Laborator 04 - Temă

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1. Să se rezolve următoarele ecuații de tip Bernoulli.

a) $x' + \underbrace{\frac{t}{6}}_{A(t)} x = \underbrace{\frac{1}{3}t}_{B(t)}\, x^{-2} \,\Big|\cdot x^2$ (ec. de tip Bernoulli)

$$x^2\cdot x' + \frac{t}{6}x^3 = \frac{1}{3}t$$

Notăm $y = x^3 \Leftrightarrow y' = 3x^2\cdot x' \Leftrightarrow x^2\cdot x' = \frac{y'}{3}$

$$\frac{y'}{3} + \frac{t}{6}y = \frac{1}{3}t \,\Big|\cdot 3 \Leftrightarrow y' + \frac{t}{2}y = t \text{ (ec. de tip afină)}$$

Etapa 1: $y' + \frac{t}{2}y = 0$

$$y' = \frac{-t}{2}y \Leftrightarrow \frac{dy}{dt} = \frac{-t}{2}y \Leftrightarrow \frac{dy}{y} = \frac{-t}{2}dt \Leftrightarrow \int\frac{dy}{y} = \int\frac{-t}{2}dt \Leftrightarrow$$

$$\Leftrightarrow \ln|y| = \frac{-t^2}{4} + C \Leftrightarrow y = e^{\frac{-t^2}{4}+C} \Leftrightarrow \boxed{y_0 = C e^{\frac{-t^2}{4}}}$$

Etapa 2: $y(t) = C(t)\cdot e^{\frac{-t^2}{4}}$

$$C'(t)\cdot e^{\frac{-t^2}{4}} + \cancel{C(t)\cdot e^{\frac{-t^2}{4}}\cdot\left(\frac{-t}{2}\right)} + \cancel{\frac{t}{2}\cdot C(t)\cdot e^{\frac{-t^2}{4}}} = t$$

$$C'(t)\cdot e^{\frac{-t^2}{4}} = t \Leftrightarrow C'(t) = t\cdot e^{\frac{t^2}{4}}$$

$$C(t) = \int t\cdot e^{\frac{t^2}{4}}dt = 2e^{\frac{t^2}{4}} + C_1$$

$$\varphi_0 = C(t)\cdot e^{\frac{-t^2}{4}} = \left(2e^{\frac{t^2}{4}} + C_1\right)\cdot e^{\frac{-t^2}{4}} = 2 + C_1\cdot e^{\frac{-t^2}{4}}$$

$$y = y_0 + \varphi_0 = C\cdot e^{\frac{-t^2}{4}} + 2 + C_1\cdot e^{\frac{-t^2}{4}} = 2 + C\cdot e^{\frac{-t^2}{4}}$$

$y = x^3 \Leftrightarrow x = \sqrt[3]{y} = \sqrt[3]{2 + C\cdot e^{\frac{-t^2}{2}}}$ (sol. generală a ec. inițiale)

b) $3tx^2\cdot x' + x^3 = 2t \,\Big|:t$

$3x^2\cdot x' + \underbrace{\frac{1}{t}}_{A(t)}x^3 = \underbrace{2}_{B(t)}$ (ec. de tip Bernoulli)

Notăm: $y = x^3 \Leftrightarrow y' = 3x^2\cdot x^2$

$y' + \frac{1}{t}y = 2$ (ec. afină)

$\underline{\text{Etapa 1}}$: $y' + \frac{1}{t} y = 0 \Leftrightarrow y' = \frac{-1}{t} y \Leftrightarrow \frac{dy}{dt} = \frac{-1}{t} y \Leftrightarrow \frac{dy}{y} = \frac{-1}{t} dt \Leftrightarrow$

$\Leftrightarrow \int \frac{1}{y} dy = \int \frac{-1}{t} dt \Leftrightarrow \ln|y| = -\ln|t| + C \Rightarrow \boxed{y_0 = C \cdot t^{-1}}$

