Seminar02 - Rezolvare

Exerciţiu01

1

$$x' = rx, \quad x(0) = x_0$$

$$\frac{dx}{dt} = r\cdot x \quad \text{(ec. cu var. separabile)}$$

$$\frac{dx}{x} = r\,dt \Rightarrow \int \frac{1}{x}\,dx = \int r\,dt$$

$$\ln x = rt + C \Rightarrow x = e^{rt+C} = e^{rt}\cdot e^{C}$$

$$x(t) = C\cdot e^{rt} \quad \text{(sol. generală)}$$

$$x(0) = C e^{r\cdot 0} = \boxed{C = x_0} \Rightarrow x_{PC}(t) = x_0 \cdot e^{rt}$$

Exerciţiu02

2

$$x' = rx\left(1 - \frac{x}{K}\right), \quad x(0) = x_0$$

$$\frac{dx}{dt} = rx\cdot\frac{K-x}{K} \quad \text{(ec. cu var. separabile)}$$

$$\frac{1}{x(K-x)}\,dx = \frac{r}{K}\,dt$$

$$\int \frac{1}{x(K-x)}\,dx = \int \frac{r}{K}\,dt$$

$$\frac{1}{x(K-x)} = \frac{A}{x} + \frac{B}{K-x} = \frac{AK - Ax + Bx}{x(K-x)} = \frac{(-A+B)x + AK}{x(K-x)}$$

$$-A+B = 0 \Rightarrow A = B \Rightarrow B = \frac{1}{K}$$

$$AK = 1 \Rightarrow A = \frac{1}{K}$$

$$\frac{1}{x(K-x)} = \frac{\frac{1}{K}}{x} + \frac{\frac{1}{K}}{K-x} = \frac{1}{K}\left(\frac{1}{x} + \frac{1}{K-x}\right)$$

$$\frac{1}{K}\int\left(\frac{1}{x} + \frac{1}{K-x}\right)dx = \frac{r}{K}\int dt \quad \Big| \cdot K$$

$$\ln x + \left(-\ln(K-x)\right) = rt + C$$

$$\ln\frac{x}{K-x} = rt + C$$

$$\frac{x}{K-x} = e^{rt+C} = \frac{C\cdot e^{rt}}{1}$$

$$\frac{x}{K-x+x} = \frac{Ce^{rt}}{1+Ce^{rt}} \Rightarrow x(t) = \frac{K\cdot Ce^{rt}}{1+Ce^{rt}}$$

$$x(0) = \frac{KC}{1+C} = x_0 \Leftrightarrow KC = x_0 + x_0 C$$

$$(K-x_0)C = x_0$$

$$\boxed{C = \frac{x_0}{K-x_0}}$$

$$x_{PC} = \frac{K\cdot\frac{x_0}{K-x_0}\cdot e^{rt}}{1 + \frac{x_0}{K-x_0}\cdot e^{rt}}$$

Exercițiu03

(3) $\dfrac{dx}{dt} = rx \ln\dfrac{K}{x}, \quad x(0) = x_0$

$$\frac{1}{x \ln\frac{K}{x}}\, dx = r\, dt$$

$$\int \frac{1}{x(\ln K - \ln x)}\, dx = \int r\, dt$$

$$u(x) = \ln x$$

$$u'(x) = \frac{1}{x}$$

$$-\int \frac{-1}{\ln K - u} \cdot u'\, dx = r\int dt$$

$$-\ln(\ln K - \ln x) = rt + C$$

$$\ln(\ln K - \ln x) = -rt + \mathcal{C}$$

$$\ln K - \ln x = e^{-rt+C} = \mathcal{C} \cdot e^{-rt}$$

$$\ln\frac{K}{x} = \mathcal{C} \cdot e^{-rt}$$

$$\frac{K}{x} = e^{Ce^{-rt}} \Rightarrow x_{(t)} = \frac{K}{e^{e^{-rt}}}$$

$$x(0) = \frac{K}{e^{C}} = x_0 \Rightarrow e^{C} = \frac{K}{x_0}$$

$$\Rightarrow C = \ln\frac{K}{x_0}$$

$$x_{PC}(t) = \frac{K}{e^{\left(\ln\frac{K}{x_0}\right) \cdot e^{-rt}}} = \frac{K}{\left(e^{\ln\frac{K}{x_0}}\right)^{e^{-rt}}}$$

$$= \frac{K}{\left(\frac{K}{x_0}\right)^{e^{-rt}}}$$

Exercițiu04

(4) $\dfrac{dx}{dt} = kx(1-x)(a-bx), \quad x_0 = \dfrac{1}{u}$

$$\frac{dx}{x(1-x)(a-bx)} = k\, dt$$

$$\int \frac{1}{x(1-x)(a-bx)}\, dx = \int k\, dt$$

$$\frac{1}{x(1-x)(a-bx)} = \frac{\overset{(1-x)(a-bx}{A}}{x} + \frac{\overset{x(a-bx)}{B}}{1-x} + \frac{\overset{x(1-x)}{C}}{a-bx}$$

$$1 = A(1-x)(a-bx) + Bx(a-bx) + Cx(1-x)$$

$x = 1 \Rightarrow B(a-b) = 1 \Rightarrow B = \dfrac{1}{a-b}$

$x = \dfrac{a}{b} \Rightarrow C \cdot \dfrac{a}{b}\left(1 - \dfrac{a}{b}\right) = 1 \Rightarrow C = \dfrac{1}{\frac{a}{b}\left(\frac{b-a}{b}\right)} = \dfrac{b^2}{a(b-a)}$

$x = 0 \Rightarrow A \cdot a = 1 \Rightarrow A = \dfrac{1}{a}$