① e) $\dfrac{dx}{x(x+y)} = \dfrac{dy}{-y(x+y)} = \dfrac{dz}{(y-x)(2x+2y+z)}$, $|x| + |y| \neq 0$

$\dfrac{dx}{x(x+y)} = \dfrac{dy}{-y(x+y)} \quad \Big| \cdot (x+y)$

$\dfrac{dx}{x} = -\dfrac{dy}{y}$

$\int \frac{1}{x} dx = -\int \frac{1}{y} dy$

$\ln|x| = -\ln|y| + C_1$

$\ln|x| + \ln|y| = C_1$

$\ln|xy| = C_1$

$\boxed{xy = C_1 = \varphi_1(x,y,z)}$

$\dfrac{dx}{x(x+y)} = \dfrac{dy}{-y(x+y)} = \dfrac{dx+dy}{x(x+y) - y(x+y)} = \dfrac{d(x+y)}{(x+y)(x-y)} = \dfrac{dz}{(y-x)(2x+2y+z)}$

$\Big| \cdot (x-y)$

$\dfrac{d(x+y)}{(x+y)(x-y)} = \dfrac{dz}{-(x-y)[2(x+y)+z]}$

$\dfrac{d(x+y)}{x+y} = \dfrac{dz}{-\boxed{2(x+y)+z}}$, Notăm $x+y = u \Rightarrow \dfrac{du}{u} = \dfrac{dz}{-2u-z}$

$\dfrac{du}{dz} = \dfrac{u}{-2u-z} \overset{|:u}{=} \dfrac{1}{-2-\frac{z}{u}}$

Fie z variabila indep.

$\dfrac{du}{dz} = \dfrac{1}{-2-\frac{1}{\frac{u}{z}}}$, $\dfrac{u}{z} = v \Rightarrow u = vz \Rightarrow u' = v'z + v$

$v'z + v = \dfrac{1}{-2-\frac{1}{v}} \Rightarrow v'z = \dfrac{v}{-2v-1} - v = \dfrac{v + 2v^2 + v}{-2v-1} = \dfrac{2v^2+2v}{-2v-1}$

$\dfrac{dv}{dz} \cdot z = \dfrac{2(v^2+v)}{-(2v+1)} \Rightarrow \dfrac{2v+1}{v^2+v} dv = -\dfrac{2}{z} dz$

$$\int \frac{2v+1}{v^2+v}\,dv = -2\int \frac{1}{z}\,dz$$

$$\ln|v^2+v| = -2\ln|z| + C_2$$

$$\ln|v^2+v| + \ln|z|^2 = C_2$$

$$\ln|z^2(v^2+v)| = C_2$$

$$z^2(v^2+v) = C_2$$

$$z^2\left(\frac{u^2}{z^2} + \frac{u}{z}\right) = C_2$$

$$\cancel{z^2}\cdot\frac{u^2+zu}{\cancel{z^2}} = C_2$$

$$(x+y)^2 + z(x+y) = C_2$$

$$\boxed{(x+y)(x+y+z) = C_2 = \psi_2(x,y,z)}$$

z – var. indep.

$$\psi_1, \psi_2 \text{ indep} \iff \frac{D(\psi_1,\psi_2)}{D(x,y)} \neq 0$$

$$\begin{vmatrix} \frac{\partial\psi_1}{\partial x} & \frac{\partial\psi_1}{\partial y} \\ \frac{\partial\psi_2}{\partial x} & \frac{\partial\psi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 2x+2y+z & 2x+2y+z \end{vmatrix} = (2x+2y+z)\begin{vmatrix} y & x \\ 1 & 1 \end{vmatrix}$$

$$= \underbrace{(2x+2y+z)}_{\neq 0}\underbrace{(y-x)}_{\neq 0}$$

$$\left[(x+y)(x+y+z)\right]'_x = x+y+z+(x+y)$$

$\Rightarrow \psi_1, \psi_2$ – indep.

Sol. implicită:

$$\begin{cases} xy = C_1 \\ (x+y)(x+y+z) = C_2 \end{cases}$$