Sisteme de ec. dif. liniare, neomogene cu coef. constanți

$$x_1' = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + f_1(t)$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + f_2(t)$$
$$\dots \qquad (1)$$
$$x_n' = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + f_n(t)$$

$\Leftrightarrow$

$$X' = AX + f(t) \quad (2)$$

Soluția generală a sistemului (1) sau (2) este

$$X = X_0 + X_p$$

Soluția particulară se poate determina prin mai multe metode

1) Metoda variației constantelor

2) Prin calcul algebric (se aplică pt anumite forme particulare ale lui $f(t)$)

I) Dacă $f(t) = \begin{pmatrix} P_{m_1}(t) \\ P_{m_2}(t) \\ \vdots \\ P_{m_n}(t) \end{pmatrix}$

P_{m_i} - pol. de grad m_i

$$X_p = \begin{pmatrix} Q_{1m}(t) \\ Q_{2m}(t) \\ Q_{nm}(t) \end{pmatrix}$$

Q_{m} = pol de grad m

$m = \max\{m_1, m_2, \dots, m_n\}$

Coeficienții polinoamelor din X_p se determină prin identificare.

__Exemplu:__

$$\begin{cases} x' = x - y + 3t^2 \\ y' = -4x - 2y + 2 + 8t \end{cases} \qquad f = \begin{pmatrix} 3t^2 \\ 2+8t \end{pmatrix}$$

__Sistem omogen:__

$$\begin{cases} x' = x - y \\ y' = -4x - 2y \end{cases} \qquad \begin{array}{l} x = \alpha_1 e^{rt} \\ y = \alpha_2 e^{rt} \end{array} \qquad \begin{cases} \alpha_1 r = \alpha_1 - \alpha_2 \\ \alpha_2 r = -4\alpha_1 - 2\alpha_2 \end{cases}$$

$$\begin{cases} (1-r)\alpha_1 - \alpha_2 = 0 \\ -4\alpha_1 - (2-r)\alpha_2 = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1-r & -1 \\ -4 & -(2-r) \end{vmatrix} = (1-r)(r-2) - 4$$

$$= -r - 2 - r^2 + 2r - 4$$

$$= +r^2 + r - 6 = 0$$

$$\Delta = \ \ + 24 = 15$$

$$r_{1,2} = \frac{-1 \pm 5}{2} \begin{cases} 2 \\ -3 \end{cases}$$

pt. $r = 2$

$$\begin{cases} -\alpha_1 - \alpha_2 = 0 \\ -4\alpha_1 - 4\alpha_2 = 0 \end{cases}$$

$\alpha_1 = 1 \Rightarrow \alpha_2 = -1$

$$x_{p_1} = \begin{pmatrix} 1 \cdot e^{2t} \\ -1 \cdot e^{2t} \end{pmatrix}$$

pt. $r = -3$

$$\begin{cases} 4\alpha_1 - \alpha_2 = 0 \\ -4\alpha_1 + \alpha_2 = 0 \end{cases} \Rightarrow \alpha_2 = 4\alpha_1$$

$\alpha_1 = 1 \Rightarrow \alpha_2 = 4$

$$x_{p_2} = \begin{pmatrix} 1 \cdot e^{-3t} \\ 4e^{-3t} \end{pmatrix}$$

$$x_0 = C_1 x_{p_1} + C_2 x_{p_2} = C_1 \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{-3t} \\ 4e^{-3t} \end{pmatrix} =$$

$$= \begin{pmatrix} C_1 e^{2t} + C_2 e^{-3t} \\ 4C_2 e^{-3t} - C_1 e^{2t} \end{pmatrix}$$

$$\boxed{\begin{aligned} x_0 &= C_1 e^{2t} + C_2 e^{-3t} \\ y_0 &= -C_1 e^{2t} + 4C_2 e^{-3t} \end{aligned}}$$

$$x_p = \begin{pmatrix} A_2 t^2 + A_1 t + A_0 \\ B_2 t^2 + B_1 t + B_0 \end{pmatrix}$$

