Laborator03

Enunţuri

1. Să se rezolve următoarele ecuaţii liniare scalare:

$$a) \begin{cases} \frac{dx}{dt} = (t^2 + 1) \cdot x, & t, x \in R_+ \\ x(0) = 2 \end{cases}$$

$$b) \begin{cases} \frac{dx}{dt} = \sqrt{t + 1} \cdot x, & t \geq -1 \\ x(0) = 2 & x \in R_+ \end{cases}$$

$$c) \begin{cases} \frac{dx}{dt} = t^2 \cdot e^t \cdot x, & t \in R \\ x(0) = 3 & x \in R_+ \end{cases}$$

$$d) \begin{cases} \frac{dx}{dt} = \frac{t^2 + 1}{t - 1} \cdot x, & t \in R \setminus \{1\} \\ x(0) = 1 & x \in R_+ \end{cases}$$

2. Să se rezolve următoarele ecuaţii diferenţiale afine:

$$a) \ x' + \frac{1 - 2t}{t^2} \cdot x = 1$$

$$b) \begin{cases} t \cdot x' + x = t \cdot sint, & t > 0 \\ x(\pi) = 2 \end{cases}$$

$$c) \begin{cases} x' + 2 \cdot t \cdot x + t - e^{-t^2} = 0 \\ x(0) = 1 \end{cases}$$

$$d) \begin{cases} \frac{dx}{dt} = x - t^2 \\ x(1) = 2 \end{cases}$$

$$e) \begin{cases} x'(t) = \frac{1}{t} \cdot x - 1 \\ x(1) = 4 \end{cases}$$

3. Să se rezolve următoarele ecuaţii reductibile la ecuaţii de tip omogen:

$\quad a)\ \ (t^2 - t \cdot x + x^2)dt + (t \cdot x - 2t^2)dx = 0$

$\quad b)\ \ x' = \dfrac{2 \cdot t \cdot x}{3t^2 - x^2}$

$\quad c)\ \ x' = \dfrac{t \cdot x + x^2}{t^2}$

$\quad d)\ \ (t + 2x)dt - tdx = 0$

$\quad e)\ \ t \cdot x \cdot x' - x^2 + 3t^2 = 0$

4. Să se rezolve următoarele ecuaţii reductibile la ecuaţii de tip omogen:

$\quad a)\ \ (t - 2x + 5)dt + (2t - x + 4)dx = 0$

$\quad b)\ \ 2 \cdot (t + 4x - 6)dt = (7t + x - 15)dx$

$\quad c)\ \ (2t - 4x + 6)dt + (t + x - 3)dx = 0$

$\quad d)\ \ (3t + 3x - 1)dt + (t + x + 1)dx = 0$

$\quad e)\ \ (t - 2x + 1)dt + (2t - 4x + 3)dx = 0$

$\quad f)\ \ (t - x - 1) + x'(x - t + 2) = 0$

Rezolvare

Exerciţiu 01

c) - **Video**

d)

① d) $\begin{cases} \dfrac{dx}{dt} = \dfrac{t^2+1}{t-1} \cdot x & , t \in \mathbb{R}\setminus\{1\} \\ x(0)=1 & x \in \mathbb{R}_+ \end{cases}$

$\dfrac{dx}{dt} = \underbrace{\dfrac{t^2+1}{t-1}}_{A(t)} \cdot x$ (ec. lin. scalară)

$\dfrac{dx}{x} = \dfrac{t^2+1}{t-1} dt$

$\int \dfrac{1}{x} dx = \int \dfrac{t^2+1}{t-1} dt$

$\ln x = \dfrac{t^2}{2} + t + 2\ln|t-1| + C$

$x = e^{\frac{t^2}{2}+t+2\ln|t-1|+C} = e^{\frac{t^2}{2}+t} \cdot e^{\ln|t+1|^2} \cdot e^{C}$

$x(t) = C \cdot |t-1|^2 \cdot e^{\frac{t^2}{2}+t}$

$x(0) = C \cdot (0-1)^2 \cdot e^0 = \boxed{C=1} \Rightarrow x_{pc}(t) = (t-1)^2 \cdot e^{\frac{t^2}{2}+2}$

