Principles of Machine Learning

Lecture 2-1: Probability and Distributions

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2. Construction of Probability Space
3. Discrete and Continuous Probabilities
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Outline

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Chapter Overview: Probability and Statistics

- **Understanding Uncertainty**
  - **Degree of Belief:** How strongly we believe an event will happen.
  - **Relative Frequency:** How often an event occurs out of a number of trials.
- **Quantifying Uncertainty:**
  - **Data:** Collecting and analyzing data.
  - **Model:** Building models to represent data.
  - **Prediction Uncertainty:** Estimating uncertainty in model predictions.

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- **Core Concepts:**
  - **Random Variables:** Variables whose values are determined by chance.
  - **Probability Distributions:** Mathematical functions that describe the likelihood of different outcomes.
- **Building Knowledge:**
  - **Probabilistic Modeling:** Creating models that incorporate uncertainty.
  - **Graphical Models:** Using graphs to represent relationships between variables.
  - **Model Selection:** Choosing the best model based on criteria like accuracy and simplicity.

Key Definitions

Definition: Random Variable

A **Random Variable** is a mapping that assigns each outcome of a random experiment to a numerical value representing a specific property or characteristic.

Definition: Probability Distribution

A **Probability Distribution** is a function that assigns a probability to each possible outcome of a random variable, indicating the likelihood of each outcome.

Applications in ML

- Data Uncertainty
- Model Uncertainty
- Prediction Uncertainty
- Basis for Advanced Topics

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Mathematical Structure of Probability

- **Need for formal structure:** Systematic framework to quantify uncertainty
- **Key observations:**
  - Individual outcomes are unpredictable (e.g., single coin toss)
  - Regular patterns emerge in aggregate (e.g., 50% heads in 1000 tosses)
- **Core components:**
  - Sample space (complete outcome catalog)
  - Event space (meaningful outcome combinations)
  - Probability measure (quantified uncertainty)
- **Why it matters:**
  - Extends Boolean logic to uncertain reasoning
  - Provides foundation for statistical inference

From Everyday Reasoning to Probability

Classical Logic Limitations

- Binary truth values (True/False)
- No gradation for uncertainty
- Example: Friend's tardiness
  - Strict logic: Either late or not
  - Reality: Multiple plausible scenarios

Probabilistic Reasoning

- Continuous plausibility scale [0,1]
- Evidence-based belief updates
- Example: Tardiness hypotheses
  - On time: 20%
  - Traffic delay: 75%
  - Alien abduction: 5%

Probability: Mathematics of plausible reasoning under uncertainty

Axiomatic Foundations (Cox-Jaynes)

Theorem (Cox-Jaynes Theorem)

Any system of plausible reasoning satisfying:

1. ***Representation:*** *Plausibilities as real numbers*
2. ***Consistency:***
   - *Non-contradiction*
   - *Honesty*
   - *Reproducibility*
3. ***Continuity:*** *Small evidence changes $\Rightarrow$ small plausibility changes*

must obey probability axioms (up to isomorphism).

Deep Insight

- Probability theory is *unique* extension of Boolean logic
- Subjectivity vs Objectivity: Personal beliefs vs physical frequencies

Probability Space Triad

Sample Space (Ω) : Elementary outcomes
(e.g., $\Omega = \{\text{H}, \text{T}\}$ for coin toss)

Event Space $(\mathcal{F})$: Space of potential results of the experiment (collection of subsets of Ω)
(e.g., $\mathcal{F} = \{\emptyset, \{\text{H}\}, \{\text{T}\}, \Omega\}$)

Probability Measure (P) : $P : \mathcal{F} \to [0, 1]$ with

- $P(\Omega) = 1$ (Certainty)
- $P(\bigcup_i A_i) = \sum_i P(A_i)$ for disjoint A_i

Kolmogorov's Axioms: Operational Perspective

Fundamental Probability Rules

1. **Non-negativity:** $P(A) \geq 0 \ \forall A \in \mathcal{F}$
2. **Normalization:** $P(\Omega) = 1$
3. **Countable Additivity:** For disjoint $\{A_i\}_{i=1}^{\infty}$,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Example

Die Roll Example

- $\Omega = \{1, 2, 3, 4, 5, 6\}$
- For fair die: $P(\{i\}) = 1/6$
- Event $A = \{2, 4, 6\}$: $P(A) = 3 \times 1/6 = 1/2$

Putting It All Together: Coin Toss Space

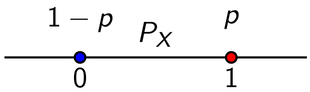
Components

- $\Omega = \{\mathrm{H}, \mathrm{T}\}$
- $\mathcal{F} = 2^{\Omega}$ (power set)
- $P(\{\mathrm{H}\}) = p$
- $P(\{\mathrm{T}\}) = 1 - p$

