Ecuații diferențiale cu derivate parțiale

Laborator 04

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1. Să se rezolve următoarele ecuații de tip Bernoulli.

$\boxed{a)}\ x' + \frac{t}{6}x = \frac{1}{3}tx^{-2}$

$\boxed{b)}\ \begin{cases} 3tx^2x' + x^3 = 2t \\ x(1) = 2 \end{cases}$

c) $\begin{cases} x' = 2tx + t\sqrt[3]{x} \\ x(0) = 4 \end{cases}$

$\boxed{d)}\ x' + 2tx = 2t^3x^3$

Rezolvare

c) $\begin{cases} x' = 2tx + t\sqrt[3]{x} \\ x(0) = 4 \end{cases}$

$x' = \underbrace{2tx}_{A(t)} + \underbrace{t}_{B(t)} x^{\frac{1}{3}} \;\Big|\, : x^{\frac{1}{3}}; \; \alpha = \frac{1}{3}$

$\frac{x'}{x^{1/3}} = 2t\,\frac{x}{x^{1/3}} + t \Leftrightarrow x'\cdot x^{\frac{-1}{3}} = 2t\,x^{1-\frac{1}{3}} + t$

$x'\cdot x^{\frac{-1}{3}} = 2t\,x^{\frac{2}{3}} + t \;\Big|\cdot \frac{2}{3}$

$\boxed{u = x^{\frac{2}{3}}} \Rightarrow u' = \frac{2}{3}x^{\frac{2}{3}-1}\cdot x' = \frac{2}{3}x^{\frac{-1}{3}}\cdot x'$

$\frac{2}{3}x'\cdot x^{\frac{-1}{3}} = \frac{4}{3}t\,x^{\frac{2}{3}} + \frac{2}{3}t$

$u' = \frac{4}{3}tu + \frac{2}{3}t$

Etapa 1

$u' = \frac{4}{3}tu \Leftrightarrow \frac{du}{dt} = \frac{4}{3}tu \Leftrightarrow \frac{du}{u} = \frac{4}{3}t\,dt \Leftrightarrow \int \frac{1}{u}\,du = \frac{4}{3}\int t\,dt$

$\ln|u| = \frac{4}{3}\cdot\frac{t^2}{2} + C = \frac{2}{3}t^2 + C$

$|u| = e^{\frac{2}{3}t^2 + C} \Rightarrow \boxed{u_0 = C\cdot e^{\frac{2}{3}t^2}}$

Etapa 2

$\varphi_0 = C(t)\cdot e^{\frac{2}{3}t^2}$

$\left(C(t)\cdot e^{\frac{2}{3}t^2}\right)' = \frac{4}{3}t\cdot C(t)\cdot e^{\frac{2}{3}t^2} + \frac{2}{3}t$

$C'(t)\cdot e^{\frac{2}{3}t^2} + C(t)\cdot e^{\frac{2}{3}t^2}\cdot\frac{2}{3}\cdot 2t = \frac{4}{3}t\cdot C(k)\cdot e^{\frac{2}{3}t^2} + \frac{2}{3}t$

$$c'(t)\, e^{\frac{2}{3}t^2} = \frac{2}{3}t$$

$$c'(t) = \frac{2}{3}t \cdot e^{\frac{-2}{3}t^2} \Rightarrow c(t) = -\frac{1}{2}\int -2\,\frac{2}{3}\, t\, e^{\frac{-2}{3}t^2}\, dt$$

$$\mu = \frac{-2}{3}t^2 \Leftrightarrow \mu' = \frac{-2}{3}t^2$$

$$\boxed{c(t) = -\frac{1}{2}e^{-\frac{2}{3}t^2} + C_1}$$

$$P_0 = \left(\frac{-1}{2}e^{\frac{-2}{3}t^2} + C_1\right) \cdot e^{\frac{2}{3}t^2}$$

