Laborator 12 - TEMĂ Petculescu Mihai-Silviu

Să se rezolve următoarele sisteme de ecuații diferențiale:

1. b) $\begin{cases} x' = -x-2y \\ y' = 2x+y \\ z' = 3x+2z \end{cases}$ $x = \alpha_1 e^{rt}$, $y = \alpha_2 e^{rt}$, $z = \alpha_3 e^{rt}$

$x' = \alpha_1 r e^{rt}$, $y = \alpha_2 r e^{rt}$, $z = \alpha_3 r e^{rt}$

$$\begin{cases} \alpha_1 r = -\alpha_1 - 2\alpha_2 \\ \alpha_2 r = 2\alpha_1 + \alpha_2 \\ \alpha_3 r = 3\alpha_1 + 2\alpha_3 \end{cases} \Leftrightarrow \begin{cases} (-1-r)\alpha_1 - 2\alpha_2 = 0 \\ 2\alpha_1 + (1-r)\alpha_2 = 0 \\ 3\alpha_1 + (2-r)\alpha_3 = 0 \end{cases} \quad (*)$$

$$D = \begin{vmatrix} -1-r & -2 & 0 \\ 2 & 1-r & 0 \\ 3 & 0 & 2-r \end{vmatrix} = (-1-r)(1-r)(2-r) - (2-r)(-2)(2)$$
$$= (-1+r^2)(2-r) + (2-r)\cdot 4$$
$$= (2-r)(r^2+3)$$

$$\Rightarrow \begin{cases} 2-r = 0 \Rightarrow r_1 = 2 \\ r^2+3 = 0 \Rightarrow r_{2,3} = \pm\sqrt{3}i \quad (\mu \pm i\beta,\ \mu = 0,\ \beta = \sqrt{3}) \end{cases}$$

• Fie $r_1 = 2$

$$(*) \Rightarrow \begin{cases} -3\alpha_1 - 2\alpha_2 = 0 \\ 2\alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_2 = 0 \\ 3\alpha_1 = 0 \Rightarrow \alpha_1 = 0,\ \alpha_3 \in R \end{cases}$$

Fie $\alpha_3 = 1 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} 0 \\ 0 \\ e^{2t} \end{pmatrix}$

- Fie $r_2 = \sqrt{3}i$

$$(*) \Rightarrow \begin{cases} (-1-\sqrt{3}i)\alpha_1 - 2\alpha_2 = 0 \\ 2\alpha_1 + (1-\sqrt{3}i)\alpha_2 = 0 \Rightarrow \alpha_1 = \frac{-1+\sqrt{3}i}{2}\alpha_2 \\ 3\alpha_1 + (2-\sqrt{3}i)\alpha_3 = 0 \Rightarrow \alpha_1 = \frac{-2+\sqrt{3}i}{3}\alpha_3,\ \alpha_1 \in R \end{cases}$$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = \frac{2}{-1+\sqrt{3}i}$, $\alpha_3 = \frac{3}{-2+\sqrt{3}i}$ $\Rightarrow \alpha_{r_2} = \begin{pmatrix} 1 \\ \frac{2}{-1+\sqrt{3}i} \\ \frac{3}{-2+\sqrt{3}i} \end{pmatrix} \Rightarrow$

$$\Rightarrow X_{r_2} = \begin{pmatrix} \cos\sqrt{3}t + i\sin\sqrt{3}t \\ \frac{2}{-1+\sqrt{3}i}(\cos\sqrt{3}t + i\sin\sqrt{3}t) \\ \frac{3}{-2+\sqrt{3}i}(\cos\sqrt{3}t + i\sin\sqrt{3}t) \end{pmatrix}$$

$$X_{n_2} = \begin{pmatrix} \cos\sqrt{3}t + i\sin\sqrt{3}t \\ \frac{2}{-1+\sqrt{3}i}\cos\sqrt{3}t + \frac{2}{-1+\sqrt{3}i} i\sin\sqrt{3}t \\ \frac{3}{-2+\sqrt{3}i}\cos\sqrt{3}t + \frac{3}{-2+\sqrt{3}i} i\sin\sqrt{3}t \end{pmatrix} = \begin{pmatrix} \cos\sqrt{3}t + i\sin\sqrt{3}t \\ \frac{-1}{2}\cos\sqrt{3}t - \frac{\sqrt{3}i}{2}\cos\sqrt{3}t + \frac{-i}{2}\sin\sqrt{3}t + \frac{\sqrt{3}}{2}\sin\sqrt{3}t \\ \frac{-6}{7}\cos\sqrt{3}t + \frac{-3\sqrt{3}i}{7}\cos\sqrt{3}t - \frac{6}{7}i\sin\sqrt{3}t + \frac{3\sqrt{3}}{7}\sin\sqrt{3}t \end{pmatrix} =$$