$\underline{\text{Etapa 2}}$: $y(t) = C(t) \cdot t^{-1}$

$C'(t) \cdot t^{-1} + \cancel{C(t) \cdot \left(\frac{-1}{t^2}\right)} + \cancel{C(t) \cdot \frac{1}{t^2}} = 2$

$C'(t) \cdot t^{-1} = 2 \Leftrightarrow C'(t) = 2t \Leftrightarrow C(t) = \int 2t\,dt = t^2 + C_1$

$\varphi_0 = (t^2 + C_1) \cdot t^{-1} = t + C_1 \cdot t^{-1}$

$y = y_0 + \varphi_0 = C \cdot t^{-1} + t + C_1 \cdot t^{-1} = t + C \cdot t^{-1}$

$y = x^3 \Leftrightarrow x = \sqrt[3]{y} = \sqrt[3]{t + C \cdot t^{-1}}$ (sol. gen. a ecuației)

$x(1) = 2 \Leftrightarrow \sqrt[3]{1 + C} = 2 \Leftrightarrow |1 + C| = 8 \Rightarrow C = 7$

$\boxed{x_{PC} = t + 7 \cdot t^{-1}}$ (sol. parțială)

d) $x' + 2tx = 2t^3 x^3 \;|: x^3$

$x^{-3} x' + 2t \cdot x^{-2} = 2t^3$ (ec. de tip Bernoulli)

Notăm: $y = x^{-2} \Leftrightarrow y' = -2 \cdot x^{-3} \cdot x' \Leftrightarrow x^{-3} \cdot x' = \frac{y'}{-2}$

$\frac{y'}{-2} + 2ty = 2t^3$ (ec. de tip afină)

$\underline{\text{Etapa 1}}$: $\frac{y'}{-2} + 2ty = 0 \Leftrightarrow \frac{y'}{-2} = -2ty \Leftrightarrow \frac{dy}{dt} = 4ty \Leftrightarrow \frac{dy}{y} = 4t\,dt \Leftrightarrow$

$\Leftrightarrow \int \frac{1}{y} dy = \int 4t\,dt \Leftrightarrow \ln|y| = 2t^2 + C \Rightarrow \boxed{y_0 = C \cdot e^{2t^2}}$

$\underline{\text{Etapa 2}}$: $y(t) = C(t) \cdot e^{2t^2}$

$\frac{-1}{2} C'(t) \cdot e^{2t^2} + \cancel{\left(\frac{-1}{2}\right) \cdot C(t) \cdot e^{2t^2} \cdot 4t} + \cancel{2t \cdot C(t) \cdot e^{2t^2}} = 2t^3$

$\frac{-1}{2} C'(t) \cdot e^{2t^2} = 2t^3 \Leftrightarrow C'(t) = 4t^3 \cdot e^{-2t^2} \Leftrightarrow C(t) = \frac{-(2t^2+1)e^{-2t^2}}{2} + C_1$

$\varphi_0 = \left(\frac{-1}{2} \cdot (2t^2+1) e^{-2t^2} + C_1\right) \cdot e^{2t^2} = \frac{-1}{2}(2t^2+1) + C_1 e^{2t^2}$

$y = y_0 + \varphi_0 = \left(\frac{-1}{2}\right)(2t^2+1) + C \cdot e^{2t^2} = -t^2 - \frac{1}{2} + C \cdot e^{2t^2}$

$y = x^{-2} \Leftrightarrow x = y^2 = \left(-t^2 - \frac{1}{2} + C \cdot e^{2t^2}\right)^2$ (sol. gen. a ecuației)

2. Să se rezolve următoarele ecuații de tip Ricati:

a) $x' + x^2 - 2x\sin t + \sin^2 t - \cos t = 0$, $\varphi_0(t) = \sin t$

$$x' = \underbrace{-}_{A(t)} x^2 + \underbrace{2x\sin t}_{B(t)} \underbrace{-\sin^2 t + \cos t}_{f(t)}$$

$$A(t) = -1;\ B(t) = 2\sin t,\ f(t) = \cos t - \sin^2 t$$

Facem schimbarea de variabilă: $x = y + \sin t$

$$x' = y' + \cancel{\cos t} = -(y + \sin t)^2 + 2\sin t(y + \sin t) - \sin^2 t + \cos t$$