$$P_0 = \frac{-1}{2} + C_1 e^{\frac{2}{3}t^2}$$

$$\mu = \mu_0 + P_0 = C e^{\frac{2}{3}t^2} - \frac{1}{2} + C_1 e^{\frac{2}{3}t^2}$$

$$\mu = C e^{\frac{2}{3}t^2} - \frac{1}{2}$$

$$\mu = x^{\frac{2}{3}} \Rightarrow \mu^3 = x^2 \Rightarrow x = \pm\sqrt{\mu^3}$$

$$x = \pm\sqrt{\left(Ce^{\frac{2}{3}t^2} - \frac{1}{2}\right)^3}, \quad Ce^{\frac{2}{3}t^2} - \frac{1}{2} \geq 0$$

$$x(0) = \sqrt{\left(C - \frac{1}{2}\right)^3} = 4 \Leftrightarrow \left(C - \frac{1}{2}\right)^3 = 16 \Leftrightarrow C - \frac{1}{2} = \sqrt[3]{16} \Rightarrow C = \frac{1}{2} + \sqrt[3]{16}$$

$$x_{PC} = \sqrt{\left(\left(\frac{1}{2} + \sqrt[3]{16}\right)e^{\frac{2}{3}t^2} - \frac{1}{2}\right)^3}$$

2. Să se rezolve următoarele ec. de tip. Ricati:

$\boxed{a)}$ $x'+x^2-2x\sin t+\sin^2 t-\cos t=0, \quad \varphi_0(t)=\sin t$

$\boxed{b)}$ $x'=tx^2-2t^2x+t^3+1, \quad \varphi_0(t)=t$

c) $\begin{cases} x'=-x^2\sin t+\dfrac{2\sin t}{\cos^2 t}, & \varphi_0(t)=\dfrac{1}{\cos t} \\ x(0)=2 \end{cases}$

Rezolvare

c) $\begin{cases} x'=-x^2\sin t+\dfrac{2\sin t}{\cos^2 t}, & \varphi_0(t)=\dfrac{1}{\cos t} \\ x(0)=2 \end{cases}$

$$x=y+\varphi_0 \Rightarrow \boxed{x=y+\frac{1}{\cos t}}$$

$$x'=y'+\cancel{\frac{\sin t}{\cos^2 t}}=-\left(y+\frac{1}{\cos t}\right)^2\cdot\sin t+\frac{2\sin t}{\cos^2 t}$$

$$=-y^2\sin t-2\frac{\sin t}{\cos t}y-\cancel{\frac{\sin t}{\cos^2 t}}+\cancel{\frac{2\sin t}{\cos^2 t}}$$

$$y'=\underbrace{-2\frac{\sin t}{\cos t}}_{A(t)}y-\underbrace{\sin t}_{B(t)}\cdot y^2 \;\Big|\, : y^2 \;;\; \alpha=2$$

$$\frac{y'}{y^2}=\frac{-2\sin t}{\cos t}\cdot\frac{y}{y^2}-\sin t$$

$$y'\cdot y^{-2}=\frac{-2\sin t}{\cos t}y^{-1}-\sin t \;\Big|\cdot(-1)$$

$$\boxed{u=y^{-1}} \Rightarrow u'=-1\cdot y^{-2}\cdot y'$$

$$-y'y^{-2}=\frac{2\sin t}{\cos t}y^{-1}+\sin t \Leftrightarrow u'=\frac{2\sin t}{\cos t}u+\sin t$$

Etapa 1

$$u'=\frac{2\sin t}{\cos t}u$$

$$\frac{du}{dt}=\frac{2\sin t}{\cos t}u \Leftrightarrow \frac{du}{u}=\frac{2\sin t}{\cos t}dt \Rightarrow \int\frac{1}{u}\,du=-2\int\frac{-\sin t}{\cos t}dt$$