Random Variable

- $X(\mathrm{H}) = 1$
- $X(\mathrm{T}) = 0$

Event	Description	Probability
$\emptyset$	Impossible	0
$\{\mathrm{H}\}$	Heads	p
$\{\mathrm{T}\}$	Tails	$1 - p$
Ω	Certain	1



Example

Two-Coin Draw (Biased Coins)

- **Sample Space** Ω: $\{(\$, \$), (\$, £), (£, \$), (£, £)\}$
- **Random Variable** X: Counts number of \$ drawn
- **Mapping:**
  - $X((\$, \$)) = 2$
  - $X((\$, £)) = 1$
  - $X((£, \$)) = 1$
  - $X((£, £)) = 0$
- **PMF:**
  - $P(X = 2) = 0.09$
  - $P(X = 1) = 0.42$
  - $P(X = 0) = 0.49$

Statistics vs. Probability in ML

- **Probability:** Models random processes; quantifies uncertainty.
- **Statistics:** Infers underlying processes from observed data.
- **Machine Learning:** Integrates both for model selection and generalization.

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Discrete vs. Continuous Distributions

- **Discrete:** Target space is countable (finite or countably infinite)
- **Continuous:** Target space is uncountably infinite (e.g., intervals in $\mathbb{R}$)
- **Nomenclature:**
  - Discrete $\Rightarrow$ Probability Mass Function (PMF)
  - Continuous $\Rightarrow$ Probability Density Function (PDF)
- **Cumulative Distribution Function (CDF):** Used for continuous RVs, but also defined for discrete

Discrete Probabilities

- **PMF:** $p(x) = P(X = x)$
- **Joint Probability:** $p(x, y)$
- **Marginal Probability:** $p(x) = \sum_y p(x, y)$
- **Conditional Probability:** $p(y \mid x) = \frac{p(x,y)}{p(x)}$
- **Applications:** Categorical features, labels, finite mixture models

Definition: Probability Mass Function (PMF)

Definition: PMF

- Let X be a discrete random variable with target space T.
- The *probability mass function* $p(x)$ assigns

$$p(x) = P(X = x), \quad x \in T.$$

- Satisfies $\sum_{x \in T} p(x) = 1$ and $p(x) \geq 0$.

Example: Joint Discrete PMF

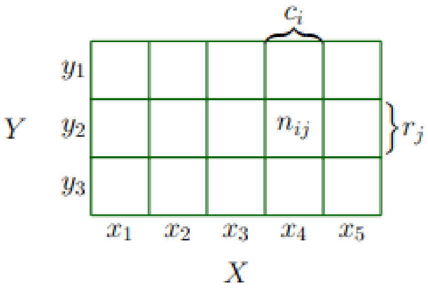
Example (Example: Bivariate Discrete Distribution)

- Two discrete random variables X and Y with states $\{x_1, \ldots, x_5\}$ and $\{y_1, y_2, y_3\}$.
- Joint probability:

$$p(x_i, y_j) = \frac{n_{ij}}{N}, \quad \text{where } n_{ij} \text{ is the count of events with } (x_i, y_j).$$

- **Marginal:** $p(x_i) = \sum_j p(x_i, y_j), \quad p(y_j) = \sum_i p(x_i, y_j)$.
- **Conditional:** $p(y_j \mid x_i) = \frac{p(x_i, y_j)}{p(x_i)}$.

Discrete Bivariate PMF



- Visualization of a discrete bivariate probability mass function.
- Random variables X and Y with a joint PMF $p(x, y)$.

Continuous Probabilities

- **Target space:** Intervals in $\mathbb{R}$ or $\mathbb{R}^D$
- **Probability of exact value:** Zero $(P(X = x) = 0)$
- **Use integrals:** $P(a \leq X \leq b) = \int_a^b f(x)\, dx$
- **Applications:** Real-valued features, Gaussian distributions

Definition: Probability Density Function (PDF)

Definition: PDF

- A function $f : \mathbb{R}^D \to \mathbb{R}$ is a *PDF* if

$$f(x) \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^D} f(x)\, dx = 1.$$

- $\forall a, b \in \mathbb{R}$ (with $a < b$):

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

Definition: Cumulative Distribution Function (CDF)

