BCQM VII Lab Note
Test 2.1 (Spectral dimension): random-walk return probability on the community super-graph (v0.1)

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Purpose

Implement Test 2.1 (spectral dimension) for Stage–2 by estimating the time-dependent spectral-dimension curve $d_s(t)$ on the *community super-graph* associated with the cloth object. This test is marked *optional* in the Phase–1 plan because it is known to be fragile at small community-graph sizes K. The objective here is therefore (i) to execute the test cleanly on the pivot baseline, (ii) to determine whether any quasi-plateau exists on current datasets, and (iii) to record finite-size limitations explicitly and without overinterpretation.

Dataset and scripts

Dataset. Pivot baseline ensemble (hits1, x10 epoch, bins=20, 5 seeds per quadrant) in:

- `outputs_cloth/ensemble_W100_N4N8_hits1_x10_bins20/`

Script.

- `bcqm_vii_cloth/analysis/spectral_dimension_supergraph.py`

CSV artefacts. Each run writes:

- `<tag>_spectral_dim_curves.csv` (per-seed $P_0(t)$ and $d_s(t)$ curves)
- `<tag>_spectral_dim_runs.csv` (per-seed window statistics over a nominated "plateau" window)
- `<tag>_spectral_dim_summary.csv` (mean over seeds for each quadrant)

Method (summary)

For each seed and quadrant (N, n):

1. Build an *undirected* cloth graph from the chosen edge set:
   - core: `core_edges_used`
   - all: `core_edges_used` ∪ `halo_edges_used`
2. Apply Louvain community detection (resolution $\gamma =$`-resolution`) on this undirected cloth graph to obtain a partition π : event $\mapsto$ community.
3. Build the *undirected* community super-graph: communities are nodes; an edge exists between communities if any cloth edge crosses between their member events.
4. Define the (simple) random-walk transition matrix P on the super-graph (degree-normalised adjacency).

5. Compute the return probability $P_0(t)$ and the spectral-dimension curve $d_s(t)$:
   - **MC mode** (`-mode mc`): Monte-Carlo estimate of $P_0(t)$ from `-n_walks` random walks up to `-t_max`.
   - **Exact mode** (`-mode exact`): noise-free evaluation using eigenvalues $\{\lambda_i\}$ of P:

$$P_0(t) = \frac{1}{K}\,\mathrm{tr}\Big(P^t\Big) = \frac{1}{K}\sum_i \lambda_i^t.$$

6. Estimate

$$d_s(t) \;=\; -2\,\frac{d\log P_0(t)}{d\log t},$$

   using a discrete log-derivative on the integer-time curve.
7. **Window summary (diagnostic only):** over $[t_{\min}, t_{\max}]$ (given by `-plateau_tmin`, `-plateau_tmax`), compute summary statistics intended to indicate whether $d_s(t)$ is approximately flat in that window. A large within-window variation indicates *no plateau*.

Runs executed

The pivot baseline was run in two stages.

Stage A: Monte-Carlo return probability (core/core default). The recommended first run (core-only) was executed to maximise K at low n:

```
mkdir -p csv/spectral_dim
python3 bcqm_vii_cloth/analysis/spectral_dimension_supergraph.py \
  --run_dir outputs_cloth/ensemble_W100_N4N8_hits1_x10_bins20 \
  --out_dir csv/spectral_dim \
  --tag pivot_core \
  --edge_source core \
  --resolution 1.0 \
  --t_max 200 \
  --n_walks 20000 \
  --plateau_tmin 10 \
  --plateau_tmax 80 \
  --rng_seed 0
```

A sensitivity run (all/all) was supported by the script via `-edge_source all`. For Gate 2 geometry diagnostics, interpretation focuses on the core/core object (consistent with A2 and the curvature/ball-growth lab notes).

