Laborator 09 - TEMĂ

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1. a) $x'' + x' = 3t + 2$

• $x'' + x' = 0$

$$\left.\begin{array}{l} x = e^{rt} \\ x' = r e^{rt} \\ x'' = r^2 e^{rt} \end{array}\right\} \Leftrightarrow r^2 e^{rt} + r e^{rt} = 0 \quad |: e^{rt}$$

$r^2 + r = 0 \Leftrightarrow r(r+1) = 0 \Rightarrow r_1 = 0, r_2 = -1$

• $e^{0t}, e^{-t}$ - sist. fund. de sol.

$$\boxed{x_0 = c_1 + c_2 e^{-t}}$$

• $a_0 = 0 \Leftrightarrow a_p \neq 0, m = 1$

$x_p = t(\lambda_1 t + \lambda_0) = \lambda_1 t^2 + \lambda_0 t$

$x_p' = 2t\lambda_1 + \lambda_0$ ; $x_p'' = 2\lambda_1$

$$2\lambda_1 + 2t\lambda_1 + \lambda_0 = 3t + 2 \Leftrightarrow \begin{cases} 2\lambda_1 = 3 \Leftrightarrow \boxed{\lambda_1 = \frac{3}{2}} \\ 2\lambda_1 + \lambda_0 = 2 \Leftrightarrow \boxed{\lambda_0 = -1} \end{cases}$$

$$\boxed{x_p = t\left(\frac{3}{2}t - 1\right) = \frac{3}{2}t^2 - t} \Leftrightarrow \underline{x(t) = x_0 + x_p = c_1 + c_2 e^{-t} + \frac{3}{2}t^2 - t}$$

b) $x'' - 4x' + 4x = t^2$

• $x'' - 4x' + 4x = 0$

$$\left.\begin{array}{l} x = e^{rt} \\ x' = r e^{rt} \\ x'' = r^2 e^{rt} \end{array}\right\} \Leftrightarrow r^2 e^{rt} - 4r e^{rt} + 4e^{rt} = 0 \quad |: e^{rt}$$

$r^2 - 4r + 4 = 0 \Leftrightarrow (r-2)^2 = 0 \Leftrightarrow r_1 = r_2 = 2$

• $e^{2t}, te^{2t}$ - sist. fund. de sol.

$$\boxed{x_0 = c_1 e^{2t} + c_2 t e^{2t}}$$

• $a_0 \neq 0 \Leftrightarrow a_p = 0$

$x_p = \lambda_2 t^2 + \lambda_1 t + \lambda_0$ ; $x_p' = 2\lambda_2 t + \lambda_1$ ; $x_p'' = 2\lambda_2$

$2\lambda_2 - 4(\underline{2\lambda_2} t + \lambda_1) + 4(\underline{\lambda_2 t^2} + \underline{\lambda_1 t} + \lambda_0) = t^2$

$$\Leftrightarrow \begin{cases} 4\lambda_2 = 1 \Leftrightarrow \boxed{\lambda_2 = \frac{1}{4}} \\ -8\lambda_2 + 4\lambda_1 = 0 \Leftrightarrow \boxed{\lambda_1 = \frac{1}{2}} \\ 2\lambda_2 - 4\lambda_1 + 4\lambda_0 = 0 \Leftrightarrow \boxed{\lambda_0 = \frac{3}{8}} \end{cases}$$

$$\boxed{x_p = \frac{1}{4}t^2 + \frac{1}{2}t + \frac{3}{8}} \Leftrightarrow \underline{x(t) = x_o + x_p = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{4}t^2 + \frac{1}{2}t + \frac{3}{8}}$$

c) $x'' - x' + x = t^3 + 6$

- $x'' - x' + x = 0$

$$\left.\begin{aligned} x &= e^{rt} \\ x' &= re^{rt} \\ x'' &= r^2 e^{rt} \end{aligned}\right\} \Leftrightarrow r^2 e^{rt} - re^{rt} + e^{rt} = 0 \;|: e^{rt}$$

$$\left.\begin{aligned} r^2 - r + 1 = 0 \\ D = 1 - 4 = -3 \end{aligned}\right\} \Rightarrow r_{1,2} = \frac{1 \pm \sqrt{3}i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\left(\alpha \pm \beta,\ \alpha = \frac{1}{2},\ \beta = \frac{\sqrt{3}}{2}\right)$$

- $e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t,\ e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$ - sist. fund. de sol.

$$\boxed{x_o = C_1 e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + C_2 e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t}$$

