Ecuații diferențiale și cu derivate parțiale
Laborator 03
27.10.2020

1. Să se rezolve următoarele ecuații liniare scalare:

$\boxed{a)}\begin{cases} \frac{dx}{dt} = (t^2+1)\cdot x, & x, t \in \mathbb{R}_+ \\ x(0) = 2 \end{cases}$

$\boxed{b)}\begin{cases} \frac{dx}{dt} = \sqrt{t+1}\cdot x, & t \geq -1 \\ x(0) = 2 & x \in \mathbb{R}_+ \end{cases}$

c) $\begin{cases} \frac{dx}{dt} = t^2 \cdot e^t \cdot x, & t \in \mathbb{R} \\ x(0) = 3 & x \in \mathbb{R}_+ \end{cases}$

d) $\begin{cases} \frac{dx}{dt} = \frac{t^2+1}{t-1}\cdot x, & t \in \mathbb{R}\setminus\{1\} \\ x(0) = 1 & x \in \mathbb{R}_+ \end{cases}$

Rezolvare

c) $\frac{dx}{dt} = \underbrace{t^2 e^t x}_{A(t)}$ (ec. liniar scalară)

$\frac{dx}{x} = t^2 e^t dt \Leftrightarrow \int \frac{1}{x} dx = \int t^2 e^t dt \Leftrightarrow \ln x = t^2 e^t - 2te^t + 2e^t + C$

$\int \underbrace{t^2}_{f} \underbrace{e^t}_{g'} dt = t^2 e^t - \int 2 \underbrace{t}_{f} \underbrace{e^t}_{g'} dt = t^2 e^t - 2\left(te^t - \int e^t dt\right) = t^2 e^t - 2te^t + 2e^t + C$

$f' = 2t$ $\quad$ $f' = 1$

$g = e^t$ $\quad$ $g = e^t$

$x = e^{(t^2-2t+2)\cdot e^t + C} \Rightarrow x(t) = C \cdot e^{(t^2-2t+2)\cdot e^t}$

$x(0) = C \cdot e^{(0-0+2)\cdot e^0} = Ce^2 = 3 \Rightarrow \boxed{C = \frac{3}{e^2}} \Rightarrow x_{PC} = \frac{3}{e^2} \cdot e^{(t^2-2t+2)e^t}$

d) $\frac{dx}{dt} = \underbrace{\frac{t^2+1}{t-1}}_{A(t)} x$ (ec. liniar scalară)

$\frac{dx}{x} = \frac{t^2+1}{t-1} dt \Leftrightarrow \int \frac{1}{x} dx = \int \frac{t^2+1}{t-1} dt \Leftrightarrow \ln x = \frac{t^2}{2} + t + 2\ln|t-1| + C$

$\int \frac{t^2+1}{t-1} dt = \int \left(t+1 + \frac{2}{t-1}\right) dt = \frac{t^2}{2} + t + 2\ln|t-1| + C$

$\boxed{\frac{f}{g} = c + \frac{r}{g}}$

$$\begin{array}{r|l} t^2+1 & t-1 = g \\ -t^2+t & t+1 = c \\ \hline / \ t+1 & \\ -t+1 & \\ \hline / \ 2 = r & \end{array}$$

$\frac{t^2+1}{t-1} = t+1 + \frac{2}{t-1}$

$x = e^{\left(\frac{t^2}{2}+t+\ln|t+1|^2\right)+C} = e^{\frac{t^2}{2}+t} \cdot e^{\ln(t-1)^2} \cdot e^C = C \cdot (t-1)^2 \cdot e^{\frac{t^2}{2}+t}$

$x(0) = C \cdot 1 \cdot e^0 \Leftrightarrow \boxed{C = 1} \Rightarrow x_{PC} = (t-1)^2 \cdot e^{\frac{t^2}{2}+t}$

2. Să se rezolve următoarele ecuații diferențiale afine:

$\boxed{a)}$ $x' + \frac{1-2t}{t^2} \cdot x = 1$

b) $\begin{cases} t \cdot x' + x = t \sin t, & t > 0 \\ x(\pi) = 2 \end{cases}$

c) $\begin{cases} x' + 2tx + t - e^{-t^2} = 0 \\ x(0) = 1 \end{cases}$

$\boxed{d)}$ $\begin{cases} \frac{dx}{dt} = x - t^2 \\ x(1) = 2 \end{cases}$

$\boxed{e)}$ $\begin{cases} x'(t) = \frac{1}{t} \cdot x - 1 \\ x(1) = 4. \end{cases}$