Stage B: Exact return probability (noise-free). An updated script added `-mode exact` (transition-matrix eigenvalues; no Monte-Carlo noise) and the pivot baseline was re-run:

```
python3 bcqm_vii_cloth/analysis/spectral_dimension_supergraph.py \
  --run_dir outputs_cloth/ensemble_W100_N4N8_hits1_x10_bins20 \
  --out_dir csv/spectral_dim \
  --tag pivot_core_exact \
  --edge_source core \
  --resolution 1.0 \
  --mode exact \
  --t_max 200 \
  --plateau_tmin 10 \
  --plateau_tmax 80
```

Results (qualitative)

No robust plateau on current super-graphs. Across the pivot baseline quadrants, the computed $d_s(t)$ curves do *not* exhibit a stable plateau over a broad diffusion-time range at the present community super-graph sizes $K \sim \mathcal{O}(10^1)$ to $\mathcal{O}(10^2)$. This remains true even in `-mode exact`, indicating that the limitation is not Monte-Carlo noise but *finite-size / fast-mixing structure.*

MC mode: consistent means without flatness. In `-mode mc`, some quadrants yield a consistent *mean* effective value (between seeds) in the nominated window $[10, 80]$, but the within-window variability remains large: $d_s(t)$ drifts across the window rather than behaving as a near-constant slope in $\log P_0(t)$ versus $\log t$. At lower n, the derivative estimator becomes particularly unstable (returns become sparse in Monte-Carlo), leading to strongly finite-size / noise dominated window statistics.

Exact mode: clarifies the mechanism (mixing-to-stationary). In `-mode exact`, $P_0(t)$ is deterministic and $d_s(t)$ is substantially less jittery, but the curve still does not plateau:

- In high-coherence cases where the super-graph is small (typically $K \approx 20$–25), the random walk mixes rapidly and $P_0(t) \to 1/K$. Consequently, $\log P_0(t)$ flattens and $d_s(t) \to 0$ at moderate t, so a late-time "plateau" window is dominated by the approach-to-stationary regime rather than by an intermediate scaling regime.
- In larger-K cases (notably the low-n core/core super-graphs), an *early-time transient* with $d_s(t)$ of order unity can be present, but it decays with t and can show even/odd oscillations (graph periodicity). This is consistent with a finite-graph crossover (diffusion $\to$ mixing), not a stable dimension-like scaling window.

Interpretation

Why spectral dimension is failing to plateau here. A clean spectral-dimension plateau typically requires a separation of scales:

$$(\text{very small } t) \quad \ll \quad (\text{intermediate diffusion regime}) \quad \ll \quad (\text{mixing time}).$$

On these community super-graphs, K is small enough that the mixing time is short, leaving little (or no) intermediate regime where $\log P_0(t)$ is well-approximated by a single power law. In other words, the diagnostic is *structurally finite-size dominated* at current K, even after removing Monte-Carlo noise.

What can be reported honestly at Stage–2. Test 2.1 can be recorded as *executed.* The correct Stage–2 reading is:

- no robust $d_s(t)$ plateau exists on current super-graphs (core/core pivot baseline),
- early-time transient "effective d_s" values exist in some regimes but are not stable,
- therefore spectral dimension is not a primary Gate 2 geometry diagnostic at this stage.

This aligns with the plan's explicit caution that Test 2.1 is informative but fragile at $N = 8$.

Next steps if spectral dimension is revisited. If Test 2.1 is later pursued as a stronger claim (Phase–2 / scaling), the most direct upgrades are:

- evaluate at larger N where K increases (e.g. $N = 16, 32, \ldots$) to open an intermediate diffusion regime;
- consider a lazy random walk (aperiodic) to suppress even/odd oscillations cleanly;
- fit $\log P_0(t)$ versus $\log t$ over an early window selected by inspection of the curve, rather than fixing a single late-time window across all K;

- report the associated K, mean degree, and a mixing-time proxy alongside d_s, to make finite-size limitations explicit.

Conclusion

Test 2.1 (spectral dimension on the community super-graph) has been implemented and executed on the pivot baseline, including a noise-free exact mode based on the transition matrix spectrum. The results show no robust plateau in $d_s(t)$ at current community sizes: the curves are dominated by finite-size and fast-mixing effects, with at most a short early-time transient. Accordingly, Test 2.1 remains an optional, non-blocking diagnostic at Stage–2, while ball-growth and curvature proxies remain the primary Gate 2 geometry diagnostics.