- $a_0 \neq 0 \Leftrightarrow a_p = 0$

$$x_p = \lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t + \lambda_0$$

$$x_p' = 3\lambda_3 t^2 + 2\lambda_2 t + \lambda_1;\quad x_p'' = 6\lambda_3 t + 2\lambda_2$$

$$(6\lambda_3 t + 2\lambda_2) - (3\lambda_3 t^2 + 2\lambda_2 t + \lambda_1) + (\lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t + \lambda_0) = t^3 + 6$$

$$\Leftrightarrow \begin{cases} \boxed{\lambda_3 = 1} \\ -3\lambda_3 + \lambda_2 = 0 \Leftrightarrow \boxed{\lambda_2 = 3} \\ 6\lambda_3 - 2\lambda_2 + \lambda_1 = 0 \Leftrightarrow \boxed{\lambda_1 = 0} \\ 2\lambda_2 - \lambda_1 + \lambda_0 = 6 \Leftrightarrow \boxed{\lambda_0 = 0} \end{cases}$$

$$\boxed{x_p = t^3 + 3t^2} \Rightarrow \underline{x(t) = x_o + x_p = c_1 e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + c_2 e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t + t^3 + 3t^2}$$

e) $x^{IV} - 2x''' + x'' = t^3$

- $x^{IV} - 2x''' + x'' = 0$

$$\left.\begin{aligned} x &= e^{rt} \\ x' &= re^{rt} \\ x'' &= r^2 e^{rt} \\ x''' &= r^3 e^{rt} \\ x^{IV} &= r^4 e^{rt} \end{aligned}\right\} \Leftrightarrow r^4 e^{rt} - 2r^3 e^{rt} + r^2 e^{rt} = 0 \;|: e^{rt}$$

$$r^4 - 2r^3 + r^2 = 0 \Leftrightarrow r^2(r^2 - 2r + 1) = 0 \Rightarrow r_1 = r_2 = 0$$

$$r^2 - 2r + 1 = 0 \Leftrightarrow (r-1)^2 = 0 \Leftrightarrow r_3 = r_4 = 1$$

- $e^{0t}, te^{0t}, e^{1t}, te^{1t}$ - sis. fund. de sol.

$$\boxed{x_o = c_1 + c_2 t + c_3 e^t + c_4 t e^t}$$

$a_0 = 0 \Rightarrow a_p \neq 0,\ r = 2$

$$x_p = t^2(\lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t + \lambda_0) = \lambda_3 t^5 + \lambda_2 t^4 + \lambda_1 t^3 + \lambda_0 t^2$$

$$x_p' = 5\lambda_3 t^4 + 4\lambda_2 t^3 + 3\lambda_1 t^2 + 2\lambda_0 t$$

$$x_p'' = 20\lambda_3 t^3 + 12\lambda_2 t^2 + 6\lambda_1 t + 2\lambda_0$$

$$x_p''' = 60\lambda_3 t^2 + 24\lambda_2 t + 6\lambda_1 t$$

$$x_p^{IV} = 120\lambda_3 t + 24\lambda_2$$

• $(120\lambda_3 t + 24\lambda_2) - 2(60\lambda_3 t^2 + 24\lambda_2 t + 6\lambda_1 t) + (20\lambda_3 t^3 + 12\lambda_2 t^2 +$

$+ 6\lambda_1 t + 2\lambda_0) = t^3$

$$\Leftrightarrow \begin{cases} 20\lambda_3 = 1 \Leftrightarrow \boxed{\lambda_3 = \frac{1}{20}} \\ -120\lambda_3 + 12\lambda_2 = 0 \Leftrightarrow \boxed{\lambda_2 = \frac{1}{2}} \\ 120\lambda_3 - 48\lambda_2 - 6\lambda_1 = 0 \Leftrightarrow \boxed{\lambda_1 = -3} \\ 24\lambda_2 + 2\lambda_0 = 0 \Leftrightarrow \boxed{\lambda_0 = -6} \end{cases}$$

$$\boxed{x_p = t^2\left(\frac{1}{20}t^3 + \frac{1}{2}t^2 - 3t - 6\right) = \frac{1}{20}t^5 + \frac{1}{2}t^4 - 3t^3 - 6t^2}$$

$$x(t) = x_o + x_p = c_1 + c_2 t + c_3 e^t + c_4 t e^t + \frac{1}{20}t^5 + \frac{1}{2}t^4 - 3t^3 - 6t^2$$

f) $x''' + 3x'' - 4x' = 2t^2 - 3t + 9$

• $x''' + 3x'' - 4x' = 0$

$x = e^{rt}$

$x' = r e^{rt}$

$x'' = r^2 e^{rt}$

$x''' = r^3 e^{rt}$

$\Leftrightarrow r^3 e^{rt} + 3r^2 e^{rt} - 4r e^{rt} = 0 \ |: e^{rt}$

$r^3 + 3r^2 - 4r = 0 \Leftrightarrow r(r^2 + 3r - 4) = 0 \Rightarrow r_1 = 0$

$D = 9 - 4(-4) = 9 + 16 = 25 \Rightarrow r_{2,3} = \frac{-3 \pm 5}{2} \Leftrightarrow \begin{cases} r_2 = 1 \\ r_3 = -4 \end{cases}$