Definition: CDF

- For a real-valued random variable $X \in \mathbb{R}^D$, the *CDF* is

$$F_X(x) = P(X_1 \leq x_1, \ldots, X_D \leq x_D).$$

- Can be written as an integral of $f(x)$ when f exists:

$$F_X(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_D} f(z)\, dz.$$

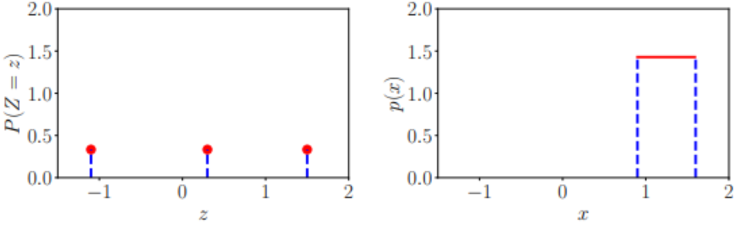
Example: Uniform Distributions

Example (Discrete vs. Continuous Uniform)

- **Discrete Uniform:** Finite states $\{z_1, z_2, z_3\}$, each with $p(z_i) = \frac{1}{3}$.
- **Continuous Uniform:** Interval $[a, b]$ with

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

- Note: PDF can exceed 1 if interval is very small, but integrates to 1.



(a) Discrete distribution (b) Continuous distribution

- Note the difference in how probabilities/densities are visualized.

- **Discrete PMF:** $\sum_{x \in T} p(x) = 1$; each $p(x) \in [0, 1]$.
- **Continuous PDF:** $\int f(x)\, dx = 1$; values of $f(x)$ can exceed 1.
- **Probability of exact point:**
  - Discrete: $P(X = x)$ can be nonzero.
  - Continuous: $P(X = x) = 0$.
- **Common Notational Overlaps:**
  - $p(x)$ used for both PMF and PDF
  - $P(X \leq x)$ also called distribution

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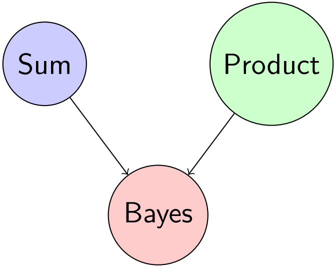
The Three Fundamental Rules

Probability Toolkit for Reasoning

Rule	**What it Does**
Sum Rule	Simplifies complex scenarios
Product Rule	Connects joint & conditional probabilities
Bayes' Theorem	Updates beliefs with new evidence



Sum Rule: Seeing the Big Picture

Weather Example

	Rainy	Sunny
Walk	0.2	0.3
Drive	0.1	0.4

Probability of walking: $0.2 + 0.3 = 0.5$

The Rule in Action

- **Discrete:** $p(\text{walk}) = \sum_{\text{weather}} p(\text{walk}, \text{weather})$
- **Continuous:** $p(\text{height}) = \int p(\text{height}, \text{weight}) d\text{weight}$

Key Idea

"Zoom out" by adding up details you don't need

Product Rule: Connecting Events

Cookie Jar

2 chocolate   3 oatmeal

Probability of first chocolate, then oatmeal:
$\frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$

The Mathematics

$p(A, B) = p(A|B)p(B) = p(B|A)p(A)$

- Joint = Conditional Œ Marginal
- Works for dependent events

Bayes' Theorem: Learning from Data

Medical Testing

- 1% disease prevalence
- 90% test accuracy if sick
- 5% false positive rate
  $\Rightarrow p(\text{Sick}|+) = \frac{0.9\times 0.01}{0.9\times 0.01 + 0.05\times 0.99} \approx 15\%$

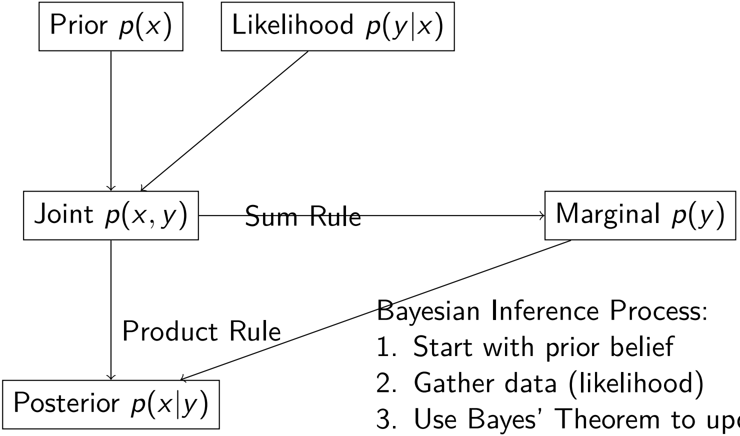
The Formula

$$\underbrace{p(\text{Hypothesis}|\text{Evidence})}_{\text{What we want}} = \frac{\overbrace{p(\text{Evidence}|\text{Hypothesis})}^{\text{Test accuracy}}\ \overbrace{p(\text{Hypothesis})}^{\text{Prior}}}{\underbrace{p(\text{Evidence})}_{\text{Normalizer}}}$$