$x_0 = A_2 t^2 + A_1 t + A_0$

$y_p = B_2 t^2 + B_1 + B_0$

$$\frac{2A_2t}{\cancel{2}} + A_1 = A_2t^2 + A_1t + A_0 - B_2t^2 - B_1t - B_0$$
$$+ 3t^2$$

$$2B_2t + B_1 = -4A_2t^2 - 4A_1t - 4A_0 - 2B_2t^2 - 2B_1t -$$
$$- 2B_0 + 2 + 8t$$

$$\begin{cases} A_2 - B_2 + 3 = 0 \Rightarrow \boxed{A_2 = 3} \\ A_1 - B_1 = 2A_2 \\ A_0 - B_0 = 1 \\ -4A_2 - 2B_2 = 0 \Rightarrow \boxed{B_2} = 2A_2 = 2\cdot 3 = \boxed{6} \\ -4A_1 - 2B_1 + 8 = 2B_2 \quad /:2 \\ -4A_0 - 2B_0 + 2 = B_2 \end{cases}$$

$$\begin{cases} A_1 - B_1 = 6 \\ -2A_1 - B_1 + 4 = 6 \end{cases}$$
$$3A_1 = 4 \Rightarrow \boxed{A_1 = \frac{4}{3}}$$

$$B_1 = \frac{4}{3} - \frac{3}{\cancel{4}\,6} = \boxed{\frac{-14}{3} = B_1}$$

$$A_0 - B_0 = 1$$
$$-4A_0 - 2B_0 = \cancel{7}4 \quad /:-2$$
$$2A_0 + B_0 = -2$$
$$3A_0 = -1 \Rightarrow \boxed{A_0 = -\frac{1}{3}}$$
$$\boxed{B_0 = -\frac{4}{3}}$$

$$x_p = \begin{pmatrix} 3t^2 + \frac{4}{3}t - \frac{1}{3} \\ 6t^2 - \frac{14}{3}t - \frac{4}{3} \end{pmatrix}$$

$$x = x_0 + x_p = \begin{pmatrix} c_1 e^{2t} + c_2 e^{-t} + 3t^2 + \frac{4}{3}t - \frac{1}{3} \\ -c_1 e^{2t} + 4 c_2 e^{-3t} + 6t^2 - \frac{14t}{3} - \frac{4}{3} \end{pmatrix}$$

2 $$f(t) = \begin{pmatrix} e^{\lambda t} P_{m_1}(t) \\ e^{\lambda t} P_{m_2}(t) \\ \vdots \\ e^{\lambda t} P_{m_n}(t) \end{pmatrix} \rightarrow x_p = \begin{pmatrix} e^{\lambda t} Q_{1m}(t) \\ e^{\lambda t} Q_{2m}(t) \\ \\ e^{\lambda t} Q_{nm}(t) \end{pmatrix}$$

<u>Caz 1</u>: λ - nu e rădăcină a ecuației caracteristice

$Q_m(t)$ - polinom de ord. $m = \max\{m_1, m_2, \dots, m_n\}$

<u>Caz 2</u>: λ - este rădăcină de ordin k a ecuației caracteristice

$$x_p = \begin{pmatrix} e^{\lambda t} Q_1(m+k)(t) \\ e^{\lambda t} Q_2(m+k)(t) \\ \dots \end{pmatrix}$$

$Q_i(m+k)$ = polinom de grad $m+k$

$m = \max\{m_1, \dots, m_n\}$

Exemplu:

$$\begin{cases} x' = 2x + y \\ y' = 3y + te^{t} \end{cases} \qquad f(t) = \begin{pmatrix} 0 \\ te^{t} \end{pmatrix}, \quad \alpha = 1$$

Sistem omogen:

$$\begin{cases} x' = 2x + y \\ y' = 3y \end{cases}$$

$$\begin{cases} \alpha_1 r = 2\alpha_1 + \alpha_2 \\ \alpha_2 r = 3\alpha_2 \end{cases} \Rightarrow \begin{cases} (2-r)\alpha_1 + \alpha_2 = 0 \\ (3-r)\alpha_2 = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 2-r & 1 \\ 0 & 3-r \end{vmatrix} = (2-r)(3-r) = 0 \quad \begin{matrix} r_1 = 2 \\ r_2 = 3 \end{matrix}$$

pt. $r = 2$

$$\begin{cases} \alpha_2 = 0 \\ \alpha_2 = 0 \end{cases} \qquad \alpha_1 \in \mathbb{R}$$