• $e^{0t}, e^{1t}, e^{-4t}$ – sist. fund. de sol.

$$\boxed{x_o = c_1 + c_2 e^t + c_3 e^{-4t}}$$

• $a_0 = 0 \Rightarrow a_p \neq 0,\ n = 1$ ;

$$x_p = t(\lambda_2 t^2 + \lambda_1 t + \lambda_0) = \lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t$$

$$x_p' = 3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0 \ ; \ x_p'' = 6\lambda_2 t + 2\lambda_1 \ ; \ x_p''' = 6\lambda_2$$

• $6\lambda_2 + 3(6\lambda_2 t + 2\lambda_1) - 4(\underline{3\lambda_2 t^2} + 2\lambda_1 t + \lambda_0) = 2t^2 - 3t + 9$

$$\Rightarrow \begin{cases} -12\lambda_2 = 2 \Leftrightarrow \boxed{\lambda_2 = \frac{-1}{6}} \\ 18\lambda_2 - 8\lambda_1 = -3 \Leftrightarrow \boxed{\lambda_1 = 0} \\ 6\lambda_2 + 6\lambda_1 - 4\lambda_0 = 9 \Leftrightarrow \boxed{\lambda_0 = \frac{-5}{2}} \end{cases}$$

$$\boxed{x_p = t\left(\frac{-1}{6}t^2 + \frac{-5}{2}\right) = \frac{-1}{6}t^3 - \frac{5}{2}t}$$

$$\underline{x(t) = x_0 + x_p = c_1 + c_2 e^{t} + c_3 e^{-4t} - \frac{1}{6}t^3 - \frac{5}{2}t}$$

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b) $x'' - 9x = 5t^2 \cdot e^{2t}$, $\alpha = 2$

• $x'' - 9x = 0$

$$\left.\begin{aligned} x &= e^{rt} \\ x' &= r e^{rt} \\ x'' &= r^2 e^{rt} \end{aligned}\right\} \Leftrightarrow r^2 e^{rt} - 9 e^{rt} = 0 \,|: e^{rt}$$

$r^2 - 9 = 0 \Leftrightarrow r^2 = 9 \Leftrightarrow r_{1,2} = \pm 3$

• $e^{3t}, e^{-3t}$ - sist. fund. de sol.

$$\boxed{x_o = C_1 e^{3t} + C_2 e^{-3t}}$$

• $\alpha = 2$ nu e răd. a ec. carac.

$x_p = e^{2t}(\lambda_2 t^2 + \lambda_1 t + \lambda_0)$

$x_p' = 2e^{2t}(\lambda_2 t^2 + \lambda_1 t + \lambda_0) + e^{2t}(2\lambda_2 t + \lambda_1) =$

$= e^{2t}(2\lambda_2 t^2 + 2\lambda_1 t + 2\lambda_2 t + 2\lambda_0 + \lambda_1)$

$x_p'' = 2e^{2t}(2\lambda_2 t^2 + 2\lambda_1 t + 2\lambda_2 t + 2\lambda_0 + \lambda_1) + e^{2t}(4\lambda_2 t + 2\lambda_1 + 2\lambda_2)$

$= e^{2t}(4\lambda_2 t^2 + 8\lambda_2 t + 4\lambda_1 t + 4\lambda_1 + 4\lambda_0 + 2\lambda_2)$

• $e^{2t}(4\lambda_2 t^2 + 8\lambda_2 t + 4\lambda_1 t + 4\lambda_1 + 4\lambda_0 + 2\lambda_2) - 9e^{2t}(\lambda_2 t^2 + \lambda_1 t + \lambda_0) =$
$= 5t^2 e^{2t} \,|: e^{2t}$

$-5\lambda_2 t^2 + 8\lambda_2 t - 5\lambda_1 t + 4\lambda_1 + 2\lambda_2 - 5\lambda_0 = 5t^2$

$$\Leftrightarrow \begin{cases} -5\lambda_2 = 5 \Leftrightarrow \boxed{\lambda_2 = -1} \\ 8\lambda_2 - 5\lambda_1 = 0 \Leftrightarrow \boxed{\lambda_1 = \frac{-8}{5}} \\ 4\lambda_1 + 2\lambda_2 - 5\lambda_0 = 0 \Leftrightarrow \boxed{\lambda_0 = \frac{-42}{25}} \end{cases}$$

$$\boxed{x_p = e^{2t}\left(-t^2 - \frac{8}{5}t - \frac{42}{25}\right)}$$

$x(t) = x_o + x_p = C_1 e^{3t} + C_2 e^{-3t} + e^{2t}\left(-t^2 - \frac{8}{5}t - \frac{42}{25}\right)$

c) $x'' + 2x' + x = e^{2t}$, $\alpha = 2$

• $x'' + 2x' + x = 0$

$$\left.\begin{aligned} x &= e^{rt} ; \; x'' = r^2 e^{rt} \\ x' &= r e^{rt} \end{aligned}\right\} \Leftrightarrow r^2 e^{rt} + 2r e^{rt} + e^{rt} = 0 \,|: e^{rt}$$