Why It Matters

Updates beliefs systematically using evidence

Putting It All Together



Bayesian Inference Process:

1. Start with prior belief
2. Gather data (likelihood)
3. Use Bayes' Theorem to update beliefs

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Expected Value: Definition

Definition (Expected Value)

For a random variable X and a function g,

$$E[g(X)] = \begin{cases} \sum_{x \in \mathcal{X}} g(x)\, p(x) & \text{(discrete)} \\ \int_{\mathcal{X}} g(x)\, p(x)\, dx & \text{(continuous)} \end{cases}$$

Linearity of Expectation

- For functions $g(x)$ and $h(x)$ with scalars a, b:

$$E[a\,g(x) + b\,h(x)] = a\,E[g(x)] + b\,E[h(x)]$$

- Key property used in variance and covariance derivations.

Mean of a Random Variable

Definition (Mean)

The mean of X is given by

$$E[X] = \begin{cases} \sum_{x \in \mathcal{X}} x\, p(x) & \text{(discrete)} \\ \int_{\mathcal{X}} x\, p(x)\, dx & \text{(continuous)} \end{cases}$$

For multivariate $X \in \mathbb{R}^D$, $E[X]$ is computed element-wise.

Other Averages: Median and Mode

- **Median:** The middle value where 50% of the data is below.
- **Mode:** The most frequently occurring value or peak in the density.
- Useful when distributions are skewed or multimodal.

Bimodal Distribution Example (Figure 6.4)

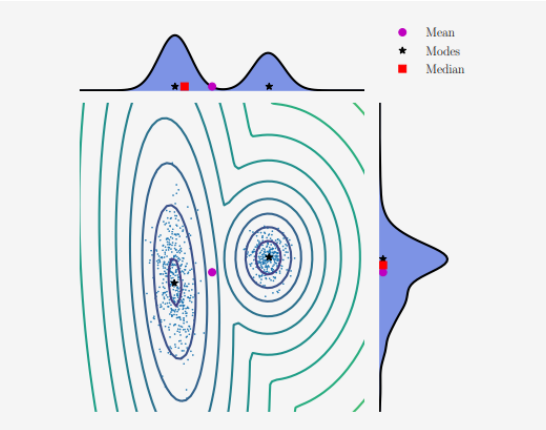
Example (Bimodal Distribution)

Consider a mixture model:

$$p(x) = 0.4\,\mathcal{N}(x \mid \mu_1, \Sigma_1) + 0.6\,\mathcal{N}(x \mid \mu_2, \Sigma_2)$$

Note: The joint distribution is bimodal, though one marginal may be unimodal.

Bimodal Example



- 2D mixture of Gaussians.
- Two distinct modes in the joint distribution.
- Marginal distributions can have different characteristics.

Covariance: Univariate Case

Definition (Covariance (Univariate))

For random variables X and Y,

$$\mathrm{Cov}[X, Y] = E\big[(X - E[X])(Y - E[Y])\big]$$

Equivalently,

$$\mathrm{Cov}[X, Y] = E[XY] - E[X]\,E[Y]$$

Variance: A Special Case of Covariance

- Variance is defined as:

$$\mathrm{Var}[X] = \mathrm{Cov}[X, X] = E\left[(X - E[X])^2\right]$$

- Standard deviation:

$$\sigma(X) = \sqrt{\mathrm{Var}[X]}$$

Covariance: Multivariate Case

Definition (Covariance (Multivariate))

For $X \in \mathbb{R}^D$ and $Y \in \mathbb{R}^E$,

$$\mathrm{Cov}[X, Y] = E\left[X\, Y^\top\right] - E[X]\, E[Y]^\top$$

The result is a $D \times E$ matrix.

Variance for Multivariate Variables

Definition (Variance (Covariance Matrix))

For a multivariate random variable $X \in \mathbb{R}^D$ with mean μ,

$$V[X] = \mathrm{Cov}[X, X] = E\left[(X - \mu)(X - \mu)^\top\right]$$

- Diagonals: individual variances.
- Off-diagonals: cross-covariances.

Definition (Empirical Mean & Covariance)

Given data $x_1, \ldots, x_N \in \mathbb{R}^D$:

$$\bar{x} = \frac{1}{N}\sum_{n=1}^{N} x_n, \quad \Sigma = \frac{1}{N}\sum_{n=1}^{N}(x_n - \bar{x})(x_n - \bar{x})^\top.$$

Note: The unbiased covariance uses $1/(N-1)$.

