Laborator 03 - Temă Petculescu Mihai-Silviu

1. Să se rezolve următoarele ecuații liniare scalare:

a) $\begin{cases} \frac{dx}{dt} = (t^2+1)\cdot x, & t, x \in \mathbb{R}_+ \\ x(0) = 2 \end{cases}$

$$\frac{dx}{dt} = \underbrace{(t^2+1)\cdot x}_{A(t)} \quad \text{(ec. liniar. scalară)}$$

$$\frac{dx}{x} = (t^2+1)\,dt \Leftrightarrow \int \frac{1}{x}\,dx = \int t^2+1\,dt \Leftrightarrow \ln x = \frac{t^3}{3} + t + \mathcal{C}$$

$$x = e^{\left(\frac{t^3}{3}+t\right)+\mathcal{C}} \Leftrightarrow \mathcal{C}\cdot e^{\left(\frac{t^3}{3}+t\right)} = x(t)$$

$$x(0) = 2 \Leftrightarrow x(0) = C\cdot e^0 \Leftrightarrow \boxed{C = 2} \Rightarrow x_{PC} = 2e^{\left(\frac{t^3}{3}+t\right)}$$

b) $\begin{cases} \frac{dx}{dt} = \sqrt{t+1}\,x, & t \geq -1 \\ x(0) = 2 & x \in \mathbb{R}_+ \end{cases}$

$$\frac{dx}{dt} = \underbrace{\sqrt{t+1}\,x}_{A(t)} \quad \text{(ec. liniar. scalară)}$$

$$\frac{dx}{x} = \sqrt{t+1}\,dt \Leftrightarrow \int \frac{1}{x}\,dx = \int \sqrt{t+1}\,dt \Leftrightarrow \ln x = \frac{2}{3}\cdot\sqrt{(t+1)^3} + \mathcal{C}$$

$$\int \sqrt{t+1}\,dt = \int (t+1)^{\frac{1}{2}}\,dt = \frac{(t+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \mathcal{C} = \frac{(t+1)^{\frac{3}{2}}}{\frac{3}{2}} + \mathcal{C}$$

$$x = e^{\frac{2}{3}\sqrt{(t+1)^3}+\mathcal{C}} \Leftrightarrow x(t) = C\cdot e^{\frac{2}{3}\sqrt{(t+1)^3}}$$

$$x(0) = 2 \Leftrightarrow x(0) = C\cdot e^{\frac{2}{3}\cdot 1} \Leftrightarrow \boxed{C = \frac{2}{e^{\frac{2}{3}}}} \Rightarrow x_{PC} = 2\cdot e^{\frac{-2}{3}}\cdot e^{\frac{2}{3}\sqrt{(t+1)^3}}$$

$$x_{PC} = 2\cdot e^{\frac{2}{3}\left(\sqrt{(t+1)^3}-1\right)}$$

2. Să se rezolve următoarele ecuații diferențiale afine:

a) $x' + \underbrace{\frac{1-2t}{t^2}}_{A(t)}\cdot x = \underbrace{1}_{B(t)}$

Etapa 1:

$$x' + \frac{1-2t}{t^2}x = 0 \Leftrightarrow \frac{dx}{dt} = \frac{2t-1}{t^2}x \Leftrightarrow \frac{1}{x}dx = \frac{2t-1}{t^2}dt \Leftrightarrow \int \frac{1}{x}dx = \int \frac{2t-1}{t^2}dt$$

$$\ln|x| = 2\ln|t| + \frac{1}{t} + C \Rightarrow \boxed{x_0 = Ct^2 + e^{\frac{1}{t}}}$$

