

# Ecuații diferențiale și cu derivate parțiale

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25.11.2020

Să se rezolve următoarele ecuații diferențiale liniare de ordin 3 omogene cu coeficienți constanți.

I. a)  $x''' - 3x'' + 2x' = 0$ ,  $x(0) = 1$ ,  $x'(0) = 2$ ,  $x''(0) = 0$

b)  $2x''' - 3x'' + x' = 0$ ,  $x(0) = -1$ ,  $x'(0) = 2$ ,  $x''(0) = 1$

c)  $x''' - 7x'' + 14x' - 8x = 0$ ,  $x(0) = 1$ ,  $x'(0) = 0$ ,  $x''(0) = 1$

d)  $x'' - 4x' + 3x = 0$ ,  $x(0) = 2$ ,  $x'(0) = 4$

e)  $x''' - x' = 0$

### Rezolvare

a)  $x''' - 3x'' + 2x' = 0$ ,  $x(0) = 1$ ,  $x'(0) = 2$ ,  $x''(0) = 0$

$$\left. \begin{array}{l} x = e^{nt} \\ x' = ne^{nt} \\ x'' = n^2 e^{nt} \\ x''' = n^3 e^{nt} \end{array} \right\} \Rightarrow n^3 e^{nt} - 3n^2 e^{nt} + 2n e^{nt} = 0 \quad | \cdot e^{-nt}$$

$$n^3 - 3n^2 + 2n = 0$$

$$n(n^2 - 3n + 2) = 0$$

$$n_1 = 0$$

$$n^2 - 3n + 2 = 0$$

$$\Delta = 9 - 8 = 1 \Rightarrow n_{2,3} = \frac{3 \pm 1}{2} \begin{cases} n_2 = 2 \\ n_3 = 1 \end{cases}$$

$e^{0 \cdot t}, e^{2 \cdot t}, e^{1 \cdot t}$  - sist. fundam. de sol.

$$\bullet x(t) = C_1 + C_2 e^{2t} + C_3 e^t \quad \bullet x'(t) = C_2 + 2e^{2t} + C_3 e^t$$

$$\boxed{x(0) = C_1 + C_2 + C_3 = 1}$$

$$x'(0) = \boxed{2C_2 + C_3 = 2}$$

$$\bullet x''(t) = 2C_2 + 2e^{2t} + C_3 e^t$$

$$x''(0) = \boxed{4C_2 + C_3 = 0}$$

$$\begin{cases} C_1 + C_2 + C_3 = 7 \\ 2C_2 + C_3 = 2 \\ 4C_2 + C_3 = 0 \end{cases} \Rightarrow \begin{cases} 2C_2 = -2 \Rightarrow C_2 = -1 \\ C_3 = 4 \end{cases}$$

$$C_1 = 7 - C_2 - C_3 = 7 + 1 - 4 = 4 \Rightarrow C_1 = 4$$

$$\underline{x_{PC} = -2e^{2t} + 4e^t}$$

d)  $x'' - 4x' + 3x = 0$ ,  $x(0) = 2$ ,  $x'(0) = 4$

$$\begin{cases} x = e^{nt} \\ x' = ne^{nt} \\ x'' = n^2 e^{nt} \end{cases} \Rightarrow n^2 e^{nt} - 4ne^{nt} + 3e^{nt} = 0 \quad | : e^{nt}$$

$$n^2 - 4n + 3 = 0$$

$$\Delta = 16 - 12 = 4 \Rightarrow n_{1,2} = \frac{4 \pm 2}{2} \begin{cases} n_1 = 3 \\ n_2 = 1 \end{cases}$$

$e^{3t}, e^{1t}$  - int. fondam. de sol.

