Definition (Group)

Consider a set $\mathbb{C}$ and an operation $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined on $\mathbb{C}$. Then $\mathrm{G} := (\mathbb{C}, \otimes)$ is called a *group* if the following hold:

1. *Closure* of C under $\otimes$: $\forall x, y \in \mathbb{C} \, : \, x \otimes y \in \mathbb{C}$
2. *Associativity*: $\forall x, y, z \in \mathbb{C} \, : \, (x \otimes y) \otimes z = x \otimes (y \otimes z)$
3. *Neutral element*: $\exists e \in \mathbb{C} \, \forall x \in \mathbb{C} \, : \, x \otimes e = x$ and $e \otimes x = x$
4. *Inverse element*: $\forall x \in \mathbb{C} \, \exists y \in \mathbb{C} \, : \, x \otimes y = e$ and $y \otimes x = e$

If additionally, $\forall x, y \in \mathbb{C} \, : \, x \otimes y = y \otimes x$, then $\mathrm{G} = (\mathbb{C}, \otimes)$ is an *Abelian group*.

Definition (Vector Space)

A real-valued *vector space* $\mathrm{V} = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

$$
\begin{aligned}
+ :&\ \mathcal{V} \times \mathcal{V} \to \mathcal{V} \\
\cdot :&\ \mathbb{R} \times \mathcal{V} \to \mathcal{V}
\end{aligned}
$$

where

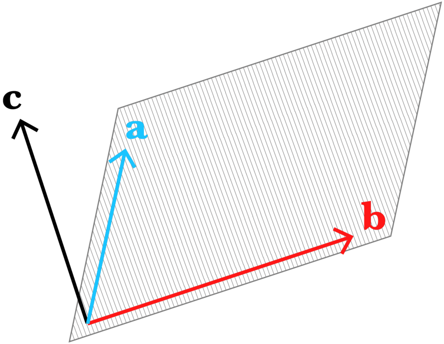
1. $(\mathcal{V}, +)$ is an Abelian Group.
2. Distributivity
   1. $\forall \lambda \in \mathbb{R}, x, y \in \mathbb{V} : \lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$
   2. $\forall \lambda, \psi \in \mathbb{R}, x \in \mathbb{V} : (\lambda + \psi) \cdot x = \lambda \cdot x + \psi \cdot x$
3. Associativity (outer operation)
4. Neutral element with respect to the outer operation

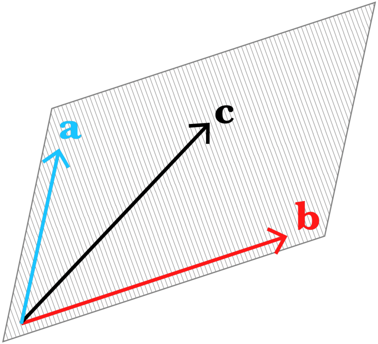
Definition (Linear (In)dependence)

Consider a vector space V with $k \in \mathbb{N}$ and $\mathbf{x_1}, \ldots, \mathbf{x_k} \in \mathrm{V}$. If there is a non-trivial linear combination, such that $0 = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i$ with at least one $\lambda_i \neq 0$, the vectors $\mathbf{x_1}, \ldots, \mathbf{x_k}$ are *linearly dependent*.
If only the trivial solution exists, i.e., $\lambda_1 = \cdots = \lambda_k = 0$ the vectors $\mathbf{x_1}, \ldots, \mathbf{x_k}$ are *linearly independent*.

- To investigate linear independency of n vectors $\longrightarrow$ solve a homogenuous linear system of n equations.

Graphical interpretation of "Linear Independence"





Definition (Generating Set and Span)

For $\mathrm{V} = (\mathcal{V}, +, \cdot)$ and $\mathcal{A} = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subseteq \mathcal{V}$, $\mathcal{A}$ is a *generating set* of V if for every $\mathbf{v} \in \mathrm{V}$:

$$\mathbf{v} = \lambda_1 \mathbf{x}_1 + \cdots + \lambda_k \mathbf{x}_k.$$

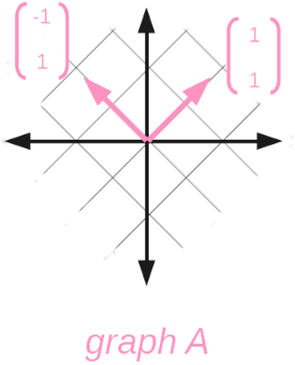
The set of all linear combinations of vectors in $\mathcal{A}$ is the *span* of the $\mathcal{A}$ and $\mathrm{V} = \text{span}[\mathcal{A}]$ if $\mathcal{A}$ spans V.