Three Expressions for Variance

- Standard definition:

$$\mathrm{Var}(X) = E[(X-\mu)^2]$$

- Raw-score formula:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

- Pairwise differences:

$$\frac{1}{N^2}\sum_{i,j}(x_i - x_j)^2 = 2\left[\frac{1}{N}\sum_i x_i^2 - \left(\frac{1}{N}\sum_i x_i\right)^2\right]$$

Affine Transformations of Random Variables

- For $y = Ax + b$:

$$E[y] = A\,E[x] + b$$

- Variance under affine transformation:

$$\mathrm{Var}(y) = A\,\mathrm{Var}(x)\,A^\top$$

- Essential in linear models and dimensionality reduction.

Statistical Independence

Definition (Statistical Independence)

Random variables X and Y are independent if

$$p(x, y) = p(x)\, p(y)$$

Equivalently, $p(x \mid y) = p(x)$.

Conditional Independence

Definition (Conditional Independence)

X and Y are conditionally independent given Z if

$$p(x, y \mid z) = p(x \mid z)\, p(y \mid z) \quad \forall z.$$

Notation: $X \perp\!\!\!\perp Y \mid Z$.

Inner Products of Random Variables

- Define inner product as:

$$\langle X, Y \rangle := \mathrm{Cov}[X, Y]$$

- Length:

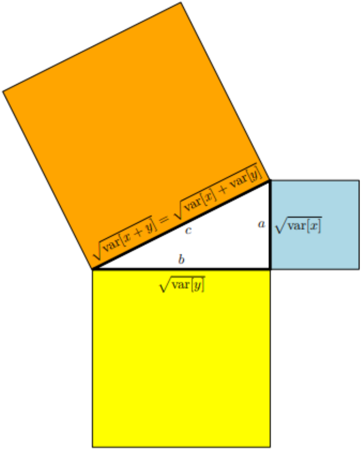
$$\|X\| = \sqrt{\mathrm{Var}[X]} = \sigma(X)$$

- Angle between X and Y:

$$\cos\theta = \frac{\mathrm{Cov}[X, Y]}{\sigma(X)\,\sigma(Y)}$$

- Interpretation: Correlation as the cosine of the angle.

Figure 6.6: Covariance as Inner Product



- Visualizes inner-product structure using covariance.
- Zero covariance corresponds to orthogonality.

Summary of Key Mathematical Relationships

- Sum Rule: $p(x) = \sum_y p(x, y)$ or $\int p(x, y)dy$
- Product Rule: $p(x, y) = p(x \mid y)p(y)$
- Bayes' Theorem: $p(x \mid y) = \frac{p(y|x)p(x)}{p(y)}$
- Linearity: $E[aX + bY] = aE[X] + bE[Y]$
- Variance: $V[X] = E[X^2] - (E[X])^2$
- Affine: $E[Ax + b] = A\,E[X] + b,\ V[Ax + b] = A\,V[X]A^\top$

Implications in Machine Learning

- I.I.D. assumption simplifies model training.
- Covariance and correlation are key in feature analysis.
- Statistical independence assumptions underpin many algorithms.
- Basis for probabilistic modeling and inference.

Closing Remarks on Summary Statistics & Independence

- Summary statistics capture essential aspects of distributions.
- Independence (and conditional independence) simplify joint models.
- Geometric interpretations (inner products, angles) provide intuition.
- These mathematical relationships are foundational in ML.

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Gaussian Distribution Overview

- Most well-studied continuous distribution
- Also called the Normal distribution
- Arises from the Central Limit Theorem
- Widely used in ML, signal processing, control, statistics

Key Properties

- Fully characterized by mean and covariance
- Closed-form expressions for marginals and conditionals
- Linear transformations preserve Gaussianity
- Computationally convenient in inference tasks

Univariate Gaussian Density

- Density formula:

$$p(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- Standard case: $\mu = 0$, $\sigma^2 = 1$

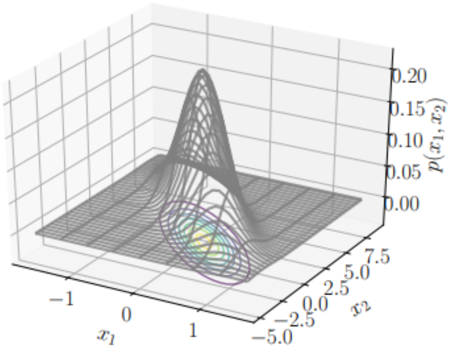
Multivariate Gaussian Density

- For $x \in \mathbb{R}^D$:

$$p(x \mid \mu, \Sigma) = (2\pi)^{-D/2} |\Sigma|^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right)$$

