$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(A - \lambda I_2) = 0$$

$$A - \lambda I_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$|A - \lambda I_2| = (a-\lambda)(d-\lambda) - bc = 0$$

$$\Leftrightarrow \lambda^2 - \underbrace{(a+d)}_{\mathrm{Tr}A}\lambda + \underbrace{ad - bc}_{\det A} = 0$$

($\lambda_1 \lambda_2$ – valori proprii ale matr. A)

$$\lambda^2 - \mathrm{Tr}A \cdot \lambda + \det A = 0$$

$\alpha_{\lambda_1}, \alpha_{\lambda_2}$ – vectori proprii matr. A

$$\alpha_{\lambda_1} \to \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix}, \quad \alpha_{\lambda_2} \to \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix}$$

$$(x, y) = (C_1\ C_2) \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^T$$

(dacă $\lambda_1 \neq \lambda_2$)

$$(x, y) = (C_1\ C_2) \begin{pmatrix} e^{\lambda_1 t} & 1 \\ 0 & e^{\lambda_1 t} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^T$$

(dacă $\lambda_1 \neq \lambda_2$)

$$(x, y) = (C_1\ C_2)\, e^{\gamma t} \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^T$$

(dacă $\lambda_{1,2} = \gamma \pm i\beta$)

$$\begin{cases} x' = 2x + y \\ y' = -3y \end{cases}$$

$x = \alpha_1 e^{rt}$, $y = \alpha_2 e^{rt}$

$x' = \alpha_1 r e^{rt}$, $y' = \alpha_2 r e^{rt}$

$$\begin{cases} \alpha_1 r = 2\alpha_1 + \alpha_2 \\ \alpha_2 r = -3\alpha_2 \end{cases} \Leftrightarrow \begin{cases} (2-r)\alpha_1 + \alpha_2 = 0 \\ (-3-r)\alpha_2 = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 2-r & 1 \\ 0 & -3-r \end{vmatrix} = (2-r)(-3-r) = 0$$

$r_1 = 2$

$r_2 = -3$

Pt. $r_1 = 2$

$\alpha_2 = 0 \Rightarrow \alpha_1 \in \mathbb{R}$

Fie $\alpha_1 = 1 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \underline{X_{r_1}} = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$

Pt. $r_2 = -3$

$5\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_2 = -5\alpha_1$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = -5 \Rightarrow \alpha_{r_2} = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \Rightarrow \underline{X_{r_2}} = \begin{pmatrix} e^{-3t} \\ -5e^{-3t} \end{pmatrix}$

$$X = C_1 X_{r_1} + C_2 X_{r_2} = C_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} e^{-3t} \\ -5e^{-3t} \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + c_2 e^{-3t} \\ -5c_2 e^{-3t} \end{pmatrix}$$

$$\begin{cases} x = c_1 e^{2t} + c_2 e^{-3t} \\ y = -5c_2 e^{-3t} \end{cases}$$

$A = \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 \\ 0 & -3-\lambda \end{pmatrix}$

$\lambda^2 - \mathrm{Tr}A \cdot \lambda + \det A = 0$

$\lambda^2 + \lambda - 6 = 0$

$\Delta = 1 + 24 = 25$

$\lambda_{1,2} = \frac{-1 \pm 5}{2}$ < -3, 2

Pt. $\lambda_1 = -3$

$(A - \lambda_1 I) \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 5 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 5u_1 + u_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$5u_1 + u_2 = 0 \Rightarrow u_2 = -5u_1$

Fie $u_1 = 1 \Rightarrow u_2 = -5$

$u_{\lambda_1} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

Pt. $\lambda_2 = 2$

$(A - \lambda_2 I) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -5u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$u_2 = 0 \Rightarrow u_1 \in \mathbb{R}$, Fie $u_1 = 1$

$\Rightarrow u_{\lambda_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}^t = (C_1, C_2) \cdot \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & -5 \\ 1 & 0 \end{pmatrix}$

$= (c_1 e^{-3t} \quad c_2 e^{2t}) \begin{pmatrix} 1 & -5 \\ 1 & 0 \end{pmatrix}$

$= (c_1 e^{-3t} + c_2 e^{2t}, \ -5c_1 e^{-3t})$

$$\Rightarrow \begin{cases} x = c_1 e^{-3t} + c_2 e^{2t} \\ y = -5c_1 e^{-3t} \end{cases}$$