$u=\cos t$

$u'=-\sin t$

$$\ln|u| = -2\ln|\cos t| + C = -\ln\cos^2 t + \ln C$$

$$u_0 = \frac{C}{\cos^2 t}$$

Etapa 2

$$y_0 = \frac{C(t)}{\cos^2 t}$$

$$\left(\frac{C(t)}{\cos^2 t}\right)' = \frac{2\sin t}{\cos t}\cdot\frac{C(t)}{\cos^2 t} + \sin t$$

$$\frac{C'(t)\cdot\cos^2 t + C(t)\cdot 2\cos t\cdot\sin t}{\cos^4 t} = \frac{2\sin t\cdot C(t)}{\cos^3 t} + \sin t$$

$$\frac{C'(t)}{\cos^2 t} + \frac{2C(t)\sin t}{\cos^3 t} = \frac{2\sin t\cdot C(t)}{\cos^3 t} + \sin t$$

$$\frac{C'(t)}{\cos^2 t} = \sin t \Rightarrow C'(t) = \sin t\cdot\cos^3 t$$

$$C(t) = \int \sin t\cos^2 t\, dt = -\frac{\cos^3 t}{3} + C_1$$

$u = \cos t$

$u' = -\sin t$

$$y_0 = \frac{\frac{-\cos^3 t}{3} + C_1}{\cos^2 t}$$

$$y_c = z + \frac{1}{\cos t} = \frac{1}{\frac{c}{\cos^2 t} - \frac{\cos t}{3}} + \frac{1}{\cos t}$$

$$u = u_0 + y_0 = \frac{c}{\cos^2 T} - \frac{\cos t}{3} + \frac{C_1}{\cos^2 t} = \frac{c}{\cos^2 t} - \frac{\cos t}{3}$$

$$u = z^{-1} = \frac{1}{z} \Rightarrow z = \frac{1}{u} = \frac{1}{\frac{c}{\cos^2 t} - \frac{\cos t}{3}}$$

$$x = z + \frac{1}{\cos t} \Leftrightarrow x = \frac{1}{\frac{c}{\cos^2 t} - \frac{\cos t}{3}} + \frac{1}{\cos t}$$

$$x(0) = \frac{1}{c - \frac{1}{3}} + 1 = 2 \Rightarrow \frac{1}{c - \frac{1}{3}} = 1 \Rightarrow C - \frac{1}{3} = 1 \Rightarrow C = \frac{1}{3} + 1 = \frac{4}{3}$$

$$x_{PC} = \frac{1}{\frac{4}{3\cos^2 t} - \frac{\cos t}{3}} + \frac{1}{\cos t}$$

3. Să se rezolve urm. ec. cu diferențiale exacte:

$\boxed{a)}$ $\frac{t}{x^2}\,dt + \frac{x^2-t^2}{x^3}\,dx = 0$

b) $\begin{cases} 2tx\,dt + (t^2+x^2)\,dx = 0 \\ x(1) = 3 \end{cases}$

$\boxed{c)}$ $(t^2+x^2+2t)\,dt + 2tx\,dx = 0$

$\boxed{d)}$ $t\,dt + x\,dx = \frac{-t\,dx - x\,dt}{t^2+x^2}$

Rezolvare

b) $\begin{cases} 2tx\,dt + (t^2+x^2)\,dx = 0 \\ x(1) = 3 \end{cases}$

$$\underbrace{2tx}_{P(t,x)}\,dt + \underbrace{(t^2+x^2)}_{Q(t,x)}\,dx = 0$$

$$\left.\begin{array}{l} \frac{\partial P}{\partial x}(t,x) = 2t \\ \frac{\partial Q}{\partial t}(t,x) = 2t \end{array}\right\} \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial t}$$

$$F(t,x) = C$$

$$\int_0^t P(\tau,0)\cdot d\tau + \int_0^x Q(t,\tau)\,d\tau = \int_0^t 2\cdot\tau\cdot 0\,d\tau + \int_0^x (t^2+\tau^2)\,d\tau =$$

$$= t^2\tau\Big|_0^x + \frac{\tau^3}{3}\Big|_0^x = t^2x + \frac{x^3}{3}$$

$t^2x + \frac{x^3}{3} = C$ (sol în formă implicită)

