Temă - Seminar 4 — Petculescu Mihai-Silviu

• Cerințe:

1) Să se verifice dacă T este o aplicație liniară

2) Să se calculeze Ker T, Im T, defectul și rangul

3) Să se stabilească dacă T este aplicație injectivă și/sau surjectivă.

d) $T: \mathbb{R}^2 \to \mathbb{R}^2,\ T(x_1,x_2) = (x_1-x_2,\ 2x_1+x_2)$

1) • $T(\vec{x}+\vec{y}) = T(\vec{x}+\vec{y}),\ \forall \vec{x},\vec{y} \in \mathbb{R}^2$

$\vec{x}+\vec{y} = (x_1+x_2)+(y_1,y_2) = (x_1+y_1,\ x_2+y_2)$

$T(\vec{x}+\vec{y}) = T(\underbrace{x_1+y_1}_{x_1},\ \underbrace{x_2+y_2}_{x_2}) = (x_1+y_1-x_2-y_2,\ 2x_1+2y_1+x_2+y_2)$

$= (x_1-x_2,\ 2x_1+x_2)+(y_1-y_2,\ 2y_1+y_2)$

$= T(\vec{x})+T(\vec{y})$ (1)

• $T(\alpha\vec{x}) = \alpha T(\vec{x}),\ \forall \alpha\in\mathbb{R},\ \vec{x}\in\mathbb{R}^2$

$T(\alpha\vec{x}) = T(\alpha(x_1,x_2)) = T(\alpha x_1,\alpha x_2) = (\alpha x_1-\alpha x_2,\ 2\alpha x_1+\alpha x_2) = \alpha(x_1-x_2,\ 2x_1+x_2)$

$= \alpha T(\vec{x})$ (2)

Din (1) și (2) $\Rightarrow$ T aplicație liniară.

2) $\operatorname{Ker} T = \{\vec{x}\in\mathbb{R}^2 \mid T(\vec{x}) = \vec{0}\}$

$T(\vec{x}) = \vec{0} \Leftrightarrow (x_1-x_2,\ 2x_1+x_2) = (0,0)$

$\Rightarrow \begin{cases} x_1-x_2=0 \\ 2x_1+x_2=0 \quad \text{"+"} \end{cases}$

$3x_1 = 0 \Leftrightarrow x_1 = 0 \Rightarrow x_2 = 0 \Rightarrow \operatorname{Ker}T = \{\vec{0}\} \Rightarrow d = 0.$

$\operatorname{Im} T = \{\vec{y}\in\mathbb{R}^2 \mid \exists\vec{x}\in\mathbb{R}^2 \text{ a.î. } T(\vec{x}) = \vec{y}\}$

$T(\vec{x}) = \vec{y} \Leftrightarrow (x_1-x_2,\ 2x_1+x_2) = (y_1,y_2) \Rightarrow \begin{cases} x_1-x_2 = y_1 \\ 2x_1+x_2 = y_2 \end{cases}$

$\Delta = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1+2 = 3 \neq 0 \Rightarrow$ Sist. are relații $\forall y_1,y_2\in\mathbb{R}$

$\Rightarrow \operatorname{Im}T = \mathbb{R}^2 \Rightarrow r = 2$

3) Cum $\operatorname{Ker}T = \{\vec{0}\} \Rightarrow$ T aplicație injectivă

Cum $\operatorname{Im}T = \mathbb{R}^2$ (= codomeniul) $\Rightarrow$ T apl. surjectivă.

e) $T: \mathbb{R}^3 \to \mathbb{R}^3,\ T(x_1,x_2,x_3) = (x_1+2x_2, -x_1+x_3, 2x_2+x_3)$

1) • $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}),\ \forall \vec{x},\vec{y} \in \mathbb{R}^3$

$$T(\vec{x}+\vec{y}) = T(\underbrace{x_1+y_1}_{x_1}, \underbrace{x_2+y_2}_{x_2}, \underbrace{x_3+y_3}_{x_3})$$

$$= (x_1+y_1+2x_2+2y_2, -x_1-y_1+x_3+y_3, 2x_2+2y_2+x_3+y_3)$$
$$= (x_1+2x_2, -x_1+x_3, 2x_2+x_3) + (y_1+2y_2, -y_1+y_3, 2y_2+y_3)$$
$$= T(\vec{x}) + T(\vec{y}) \quad (1)$$

• $T(\alpha\vec{x}) = T(\alpha(x_1,x_2,x_3)) = T(\alpha x_1, \alpha x_2, \alpha x_3)$
$$= (\alpha x_1+2\alpha x_2, -\alpha x_1+\alpha x_3, 2\alpha x_2+\alpha x_3)$$
$$= \alpha(x_1+2x_2, -x_1+x_3, 2x_2+x_3) = \alpha T(\vec{x}) \quad (2)$$

Din (1) și (2) $\Rightarrow T$ aplicație liniară.