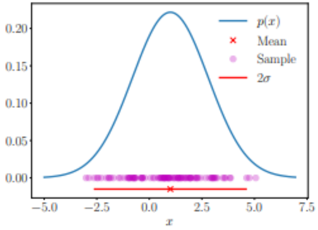
- Denoted as $N(x \mid \mu, \Sigma)$

Bivariate Gaussian Mesh

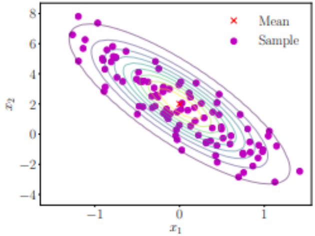


- Mesh plot of a bivariate Gaussian
- Contour lines illustrate elliptical density shapes

Gaussian Samples



(a) Univariate (one-dimensional) Gaussian; The red cross shows the mean and the red line shows the extent of the variance.



(b) Multivariate (two-dimensional) Gaussian, viewed from top. The red cross shows the mean and the colored lines show the contour lines of the density.

- Left: Univariate Gaussian with samples
- Right: Bivariate Gaussian with overlaid samples

Marginals of Joint Gaussian

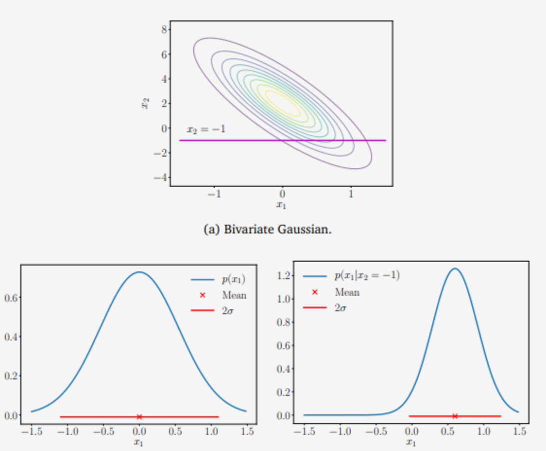
- Joint Gaussian:

$$p(x, y) = N\left(\begin{bmatrix}\mu_x \\ \mu_y\end{bmatrix}, \begin{bmatrix}\Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix}\right)$$

- Marginal of x:

$$p(x) = N(x \mid \mu_x, \Sigma_{xx})$$

Joint, Marginal, and Conditional



(a) Bivariate Gaussian.

(b) Marginal distribution.

(c) Conditional distribution.

- (a) Joint bivariate Gaussian
- (b) Marginal of joint Gaussian is Gaussian
- (c) Conditional distribution is also Gaussian

Product of Gaussian Densities

Definition (Product of Gaussians)

The product of two Gaussian densities is proportional to a Gaussian:

$$N(x \mid a, A)\, N(x \mid b, B) = c\, N(x \mid c, C)$$

with

$$C = (A^{-1} + B^{-1})^{-1}, \quad c = C\,(A^{-1}a + B^{-1}b),$$

and scaling constant

$$c = (2\pi)^{-D/2}|A + B|^{-1/2} \exp\left(-\frac{1}{2}(a - b)^\top (A + B)^{-1}(a - b)\right).$$

Scaling Constant for Product

- Expressible as a Gaussian:

$$c = N(a \mid b, A + B) = N(b \mid a, A + B)$$

- Compact notation for Gaussian products

Sums of Independent Gaussians

- If $X \sim N(\mu_x, \Sigma_x)$ and $Y \sim N(\mu_y, \Sigma_y)$ are independent,

$$X + Y \sim N(\mu_x + \mu_y, \Sigma_x + \Sigma_y)$$

- Follows from linearity of expectation and variance additivity

Linear Transformations

- For $Y = AX + b$ and $X \sim N(\mu, \Sigma)$,

$$Y \sim N(A\mu + b,\ A\Sigma A^\top)$$

- Affine transformations preserve Gaussianity

Example: Weighted Sum

Example (Weighted Sum)

For independent Gaussian random variables:

$$p(ax + by) = N(a\mu_x + b\mu_y,\ a^2\Sigma_x + b^2\Sigma_y)$$

Gaussian Applications in ML

- Likelihoods and priors in linear regression
- Mixture models for density estimation
- Gaussian processes for regression and classification
- Kalman filters in signal processing and control

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Why Special Distributions Matter

The Challenge:

- Bayesian updating often changes distribution forms
- Want tractable math for:
  - Posterior calculations
  - Predictive distributions

The Solution:

