci flow 与 Kahler-Ricci flow, Perelman 的庞加莱猜想的证明和 Hamilton-Tian 猜想及其进展) 没 (兴趣) 跟上. 这个 review 会只保留我喜欢的证明细节以及完全按照授课的顺序并真包含于授课内容 (闲扯都在 remark 里面, notation 可能会有些乱,anyway).

码完就回归老本行. 感谢 BICMR(让我多学半个月几何:P).

结果一做题发现根本写不出 Levi-Civita 长啥样, 全白学了.note 可能真的写的很烂, 毕竟黎曼几何练习时长可能只有十二三天... 中间还在复习复几何 (好玩多了!).

无特别说明, ∇ 就是 Levi-Civita connection.

1 Differential Manifold

2 Levi-Civita connection, Riem. curvature

Theorem 1. *设 (M, g) 为黎曼流形, 则满足下面条件的仿射联络 ∇ 是存在且惟一的:*

(i) $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle, \quad \forall X, Y, Z \in \Gamma(TM)$;

(ii) $\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall X, Y \in \Gamma(TM)$.

在局部坐标系 $\{x^i\}$ 中, 黎曼度量 g 可表示为

$$g = g_{ij} dx^i \otimes dx^j,$$

Levi-Civita 联络的 Christoffel 系数在局部坐标下的表达式

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

其中, g^{kl} 表示 $(g_{ij})_{n\times n}$ 的逆矩阵在 (k, l) 位置的元素.

Remark. 这里可以有一个合适的解释, 关于 the 2nd condition above (梅老师的书叫它无挠, 下面的引用是来自 milnor 的 morse theory, 叫 symmetric property):

An alternative characterization of symmetry will be very useful later. Consider a parametrized surface in M : that is a smooth function

$$s : \mathbb{R}^2 \longrightarrow M.$$

By a vector field V along s is meant a function which assigns to each $(x, y) \in \mathbb{R}^2$ a tangent vector

$$V_{(x,y)} \in TM_{s(x,y)}.$$

As examples, the two standard vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ give rise to vector fields $s_*\frac{\partial}{\partial x}$, and $s_*\frac{\partial}{\partial y}$ along s. These will be denoted briefly by $\frac{\partial s}{\partial x}$ and $\frac{\partial s}{\partial y}$; and called the velocity vector fields of s. For any smooth vector field V along s the *covariant derivatives* $\frac{\mathrm{D}V}{\mathrm{d}x}$ and $\frac{\mathrm{D}V}{\mathrm{d}y}$ are new vector fields, constructed as follows. For each fixed y_0, restricting V to the curve

$$x \longmapsto s(x, y_0)$$

one obtains a vector field along this curve. Its covariant derivative with respect to x is defined to be $\left(\frac{\mathrm{D}V}{\mathrm{d}x}\right)_{(x,y_0)}$. This defines $\frac{\mathrm{D}V}{\mathrm{d}x}$ along the entire parametrized surface s.

Lemma 2.1. *If the connection is symmetric then*

$$\frac{\mathrm{D}}{\mathrm{d}x}\frac{\partial s}{\partial y} = \frac{\mathrm{D}}{\mathrm{d}y}\frac{\partial s}{\partial x}.$$

课上讲的 riem. curvature 定义和梅老师的书差了一个符号:

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$$

$$(X, Y, Z) \mapsto R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X)Z - \nabla_{[X,Y]}Z$$

Remark. 我们理解这玩意是需要用 curve 上的 vector field 的:

Now consider a parametrized surface

$$s : \mathbb{R}^2 \longrightarrow M.$$

Given any vector field V along s. One can apply the two covariant differentiation operators $\frac{\mathrm{D}}{\mathrm{d}x}$ and $\frac{\mathrm{D}}{\mathrm{d}y}$ to V. In general these operators will not commute with each other. However, we have:

$$\frac{\mathrm{D}}{\mathrm{d}y}\frac{\mathrm{D}}{\mathrm{d}x}V - \frac{\mathrm{D}}{\mathrm{d}x}\frac{\mathrm{D}}{\mathrm{d}y}V = R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right)V$$

不过在把逆变指标 (首先需要说明这是一个张量场) 降为协变指标的时候又一致了 (用度量 pairing, R_m). 我们依次定义 (证明 or 计算, 反正都是算)

- 1st/2nd Bianchi identity and other symmetric property;
- In local coordinate, we have...;
- Define section/ricci/scalar curvature;
- If for some $K_0 \in \mathbb{R}$, $K \equiv K_0$ at all $p \in M$, then (e.g. space form):

$$R_m(X, Y, Z, W) = K_0\{\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle\}$$

- Example: Einstein Manifold.