$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Leftrightarrow r_1 = r_2 = -1.$

• $e^{-t}, te^{-t}$ - sist. fund de sol.

$$\boxed{x_0 = c_1 e^{-t} + c_2 t e^{-t}}$$

• $\alpha = 2$ nu e rădăcină a ec. caracteristice.

$$x_p = e^{2t}(\lambda_2 t^2 + \lambda_1 t + \lambda_0)$$

$$x_p' = 2e^{2t}(\lambda_2 t^2 + \lambda_1 t + \lambda_0) + e^{2t}(2\lambda_2 t + \lambda_1)$$

$$= e^{2t}(2\lambda_2 t^2 + 2\lambda_1 t + 2\lambda_2 t + 2\lambda_0 + \lambda_1)$$

$$x_p'' = 2e^{2t}(2\lambda_2 t^2 + 2\lambda_1 t + 2\lambda_2 t + 2\lambda_0 + \lambda_1) + e^{2t}(4\lambda_2 t + 2\lambda_1 + 2\lambda_2)$$

$$= e^{2t}(4\lambda_2 t^2 + 8\lambda_2 t + 4\lambda_1 t + 4\lambda_0 + 4\lambda_1 + 2\lambda_2)$$

• $e^{2t}(4\lambda_2 t^2 + 8\lambda_2 t + 4\lambda_1 t + 4\lambda_0 + 4\lambda_1 + 2\lambda_2) + 2e^{2t}(2\lambda_2 t^2 + 2\lambda_1 t +$
$+ 2\lambda_2 t + 2\lambda_0 + \lambda_1) + e^{2t}(\lambda_2 t^2 + \lambda_1 t + \lambda_0) = e^{2t} \; |: e^{2t}$

$$9\lambda_2 t^2 + 12\lambda_2 t + 9\lambda_1 t + 2\lambda_2 + 6\lambda_1 + 9\lambda_0 = 1$$

$$\Rightarrow \begin{cases} 9\lambda_2 = 0 \iff \boxed{\lambda_2 = 0} \\ 12\lambda_2 + 9\lambda_1 = 0 \iff \boxed{\lambda_1 = 0} \\ 2\lambda_2 + 6\lambda_1 + 9\lambda_0 = 1 \iff \boxed{\lambda_0 = \frac{1}{9}} \end{cases}$$

$$\boxed{x_p = e^{2t}\frac{1}{9}} \iff \underline{x(t) = x_0 + x_p = c_1 e^{-t} + c_2 t e^{-t} + e^{2t}\frac{1}{9}}$$

e) $x'' - 3x' + 2x = 3t^2 e^t, \; \alpha = 1$

• $x'' - 3x' + 2x = 0$

$$\left.\begin{array}{l} x = e^{rt} \\ x' = re^{rt} \\ x'' = r^2 e^{rt} \end{array}\right\} \iff r^2 e^{rt} - 3re^{rt} + 2e^{rt} = 0 \; |: e^{rt}$$

$$\left.\begin{array}{l} r^2 - 3r + 2 = 0 \\ \Delta = 9 - 4\cdot 2 = 1 \end{array}\right\} \Rightarrow r_{1,2} = \frac{3 \pm 1}{2} \iff \begin{cases} r_1 = 2 \\ r_2 = 1 \end{cases}$$

• $e^t, e^{2t}$ - sist. fund. de sol

$$\boxed{x_0 = c_1 e^t + c_2 e^{2t}}$$

• $\alpha = 1$ e răd. a ec. caract.

$$x_p = te^t(\lambda_2 t^2 + \lambda_1 t + \lambda_0) = e^t(\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t)$$

$$x_p' = e^t(\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t) + e^t(3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0)$$

$$= e^t(\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t + 3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0)$$

$$x_p'' = e^t(\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t + 3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0) + e^t(3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0 t + 6\lambda_2 t + 2\lambda_1)$$

$$= e^t(\lambda_2 t^3 + \lambda_1 t^2 + 6\lambda_2 t^2 + \lambda_0 t + 4\lambda_1 t + 6\lambda_2 t + \lambda_0 + 2\lambda_1)$$

