Ecuații diferențiale și cu derivate parțiale

Laborator 05

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1. $\boxed{a)}$ $x'' + x' = 3t + 2$
   $\boxed{b)}$ $x'' - 4x' + 4x = t^2$
   $\boxed{c)}$ $x'' - x' + x = t^3 + 6$
   d) $3x'' - 2x' - x = t^2 - 1$
   $\boxed{e)}$ $x^{IV} - 2x''' + x'' = t^3$
   $\boxed{f)}$ $x''' + 3x'' - 4x' = 2t^2 - 3t + 9$
   g) $x^{V} - 4x^{IV} + 5x''' = 600t^3 - 240t^2 + 120$

<u>Rezolvare</u>

d) $3x'' - 2x' - x = t^2 - 1$

• $3x'' - 2x' - x = 0$

$$\left.\begin{aligned} x &= e^{rt} \\ x' &= re^{rt} \\ x'' &= r^2 e^{rt} \end{aligned}\right\} \Rightarrow 3r^2 e^{rt} - 2re^{rt} - e^{rt} = 0 \;\; |: e^{rt}$$

$3r^2 - 2r - 1 = 0$; $r_{1,2} = \dfrac{2 \pm 4}{6} \Leftrightarrow \begin{cases} r_1 = 1 \\ r_2 = -\frac{1}{3} \end{cases}$

$D = 4 + 12 = 16$

• $e^{1t}, e^{-\frac{1}{3}t}$ – sist. fund de soluții

$$\boxed{x_0 = c_1 e^{t} + c_2 e^{\frac{-1}{3}t}}$$

• $$\left.\begin{aligned} x_p &= \lambda_2 t^2 + \lambda_1 t + \lambda_0 \\ x'_p &= 2\lambda_2 t + \lambda_1 \\ x''_p &= 2\lambda_2 \end{aligned}\right\} \Leftrightarrow 3\cdot 2\lambda_2 - 2(2\lambda_2 t + \lambda_1) - (\lambda_2 t^2 + \lambda_1 t + \lambda_0) = t^2 - 1$$

$-\lambda_2 t^2 - 4\lambda_2 t - \lambda_1 t + 6\lambda_2 - 2\lambda_1 - \lambda_0 = t^2 - 1 \Rightarrow$

$$\Rightarrow \begin{cases} -\lambda_2 = 1 \\ -4\lambda_2 - \lambda_1 = 0 \\ 6\lambda_2 - 2\lambda_1 - \lambda_0 = -1 \end{cases} \Leftrightarrow \begin{cases} \lambda_2 = -1 \\ \lambda_1 = 4 \\ \lambda_0 = +1 + 6\lambda_2 - 2\lambda_1 = 1 - 6 - 8 = -13 \end{cases}$$

• $\boxed{x_p = -t^2 + 4t - 13}$

• $x(t) = x_0 + x_p = c_1 e^{t} + c_2 e^{\frac{-1}{3}t} - t^2 + 4t - 13.$

g) $x^{V} - 4x^{IV} + 5x''' = 600t^3 - 240t^2 + 720.$

• $x^{V} - 4x^{IV} + 5''' = 0$

$x = e^{nt}$

$x' = ne^{nt}$

$x'' = n^2 e^{nt}$

$x''' = n^3 e^{nt}$

$x^{IV} = n^4 e^{nt}$

$x^{V} = n^5 e^{nt}$

$\Rightarrow n^5 e^{nt} - 4n^4 e^{nt} + 5n^3 e^{nt} = 0 \quad | : e^{nt}$

$n^5 - 4n^4 + 5n^3 = 0$

$n^3(n^2 - 4n + 5) = 0 \Rightarrow n_1 = n_2 = n_3 = 0$

$n^2 - 4n + 5 = 0$

$D = 16 - 4\cdot 5 = -4$

$n_{4,5} = \frac{4 \pm 2i}{2} = 2 \pm i \ (\alpha \pm i\beta, \alpha = 2, \beta = 1)$

• $e^{0t}, te^{0t}, t^2 e^{0t}, e^{2t}\cos 1t, e^{2t}\sin 1t$ – sis. fund de s.

