L.

Model ex. examen

53. Se consideră conica

$\Gamma: 2x^2-4xy+5y^2+4x-16y-22=0.$

Să se determine:

a) invarianții metrici ai lui Γ

b) natura și genul conicei

c) valorile și vectorii proprii corespunzători matricei A, a părții principale a lui Γ.

d) centrul conicei și ec. redusă la centru a conicei Γ.

Rezolvare

a) $A_\Gamma = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 5 & -8 \\ 2 & -8 & -22 \end{pmatrix}$ $\qquad A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$

$$\Delta = \begin{vmatrix} 2 & -2 & 2 \\ -2 & 5 & -8 \\ 2 & -8 & -22 \end{vmatrix} = -220+32+32-20-128+88 = -216 < 0$$

$\neq 0$

$$\delta = \begin{vmatrix} 2 & -2 \\ -2 & 5 \end{vmatrix} = 10-4=6 > 0.$$

$I = 2+5=7$

b) Cum $\Delta \neq 0 \Rightarrow$ conică nedegenerată

Cum $\delta \neq 0 \Rightarrow$ conică cu centru

Cum $\delta > 0 \Rightarrow$ conică de tip eliptic.

c) $A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$

$$A - \lambda I_2 = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix}$$

$$\det(A - \lambda I_2) = (2-\lambda)(5-\lambda) - 4$$
$$= 10 - 7\lambda + \lambda^2 - 4$$
$$= \lambda^2 - 7\lambda + 6$$

$\lambda^2 - 7\lambda + 6 = 0$

$\Delta = 49 - 24 = 25 \Rightarrow \lambda_{1,2} = \dfrac{7 \pm 5}{2}$ $\begin{array}{l} 6 \\ 1 \end{array}$

Pt. $\lambda = 6$

$$(A - 6 I_2) \cdot \vec{u}_{\lambda_1} = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -4u_1 - 2u_2 \\ -2u_1 - u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-2u_1 - u_2 = 0 \Rightarrow u_2 = -2u_1$

Fie $u_1 = 1 \Rightarrow u_2 = -2 \Rightarrow \vec{u}_{\lambda_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\|\vec{u}_{\lambda_1}\| = \sqrt{5}$

Pt. $\lambda = 1$.

$$(A - 1 \cdot I_2) \vec{u}_{\lambda_2} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 - 2u_2 \\ -2u_1 + 4u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$u_1 - 2u_2 = 0 \Rightarrow u_1 = 2u_2$

Fie $u_2 = 1 \Rightarrow u_1 = 2. \Rightarrow u_{\lambda_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $\|\vec{u}_{\lambda_2}\| = \sqrt{5}$

d) $\begin{cases} \dfrac{1}{2} \dfrac{\partial \Gamma}{\partial x} = 0 \\ \dfrac{1}{2} \dfrac{\partial \Gamma}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{2}(4x - 4y + 4) = 0. \\ \dfrac{1}{2}(-4x + 10y - 16) = 0. \end{cases}$

$\begin{cases} 2x - 2y + 2 = 0 \\ -2x + 5y - 8 = 0. \end{cases} \Leftrightarrow \begin{cases} 2x - 2y = -2 \\ -2x + 5y = 8. \end{cases}$

$\overline{ 3y = 6} \Rightarrow y = 2$

$$2x - 2y + 2 = 0$$

$$x - y + 1 = 0 \Rightarrow x = y - 1 = 2 - 1 = 1$$

$$\Rightarrow C(1, 2)$$

Vectorii ortonormați sunt:

$$\vec{u}^{\,n}_{\lambda_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \vec{u}_{\lambda_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$T = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

Aplicăm rotația $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y' \\ -\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y' \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y' \\ y = -\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y' \end{cases}$$

Înlocuim în ec. conicei:

$$2\left(\frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'\right)^2 - 4\left(\frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'\right)\left(-\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'\right)$$

$$+ 5\left(-\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'\right)^2 + 4\left(\frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'\right) - 16\left(-\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'\right)$$

$$- 22 = 0.$$

$$2\left(\frac{1}{5}(x')^2+\frac{4}{5}x'y'+\frac{4}{5}(y')^2\right)-4\left(-\frac{2}{5}(x')^2+\frac{1}{5}x'y'\right.$$

$$\left.-\frac{4}{5}x'y'+\frac{2}{5}(y')^2\right)+5\left(\frac{4}{5}(x')^2-\frac{4}{5}x'y'+\frac{1}{5}(y')^2\right)$$

$$+\frac{4}{\sqrt{5}}x'+\frac{8}{\sqrt{5}}y'+\frac{32}{\sqrt{5}}x'-\frac{16}{\sqrt{5}}y'-22=0.$$

$$\underline{\frac{2}{5}(x')^2}+\cancel{\frac{8}{5}x'y'}+\cancel{\frac{8}{5}(y')^2}+\underline{\frac{8}{5}(x')^2}\cancel{-\frac{4}{5}x'y'}+\cancel{\frac{16}{5}x'y'}\cancel{-\frac{8}{5}(y')^2}$$

$$+\underline{\frac{20}{5}(x')^2}-\cancel{\frac{20}{5}x'y'}+\frac{5}{5}(y')^2+\frac{36}{\sqrt{5}}x'-\frac{8}{\sqrt{5}}y'-22=0.$$

$$6(x')^2+(y')^2+\frac{36}{\sqrt{5}}x'-\frac{8}{\sqrt{5}}y'-22=0.$$

$$6\left((x')^2+\frac{6}{\sqrt{5}}x'\right)+(y')^2-\frac{8}{\sqrt{5}}y'-22=0.$$

$$6\left((x')^2+2\cdot x'\cdot\frac{3}{\sqrt{5}}+\frac{9}{5}\right)+\left((y')^2-2\cdot y'\cdot\frac{4}{\sqrt{5}}+\frac{16}{5}\right)-22$$

$$-\frac{54}{5}-\frac{16}{5}=0$$

$$6\left(x'+\frac{3}{\sqrt{5}}\right)^2+\left(y'+\frac{4}{\sqrt{5}}\right)^2=22+\frac{70}{5}=22+14=36$$

Facem translatia $X=x'+\frac{3}{\sqrt{5}}$

$Y=y'+\frac{4}{\sqrt{5}}$

$\Rightarrow 6X^2+Y^2=36$

$$\frac{X^2}{6}+\frac{Y^2}{36}=1$$

$$\frac{X^2}{(\sqrt{6})^2}+\frac{Y^2}{6^2}=1 \quad (\text{Elipsa})$$