我还是 copy 一下吧:

$$\begin{aligned} R_{ijkl} =& \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right) \\ & + g_{rs}\Gamma^r_{jk}\Gamma^s_{il} - g_{rs}\Gamma^r_{jl}\Gamma^s_{ik} \end{aligned}$$

3 Geodesics, Variation formula

定义曲线的长度: 设 $\sigma : [a,b] \to M$ 为黎曼流形 (M,g) 上的 C^1 曲线, 定义其长度 $L(\sigma)$ 为

$$L(\sigma) = \int_a^b \|\dot\sigma(t)\| \mathrm{d}t$$

其中 $\|\dot\sigma(t)\| = (g(\dot\sigma(t),\dot\sigma(t)))^{1/2}$ 是切向量 $\dot\sigma(t)$ 的长度. 这玩意与重新参数化无关以及在等距变换下不变.

Definition 1 (Variation of curves)**.** Let $\gamma : [0,l] \times (-\epsilon,\epsilon) \to M$, $\gamma_u(t) = \gamma(t,u)$, diffeomorphism. Denote

$$X = \frac{\partial\gamma}{\partial t} = \frac{\partial\gamma_u}{\partial t},\ Y = \frac{\partial\gamma}{\partial u}$$

be the tangent field and variation field of γ, respectively.

Proposition 3.1 (1st variation formula)**.** *Denote $(\nabla_X X)^\perp$ as the projection of $\nabla_X X$ to X, then*

$$\frac{\mathrm{d}}{\mathrm{d}u} L(\gamma_u) = \langle Y, \frac{X}{|X|}\rangle\Big|_{t=0}^{l} - \int_0^l \frac{1}{|X|}\cdot\langle Y, (\nabla_X X)^\perp\rangle \mathrm{d}t$$

Thus if we fix $\gamma(0)$ and $\gamma(u)$, then the boundary will be vanished.

A geodesic on a smooth manifold M with an affine connection ∇ is defined as a curve $\gamma(t)$ such that parallel transport along the curve preserves the tangent vector to the curve, i.e.

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

It's a ODE. Then we define the exponential map:

$$\exp_p(v) = \gamma_v(1) \text{ once the RHS exists}, \quad \forall p \in M,\ v \in T_pM$$

Let $\alpha : I \to T_pM$ be a curve differentiable in T_pM such that $\alpha(0) = 0$ and $\alpha'(0) = v$. Since $T_pM \cong \mathbb{R}^n$, we can choose $\alpha(t) := vt$. In this case, by the definition of the differential of the exponential in 0 applied over v, we obtain:

$$T_0 \exp_p(v) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp_p \circ \alpha(t)\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp_p(vt)\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma(1,p,vt))\Big|_{t=0} = v.$$

So (with the right identification $T_0T_pM \cong T_pM$) the differential of $\exp_p$ is the identity. By the implicit function theorem, It's a diffeomorphism on neighborhood of $0 \in T_pM$. The Gauss Lemma now tells that $\exp_p$ is also a *radial isometry*.

Lemma 3.1. *Let $v, w \in B_\epsilon(0) \subset T_vT_pM \cong T_pM$ and $M \ni q = \exp_p(v)$. Then,*

$$\left\langle T_v \exp_p(v), T_v \exp_p(w)\right\rangle_q = \langle v, w\rangle_p.$$

It shows that locally $\exp_p$ is well-defined, and the line is the unique minimal geodesic joining p and q. i.e. in all the directions permitted by the domain of definition of $\exp_p$, it remains an isometry. The proof needs (at least the idea) of Jacobi field.

Theorem 2 (Hopf-Rinow). *Let (M, g) be a connected Riemannian manifold. Then the following statements are equivalent:*

- *The closed and bounded subsets of M are compact;*
- *M is a complete metric space;*
- *M is geodesically complete; that is, for every $p \in M$, the exponential map $\exp_p$ is defined on the entire tangent space T_pM.*

The following is corollary of the Hopf-Rinow theorem:

- For a complete metric space, every two points can be joined by a minimizing geodesic.
- cpt. connected Riem. mfd. must be complete.
- Isometry will keep the completeness of a Riem. mfd..

4 2nd variation fields, Jacobi fields

由于是个变分, 下面的 X, Y 不仅依赖于 γ_0 的信息, 实际上是 field 的 restriction. 这里偷个懒, 就不记成 $\tilde{X}, \tilde{Y}$. 计算也没啥技术含量, 无非是用 Levi-Civita 的与度量相容性把求导 (向量场) 放到 $\langle\cdot,\cdot\rangle$ 里面去, 全部导干净了就往里带.

Proposition 4.1 (2nd variation formula).

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\bigg|_{u=0} L(\gamma_u) = \langle \nabla_Y Y, X\rangle\bigg|_0^l - \int_0^l [|(\nabla_X Y)^\perp|^2 - R_m(X, Y, X, Y)]\mathrm{d}t$$

Theorem 3 (Bonnet-Myers). *Let (M^n, g) be a complete connected Riem. manifold of dimension n whose Ricci curvature satisfies $Ric \geq (n-1)K_0 \cdot g$ for some positive real number K_0. Then*

$$\mathrm{Diam}(M, g) = \sup\{d(p, q) | p, q \in M\} \leq \frac{\pi}{\sqrt{K_0}}$$

因为作业里有道题是把 $Ric \geq (n-1)K_0 \cdot g$ 换成

$$Ric(\dot{\gamma}(t), \dot{\gamma}(t)) \geq a + \frac{\mathrm{d}f}{\mathrm{d}t},\ |f(t)| \leq const, \quad along\ \gamma$$

然后要求证明一个 Diam 的估计, 所以我就打一遍证明吧.