Etapa 2:

$$\varphi_0 = C(t)\,t^2 + e^{\frac{1}{t}}$$

$$\left(C(t)t^2+e^{\frac{1}{t}}\right)' + \frac{1-2t}{t^2}\left(C(t)t^2+e^{\frac{1}{t}}\right)=1$$

$$C'(t)t^2+2tC(t)+\frac{-e^{\frac{1}{t}}}{t^2}+(1-2t)C(t)+\frac{(1-2t)e^{\frac{1}{t}}}{t^2}=1$$

$$C'(t)t^2 = 1+\frac{2e^{\frac{1}{t}}}{t} \Leftrightarrow C'(t)=\frac{2e^{\frac{1}{t}}+t}{t^3} \Leftrightarrow C(t)=\int \frac{2e^{\frac{1}{t}}+t}{t^3}\,dt$$

$$C(t)=\frac{2(t-1)e^{\frac{1}{t}}-1}{t}+C_1 \Rightarrow \boxed{\varphi_0 = t\left(2(t-1)e^{\frac{1}{t}}-1\right)+e^{\frac{1}{t}}}$$

$$x=x_0+\varphi_0 = ct^2+2e^{\frac{1}{t}}+t\left(2(t-1)e^{\frac{1}{t}}-1\right)$$

d) $\frac{dx}{dt}=x-t^2 \Leftrightarrow x'\underbrace{-x}_{A(t)}=\underbrace{-t^2}_{B(t)}$ (ec. dif. afină)

<u>Etapa 1</u>

$$\frac{dx}{dt}=x \Leftrightarrow \frac{1}{x}dx=dt \Leftrightarrow \int\frac{1}{x}dx=\int dt \Leftrightarrow \ln|x|=t+C \Rightarrow \boxed{x_0=e^tc}$$

<u>Etapa 2</u>

$$\varphi_0=e^tc(t)$$

$$\left(e^tc(t)\right)'-e^tc(t)=-t^2 \Leftrightarrow e^tc(t)+e^tc'(t)-e^tc(t)=-t^2 \Leftrightarrow c'(t)=\frac{-t^2}{e^t}$$

$$c(t)=\int\frac{-t^2}{e^t}dt = \boxed{(t^2+2t+2)e^{-t}=\varphi_0}$$

$$\bullet \int \underbrace{t^2}_{f}\underbrace{e^{-t}}_{g'}dt = -\left(-t^2e^{-t}+\int 2\underbrace{t}_{f}\underbrace{e^{-t}}_{g'}dt\right) = -\left(-t^2e^{-t}+2\left(-te^{-t}+\int e^{-t}dt\right)\right)$$

$$f'=2t,\ g=-e^{-t} \qquad f'=1;\ g'=-e^{-t} \qquad =(t^2+2t+2)e^{-t}$$

$$x=x_0=\varphi_0=e^tc+(t^2+2t+2)e^{-t} \Leftrightarrow x(t)=e^tc+(t^2+2t+2)e^{-t}$$

$$x(1)=2 \Rightarrow eC+5e^{-1}=2 \Leftrightarrow C=\frac{2}{e}-\frac{5}{e^2}=\frac{2e-5}{e^2} \Rightarrow \boxed{x_{PC}=e^t\left(\frac{2e-5}{e^2}\right)+(t^2+2t+2)e^{-t}}$$

e) $x'\underbrace{-\frac{1}{t}}_{A(t)}x=\underbrace{-1}_{B(t)}$

<u>Etapa 1</u>: $\frac{dx}{dt}=\frac{1}{t}x \Leftrightarrow \int\frac{1}{x}dx=\int\frac{1}{t}dt \Leftrightarrow \boxed{x_0=t\cdot c}$

<u>Etapa 2</u> $\varphi_0=tc(t)$; $\left(tc(t)\right)'-\frac{1}{t}\cdot tc(t)=-1 \Leftrightarrow tc'(t)=-1 \Leftrightarrow c'(t)=\frac{-1}{t}$

$$c(t)=\int\frac{-1}{t}dt=-\ln|t|+C_1 \Rightarrow \boxed{\varphi_0=t(-\ln t+C_1)}$$

$$x=x_0+\varphi_0=tc+t(-\ln t+C_1) \Leftrightarrow x(t)=t(-\ln t+c)$$

$$x(1)=1(-0+c) \Rightarrow \boxed{C=4} \Rightarrow x_{PC}=t(-\ln t+4)$$

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3. Să se rezolve următoarele ecuații reductibile la ec. de tip omogen:

a) $(t^2 - tx + x^2)\,dt + (tx - 2t^2)\,dx = 0$

$(t^2 - tx + x^2)\,dt = -(tx - 2t^2)\,dx \Leftrightarrow t^2 - tx + x^2 = -(tx - 2t^2)\frac{dx}{dt}$