• $\boxed{x_0 = c_1 + c_2 t + c_3 t^2 + c_4 e^{2t}\cos t + c_5 e^{2t}\sin t}$

$x_p = t^3(\lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t + \lambda_0) = \lambda_3 t^6 + \lambda_2 t^5 + \lambda_1 t^4 + \lambda_0 t^3$

$x_p' = 6\lambda_3 t^5 + 5\lambda_2 t^4 + 4\lambda_1 t^3 + 3\lambda_0 t^2$

$x_p'' = 30\lambda_3 t^4 + 20\lambda_2 t^3 + 12\lambda_1 t^2 + 6\lambda_0 t$

$x_p''' = 120\lambda_3 t^3 + 60\lambda_2 t^2 + 24\lambda_1 t + 6\lambda_0$

$x_p^{IV} = 360\lambda_3 t^2 + 120\lambda_2 t + 24\lambda_1$

$x_p^{V} = 720\lambda_3 t + 120\lambda_2$

$\underline{720\lambda_3 t} + 120\lambda_2 - \underline{4\cdot 360\lambda_3 t^2} - \underline{4\cdot 120\lambda_2 t} - 4\cdot 24\lambda_1 + \underline{5\cdot 120\lambda_3 t^3} + \underline{300\lambda_2 t^2} +$

$+ \underline{5\cdot 24\lambda_1 t} + 30\lambda_0 = \underline{600t^3} - \underline{240t^2} + 720$

$$\Rightarrow \begin{cases} 600\lambda_3 = 600 \Leftrightarrow \boxed{\lambda_3 = 1} \\ -4\cdot 360\lambda_3 + 300\lambda_2 = -240 \Leftrightarrow 300\lambda_2 = -240 + 1440 \Leftrightarrow \boxed{\lambda_2 = 4} \\ 720\lambda_3 - 480\lambda_2 + 120\lambda_1 = 0 \Leftrightarrow 120\lambda_1 = 480\cdot 4 - 720 \Leftrightarrow \boxed{\lambda_1 = 10} \\ 120\lambda_2 - 96\lambda_1 + 30\lambda_0 = 720 \Leftrightarrow 30\lambda_0 = 720 - 120\cdot 4 + 96\cdot 10 \Leftrightarrow \boxed{\lambda_0 = 20} \end{cases}$$

• $\boxed{x_p = t^3(t^3 + 4t^2 + 10t + 20)}$

$x(t) = x_0 + x_p = c_1 + c_2 t + c_3 t^2 + c_4 e^{2t}\cos t + c_5 e^{2t}\sin t + t^3(t^3 + 4t^2 + 10t + 20).$

2. a) $x'' - 4x = te^{3t}$

$\boxed{b)}$ $x'' - 5x = 5t^2 e^{2t}$

$\boxed{c)}$ $x'' + 2x' + x = e^{2t}$

d) $x'' - 4x = t^2 e^{2t}$

$\boxed{e)}$ $x'' - 3x' + 2x = 3t^2 e^{t}$

$\boxed{f)}$ $x'' + 2x' - 3x = 4te^{t}$

Rezolvare

a) $x'' - 4x = te^{3t}$, $\alpha = 3$

• $x'' - 4x = 0$

$$\left.\begin{aligned} x &= e^{rt} \\ x' &= re^{rt} \\ x'' &= r^2 e^{rt} \end{aligned}\right\} \Leftrightarrow r^2 e^{rt} - 4 \cdot e^{rt} = 0 \;|: e^{rt}$$

$r^2 - 4 = 0 \Leftrightarrow r^2 = 4 \Rightarrow r_{1,2} = \pm 2$

• e^{2t}, e^{-2t} – sist. fund de soluții

• $\boxed{x_0 = C_1 e^{2t} + C_2 e^{-2t}}$

$x_p = e^{3t}(\lambda_1 t + \lambda_0)$

$x_p' = 3e^{3t}(\lambda_1 t + \lambda_0) + e^{3t}\lambda_1 = e^{3t}(3\lambda_1 t + 3\lambda_0 + \lambda_1)$

$x_p'' = 3e^{3t}(3\lambda_1 t + 3\lambda_0 + \lambda_1) + e^{3t} \cdot 3\lambda_1 = e^{3t}(9\lambda_1 t + 9\lambda_0 + 6\lambda_1)$

• $e^{3t}(9\lambda_1 t + 9\lambda_0 + 6\lambda_1) - 4e^{3t}(\lambda_1 t + \lambda_0) = te^{3t} \;|: e^{3t}$

$\underline{9\lambda_1 t} + 9\lambda_0 + 6\lambda_1 - \underline{4\lambda_1 t} - 4\lambda_0 = \underline{t}$

$$\Leftrightarrow \begin{cases} 5\lambda_1 = 1 \Leftrightarrow \boxed{\lambda_1 = \frac{1}{5}} \\ 5\lambda_0 + 6\lambda_1 = 0 \Leftrightarrow \lambda_0 = \frac{-6}{5} \cdot \frac{1}{5} = \frac{-6}{25} \end{cases}$$

