Laboratör02

Enunţuri

1. Să se rezolve următoarele ecuaţii diferenţiale direct integrabile:

$$
\begin{array}{l}
a) \begin{cases} x'(t) = t^2 + 2\,, & x, t \in R \\ x(1) = 1 \end{cases} \\
b) \begin{cases} x'(t) = \frac{1}{t^2-1}\,, & t \in (-1, 1) \\ & x \in R \end{cases} \\
c) \begin{cases} x'(t) = \frac{1}{1+t^2}\,, & t, x \in R \\ x(-1) = -2 \end{cases} \\
d) \begin{cases} x'(t) = sin\,t + 4t^2 \\ x(\frac{\pi}{3}) = -\frac{1}{3} \end{cases} \\
e) \begin{cases} x'(t) = \frac{\sqrt{lnt}}{t}\,, & t > 1 \\ x(e) = \frac{5}{3} & x \in R \end{cases}
\end{array}
$$

2. Să se rezolve următoarele ecuaţii diferenţiale cu variabile separate:

$$
\begin{array}{ll}
a) \dfrac{1}{1+t^2}dt + \dfrac{1}{x}dx = 0, & x > 0,\ t \in R \\
b)\ dx + \dfrac{1}{t^2-9}dt = 0, & t > 3,\ x \in R \\
c) \sqrt{x}\,dx = \sqrt{t}\,dt, & t > 0,\ x > 0 \\
\quad x(1) = 1 & \\
d)\ sint\ dt - cosx\ dx = 0, & x \in [0, \pi] \\
\quad x(0) = \dfrac{\pi}{2} & \\
e) \dfrac{x}{1-x^2}dx = \dfrac{1}{1-t}dt, & t < 1,\ x \in (0, 1) \\
\quad x(0) = \dfrac{1}{2} &
\end{array}
$$

3. Să se rezolve următoarele ecuaţii diferenţiale cu variabile separabile:

$a)\ (t+1)\cdot x'(t) = 2x-3$
$b)\ (t^2-1)\cdot x'(t) + 2tx^2 = 0$
$c)\ x'(t) = \dfrac{-t}{\sqrt{1+t^2}}\cdot\dfrac{\sqrt{1+x^2}}{x}, \qquad x<0,\ t\in R$
$d)\ \dfrac{dx}{dt} = \dfrac{t}{1+t}(1-x), \qquad t>-1,\ x>1$
$x(0)=5$

Rezolvare

Exercitiul 1

a) - [Video]

1 a) $\begin{cases} x'(t) = t^2+2 \\ x(1) = 1 \end{cases}, \ t, x \in \mathbb{R}$

$x'(t) = \underbrace{t^2+2}_{f(t)}$ (ec. direct integrabilă)

$x(t) = \int (t^2+2)\,dt = \int t^2\,dt + \int 2\,dt = \frac{t^3}{3} + 2t + C$

$x(1) = \frac{1^3}{3} + 2\cdot 1 + C = \frac{1}{3}+2+C = 1 \Rightarrow C = \frac{3}{1} - \frac{1}{3} - \frac{3}{2} = \frac{3-1-6}{3} = \frac{-4}{3} \Rightarrow \boxed{C = -\frac{4}{3}}$

$\Rightarrow x_{pc}(t) = \frac{t^3}{3} + 2t - \frac{4}{3}$

e)

1 e) $\begin{cases} x'(t) = \frac{\sqrt{\ln t}}{t} \\ x(e) = \frac{5}{3} \end{cases}, \ t>1, \ x\in\mathbb{R}$

$x'(t) = \underbrace{\frac{\sqrt{\ln t}}{t}}_{f(t)}$ (ec. direct integr.)

$x(t) = \int \frac{\sqrt{\ln t}}{t}\,dt = \int \sqrt{\ln t}\cdot\frac{1}{t}\,dt = \int \sqrt{u}\cdot u'\,dt = \int u^{\frac{1}{2}}\cdot u'\,dt = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$u(t) = \ln t$
$u'(t) = \frac{1}{t}$

$= \frac{(\ln t)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}\sqrt{(\ln t)^3} + C$

$x(e) = \frac{2}{3}\sqrt{(\underbrace{\ln e}_{1})^3} + C = \frac{2}{3} + C = \frac{5}{3} \Rightarrow C = \frac{5}{3} - \frac{2}{3} = \frac{3}{3} = 1 \Rightarrow \boxed{C=1}$

$x_{pc}(t) = \frac{2}{3}\sqrt{(\ln t)^3} + 1$

Exercitiul 2

c) - Video

2 c) $\begin{cases} \sqrt{x}\,dx = \sqrt{t}\,dt \\ x(1) = 1 \end{cases}$, $t>0$, $x>0$

$\sqrt{x}\,dx = \sqrt{t}\,dt$ (ec. cu variabile separate)

$\int \sqrt{x}\,dx = \int \sqrt{t}\,dt \iff \int x^{\frac{1}{2}}\,dx = \int t^{\frac{1}{2}}\,dt \iff \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \mathcal{C}$