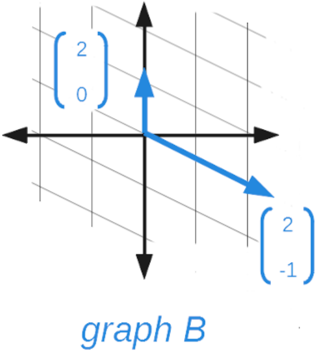
Definition (Basis)

For $\mathrm{V} = (\mathcal{V}, +, \cdot)$ and $\mathcal{A} = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subseteq \mathcal{V}$,
a generating set $\mathcal{A}$ of V is *minimal* if there exists no smaller set $\tilde{\mathcal{A}} \not\subseteq \mathcal{A} \subseteq \mathcal{V}$ that spans V.

Every linearly independent generating set of V that is minimal, is a *basis* of V.

Graphical interpretation of “Basis Vectors”





Example 3. The first two sets are both bases in $\mathbb{R}^3$, however the third set is not a base in $\mathbb{R}^4$. (Why?)

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ 0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\}$$

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix} \right\}$$

Mathematical Foundations: Linear Algebra

- No unique basis
- All bases have the same number of elements, called the *basis vector*
- *Dimension* of V, dim(V): The number of basis vectors of V
- $\mathrm{U} \subseteq \mathrm{V} \longrightarrow \dim(\mathrm{U}) \leq \dim(\mathrm{V})$ & $\mathrm{U} = \mathrm{V} \longrightarrow \dim(\mathrm{U}) = \dim(\mathrm{V})$
- Intuitively, dim(V) is the number of independent directions in V.

Definition (Rank)

The *rank* of **A** is the number of linearly independent columns (or rows) of $\mathbf{A} \in \mathbb{R}^{m \times n}$.

- $\text{rk}(\mathbf{A}) = \text{rk}(\mathbf{A}^\top)$
- The columns of **A** span a subspace $U \subseteq \mathbb{R}^m$ with $\dim(U) = \text{rk}(\mathbf{A})$
- The rows of **A** span a subspace $W \subseteq \mathbb{R}^n$ with $\dim(W) = \text{rk}(\mathbf{A})$
- $\mathbf{A} \in \mathbb{R}^{n \times n}$ is regular $\iff$ $\text{rk}(\mathbf{A}) = n$
- $\mathbf{Ax} = \mathbf{b}$ can be solved $\iff$ $\text{rk}(\mathbf{A}) = \text{rk}(\mathbf{A}|\mathbf{b})$
- Subspace of solutions for $\mathbf{Ax} = \mathbf{0}$ (*kernel* or *null space*) have dimension $n - \text{rk}(\mathbf{A})$
- **A** has *full rank* if $\text{rk}(\mathbf{A}) = \min(m, n)$, otherwise has *rank deficiency*

Definition (Linear Mapping)

For vector spaces V, W, a mapping $\Phi : \mathrm{V} \to \mathrm{W}$ is a *linear mapping/ vector space homomorphism/linear transformation* if

$$\Phi(\lambda \mathbf{x} + \psi \mathbf{y}) = \lambda \Phi(\mathbf{x}) + \psi \Phi(\mathbf{y})$$

Definition (Injective, Surjective, Bijective)

For sets $\mathcal{V}, \mathcal{W}$, a mapping $\Phi : \mathcal{V} \to \mathcal{W}$ is

- *Injective* if $\Phi(\mathbf{x}) = \Phi(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}$
- *Surjective* if $\Phi(\mathcal{V}) = \mathcal{W}$
- *Bijective* if satisfies both of above

Definition (Transformation Matrix)

For

- vector spaces V and W with corresponding (ordered) bases $\mathrm{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ and $\mathrm{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$.
- a *linear mapping* $\Phi : \mathrm{V} \to \mathrm{W}$ and $j = \{1, \ldots, n\}$, $\Phi(\mathbf{b}_j) = \alpha_{1j}\mathbf{c}_1 + \cdots + \alpha_{mj}\mathbf{c}_m = \sum_{i=1}^{m} \alpha_{ij}\mathbf{c}_i$ is the unique representation of $\Phi(\mathbf{b}_j)$ with respect to C.

$m \times n-$matrix $\mathbf{A}_\Phi$, whose elements are given by

$$A_\Phi(i, j) = \alpha_{ij},$$

called *transformation matrix* of Φ (w.r.t. the ordered bases of B of V and C of W.)

Examples of Transformation of Vectors