• $\frac{2}{-1+\sqrt{3}i} = \frac{-1-\sqrt{3}i}{2}$; $\frac{3}{-2+\sqrt{3}i} = \frac{-6-3\sqrt{3}i}{7}$

$$= \underbrace{\begin{pmatrix} \cos\sqrt{3}t \\ \frac{-1}{2}\cos\sqrt{3}t + \frac{\sqrt{3}}{2}\sin\sqrt{3}t \\ \frac{-6}{7}\cos\sqrt{3}t + \frac{3\sqrt{3}}{7}\sin\sqrt{3}t \end{pmatrix}}_{\tilde{X}_{n_2}} + i \underbrace{\begin{pmatrix} \sin\sqrt{3}t \\ \frac{-\sqrt{3}}{2}\cos\sqrt{3}t + \frac{-1}{2}\sin\sqrt{3}t \\ \frac{-3\sqrt{3}}{7}\cos\sqrt{3}t - \frac{6}{7}\sin\sqrt{3}t \end{pmatrix}}_{\tilde{\tilde{X}}_{n_2}}$$

$$X = c_1 X_1 + c_2 \tilde{X}_{n_2} + c_3 \tilde{\tilde{X}}_{n_2} = c_1 \begin{pmatrix} 0 \\ 0 \\ e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} \cos\sqrt{3}t \\ -\frac{1}{2}\cos\sqrt{3}t + \frac{\sqrt{3}}{2}\sin\sqrt{3}t \\ -\frac{6}{7}\cos\sqrt{3}t + \frac{3\sqrt{3}}{7}\sin\sqrt{3}t \end{pmatrix} +$$

$$+ c_3 \begin{pmatrix} \sin\sqrt{3}t \\ -\frac{\sqrt{3}}{2}\cos\sqrt{3}t - \frac{1}{2}\sin\sqrt{3}t \\ -\frac{3\sqrt{3}}{7}\cos\sqrt{3}t - \frac{6}{7}\sin\sqrt{3}t \end{pmatrix} = \begin{pmatrix} c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t \\ c_2\frac{-1}{2}\cos\sqrt{3}t + c_2\frac{\sqrt{3}}{2}\sin\sqrt{3}t + c_3\frac{-\sqrt{3}}{2}\cos\sqrt{3}t + c_3\frac{-1}{2}\sin\sqrt{3}t \\ c_1 e^{2t} + c_2\frac{-6}{7}\cos\sqrt{3}t + c_2\frac{3\sqrt{3}}{7}\sin\sqrt{3}t + c_3\frac{-3\sqrt{3}}{7}\cos\sqrt{3}t + c_3\frac{-6}{7}\sin\sqrt{3}t \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$x = c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t$

$y = c_2\frac{-1}{2}\cos\sqrt{3}t + c_2\frac{\sqrt{3}}{2}\sin\sqrt{3}t + c_3\frac{-\sqrt{3}}{2}\cos\sqrt{3}t + c_3\frac{-1}{2}\sin\sqrt{3}t$

$z = c_1 e^{2t} + c_2\frac{-6}{7}\cos\sqrt{3}t + c_2\frac{3\sqrt{3}}{7}\sin\sqrt{3}t + c_3\frac{-3\sqrt{3}}{7}\cos\sqrt{3}t + c_3\frac{-6}{7}\sin\sqrt{3}$

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1.c) $\begin{cases} x' = x - 2y + 3z \\ y' = y \\ z' = -3x + z \end{cases}$ $\quad x = \alpha_1 e^{rt},\ y = \alpha_2 e^{rt},\ z = \alpha_3 e^{rt}$

$\quad x' = \alpha_1 r e^{rt},\ y' = \alpha_2 r e^{rt},\ z = \alpha_3 r e^{rt}$

$$\begin{cases} \alpha_1 r = \alpha_1 - 2\alpha_2 + 3\alpha_3 \\ \alpha_2 r = \alpha_2 \\ \alpha_3 r = -3\alpha_1 + \alpha_3 \end{cases} \Leftrightarrow \begin{cases} (1-r)\alpha_1 - 2\alpha_2 + 3\alpha_3 = 0 \\ (1-r)\alpha_2 = 0 \\ -3\alpha_1 + (1-r)\alpha_3 = 0 \end{cases} \quad (*)$$