$$x(t) = C_1 e^{3t} + C_2 e^t \Rightarrow x(0) = C_1 + C_2 = 2$$

$$x'(t) = C_1 \cdot 3e^{3t} + C_2 e^t \Rightarrow x'(0) = 3C_1 + C_2 = 4$$

$$\begin{cases} C_1 + C_2 = 2 \\ 3C_1 + C_2 = 4 \end{cases}$$

$$2C_1 = 2 \Rightarrow C_1 = 1 \Rightarrow \underline{x_{PC} = e^{3t} + e^t}$$

$$C_2 = 1$$

II. a)  $x'' + 2x' + x = 0$ ,  $x(0) = 1$ ,  $x'(0) = 3$

b)  $x^{IV} - 5x'' + 4x = 0$

c)  $x''' - 6x'' + 12x' - 8x = 0$

d)  $x^{IV} - 2x'' + x = 0$

e)  $x^{IV} + 2x''' + x'' = 0$

f)  $x^{(6)} - x^{(5)} - 4x^{(4)} + 2x''' + 5x'' - x' - 2x = 0$

g)  $x^{(7)} + 3x^{(6)} + 3x^{(5)} + x^{(4)} = 0$

II a)  $x'' + 2x' + x = 0$ ,  $x(0) = 7$ ,  $x'(0) = 3$

$$\left. \begin{array}{l} x = e^{\lambda t} \\ x' = \lambda e^{\lambda t} \\ x'' = \lambda^2 e^{\lambda t} \end{array} \right\} \Rightarrow \lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + e^{\lambda t} = 0 \quad | : e^{\lambda t}$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)^2 = 0$$

$$\underline{\lambda_1 = \lambda_2 = -1}$$

$e^{-t}$ ,  $t e^{-t}$  - int. fundam. de sol

$$x(t) = C_1 e^{-t} + C_2 t e^{-t} \Rightarrow x(0) = \boxed{C_1 = 7}$$

$$x'(t) = C_1(-e^{-t}) + C_2(e^{-t} + t(-e^{-t})) \Rightarrow x'(0) = \boxed{-C_1 + C_2 = 3}$$

$$C_2 = 3 + C_1 = 3 + 7 = 10 \Rightarrow \boxed{C_2 = 10}$$

$$\underline{x_{PC} = e^{-t} + 10t e^{-t}}$$

II b)  $x^{(6)} - x^{(5)} - 4x^{(4)} + 2x^{(3)} + 5x'' - x' - 2x = 0$

$$\left. \begin{array}{l} x = e^{\lambda t} \\ x' = \lambda e^{\lambda t} \\ x'' = \lambda^2 e^{\lambda t} \\ x^{(3)} = \lambda^3 e^{\lambda t} \\ x^{(4)} = \lambda^4 e^{\lambda t} \\ x^{(5)} = \lambda^5 e^{\lambda t} \\ x^{(6)} = \lambda^6 e^{\lambda t} \end{array} \right\} \Rightarrow \lambda^6 e^{\lambda t} - \lambda^5 e^{\lambda t} - 4\lambda^4 e^{\lambda t} + 2\lambda^3 e^{\lambda t} + 5\lambda^2 e^{\lambda t} - \lambda e^{\lambda t} - 2e^{\lambda t} = 0 \quad | : e^{\lambda t}$$

$$\lambda^6 - \lambda^5 - 4\lambda^4 + 2\lambda^3 + 5\lambda^2 - \lambda - 2 = 0$$

$$\begin{array}{r|l} \lambda^6 - \lambda^5 - 4\lambda^4 + 2\lambda^3 + 5\lambda^2 - \lambda - 2 & \lambda - 1 \\ \underline{-\lambda^6 + \lambda^5} & \lambda^5 - 4\lambda^3 - 2\lambda^2 + 3\lambda + 2 \quad | \lambda - 1 \\ 1 \quad 1 & \underline{-\lambda^5 + \lambda^4} \\ & \lambda^4 - 4\lambda^3 - 2\lambda^2 + 3\lambda + 2 \\ & \underline{+\lambda^4 - 4\lambda^3} \\ & 1 \quad -2\lambda^3 + 5\lambda^2 - \lambda - 2 \\ & \underline{+2\lambda^3 - 2\lambda^2} \\ & 1 \quad 3\lambda^2 - \lambda - 2 \\ & \underline{-3\lambda^2 + 3\lambda} \\ & 1 \quad 2\lambda - 2 \\ & \underline{-2\lambda + 2} \\ & 1 \quad 1 \end{array}$$