• $e^t(\lambda_2 t^3 + \lambda_1 t^2 + 6\lambda_2 t^2 + \lambda_0 t + 4\lambda_1 t + 6\lambda_2 t + \lambda_0 + 2\lambda_1) - 3e^t(\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t + 3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0) + 2e^t(\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t) = 3t^2 e^t \,|\, : e^t$

$$-3\lambda_2 t^2 - 2\lambda_1 t + 6\lambda_2 t - 2\lambda_0 + 2\lambda_1 = 3t^2$$

$$\Leftrightarrow \begin{cases} -3\lambda_2 = 3 \Leftrightarrow \boxed{\lambda_2 = -1} \\ -2\lambda_1 + 6\lambda_2 = 0 \Leftrightarrow -2\lambda_1 - 6 = 0 \Leftrightarrow \boxed{\lambda_1 = -3} \\ -2\lambda_0 + 2\lambda_1 = 0 \Leftrightarrow -2\lambda_0 - 6 = 0 \Leftrightarrow \boxed{\lambda_0 = -3} \end{cases}$$

$$\boxed{x_p = e^t(-t^2 - 3t - 3)}$$

$$x(t) = x_o + x_p = c_1 e^t + c_2 e^{2t} + e^t(-t^2 - 3t - 3)$$

f) $x'' + 2x' - 3x = 4te^t,\ \alpha = 1$

• $x'' + 2x' - 3x = 0$

$$\left.\begin{aligned} x &= e^{rt} \\ x' &= re^{rt} \\ x'' &= r^2 e^{rt} \end{aligned}\right\} \Leftrightarrow r^2 e^{rt} + 2re^{rt} - 3e^{rt} = 0 \,|\, : e^{rt}$$

$$\left.\begin{aligned} r^2 + 2r - 3 &= 0 \\ D = 4 - 4\cdot(-3) &= 16 \end{aligned}\right\} \Leftrightarrow r_{1,2} = \frac{-2 \pm 4}{2} \Leftrightarrow \left[\begin{aligned} r_1 &= 1 \\ r_2 &= -3 \end{aligned}\right.$$

• $e^{1t}, e^{-3t}$ - sist. fund. de sol.

$$\boxed{x_o = c_1 e^t + c_2 e^{-3t}}$$

• $\alpha = 1$ e rădăc. a ec. caract.

$$x_p = t(e^t(\lambda_1 t + \lambda_0)) = e^t(\lambda_1 t^2 + \lambda_0 t)$$

$$x_p' = e^t(\lambda_1 t^2 + \lambda_0 t) + e^t(2\lambda_1 t + \lambda_0) = e^t(\lambda_1 t^2 + \lambda_0 t + 2\lambda_1 t + \lambda_0)$$

$$x_p'' = e^t(\lambda_1 t^2 + \lambda_0 t + 2\lambda_1 t + \lambda_0) + e^t(2t\lambda_1 + \lambda_0 + 2\lambda_1)$$

$$= e^t(\lambda_1 t^2 + \lambda_0 t + 4\lambda_1 t + 2\lambda_0 + 2\lambda_1)$$

$$e^t(\lambda_1 t^2 + \lambda_0 t + 4\lambda_1 t + 2\lambda_0 + 2\lambda_1) + 2e^t(\lambda_1 t^2 + \lambda_0 t + 2\lambda_1 t + \lambda_0) - 3e^t$$

$$(\lambda_1 t^2 + \lambda_0 t) = 4te^t \;\Big|\; : e^t$$

$$\Leftrightarrow \begin{cases} 8\lambda_0 = 4 \Leftrightarrow \boxed{\lambda_0 = \frac{1}{2}} \\ 2\lambda_1 + 4\lambda_0 = 0 \Leftrightarrow \boxed{\lambda_1 = -1} \end{cases}$$

$$\boxed{x_p = te^t\left(-t + \frac{1}{2}\right) = e^t\left(-t^2 + \frac{1}{2}t\right)}$$

$$\underline{x(t) = x_0 + x_p = c_1 e^t + c_2 e^{-3t} + e^t\left(-t^2 + \frac{1}{2}\right)t.}$$

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3. b) $x'' - 4x = e^{2t}\cos 2t$, $\alpha = 2$, $\beta = 2$

• $x'' - 4x = 0$

$$\left.\begin{aligned} x &= e^{nt} \\ x' &= ne^{nt} \\ x'' &= n^2e^{nt}\end{aligned}\right\} \Leftrightarrow n^2e^{nt} - 4e^{nt} = 0 \mid : e^{nt}$$

$n^2 - 4 = 0 \Leftrightarrow n^2 = 4 \Leftrightarrow n_{1,2} = \pm 2$

• $e^{-2t}$, $e^{2t}$ - sist. fund. de soluții

$$\boxed{x_0 = C_1 e^{-2t} + C_2 e^{2t}}$$

• $x_p = e^{2t}(\lambda_0)\sin 2t + e^{2t}(\beta_0)\cos 2t$

$x_p' = \lambda_0(2e^{2t}\sin 2t + 2e^{2t}\cos 2t) + \beta_0(2e^{2t}\cos 2t - 2e^{2t}\sin 2t)$

$= 2e^{2t}\sin 2t(\lambda_0 - \beta_0) + 2e^{2t}\cos 2t(\lambda_0 + \beta_0)$

$x_p'' = 2(\lambda_0 - \beta_0)(2e^{2t}\sin 2t + e^{2t}\cdot 2\cos 2t) + 2(\lambda_0 + \beta_0)(2e^{2t}\cos 2t -$