$$D = \begin{vmatrix} 1-r & -2 & 3 \\ 0 & 1-r & 0 \\ -3 & 0 & 1-r \end{vmatrix} = (1-r)^3 + 9(1-r) = (1-r)(r^2 - 2r + 10) = 0$$

$$\Leftrightarrow \left[\begin{array}{l} 1 - r = 0 \Rightarrow r_1 = 1 \\ r^2 - 2r + 10 = 0 \Rightarrow r_{2,3} = 1 \pm 3i \end{array} \right.$$

$$D = 4 - 4\cdot 10 = -36$$

• Pt $r_1 = 1$

$(*) \Rightarrow \begin{cases} -2\alpha_2 + 3\alpha_3 = 0 \Rightarrow 2\alpha_2 = 3\alpha_3 \Leftrightarrow \alpha_2 = \frac{3}{2}\alpha_3,\ \alpha_3 \in R \\ -3\alpha_1 = 0 \Rightarrow \alpha_1 = 0 \end{cases}$

Fie $\alpha_3 = 1 \Rightarrow \alpha_2 = \frac{3}{2} \Rightarrow \alpha_{r_1} = \begin{pmatrix} 0 \\ \frac{3}{2} \\ 1 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} 0 \\ \frac{3}{2} e^t \\ e^t \end{pmatrix}$

• Pt $r_2 = 1 + 3i$ $\quad (\mu + i\beta,\ \mu = 1,\ \beta = 3)$

$(*) \Rightarrow \begin{cases} -3i\alpha_1 - 2\alpha_2 = 0 \Leftrightarrow \alpha_2 = \frac{-3i}{2}\alpha_1 \\ 2\alpha_1 - 3i\alpha_2 = 0 \\ 3\alpha_1 + (1-3i)\alpha_3 = 0 \Leftrightarrow \alpha_3 = \frac{\overset{1+3i}{3}}{1-3i}\alpha_1 = \frac{3+9i}{10}\alpha_1,\ \alpha_1 \in R \end{cases}$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = \frac{-3i}{2},\ \alpha_3 = \frac{3+9i}{10} \Rightarrow \alpha_{r_2} = \begin{pmatrix} 1 \\ \frac{-3i}{2} \\ \frac{3+9i}{10} \end{pmatrix}$

$$X_{r_2} = \begin{pmatrix} (\cos 3t + i \sin 3t) e^t \\ \left(\frac{-3i}{2}\cos 3t + \frac{3}{2}\sin 3t\right) e^t \\ \left(\frac{3+9i}{10}\cos 3t + \frac{3i-9}{10}\sin 3t\right) e^t \end{pmatrix} = \underbrace{\begin{pmatrix} \cos 3t \cdot e^t \\ e^t \frac{3}{2}\sin 3t \\ e^t \frac{3}{10}\cos 3t - \frac{9}{10} e^t \sin 3t \end{pmatrix}}_{\tilde{X}_{r_2}} +$$

$$+ i \underbrace{\begin{pmatrix} e^t \sin 3t \\ e^t \frac{-3}{2}\cos 3t \\ e^t \frac{9}{10}\cos 3t + e^t \frac{3}{10}\sin 3t \end{pmatrix}}_{\tilde{\tilde{X}}_{r_2}}$$

$$X = C_1 X_1 + C_2 \widetilde{X_{12}} + C_3 \widetilde{\widetilde{X_{12}}} = C_1 \begin{pmatrix} 0 \\ \frac{3}{2} e^t \\ e^t \end{pmatrix} + C_2 \begin{pmatrix} e^t \cos 3t \\ e^t \frac{3}{2} \sin 3t \\ e^t \frac{3}{10} \cos 3t - \frac{9}{10} e^t \sin 3t \end{pmatrix} +$$

$$+ C_3 \begin{pmatrix} e^t \sin 3t \\ \frac{-3}{2} e^t \cos 3t \\ \frac{9}{10} e^t \cos 3t + e^t \frac{3}{10} \sin 3t \end{pmatrix} = \begin{pmatrix} C_2 e^t \cos 3t + C_3 e^t \sin 3t \\ \frac{3}{2} C_1 e^t + \frac{3}{2} C_2 e^t \sin 3t - \frac{3}{2} C_3 e^t \cos 3t \\ C_1 e^t + C_2 e^t \frac{3}{10} \cos 3t - C_2 \frac{9}{10} e^t \sin 3t + \\ + C_3 \frac{9}{10} e^t \cos 3t + C_3 e^t \frac{3}{10} \sin 3t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \quad x = C_2 e^t \cos 3t + C_3 e^t \sin 3t$$

$$y = \frac{3}{2} C_1 e^t + \frac{3}{2} C_2 e^t \sin 3t - \frac{3}{2} C_3 e^t \cos 3t$$

$$z = C_1 e^t + \frac{3}{10} C_2 e^t \cos 3t - \frac{9}{10} C_2 e^t \sin 3t + \frac{9}{10} C_3 e^t \cos 3t + C_3 e^t \frac{3}{10} \sin 3t$$