Fie $\alpha_1 = 1$

$$x_{p_1} = \begin{pmatrix} 1 \cdot e^{2t} \\ 0 \cdot e^{2t} \end{pmatrix}$$

pt $r = 3$

$$\begin{cases} -\alpha_1 + \alpha_2 = 0 \\ 0 = 0 \end{cases} \qquad \alpha_1 = \alpha_2$$

$$x_{r2} = \begin{pmatrix} 1 \cdot e^{3t} \\ 1 \cdot e^{3t} \end{pmatrix}$$

$$x_0 = C_1 x_{r1} + C_2 x_{r2} = C_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix} =$$

$$= \begin{pmatrix} C_1 e^{2t} + C_2 e^{3t} \\ C_2 e^{3t} \end{pmatrix}$$

$$x_p = \begin{pmatrix} (A_1 t + A_0) e^t \\ (B_1 t + B_0) e^t \end{pmatrix}$$

$$\begin{cases} A_1 e^t + (A_1 t + A_0) e^t = 2(A_1 t + A_0) e^t + (B_1 t + B_0) e^t \\ B_1 e^t + (B_1 t + B_0) e^t = 3(B_1 t + B_0) e^t + t e^t \end{cases}$$

$$\begin{cases} A_1 + A_1 t + A_0 = 2A_1 t + 2A_0 + B_1 t + B_0 \\ B_1 + B_1 t + B_0 = 3 B_1 t + 3B_0 + t \end{cases}$$

$$\begin{cases} B_1 = -A_1 \rightarrow \boxed{A_1 = \frac{1}{2}} \\ A_1 + A_0 = 2A_0 + B_0 \Rightarrow A_0 = A_1 - B_0 = \frac{1}{2} + \frac{1}{4} \Rightarrow \boxed{A_0 = \frac{3}{4}} \\ B_1 = 3B_1 + 1 \rightarrow 2B_1 = -1 \Rightarrow \boxed{B_1 = -\frac{1}{2}} \\ B_1 + B_0 = 3B_0 \Rightarrow B_0 = \frac{B_1}{2} = -\frac{1}{4} \Rightarrow \boxed{B_0 = -\frac{1}{4}} \end{cases}$$

$$x_p = \begin{pmatrix} (\frac{1}{2}t + \frac{3}{4}) e^t \\ (-\frac{1}{2}t - \frac{1}{4}) e^t \end{pmatrix}$$

Soluţia generală este:

$$X = X_0 + X_p = \begin{pmatrix} c_1 e^{2t} + c_2 e^{2t} + \left(\frac{1}{2}t + \right. \\ c_2 e^{3t} + \left(-\frac{1}{2}t - \frac{1}{4}\right) \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \end{pmatrix}$$

Exemplu.

$$\begin{cases} x' = x + e^t \\ y' = x + y - e^t \end{cases}$$

$$\begin{cases} \alpha_1 r = \alpha_1 \\ \alpha_2 r = \alpha_1 + \alpha_2 \end{cases} \Rightarrow \begin{cases} (1-r)\alpha_1 = 0 \\ \alpha_1 + (1-r)\alpha_2 = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1-r & 0 \\ 1 & 1+r \end{vmatrix} = (1-r)^2$$

$$r_{1,2} = 1$$

Pt. $\underline{\underline{r = 1}}$

$$\begin{cases} 0 = 0 \\ \alpha_1 = 0 \end{cases} \quad \alpha_2 \in \mathbb{R}$$

$$X_{01} = \begin{pmatrix} 0 e^t \\ 1 \cdot e^t \end{pmatrix}$$

$$\begin{cases} x = (A_1 t + A_0)e^t \\ y = (B_1 t + B_0)\, e^t \end{cases}$$

$$\begin{cases} A_1 e^t + (A_1 t + A_0)e^t = (A_1 t + A_0)\, e^t \\ B_1 e^t + (B_1 t + B_0)e^t = (A_1 t + A_0)e^t + \end{cases}$$