2) $\operatorname{Ker} T = \{\vec{x} \in \mathbb{R}^3 \mid T(\vec{x}) = \vec{0}\}$

$T(\vec{x}) = \vec{0} \Leftrightarrow (x_1+2x_2, -x_1+x_3, 2x_2+x_3) = (0,0,0)$

$$\Rightarrow \begin{cases} x_1+2x_2=0 \\ -x_1+x_3=0 \\ 2x_2+x_3=0 \end{cases} \qquad D = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = -2+2=0.$$

Fie x_3 necunoscută secundară, $x_3 = \alpha$

$$\begin{cases} x_1+2x_2=0 \\ -x_1=-x_3 \Leftrightarrow x_1=x_3 \\ 2x_2=-x_3 \Leftrightarrow x_2=-x_3/2 \end{cases} \Rightarrow \operatorname{Ker} T = \{(\alpha, -\alpha/2, \alpha) \mid \alpha \in \mathbb{R}\}$$
$$= \{\alpha(1, -1/2, 1) \mid \alpha \in \mathbb{R}\}$$
$$\Rightarrow d = 1.$$

$\operatorname{Im} T = \{\vec{y} \in \mathbb{R}^3 \mid \exists\, \vec{x} \in \mathbb{R}^3 \text{ a.î. } T(\vec{x}) = \vec{y}\}$

$T(\vec{x}) = \vec{y} \Leftrightarrow (x_1+2x_2, -x_1+x_3, 2x_2+x_3) = (y_1,y_2,y_3)$

$$\begin{cases} x_1+2x_2=y_1 \\ -x_1+x_3=y_2 \\ 2x_2+x_3=y_3 \end{cases} \quad D=0;\ D_p = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2 \neq 0;\ D_c = \begin{vmatrix} 1 & 2 & y_1 \\ -1 & 0 & y_2 \\ 0 & 2 & y_3 \end{vmatrix} = -2y_1-2y_2+2y_3 = -2(y_1+y_2-y_3) = 0$$

$\Rightarrow -y_1-y_2+y_3=0 \Rightarrow y_3 = y_1+y_2$; $\operatorname{Im} T = \{(y_1, y_2, y_1+y_2)\}$

$(y_1, y_2, y_1+y_2) = (y_1, 0, y_1) + (0, y_2, y_2) = y_1(1,0,1) + y_2(0,1,1)$

$\Rightarrow n = \dim(\operatorname{Im} T) = 2$

3) Cum $\operatorname{Ker} T \neq \{\vec{0}\} \Rightarrow T$ nu e inj

Cum $\operatorname{Im} T \neq \mathbb{R}^3 \Rightarrow T$ nu e surj

f) $T:\mathbb{R}^3 \to \mathbb{R}^3$, $T(x_1,x_2,x_3) = (x_1, x_2+x_3, x_1+x_2+x_3)$.

1) • $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}), \ \forall \vec{x},\vec{y} \in \mathbb{R}^3$

$$T(\vec{x}+\vec{y}) = T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (x_1+y_1, x_2+y_2+x_3+y_3, x_1+y_1+x_2+y_2+x_3+y_3)$$
$$= (x_1, x_2+x_3, x_1+x_2+x_3) + (y_1, y_2+y_3, y_1+y_2+y_3)$$
$$= T(\vec{x}) + T(\vec{y}) \quad (1)$$

• $T(\alpha\vec{x}) = \alpha T(\vec{x}), \alpha \in \mathbb{R}, \vec{x} \in \mathbb{R}^3$

$$T(\alpha\vec{x}) = T(\alpha(x_1,x_2,x_3)) = T(\alpha x_1, \alpha x_2, \alpha x_3)$$
$$= (\alpha x_1, \alpha x_2 + \alpha x_3, \alpha x_1 + \alpha x_2 + \alpha x_3).$$
$$= \alpha(x_1, x_2+x_3, x_1+x_2+x_3) = \alpha T(\vec{x}) \quad (2)$$

Din (1) și (2) $\Rightarrow T$ aplicație liniară

2) $\operatorname{Ker} T = \{\vec{x} \in \mathbb{R}^3 \mid T(\vec{x}) = \vec{0}\}$

$T(\vec{x}) = \vec{0} \Leftrightarrow (x_1, x_2+x_3, x_1+x_2+x_3) = (0,0,0)$

$$\begin{cases} x_1 = 0 \\ x_2+x_3 = 0 \\ x_1+x_2+x_3 = 0 \end{cases} \Rightarrow x_2+x_3 = 0 \Rightarrow x_2 = -x_3$$