• $\boxed{x_p = e^{3t}\left(\frac{1}{5}t - \frac{6}{25}\right)}$

• $x(t) = x_0 + x_p = C_1 e^{2t} + C_2 e^{-2t} + e^{3t}\left(\frac{1}{5}t - \frac{6}{25}\right)$

d) $x'' - 4x = t^2 e^{2t}$, $\alpha = 2$

• $x'' - 4x = 0$

$$\left.\begin{aligned} x &= e^{rt} \\ x' &= re^{rt} \\ x'' &= r^2 e^{rt} \end{aligned}\right\} \Rightarrow r^2 e^{rt} - 4e^{rt} = 0 \;|: e^{rt}$$

$r^2 - 4 = 0 \Rightarrow r_{1,2} = \pm 2$

• e^{2t}, e^{-2t} – sist. fund de soluții

• $\boxed{x_0 = C_1 e^{2t} + C_2 e^{-2t}}$

$$\bullet\ x_p = t e^{2t}(\lambda_2 t^2 + \lambda_1 t + \lambda_0) = e^{2t}(\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t)$$

$$x'_p = 2e^{2t}(\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t) + e^{2t}(3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0)$$

$$= e^{2t}(2\lambda_2 t^3 + 2\lambda_1 t^2 + 2\lambda_0 t + 3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0)$$

$$x''_p = 2e^{2t}(2\lambda_2 t^3 + 2\lambda_1 t^2 + 2\lambda_0 t + 3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0) + e^{2t}(6\lambda_2 t^2 + 4\lambda_1 t +$$

$$2\lambda_0 + 6\lambda_2 t + 2\lambda_1)$$

$$= e^{2t}(\underline{4\lambda_2 t^3} + \underline{4\lambda_1 t^2} + \underline{4\lambda_0} t + \underline{6\lambda_2 t^2} + \underline{4\lambda_1} t + 2\lambda_0 + \underline{6\lambda_2 t^2} + \underline{4\lambda_1 t} +$$

$$+ 2\lambda_0 + \underline{6\lambda_2 t} + 2\lambda_1)$$

$$\bullet\ 4\lambda_2 t^3 + 4\lambda_1 t^2 + 12\lambda_2 t^2 + 4\lambda_0 t + 8\lambda_1 t + 6\lambda_2 t + 4\lambda_0 + 2\lambda_1 - 4\lambda_2 t^3 - 4\lambda_1 t - 4\lambda_0 t = \underline{t^2}$$

$$\Rightarrow \begin{cases} 12\lambda_2 = 1 \\ 8\lambda_1 + 6\lambda_2 = 0 \\ 4\lambda_0 + 2\lambda_1 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_2 = \frac{1}{12} \\ \lambda_1 = \frac{-6}{8} \cdot \frac{1}{12} = \frac{-1}{16} \\ \lambda_0 = \frac{-2}{4} \cdot \frac{-1}{16} = \frac{1}{32} \end{cases}$$

$$\Rightarrow \boxed{x_p = t e^{2t}\left(\frac{1}{12}t^2 - \frac{1}{16}t + \frac{1}{32}\right)}$$

$$\bullet\ x(t) = x_o + x_p = c_1 e^{2t} + c_2 e^{-2t} + t e^{2t}\left(\frac{1}{12}t^2 - \frac{1}{16}t + \frac{1}{32}\right)$$

3. a) $x'' + 4x' - 3x = t\sin 2t$

$\boxed{b)}$ $x'' - 4x = e^{2t}\cos 2t$

$\boxed{c)}$ $x'' - 2x' + 5x = te^{t}\sin t$

$\boxed{d)}$ $x'' - 2x' + 5x = e^{t}\cos 2t$

$\boxed{e)}$ $x'' - x = 2t\sin t$

$\boxed{f)}$ $x'' + 4x = e^{2t}\sin 2t$

Rezolvare

a) $x'' + 4x' - 3x = t\sin 2t$, $\alpha = 0$, $\beta = 2$

• $x'' + 4x' - 3x = 0$

$$\left.\begin{array}{l} x = e^{rt} \\ x' = re^{rt} \\ x'' = r^2 e^{rt} \end{array}\right\} \Leftrightarrow r^2 e^{rt} + 4re^{rt} - 3e^{rt} = 0 \;|:e^{rt}$$

$$\left.\begin{array}{l} r^2 + 4r - 3 = 0 \\ \Delta = 16 + 4\cdot 3 = 28 \end{array}\right\} \Leftrightarrow r_{1,2} = \frac{-4 \pm 2\sqrt{7}}{2} = -2 \pm \sqrt{7}$$

• $e^{(-2+\sqrt{7})t}$, $e^{(-2-\sqrt{7})t}$ - sist. fund. de sol.

