Laborator04

Enunţuri

1. Să se rezolve următoarele ecuații de tip Bernoulli.

$$\begin{aligned}
&a)\ \ x' + \frac{t}{6} \cdot x = \frac{1}{3} \cdot t \cdot x^{-2} \\
&b)\ \begin{cases} 3 \cdot t \cdot x^2 \cdot x' + x^3 = 2t \\ x(1) = 2 \end{cases} \\
&c)\ \begin{cases} x' = 2 \cdot t \cdot x + t \cdot \sqrt[3]{x} \\ x(0) = 4 \end{cases} \\
&d)\ \ x' + 2 \cdot t \cdot x = 2 \cdot t^3 \cdot x^3
\end{aligned}$$

2. Să se rezolve următoarele ecuații de tip Riccati.

$$\begin{aligned}
&a)\ \ x' + x^2 - 2 \cdot x \cdot sint + sin^2 t - cost = 0, && \rho_0(t) = sint \\
&b)\ \ x' = t \cdot x^2 - 2 \cdot t^2 \cdot x + t^3 + 1, && \rho_0(t) = t \\
&c)\ \begin{cases} x' = -x^2 \cdot sint + \frac{2 \cdot sint}{cos^2 t}, \\ x(0) = 2 \end{cases} && \rho_0(t) = \frac{1}{cost}
\end{aligned}$$

3. Să se rezolve următoarele ecuații cu diferențiale exacte.

$$\begin{aligned}
&a)\ \ \frac{t}{x^2} dt + \frac{x^2 - t^2}{x^3} dx = 0 \\
&b)\ \begin{cases} 2 \cdot t \cdot x\ dt + (t^2 + x^2) dx = 0 \\ x(1) = 3 \end{cases} \\
&c)\ \ (t^2 + x^2 + 2t) dt + 2 \cdot t \cdot x\ dx = 0 \\
&d)\ \ t\ dt + x\ dx = \frac{-t\ dx - x\ dt}{t^2 + x^2}
\end{aligned}$$

4. Să se rezolve următoarele ecuații căutând un factor integrant.

$$\begin{aligned}
&a)\ \ (x^2 - 2 \cdot t \cdot x) dt + t^2 dx = 0 \\
&b)\ \ 2 \cdot t \cdot x\ dt = (t^2 - x^2) dx \\
&c)\ \ (2 \cdot t \cdot x - t) dt + (x^2 + x + 2t^2) dx = 0
\end{aligned}$$

Rezolvare

Exerciţiu 01

c) - Video

1. c) $\begin{cases} x' = 2tx + t\sqrt[3]{x} \\ x(0) = 4 \end{cases}$

$x' = \underbrace{2t}_{A(t)} x + \underbrace{t}_{B(t)} x^{\frac{1}{3}} \quad \Big| : x^{\frac{1}{3}}, \ \alpha = \frac{1}{3}$

$\frac{x'}{x^{1/3}} = 2t \frac{x}{x^{1/3}} + t$

$x' \cdot x^{-\frac{1}{3}} = 2t x^{1-\frac{1}{3}} + t = 2t x^{\frac{2}{3}} + t$

$u = x^{\frac{2}{3}} \Rightarrow u' = \frac{2}{3} x^{\frac{2}{3}-1} \cdot x' = \frac{2}{3} x^{-\frac{1}{3}} \cdot x'$

$x' \cdot x^{-\frac{1}{3}} = 2t x^{\frac{2}{3}} + t \quad \Big| \cdot \frac{2}{3}$

$\frac{2}{3} x' x^{-1/3} = \frac{4}{3} t x^{2/3} + \frac{2}{3} t$

$u' = \underbrace{\frac{4}{3} t}_{A(t)} u + \underbrace{\frac{2}{3} t}_{B(t)}$

Etapa 1: $u' = \frac{4}{3} t u \Leftrightarrow \frac{du}{dt} = \frac{4}{3} t u$

$\Leftrightarrow \frac{du}{u} = \frac{4}{3} t \, dt \Leftrightarrow \int \frac{1}{u} du = \frac{4}{3} \int t \, dt$

$\Leftrightarrow \ln|u| = \frac{4}{3} \cdot \frac{t^2}{2} + C = \frac{2t^2}{3} + C \Rightarrow u = e^{\frac{2t^2}{3} + C}$