Fie $x_3 = \alpha \in \mathbb{R} \Rightarrow \operatorname{Ker} T = \{(0,-\alpha,\alpha) \mid \alpha \in \mathbb{R}\} = \{\alpha(0,-1,1) \mid \alpha \in \mathbb{R}\}$

$\Leftrightarrow d = \dim(\operatorname{Ker} T) = 1.$

$\operatorname{Im} T = \{\vec{y} \in \mathbb{R}^3 \mid \exists \vec{x} \in \mathbb{R}^3 \text{ a.î. } T(\vec{x}) = \vec{y}\}.$

$T(\vec{x}) = \vec{y} \Leftrightarrow (x_1, x_2+x_3, x_1+x_2+x_3) = (y_1,y_2,y_3)$

$$\begin{cases} x_1 = y_1 \\ x_2+x_3 = y_2 \\ x_1+x_2+x_3 = y_3 \end{cases} ; \quad D = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1-1 = 0. ; \quad D_p = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

$$D_c = \begin{vmatrix} 1 & 0 & y_1 \\ 0 & 1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = y_3 - y_1 - y_2 = 0 \Rightarrow y_3 = y_1 + y_2$$

$\operatorname{Im} T = \{(y_1, y_2, y_1+y_2) \mid y_1, y_2 \in \mathbb{R}\}$

$(y_1, y_2, y_1+y_2) = (y_1, 0, y_1) + (0, y_2, y_2) = y_1(1,0,1) + y_2(0,1,1) \Rightarrow n = \dim(\operatorname{Im} T) = 2.$

3) Cum $\operatorname{Ker} T \neq \{\vec{0}\} \Rightarrow T$ nu e inj.

Cum $\operatorname{Im} T \neq \mathbb{R}^3 \Rightarrow T$ nu e surj.

g) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x_1, x_2, x_3) = (x_2 - x_3, x_1 + x_2)$

1) • $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}), \forall \vec{x}, \vec{y} \in \mathbb{R}^3$

$$T(\vec{x} + \vec{y}) = (x_2 + y_2 - x_3 - y_3, x_1 + y_1 + x_2 + y_2)$$
$$= (x_2 - x_3, x_1 + x_2) + (y_2 - y_3, y_1 + y_2) = T(\vec{x}) + T(\vec{y}) \quad (1)$$

• $T(\alpha\vec{x}) = \alpha T(\vec{x}), \forall \alpha \in \mathbb{R}, \vec{x} \in \mathbb{R}^3$.

$$T(\alpha\vec{x}) = T(\alpha x_1, \alpha x_2, \alpha x_3) = (\alpha x_2 - \alpha x_3, \alpha x_1 + \alpha x_2) = \alpha(x_2 - x_3, x_1 + x_2) = \alpha T(\vec{x}) \quad (2)$$

Din (1) și (2) $\Rightarrow$ T aplicație liniară.

2) $\ker T = \{\vec{x} \in \mathbb{R}^3 \mid T(\vec{x}) = \vec{0}\}$

$T(\vec{x}) = \vec{0} \Leftrightarrow (x_2 - x_3, x_1 + x_2) = \vec{0}$

$$\begin{cases} x_2 - x_3 = 0 \\ x_1 + x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = x_2 \\ x_1 = -x_2 \end{cases}$$

Fie $x_2 = \alpha \Rightarrow \ker T = \{(-\alpha, \alpha, \alpha) \mid \alpha \in \mathbb{R}\} = \{\alpha(-1, 1, 1) \mid \alpha \in \mathbb{R}\}. \Rightarrow d = 1.$

$\operatorname{Im} T = \{\vec{y} \in \mathbb{R}^2 \mid \exists \vec{x} \in \mathbb{R}^3 \text{ a.î. } T(\vec{x}) = \vec{y}\}$

$T(\vec{x}) = \vec{y}$

$$\begin{cases} x_2 - x_3 = y_1 \\ x_1 + x_2 = y_2 \end{cases} \Rightarrow \begin{cases} x_2 = y_1 + x_3 \\ x_1 + x_2 = y_2 \end{cases}$$

$D = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \neq 0 \Rightarrow$ sist. are soluții $\forall y_1, y_2 \in \mathbb{R} \Rightarrow \operatorname{Im} T = \mathbb{R}^2$

$\Leftrightarrow n = 2$.

3) Cum $\ker T \neq \{0\} \Rightarrow T$ nu e inj.

Cum $\operatorname{Im} T = \mathbb{R}^2$ (= codomeniul) $\Rightarrow T$ e surj.