$\frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \mathcal{C} \iff x^{\frac{3}{2}} = t^{\frac{3}{2}} + \mathcal{C}$

$\sqrt{x^3} = \sqrt{t^3} + \mathcal{C} \iff x^3 = (\sqrt{t^3} + \mathcal{C})^2 \Rightarrow x = \sqrt[3]{(\sqrt{t^3} + \mathcal{C})^2}$

$x(1) = 1$
$x(1) = \sqrt[3]{(1+C)^2} = 1 \Rightarrow (1+C)^2 = 1 \Rightarrow 1+C = \pm 1 < \begin{matrix} c = 0 \\ c = -2 \end{matrix}$

$c = 0 \Rightarrow x = \sqrt[3]{(\sqrt{t^3})^2} = \sqrt[3]{t^3} = t \Rightarrow x_{PC}(t) = t$

$c = -2 \Rightarrow x_{PC} = \sqrt[3]{(\sqrt{t^3} - 2)^2}$

e)

2 e) $\begin{cases} \frac{x}{1-x^2}\,dx = \frac{1}{1-t}\,dt \\ x(0) = \frac{1}{2} \end{cases}$, $t<1$, $x \in (0,1)$

$\frac{x}{1-x^2}\,dx = \frac{1}{1-t}\,dt$ (ec. cu var. separate)

$-\frac{1}{2}\int \frac{-2x}{1-x^2}\,dx = -\int \frac{-1}{1-t}\,dt \iff -\frac{1}{2}\int \frac{u'}{u}\,dx = -\int \frac{v'}{v}\,dt$

$u(x) = 1-x^2$ $\quad v(t) = 1-t$
$u'(x) = -2x$ $\quad v'(t) = -1$

$-\frac{1}{2}\ln|u| = -\ln|v| + \mathcal{C} \quad / \cdot (-2)$

$\ln|1-x^2| = 2\ln|1-t| + \mathcal{C}$

$\ln(1-x^2) = \ln(1-t)^2 + \ln C$

$\ln(1-x^2) = \ln[(1-t)^2 \cdot C]$

$1-x^2 = C(1-t)^2$

$x^2 = 1 - C(1-t)^2 \Rightarrow x = \sqrt{1-C(1-t)^2}$, $1-C(1-t)^2 > 0$

$x(0) = \sqrt{1-C} = \frac{1}{2} \Rightarrow 1-C = \frac{1}{4} \Rightarrow C = \frac{3}{4}$

$x_{PC} = \sqrt{1 - \frac{3}{4}(1-t)^2}$, $1 - \frac{3}{4}(1-t)^2 > 0$.

Exercitiul 3

b) - Video

3 b) $(t^2-1)\cdot x'(t)+2tx^2=0$

$$x'(t)=\frac{dx}{dt}$$

$$(t^2-1)\frac{dx}{dt}+2tx^2=0$$

$$\underbrace{(t^2-1)}_{A_1(t)}dx+\underbrace{2t}_{A_2(t)}\underbrace{x^2}_{B_2(x)}dt=0 \text{ (ec. cu variabile separabile)}$$

$$(t^2-1)dx=-2tx^2dt$$

$$-\frac{1}{x^2}dx=\frac{2t}{t^2-1}dt$$

$$\int -\frac{1}{x^2}dx=\int\frac{2t}{t^2-1}dt \Leftrightarrow \frac{1}{x}=\ln|t^2-1|+C \Rightarrow x=\frac{1}{\ln|t^2-1|+C},$$

$u(t)=t^2-1$
$u'(t)=2t$

$$\ln|t^2-1|+C\neq 0$$

c)

3 c) $\dot{x}(t)=\frac{-t}{\sqrt{1+t^2}}\cdot\frac{\sqrt{1+x^2}}{x}$, $x<0$, $t\in\mathbb{R}$

$$\frac{dx}{dt}=\frac{-t}{\sqrt{1+t^2}}\cdot\frac{\sqrt{1+x^2}}{x}$$

$$\frac{x}{\sqrt{1+x^2}}dx=-\frac{t}{\sqrt{1+t^2}}dt$$

$$\int\frac{2x}{2\sqrt{1+x^2}}dx=-\int\frac{2t}{2\sqrt{1+t^2}}dt \Leftrightarrow \int\frac{u'}{2\sqrt{u}}dx=-\int\frac{v'}{2\sqrt{v}}dt$$

$u(x)=1+x^2$
$u'(x)=2x$

$$\sqrt{1+x^2}=-\sqrt{1+t^2}+C$$

$$1+x^2=\left(C-\sqrt{1+t^2}\right)^2$$

$$x^2=\left(C-\sqrt{1+t^2}\right)^2-1$$

$$x=-\sqrt{\left(C-\sqrt{1+t^2}\right)^2-1},\quad \left(C-\sqrt{1+t^2}\right)^2-1\geq 0$$