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Să se rezolve următoarele sisteme de ecuații diferențiale:

2. b) $\begin{cases} x' = 7x - 5y + 10z \\ y' = 4x - 2y + 8z \\ z' = x - y + 4z \end{cases}$ $x = \alpha_1 e^{rt}$, $y = \alpha_2 e^{rt}$, $z = \alpha_3 e^{rt}$
$x' = \alpha_1 r e^{rt}$, $y' = \alpha_2 r e^{rt}$, $z' = \alpha_3 r e^{rt}$

$$\begin{cases} \alpha_1 r = 7\alpha_1 - 5\alpha_2 + 10\alpha_3 \\ \alpha_2 r = 4\alpha_1 - 2\alpha_2 + 8\alpha_3 \\ \alpha_3 r = \alpha_1 - \alpha_2 + 4\alpha_3 \end{cases} \Rightarrow \begin{cases} (7-r)\alpha_1 - 5\alpha_2 + 10\alpha_3 = 0 \\ 4\alpha_1 + (-2-r)\alpha_2 + 8\alpha_3 = 0 \\ \alpha_1 - \alpha_2 + (4-r)\alpha_3 = 0 \end{cases} \quad (*)$$

$$D = \begin{vmatrix} 7-r & -5 & 10 \\ 4 & -2-r & 8 \\ 1 & -1 & 4-r \end{vmatrix} = (7-r)(-2-r)(4-r) - 40 - 40 - 10(-2-r) + 8(7-r) + 20(4-r)$$

$$= (7-r)(r^2-2r) + 20(4-r) - 10(-r-2) - 80$$

$$= (7-r)r(r-2) + 10(-2r+8+r+2-8)$$

$$= (r-2)(-r^2+7r-10) = 0 \Leftrightarrow \begin{cases} r-2=0 \Rightarrow r_1 = 2 \\ -r^2+7r-10=0 \Leftrightarrow r^2-7r+10=0 \end{cases}$$

$$(r-5)(r-2)=0 \Rightarrow \begin{cases} r_2 = 5 \\ r_3 = 2 \end{cases}$$

• Pt $r_1 = r_3 = 2$

$$(*) \Rightarrow \begin{cases} 5\alpha_1 - 5\alpha_2 + 10\alpha_3 = 0 \\ 4\alpha_1 - 4\alpha_2 + 8\alpha_3 = 0 \\ \alpha_1 - \alpha_2 + 2\alpha_3 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 - 2\alpha_3,\ \alpha_2, \alpha_3 \in R$$

$$(\alpha_2 - 2\alpha_3, \alpha_2, \alpha_3) = (\alpha_2, \alpha_2, 0) + (-2\alpha_3, 0, \alpha_3) = \alpha_2 \underbrace{(1,1,0)}_{\alpha_{r_2}} + \alpha_3 \underbrace{(-2,0,1)}_{\alpha_{r_3}} \Rightarrow$$

$$\Rightarrow X_{r_2} = \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix}, \quad X_{r_3} = \begin{pmatrix} -2e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}$$

• Pt $r_2 = 5$

$$(*) \Rightarrow \begin{cases} 2\alpha_1 - 5\alpha_2 + 10\alpha_3 = 0 \Leftrightarrow \alpha_1 = \dfrac{5\alpha_2 - 10\alpha_3}{2} \\ 4\alpha_1 - 7\alpha_2 + 8\alpha_3 = 0 \Leftrightarrow \alpha_1 = \dfrac{7\alpha_2 - 8\alpha_3}{4} \\ \alpha_1 - \alpha_2 - \alpha_3 = 0 \Leftrightarrow \alpha_1 = \alpha_2 + \alpha_3 \end{cases} \Rightarrow \alpha_2 = 4\alpha_3 \Bigg\} \Rightarrow \alpha_1 = 5\alpha_3,\ \alpha_3 \in R$$