- Conjugate families that:
  - Keep same distribution type after updating
  - Maintain fixed number of parameters
- Exponential families provide foundation

Key Distributions for Binary Outcomes

The Bernoulli-Beta Family

	Likelihood	**Conjugate Prior**
Single trial	Bernoulli	Beta
Multiple trials	Binomial	Beta

Coin Flip Analogy

- Bernoulli: Single flip result (H/T)
- Binomial: Count of heads in 10 flips
- Beta: Describes our belief about fairness

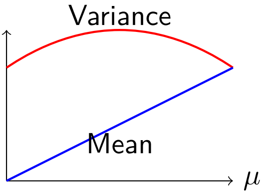
Bernoulli Distribution Demystified

The Coin Flip Distribution

$$p(x \mid \mu) = \underbrace{\mu^x}_{\text{Success if } x=1} \underbrace{(1-\mu)^{1-x}}_{\text{Failure if } x=0}$$

Key Properties:

- $\mathbb{E}[X] = \mu$
- $\text{Var}[X] = \mu(1-\mu)$
- Maximum entropy for binary



From Single Flips to Multiple Trials: Binomial

Counting Successes

$$p(m \mid N, \mu) = \underbrace{\binom{N}{m}}_{\text{Count arrangements}} \underbrace{\mu^m}_{m \text{ successes}} \underbrace{(1-\mu)^{N-m}}_{N-m \text{ failures}}$$

Dice Example

Probability of rolling exactly 3 sixes in 10 rolls:

$$\binom{10}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^7 \approx 0.155$$

Exponential Family: The Unified Framework

Canonical Form

$$p(x \mid \eta) = h(x) \exp\left(\eta^\top T(x) - A(\eta)\right)$$

- η: Natural parameters
- $T(x)$: Sufficient statistics
- $A(\eta)$: Log-normalizer

Why It Matters:

- Guarantees conjugacy
- Enables efficient computation
- Unifies discrete/continuous

Examples:

- Bernoulli
- Gaussian
- Poisson
- Beta

Conjugacy in Action: Beta-Bernoulli

Prior: $\mathrm{Beta}(\alpha, \beta)$

$$p(\mu) \propto \mu^{\alpha-1}(1-\mu)^{\beta-1}$$

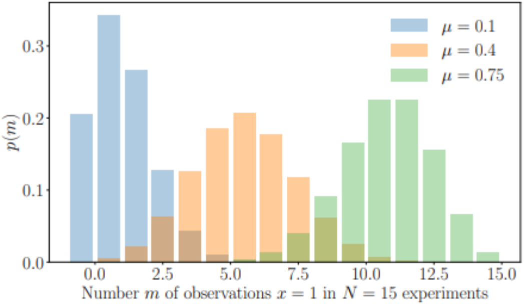
Likelihood: $\mathrm{Bern}(x \mid \mu)$

$$\mu^{x}(1-\mu)^{1-x}$$

Posterior: $\mathrm{Beta}(\alpha + x, \beta + 1 - x)$

$$p(\mu \mid x) \propto \mu^{\alpha+x-1}(1-\mu)^{\beta+(1-x)-1}$$

Binomial Distribution



- Illustrates probability mass vs. number of successes.
- Typical for coin-flip experiments.

Example: Beta Distribution

Example (Beta Distribution)

For $\mu \in [0, 1]$ with parameters $\alpha, \beta > 0$,

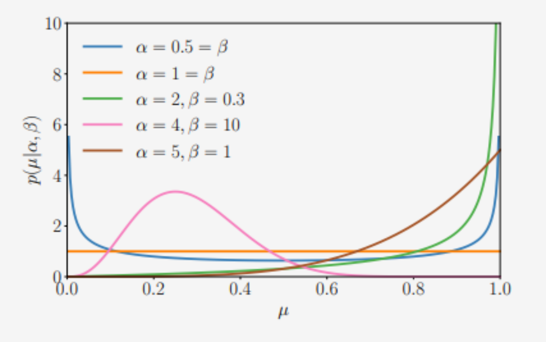
$$p(\mu \mid \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \mu^{\alpha-1}(1 - \mu)^{\beta-1}.$$

Also,

$$E[\mu] = \frac{\alpha}{\alpha + \beta}, \quad V[\mu] = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

- Used to model uncertainty in a probability parameter.

Beta Distribution



- Shows effects of varying α and β.
- Special cases: Uniform ($\alpha = \beta = 1$), bimodal ($\alpha, \beta < 1$), unimodal ($\alpha, \beta > 1$).