$- e^{2t}\sin 2t \cdot 2)$

$= 8e^{2t}\lambda_0\cos 2t - 8e^{2t}\beta_0\sin 2t$

• $8e^{2t}(\lambda_0\cos 2t - \beta_0\sin 2t) - 4e^{2t}(\lambda_0\sin 2t + \beta_0\cos 2t) = e^{2t}\cos 2t \mid : e^{2t}$

$$\Leftrightarrow \begin{cases} 8\lambda_0 - 4\beta_0 = 1 \\ -4\lambda_0 - 8\beta_0 = 0 \mid \cdot 2 \end{cases} \Leftrightarrow \begin{cases} 8\lambda_0 - 4\beta_0 = 1 \\ -8\lambda_0 - 16\beta_0 = 0 \quad "+" \end{cases}$$

$-20\beta_0 = 1 \Rightarrow \boxed{\beta_0 = \frac{-1}{20}} \Rightarrow \boxed{\lambda_0 = \frac{1}{10}}$

$$\boxed{x_p = e^{2t}\frac{1}{10}\sin 2t - e^{2t}\frac{1}{20}\cos 2t}$$

$\underline{x(t) = x_0 + x_p = C_1e^{2t} + C_2e^{-2t} + e^{2t}\frac{1}{10}\sin 2t - e^{2t}\frac{1}{20}\cos 2t}$

c) $x'' - 2x' + 5x = te^t\sin t$, $\alpha = 1$, $\beta = 1$.

• $x'' - 2x' + 5x = 0$

$$\left.\begin{aligned} x &= e^{nt} \\ x' &= ne^{nt} \\ x'' &= n^2e^{nt}\end{aligned}\right\} \Leftrightarrow n^2e^{nt} - 2ne^{nt} + 5e^{nt} = 0 \mid : e^{nt}$$

$$\left.\begin{aligned} &n^2 - 2n + 5 = 0 \\ &\Delta = 4 - 4\cdot 5 = 4 - 20 = -16 \end{aligned}\right\} \Rightarrow n_{1,2} = \frac{2 \pm 4i}{2} \Leftrightarrow \begin{cases} n_1 = 1 + 2i \\ n_2 = 1 - 2i \end{cases}$$

• $e^t \sin 2t,\ e^t \cos 2t$ - sist. fund. de sol.

$$\boxed{x_o = C_1 e^t \sin 2t + C_2 e^t \cos 2t}$$

• $x_p = e^t(\lambda_1 t + \lambda_0)\sin t + e^t(\beta_1 t + \beta_0)\cos t$

$$x_p' = e^t(\lambda_1 t + \lambda_0)\sin t + e^t \lambda_1 \sin t + e^t(\lambda_1 t + \lambda_0)\cos t + e^t(\beta_1 t + \beta_0)\cos t +$$
$$+ e^t \cdot \beta_1 \cdot \cos t - e^t(\beta_1 t + \beta_0)\sin t$$
$$= \sin t\, e^t(\lambda_1 t + \lambda_0 + \lambda_1 - \beta_1 t - \beta_0) + e^t \cos t(\lambda_1 t + \lambda_0 + \beta_1 t + \beta_0 + \beta_1)$$

$$x_p'' = e^t \sin t(\lambda_1 t + \lambda_0 + \lambda_1 - \beta_1 t - \beta_0) + e^t \cos t(\lambda_1 t + \lambda_0 + \lambda_1 - \beta_1 t - \beta_0) +$$
$$+ e^t \sin t(\lambda_1 - \beta_1) + e^t \cos t(\lambda_1 t + \lambda_0 + \beta_1 t + \beta_0 + \beta_1) - e^t \sin t(\lambda_1 t + \lambda_0 +$$
$$+ \beta_1 t + \beta_0 + \beta_1) + e^t \cos t(\lambda_1 + \beta_1).$$

$$= -2e^t \sin t(\beta_1 t + \beta_0) + 2e^t \cos t(\lambda_1 t + \lambda_0 + \lambda_1 + \beta_1)$$

• $\left(-2e^t \sin t(\beta_1 t + \beta_0) + 2e^t \cos t(\lambda_1 t + \lambda_0 + \lambda_1 + \beta_1)\right) - 2\left(e^t \sin t(\lambda_1 t + \lambda_0 + \right.$
$+ \lambda_1 - \beta_1 t - \beta_0) + e^t \cos t(\lambda_1 t + \lambda_0 + \beta_1 t + \beta_0 + \beta_1)\big) + 5\left(e^t(\lambda_1 t + \lambda_0)\sin t + \right.$
$\left. + e^t(\beta_1 t + \beta_0)\cos t\right) = te^t \sin t \quad | : e^t$