$u_0 = C \cdot e^{2t^2/3}$

Etapa 2: $\varphi_0 = C(t) \cdot e^{\frac{2t^2}{3}}$

$\left(C(t) \cdot e^{\frac{2t^2}{3}}\right)' = \frac{4}{3} t \cdot C(t) \cdot e^{\frac{2t^2}{3}} + \frac{2}{3} t$

$C'(t) \cdot e^{\frac{2t^2}{3}} + \cancel{C(t) \cdot e^{\frac{2t^2}{3}} \cdot \frac{4}{3} t} = \cancel{\frac{4}{3} t \, C(t) e^{\frac{2t^2}{3}}} + \frac{2}{3} t$

$C'(t) \cdot e^{\frac{2t^2}{3}} = \frac{2}{3} t \quad \Big| \cdot e^{-\frac{2t^2}{3}}$

$C'(t) = \frac{2}{3} t e^{-\frac{2t^2}{3}} \Rightarrow C(t) = \frac{2}{3} \cdot \left(-\frac{3}{4}\right) \int -\frac{4}{3} t e^{-\frac{2t^2}{3}} dt$

$v = -\frac{2t^2}{3}$

$v' = -\frac{4}{3} t$

$C(t) = -\frac{1}{2} \cdot e^{-\frac{2t^2}{3}} + C_1$

$\varphi_0 = \left(-\frac{1}{2} e^{-\frac{2t^2}{3}} + C_1\right) \cdot e^{\frac{2t^2}{3}}$

$\varphi_0 = -\frac{1}{2} + C_1 e^{2t^2/3}$

$u = u_0 + \varphi_0$

$= C \cdot e^{2t^2/3} - \frac{1}{2} + C_1 e^{2t^2/3}$

$u = C \cdot e^{2t^2/3} - \frac{1}{2}$

$u = x^{\frac{2}{3}} \Rightarrow u^3 = x^2 \Rightarrow x = \sqrt{u^3}$

$x = \sqrt{\left(C e^{2t^2/3} - \frac{1}{2}\right)^3},$

$C e^{2t^2/3} - \frac{1}{2} \geq 0$

$x(0) = \sqrt{\left(C - \frac{1}{2}\right)^3} = 4$

$\left(C - \frac{1}{2}\right)^3 = 16$

$C - \frac{1}{2} = \sqrt[3]{16}$

$C = \frac{1}{2} + \sqrt[3]{16}$

$x_{PC} = \sqrt{\left[\left(\frac{1}{2} + \sqrt[3]{16}\right) \cdot e^{\frac{2t^2}{3}} - \frac{1}{2}\right]^3}$

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Exerciţiu 02

a) - Video

2 a) Laborator 4.

$\underbrace{x'}_{}+\underbrace{x^2}_{A(t)} - 2x\underbrace{\sin t}_{B(t)} + \underbrace{\sin^2 t - \cos t}_{f(t)} = 0, \quad \varphi_0(t) = \sin t$

$\boxed{x = y + \varphi_0} \Rightarrow x = y + \sin t \Rightarrow x' = y' + \cos t$

$y' + \cancel{\cos t} + (y + \sin t)^2 - 2\sin t\,(y + \sin t) + \sin^2 t - \cancel{\cos t} = 0$

$y' + y^2 + \cancel{2y\sin t} + \cancel{\sin^2 t} - \cancel{2y\sin t} - \cancel{2\sin^2 t} + \cancel{\sin^2 t} = 0$

$y' + y^2 = 0$

$\frac{dy}{dt} = -y^2$

$\frac{dy}{-y^2} = dt \Rightarrow \int -\frac{1}{y^2}dy = \int dt \Leftrightarrow \frac{1}{y} = t + C \Rightarrow y = \frac{1}{t+C}, \; t + C \neq 0$

$\Rightarrow x = \frac{1}{t+C} + \sin t$

c)

c (2) c)

$x' = -x^2\underbrace{\sin t}_{A(t)} + \underbrace{\frac{2\sin t}{\cos^2 t}}_{f(t)}, \quad \varphi_0(t) = \frac{1}{\cos t}$

$x(0) = 2$

$x = y + \varphi_0 \Rightarrow x = y + \frac{1}{\cos t} \Rightarrow x' = y' + \frac{\sin t}{\cos^2 t}$