$$+ (B_1 t + B_0)\, e^t$$

$$\begin{cases} A_1 + A_1 t + A_0 = A_1 t + A_0 \quad \Rightarrow A_1 = 0 \\ \underline{B_1} + \cancel{B_1 t} + \cancel{B_0} = A_1 t + A_0 + \cancel{B_1 t} + \cancel{B_0} \end{cases} \quad , B_1 = A_0$$

$$x_0 = e^t$$
$$y_0 = (t+1)\, e^t$$

$$x_p = \begin{pmatrix} (A_2 t^2 + A_1 t + A_0)e^t \\ (B_2 t^2 + B_2 t + B_0)e^t \end{pmatrix}$$

$$\begin{cases} 2t A_2 + A_1 + \cancel{A_2 t^2} + \cancel{A_1 t} + \cancel{A_0} = \cancel{A_2 t^2} + \cancel{A_1 t} + \cancel{A_0} + 1 \\ 2B_2 + B_2 + \cancel{B_2 t^2} + \cancel{B_1 t} + \cancel{B_0} = \cancel{A_2 t^2} + A_1 t + A_0 \\ \qquad + \cancel{B_2 t^2} + \cancel{B_1 t} + \\ \qquad \cancel{B_0} - 1 \end{cases}$$

$$A_2 = 0$$
$$A_1 = 1$$
$$2B_2 = A_1 \Rightarrow B_2 = \frac{1}{2}$$
$$B_2 = A_0 - 1 \Rightarrow A_0 = \frac{3}{2}$$

$$x_p = \begin{pmatrix} (t + \frac{3}{2})\, e^t \\ (\frac{t^2}{2} + t + 1)e^t \end{pmatrix}$$

$$X = \begin{pmatrix} e^t \\ (t+1)e^t \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} e^t \\ e^t \end{pmatrix}}_{X_{p1}} + \underbrace{\begin{pmatrix} 0 \\ te^t \end{pmatrix}}_{X_{p2}}$$

$$X_0 = c_1 X_{p1} \begin{pmatrix} e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$

$$X = \begin{pmatrix} c_1 e^t + \left(t + \frac{3}{2}\right) e^t \\ c_1 e^t + c_2 t e^t + \left(\frac{t^2}{2} + t + 1\right) e^t \end{pmatrix}$$

$$f_i = e^{\alpha t}\left(P_{m_i}(t) \cdot \cos\beta t + Q_{p_i}(t) \cdot \sin\beta t \right)$$

a) Dacă $\alpha + i\beta$ nu este rădăcină a ecuației caracteristice, atunci soluția caracteristică se ia de forma:

$$x_p = e^{\alpha t}\left(P_m(t)\cos\beta t + Q_m(t)\sin\beta t \right)$$

$$m = \max\{ m_1, m_2, \dots, m_m, p_1, p_2 \dots p_i$$

b) Dacă $\alpha + i\beta$ este rădăcină a ecuației caracteristice, atunci:

$$x_p = e^{\alpha t}\left(P_{m+k}(t)\cos\beta t + Q_{m+k}\sin \right.$$

Exemplu:

1.

$$\begin{cases} x' = x + y + \cos t \\ y' = -x + y - \sin t \end{cases}$$

$x' = x + y$

$y' = -x + y$

$x = \alpha_1 e^{rt} \Rightarrow x' = \alpha_1 r e^{rt}$

$y = \alpha_2 e^{rt} \Rightarrow y' = \alpha_2 r e^{rt}$

$$\begin{cases} \alpha_1 r = \alpha_1 + \alpha_2 \\ \alpha_2 r = -\alpha_1 + \alpha_2 \end{cases} \Rightarrow \begin{matrix} (1-r)\alpha_1 + \alpha_2 = 0 \\ -\alpha_1 + (1-r)\alpha_2 = 0 \end{matrix}$$

$$\Delta = \begin{vmatrix} 1-r & -1 \\ +1 & 1-r \end{vmatrix} = (1-r)^2 - 1 = 0$$