$$= -y^2 - \cancel{2y\sin t} - \cancel{\sin^2 t} + \cancel{2\sin t \cdot y} + \cancel{2\sin^2 t} - \cancel{\sin^2 t} + \cancel{\cos t}$$

$$y' = -y^2 \iff y' + y^2 = 0$$

$$\frac{dy}{dt} = -y^2 \iff \frac{dy}{-y^2} = dt \iff \int \frac{-1}{y^2}\,dy = \int dt \iff \frac{1}{y} = t + C \iff$$

$$\iff y = \frac{1}{t+C},\ t, C \neq 0. \Rightarrow x = \frac{1}{t+C} + \sin t$$

b) $x' = \underbrace{t}_{A(t)} x^2 \underbrace{-2t^2 x}_{B(t)} + \underbrace{t^3 + 1}_{f(t)}$, $\varphi_0(t) = t$

$$A(t) = t;\ B(t) = -2t^2;\ f(t) = t^3 + 1$$

Facem schimbarea de variabilă: $x = y + t$

$$x' = y' + \cancel{1} = t(y + t)^2 - 2t^2(y + t) + t^3 + 1$$

$$= t(y^2 + 2yt + t^2) - 2t^2 y - 2t^3 + t^3 + 1$$

$$= y^2 t + \cancel{2yt^2} + \cancel{t^3} - \cancel{2t^2 y} - \cancel{2t^3} + \cancel{t^3} \cancel{+1}$$

$$y' = y^2 t$$

$$\frac{dy}{dt} = y^2 t \iff \frac{dy}{y^2} = t\,dt \iff \int \frac{1}{y^2}\,dy = \int t\,dt \iff \frac{-1}{y} = \frac{t^2}{2} + C \iff$$

$$\iff \frac{y}{-1} = \frac{2}{t^2 + C} \iff y = \frac{-2}{t^2 + C},\ t, C \neq 0$$

$$\Rightarrow x = \frac{-2}{t^2 + C} + t$$

3. Să se rezolve următoarele ecuații cu diferențiale exacte:

a) $\underbrace{\frac{t}{x^2}}_{P(t,x)} dt + \underbrace{\frac{x^2-t^2}{x^3}}_{Q(t,x)} dx = 0$

$$\left.\begin{aligned} \frac{\partial P}{\partial x}(t,x) &= -2t \cdot x^{-3} \\ \frac{\partial Q}{\partial t}(t,x) &= -2t \cdot x^{-3} \end{aligned}\right\} \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial t} \Leftrightarrow F(t,x) = c$$

$$F(t,x) = \int_{t_0}^{t} P(\tau, x_0)\, d\tau + \int_{x_0}^{x} Q(t, \nu)\, d\nu$$

$$= \int_{t_0}^{t} \frac{\tau}{x_0^2} d\tau + \int_{x_0}^{x} \frac{\nu^2 - t^2}{\nu^3} d\nu$$

$$= \left(\frac{\tau^2}{2x_0^2}\right)\Bigg|_{t_0}^{t} + \left(\ln|\nu| + \frac{t^2}{2\nu^2}\right)\Bigg|_{x_0}^{x}$$

$$= \left(\cancel{\frac{t^2}{2x_0^2}} - \frac{t_0^2}{2x_0^2}\right) + \left(\ln|x| + \frac{t^2}{2x^2} - \ln|x_0| - \cancel{\frac{t^2}{2x_0^2}}\right)$$

$$= \ln|x| + \frac{t^2}{2x^2} + c$$

• $\ln|x| + \frac{t^2}{2x^2} = c$ (sol. în formă implicită)

c) $\underbrace{(t^2+x^2+2t)}_{P(t,x)} dt + \underbrace{2tx}_{Q(t,x)} dx = 0$

$$\left.\begin{aligned} \frac{\partial P}{\partial x}(t,x) &= 2x \\ \frac{\partial Q}{\partial t}(t,x) &= 2x \end{aligned}\right\} \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial t} \Leftrightarrow F(t,x) = c.$$