证明. Let p and q be any two points in M. Since M is complete, there exists a minimizing geodesic $\gamma : [0,1] \to M$ joining p to q. Let us consider parallel fields $e_1(t), \ldots, e_{n-1}(t)$ along γ which are orthonormal, for each $t \in [0,1]$ (I forgot to take unit speed here (I apologize)), belong to the orthogonal complement of $\gamma'(t)$. Specifically, after choosing the orthogonal basis in T_pM, the parallel translation, $\nabla_{\dot\gamma(t)} e_i(t) = 0$, $\exists!$ ODE solution. Let $e_n(t) = \frac{\gamma'(t)}{l}$ and let Y_j be a vector field along γ given by

$$Y_j(t) = (\sin \pi t) e_j(t), \quad j = 1, \ldots, n-1$$

It is clear that $Y_j(0) = Y_j(1) = 0$, therefore Y_j generates a proper variation of γ, whose energy we denote by E_j. Using the formula above, we obtain

$$\begin{aligned} \frac{1}{2} E_j''(0) &= -\int_0^1 \left\langle Y_j, Y_j'' + R\left(\gamma', Y_j\right)\gamma'\right\rangle \mathrm{d}t \\ &= \int_0^1 \sin^2 \pi t \left(\pi^2 - l^2 K\left(e_n(t), e_j(t)\right)\right) \mathrm{d}t \end{aligned}$$

where $K\left(e_n(t), e_j(t)\right)$ is the sectional curvature. Summing on j and using the definition of the Ricci curvature and the given estimate, we get

$$\begin{aligned} \frac{1}{2} \sum_{j=1}^{n-1} E_j''(0) &= \int_0^1 \left\{\sin^2 \pi t \left((n-1)\pi^2 - (n-1) l^2 \operatorname{Ric}_{\gamma(t)}\left(e_n(t)\right)\right)\right\} \mathrm{d}t \\ &\leq \int_0^1 \left\{\sin^2 \pi t \left((n-1)\pi^2 - (n-1) l^2 K_0\right)\right\} \mathrm{d}t \end{aligned}$$

Then the conclusion. □

Remark. 反正打了也没人看我就先写在这个里面了. 最后一个式子换成:

$$\begin{aligned} \frac{1}{2} \sum_{j=1}^{n-1} E_j''(0) &= \int_0^1 \left\{\sin^2 \pi t \left((n-1)\pi^2 - (n-1) l^2 \operatorname{Ric}_{\gamma(t)}\left(e_n(t)\right)\right)\right\} \mathrm{d}t \\ &\leq (n-1)\left[\frac{\pi^2 - l^2 a}{2} - l \int_0^1 \sin^2(\pi s) \frac{\mathrm{d}f}{\mathrm{d}s} \mathrm{d}s\right] \\ &= (n-1)\left[\frac{\pi^2 - l^2 a}{2} + \pi l \int_0^1 \sin(2\pi s) f(s) \mathrm{d}s\right] \\ &\leq (n-1)\left[\frac{\pi^2 - l^2 a}{2} + c\pi l \int_0^1 |\sin(2\pi s)| \mathrm{d}s\right] \\ &= (n-1)\left[\frac{\pi^2 - l^2 a}{2} + 2cl\right]. \end{aligned}$$

We apply integration by parts here (the ommitance of ^{2} appears in re-scalar). Then solve the one-variable quadratic inequality, we get the final estimate:

$$\operatorname{Diam}(M, g) \leq \frac{\pi^2}{\sqrt{c^2 + \pi^2 a} - c}$$

Since M is complete and bounded, then compact.

下面一个定理我们课上只证明了推论里的偶情形:

Theorem 4 (Weinstein and Synge)**.** *Let f be an isometry of a compact oriented Riemannian manifold M^n. Suppose that M has positive sectional curvature and that f preserves the orientation of M if n is even, and reverses it if n is odd. Then f has a fixed point.*

Proposition 4.2 (Synge)**.** *Let M^n be a compact manifold with positive sectional curvature. Then if M^n is orientable and n is even, then M is simply connected; If n is odd, then M^n is orientable.*

Definition 2 (Jacobi field)**.** 一组测地线 $(\gamma_u(t))$ 的变分.

In other words, the Jacobi fields along a geodesic form the tangent space to the geodesic in the space of all geodesics. Obviously, a vector field J along a geodesic γ is said to be a *Jacobi* field if it satisfies the *Jacobi* equation:

$$\frac{D^2}{dt^2}J(t) - R(J(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0,$$

(注意这里这个正负号和 R 的定义有关... 我晕了好几次) 这是一个二阶 ODE, 给了两个初值就有局部解.