$y' + \cancel{\frac{\sin t}{\cos^2 t}} = -\left(y + \frac{1}{\cos t}\right)^2 \cdot \sin t + \frac{2\sin t}{\cos^2 t}$

$= -y^2\sin t - 2y\frac{1}{\cos t}\sin t - \cancel{\frac{\sin t}{\cos^2 t}} + \cancel{\frac{2\sin t}{\cos^2 t}}$

$y' = \underbrace{-\sin t}_{B(t)} \cdot y^2 - 2\underbrace{\frac{\sin t}{\cos t}}_{A(t)} \cdot y, \quad \alpha = 2$

$| : y^2$

$\frac{y'}{y^2} = -\sin t - 2\frac{\sin t}{\cos t}\cdot\frac{y}{y^2}$

$y' \cdot y^{-2} = -\sin t - 2\frac{\sin t}{\cos t}\,y^{-1}$

$u = y^{-1} \Rightarrow u' = -1 \cdot y^{-2} \cdot y'$

$-y'y^{-1} = \sin t + 2\frac{\sin t}{\cos t}y^{-1}$

$u' = \sin t + \frac{2\sin t}{\cos t}u$

$u = \frac{1}{y} \Rightarrow y = \frac{1}{u} = \frac{1}{-\frac{\cos t}{3} + \frac{C}{\cos^2 t}}$

$x_{sg} = y + \frac{1}{\cos t} = \frac{1}{\frac{C}{\cos^2 t} - \frac{\cos t}{3}} + \frac{1}{\cos t}$

$x(0) = \frac{1}{C - \frac{1}{3}} + 1 = 2 \Rightarrow \frac{1}{C - \frac{1}{3}} = 1 \Rightarrow C - \frac{1}{3} = 1 \Rightarrow C = 1 + \frac{1}{3}$

$C = \frac{4}{3}$

$x_{pc} = \frac{1}{\frac{4}{3\cos^2 t} - \frac{\cos t}{3}} + \frac{1}{\cos t}$

Exerciţiu 03

a)

3.a. Sem 4.

$$\underbrace{\frac{t}{x^2}}_{P(t,x)}dt+\underbrace{\frac{x^2-t^2}{x^3}}_{Q(t,x)}dx=0$$

$$\left.\begin{array}{l}\frac{\partial P}{\partial x}=\frac{-t\cdot 2x}{x^4}=\frac{-2t}{x^3}\\ \frac{\partial Q}{\partial t}=\left(\frac{1}{x}-\frac{t^2}{x^3}\right)'_t=-\frac{2t}{x^3}\end{array}\right\}\Rightarrow\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial t}$$

$F(t,x)=C$,

$$F(t,x)=\int_0^t P(\tau,x_0)\,d\tau+\int_{x_0}^x Q(t,\sigma)\,d\sigma$$

$$=\int_0^t\frac{\tau}{x_0^2}\,d\tau+\int_{x_0}^x\frac{\sigma^2-t^2}{\sigma^3}\,d\sigma=\frac{1}{x_0^2}\cdot\frac{\tau^2}{2}\Big|_0^t+\int_{x_0}^x\left(\frac{1}{\sigma}-\frac{t^2}{\sigma^3}\right)d\sigma$$

$$=\frac{t^2}{2x_0^2}+\ln|\sigma|\Big|_{x_0}^x-t^2\int_{x_0}^x\sigma^{-3}\,d\sigma=\frac{t^2}{2x_0^2}+\ln|x|-\ln|x_0|-t^2\,\frac{\sigma^{-2}}{-2}\Big|_{x_0}^x$$

$$\Rightarrow F(t,x)=\cancel{\frac{t^2}{2x_0^2}}+\ln|x|\underbrace{-\ln|x_0|}_{+C}+\frac{t^2}{2x^2}-\cancel{\frac{t^2}{2x_0^2}}$$

$$\ln|x|+\frac{t^2}{2x^2}=C$$

b) - [Video]()

(1) b) $\begin{cases}2tx\,dt+(t^2+x^2)dx=0\\ x(1)=3\end{cases}$

$$\underbrace{2tx}_{P(t,x)}\,dt+(\underbrace{t^2+x^2}_{Q(t,x)})dx=0$$

$$\left.\begin{array}{l}\frac{\partial P}{\partial x}(t,x)=2t\\ \frac{\partial Q}{\partial t}(t,x)=2t\end{array}\right\}\Rightarrow\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial t}$$