$$\begin{array}{r}
 n^4 + n^3 - 3n^2 - 5n - 2 \quad | \quad n-2 \\
 \underline{-4n^4 + 2n^3} \phantom{-3n^2 - 5n - 2} \\
 3n^3 - 3n^2 - 5n - 2 \\
 \underline{-3n^3 + 6n^2} \phantom{-5n - 2} \\
 3n^2 - 5n - 2 \\
 \underline{-3n^2 + 6n} \phantom{-2} \\
 n - 2 \\
 \underline{-n + 2} \\
 1 \quad 1
 \end{array}$$

$$(n-1)^2 (n-2)(n+1)^3 = 0$$

$$n_1 = n_2 = 1$$

$$n_3 = 2$$

$$n_4 = n_5 = n_6 = -1$$

$$\left. \begin{array}{l} n_1 = n_2 = 1 \\ n_3 = 2 \\ n_4 = n_5 = n_6 = -1 \end{array} \right\} \Rightarrow e^{1t}, t e^{1t}, e^{2t}, e^{-1t}, t e^{-1t}, t^2 e^{-1t}$$

-int. fond de sol.

$$x(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t} + c_4 e^{-t} + c_5 t e^{-t} + c_6 t^2 e^{-t}$$

$$\text{III a) } x'' + x = 0, x(0) = 3, x'(0) = 5$$

$$\boxed{\text{b)}} x^{IV} + 4x = 0$$

$$\text{c) } x^{IV} + 8x'' + 76x = 0$$

$$\boxed{\text{d)}} x'' + 4x' + 13x = 0$$

$$\boxed{\text{e)}} x'' + 4x' + 5x = 0$$

Recherche

$$\text{a) } x'' + x = 0, x(0) = 3, x'(0) = 5$$

$$x = e^{nt} \quad \Rightarrow n^2 e^{nt} + e^{nt} = 0 \quad | : e^{nt}$$

$$x' = n e^{nt} \quad n^2 + 1 = 0$$

$$x'' = n^2 e^{nt} \quad n^2 = -1$$

$$n = \pm i \quad (\alpha \pm i\beta, \alpha=0, \beta=1)$$

$$e^{0t} \cos t, e^{0t} \sin t \text{ - int. fondem. de sol.}$$

$$x(t) = c_1 \cos t + c_2 \sin t \quad (\Rightarrow x(0) = c_1 \cos 0 + c_2 \sin 0 = \boxed{c_1 = 3})$$

$$x'(t) = c_1 (-\sin t) + c_2 \cos t \quad (\Rightarrow x'(0) = c_1 (-\sin 0) + c_2 \cos 0 = \boxed{c_2 = 5})$$

$$x_{\text{PC}} = 3 \cos t + 5 \sin t$$

$$\text{III c) } x^{IV} + 8x'' + 76x = 0$$

$$\left. \begin{array}{l} x = e^{nt} \\ x' = n e^{nt} \\ x'' = n^2 e^{nt} \\ x''' = n^3 e^{nt} \\ x^{IV} = n^4 e^{nt} \end{array} \right\} \Rightarrow n^4 e^{nt} + 8n^2 e^{nt} + 76 e^{nt} = 0 \quad | : e^{nt}$$

$$n^4 + 8n^2 + 76 = 0$$

$$\text{Not } n^2 = p \Rightarrow p^2 + 8p + 76 = 0$$

$$(p+4)^2 = 0$$

$$\Rightarrow p = -4 \text{ (de 2 ori)}$$

$$n^2 = -4 \text{ (de 2 ori)}$$

$$n = \pm 2i \text{ (de 2 ori)} \quad (\alpha \pm i\beta, \alpha=0, \beta=2)$$

$$e^{0 \cdot t} \cos 2t, e^{0t} \sin 2t, t e^{0t} \cos 2t, t e^{0t} \sin 2t$$

- int. fund. de sol.