构造 Jacobi field 的方法是重要的: 对 $v, w \in T_pM$, 定义

$$\gamma_u(t) = \gamma(t,u) = \exp_p(t \cdot (v + u \cdot w)),\ J(t) = \frac{\partial}{\partial u}\bigg|_{u=0} \gamma(t,u)$$

这就是满足 $J(0) = 0, J'(0) = \nabla_{\dot{\gamma}(0)} J(0) = w$ 的场 (与 v 无关). 这个观察和下面这个 lemma 可以告诉我们考虑 $J(0) = 0,\ J \perp \gamma'$ 的场足够了:

Lemma 4.1. *For any Jacobi field J along geodesic γ,*

$$\langle J, \gamma' \rangle(t) = \langle J, \gamma' \rangle(0) + t \cdot \langle J', \gamma' \rangle(0)$$

把这族 Jacobi field 记为 $J_0^\perp$, 这是一个 $n-1$ 维的向量空间, 取 t 值后与 $\dot{\gamma}(t)$ 张成该点处的切空间.

下面解释 conjugate point: 取初始点 p, $q = \gamma(t_0)$,γ 是测地线. q 是 conjugate point, 如果存在非平凡 Jacobi field along γ, 而且在 p, q 都消失; Then $\exp_p$is degenerated at t_0; Another viewpoint is that conjugate points tell when the geodesics fail to be length-minimizing. 但这不代表一定有两条或以上测地线连着这两个点, 只能说 start at p and *almost* end at q.

Hyperbolic space H^n (Poincaré/Upper half space model), $\mathbb{R}^n$, $\mathbb{S}^n$ are all constant sectional curvature riem. mfds.,(called the space form?). We can uniform all constant sectional curvature cases as the following:

Theorem 5. *The universal cover of an n-dimensional space form M^n with negative constant sectional curvature is isometric to H^n, with curvature $K = 0$ is isometric to $\mathbb{R}^n$, and with positive constant curvature is isometric to $\mathbb{S}^n$.*

In the constant sectional curvature situation, Jacobi equation can be written as:

$$\frac{D^2}{dt^2}J - K_0(\langle J, \dot\gamma\rangle\dot\gamma - J) = 0.$$

If $J \in J_0^\perp$, then (given J, J')

$$J(t) = \begin{cases} a \cdot \frac{\sin(\sqrt{K_0}\cdot t)}{\sqrt{K_0}} \cdot e(t),\ a \in \mathbb{R} & \text{if } K_0 > 0 \\ a \cdot t \cdot e,\ a \in \mathbb{R} & \text{if } K_0 = 0 \\ a \cdot \frac{\sinh(\sqrt{-K_0}\cdot t)}{\sqrt{-K_0}} \cdot e(t),\ a \in \mathbb{R} & \text{if } K_0 < 0 \end{cases}$$

5 Rauch comparison, energy functional

Let M be a Riemannian manifold and let $\gamma : [0, l] \to M$ be a unit speed geodesic of M. Let X be a piecewise differentiable vector field along γ. Define the index:

$$\int_0^l [\langle \nabla_{\dot\gamma}X, \nabla_{\dot\gamma}X\rangle - R_m(\gamma', X, \gamma', X)]\mathrm{d}t = I(X, X)$$

I also (can be extended to) define a symmetric form on the space of vector fields on γ:

$$\int_0^l [\langle \nabla_{\dot\gamma}X, \nabla_{\dot\gamma}Y\rangle - R_m(\gamma', X, \gamma', Y)]\mathrm{d}t = I(X, Y)$$

Lemma 5.1. *Let $\gamma : [0, l] \to M$ be a geodesic without conjugate points to $\gamma(0)$ in the interval $[0, l]$. Let $J \in J_0^\perp$, and let X be a vector field along γ, with $\langle X, \gamma'\rangle = 0,\ X(0) = 0,\ X(l) = J(l)$. Then*

$$I(X, X) \geq I(J, J)$$

and equality occurs if and only if $X = J$ on $[0, l]$.

做法就是取 $J_0^\perp$ 的基之后展开使劲整理算.

Theorem 6 (Rauch)**.** *Let $\gamma/\underline{\gamma} : [0, l] \to M^{n+k}/\underline{M}^n$ be unit speed geodesics of two complete riemannian manifolds and let $J/\underline{J}$ be Jacobi fields along $\gamma/\underline{\gamma}$, respectively, such that*

$$J(0) = \underline{J}(0) = 0, \quad \langle J'(0), \gamma'(0)\rangle = \langle \underline{J}'(0), \tilde\gamma'(0)\rangle$$
$$|J'(0)| = |\underline{J}'(0)|$$

Assume that γ does not have conjugate points on $[0, l]$ and that, for all t and all $x \in T_{\gamma(t)}(M), \underline{x} \in T_{\underline{\gamma}(t)}(\underline{M})$, we have

$$\underline{K}\left(\underline{x}, \underline{\gamma}'(t)\right) \leq K\left(x, \gamma'(t)\right),$$

where $K(x, y)$ denotes the sectional curvature with respect to the plane generated by x and y. Then

$$|\underline{J}| \geq |J|$$

In addition, if for some $t_0 \in [0, l]$, we have $|\underline{J}(t_0)| = |J(t_0)|$, then $\underline{K}\left(\underline{J}(t), \underline{\gamma}'(t)\right) = K(J(t), \gamma'(t))$, for all $t \in [0, t_0]$.

