Laborator05

Enunţuri

Să se rezolve următoarele ecuaţii:

$$
\begin{aligned}
&1.\ \ t \cdot x' = x^3 + x \\
&2.\ \ x' = \frac{t + 2x + 1}{2t + 4x + 3} \\
&3.\ \ x' = 4\frac{x}{t} + t\sqrt{x} \\
&4.\ \ x' = \frac{x}{t} + tg\frac{x}{t} \\
&5.\ \ (t^2 + tx + x^2)dt - t^2 dx = 0
\end{aligned}
$$

Rezolvare

Exerciţiu 01

(1) $tx' = x^3 + x$

$t \cdot \frac{dx}{dt} = x^3 + x$ (ec. cu variabile separabile)

$\frac{dx}{x^3 + x} = \frac{1}{t}dt$

$\int \frac{1}{x^3+x}dx = \int \frac{1}{t}dt$

$\int \frac{1}{x}dx - \frac{1}{2}\int \frac{2x}{x^2+1}dx = \int \frac{1}{t}dt$

$\ln|x| - \frac{1}{2}\ln(x^2+1) = \ln|t| + C \quad |\cdot 2$

$2\ln|x| - \ln(x^2+1) = 2\ln t + \ln c$

$\ln\frac{x^2}{x^2+1} = \ln ct^2 \Rightarrow \frac{x^2}{x^2+1} = ct^2 \Leftrightarrow x^2 = ct^2x^2 + ct^2$

$x^2(1 - ct^2) = ct^2 \Rightarrow x^2 = \frac{ct^2}{1-ct^2} \Leftrightarrow x = \pm\sqrt{\frac{ct^2}{1-ct^2}}$, $\frac{ct^2}{1-ct^2} > 0$, $1 - ct^2 \neq 0$

$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$= \frac{Ax^2 + A + Bx^2 + Cx}{x(x^2+1)}$

$= \frac{(A+B)x^2 + Cx + A}{x(x^2+1)}$

$\begin{cases} A + B = 0 \\ C = 0 \\ A = 1 \end{cases} \Rightarrow B = -1$

$\frac{1}{x^3+x} = \frac{1}{x} - \frac{x}{x^2+1}$

Exerciţiu 02

(2) $\boxed{x' = \dfrac{t+2x+1}{2t+4x+3}}$ [ec. reductibilă la ec. de tip omogen]

$\Delta = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4-4=0$

$x' = \dfrac{t+2x+1}{2(t+2x)+3}$

$u = t+2x \Rightarrow u' = 1+2x'$

$u' = 1+2x' = 1+2\cdot\dfrac{u+1}{2u+3} = \dfrac{2u+3+2u+2}{2u+3}$

$\dfrac{du}{dt} = \dfrac{4u+5}{2u+3} \Leftrightarrow \dfrac{2u+3}{4u+5}du = dt$

$\int \dfrac{2u+3}{4u+5}du = \int dt$

$\dfrac{1}{2}\int\dfrac{2(2u+3)}{4u+5}du = \dfrac{1}{2}\int\dfrac{4u+6}{4u+5}du$

$= \dfrac{1}{2}\int\dfrac{4u+5+1}{4u+5}du = \dfrac{1}{2}\int 1+\dfrac{1}{4u+5}du$

$= \dfrac{1}{2}u + \dfrac{1}{4}\dfrac{1}{2}\int\dfrac{4}{4u+5}du$

$= \dfrac{1}{2}u + \dfrac{1}{8}\ln|4u+5|$

$\Rightarrow \dfrac{1}{2}u+\dfrac{1}{8}\ln|4u+5| = t+C$

$\dfrac{1}{2}(t+2x)+\dfrac{1}{8}\ln|4(t+2x)+5| = t+C$

$\dfrac{1}{2}t + x + \dfrac{1}{8}\ln(4t+8x+5) = t+C$

$x+\dfrac{1}{8}\ln|4t+8x+5| = \dfrac{1}{2}t + C$

(sol. în formă implicită)

Exerciţiu 03

(3) $x' = 4\cdot\dfrac{x}{t}+t\sqrt{x}$

$x' = \underbrace{\dfrac{4}{t}}_{A(t)}x + \underbrace{t}_{b(t)}\cdot x^{\frac{1}{2}}$, $\alpha = \dfrac{1}{2}$ [ec. Bernoulli cu $\alpha = 1/2$]