$\int \dfrac{t^2+1}{t-1} dt = \int \left(t+1+\dfrac{2}{t-1}\right) dt$

$\begin{array}{r} t^2 \quad +1 \\ -t^2+t \\ \hline t+1 \\ -t+1 \\ \hline 2 \end{array}$ $\begin{array}{l} \boxed{t-1} \to g \\ \boxed{t+1} \, C \end{array}$ $\dfrac{f}{g} = C + \dfrac{r}{g}$

Exerciţiu 02

b) - [Video](#)

② h) $\begin{cases} tx' + x = t\sin t & , t > 0 \\ x(\pi) = 2 \end{cases}$

$t \cdot \dfrac{dx}{dt} + x = t\sin t \quad | :t$

$\dfrac{dx}{dt} + \underbrace{\dfrac{1}{t}}_{A(t)} x = \underbrace{\sin t}_{B(t)}$ (ec. afină)

Etapa 1 $\dfrac{dx}{dt} + \dfrac{1}{t} x = 0$

$\dfrac{dx}{dt} = -\dfrac{1}{t} x$

$\dfrac{dx}{x} = -\dfrac{1}{t} dt \iff \int \dfrac{1}{x} dx = -\int \dfrac{1}{t} dt$

$\ln|x| = -\ln t + C$

$\ln|x| = -\ln t + \ln C = \ln \dfrac{C}{t}$

$\boxed{x_0 = \dfrac{C}{t}}$

Etapa 2 $\quad x_0 = \dfrac{C(t)}{t}$

$\left(\dfrac{C(t)}{t}\right)' + \dfrac{1}{t} \cdot \dfrac{C(t)}{t} = \sin t$

$\dfrac{C'(t)\cdot t - C(t)}{t^2} + \dfrac{C(t)}{t^2} = \sin t$

$\dfrac{C'(t)}{t} - \dfrac{C(t)}{t^2} + \dfrac{C(t)}{t^2} = \sin t$

$C'(t) = \underbrace{t\sin t}_{f(t)}$

$C(t) = \int \underbrace{t}_{f} \underbrace{\sin t}_{g'} dt = -t\cos t + \int \cos t \, dt$
$\quad f'=1 \quad g = -\cos t \qquad = -t\cos t + \sin t + C_1$

$\Rightarrow \boxed{x_0 = \dfrac{-t\cos t + \sin t + C_1}{t}}$

$\Rightarrow X(t) = x_0 + x_0 = \dfrac{-t\cos t + \sin t + C_1}{t}$

$x(\pi) = \dfrac{\pi + C}{\pi} = 2 \Rightarrow \boxed{C = \pi}$

$x_{pc} = \dfrac{-t\cos t + \sin t + \pi}{t}$

c)

② c) $\begin{cases} x' + 2tx + t - e^{-t^2} = 0 \\ x(0) = 1 \end{cases}$

$\dfrac{dx}{dt} + \underbrace{2t}_{A(t)} x = \underbrace{e^{-t^2} - t}_{B(t)}$

Etapa 1

$\dfrac{dx}{dt} + 2tx = 0$

$\dfrac{dx}{dt} = -2tx$

$\dfrac{dx}{x} = -2t\,dt$

$\int \dfrac{1}{x}\,dx = -\int 2t\,dt$

$\ln|x| = -t^2 + C$

$x = e^{-t^2+C} \Rightarrow \boxed{x_0 = C \cdot e^{-t^2}}$

Etapa 2

$x_0 = C(t) \cdot e^{-t^2}$

$\left(C(t) \cdot e^{-t^2}\right)' + 2t \cdot C(t) e^{-t^2} = e^{-t^2} - t$

$C'(t) \cdot e^{-t^2} + C(t) \cdot e^{-t^2} \cdot (-2t) + 2t C(t) e^{-t^2} = e^{-t^2} - t$

$C'(t) = 1 - t e^{t^2}$

$C(t) = \int \left(1 - t e^{t^2}\right) dt = \int 1\,dt - \frac{1}{2}\int 2t\, e^{t^2} dt$