The idea is to construct a comparison vector field along γ, and apply the index lemma.

For (M, g) a complete riem. mfd., curve γ, define the energy functional:

$$E(\gamma) = \frac{1}{2} \int_0^l \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle \mathrm{d}t$$

我们和前面一样取 γ 的一组变分, 算一下: (when $u = 0$)

$$\frac{\partial}{\partial u} E(\gamma_u) = \int_0^l \langle \nabla_X Y, X \rangle \mathrm{d}t = -\int_0^l \langle \nabla_X X, Y \rangle \mathrm{d}t$$

Then γ_0 is a critical point of E of this variation iff γ_0 is a geodesic. Then we say that Jacobi fields are determined by the energy functional:

$$\frac{\partial^2}{\partial u^2} E(\gamma_u) = \langle \nabla_X Y, Y \rangle(l) + \langle X(l), \nabla_Y \tilde{Y}(l, 0) \rangle - \int_0^l \langle \mathrm{Jac}(Y), Y \rangle \mathrm{d}t$$

where $\mathrm{Jac}(Y) = \nabla_X \nabla_X Y - R(X, Y)X$.

Similarly, for a two-parameter families:

$$\tilde{\gamma}(t, u_1, u_2) : [0, l] \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to M$$

with $\tilde{\gamma}(t, 0, 0) = \gamma(t)$ being geodesic, $\tilde{\gamma}(0, u_1, u_2) \equiv p$, set:

$$\tilde{Y}_1 = \frac{\partial \tilde{\gamma}}{\partial u_1}, \quad \tilde{Y}_2 = \frac{\partial \tilde{\gamma}}{\partial u_2}, \quad Y_{1/2} = \tilde{Y}_{1/2} \Big|_{\gamma}$$

then (evaluated at $u_1 = u_2 = 0$)

$$\frac{\partial^2}{\partial u_1 \partial u_2} E(\tilde{\gamma}) = \langle \nabla_X Y_1, Y_2 \rangle(l) + \langle X(l), \nabla_{Y_1} \tilde{Y}_2(l, 0) \rangle - \int_0^l \langle \mathrm{Jac}(Y_1), Y_2 \rangle \mathrm{d}t$$

If $Y_1(l) = Y_2(l) = 0$, then the boundary terms are vanished, i.e.

$$\frac{\partial^2}{\partial u_1 \partial u_2} E(\tilde{\gamma}) \Big|_{u_1 = u_2 = 0} = -\int_0^l \langle \mathrm{Jac}(Y_1), Y_2 \rangle \mathrm{d}t$$

Denote it as $B_\gamma(Y_1, Y_2)$. (Obviously $= B_\gamma(Y_2, Y_1)$).

The following setting is 1-parameter variation.

Lemma 5.2.

$$\frac{\partial^2}{\partial u^2} \Big|_{u=0} E(\gamma_u) \text{ iff } Y \text{ is a Jacobi field.}$$

Proposition 5.1. *For γ a minimal geodesic, There is no conjugate point to $\gamma(0)$ on $\gamma|_{[0,l]}$.*

6 Manifolds of non-positive curvature

Definition 3 (local isometry)**.** A map f between two riem. mfds, $M \to N$, is a local diffeomorphism ($\forall p \in M$) also a isometry locally, i.e.

$$\langle u, v\rangle = \langle \mathrm{d}f_p(u), \mathrm{d}f_p(v)\rangle, \quad \forall u, v \in T_pM.$$

We have the following properties:

- The local isometry induces isomorphism of Levi-Civita connection and curvatures.
- The local isometry maps geodesics to geodesics with the same speed.
- Gauss Lemma:
$$\left\langle \mathrm{d}(\exp_p)_v(w), \mathrm{d}(\exp_p)_v(v)\right\rangle_{\exp_p(v)} = \langle w, w\rangle_p.$$
- A surj. local isometry from a complete mfd is a covering map.

Theorem 7 (Cartan-Hadamard)**.** *Suppose (M^n, g) is a complete connected riem. mfd with sectional curvature $K \le 0$. Then $\forall p \in M$, $\exp_p : T_pM \to M$ is a covering map. In particular, if M is simply connected, then $\exp_p$ is a diffeomorphism.*

截面曲率是用来和 space form ($\mathbb{R}^n$) 做 Rauch comparison 的, 这我们就说明了没 conjugate pts, 就说明了 local isometry, 再用 Hopf-Rinow 说明在 pullback 的度量下切空间是完备的, 再用上面列出的最后一个观察就可以了.