$F(t,x)=C$

$$\int_0^t P(\tau,0)\,d\tau+\int_0^x Q(t,\sigma)\,d\sigma=$$

$$=\int_0^t 2\tau\cdot 0\,d\tau+\int_0^x(t^2+\sigma^2)\,d\sigma$$

$$=t^2\sigma\Big|_0^x+\frac{\sigma^3}{3}\Big|_0^x=t^2x+\frac{x^3}{3}$$

$t^2x+\frac{x^3}{3}=C$ (sol. in formă implicită)

$1^2\cdot 3+\frac{3^3}{3}=C$

$\Rightarrow C=3+9=12$

PC: $t^2x+\frac{x^3}{3}=12$

Exerciţiu 04

a) - [Video]()

④ a) $\underbrace{(x^2-2tx)}_{P(t,x)}dt + \underbrace{t^2}_{Q(t,x)}dx = 0 \quad | \mu(x)$

$$\left.\begin{aligned} \frac{\partial P}{\partial x}(t,x) &= 2x-2t \\ \frac{\partial Q}{\partial t}(t,x) &= 2t \end{aligned}\right\} \Rightarrow \frac{\partial P}{\partial x} \neq \frac{\partial Q}{\partial t}$$

Fie $\mu = \mu(x)$

$$\underbrace{\mu(x)\cdot(x^2-2tx)}_{P^*(t,x)}dt + \underbrace{\mu(x)\cdot t^2}_{Q^*(t,x)}dx = 0$$

$$\frac{\partial P^*}{\partial x}(t,x) = \mu'(x)(x^2-2tx) + \mu(x)(2x-2t)$$

$$\frac{\partial Q^*}{\partial t}(t,x) = \mu(x)\cdot 2t$$

$$\mu'(x)(x^2-2tx) + \mu(x)(2x-2t) = \mu(x)\cdot 2t$$

$$\mu'(x)(x^2-2tx) = \mu(x)(2t-2x+2t)$$

$$\mu'(x)\cdot x(x-2t) = \mu(x)(4t-2x)$$

$$\mu'(x)\cdot x\cancel{(x-2t)} = -2\mu(x)\cancel{(x-2t)}$$

$$\frac{d\mu}{dx}\cdot x = -2\mu$$

$$\frac{d\mu}{\mu} = -\frac{2}{x}dx \Rightarrow \int \frac{1}{\mu} = -2\int\frac{1}{x}dx$$

$$\ln|\mu| = -2\ln|x| + C$$

$$\ln|\mu| = -\ln x^2 + \ln C$$

$$\mu = \frac{c}{x^2} \qquad \text{Pp. } c = 1$$

$$(x^2-2tx)\cdot\frac{c}{x^2}dt + t^2\cdot\frac{c}{x^2}dx = 0$$

$F(t,x) = C$

$$F(t,x) = \int_0^t P(\tau,x_0)\,d\tau + \int_{x_0}^x Q(t,\sigma)\,d\sigma$$

$$= \int_0^t \frac{x_0^2-2\tau x_0}{x_0^2}d\tau + \int_{x_0}^x t^2\cdot\frac{1}{\sigma^2}d\sigma$$

$$= \int_0^t \left(1-\frac{2\tau}{x_0}\right)d\tau + t^2\int_{x_0}^x \frac{1}{\sigma^2}d\sigma$$

$$= \tau\Big|_0^t - \frac{\tau^2}{x_0}\Big|_0^t + t^2\cdot\left(-\frac{1}{\sigma}\right)\Big|_{x_0}^x$$

$$= t - \cancel{\frac{t^2}{x_0}} - \frac{t^2}{x} + \cancel{\frac{t^2}{x_0}} + C$$

$$t - \frac{t^2}{x} = C$$

④ a)

$\underbrace{(x^2-2tx)}_{P(t,x)}dt + \underbrace{t^2}_{Q(t,x)}dx = 0 \quad | \mu(t)$

$$\left.\begin{aligned} \frac{\partial P}{\partial x}(t,x) &= 2x-2t \\ \frac{\partial Q}{\partial t}(t,x) &= 2t \end{aligned}\right\} \Rightarrow \frac{\partial P}{\partial x} \neq \frac{\partial Q}{\partial y}$$

