

Laborator05

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Enunțuri

Să se rezolve următoarele ecuații:

1. $t \cdot x' = x^3 + x$
2. $x' = \frac{t + 2x + 1}{2t + 4x + 3}$
3. $x' = 4\frac{x}{t} + t\sqrt{x}$
4. $x' = \frac{x}{t} + tg\frac{x}{t}$
5. $(t^2 + tx + x^2)dt - t^2 dx = 0$

Rezolvare

Exercițiu 01

① $t \cdot x' = x^3 + x$

$t \cdot \frac{dx}{dt} = x^3 + x$ (ec. cu variabile separabile)

$\frac{dx}{x^3 + x} = \frac{1}{t} dt$

$\int \frac{1}{x^3 + x} dx = \int \frac{1}{t} dt$

$\int \frac{1}{x} dx - \frac{1}{2} \int \frac{2x}{x^2 + 1} dx = \int \frac{1}{t} dt$

$\ln|x| - \frac{1}{2} \ln|x^2 + 1| = \ln|t| + C \quad | \cdot 2$

$2 \ln|x| - \ln|x^2 + 1| = 2 \ln t + \ln C$

$\ln \frac{x^2}{x^2 + 1} = \ln C t^2 \Rightarrow \frac{x^2}{x^2 + 1} = C t^2$

$x^2 = C t^2 + C t^2$

$x^2 (1 - C t^2) = C t^2 \Rightarrow x^2 = \frac{C t^2}{1 - C t^2} \Rightarrow x = \pm \sqrt{\frac{C t^2}{1 - C t^2}}$

$\frac{C t^2}{1 - C t^2} \geq 0, 1 - C t^2 \neq 0$

$\frac{1}{x^3 + x} = \frac{1}{x(x^2 + 1)} = \frac{\frac{1}{x^2}}{x^2 + 1} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$

$= \frac{A x^2 + A + Bx^2 + Cx}{x(x^2 + 1)} = \frac{(A+B)x^2 + Cx + A}{x(x^2 + 1)}$

$\begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \Rightarrow B=-1$

$\frac{1}{x^3 + x} = \frac{1}{x} - \frac{x}{x^2 + 1}$

Exercițiu 02

②

$$x' = \frac{t+2x+1}{2t+4x+3}$$

[ec. reducibilă la ec. de tip omogen]

$$\Delta = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$$

$$x' = \frac{t+2x+1}{2(t+2x)+3}$$

$$u = t+2x \Rightarrow u' = 1+2x'$$

$$u' = 1+2x' = 1+2 \cdot \frac{u+1}{2u+3} = \frac{2u+3+2u+2}{2u+3}$$

$$\frac{du}{dt} = \frac{4u+5}{2u+3} \Leftrightarrow \frac{2u+3}{4u+5} du = dt$$

$$\int \frac{2u+3}{4u+5} du = \int dt$$

$$\frac{1}{2} \int \frac{2(2u+3)}{4u+5} du = \frac{1}{2} \int \frac{4u+6}{4u+5} du$$

$$= \frac{1}{2} \int \frac{4u+5+1}{4u+5} du = \frac{1}{2} \int 1 + \frac{1}{4u+5} du$$

$$= \frac{1}{2} u + \frac{1}{8} \int \frac{4}{4u+5} du$$

$$= \frac{1}{2} u + \frac{1}{8} \ln|4u+5|$$

$$\Rightarrow \frac{1}{2} u + \frac{1}{8} \ln|4u+5| = t + C$$

$$\frac{1}{2} (t+2x) + \frac{1}{8} \ln|4(t+2x)+5| = t + C$$

$$\frac{1}{2} t + x + \frac{1}{8} \ln(4t+8x+5) = t + C$$

$$x + \frac{1}{8} \ln(4t+8x+5) = \frac{1}{2} t + C$$

(sol. în formă implicită)