Fie $\alpha_3 = 1 \Rightarrow \alpha_1 = 5, \alpha_2 = 4 \Rightarrow \alpha_{11} = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} \Rightarrow X_{11} = \begin{pmatrix} 5e^{5t} \\ 4e^{5t} \\ e^{5t} \end{pmatrix}$

$$X = C_1 X_{11} + C_2 X_{12} + C_3 X_{13} = C_1 \begin{pmatrix} 5e^{5t} \\ 4e^{5t} \\ e^{5t} \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} -2e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}$$

$$X = \begin{pmatrix} 5C_1 e^{5t} + C_2 e^{2t} + C_3 - 2e^{2t} \\ 4C_1 e^{5t} + C_2 e^{2t} \\ C_1 e^{5t} + C_3 e^{2t} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = 5 \cdot C_1 e^{5t} + C_2 e^{2t} - C_3 2e^{2t} \\ y = 4C_1 e^{5t} + C_2 e^{2t} \\ z = C_1 e^{5t} + C_3 e^{2t} \end{cases}$$

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Să se rezolve următoarele sisteme de ecuații diferențiale:

3. c) $\begin{cases} x' = x + y - 2z \\ y' = 4x + y \\ z' = 2x + y - z \end{cases}$ $\quad x = \alpha_1 e^{rt},\ y = \alpha_2 e^{rt},\ z = \alpha_3 e^{rt}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad x' = \alpha_1 r e^{rt},\ y' = \alpha_2 r e^{rt},\ z' = \alpha_3 r e^{rt}$

$$\begin{cases} \alpha_1 r = \alpha_1 + \alpha_2 - 2\alpha_3 \\ \alpha_2 r = 4\alpha_1 + \alpha_2 \\ \alpha_3 r = 2\alpha_1 + \alpha_2 - \alpha_3 \end{cases} (*') \Rightarrow \begin{cases} (1-r)\alpha_1 + \alpha_2 - 2\alpha_3 = 0 \\ 4\alpha_1 + (1-r)\alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 + (-1-r)\alpha_3 = 0 \end{cases} (*)$$

$$D = \begin{vmatrix} 1-r & 1 & -2 \\ 4 & 1-r & 0 \\ 2 & 1 & -1 \end{vmatrix} = (1-r)(1-r)(-1-r) - 8 + 4(1-r) - 4(-1-r)$$

$$= (1-r)^2(-1-r) + 4(-2 + 1 - r + 1 + r)$$

$$= (1-r)^2(-1-r) = 0 \Rightarrow \begin{cases} -1-r = 0 \Rightarrow r_1 = -1 \\ (1-r)^2 = 0 \Rightarrow r_2 = r_3 = 1 \end{cases}$$

• Pt $r_1 = -1$

$$(*) \Rightarrow \begin{cases} 2\alpha_1 + \alpha_2 - 2\alpha_3 = 0 \Rightarrow \alpha_3 = 0 \\ 4\alpha_1 + 2\alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_1 = \frac{-1}{2}\alpha_2,\ \alpha_2 \in R \end{cases}$$

Fie $\alpha_2 = 1 \Rightarrow \alpha_1 = \frac{-1}{2} \Rightarrow \alpha_{r_1} = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} -1/2\, e^{-t} \\ e^{-t} \\ 0 \end{pmatrix}$

• Pt $r_2 = r_3 = 1$

$$(*) \Rightarrow \begin{cases} \alpha_2 - 2\alpha_3 = 0 \Rightarrow \alpha_2 = 2\alpha_3,\ \alpha_3 \in R \\ 4\alpha_1 = 0 \Rightarrow \alpha_1 = 0 \\ 2\alpha_1 + \alpha_2 - 2\alpha_3 = 0 \end{cases}$$

Fie $\alpha_3 = 1 \Rightarrow \alpha_2 = 2 \Rightarrow \alpha_{r_{2,3}} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

• Pt $r = 1$ căutăm: $x = (A_1 t + A_0)e^t,\ y = (B_1 t + B_0)e^t,\ z = (C_1 t + C_0)e^t$

$x' = A_1 e^t + (A_1 t + A_0)e^t,\ y' = B_1 e^t + (B_1 t + B_0)e^t,\ z' = C_1 e^t + (C_1 t + C_0)e^t$

$$(*') \Rightarrow \begin{cases} A_1 + A_1 t + A_0 = A_1 t + A_0 + B_1 t + B_0 - 2(C_1 t + C_0) \\ B_1 + B_1 t + B_0 = 4(A_1 t + A_0) + B_1 t + B_0 \\ C_1 + C_1 t + C_0 = 2(A_1 t + A_0) + B_1 t + B_0 - (C_1 t + C_0) \end{cases}$$

$$\begin{cases} A_1 = A_1 + B_1 - 2C_1 \\ A_1 + A_0 = A_0 + B_0 - 2C_0 \\ B_1 = 4A_1 + B_1 \\ B_1 + B_0 = 4A_0 + B_0 \\ C_1 = 2A_1 + B_1 - C_1 \\ C_1 + C_0 = 2A_0 + B_0 - C_0 \end{cases}$$

$$\Leftrightarrow \begin{cases} A_1 = A_1 + B_1 - 2C_1 \Rightarrow B_1 = 2C_1,\ C_1 \in R \\ B_1 = 4A_1 + B_1 \Rightarrow A_1 = 0 \\ C_1 = 2A_1 + B_1 - C_1 \end{cases}$$