这章有点短, 打个习题吧:

Remark. Let (M, g) be a Riemannian manifold with non-positive sectional curvature. Show that, for all $p \in M$, there is no conjugate point of p on M.

证明. Let $\gamma : [0, \epsilon] \to M$ be a geodesic curve and (assume) J (to) be a Jacobi vector field along γ with $J(0) = J(\epsilon) = 0$. Now look at the map $\phi : t \mapsto \|J(t)\|^2$. Its second derivative is easy to compute and using the definitions:

$$\begin{aligned}\phi''(t) &= \left\langle \frac{D^2}{dt^2}J, J\right\rangle + \left\langle \frac{D}{dt}J, \frac{D}{dt}J\right\rangle \\ &= -\langle R(\gamma', J)\gamma', J\rangle + \left\|\frac{D}{dt}J\right\|^2 \\ &= -\left[\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2\right]\kappa(\gamma', J) + \left\|\frac{D}{dt}J\right\|^2.\end{aligned}$$

In particular, if all sectional curvatures are non-positive then $\phi'' \ge 0$. So ϕ is a convex map and since $\phi(0) = \phi(\epsilon) = 0, \phi(t) = 0$ for any $t \in [0, \epsilon]$. It follows that J has to be trivial. So there is no conjugate point of p on M. □

7 Bishop-Gromov relative volume comparison

First we fix a nowhere vanishing, global defined n-form, ω, in M, which is also compatible with the orientation: $\omega_g = \sqrt{\det g_{ij}} \cdot \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. Define the volume of cpt subset K:

$$\mathrm{Vol}(K) = \int_K \omega_g$$

and the volume for open set is defined by cpt. exhaustion.

Definition 4 (Cut locus)**.** Define the cut points on direction $v \in T_pM$, $|v| = 1$:

$$t_v = \sup\{t_0 > 0 : \exp_p(t \cdot v) \text{ is minimal up tp } t_0\}$$

By Gauss lemma, t_v must be a positive number (could be ∞), Then set

$$U_p = \{t \cdot v : v \in T_pM, |v| = 1, 0 \le t < t_v\}$$

Define the cut point in the direction $v \in T_pM$ to be $\exp_p(t_v \cdot v)$. It always can be characterized a conjugate point or there exists at least two minimal geodesics connect them.

Define $S_pM = \{v \in T_pM : |v| = 1\}$, the cut locus is the set $C_p = \{\exp_p(t_v \cdot v) : v \in S_pM\}$. Obviously, $\mathrm{Vol}(C_p) = 0$. When we integral, we will minus the intersect part.

U_p is a star shaped open domain of T_pM, $\exp_p$ is a diffeomorphism defined on it.

The volume can be calulated in the normal coordinate (as the following). Fix $\{e_i\}$ as an orthonormal frame at p, compatible with orientation. Then give T_pM a coordinate: $x = x^i e_i$.

For $v \in T_pM$, at $\exp_p(v)$, define:

$$\frac{\partial}{\partial x^i} = \mathrm{d}(\exp_p)_v(e_i) \in T_{\exp_p(v)}M, \ then \frac{\partial}{\partial x^i}\bigg|_p$$

Construct the variations of geodesics:

$$\gamma_{i,u}(t) = \gamma_i(t,u) = \exp_p(t \cdot (x_0 + ue_i)), \quad i = 1, 2, \ldots, n$$

Construct the Jacobi field:

$$J_i(t,u) = \frac{\partial}{\partial u}\gamma_i = t\frac{\partial}{\partial x^i}, \ then \ \frac{\partial}{\partial x^i} = t^{-1} J_i$$

Alobg the geodesic $\gamma_{x_0} = \gamma_{i,0}$, (assume $x_0 = |x_0| \cdot e_n$)

$$g_{ij} = t^{-2} \cdot g_{\gamma_{x_0}(t)}(J_i, J_j). \ Then \ g_{in} = \delta_{in}$$

At $p = \gamma_{x_0}$, $g_{ij}(p) = \delta_{ij}$. Along the geodesic $\gamma_{x_0} = \gamma_{i,0}$, we compute:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\sqrt{\det(g_{ij})} &= \frac{1}{2}g^{ij}\frac{\partial}{\partial t}g_{ij}\\ &= t^{-2}g^{ij}\langle J_i', J_j\rangle - nt^{-1}\\ &= t^{-2}\sum_{i,j\le n-1} g^{ij}\langle J_i', J_j\rangle - (n-1)t^{-1}\end{aligned}$$