$$\Leftrightarrow \begin{cases} 3\lambda_1 t + 3\lambda_0 - 2\lambda_1 = t \\ t(4\lambda_1 + 7\beta_1) + 4\lambda_0 + 2\lambda_1 + 4\beta_1 + 7\beta_0 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 3\lambda_1 = 1 \\ 3\lambda_0 - 2\lambda_1 = 0 \\ 4\lambda_1 + 7\beta_1 = 0 \\ 4\lambda_0 + 2\lambda_1 + 4\beta_1 + 7\beta_0 = 0 \end{cases} \qquad \Leftrightarrow \begin{cases} \lambda_1 = \frac{1}{3} \\ \lambda_0 = \frac{2}{9} \\ \beta_1 = \frac{-4}{21} \\ \beta_0 = \frac{-50}{441} \end{cases}$$

$$\boxed{x_p = e^t\left(\frac{1}{3}t + \frac{2}{9}\right)\sin t + e^t\left(\frac{-4}{21}t + \frac{-50}{441}\right)\cos t}$$

• $x(t) = x_o + x_o = C_1 e^t \sin 2t + C_2 e^t \cos 2t + e^t\left(\frac{1}{3}t + \frac{2}{9}\right)\sin t +$
$+ e^t\left(\frac{-4}{21}t + \frac{-50}{441}\right)\cos t.$

d) $x'' - 2x' + 5x = e^t \cos 2t$, $\alpha = 1$, $\beta = 2$

• $x'' - 2x' + 5x = 0$

$$\left.\begin{aligned} x &= e^{nt} \\ x' &= n e^{nt} \\ x'' &= n^2 e^{nt} \end{aligned}\right\} \Leftrightarrow n^2 e^{nt} - 2n e^{nt} + 5 e^{nt} = 0 \mid : e^{nt}$$

$$\left.\begin{aligned} & n^2 - 2n + 5 = 0 \\ & D = 4 - 4 \cdot 5 = -16 \end{aligned}\right\} \Leftrightarrow n_{1,2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

• $e^t \sin 2t$, $e^t \cos 2t$ - sist. fund. de sol.

$$\boxed{x_0 = c_1 e^t \sin 2t + c_2 e^t \cos 2t}$$

• $x_p = e^t \lambda_0 \sin 2t + e^t \beta_0 \cos 2t$

$x_p' = e^t \lambda_0 \sin 2t + e^t \lambda_0 2 \cos 2t + e^t \beta_0 \cos 2t + e^t \beta_0 (-2) \sin 2t$

$= e^t \sin 2t (\lambda_0 - 2\beta_0) + e^t \cos 2t (2\lambda_0 + \beta_0)$

$x_p'' = (\lambda_0 - 2\beta_0) e^t \sin 2t + (\lambda_0 - 2\beta_0) e^t \cos 2t \cdot 2 + (2\lambda_0 + \beta_0) e^t \cos 2t +$

$+ (-2)(2\lambda_0 + \beta_0) e^t \sin 2t$

$= e^t \sin 2t (-3\lambda_0 - 4\beta_0) + e^t \cos 2t (4\lambda_0 - 3\beta_0)$

• $e^t \sin 2t (-3\lambda_0 - 4\beta_0) + e^t \cos 2t (4\lambda_0 - 3\beta_0) - 2(e^t \sin 2t (\lambda_0 - 2\beta_0) + e^t \cdot$
$\cos 2t (2\lambda_0 + \beta_0)) + 5(e^t \lambda_0 \sin 2t + e^t \beta_0 \cos 2t) = e^t \cos 2t \mid : e^t$

$$\Leftrightarrow \begin{cases} -3\lambda_0 - 4\beta_0 - 2\lambda_0 + 4\beta_0 + 5\lambda_0 = 0 \\ 4\lambda_0 - 3\beta_0 - 4\lambda_0 - 2\beta_0 + 5\beta_0 = 1 \end{cases}$$

e) $x'' - x = 2t \sin t$, $\alpha = 0$, $\beta = 1$

• $x'' - x = 0$

$$\left.\begin{aligned} x &= e^{nt} \\ x' &= n e^{nt} \\ x'' &= n^2 e^{nt} \end{aligned}\right\} \Leftrightarrow n^2 e^{nt} - e^{nt} = 0 \mid : e^{nt}$$

$n^2 - 1 = 0 \Rightarrow n_{1,2} = \pm 1$

• $e^t, e^{-t}$ - sist. fund. de sol.

$$\boxed{x_0 = c_1 e^t + c_2 e^{-t}}$$

• $x_p = (\lambda_1 t + \lambda_0) \sin t + (\beta_1 t + \beta_0) \cos t$

$x_p' = \lambda_1 \sin t + (\lambda_1 t + \lambda_0) \cos t + \beta_1 \cos t + (\beta_1 t + \beta_0)(-\sin t)$

$x_p'' = \sin t (\lambda_1 - \beta_1 t - \beta_0) + \cos t (\lambda_1 t + \lambda_0 + \beta_1)$