Fie $C_1 = 1 \Rightarrow B_1 = 2$

$$\Leftrightarrow \begin{cases} A_1 + A_0 = A_0 + B_0 - 2C_0 \Rightarrow B_0 = 2C_0,\ C_0 \in R \\ B_1 + B_0 = 4A_0 + B_0 \Rightarrow A_0 = \frac{1}{2} \\ C_1 + C_0 = 2A_0 + B_0 - C_0 \end{cases}$$

Fie $C_0 = 1 \Rightarrow B_0 = 2$

$$X_{n_{2,3}} = \begin{pmatrix} \frac{1}{2}e^t \\ 2te^t + 2e^t \\ te^t + e^t \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2}e^t \\ 2e^t \\ e^t \end{pmatrix}}_{X_{n_2}} + \underbrace{\begin{pmatrix} 0 \\ 2te^t \\ te^t \end{pmatrix}}_{X_{n_3}}$$

$$X = C_1 X_{n_1} + C_2 X_{n_2} + C_3 X_{n_3}$$

$$= C_1 \begin{pmatrix} \frac{-1}{2}e^{-t} \\ e^{-t} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} \frac{1}{2}e^t \\ 2e^t \\ e^t \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 2te^t \\ te^t \end{pmatrix}$$

$$= \begin{pmatrix} C_1 \cdot \frac{-1}{2}e^{-t} + C_2 \frac{1}{2}e^t + C_3 \cdot 0 \\ C_1 e^{-t} + C_2 \cdot 2e^t + C_3 \cdot 2te^t \\ C_2 e^t + C_3 te^t \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = \frac{-1}{2} C_1 e^{-t} + \frac{1}{2} C_2 e^t \\ y = C_1 e^{-t} + 2C_2 e^t + 2C_3 te^t \\ z = C_2 e^t + C_3 te^t \end{cases}$$

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3. d) $\begin{cases} x' = 2x - y - z \\ y' = 2x - y - 2z \\ z' = -x + y + 2z \end{cases}$

$x = \alpha_1 e^{rt}$, $y = \alpha_2 e^{rt}$, $z = \alpha_3 e^{rt}$

$x' = \alpha_1 e^{rt} r$, $y' = \alpha_2 r e^{rt}$, $z' = \alpha_3 r e^{rt}$

$$\begin{cases} \alpha_1 r = 2\alpha_1 - \alpha_2 - \alpha_3 \\ \alpha_2 r = 2\alpha_1 - \alpha_2 - 2\alpha_3 \quad (*') \\ \alpha_3 r = -\alpha_1 + \alpha_2 + 2\alpha_3 \end{cases} \Rightarrow \begin{cases} (2-r)\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ 2\alpha_1 + (-1-r)\alpha_2 - 2\alpha_3 = 0 \quad (*) \\ -\alpha_1 + \alpha_2 + (2-r)\alpha_3 = 0 \end{cases}$$

$$D = \begin{vmatrix} 2-r & -1 & -1 \\ 2 & -1-r & -2 \\ -1 & 1 & 2-r \end{vmatrix} = (2-r)(-1-r)(2-r) - 4 - (-1-r) + 4(2-r)$$

$$= (-1-r)(-r+3)(-r+1) + 4(-r+1)$$

$$= (r-1)^2(-r+1) = 0 \Leftrightarrow \begin{cases} r_1 = 1 \\ r_2 = r_3 = 1 \end{cases}$$

• Pt $r_1 = r_2 = r_3 = 1$

$$(*) \Rightarrow \begin{cases} \alpha_1 - \alpha_2 - \alpha_3 = 0 \Rightarrow \alpha_1 = \alpha_2 + \alpha_3 \\ 2\alpha_1 - 2\alpha_2 - 2\alpha_3 = 0 \\ -\alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases}$$

