Curs09 - Rezolvare

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- Exerciţiu 5. a)
- Exerciţiu 5. b)

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5 a) $x'' - 5x' + 6x = e^{2t}(\sin t + t\cos t)$ $\quad \begin{matrix}\alpha = 2\\ \beta = 1\end{matrix}$ $\quad \alpha \pm i\beta = 2 \pm i$

$x'' - 5x' + 6x = 0$

$x = e^{rt},\ x' = re^{rt},\ x'' = r^2 e^{rt}$

$r^2 e^{rt} - 5re^{rt} + 6e^{rt} = 0 \quad |: e^{rt}$

$r^2 - 5r + 6 = 0$

$\Delta = 25 - 24 = 1$

$r_{1,2} = \frac{5 \pm 1}{2} \begin{cases} r_1 = 3 \\ r_2 = 2 \end{cases}$

$e^{3\cdot t},\ e^{2\cdot t} \to$ S.f.s.

$\boxed{x_o = c_1 e^{3t} + c_2 e^{2t}}$

$x_p = e^{2t}\left((\lambda_1 t + \lambda_0)\cos t + (\beta_1 t + \beta_0)\sin t\right)$

$x_p' = 2e^{2t}\left((\lambda_1 t + \lambda_0)\cos t + (\beta_1 t + \beta_0)\sin t\right) + e^{2t}\left(\lambda_1 \cos t + (\lambda_1 t + \lambda_0)\cdot(-\sin t) + \beta_1 \sin t + (\beta_1 t + \beta_0)\cos t\right)$

$= e^{2t}\left[(2\lambda_1 t + 2\lambda_0 + \lambda_1 + \beta_1 t + \beta_0)\cos t + (2\beta_1 t + 2\beta_0 - \lambda_1 t - \lambda_0 + \beta_1)\sin t\right]$

$x_p'' = 2e^{2t}\left[(2\lambda_1 t + 2\lambda_0 + \lambda_1 + \beta_1 t + \beta_0)\cos t + (2\beta_1 t + 2\beta_0 - \lambda_1 t - \lambda_0 + \beta_1)\sin t\right]$

$+ e^{2t}\left[(2\lambda_1 + \beta_1)\cos t + (2\lambda_1 t + 2\lambda_0 + \lambda_1 + \beta_1 t + \beta_0)\cdot(-\sin t) + (2\beta_1 - \lambda_1)\sin t + (2\beta_1 t + 2\beta_0 - \lambda_1 t - \lambda_0 + \beta_1)\cos t\right]$

$x_p'' = e^{2t}\left[(4\lambda_1 t + 4\lambda_0 + 2\lambda_1 + 2\beta_1 t + 2\beta_0 + 2\lambda_1 + \beta_1 + 2\beta_1 t + 2\beta_0 - \lambda_1 t - \lambda_0 + \beta_1)\cos t\right.$

$\left. + (4\beta_1 t + 4\beta_0 - 2\lambda_1 t - 2\lambda_0 + 2\beta_1 - 2\lambda_1 t - 2\lambda_0 - \lambda_1 - \beta_1 t - \beta_0 + 2\beta_1 - \lambda_1)\sin t\right]$

$(3\lambda_1 t + 4\beta_1 t + 3\lambda_0 + 4\lambda_1 + 4\beta_0 + 2\beta_1)\cos t + (3\beta_1 t - 4\lambda_1 t + 3\beta_0 + 4\beta_1 - 4\lambda_0 - 2\lambda_1)\sin t$

$- 5(2\lambda_1 t + \beta_1 t + 2\lambda_0 + \lambda_1 + \beta_0)\cos t - 5(2\beta_1 t - \lambda_1 t + 2\beta_0 - \lambda_0 + \beta_1)\sin t$

$+ 6(\lambda_1 t + \lambda_0)\cos t + 6(\beta_1 t + \beta_0)\sin t = \sin t + t\cos t$

$$\begin{cases} 3\lambda_1 t + 4\beta_1 t + 3\lambda_0 + 4\lambda_1 + 4\beta_0 + 2\beta_1 - 10\lambda_1 t - 5\beta_1 t - 10\lambda_0 - 5\lambda_1 - 5\beta_0 + 6\lambda_1 t + 6\lambda_0 = t \\ 3\beta_1 t - 4\lambda_1 t + 3\beta_0 + 4\beta_1 - 4\lambda_0 - 2\lambda_1 - 10\beta_1 t + 5\lambda_1 t - 10\beta_0 + 5\lambda_0 - 5\beta_1 + 6\beta_1 t + 6\beta_0 = 1 \end{cases}$$

$$\begin{cases} -\lambda_1 - \beta_1 = 1 \\ -\lambda_0 - \lambda_1 - \beta_0 + 2\beta_1 = 0 \\ -\beta_1 + \lambda_1 = 0 \\ \lambda_0 - 2\lambda_1 - \beta_0 - \beta_1 = 1 \end{cases}$$

$$\begin{cases} -\lambda_1 - \beta_1 = 1 \\ \lambda_1 - \beta_1 = 0 \end{cases}$$

$-2\beta_1 = 1 \Rightarrow \boxed{\beta_1 = -\frac{1}{2}} \Rightarrow \boxed{\lambda_1 = \frac{1}{2}}$

$-\lambda_0 - \beta_0 = \lambda_1 - 2\beta_1 = \frac{1}{2} - 2\left(-\frac{1}{2}\right) = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}$