$$F(t,x) = \int_0^t P(\tau, 0)\, d\tau + \int_0^x Q(t,\nu)\, d\nu$$

$$= \int_0^t (\tau^2 + 2\tau)\, d\tau + \int_0^x (2t\nu)\, d\nu$$

$$= \left(\frac{\tau^3}{3} + \tau^2\right)\Bigg|_0^t + (t\nu^2)\Bigg|_0^x$$

$$= \frac{t^3}{3} + t^2 + tx^2 + c$$

$\frac{t^3}{3} + t^2 + tx^2 = c$ (sol în formă implicită)

d) $t\,dt + x\,dx = \frac{-t\,dx - x\,dt}{t^2+x^2} \quad \Big| \cdot (t^2+x^2)$

$$t(t^2+x^2)dt + x(t^2+x^2)dx = -t\,dx - x\,dt$$

$$\underbrace{(t^3 + x^2t + x)}_{P(t,x)}dt + \underbrace{(x^3 + t^2x + t)}_{Q(t,x)}dx = 0$$

$$\left.\begin{array}{l} \frac{\partial P}{\partial x}(t,x) = 2xt + 1 \\ \frac{\partial Q}{\partial t}(t,x) = 2xt + 1 \end{array}\right\} \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial t} \Leftrightarrow F(t,x) = c$$

$$F(t,x) = c \Leftrightarrow \int_0^t P(z,0)\,dz + \int_0^x Q(t,v)\,dv =$$

$$= \int_0^t (z^3)\,dz + \int_0^x (v^3 + t^2 v + \cancel{0})\,dv =$$

$$= \frac{z^4}{4}\Big|_0^t + \left(\frac{v^4}{4} + t^2 \cdot \frac{v^2}{2}\right)\Big|_0^x$$

$$= \frac{t^4}{4} + \frac{x^4}{4} + \frac{x^2t^2}{2} + C$$

$$= \frac{t^4 + 2x^2t^2 + x^4}{4} + C = \frac{1}{4}(t^2+x^2)^2 + C$$

- $\frac{1}{4}(t^2+x^2)^2 = C$ (sol în formă implicită)

4. Să se rezolve următoarele ecuații căutând un factor integrant

b) $2tx\,dt = (t^2-x^2)\,dx$

$$2tx\,dt-(t^2-x^2)\,dx \Leftrightarrow \underbrace{2tx}_{P(t,x)}\,dt+\underbrace{(x^2-t^2)}_{Q(t,x)}\,dx=0$$

$$\left.\begin{aligned}\frac{\partial P}{\partial x}(t,x)&=2t\\ \frac{\partial Q}{\partial t}(t,x)&=-2t\end{aligned}\right\} \Rightarrow \frac{\partial P}{\partial x}\neq\frac{\partial Q}{\partial t}$$

Fie $\mu=\mu(x)$

$$\underbrace{\mu(x)\cdot(2tx)}_{P^*(t,x)}dt+\underbrace{\mu(x)\,(x^2-t^2)}_{Q^*(t,x)}dx=0$$

$$\frac{\partial P^*}{\partial x}(t,x)=\mu'(x)\,(2tx)+\mu(x)\cdot 2t$$

$$\frac{\partial Q}{\partial t}(t,x)=\mu(x)\cdot(-2t)$$

$$\mu'(x)(2tx)+\mu(x)\cdot 2t=-2t\mu(x)$$

$$\mu'(x)\cdot 2tx=-4t\,\mu(x)\ \Big|:2t$$

$$\mu'(x)\cdot x=-2\mu(x)$$

$$\frac{d\mu}{dx}\cdot x=-2\mu \Leftrightarrow \frac{d\mu}{\mu}=\frac{-2}{x}dx \Rightarrow \int\frac{1}{\mu}d\mu=\int\frac{-2}{x}dx \Leftrightarrow$$

$$\Leftrightarrow \ln|\mu|=-2\ln|x|+c=\ln x^{-2}+\ln c \Rightarrow \mu=\frac{c}{x^2}$$