$x' = \frac{t^2 - tx + x^2}{-(tx - 2t^2)} = \frac{1 - \frac{x}{t} + \left(\frac{x}{t}\right)^2}{-\frac{x}{t} + 2} = \frac{1 - \frac{x}{t} + \left(\frac{x}{t}\right)^2}{2 - \frac{x}{t}} = f\left(\frac{x}{t}\right)$ (ec. de tip omogen)

$\boxed{u = \frac{x}{t}} \Rightarrow x = u \cdot t \Leftrightarrow x' = u't + u \Leftrightarrow u't + u = \frac{1 - u + u^2}{2 - u}$

$u't = \frac{1 - u + u^2 - u(2-u)}{2 - u} = \frac{1 - u + u^2 - 2u + u^2}{2 - u} = \frac{1 - 3u + 2u^2}{2 - u}$

$\frac{du}{dt}t = \frac{1 - 3u + 2u^2}{2 - u} \Leftrightarrow \frac{2 - u}{1 - 3u + 2u^2}\,du = \frac{1}{t}\,dt \Leftrightarrow \int \frac{2 - u}{1 - 3u + 2u^2}\,du = \int \frac{1}{t}\,dt$

- $\frac{2 - u}{1 - 3u + 2u^2} = \frac{2 - u}{(1-u)(1-2u)} = \frac{A}{1-u} + \frac{B}{1-2u} \quad \Big| \cdot 1 - u \quad \Big| \cdot 1 - 2u$

$\frac{2 - u}{1 - 2u} = A + \frac{(1-u)B}{1-2u} \quad \Big| u = 1 \Rightarrow A = \frac{1}{-1} = -1$

$\frac{2 - u}{1 - u} = \frac{(1-2u)A}{1-u} + B \quad \Big| u = \frac{1}{2} \Rightarrow B = \frac{2 - \frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3$

$\int \frac{2 - u}{1 - 3u + 2u^2}\,du = \int\left(\frac{-1}{1-u} + \frac{3}{1-2u}\right)du = -\int \frac{1}{1-u}\,du + 3\int \frac{1}{1-2u}\,du$

$= \ln|1-u| - \frac{3}{2}\ln|1-2u| = \ln\left|\frac{1-u}{(1-2u)^{\frac{3}{2}}}\right|$

$\int \frac{1}{t}\,dt = \ln|t| + C \Rightarrow \ln|t| + C = \ln\left|\frac{1-u}{(1-2u)^{\frac{3}{2}}}\right| \Leftrightarrow |t|C = \left|\frac{1-u}{(1-2u)^{\frac{3}{2}}}\right|$

$\Leftrightarrow tC = \frac{1 - \frac{x}{t}}{\left(1 - 2\frac{x}{t}\right)^{\frac{3}{2}}}$ (sol în formă implicită)

c) $x' = \frac{tx + x^2}{t^2} \Leftrightarrow x' = \frac{x}{t} + \frac{x^2}{t^2} = \left(\frac{x}{t}\right)\left(1 + \frac{x}{t}\right) = f\left(\frac{x}{t}\right)$ (ec. de tip omogen)

$\boxed{u = \frac{x}{t}} \Rightarrow x = u \cdot t \Leftrightarrow x' = u't + u \Leftrightarrow u't + u = u + u^2 \Leftrightarrow u't = u^2$