$1^2\cdot 3 + \frac{3^3}{3} = C \Rightarrow C = 3+9 = 12$

PC: $t^2x + \frac{x^3}{3} = 12$

4. Să se rez. urm. ec. căutând un factor integrant:

a) $(x^2-2tx)\,dt + t^2\,dx = 0$

$\boxed{b)}$ $2tx\,dt = (t^2-x^2)\,dx$

$\boxed{c)}$ $(2tx-t)\,dt + (x^2+x+2t^2)\,dx = 0$

<u>Rezolvare</u>

a) $\underbrace{(x^2-2tx)}_{P(t,x)}\,dt + \underbrace{t^2}_{Q(t,x)}\,dx = 0$

$$\left.\begin{aligned} &\frac{\partial P}{\partial x}(t,x) = 2x-2t \\ &\frac{\partial Q}{\partial t}(t,x) = 2t \end{aligned}\right\} \Rightarrow \frac{\partial P}{\partial x} \neq \frac{\partial Q}{\partial t}$$

Fie $\mu = \mu(x)$

$$\underbrace{\mu(x)\cdot(x^2-2tx)}_{P^*(t,x)}\,dt + \underbrace{\mu(x)\cdot t^2}_{Q^*(t,x)}\,dx = 0$$

$$\frac{\partial P^*}{\partial x}(t,x) = \mu'(x)(x^2-2tx) + \mu(x)(2x-2t)$$

$$\frac{\partial Q^*}{\partial t}(t,x) = \mu(x)\cdot 2t$$

$$\mu'(x)(x^2-2tx) + \mu(x)(2x-2t) = \mu(x)\cdot 2t$$

$$\mu'(x)(x^2-2tx) = \mu(x)(2t-2x+2t)$$

$$\mu'(x)\cdot x(x-2t) = \mu(x)(4t-2x)$$

$$\mu'(x)\cdot \cancel{x(x-2t)} = -2\mu(x)\cancel{(x-2t)}$$

$$\frac{d\mu}{dx}\cdot x = -2\mu \Leftrightarrow \frac{d\mu}{\mu} = \frac{-2}{x}\,dx \Rightarrow \int \frac{1}{\mu} = -2\int \frac{1}{x}\,dx$$

$$\ln|\mu| = -2\ln|x| + C = -\ln x^2 + \ln C \Rightarrow \mu = \frac{C}{x^2} \quad ; \text{ P.P } c=1.$$

$$\underbrace{(x^2-2tx)\cdot\frac{1}{x^2}}_{P^*(t,x)}\,dt + \underbrace{t^2\cdot\frac{1}{x^2}}_{Q^*(t,x)}\,dx = 0$$

$$F(t,x) = C$$

$$F(t,x) = \int_0^t P(\zeta, x_0)\, d\zeta + \int_{x_0}^{x} Q(t,\tau)\, d\tau$$

$$= \int_0^t \frac{x_0^2 - 2\zeta x_0}{x_0^2}\, d\zeta + \int_{x_0}^{x} t^2 \cdot \frac{1}{\tau^2}\, d\tau$$

$$= \int_0^t \left(1 - \frac{2\zeta}{x_0}\right) d\zeta + t^2 \int_{x_0}^{x} \frac{1}{\tau^2}\, d\tau$$

$$= \zeta \Big|_0^t - \frac{\zeta^2}{x_0}\Big|_0^t + t^2 \cdot \left(\frac{-1}{\tau}\right)\Big|_{x_0}^{x}$$

$$= t - \frac{t^2}{x_0} - \frac{t^2}{x} + \frac{t^2}{x_0} + C$$

$$= t - \frac{t^2}{x} = C$$ (sol. ec. în formă implicită)