$\qquad = t - \frac{1}{2} e^{t^2} + C_1$

$\begin{aligned} u(t) &= t^2 \\ u'(t) &= 2t \end{aligned}$

$\Rightarrow x_0 = \left(t - \frac{1}{2} e^{t^2} + C_1\right) \cdot e^{-t^2} = \boxed{t e^{-t^2} - \frac{1}{2} + C_1 e^{-t^2}}$

$x(t) = x_0 + x_0 = t e^{-t^2} - \frac{1}{2} + C e^{-t^2}$

$x(0) = -\frac{1}{2} + C = 1 \Rightarrow C = 1 + \frac{1}{2} = \frac{3}{2}$

$x_{pc} = t e^{-t^2} - \frac{1}{2} + \frac{3}{2} e^{-t^2}$

Exerciţiu 03

b) - [Video](#)

③ h) $x' = \dfrac{2tx}{3t^2 - x^2}$

$x' = \dfrac{\frac{2tx}{t^2}}{\frac{3t^2}{t^2} - \frac{x^2}{t^2}} = \dfrac{2 \cdot \frac{x}{t}}{3 - \left(\frac{x}{t}\right)^2} = f\left(\frac{x}{t}\right)$ (ec. de tip omogen)

$\boxed{u = \dfrac{x}{t}} \Rightarrow x = u \cdot t \Rightarrow x' = u' \cdot t + u$

$u't + u = \dfrac{2u}{3 - u^2}$

$u't = \dfrac{2u}{3 - u^2} - u = \dfrac{2u - 3u + u^3}{3 - u^2} = \dfrac{u^3 - u}{3 - u^2}$

$\dfrac{du}{dt} \cdot t = \dfrac{u^3 - u}{3 - u^2}$

$\dfrac{3 - u^2}{u^3 - u} du = \dfrac{1}{t} dt$

$\int \dfrac{3 - u^2}{u^3 - u} du = \int \dfrac{1}{t} dt$

$\int \dfrac{3 - u^2}{u^3 - u} du = \int \dfrac{3 - u^2}{u(u^2 - 1)} du = \int \dfrac{3 - u^2}{u(u-1)(u+1)} du$

$\dfrac{3 - u^2}{u(u-1)(u+1)} = \dfrac{A}{u} + \dfrac{B}{u-1} + \dfrac{C}{u+1} \quad \Big| \cdot u \quad \Big| \cdot (u-1) /_{u+1}$

$\dfrac{3 - u^2}{(u-1)(u+1)} = A + \dfrac{Bu}{u-1} + \dfrac{Cu}{u+1} \quad (u = 0 \Rightarrow A = -3)$

$\dfrac{3 - u^2}{u(u+1)} = \dfrac{A(u-1)}{u} + B + \dfrac{C(u-1)}{u+1} \quad (u = 1 \Rightarrow B = 1)$

$\dfrac{3 - u^2}{u(u-1)} = \dfrac{A(u+1)}{u} + \dfrac{B(u+1)}{u-1} + C \quad (u = -1 \Rightarrow C = 1)$

$\int \dfrac{3 - u^2}{u^3 - u} du = \int \left(\dfrac{-3}{u} + \dfrac{1}{u-1} + \dfrac{1}{u+1}\right) du = -3\ln|u| + \ln|u-1| + \ln|u+1|$

$\Rightarrow \ln\left|\dfrac{u^2 - 1}{u^3}\right| = \ln|t| + C = \ln|t| + \ln C \Rightarrow \dfrac{u^2 - 1}{u^3} = C \cdot |t|$

sol. în formă implicită $\boxed{\dfrac{\left(\frac{x}{t}\right)^2 - 1}{\left(\frac{x}{t}\right)^3} = C \cdot |t|}$

e)

3) e) $txx' - x^2 + 3t^2 = 0$ | $:tx$

$x' - \dfrac{x^2}{tx} + \dfrac{3t^2}{tx} = 0$

$x' - \dfrac{x}{t} + 3\dfrac{t}{x} = 0$

$x' = \dfrac{x}{t} - \dfrac{3}{\frac{x}{t}}$ — (ec. de tip omogen)