$\frac{du}{dt}t = \frac{u^2}{1} \Leftrightarrow \frac{1}{u^2}\,du = \frac{1}{t}\,dt \Leftrightarrow \int \frac{1}{u^2}\,du = \int \frac{1}{t}\,dt \Leftrightarrow \frac{-1}{u} = \ln|t| + C \Leftrightarrow$

$\Leftrightarrow C \cdot |t| = e^{\frac{-1}{u}} \Rightarrow \boxed{Ct = e^{\frac{-t}{x}}}$

d) $(t+2x)\,dt - t\,dx = 0$

$$(t+2x)\,dt = t\,dx \Leftrightarrow t+2x = t\,\frac{dx}{dt} \Leftrightarrow t+2x = t\,x' \Leftrightarrow x' = 1+2\,\frac{x}{t} = f\left(\frac{x}{t}\right)$$

(ec. de tip omogen)

$$\boxed{u = \frac{x}{t}} \Leftrightarrow x = u\cdot t \Rightarrow x' = u't + u \Leftrightarrow u't + u = 1+2u \Leftrightarrow u't = 1+u$$

$$\frac{du}{dt}\,t = 1+u \Leftrightarrow \frac{1}{1+u}\,du = \frac{1}{t}\,dt \Leftrightarrow \int \frac{1}{1+u}\,du = \int \frac{1}{t}\,dt$$

$$\ln|1+u| = \ln|t| + c \Leftrightarrow 1+u = C\cdot t \Leftrightarrow 1+\frac{x}{t} = C\cdot t \Leftrightarrow \boxed{x(t) = Ct^2 - t}$$

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4. Să se rezolve urm. ecuații reductibile la ec. de tip omogen:

b) $2(t+4x-6)\,dt = (7t+x-15)\,dx$

$$\frac{dx}{dt} = \frac{2(t+4x-6)}{7t+x-15} = f\left(\frac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2}\right)$$

$$D = \begin{vmatrix} 2 & 8 \\ 7 & 1 \end{vmatrix} = 2-56 = -54 \neq 0 \Rightarrow \begin{cases} 2t+8x=12 \\ 7x+x=15 \mid \cdot 8 \end{cases} \Leftrightarrow \begin{cases} 2t+8x=12 \\ 56t+8x=120 \mid \cdot(-1) \end{cases}$$

$$54t = 108 \Rightarrow \boxed{\begin{cases} t_0 = 2 \\ x_0 = 1 \end{cases}}$$

$$\boxed{\begin{matrix} x = u+1 \\ t = z+2 \end{matrix}}$$

$$\frac{du}{dz} = \frac{2(z+2)+8(u+1)-12}{7(z+2)+(u+1)-15} = \frac{2z+4+8u+8-12}{7z+14+u+1-15} = \frac{2z+8u}{7z+u} = \frac{2+8\frac{u}{z}}{7+\frac{u}{z}} = g\left(\frac{u}{z}\right)$$

$$\boxed{v = \frac{u}{z}} \Leftrightarrow u = vz \Leftrightarrow u' = v'z + v$$

$$v'z + v = \frac{2+8v}{7+v} \Leftrightarrow v'z = \frac{2+8v}{7+v} - v^{(7+v)} = \frac{2+8v-7v-v^2}{7+v} = \frac{-v^2+v+2}{7+v}$$

$$\frac{dv}{dz} z = \frac{-v^2+v+2}{7+v} \Leftrightarrow \frac{7+v}{-v^2+v+2}\,dv = \frac{1}{z}\,dz \Leftrightarrow \int \frac{7+v}{-v^2+v+2}\,dv = \int \frac{1}{z}\,dz$$