$$\boxed{x_0 = C_1 e^{(-2+\sqrt{7})t} + C_2 e^{(-2-\sqrt{7})t}}$$

• $x_p = (\lambda_1 t + \lambda_0)\sin 2t + (\beta_1 t + \beta_0)\cos 2t$

$x_p' = \lambda_1 \sin 2t + (\lambda_1 t + \lambda_0) 2\cos 2t + \beta_1 \cos 2t + (\beta_1 t + \beta_0)\cdot(-2)\sin 2t$

$= \sin 2t(\lambda_1 - 2\beta_1 t - 2\beta_0) + \cos 2t(2\lambda_1 t + 2\lambda_0 + \beta_1)$

$x_p'' = (-2\beta_1)\sin 2t + (\lambda_1 - 2\beta_1 t - 2\beta_0)2\cos 2t + 2\lambda_1\cdot\cos 2t +$

$+ (2\lambda_1 t + 2\lambda_0 + \beta_1)\cdot(-2)\cdot\sin 2t$

$= \sin 2t(-4\beta_1 - 4\lambda_1 t - 4\lambda_0) + \cos 2t(4\lambda_1 - 4\beta_1 t - 4\beta_0)$

• $\left(\sin 2t(-4\beta_1 - 4\lambda_1 t - 4\lambda_0) + \cos 2t(4\lambda_1 - 4\beta_1 t - 4\beta_0)\right) + 4\left(\sin 2t(\lambda_1 - 2\beta_1 t -\right.$

$\left.- 2\beta_0) + \cos 2t(2\lambda_1 t + 2\lambda_0 + \beta_1)\right) - 3\left((\lambda_1 t + \lambda_0)\sin 2t + (\beta_1 t + \beta_0)\cos 2t\right) =$

$= t\sin 2t$

$$\Leftrightarrow \begin{cases} -4\beta_1 - 4\lambda_1 t - 4\lambda_0 + 4\lambda_1 - 8\beta_1 t - 8\beta_0 - 3\lambda_1 t - 3\lambda_0 = t \\ 4\lambda_1 - 4\beta_1 t - 4\beta_0 + 8\lambda_1 t + 8\lambda_0 + 4\beta_1 - 3\beta_1 t - 3\beta_0 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -7\lambda_1 - 8\beta_1 = 1 \\ 4\lambda_1 - 4\lambda_0 - 4\beta_1 - 8\beta_0 = 0 \\ 8\lambda_1 - 7\beta_1 = 0 \\ 4\lambda_1 + 8\lambda_0 + 4\beta_1 - 7\beta_0 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -7\lambda_1 - 8\beta_1 = 1 \mid \cdot 8 \\ 8\lambda_1 - 7\beta_1 = 0 \mid \cdot 7 \end{cases} \Leftrightarrow \begin{cases} -56\lambda_1 - 64\beta_1 = 8 \\ 56\lambda_1 - 49\beta_1 = 0 \quad \text{,,+"} \end{cases} \Leftrightarrow \boxed{\lambda_1 = \frac{-7}{113}}$$

$$-113\beta_1 = 8 \Leftrightarrow \boxed{\beta_1 = \frac{-8}{113}}$$

$$\Leftrightarrow \begin{cases} 4\lambda_1 - 4\lambda_0 - 4\beta_1 - 8\beta_0 = 0 \mid \cdot 2 \\ 4\lambda_1 + 8\lambda_0 + 4\beta_1 - 7\beta_0 = 0 \end{cases} \Leftrightarrow \begin{cases} 8\lambda_1 - 8\lambda_0 - 8\beta_1 - 16\beta_0 = 0 \\ 4\lambda_1 + 8\lambda_0 + 4\beta_1 - 7\beta_0 = 0 \quad \text{,,+"} \end{cases}$$

$$12\lambda_1 - 4\beta_1 - 23\beta_0 = 0$$

$$\Leftrightarrow \boxed{\beta_0 = \frac{-52}{113 \cdot 23}}\ ; \boxed{\lambda_0 = \frac{127}{113 \cdot 23}}$$

- $\boxed{x_p = \left(\frac{-7}{113}t + \frac{127}{113 \cdot 23}\right)\sin 2t + \left(\frac{-8}{113}t + \frac{-52}{113 \cdot 23}\right)\cos 2t}$

$$x(t) = x_0 + x_p = C_1 e^{(-2+\sqrt{7})t} + C_2 e^{(-2-\sqrt{7})t} + \left(\frac{-7}{113}t + \frac{127}{113 \cdot 23}\right)\sin 2t +$$

$$+ \left(\frac{-8}{113}t - \frac{52}{113 \cdot 23}\right)\cos 2t$$