Curs10 - Rezolvare

1. Ecuaţia caracteristică are toate rădăcinile reale şi distincte

Exemplu: Să se rezolve sistemul de ecuaţii diferenţiale:

$$\begin{cases} x_1' = 3 \cdot x_1 - x_2 + x_3 \\ x_2' = -x_1 + 5 \cdot x_2 - x_3 \\ x_3' = x_1 - x_2 + 3 \cdot x_3 \end{cases}$$

Ec. 1

$$\Delta = \begin{vmatrix} 3-r & -1 & 1 \\ -1 & 5-r & -1 \\ 1 & -1 & 3-r \end{vmatrix} \overset{-L_3+L_1}{=} \begin{vmatrix} 2-r & 0 & -2+r \\ -1 & 5-r & -1 \\ 1 & -1 & 3-r \end{vmatrix} = (r-2) \begin{vmatrix} -1 & 0 & 1 \\ -1 & 5-r & -1 \\ 1 & -1 & 3-r \end{vmatrix}$$

$$\underset{L_1+L_3}{\overset{L_1+L_2}{=}} (r-2) \begin{vmatrix} -1 & 0 & 1 \\ -2 & 5-r & 0 \\ 0 & -1 & 4-r \end{vmatrix} \begin{matrix} -1 & 0 & 1 \\ -2 & 5-r & 0 \end{matrix} = (r-2)\left[-(5-r)(4-r)+2\right] = (r-2)(-20+5r+4r-r^2+2)$$

$$= (r-2)(-r^2+9r-18) = 0$$

$r-2=0 \Rightarrow r_1 = 2$

$-r^2+9r-18=0 \quad | \cdot (-1)$

$r^2-9r+18=0$

$\Delta_r = 81-72 = 9 \Rightarrow r_{2,3} = \frac{9 \pm 3}{2}$ $\begin{cases} r_2 = 6 \\ r_3 = 3 \end{cases}$

Ex 1

$$\begin{cases} x_1' = 3x_1 - x_2 + x_3 \\ x_2' = -x_1 + 5x_2 - x_3 \\ x_3' = x_1 - x_2 + 3x_3 \end{cases} \qquad \begin{aligned} x_1(0) &= 2 \\ x_2(0) &= 1 \\ x_3(0) &= -4 \end{aligned}$$

Sol. sist. este:
$$\begin{cases} x_1 = C_1 e^{2t} + C_2 e^{3t} + C_3 e^{6t} \\ x_2 = C_2 e^{3t} - 3C_3 e^{6t} \\ x_3 = -C_1 e^{2t} + C_2 e^{3t} + C_3 e^{6t} \end{cases}$$

$$\begin{cases} x_1(0) = C_1 + C_2 + C_3 = 2 \\ x_2(0) = C_2 - 3C_3 = 1 \\ x_3(0) = -C_1 + C_2 + C_3 = 4 \end{cases} \Rightarrow \begin{cases} C_1 + C_2 + C_3 = 2 \\ C_2 - 3C_3 = 1 \\ -C_1 + C_2 + C_3 = -4 \end{cases} \overset{(-)}{\Longrightarrow} 2C_1 = 6 \Rightarrow \boxed{C_1 = 3}$$

$$\begin{cases} C_2 + C_3 = -1 \\ C_2 - 3C_3 = 1 \end{cases}$$

$(-) \quad 4C_3 = -2 \Rightarrow C_3 = -\frac{2}{4} \rightarrow \boxed{C_3 = -\frac{1}{2}}$ $\qquad C_2 = 1 + 3C_3 = 1 - \frac{3}{2} = \boxed{-\frac{1}{2} = C_2}$

$$\Rightarrow PC: \begin{cases} x_1 = 3e^{2t} - \frac{1}{2}e^{3t} - \frac{1}{2}e^{6t} \\ x_2 = -\frac{1}{2}e^{3t} - \frac{3}{2}e^{6t} \\ x_3 = -3e^{2t} - \frac{1}{2}e^{3t} - \frac{1}{2}e^{6t} \end{cases}$$

$$r_1 = 2 \Rightarrow \begin{cases} \alpha_1 - \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + 3\alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_2 + \alpha_3 = 0 \end{cases} \overset{(+)}{\Longrightarrow} 2\alpha_2 = 0 \Rightarrow \boxed{\alpha_2 = 0}$$

$$X_r = \alpha_1 e^{rt}$$

$$\alpha_1 + \alpha_3 = 0 \Rightarrow \boxed{\alpha_3 = -\alpha_1}$$

$$\text{Fie } \alpha_1 = 1 \Rightarrow \alpha_3 = -1 \Rightarrow \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{2t} = \begin{pmatrix} e^{2t} \\ 0 \\ -e^{2t} \end{pmatrix}$$

$$r_3 = 6$$

$$\begin{cases} -3\alpha_1 - \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_2 - 3\alpha_3 = 0 \end{cases} \overset{(-)}{\Longrightarrow} -4\alpha_1 + 4\alpha_3 \Rightarrow \boxed{\alpha_3 = \alpha_1}$$

$$-\alpha_1 - \alpha_2 - \alpha_1 = 0$$

$$-2\alpha_1 - \alpha_2 = 0 \Rightarrow \boxed{\alpha_2 = -2\alpha_1}$$

$$\text{Fie } \alpha_1 = 1 \Rightarrow \alpha_2 = -2, \alpha_3 = 1 \Rightarrow \alpha_{r_3} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

2. Ecuaţia caracteristică are toate rădăcinile complexe şi distincte

Exemplu: Să se rezolve sistemul de ecuaţii diferenţiale.

$$\begin{cases} x' = -9 \cdot y \\ y' = 4 \cdot x \end{cases}$$

Ex 2

$$\begin{cases} x' = -9y \\ y' = 4x \end{cases} \qquad \begin{matrix} x(0) = 6 \\ y(0) = 2 \end{matrix}$$

Sol. generală:
$$\begin{cases} x = 3C_1 \cos 6t + 3C_2 \sin 6t \\ y = 2C_1 \sin 6t - 2C_2 \cos 6t \end{cases}$$

$$x(0) = 3C_1 \underbrace{\cos 0}_{1} + 3C_2 \underbrace{\sin 0}_{0} = 3C_1 = 6 \Rightarrow \boxed{C_1 = 2}$$

$$y(0) = 2C_1 \underbrace{\sin 0}_{0} - 2C_2 \underbrace{\cos 0}_{1} = -2C_2 = 2 \Rightarrow \boxed{C_2 = -1}$$

PC:
$$\begin{cases} x = 6\cos 6t - 3 \sin 6t \\ y = 4 \cos 6t + 2 \sin 6t \end{cases}$$