Rezolvare

b) $t x' + x = t \sin t \; |:t$

$x' + \underbrace{\frac{1}{t}}_{A(t)} x = \underbrace{\sin t}_{B(t)}$ (ec. diferențială afină)

Etapa 1

$x' + \frac{1}{t} x = 0 \Leftrightarrow \frac{dx}{dt} = \frac{-1}{t} x \Leftrightarrow \frac{dx}{x} = \frac{-1}{t} dt \Leftrightarrow \int \frac{1}{x} dx = -\int \frac{1}{t} dt$

$\ln |x| = -\ln t + \mathcal{C} = -\ln t + \ln C = \ln \frac{C}{t} \Rightarrow \boxed{x_0 = \frac{C}{t}}$

Etapa 2

$\varphi_0 = \frac{C(t)}{t}$

$\left(\frac{C(t)}{t}\right)' + \frac{1}{t} \cdot \frac{C(t)}{t} = \sin t \Leftrightarrow \frac{C'(t) \cdot t - \cancel{C(t)}}{t^2} + \frac{\cancel{C(t)}}{t^2} = \sin t$

$\frac{C'(t)}{t} = \sin t \Leftrightarrow C'(t) = t \sin t \Leftrightarrow C(t) = \int \underbrace{t}_{f} \underbrace{\sin t}_{g'} dt = -t\cos t + \sin t + C_1$

$f' = 1; \; g = -\cos t$

$\Rightarrow \boxed{\varphi_0 = \frac{-t\cos t + \sin t + C_1}{t}}$

$x = x_0 + \varphi_0 = \frac{C}{t} + \frac{-t\cos t + \sin t + C_1}{t}$

$x(t) = \frac{-t\cos t + \sin t + \mathcal{C}}{t}$

$x(\pi) = \frac{\pi + C}{\pi} = 1 + \frac{C}{\pi} = 2 \Leftrightarrow \frac{C}{\pi} = 1 \Rightarrow \boxed{C = \pi} \Rightarrow x_{PC} = \frac{-t\cos t + \sin t + \pi}{t}$

c) $x' + 2tx + t - e^{-t^2} = 0$

$\underbrace{x' + 2tx}_{A(t)} = \underbrace{e^{-t^2} - t}_{B(t)}$ (ec. diferențială afină)

Etapa 1

$x' + 2tx = 0 \Leftrightarrow \frac{dx}{dt} = -2tx \Leftrightarrow \frac{dx}{x} = -2t\,dt \Leftrightarrow \int \frac{1}{x}\,dx = -2\int t\,dt$

$\ln|x| = -t^2 + \mathcal{C} \Leftrightarrow |x| = e^{-t^2+\mathcal{C}} = e^{-t^2+\mathcal{C}} \cdot e^{\mathcal{C}} = c\cdot e^{-t^2} \Rightarrow \boxed{x_0 = c\cdot e^{-t^2}}$

Etapa 2

$y_0 = C(t)\cdot e^{-t^2}$

$(C(t)\cdot e^{-t^2})' + 2t\,C(t)\cdot e^{-t^2} = e^{-t^2} - t$

$C'(t)\cdot e^{-t^2} + \cancel{C(t)\cdot e^{-t^2}\cdot(-2t)} + \cancel{C(t)\cdot e^{-t^2}\cdot 2t} = e^{-t^2} - t$

$C'(t)\cdot e^{-t^2} = e^{-t^2} - t \Leftrightarrow C'(t) = 1 - te^{t^2}$

$C(t) = \int (1 - te^{t^2})\,dt = \int 1\,dt - \frac{1}{2}\int 2te^{t^2}\,dt = t - \frac{1}{2}\int u'\cdot e^{u}\,dt$

$u = t^2$

$u' = 2t$

$= t - \frac{1}{2}e^{t^2} + C_1$

$\Rightarrow y_0 = (t - \frac{1}{2}e^{t^2} + C_1)\cdot e^{-t^2} = \boxed{te^{-t^2} - \frac{1}{2} + C_1 e^{-t^2}}$