- $$\frac{7+v}{-v^2+v+2} = \frac{7+v}{(2-v)(v+1)} = \frac{A}{2-v} + \frac{B}{v+1} \quad \Big| \cdot 2-v \quad \Big| \cdot v+1$$

$$\frac{7+v}{v+1} = A + \frac{B(2-v)}{v+1} \quad \Big| v=2 \Rightarrow A = \frac{9}{3} = 3$$

$$\frac{7+v}{2-v} = \frac{A(v+1)}{2-v} + B \quad \Big| v=-1 \Rightarrow B = \frac{6}{3} = 2$$

$$\int \frac{1}{z}\,dz = \ln|z| \Leftrightarrow \ln|z| = -3\ln|2-v| + 2\ln|v+1| + C \Leftrightarrow \ln|z| = \ln\left|\frac{C(v+1)^2}{(2-v)^3}\right|$$

$$z = \frac{C(v+1)^2}{|(2-v)^3|} \Rightarrow t-2 = \frac{C\left(\frac{x-1}{t-2}+1\right)^2}{\left|\left(2-\frac{x-1}{t-2}\right)^3\right|}$$ (sol în formă implicită)

c) $(2t-4x+6)\,dt = -(t+x-3)\,dx$

$$\frac{dx}{dt} = \frac{2t-4x+6}{-t-x+3} = f\left(\frac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2}\right)$$

$$D = \begin{vmatrix} 2 & -4 \\ -1 & -1 \end{vmatrix} = -2+4 = 2 \neq 0 \Rightarrow \begin{cases} 2t-4x=-6 \\ -t-x=-3 \mid \cdot 2 \end{cases} \Leftrightarrow \begin{cases} 2t-4x=-6 \\ -2t-2x=-6 \mid \cdot(+) \end{cases} \Rightarrow \boxed{\begin{cases} x_0 = 2 \\ t_0 = 1 \end{cases}}$$

$$\boxed{\begin{aligned} x &= u+2 \\ t &= z+1 \end{aligned}}$$

$$\frac{du}{dz} = \frac{2(z+1)-4(u+2)+6}{-(z+1)-(u+2)+3} = \frac{2z+2-4u-8+6}{-z-1-u-2+3} = \frac{2z-4u}{-z-u} = \frac{2-4\frac{u}{z}}{-1-\frac{u}{z}} = g\left(\frac{u}{z}\right)$$

$$\boxed{v = \frac{u}{z}} \Leftrightarrow u = vz \Rightarrow u' = v'z + v$$

$$v'z + v = \frac{2-4v}{-1-v} \Leftrightarrow v'z = \frac{2-4v}{-1-v} - v = \frac{2-4v+v+v^2}{-1-v} = \frac{v^2-3v+2}{-1-v}$$

$$\frac{dv}{dz}z = \frac{v^2-3v+2}{-1-v} \Leftrightarrow \frac{-1-v}{v^2-3v+2}dv = \frac{1}{z}dz \Leftrightarrow \int \frac{-1-v}{v^2-3v+2}dv = \int \frac{1}{z}dz$$

$$\cdot \frac{-1-v}{v^2-3v+2} = \frac{-1-v}{(v-1)(v-2)} = \frac{A}{v-1} + \frac{B}{v-2} \quad \Big| \cdot (v-1) \quad \Big| (v-2)$$

$$\frac{-1-v}{v-2} = A + \frac{B(v-1)}{v-2} \quad \Big| v=1 \Rightarrow A = \frac{-2}{-1} = 2$$

$$\frac{-1-v}{v-1} = \frac{A(v-2)}{v-1} + B \quad \Big| v=2 \Rightarrow B = \frac{-3}{1} = -3$$

$$\int \frac{1}{z}dz = \ln|z| \Leftrightarrow \ln|z| = \ln\left|\frac{c(v-1)^2}{(v-2)^3}\right| \Rightarrow |z| = \left|\frac{c(v-1)^2}{(v-2)^3}\right|$$

$$t-1 = \frac{c\left(\frac{x-2}{t-1}-1\right)^2}{\left(\frac{x-2}{t-1}-2\right)^3}$$ (sol în formă implicită)

d) $(3t+3x-1)\,dt = -(t+x+1)\,dx$

$$\frac{dx}{dt} = \frac{3t+3x-1}{-t-x-1} = f\left(\frac{a_1t+b_1x+c_1}{a_2t+b_2x+c_2}\right)$$