• Pt $r = 1$, alegem $x = (A_2 t^2 + A_1 t + A_0)e^t$, $y = (B_2 t^2 + B_1 t + B_0)e^t$, $z = (C_2 t^2 + C_1 t + C_0)e^t$, $x' = (2A_2 t + A_1)e^t + (A_2 t^2 + A_1 t + A_0)e^t$, $y' = (2B_2 t + B_1)e^t + (B_2 t^2 + B_1 t + B_0)e^t$, $z' = (2C_2 t + C_1)e^t + (C_2 t^2 + C_1 t + C_0)e^t$

$$(*') \Rightarrow \begin{cases} 2A_2 t + A_1 + A_2 t^2 + A_1 t + A_0 = 2A_2 t^2 + 2A_1 t + 2A_0 - B_2 t^2 - B_1 t - B_0 - C_2 t^2 - C_1 t - C_0 \\ 2B_2 t + B_1 + B_2 t^2 + B_1 t + B_0 = 2A_2 t^2 + 2A_1 t + 2A_0 - B_2 t^2 - B_1 t - B_0 - 2C_2 t^2 - 2C_1 t - 2C_0 \\ 2C_2 t + C_1 + C_2 t^2 + C_1 t + C_0 = -A_2 t^2 - A_1 t - A_0 + B_2 t^2 + B_1 t + B_0 + 2C_2 t^2 + 2C_1 t + 2C_0 \end{cases}$$

$$\Leftrightarrow \begin{cases} A_2 = 2A_2 - B_2 - C_2 \\ 2A_2 + A_1 = 2A_1 - B_1 - C_1 \\ A_1 + A_0 = 2A_0 - B_0 - C_0 \\ \hline B_2 = 2A_2 - B_2 - 2C_2 \\ 2B_2 + B_1 = 2A_1 - B_1 - 2C_1 \\ B_1 + B_0 = 2A_0 - B_0 - 2C_0 \\ \hline C_2 = -A_2 + B_2 + 2C_2 \\ 2C_2 + C_1 = -A_1 + B_1 + 2C_1 \\ -C_1 + C_0 = -A_0 + B_0 + 2C_0 \end{cases}$$

$$\Leftrightarrow \begin{cases} A_2 = 2A_2 - B_2 - C_2 \Leftrightarrow A_2 - B_2 - C_2 = 0 \\ B_2 = 2A_2 - B_2 - 2C_2 \\ C_2 = -A_2 + B_2 + 2C_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2A_2 + A_1 = 2A_1 - B_1 - C_1 \Leftrightarrow B_1 = 2A_2 \\ 2B_2 + B_1 = 2A_1 - B_1 - 2C_1 \\ 2C_2 + C_1 = -A_1 + B_1 + 2C_1 \Rightarrow C_2 = -A_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} A_1 + A_0 = 2A_0 - B_0 - C_0 \Leftrightarrow B_1 = A_1 \\ B_1 + B_0 = 2A_0 - B_0 - 2C_0 \\ C_1 + C_0 = -A_0 + B_0 + 2C_0 \Rightarrow -C_1 = A_1 \end{cases}$$

Fie $A_2 = 1 \Rightarrow B_2 = 2, C_2 = -1$

$A_1 = 1 \Rightarrow B_1 = 1, C_1 = -1$

$A_0 = 1 \Rightarrow B_0 = 0, C_0 = 0$

$x = (t^2 + t + 1) e^t$

$y = (2t^2 + t) e^t$

$z = (-t^2 - t) e^t$

$$X_{1,2,3} = \begin{pmatrix} (t^2+t+1)e^t \\ (2t^2+t)e^t \\ (-t^2-t)e^t \end{pmatrix} = \underbrace{\begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}}_{X_{1}} + \underbrace{\begin{pmatrix} te^t \\ te^t \\ -te^t \end{pmatrix}}_{X_{2}} + \underbrace{\begin{pmatrix} t^2e^t \\ 2t^2e^t \\ -t^2e^t \end{pmatrix}}_{X_{3}}$$

- $X = c_1 X_{1} + c_2 X_{2} + c_3 X_{3} = c_1 \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} te^t \\ te^t \\ -te^t \end{pmatrix} + c_3 \begin{pmatrix} t^2e^t \\ 2t^2e^t \\ -t^2e^t \end{pmatrix}$

$$= \begin{pmatrix} c_1 e^t + c_2 t e^t + c_3 t^2 e^t \\ c_2 t e^t + 2 c_3 t^2 e^t \\ -c_2 t e^t - c_3 t^2 e^t \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x = c_1 e^t + c_2 t e^t + c_3 t^2 e^t \\ y = c_2 t e^t + 2 c_3 t^2 e^t \\ z = -c_2 t e^t - c_3 t^2 e^t \end{cases}$$