$(1-r-1)(1-r+1) = 0$

$-r(2-r) = 0$

$r_1 = 0$

$r_2 = 2$

Pentru $r_1 = 0$

$$\begin{cases} \alpha_1 - \alpha_2 = 0 \\ -\alpha_1 + \alpha_2 = 0 \end{cases} \Rightarrow \alpha_2 = \alpha_1$$

Fie $\alpha_1 = 1 \Rightarrow$ şi $\alpha_2 = 1 \Rightarrow$

$\Rightarrow \quad x_{r_1} = \begin{pmatrix} 1 \cdot e^{0t} \\ 1 \cdot e^{0 \cdot t} \end{pmatrix}$

Pentru $r_2 = 2$

$$\begin{cases} -\alpha_1 - \alpha_2 = 0 \\ -\alpha_1 - \alpha_2 = 0 \end{cases} \Rightarrow \alpha_2 = -\alpha_1$$

$$\alpha_1 = 1 \Rightarrow \alpha_2 = -1$$

$$x_{r_2} = \begin{pmatrix} 1 \cdot e^{2t} \\ -1 \cdot e^{2t} \end{pmatrix}$$

$$x_0 = C_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} =$$

$$= \begin{pmatrix} C_1 + C_2 e^{2t} \\ C_1 - C_2 e^{2t} \end{pmatrix}$$

Observăm că:

$$f = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \qquad \begin{matrix} \alpha = 0 \\ \beta = 1 \end{matrix}$$

$\alpha + i\beta = i \Rightarrow$ nu este rădăcină a ecuației caracteristice.

$$x_p = \begin{pmatrix} A\cos t + B\sin t \\ C\cos t + D\sin t \end{pmatrix}$$

$x_p = A\cos t + B\sin t \Rightarrow x_p' = -A\sin t + B\cos t$

$y_p = C\cos t + D\sin t \qquad y_p' = -C\sin t + D\cos t$

$-A\sin t + B\cos t = A\cos t + B\sin t + C\cos t + D\sin t +$
$+ \cos t$

$-C\sin t + D\cos t = -A\cos t - B\sin t + C\cos t + D\sin t - \sin t$

$$\begin{cases} -A = B - D \\ B = A - C + 1 \\ -C = -B + D - 1 \\ D = -A + C \end{cases} \Rightarrow \begin{cases} -A - B + D = 0 \Rightarrow D = A + B \\ B - A + C - 1 = 0 \Rightarrow C = 1 + \\ -C + B - D + 1 = 0 \\ D + A - C = 0 \end{cases}$$

$$\begin{cases} B - 1 - A + \cancel{B} - A - \cancel{B} = -1 \\ \cancel{+A} - A - \cancel{B} + B + A + B = 0 \end{cases}$$

$$\begin{cases} -2A + B = 0 \\ A + 2B = 1/2 \end{cases} \Rightarrow \begin{cases} -2A + B = 0 \\ 2A + 4B = 2 \end{cases}$$

$$5B = 2$$

$$\boxed{B = \frac{2}{5}}$$

$$\boxed{A = \frac{1}{5}}$$

$$C = 1 + A - B$$

$$= \frac{5}{1} + \frac{1}{5} - \frac{2}{5} = \frac{4}{5}$$

$$\Rightarrow \boxed{C = \frac{4}{5}}$$

$$\boxed{D = \frac{3}{5}}$$

$$x_p = \begin{pmatrix} \frac{1}{5}\cos t + \frac{2}{5}\sin t \\ \frac{4}{5}\cos t + \frac{4}{5}\sin t \end{pmatrix}$$

$$x = x_0 + x_p = \begin{pmatrix} c_1 + c_2 e^{2t} + \frac{1}{5}\cos t + \frac{2}{5} s \\ c_1 - c_2 e^{2t} + \frac{4}{5}\cos t + \frac{3}{5} s \end{pmatrix}$$

$$x = c_1 + c_2 e^{2t} + \frac{1}{5}\cos t + \frac{2}{5}\sin t$$

$$y = c_1 - c_2 e^{2t} + \frac{4}{5}\cos t + \frac{3}{5}\sin t$$

Exemplul 2:

$$x' = x + y + e^t \cos t$$
$$y' = -x + y - e^t \sin t$$