Note that $g^{ij}\langle J_i', J_j\rangle$ is indep. of linear transformmation of the Jacobi field, we may rescalar such that at $\gamma_{x_0}(t_0)$,

$$g_{ij} = t_0^{-2}|J_i(t_0)||J_j(t_0)|\delta_{ij}$$

(We may assume $\{\frac{J_i(t_0)}{|J_i(t_0)|}\}$) is the orthonormal basis of $T_{\gamma_{x_0}(t_0)}M$. Denote $\tilde{J}_i = \frac{J_i}{|J_i(t_0)|}$ is a normalized Jacobi field (basis of $J_0^{\perp}$). Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\sqrt{\det(g_{ij})}(t_0) = \sum_{i\le n-1}\langle \tilde{J}_i', \tilde{J}_i\rangle(t_0) - (n-1)t_0^{-1}$$

We will give a polar coordinate structure of T_pM:

$$\Phi : U_p \to \exp_p(U_p) \subset M, \quad (t,v)\mapsto \exp_p(t\cdot v)$$

where $v \in S_pM$.

Let $\mathcal{A}(t,v)\wedge\omega_{S^{n-1}} = \omega_g$ be the riem. volume form (define $\mathcal{A}(t,v)$). Then for any subset A, by Fubini theorem:

$$\mathrm{Vol}(A) = \mathrm{Vol}(A\setminus C_p) = \int_{S^{n-1}}(\int_{A_v}\mathcal{A}(t,v)\ \mathrm{d}t)\ \omega_{S^{n-1}}$$

where $A_v = \{0 < t < t_v : t\cdot v \in A\}$, $\forall v \in S_pM$. Use the index lemma, we can prove that

Theorem 8 (Bishop-Gromov)**.** *(M^n, g) a connected complete Riem. mfd., Ric $\ge (n-1)K_0$, $K_0\in\mathbb{R}$. Then along any minimal geodesic $\gamma : [0,l]\to M$, suppose ω_g in normal coord., we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\sqrt{\det(g_{ij})}}{\sqrt{\det(\tilde{g}_{ij})}} \le 0$$

where $\tilde{g}$ is the corresponding metric on space form of constant sectional curvature K_0.

Define annuls:

$$A_{s,r}(p) = \{\exp_p(t\cdot v) : s < t < \min\{r, t_v\}, v\in S_pM\}$$

Then for any $0 \le r_1 \le \min\{r_2, r_3\} \le \max\{r_2, r_3\} \le r_4$,

$$\frac{\mathrm{Vol}(A_{r_3,r_4}(p))}{\tilde{\mathrm{Vol}}(\tilde{A}_{r_3,r_4}(\tilde{p}))} \le \frac{\mathrm{Vol}(A_{r_1,r_2}(p))}{\tilde{\mathrm{Vol}}(\tilde{A}_{r_1,r_2}(\tilde{p}))}$$

定理里面的 $\tilde{p}, \tilde{\gamma}$ 都是任意的.

8 Bochner formula, Splitting results

我们需要把 Levi-Civita 定义到 (r,s) 型张量场上:

(1) 如果 f 为光滑函数, 则令 $\nabla_X f = Xf$, 这也就是方向导数;

(2) 如果 ω 为 1 形式, 定义 $\nabla_X\omega$ 如下: 任给光滑向量场 Y, 令

$$\nabla_X\omega(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$$

设 θ 为 (r,s) 型的张量场, 任给 1 形式 $\{\eta_i\}_{i=1}^r$ 以及向量场 $\{Y_j\}_{j=1}^s$, 令

$$\begin{aligned}\nabla_X\theta(\eta_1,\cdots,\eta_r;Y_1,\cdots,Y_s) =& X(\theta(\eta_1,\cdots,\eta_r;Y_1,\cdots,Y_s)) \\ &-\sum_{i=1}^r\theta(\eta_1,\cdots,\eta_{i-1},\nabla_X\eta_i,\eta_{i+1},\cdots\eta_r;Y_1,\cdots,Y_s) \\ &-\sum_{j=1}^s\theta(\eta_1,\cdots,\eta_r;Y_1,\cdots,Y_{j-1},\nabla_X Y_j,Y_{j+1},\cdots,Y_s),\end{aligned}$$

这样定义的 $\nabla_X\theta$ 是 (r,s) 型的场张量 (张量场). 如果此协变导数为零, 则称 θ 关于 X 平行. 会发现在 Levi-Civita 联络下, 黎曼度量总是平行的. 这个定义的好处是求导操作和取张量, 外积, 缩并都是交换的.

进一步, 由于协变求导算子 ∇_X 关于 X 函数线性, 因此我们可以象对函数求全微分那样, 对张量场求一种全微分. 具体来说, 设 θ 为 (r,s) 型张量场, 定义 $(r,s+1)$ 型的张量场 $\nabla\theta$ 如下: 任给 1 形式 $\{\eta_i\}_{i=1}^r$ 以及向量场 $\{Y_j\}_{j=1}^s, X$, 令

$$\nabla\theta(\eta_1,\cdots,\eta_r;Y_1,\cdots,Y_s,X) = \nabla_X\theta(\eta_1,\cdots,\eta_r;Y_1,\cdots,Y_s)$$

这样定义的 $\nabla\theta$ 是 $(r,s+1)$ 型的场张量 (张量场), 称为 θ 的协变微分. 老师还算了一般的局部坐标表示, 太猛了.