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3. l) $\begin{cases} x' = x - y \\ y' = -y - z \\ z' = -z \end{cases}$ $\quad x = \alpha_1 e^{rt},\ y = \alpha_2 e^{rt},\ z = \alpha_3 e^{rt}$

$x' = \alpha_1 r e^{rt},\ y' = \alpha_2 r e^{rt},\ z' = \alpha_3 r e^{rt}$

$$\begin{cases} \alpha_1 \cdot r = \alpha_1 - \alpha_2 \\ \alpha_2 r = -\alpha_2 - \alpha_3 \quad (*') \\ \alpha_3 r = -\alpha_3 \end{cases} \Rightarrow \begin{cases} (1-r)\alpha_1 - \alpha_2 = 0 \\ (-1-r)\alpha_2 - \alpha_3 = 0 \quad (*) \\ (-1-r)\alpha_3 = 0 \end{cases}$$

$$D = \begin{vmatrix} 1-r & -1 & 0 \\ 0 & -1-r & -1 \\ 0 & 0 & -1-r \end{vmatrix} = (1-r)(-1-r)^2 = 0 \Rightarrow \begin{cases} r_1 = 1 \\ r_2 = r_3 = -1 \end{cases}$$

• Pt $r_1 = 1$

$$(*) \Rightarrow \begin{cases} -\alpha_2 = 0 \\ -2\alpha_2 - \alpha_3 = 0 \\ -2\alpha_3 = 0 \end{cases} \Rightarrow \alpha_2 = \alpha_3 = 0,\ \alpha_1 \in R$$

Fie $\alpha_1 = 1 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}$

• Pt $r_2 = r_3 = -1$

$$(*) \Rightarrow \begin{cases} 2\alpha_1 - \alpha_2 = 0 \Rightarrow 2\alpha_1 = \alpha_2,\ \alpha_1 \in R \\ -\alpha_3 = 0 \Rightarrow \alpha_3 = 0 \end{cases}$$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = 2 \Rightarrow \alpha_{2,3} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

• Pt $r = 0$, căutăm $x = (A_1 t + A_0) e^{-t},\ y = (B_1 t + B_0) e^{-t},\ z = (C_1 t + C_0) e^{-t}$,

$x' = A_1 e^{-t} - (A_1 t - A_0) e^{-t},\ y' = B_1 e^{-t} - (B_1 t + B_0) e^{-t},\ z' = C_1 e^{-t} - (C_1 t + C_0) e^{-t}$

$$(*') \Rightarrow \begin{cases} A_1 - A_1 t - A_0 = A_1 t + A_0 - B_1 t - B_0 \\ B_1 - B_1 t - B_0 = -B_1 t - B_0 - C_1 t - C_0 \\ C_1 - C_1 t - C_0 = -C_1 t - C_0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -A_1 = A_1 - B_1 \Leftrightarrow 2A_1 = B_1 \\ A_1 - A_0 = A_0 - B_0 \Leftrightarrow A_1 = 2A_0 - B_0 \\ -B_1 = -B_1 - C_1 \\ B_1 - B_0 = -B_0 - C_0 \Leftrightarrow B_1 = C_0 \\ C_1 - C_0 = -C_0 \Leftrightarrow C_1 = 0 \end{cases}$$

Fie $A_1 = 1 \Rightarrow B_1 = C_0 = 2$, $A_0 = 1 \Rightarrow B_0 = 1$

$$x_{n_{2,3}} = \begin{pmatrix} te^{-t} + e^{-t} \\ 2te^{-t} + e^{-t} \\ 2e^{-t} \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-t} \\ e^{-t} \\ 2e^{-t} \end{pmatrix}}_{x_{n_2}} + \underbrace{\begin{pmatrix} te^{-t} \\ 2te^{-t} \\ 0 \end{pmatrix}}_{x_{n_3}}$$

$$X = C_1 x_{n_1} + C_2 x_{n_2} + C_3 x_{n_3} = C_1 \begin{pmatrix} e^{t} \\ 0 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} e^{-t} \\ e^{-t} \\ 2e^{-t} \end{pmatrix} + C_3 \begin{pmatrix} te^{-t} \\ 2te^{-t} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} C_1 e^{t} + C_2 e^{-t} + C_3 te^{-t} \\ C_2 e^{-t} + 2C_3 e^{-t} \\ 2C_2 e^{-t} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = C_1 e^{t} + C_2 e^{-t} + C_3 te^{-t} \\ y = C_2 e^{-t} + 2C_3 e^{-t} \\ z = 2C_2 e^{-t} \end{cases}$$