Seminar14

Exerciţii

$$d)\begin{cases} x' &= x \\ y' &= y \\ z' &= -2xy \end{cases}, \ x \neq 0, y \neq 0$$

$$e)\begin{cases} x' &= y \\ y' &= x \\ z' &= x - y \end{cases}, \ y \neq x \neq 0$$

Rezolvare

Exerciţiu d)

$$\begin{vmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_1}{\partial z} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} & \frac{\partial \psi_2}{\partial z} \\ \frac{\partial \psi_3}{\partial x} & \frac{\partial \psi_3}{\partial y} & \frac{\partial \psi_3}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} & 0 \\ \frac{1}{e^t} & 0 & 0 \\ y & x & 1 \end{vmatrix} = \frac{x}{e^t y^2} \neq 0 \Rightarrow \psi_1, \psi_2, \psi_3 \text{ - indep.}$$

Sol. implicită:

$$\begin{cases} \frac{x}{y} = c_1 \\ \frac{x}{e^t} = c_2 \\ xy + z = c_3 \end{cases} \quad \begin{matrix} \Rightarrow y = \frac{x}{c_1} \Rightarrow y = \frac{c_2 e^t}{c_1} \\ \Rightarrow x = c_2 e^t \\ \Rightarrow z = c_3 - xy = c_3 - c_2 e^t \cdot \frac{c_2 e^t}{c_1} \end{matrix}$$

Sol. explicită:

$$(\Leftrightarrow) \begin{cases} x = c_2 e^t \\ y = \frac{c_2}{c_1} e^t \\ z = c_3 - \frac{c_2^2 e^{2t}}{c_1} \end{cases}$$

d) $\begin{cases} x' = x \\ y' = y \\ z' = -2xy \end{cases}$ $x, y \neq 0$ $\Rightarrow \begin{cases} \frac{dx}{dt} = x \\ \frac{dy}{dt} = y \\ \frac{dz}{dt} = -2xy \end{cases} \Rightarrow \begin{cases} \frac{dx}{x} = dt \\ \frac{dy}{y} = dt \\ \frac{dz}{-2xy} = dt \end{cases} \Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{-2xy} = dt$

$\frac{dx}{x} = \frac{dy}{y}$

$\int \frac{1}{x} dx = \int \frac{1}{y} dy$

$\ln|x| = \ln|y| + C_1$

$\ln|x| - \ln|y| = C_1$

$\ln\left|\frac{x}{y}\right| = C_1$

$\boxed{\frac{x}{y} = C_1 = \varphi_1(t, x, y, z)}$

$\frac{dx}{x} = dt$

$\int \frac{1}{x} dx = \int dt$

$\ln|x| = t + C_2$

$|x| = e^{t + C_2} = C_2 e^t$

$\boxed{\frac{x}{e^t} = C_2 = \varphi_2(t, x, y, z)}$

$\frac{y\,dx}{x} = \frac{x\,dy}{y} = \underbrace{\frac{y\,dx}{xy} = \frac{x\,dy}{xy}}_{} = \frac{y\,dx + x\,dy}{2xy} = \frac{d(xy)}{2xy} = \frac{dz}{-2xy}$

$\frac{d(xy)}{2xy} = \frac{dz}{-2xy} \quad | \cdot 2xy$

$d(xy) = -dz$

Notăm $xy = u$ $\Rightarrow du = -dz$

$\int du = -\int dz$

$u = -z + C_3$

$u + z = C_3$

$\boxed{xy + z = C_3 = \varphi_3(t, x, y, z)}$

Verificăm dacă $\varphi_1, \varphi_2, \varphi_3$ sunt indep. $\Leftrightarrow \frac{D(\varphi_1, \varphi_2, \varphi_3)}{D(x, y, z)} \neq 0$