$\bullet \sin t(\lambda_1 - \beta_1 t - \beta_0) + \cos t(\lambda_1 t + \lambda_0 + \beta_1) - (\lambda_1 t + \lambda_0)\sin t - (\beta_1 t + \beta_0)\cos t =$
$= 2t \sin t$

$\Leftrightarrow \begin{cases} \lambda_1 - \beta_1 t - \beta_0 - \lambda_1 t - \lambda_0 = 2t \\ \lambda_1 t + \lambda_0 + \beta_1 - \beta_1 t - \beta_0 = 0 \end{cases} \Leftrightarrow \begin{cases} -\beta_1 - \lambda_1 = 2 \\ \lambda_1 - \beta_0 - \lambda_0 = 0 \\ \lambda_1 - \beta_1 = 0 \\ \lambda_0 + \beta_1 - \beta_0 = 0 \end{cases}$

$\Leftrightarrow \begin{cases} -\beta_1 - \lambda_1 = 2 \\ \lambda_1 - \beta_1 = 0 \end{cases}$ "+"

$-2\beta_1 = 2 \Rightarrow \boxed{\beta_1 = -1} \Leftrightarrow \boxed{\lambda_1 = -1}$

$\Leftrightarrow \begin{cases} \lambda_1 - \beta_0 - \lambda_0 = 0 \\ \lambda_0 + \beta_1 - \beta_0 = 0 \end{cases}$ "+"

$\lambda_1 + \beta_1 - 2\beta_0 = 0 \Leftrightarrow -2\beta_0 = 2 \Rightarrow \boxed{\beta_0 = -1} \Leftrightarrow \boxed{\lambda_0 = 0}$

$\boxed{x_p = -t \sin t + (-t-1)\cos t}$

$x(t) = x_0 + x_p = c_1 e^t + c_2 e^{-t} + (-t)\sin t + (-t-1)\cos t$

f) $x'' + 4x = e^{2t} \sin 2t$, $\alpha = 2$, $\beta = 2$

$\bullet\ x'' + 4x = 0$

$\left.\begin{array}{l} x = e^{nt} \\ x' = n e^{nt} \\ x'' = n^2 e^{nt} \end{array}\right\} \Leftrightarrow n^2 e^{nt} + 4e^{nt} = 0 \mid : e^{nt}$

$n^2 + 4 = 0 \Leftrightarrow n^2 = -4 \Leftrightarrow n_{1,2} = \pm 2i$

$\bullet\ e^{0t} \sin 2t,\ e^{0t} \cos 2t$ - sist. fund. de sol.

$\boxed{x_0 = c_1 \sin 2t + c_2 \cos 2t}$

$\bullet\ x_p = e^{2t} \lambda_0 \sin 2t + e^{2t} \beta_0 \cos 2t$

$x_p' = 2e^{T} \lambda_0 \sin 2t + e^{2t} \lambda_0 \cdot 2\cos 2t + 2e^{2t} \beta_0 \cos 2t + e^{2t} \beta_0 (-2)\sin 2t$

$= e^{2t} \sin 2t(2\lambda_0 - 2\beta_0) + e^{2t} \cos 2t (2\lambda_0 + 2\beta_0)$

$x_p'' = 2e^{2t} \sin 2t(2\lambda_0 - 2\beta_0) + e^{2t} \cdot 2\cos 2t(2\lambda_0 - 2\beta_0) + 2e^{2t} \cos 2t(2\lambda_0 + 2\beta_0) +$
$+ e^{2t}(-2)\sin 2t(2\lambda_0 + 2\beta_0)$

$= e^{2t} \sin 2t(-8\beta_0) + e^{2t} \cos 2t(8\lambda_0)$

$\cdot\ e^{2t}\sin 2t(-8\beta_0) + e^{2t}\cos 2t(8\lambda_0) + 4e^{2t}\lambda_0\sin 2t + 4e^{2t}\beta_0\cos 2t =$

$= e^{2t}\sin 2t \quad |: e^{2t}$

$$\Leftrightarrow \begin{cases} -8\beta_0 + 4\lambda_0 = 1 \\ 8\lambda_0 + 4\beta_0 = 0 \quad |\cdot 2 \end{cases} \Leftrightarrow \begin{cases} -8\beta_0 + 4\lambda_0 = 1 \\ \underline{8\beta_0 + 16\lambda_0 = 0} \quad \text{,,+''} \end{cases}$$

$$20\lambda_0 = 1 \Rightarrow \boxed{\lambda_0 = \frac{1}{20}} \Leftrightarrow \boxed{\beta_0 = \frac{-1}{10}}$$

$$\boxed{x_p = \frac{1}{20}e^{2t}\sin 2t - \frac{1}{10}e^{2t}\cos 2t}$$

$$\underline{x(t) = x_0 + x_p = c_1\sin 2t + c_2\cos 2t + \frac{1}{20}e^{2t}\sin 2t - \frac{1}{10}e^{2t}\cos 2t}$$