$$x(t) = C_1 \cos 2t + C_2 \sin 2t + C_3 t \cos 2t + C_4 t \sin 2t$$

$$\text{IV d) } x^{IV} + 2x''' + 4x'' - 2x' - 5x = 0$$

$$\boxed{b)} x''' - 3x'' + 9x' + 73x = 0$$

$$c) x^{(7)} - x^{(6)} + x^{(5)} - x^{(4)} = 0$$

$$\boxed{d)} x''' - 5x'' + 17x' - 73x = 0$$

$$\boxed{e)} x^V + 4x^{IV} + 3x''' - 6x' - 2x = 0$$

Rezolvare

$$a) x^{IV} + 2x''' + 4x'' - 2x' - 5x = 0$$

$$\left. \begin{array}{l} x = e^{nt} \\ x' = n e^{nt} \\ x'' = n^2 e^{nt} \\ x''' = n^3 e^{nt} \\ x^{IV} = n^4 e^{nt} \end{array} \right\} \Rightarrow n^4 e^{nt} + 2n^3 e^{nt} + 4n^2 e^{nt} - 2n e^{nt} - 5 e^{nt} = 0 \quad | : e^{nt}$$

$$n^4 + 2n^3 + 4n^2 - 2n - 5 = 0.$$

$$\begin{array}{r}
 n^4 + 2n^3 + 4n^2 - 2n - 5 \quad | \quad (n-1) \\
 \underline{4n^4 + n^3} \\
 3n^3 + 4n^2 - 2n - 5 \\
 \underline{-3n^3 + 3n^2} \\
 7n^2 - 2n - 5 \\
 \underline{-7n^2 + 7n} \\
 5n - 5 \\
 \underline{-5n + 5} \\
 1
 \end{array}
 \quad
 \begin{array}{r}
 n^3 + 3n^2 + 7n + 5 \quad | \quad (n+1) \\
 \underline{-n^3 - n^2} \\
 2n^2 + 7n + 5 \\
 \underline{-2n^2 - 2n} \\
 5n + 5 \\
 \underline{-5n - 5} \\
 1
 \end{array}$$

$$\begin{aligned}
 & (n-1)(n+1)(n^2+2n+5) = 0 \\
 & n_1 = 1 \\
 & n_2 = -1 \\
 & \left\{ \begin{array}{l} n^2 + 2n + 5 = 0 \\ D = 4 - 20 = -16 \\ n_{1,2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i \\ (\alpha \pm i\beta, \alpha = -1, \beta = 2) \end{array} \right.
 \end{aligned}$$

$e^{1t}, e^{-1t}, e^{-1t} \cos 2t, e^{-1t} \sin 2t$  - int. fond de sol.

$$x(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{-t} \cos t + c_4 e^{-t} \sin t.$$

$$c) x^{(7)} - x^{(6)} + x^{(5)} - x^{(4)} = 0$$

$$\begin{array}{l}
 x = e^{nt} \\
 x' = n e^{nt} \\
 x'' = n^2 e^{nt} \\
 x^{(3)} = n^3 e^{nt} \\
 x^{(4)} = n^4 e^{nt} \\
 x^{(5)} = n^5 e^{nt} \\
 x^{(6)} = n^6 e^{nt} \\
 x^{(7)} = n^7 e^{nt}
 \end{array}
 \quad
 \Rightarrow
 \begin{array}{l}
 n^7 e^{nt} - n^6 e^{nt} + n^5 e^{nt} - n^4 e^{nt} = 0 \quad | : e^{nt} \\
 n^7 - n^6 + n^5 - n^4 = 0 \\
 n^4 (n^3 - n^2 + n - 1) = 0 \\
 n^4 (n-1)(n^2+1) = 0 \\
 n_1 = n_2 = n_3 = n_4 = 0 \\
 n-1 = 0 \Rightarrow n_5 = 1 \\
 n^2+1 = 0 \Rightarrow n^2 = -1 \Rightarrow n_{6,7} = \pm i \quad (\alpha \pm i\beta, \alpha = 0, \beta = 1)
 \end{array}$$

$e^{0t}, t e^{0t}, t^2 e^{0t}, t^3 e^{0t}, e^{1t}, e^{0t} \cos t, e^{0t} \sin t$   
- int. fond de sol.

$$x(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t + c_6 \cos t + c_7 \sin t$$