$\dfrac{x'}{x^{1/2}} = \dfrac{4}{t}\cdot\dfrac{x}{x^{1/2}}+t$

$\dfrac{x'}{\sqrt{x}} = \dfrac{4}{t}\sqrt{x}+t \quad \Big| \cdot\dfrac{1}{2}$

$\boxed{u=\sqrt{x}} \Rightarrow u' = \dfrac{1}{2\sqrt{x}}\cdot x'$

$\dfrac{x'}{2\sqrt{x}} = \dfrac{2}{t}\sqrt{x}+\dfrac{1}{2}t$

$u' = \dfrac{2}{t}u+\dfrac{t}{2}$ (ec. afină)

Etapa 1 $\quad u' = \dfrac{2}{t}u$

$\dfrac{du}{dt} = \dfrac{2}{t}u$

$\dfrac{du}{u} = \dfrac{2}{t}dt$

$\int\dfrac{1}{u}du = 2\int\dfrac{1}{t}dt$

$\ln|u| = 2\ln|t|+C$

$\ln|u| = \ln t^2+\ln C$

$\boxed{u_0 = C\,t^2}$

Etapa 2 $\quad \varphi_0 = C(t)\cdot t^2$

$(C(t)\cdot t^2)' = \dfrac{2}{t}C(t)\cdot t^2+\dfrac{t}{2}$

$C'(t)t^2+\cancel{C(t)2t} = \cancel{2C(t)\cdot t}+\dfrac{t}{2}$

$C'(t)\cdot t^2 = \dfrac{t}{2} \quad \Big| : t^2$

$C'(t) = \dfrac{1}{2t} \Rightarrow C(t) = \dfrac{1}{2}\int\dfrac{1}{t}dt$

$C(t) = \dfrac{1}{2}\ln|t|+C_1$

$\Rightarrow \varphi_0 = \left(\dfrac{1}{2}\ln|t|+C_1\right)t^2$

$\boxed{\varphi_0 = \dfrac{1}{2}t^2\ln|t|+C_1t^2}$

$u = u_0+\varphi_0 = Ct^2+\dfrac{1}{2}t^2\ln|t| = \sqrt{x}$

$x = \left(Ct^2+\dfrac{1}{2}t^2\ln|t|\right)^2$

Exerciţiu 04

④ $x' = \frac{x}{t} + tg\frac{x}{t}$ (ec. de tip omogen)

$\boxed{u = \frac{x}{t}} \Rightarrow x = ut \Rightarrow x' = u't + u$

$u't + \cancel{u} = \cancel{u} + tg\, u$

$\frac{du}{dt} \cdot t = tg\, u$

$\frac{du}{tg\, u} = \frac{1}{t} dt$

$\int \frac{1}{tg\, u} du = \int \frac{1}{t} dt$

$\int \frac{\cos u}{\sin u} du = \int \frac{1}{t} dt$

$\ln|\sin u| = \ln|t| + C$

$\ln|\sin u| = \ln|t| + \ln C$

$\sin u = tC$

$\sin\frac{x}{t} = tC$ (sol. implicită)

$\frac{x}{t} = (-1)^k \arcsin(tC) + k\pi$

$x = (-1)^k \cdot t \arcsin(tC) + tk\pi$ (sol. generală explicită)

Exerciţiu 05

⑤ $(t^2 + tx + x^2)dt - t^2 dx = 0 \quad /: t^2 dt$

$\frac{t^2 + tx + x^2}{t^2} - \frac{dx}{dt} = 0$

$\frac{dx}{dt} = 1 + \frac{x}{t} + \left(\frac{x}{t}\right)^2$ (ec. de tip omogen)

$\boxed{u = \frac{x}{t}} \Rightarrow x = ut \Rightarrow x' = u't + u$

$u't + \cancel{u} = 1 + \cancel{u} + u^2$

$u't = 1 + u^2$

$\frac{du}{dt} \cdot t = 1 + u^2$

$\frac{du}{u^2+1} = \frac{1}{t} dt$

$\int \frac{1}{u^2+1} du = \int \frac{1}{t} dt$

$\operatorname{arctg} u = \ln|t| + C$

$\operatorname{arctg}\frac{x}{t} = \ln|t| + C$ (sol. în formă implicită)

$\frac{x}{t} = tg(\ln|t| + C)$

$x = t\, tg(\ln|t| + C)$ (sol. generală în formă explicită)