Intuition: Beta Parameters

- α: Shifts mass toward 1.
- β: Shifts mass toward 0.
- Special cases yield uniform, bimodal, or symmetric unimodal shapes.

Conjugacy: Motivation

- Prior knowledge should be updated analytically.
- Desire for the posterior to be in the same family as the prior.
- Simplifies computation in Bayesian inference.

Definition: Conjugate Prior

Definition (Conjugate Prior)

A prior $p(\theta)$ is *conjugate* to a likelihood $p(x \mid \theta)$ if the posterior $p(\theta \mid x)$ is in the same family as $p(\theta)$.

Example: Beta-Binomial Conjugacy

Example (Beta-Binomial Conjugacy)

For $x \sim \text{Bin}(N, \mu)$,

$$p(x \mid N, \mu) = \binom{N}{x} \mu^x (1 - \mu)^{N-x}.$$

With prior $\mu \sim \text{Beta}(\alpha, \beta)$,

$$p(\mu \mid \alpha, \beta) \propto \mu^{\alpha-1}(1 - \mu)^{\beta-1}.$$

Then the posterior is

$$p(\mu \mid x, N, \alpha, \beta) \propto \mu^{x+\alpha-1}(1 - \mu)^{N-x+\beta-1},$$

i.e., $\mu \mid x \sim \text{Beta}(x + \alpha, N - x + \beta)$.

Beta-Binomial Posterior Update

- Posterior parameters: $\alpha' = x + \alpha, \quad \beta' = N - x + \beta.$
- Conjugacy simplifies parameter updates.

Example: Beta-Bernoulli Conjugacy

Example (Beta-Bernoulli Conjugacy)

For $x \in \{0, 1\}$ with

$$p(x \mid \theta) = \theta^x (1 - \theta)^{1-x},$$

and prior $\theta \sim \text{Beta}(\alpha, \beta)$,

$$p(\theta \mid \alpha, \beta) \propto \theta^{\alpha-1} (1 - \theta)^{\beta-1}.$$

Then,

$$p(\theta \mid x, \alpha, \beta) \propto \theta^{\alpha+x-1} (1 - \theta)^{\beta+(1-x)-1},$$

i.e., $\theta \mid x \sim \text{Beta}(\alpha + x, \beta + 1 - x)$.

Conjugate Priors in ML

- Conjugacy yields closed-form posteriors.
- Common pairs: Beta-Binomial/Bernoulli, Gaussian-Gaussian, Gamma-Poisson.
- Facilitates iterative updates with new data.

Sufficient Statistics: Motivation

- Statistics that capture all information about parameters.
- Enable data reduction without loss of inferential power.
- Underpin conjugacy and exponential families.

Theorem: Fisher-Neyman Factorization

Theorem (Fisher-Neyman)

Let X have density $p(x \mid \theta)$. Then a statistic $\phi(x)$ is sufficient for θ if and only if

$$p(x \mid \theta) = h(x)\, g_\theta(\phi(x)),$$

where $h(x)$ is independent of θ.

Sufficient Statistics in ML

- Finite-dimensional summaries even for infinite data.
- Key for efficient maximum likelihood estimation.
- Basis for the exponential family formulation.

Definition: Exponential Family

Definition (Exponential Family)

A family of distributions is in the exponential family if it can be written as

$$p(x \mid \theta) = h(x)\, \exp\Big(\langle \theta, \phi(x) \rangle - A(\theta)\Big),$$

where:

- $\phi(x)$: vector of sufficient statistics,
- θ: natural parameters,
- $A(\theta)$: log-partition function.

Exponential Family: Features

- Finite-dimensional sufficient statistics.
- Conjugate priors are easy to derive.
- Log-likelihood is concaveefficient optimization.
- Unifies many common distributions (e.g., Gaussian, Bernoulli, Poisson).

Natural Parameters and Sigmoid

- In the Bernoulli, relate μ and θ via:

$$\mu = \frac{1}{1 + \exp(-\theta)}.$$

- Sigmoid (logistic) function: maps $\theta \in \mathbb{R}$ to $\mu \in (0, 1)$.
- Crucial for logistic regression and neural network activations.

Exponential Families & Conjugacy

- Every exponential family member has a conjugate prior.
- Posterior update involves only sufficient statistics.
- Simplifies Bayesian inference and parameter estimation.

Summary: Conjugacy & Exponential Family

- Conjugate priors yield posteriors of the same form.
- Sufficient statistics capture all necessary data information.
- Exponential families unify many common distributions.
- These properties enable efficient inference in ML.

Closing Remarks

- Understanding these concepts aids in selecting proper models.
- Conjugacy and exponential families simplify Bayesian updates.
- Fundamental for many advanced ML algorithms.