$x = x_0 + y_0 = c\cdot e^{-t^2} + te^{-t^2} - \frac{1}{2} + C_1 e^{-t^2}$

$x(t) = te^{-t^2} - \frac{1}{2} + \mathcal{C}\cdot e^{-t^2}$

$x(0) = 1 \Leftrightarrow x(0) = \frac{-1}{2} + C \Rightarrow \boxed{C = 1 + \frac{1}{2} = \frac{3}{2}} \Rightarrow x_{PC} = te^{-t^2} - \frac{1}{2} + \frac{3}{2}e^{-t^2}$

3. Să se rezolve următoarele ecuații reductibile la ecuații de tip omogen:

$\boxed{a)}$ $(t^2 - t\cdot x + x^2)\,dt + (tx - 2t^2)\,dx = 0$

b) $x' = \frac{2tx}{3t^2 - x^2}$

$\boxed{c)}$ $x' = \frac{tx + x^2}{t^2}$

$\boxed{d)}$ $(t + 2x)\,dt - t\,dx = 0$

e) $txx' - x^2 + 3t^2 = 0$

Rezolvare

b) $x' = \frac{2tx}{3t^2 - x^2} \;\backslash t^2 \Leftrightarrow x' = \frac{\frac{2tx}{t^2}}{\frac{3t^2}{t^2} - \frac{x^2}{t^2}} = \frac{2\cdot\frac{x}{t}}{3 - (\frac{x}{t})^2} = f(\frac{x}{t})$ (ec. de tip omogen)

$u = \frac{x}{t} \Rightarrow x = u\cdot t \Rightarrow x' = u't + u \Leftrightarrow u't + u = \frac{2u}{3 - u^2}$

$u't = \frac{2u}{3-u^2} - u = \frac{2u - 3u + u^3}{3 - u^2} = \frac{u^3 - u}{3 - u^2} \Leftrightarrow \frac{du}{dt}\,t = \frac{u^3 - u}{3 - u^2} \Leftrightarrow \frac{3 - u^2}{u^3 - u}\,du = \frac{1}{t}\,dt$

$\int \frac{3 - u^2}{u^3 - u}\,du = \int \frac{1}{t}\,dt$

$$\int \frac{3-u^2}{u^3-u}du = \int \frac{3-u^2}{u(u^2-1)}du = \int \frac{3-u^2}{u(u+1)(u-1)}du$$

- $\frac{3-u^2}{u(u-1)(u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1} \quad |\cdot u \quad |\cdot u-1 \quad |\cdot u+1$

$$\frac{3-u^2}{(u-1)(u+1)} = A + \frac{Bu}{u-1} + \frac{Cu}{u+1} \quad \Big| u=0 \Rightarrow A=-3.$$

$$\frac{3-u^2}{u(u+1)} = \frac{(u-1)A}{u} + B + \frac{C(u-1)}{u+1} \quad \Big| u=1 \Rightarrow B=\frac{2}{2}=1$$

$$\frac{3-u^2}{u(u-1)} = \frac{(u+1)A}{u} + \frac{(u+1)B}{u-1} + C \quad \Big| u=-1 \Rightarrow C=\frac{2}{2}=1.$$

- $\int \frac{3-u^2}{u^3-u}du = \int \left(\frac{-3}{u} + \frac{1}{u-1} + \frac{1}{u+1}\right)du = -3\ln|u| + \ln|u-1| + \ln|u+1|$

$$\Rightarrow \ln\left|\frac{u^2-1}{u^3}\right| = \ln|t| + \mathcal{C} = \ln|t| + \ln C \Rightarrow \frac{u^2-1}{u^3} = C\cdot|t|$$

$$\boxed{\frac{\left(\frac{x}{t}\right)^2-1}{\left(\frac{x}{t}\right)^3} = C\cdot|t|}$$ (sol. în formă implicită)

e) $txx' - x^2 + 3t^2 = 0 \quad |:tx$

$$x' - \frac{x^2}{tx} + \frac{3t^2}{tx} = 0 \Leftrightarrow x' - \frac{x}{t} + 3\frac{t}{x} = 0 \Leftrightarrow x' = \frac{x}{t} - \frac{3}{\frac{x}{t}} = f\left(\frac{x}{t}\right)$$

$$u = \frac{x}{t} \Rightarrow x = u\cdot t \Rightarrow x' = u't + u.$$

$$u't + \cancel{u} = \cancel{u} - \frac{3}{u} \Leftrightarrow \frac{du}{dt}\cdot t = \frac{-3}{u} \Leftrightarrow \frac{du}{u} = \frac{-3}{t}dt \Leftrightarrow \int \frac{1}{u}du = -3\int\frac{1}{t}dt$$

$$\ln|u| = -3\ln|t| + C = -\ln|t|^3 + \ln C = \ln\frac{C}{|t|^3}$$

$$|u| = \frac{C}{|t|^3} \Leftrightarrow u = \frac{C}{t^3} \Leftrightarrow \frac{x}{t} = \frac{C}{t^3} \Rightarrow \boxed{x(t) = \frac{C}{t^2}}$$