$$D = \begin{vmatrix} 3 & 3 \\ -1 & -1 \end{vmatrix} = -3+3 = 0 \Leftrightarrow \frac{dx}{dt} = \frac{3(t+x)-1}{-(t+x)-1}$$

$$\boxed{u = t+x} \Leftrightarrow u' = 1+x' \Rightarrow \frac{du}{dt} = 1 + \frac{3u-1}{-u-1} = \frac{2u-2}{-u-1} \Rightarrow \frac{-u-1}{2u-2}du = dt$$

$$\int \frac{-u-1}{2u-2}du = \int dt \Leftrightarrow t + C = \frac{-1}{2}\left(u + \frac{1}{2}\ln|u-1|\right)$$

$$\int \frac{-u-1}{2u-2}du = \frac{-1}{2}\int \frac{u+1}{u-1}du = \frac{-1}{2}\left(\int 1\,du + \int \frac{2}{u-1}du\right) = \frac{-1}{2}\left(u + \frac{1}{2}\ln|u-1|\right)$$

$$t + C = \frac{-1}{2}\left(t+x+\frac{1}{2}\ln|t+x-1|\right)$$ (sol. sub formă implicită)

e) $(t-2x+1)\,dt = -(2t-4x+3)\,dx$

$$\frac{dx}{dt} = \frac{t-2x+1}{-2(t-2x)+3} = f\begin{pmatrix} a_1 t + b_1 x + c_1 \\ a_2 t + b_2 x + c_2 \end{pmatrix}$$

$$D = \begin{vmatrix} +1 & -2 \\ -2 & 4 \end{vmatrix} = 4-4=0 \Leftrightarrow \boxed{u = t-2x} \Leftrightarrow u' = 1-2x'$$

$$\frac{dx}{dt} = 1 - 2\frac{u+1}{-2u+3} = \frac{-2u+3-2u-2}{-2u+3} = \frac{1-4u}{-2u+3} \Leftrightarrow \frac{-2u+3}{1-4u}\,du = dt \Leftrightarrow \int \frac{-2u+3}{1-4u}\,du = \int dt$$

$$\int \frac{2u-3}{4u-1}\,du = \frac{1}{2}\int \frac{4u-6}{4u-1} = \frac{1}{2}\left(\int \frac{4u-1}{4u-1}\,du - \int \frac{5}{4u-1}\,du\right) = \frac{1}{2}\left(u - \frac{5\ln|4u-1|}{4}\right) =$$

$$= \frac{1}{2}u - \frac{5}{8}\ln|4u-1|$$

$$\int dt = t + C \Leftrightarrow t + C = \frac{1}{2}u - \frac{5}{8}\ln|4u-1| = \frac{1}{2}(t-2x) - \frac{5}{8}\ln|4t-8x-1|$$

(sol sub formă implicită)

f) $(t-x-1) + x'(x-t+2) = 0$

$$\frac{dx}{dt} = \frac{t-x-1}{-x+t-2} = f\begin{pmatrix} a_1 t + b_1 x + c_1 \\ a_2 t + b_2 x + c_2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 0$$

$$\frac{dx}{dt} = \frac{(t-x)-1}{(t-x)-2}; \quad \boxed{u = t-x} \Leftrightarrow u' = 1-x'$$

$$\frac{du}{dt} = 1 - \frac{u-1}{u-2} = \frac{-1}{u-2} \Rightarrow (-u+2)\,du = dt$$

$$\int (-u+2)\,du = \int dt$$

$$\int (-u+2)\,du = -\int u\,du + 2\int du = -\frac{u^2}{2} + 2u = \frac{4u-u^2}{2}$$

$$\int dt = t + C$$

$$t + C = \frac{4u-u^2}{2} \Rightarrow \boxed{t + C = \frac{4(t-x)-(t-x)^2}{2}}$$ (sol sub formă implicită)