$\underbrace{\qquad\qquad}_{f\left(\frac{x}{t}\right)}$

$\boxed{u = \dfrac{x}{t}} \Rightarrow x = u\cdot t \Rightarrow x' = u't + u$

$u't + u = u - \dfrac{3}{u}$

$\dfrac{du}{dt}\, t = -\dfrac{3}{u}$ (ec. cu var. separabile)

$u\,du = \dfrac{-3}{t}\,dt$

$\displaystyle\int u\,du = -3\int \dfrac{1}{t}\,dt$

$\dfrac{u^2}{2} = 3\ln|t| + C$

$u^2 = -6\ln|t| + C$

$u = \pm\sqrt{-6\ln|t| + C}$; $-6\ln|t| + C \geq 0$

$\dfrac{x}{t} = \pm\sqrt{-6\ln|t| + C}$

$x_{u} = \pm t\sqrt{-6\ln|t| + C}.$

Exerciţiu 04

a)

4) a) $(t - 2x + 5)dt + (2t - x + 4)dx = 0$

$(2t - x + 4)dx = (-t + 2x - 5)dt$

$\dfrac{dx}{dt} = \dfrac{-t + 2x - 5}{2t - x + 4} = f\left(\dfrac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2}\right)$

$\Delta = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1 - 4 = -3 \neq 0$

$\begin{cases} -t + 2x = 5 \\ 2t - x = -4 \end{cases} \Big|_2 \Leftrightarrow \begin{cases} -t + 2x = 5 \\ 4t - 2x = -8 \end{cases}$

$\overline{\qquad 3t = -3 \qquad}$

$\boxed{t_0 = -1}$
$\boxed{x_0 = 2}$

$\boxed{\begin{aligned} x &= u + 2 \\ t &= z - 1 \end{aligned}}$

$\dfrac{du}{dz} = \dfrac{-(z-1) + 2(u+2) - 5}{2(z-1) - (u+2) + 4} = \dfrac{-z + 1 + 2u + 4 - 5}{2z - 2 - u - 2 + 4}$

$= \dfrac{-z + 2u}{2z - u} = \dfrac{-1 + 2\frac{u}{z}}{2 - \frac{u}{z}} = g\left(\dfrac{u}{z}\right)$

$\boxed{v = \dfrac{u}{z}} \Rightarrow u = v\cdot z \Rightarrow u' = v'z + v$

$v'z + v = \dfrac{-1 + 2v}{2 - v} \Rightarrow v'z = \dfrac{-1 + 2v}{2 - v} - v$

$v'z = \dfrac{-1 + 2v - 2v + v^2}{2 - v} = \dfrac{v^2 - 1}{2 - v}$

$\dfrac{dv}{dz} = \dfrac{v^2 - 1}{2 - v}$

$\dfrac{2 - v}{v^2 - 1}\,dv = \dfrac{1}{z}\,dz \Leftrightarrow \displaystyle\int \dfrac{2 - v}{v^2 - 1}\,dv = \int \dfrac{1}{z}\,dz$

$$\int \frac{2-v}{v^2-1}\,dv = \int \frac{2}{v^2-1}\,dv - \frac{1}{2}\int \frac{2v}{v^2-1}\,dv$$

$$= 2 \cdot \frac{1}{2}\ln\left|\frac{v-1}{v+1}\right| - \frac{1}{2}\ln|v^2-1|$$

$$\Rightarrow \ln\left|\frac{v-1}{v+1}\right| - \frac{1}{2}\ln|v^2-1| = t + C \quad \big/ \cdot 2$$

$$\textcircled{2}\ln\left|\frac{v-1}{v+1}\right| - \ln|v^2-1| = 2t + C$$

$$\ln\left|\frac{v-1}{v+1}\cdot\frac{1}{v^2-1}\right| = 2t + c$$

$$\frac{1}{|v+1|^2} = e^{2t+c} = C \cdot e^{2t}$$

$$\frac{1}{\left(\frac{u}{6}+1\right)^2} = C \cdot e^{2t}$$

$$\frac{1}{\left(\frac{x-2}{t+1}+1\right)^2} = C e^{2t}$$

(sol. în forma
   implicită)