4. Să se rezolve următoarele ecuații reductibile la ecuații de tip omogen:

a) $(t-2x+5)dt + (2t-x+4)dx = 0$

$\boxed{b)}$ $2(t+4x-6)\,dt = (7t+x-15)dx$

$\boxed{c)}$ $(2t-4x+6)dt + (t+x-3)dx = 0$

$\boxed{d)}$ $(3t+3x-1)dt + (t+x+1)dx = 0$

$\boxed{e)}$ $(t-2x+1)dt + (2t-4x+3)dx = 0$

$\boxed{f)}$ $(t-x-1) + x'(x-t+2) = 0$

Rezolvare

a) $(2t-x+4)dx = (-t+2x-5)dt$

$$\frac{dx}{dt} = \frac{-t+2x-5}{2t-x+4} = f\begin{pmatrix} a_1t+b_1x+c_1 \\ a_2t+b_2x+c_2 \end{pmatrix}$$

$$D = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1-4 = -3 \neq 0 \Leftrightarrow \begin{cases} -t+2x=5 \\ 2t-x=-4 \,|\cdot 2 \end{cases} \Leftrightarrow \begin{cases} -t+2x=5 \\ 4t-2x=-8 \;\;\text{"+"} \end{cases}$$

$$3t = -3 \Rightarrow \boxed{\begin{cases} t_0 = -1 \\ x_0 = 2 \end{cases}}$$

$$\boxed{\begin{array}{l} x = u + 2 \\ t = z - 1 \end{array}}$$

$$\frac{du}{dz} = \frac{-(z-1) + 2(u+2) - 5}{2(z-1) - (u+2) + 4} = \frac{-z + 1 + 2u + 4 - 5}{2z - 2 - u - 2 + 4} = \frac{-z + 2u}{2z - u} = \frac{-1 + 2\frac{u}{z}}{2 - \frac{u}{z}} = g\left(\frac{u}{z}\right)$$

$$\boxed{v = \frac{u}{z}} \Rightarrow u = v \cdot z \Rightarrow u' = v' \cdot z + v$$

$$v' \cdot z + v = \frac{-1 + 2v}{2 - v} \Rightarrow v' z = \frac{-1 + 2v}{2 - v} - \frac{2-v}{v} = \frac{-1 + 2v - 2v + v^2}{2 - v} = \frac{v^2 - 1}{2 - v}$$

$$\frac{dv}{dz} = \frac{v^2 - 1}{2 - v} \Leftrightarrow \frac{2 - v}{v^2 - 1} dv = dz \Rightarrow \int \frac{2 - v}{v^2 - 1} dv = \int dz$$

- $\int \frac{2 - v}{v^2 - 1} dv = \int \frac{2}{v^2 - 1} dv - \frac{1}{2} \int \frac{2v}{v^2 - 1} dv = 2 \cdot \frac{1}{2} \ln \left| \frac{v - 1}{v + 1} \right| - \frac{1}{2} \ln |v^2 - 1|$

$$\Rightarrow \ln \left| \frac{v - 1}{v + 1} \right| - \frac{1}{2} \ln |v^2 - 1| = t + C \mid \cdot 2$$

$$2 \ln \left| \frac{v - 1}{v + 1} \right| - \ln |v^2 - 1| = 2t + C \Leftrightarrow \ln \left| \frac{v - 1}{v + 1} \cdot \frac{1}{v^2 - 1} \right| = 2t + C$$

$$\frac{1}{(v + 1)^2} = e^{2t + C} = \mathcal{C} \cdot e^{2t} \Leftrightarrow \frac{1}{\left(\frac{u}{z} + 1\right)^2} = C \cdot e^{2t}$$

$$\boxed{\frac{1}{\left(\frac{x - 2}{t + 1} + 1\right)^2} = C e^{2t}}$$ (sol. în formă implicită)