Pp. $\mu = \mu(t)$

$$\underbrace{\mu(t)(x^2-2tx)}_{P^*(t,x)}dt + \underbrace{\mu(t)\,t^2}_{Q^*(t,x)}dx = 0$$

$$\frac{\partial P^*}{\partial x}(t,x) = \mu(t)(2x-2t)$$

$$\frac{\partial Q^*}{\partial t}(t,x) = \mu'(t)\cdot t^2 + \mu(t)\cdot 2t$$

$$\mu'(t)\cdot t^2 + \mu(t)\cdot 2t = \mu(t)(2x-2t)$$

$$\mu'(t)\cdot t^2 = \mu(t)(2x-2t-2t)$$

$\mu'(t)\cdot t^2 = \mu(t)(2x-4t)$ (nu e bine)

c)

4) c) Sem 4

$$\underbrace{(2tx-t)}_{P(t,x)}dt + \underbrace{(x^2+x+2t^2)}_{Q(t,x)}dx = 0$$

$$\left.\begin{aligned}\frac{\partial P}{\partial x}(t,x) &= 2t\\ \frac{\partial Q}{\partial t}(t,x) &= 4t\end{aligned}\right\} \frac{\partial P}{\partial x} \neq \frac{\partial Q}{\partial t}$$

Pp. $\mu = \mu(t)$

$$\underbrace{\mu(t)(2tx-t)}_{P^*(t,x)}dt + \underbrace{\mu(t)(x^2+x+2t^2)}_{Q^*(t,x)}dx = 0$$

$$\frac{\partial P^*}{\partial x} = \mu(t)\cdot 2t \quad ; \quad \frac{\partial Q^*}{\partial t} = \mu'(t)(x^2+x+2t^2)+\mu(t)\cdot 4t$$

$$\mu'(t)(x^2+x+2t^2) + 4t\cdot\mu(t) = 2t\mu(t)$$

$$\mu'(t)(x^2+x+2t^2) = -2t\mu(t)$$

(nu e bine!)

Pp. $\mu = \mu(x)$

$$\underbrace{\mu(x)(2tx-t)}_{P^*(t,x)}dt + \underbrace{\mu(x)(x^2+x+2t^2)}_{Q^*(t,x)}dx = 0$$

$$\frac{\partial P^*}{\partial x} = \mu'(x)(2tx-t)+\mu(x)\cdot 2t$$

$$\frac{\partial Q^*}{\partial t} = \mu(x)\cdot 4t$$

$$\mu'(x)(2tx-t)+2t\mu(x) = 4t\mu(x)$$

$$\mu'(x)\cdot t(2x-1) = 2t\mu(x)$$

$$\mu'(x)(2x-1) = 2\mu(x)$$

$$\frac{d\mu}{dx}(2x-1) = 2\mu$$

$$\frac{d\mu}{\mu} = \frac{2}{2x-1}dx$$

$$\int \frac{1}{\mu}d\mu = \int \frac{2}{2x-1}dx$$

$$\ln|\mu| = \ln|2x-1| + C$$

$$\ln|\mu| = \ln C|2x-1|$$

$$\mu = C(2x-1)$$

Fie $C = 1 \Rightarrow \mu(x) = 2x-1$

$$\underbrace{(2x-1)(2tx-t)}_{P^*(t,x)}dt + \underbrace{(2x-1)(x^2+x+2t^2)}_{Q^*(t,x)}dx = 0$$

$F(t,x) = C$. t

$$F(t,x) = \int_0^t P^*(\tau,0)\,d\tau + \int_0^x Q^*(t,\sigma)\,d\sigma$$

$$= \int_0^t (-1)(-\tau)\,d\tau + \int_0^x (2\sigma-1)(\sigma^2+\sigma+2t^2)\,d\sigma$$

$$= \int_0^t \tau\,d\tau + \int_0^x (2\sigma^3+2\sigma^2+4\sigma t^2-\sigma^2-\sigma-2t^2)\,d\sigma$$

$$= \frac{\tau^2}{2}\Big|_0^t + \left(2\cdot\frac{\sigma^4}{4}+2\cdot\frac{\sigma^3}{3}+4t^2\cdot\frac{\sigma^2}{2}-\frac{\sigma^3}{3}-\frac{\sigma^2}{2}-2t^2\sigma\right)\Big|_0^x$$

$$= \frac{t^2}{2}+\frac{x^2}{2}+\frac{2x^3}{3}+2t^2x^2-\frac{x^3}{3}$$