Exerciţiu e)

e) $\begin{cases} x' = y \\ y' = x \\ z' = x - y \end{cases}$, $x \neq y \neq 0 \Leftrightarrow \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = x \\ \frac{dz}{dt} = x - y \end{cases} \Leftrightarrow \begin{cases} \frac{dx}{y} = dt \\ \frac{dy}{x} = dt \\ \frac{dz}{x - y} = dt \end{cases} \Rightarrow \frac{dx}{y} = \frac{dy}{x} = \frac{dz}{x - y} = dt$

$\frac{dx}{y} = \frac{dy}{x}$

$x\,dx = y\,dy$

$\int x\,dx = \int y\,dy$

$\frac{x^2}{2} = \frac{y^2}{2} + C_1$

$\boxed{x^2 - y^2 = C_1 = \varphi_1(t, x, y, z)}$

$\frac{dx}{y} = \frac{dy}{x} = \frac{dx - dy}{y - x} = \underbrace{\frac{d(x - y)}{y - x} = \frac{dz}{x - y}}_{}$

$\frac{d(x - y)}{y - x} = \frac{dz}{-(y - x)} \quad | \cdot (y - x)$

$d(x - y) = -dz$

Notăm $x - y = u$ $\Rightarrow du = -dz$

$\int du = -\int dz$

$u = -z + C_2$

$u + z = C_2$

$\boxed{x - y + z = C_2 = \varphi_2(t, x, y, z)}$

$x - y = C_2 - z$

$\frac{dz}{x - y} = \underbrace{\frac{dz}{C_2 - z} = dt}_{}$

$-\int \frac{-1}{C_2 - z} dz = \int dt$

$-\ln|C_2 - z| = t + C_3$

$\ln|C_2 - z| = -t + C_3$

$|C_2 - z| = e^{-t + C_3} = C_3 e^{-t}$

$\boxed{(x - y) \cdot e^t = C_3 = \varphi_3(t, x, y, z)}$

Verif. dc. $\varphi_1, \varphi_2, \varphi_3$ sunt indep.

$$\begin{vmatrix} \frac{\partial \varphi_1}{\partial x} & \frac{\partial \varphi_1}{\partial y} & \frac{\partial \varphi_1}{\partial z} \\ \frac{\partial \varphi_2}{\partial x} & \frac{\partial \varphi_2}{\partial y} & \frac{\partial \varphi_2}{\partial z} \\ \frac{\partial \varphi_3}{\partial x} & \frac{\partial \varphi_3}{\partial y} & \frac{\partial \varphi_3}{\partial z} \end{vmatrix} = \begin{vmatrix} 2x & -2y & 0 \\ 1 & -1 & 1 \\ e^t & -e^t & 0 \end{vmatrix} = -2y e^t + 2x e^t = 2e^t(\underbrace{x - y}_{\neq 0}) \neq 0$$

$\Rightarrow \varphi_1, \varphi_2, \varphi_3$ indep.

Sol. implicită:

$\begin{cases} x^2 - y^2 = C_1 \\ x - y + z = C_2 \\ (x - y) e^t = C_3 \end{cases}$

$\Rightarrow (x - y)(x + y) = C_1 \Rightarrow x + y = \frac{C_1 e^t}{C_3}$

$\Rightarrow z = C_2 - \frac{C_3}{e^t}$

$\Rightarrow x - y = \frac{C_3}{e^t}$

$\Rightarrow \begin{cases} x + y = \frac{C_1 e^t}{C_3} \\ x - y = \frac{C_3}{e^t} \end{cases}$

$2x = \frac{C_1 e^t}{C_3} + \frac{C_3}{e^t} = \frac{C_1 e^{2t} + C_3^2}{C_3 e^t}$

$x = \frac{C_1 e^{2t} + C_3^2}{2 C_3 e^t}$

$y = \frac{C_1 e^{2t} + C_3^2}{2 C_3 e^t} - \frac{C_3}{e^t} = \frac{C_1 e^{2t} - C_3^2}{2 C_3 e^t}$

Sol. explicită:

$x = \frac{C_1 e^{2t} + C_3^2}{2 C_3 e^t}$, $y = \frac{C_1 e^{2t} - C_3^2}{2 C_3 e^t}$, $z = C_2 - \frac{C_3}{e^t}$