一堆算子懒得抄了, 就写一下 Laplace 算子的局部:

$$\nabla f = \mathrm{divgrad}(f) = \mathrm{tr}_g\mathrm{Hess}(f) = g^{ij}\cdot\nabla^2 f\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)$$

其中 $\mathrm{Hess}(f) = \nabla(\mathrm{d}f)$. 再把 Christoffel 符号带进去算一下:

$$= \frac{1}{\sqrt{\det(g_{ij})}}\frac{\partial}{\partial x^k}\left(g^{ik}\sqrt{\det(g_{ij})}\frac{\partial f}{\partial x^i}\right)$$

然后我们希望给张量一个范数. 取 $\{e_i\}$ 是 TM 的基,$\{e^i\}$ 是它的对偶基. 记 $g_{ij} = g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})$, 记 $g^{ij} = g(\mathrm{d}x^i,\mathrm{d}x^j)$. 给个范数:

$$\begin{aligned}&\langle \mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_r}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}},\mathrm{d}x^{k_1}\otimes\cdots\otimes\mathrm{d}x^{k_r}\otimes\frac{\partial}{\partial x^{l_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{l_s}}\rangle \\ =& g^{i_1k_1}\cdots g^{i_rk_r}\cdot g_{j_1l_1}\cdots g_{j_sl_s}\end{aligned}$$

这就给 $e^{i_1}\otimes\cdots\otimes e^{i_r}\otimes e^{j_1}\otimes\cdots\otimes e^{j_s}$ 一个正交基. 上课的时候算了一下几个算子的 norm. 可以算的是, 上面定义的 ∇ 和张量丛上的内积是相容的,i.e.

$$\nabla_X\langle T_1,T_2\rangle = \langle\nabla_X T_1,T_2\rangle + \langle T_1,\nabla_X T_2\rangle$$

Lemma 8.1 (Bochner formula)**.** *For $f \in C^\infty(M)$, we have*

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess}(f)|^2 + \langle \nabla\Delta f, \nabla f\rangle + Ric(\nabla f, \nabla f)$$

Theorem 9 (Laplacian comparison)**.** *(M^n, g) a complete Riem. mfd. with $Ric \geq -(n-1)k^2, k \geq 0$. Let $\tilde{M}$ the space form of sectional curvature $\equiv -k^2$ of dimension n. For $p \in M, \tilde{p} \in \tilde{M}$, define*

$$\rho_M(\cdot) = d(p, \cdot), \rho_{\tilde{M}}(\cdot) = d(\tilde{p}, \cdot)$$

If $x \in M, y \in \tilde{M}$ satisfy $\rho_M(x) = \rho_{\tilde{M}}(y)$, and x is a smooth point of ρ_M. Then

$$\Delta\rho_M(x) \leq \tilde{\Delta}\rho_{\tilde{M}}(y)$$

关键就是找 (算) 向量场和 Jacobi field, 用 index lemma:

$$\Delta\rho_M(x) \leq \sum_{i \leq n-1} \mathrm{Hess}(\rho_M)(e_i, e_i) = \sum_{i \leq n-1} I(J_i, J_i) \leq ... \leq \tilde{\Delta}\rho_{\tilde{M}}(y)$$

If $k = 0$, then we obtain: of ρ_M is smooth at x

$$\Delta\rho_M(x) \leq \frac{n-1}{\rho_M(x)}$$

Globally, for complete (M^n, g), $Ric \geq 0$, the inequality holds in the weak sense, i.e. for any $\phi \in C_0^\infty(M)$, $\phi \geq 0$,

$$\int_M \rho \cdot \Delta\phi \, \mathrm{dvol} \leq \int_M \frac{n-1}{\rho} \cdot \phi \, \mathrm{dvol}$$

It can be proved using Stoke's formula and domain exhaustion.

对 Busemann function 用这个定理 (which 我懒得誊抄维基的定义),

Theorem 10 (Cheeger-Gromoll, Splitting theorem)**.** *(M^n, g) a complete riem. mfd., $Ric \geq 0$. If M contains a minimizing geodesic line, then isometrically,*

$$(M, g) \cong (N, h) \times (\mathbb{R}, \mathrm{d}s^2)$$

where N is a riem. mfd. of dim $n-1$.

The proof can be summarized as follows. The fundamental Laplacian comparison theorem proved earlier that Busemann functions are both superharmonic under the Ricci curvature assumption. Either of these functions could be negative at some points, but the triangle inequality implies that their sum is nonnegative. The strong maximum principle implies that the sum is identically zero and hence that each Busemann function is in fact (weakly) a harmonic function. The standard elliptic regularity implies the smoothness of the Busemann functions. Then, the proof can be finished by using Bochner's formula, setting up the de Rham decomposition theorem (抄 wiki 的).

剩下的我就不学啦! 晚安!