$\lambda_0 - \beta_0 = 1 + 2\lambda_1 + \beta_1 = 1 + 1 - \frac{1}{2} = -\frac{1}{2}$

$$\begin{cases} -\lambda_0 - \beta_0 = \frac{3}{2} \\ \lambda_0 - \beta_0 = -\frac{1}{2} \end{cases}$$

$-2\beta_0 = 1 \Rightarrow \boxed{\beta_0 = -\frac{1}{2}} \Rightarrow \boxed{\lambda_0 = -1}$

$$\boxed{x_p = e^{2t}\left[\left(\tfrac{1}{2}t - 1\right)\cos t + \left(-\tfrac{1}{2}t - \tfrac{1}{2}\right)\sin t\right]}$$

$x(t) = x_o + x_p = C_1 e^{3t} + C_2 e^{2t} + e^{2t}\left[\left(\frac{1}{2}t - 1\right)\cos t + \left(-\frac{1}{2}t - \frac{1}{2}\right)\sin t\right]$

Exerciţiu 5. b)

(5) b) $x'' + 9x = t \sin 3t + \cos 3t$ $\qquad \alpha = 0,\ \beta = 3 \quad \alpha \pm i\beta = \pm 3i$

$x'' + 9x = 0$

$x = e^{rt},\ x' = r e^{rt},\ x'' = r^2 e^{rt}$

$r^2 e^{rt} + 9 e^{rt} = 0 \quad | : e^{rt}$

$r^2 + 9 = 0 \Rightarrow r^2 = -9$

$r_{1,2} = \pm 3i \quad (\alpha \pm i\beta,\ \alpha = 0,\ \beta = 3)$

$e^{0\cdot t} \cos 3\cdot t,\ e^{0\cdot t} \sin 3t$ - s.f.s.

$$\boxed{x_0 = c_1 \cos 3t + c_2 \sin 3t}$$

$x_p = t \cdot e^{0\cdot t} \left((\lambda_1 t + \lambda_0) \cos 3t + (\beta_1 t + \beta_0) \sin 3t \right)$

$x_p = (\lambda_1 t^2 + \lambda_0 t) \cos 3t + (\beta_1 t^2 + \beta_0 t) \sin 3t$

$x_p' = (2\lambda_1 t + \lambda_0) \cos 3t + (\lambda_1 t^2 + \lambda_0 t)(-3) \sin 3t + (2\beta_1 t + \beta_0) \sin 3t$
$\quad + (\beta_1 t^2 + \beta_0 t) \cdot 3 \cos 3t$
$\quad = (2\lambda_1 t + \lambda_0 + 3\beta_1 t^2 + 3\beta_0 t) \cos 3t + (-3\lambda_1 t^2 - 3\lambda_0 t + 2\beta_1 t + \beta_0) \sin 3t$

$x_p'' = (2\lambda_1 + 6\beta_1 t + 3\beta_0) \cos 3t + (2\lambda_1 t + \lambda_0 + 3\beta_1 t^2 + 3\beta_0 t) \cdot (-3) \sin 3t$
$\quad + (-6\lambda_1 t - 3\lambda_0 + 2\beta_1) \sin 3t + (-3\lambda_1 t^2 - 3\lambda_0 t + 2\beta_1 t + \beta_0) \cdot 3 \cos 3t$

$x_p'' = (2\lambda_1 + 6\beta_1 t + 3\beta_0 - 9\lambda_1 t^2 - 9\lambda_0 t + 6\beta_1 t + 3\beta_0) \cos 3t$
$\quad + (-6\lambda_1 t - 3\lambda_0 - 9\beta_1 t^2 - 9\beta_0 t - 6\lambda_1 t - 3\lambda_0 + 2\beta_1) \sin 3t$

$(-9\lambda_1 t^2 + 12\beta_1 t - 9\lambda_0 t + 2\lambda_1 + 6\beta_0) \cos 3t + (-9\beta_1 t^2 - 12\lambda_1 t - 9\beta_0 t - 6\lambda_0 + 2\beta_1) \sin 3t$
$+ (9\lambda_1 t^2 + 9\lambda_0 t) \cos 3t + (9\beta_1 t^2 + 9\beta_0 t) \sin 3t = t \sin 3t + \cos 3t$

$$\begin{cases} -\cancel{9\lambda_1 t^2} + 12\beta_1 t - \cancel{9\lambda_0 t} + 2\lambda_1 + 6\beta_0 + \cancel{9\lambda_1 t^2} + \cancel{9\lambda_0 t} = 1 \\ -\cancel{9\beta_1 t^2} - 12\lambda_1 t - \cancel{9\beta_0 t} - 6\lambda_0 + 2\beta_1 + \cancel{9\beta_1 t^2} + \cancel{9\beta_0 t} = t \end{cases}$$

$$\begin{cases} 12\beta_1 = 0 \\ 2\lambda_1 + 6\beta_0 = 1 \\ -12\lambda_1 = 1 \\ -6\lambda_0 + 2\beta_1 = 0 \end{cases}$$

$\beta_1 = 0 \quad \Rightarrow \lambda_0 = 0.$

$\lambda_1 = -\frac{1}{12}$

$6\beta_0 = 1 - 2\cdot\lambda_1 = 1 - 2\left(\frac{-1}{12}\right) = 1 + \frac{1}{6} = \frac{7}{6}.$

$$\boxed{x_p = t\left(-\frac{1}{12} t \cos 3t + \frac{7}{6} \sin 3t\right)}$$

$x(t) = x_0 + x_p = c_1 \cos 3t + c_2 \sin 3t + t\left(-\frac{1}{12} t \cos 3t + \frac{7}{6} \sin 3t\right)$