Presupunem $c=1$

$$\underbrace{\frac{1}{x^2}\cdot 2tx}_{P^*(t,x)}\,dt+\underbrace{\frac{1}{x^2}\,(x^2-t^2)}_{Q^*(t,x)}\,dx=0$$

$$F(t,x)=c=\int_0^t P^*(\tau,x_0)\,d\tau+\int_{x_0}^x Q^*(t,\nabla)\,d\nabla$$

$$=\int_0^t\left(\frac{1}{x_0}\cdot 2\tau\right)d\tau+\int_{x_0}^x\left(1-\frac{t^2}{\nabla^2}\right)d\nabla$$

$$=\frac{\tau^2}{x_0}\Big|_0^t+\frac{t^2}{\nabla}\Big|_{x_0}^x=\cancel{\frac{t^2}{x_0}}+\frac{t^2}{x}-\cancel{\frac{t^2}{x_0}}=\frac{t^2}{x}$$

$\frac{t^2}{x}=c$ (sol în formă implicită)

c) $\underbrace{(2tx-t)}_{P(t,x)}\,dt + \underbrace{(x^2+x+2t^2)}_{Q(t,x)}\,dx = 0$

$$\left.\begin{aligned} \frac{\partial P}{\partial x}(t,x) &= 2t \\ \frac{\partial Q}{\partial t}(t,x) &= 4t \end{aligned}\right\} \Rightarrow \frac{\partial P}{\partial x} \neq \frac{\partial Q}{\partial t}$$

Presupunem $\mu = \mu(x)$

$$\underbrace{\mu(x)(2tx-t)}_{P^*(t,x)}\,dt + \underbrace{\mu(x)(x^2+x+2t^2)}_{Q^*(t,x)}\,dx = 0$$

$$\left.\begin{aligned} \frac{\partial P^*}{\partial x}(t,x) &= \mu'(x)(2tx-t) + \mu(x)\cdot 2t \\ \frac{\partial Q}{\partial t}(t,x) &= \mu(x)\cdot 4t \end{aligned}\right\} \Leftrightarrow$$

$\Leftrightarrow \mu'(x)\cdot(2tx-t) + \mu(x)(2t) = \mu(x)\cdot 4t$

$\mu'(x)\cdot t(2x-1) = \mu(x)\cdot 2t \quad |:t$

$\mu'(x)(2x-1) = 2\mu(x)$

$\frac{d\mu}{dx}(2x-1) = 2\mu \Leftrightarrow \frac{d\mu}{\mu} = \frac{2}{2x-1}\,dx \Rightarrow \ln|\mu| = \ln|2x-1| + C \Leftrightarrow$

$\Leftrightarrow \ln|\mu| = \ln|C\cdot(2x-1)| \Leftrightarrow \mu = C(2x-1)$

Fie $C=1 \Rightarrow \mu = 2x-1$

$$\underbrace{(2x-1)(2tx-t)}_{P^*(t,x)}\,dt + \underbrace{(2x-1)(x^2+x+2t^2)}_{Q^*(t,x)}\,dx = 0$$

$$\begin{aligned} F(t,x) = c &= \int_0^t P^*(z,0)\,dz + \int_0^x Q^*(t,v)\,dv \\ &= \int_0^t z\,dz + \int_0^x (2v-1)(v^2+v+2t^2)\,dv \\ &= \frac{z^2}{2}\Big|_0^t + \int_0^x \left(2v^3 + v^2 + 4t^2v - v - 2t^2\right) dv \\ &= \frac{t^2}{2} + \left(\frac{2v^4}{4} + \frac{v^3}{3} + \frac{4t^2v^2}{2} - \frac{v^2}{2}\right)\Big|_0^x \\ &= \frac{t^2}{2} + \cancel{\frac{x^4}{2}} + \frac{x^3}{3} + \frac{2t^2x^2}{1} - \cancel{\frac{x^2}{2}} = c \end{aligned}$$

$\frac{t^2}{2} + \frac{x^3}{3} + 2t^2x^2 = c$ (sol în formă implicită)