Ecuații diferențiale și cu derivate parțiale
Laborator 01
07.10.2020

1. Să se calculeze următoarele primitive:

1) $\int (x^2+3x-2)\,dx$

2) $\int x^2 \ln x \, dx$

3) $\int \frac{3x^2+4}{x^3+4x}\,dx$

Rezolvare:

1) $\int (x^2+3x-2)\,dx = \int x^2 dx + 3\int x\,dx - 2\int dx = \frac{x^3}{3} + 3\cdot\frac{x^2}{2} - 2x + \mathcal{C}$

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + \mathcal{C}}$$

2) $\int \underbrace{x^2}_{g'} \underbrace{\ln x}_{f} \, dx = \frac{x^3}{3}\ln x - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3} + \mathcal{C}$

$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + \mathcal{C}$

$f' = \frac{1}{x}, \; g = \frac{x^3}{3}$

$$\boxed{\int f\cdot g'\,dx = f\cdot g - \int f'\cdot g\,dx}$$

3) $\int \frac{3x^2+4}{x^3+4x}\,dx = \int \frac{u'}{u}\,dx = \ln|u| = \ln(x^3+4x) + \mathcal{C}$

$u(x) = x^3+4x$

$u'(x) = 3x^2+4$

2. Să se determine primitiva $F: \mathbb{R} \to \mathbb{R}$ a funcției $f: \mathbb{R}\to\mathbb{R}$, $f(x) = \frac{x}{x^2+1}$ cu proprietatea că $F(0) = 3$.

$F(x) = \int f(x)\,dx$

$\frac{1}{2}\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{u'}{u}\,dx = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(x^2+1) + \mathcal{C}$

$u(x) = x^2+1$

$u'(x) = 2x$

$F(0) = 3 = \frac{1}{2}\ln 1 + \mathcal{C} \Leftrightarrow \mathcal{C} = 3 \Rightarrow F(x) = \frac{1}{2}\ln(x^2+1) + 3$

3. Să se determine primitivele următoarelor funcții, care îndeplinesc condițiile precizate:

a) $\int \ln x \, dx$, $F(1)=3$; c) $\int \sin^2 x \, dx$, $F(0)=1$

b) $\int \frac{5}{x^2+3x+2} dx$, $F(0)=5$; d) $\int \frac{3}{x^2+4x+5} dx$, $F(-2)=2$

Rezolvare

a) $\int \underset{g'}{1} \cdot \underset{f}{\ln x} \, dx = x \ln x - \int \frac{1}{x} \cdot x \, dx = x \ln x - \int dx = \underbrace{x \ln x - x + \mathcal{C}}_{F(x)}$

$f' = \frac{1}{x}$, $g = x$

$F(1) = 1 \cdot \ln 1 - 1 + C = -1 + C = 3 \Rightarrow C = 4 \Leftrightarrow F(x) = x \ln x - x + 4$

b) $\int \frac{5}{x^2+3x+2} dx = 5 \cdot \int \frac{1}{x^2+3x+2} dx$

$$\boxed{ax^2+bx+c=0,\ x_1, x_2 \text{ răd.} \Rightarrow ax^2+bx+c = a(x-x_1)(x-x_2)}$$

$x^2+3x+2=0$

$D = 9-8 = 1 \Rightarrow x_{1,2} = \frac{-3 \pm 1}{2} \begin{cases} -1 \\ -2 \end{cases} \Rightarrow x^2+3x+2 = (x+1)(x+2)$

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{Ax+2A+Bx+B}{(x+1)(x+2)} = \frac{(A+B)x+2A+B}{(x+1)(x+2)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ 2A+B=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \end{cases}$$

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\int \frac{1}{(x+1)(x+2)} dx = \int \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx = \underbrace{5\left(\ln|x+1| - \ln|x+2| + \mathcal{C}\right)}_{F(x)}$$

$F(0) = 5 = 5(\ln 1 - \ln 2) + \mathcal{C} = -5\ln 2 + \mathcal{C} \Rightarrow C = 5 + 5\ln 2$

$F(x) = 5(\ln|x+1| - \ln|x+2|) + 5 + 5\ln 2.$

$\underline{M_1}$

c) $\underset{I}{\int \sin^2 dx} = \int \underset{f}{\sin x} \cdot \underset{g'}{\sin x} \, dx = -\sin x \cos x + \int \cos^2 x \, dx$

$= -\sin x \cos x + \int (1 - \sin^2 x) \, dx$

$= -\sin x \cos x - \int \sin^2 x \, dx + x$

$f' = \cos x$

$g = -\cos x$

continuare c)

$2I = -\sin x + \cos x + x$

$I = \dfrac{-\sin x + \cos x + x}{2} + \mathscr{C}$

$F(x) = \dfrac{-\sin x + \cos x + x}{2} + 1$

$F(0) = 1$

$F(0) = \dfrac{-\sin 0 + \cos x + 0}{2} + \mathscr{C} = 1 \Rightarrow$

$\Rightarrow C = 1$

M2

$\cos 2x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \dfrac{1-\cos 2x}{2}$

$I = \int \left(\frac{1}{2} - \frac{\cos 2x}{2}\right) dx = \int \frac{1}{2} dx - \frac{1}{2}\int \cos 2x \, dx = \frac{1}{2}x - \frac{1}{2}\cdot\frac{\sin 2x}{2} + \mathscr{C}$

$= \dfrac{x}{2} - \dfrac{\sin 2x}{4} + \mathscr{C}$

d) $\int \dfrac{3}{x^2+4x+5} dx = 3\int \dfrac{1}{x^2+4x+5} dx$

$x^2+4x+5=0$

$D = 16-20 = -4 < 0 \Rightarrow x^2+4x+5 = (x+2)^2+1$

$3\int \dfrac{1}{(x+2)^2+1} dx = 3\,\mathrm{arc\,tg}(x+2) + \mathscr{C}$

$\int \dfrac{1}{x^2+a^2} dx = \dfrac{1}{a} \mathrm{arctg}\, x + \mathscr{C}$

$F(-2) = 2 = 3\,\mathrm{arctg}\, 0 + \mathscr{C} \Leftrightarrow \boxed{C = 2}$

$F(x) = 3\,\mathrm{arctg}(x+2) + 2.$