Exercițiu 03

③ $x' = \frac{4}{t} \cdot \frac{x}{t} + t \sqrt{x}$

$$x' = \frac{4}{t} x + t \cdot x^{\frac{1}{2}}, \quad \alpha = \frac{1}{2} \text{ [ec. Bernoulli: cu } \alpha = 1/2]$$

$$\frac{x'}{x^{\frac{1}{2}}} = \frac{4}{t} \cdot \frac{x}{x^{\frac{1}{2}}} + t$$

$$\frac{x'}{\sqrt{x}} = \frac{4}{t} \sqrt{x} + t \quad \left\{ \cdot \frac{1}{2} \right.$$

$$u = \sqrt{x} \Rightarrow u' = \frac{1}{2\sqrt{x}} \cdot x'$$

$$\frac{x'}{2\sqrt{x}} = \frac{2}{t} \sqrt{x} + \frac{1}{2} t$$

$$u' = \frac{2}{t} u + \frac{t}{2} \text{ [ec. afină]}$$

[etapa 1] $u' = \frac{2}{t} u$

$$\frac{du}{dt} = \frac{2}{t} u$$

$$\frac{du}{u} = \frac{2}{t} dt$$

$$\int \frac{1}{u} du = 2 \int \frac{1}{t} dt$$

$$\ln|u| = 2 \ln|t| + C$$

$$\ln|u| = \ln t^2 + \ln C$$

$$u_0 = C t^2$$

[etapa 2] $P_0 = C(t) \cdot t^2$

$$(C(t) \cdot t^2)' = \frac{2}{t} C(t) \cdot t^2 + \frac{t}{2}$$

$$C'(t) t^2 + C(t) \cdot 2t = 2 C(t) \cdot t + \frac{t}{2}$$

$$C'(t) \cdot t^2 = \frac{t}{2} \quad | : t^2$$

$$C'(t) = \frac{1}{2t} \Rightarrow C(t) = \frac{1}{2} \int \frac{1}{t} dt$$

$$C(t) = \frac{1}{2} \ln|t| + C_1$$

$$\Rightarrow P_0 = \left(\frac{1}{2} \ln|t| + C_1 \right) t^2$$

$$P_0 = \frac{1}{2} t^2 \ln|t| + C_1 t^2$$

$$u = u_0 + P_0 = C t^2 + \frac{1}{2} t^2 \ln|t| = \sqrt{x}$$

$$x = \left(C t^2 + \frac{1}{2} t^2 \ln|t| \right)^2$$

Exercițiu 04

$$4) x' = \frac{x}{t} + \operatorname{tg} \frac{x}{t} \quad \left\{ \begin{array}{l} \text{ec. de tip} \\ \text{omogen} \end{array} \right\}$$

$$u = \frac{x}{t} \Rightarrow x = ut \Rightarrow x' = u't + u$$

$$u't + u = x' = \frac{x}{t} + \operatorname{tg} u$$

$$\frac{du}{dt} \cdot t = \operatorname{tg} u$$

$$\frac{du}{\operatorname{tg} u} = \frac{1}{t} dt$$

$$\int \frac{1}{\operatorname{tg} u} du = \int \frac{1}{t} dt$$

$$\int \frac{\cos u}{\sin u} du = \int \frac{1}{t} dt$$

$$\ln |\sin u| = \ln |t| + C$$

$$\ln |\sin u| = \ln |t| + \ln C$$

$$\sin u = tC$$

$$\sin \frac{x}{t} = tC \quad (\text{sol. implicită})$$

$$\frac{x}{t} = (-1)^k \arcsin(tC) + k\pi$$

$$x = (-1)^k t \arcsin(tC) + t k \pi \quad (\text{sol. generală explicită})$$

Exercițiu 05

$$5) (t^2 + tx + x^2) dt - t^2 dx = 0 \quad / : t^2 dt$$

$$\frac{t^2 + tx + x^2}{t^2} - \frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = 1 + \frac{x}{t} + \left(\frac{x}{t}\right)^2 \quad \left\{ \begin{array}{l} \text{ec. de tip} \\ \text{omogen} \end{array} \right\}$$

$$u = \frac{x}{t} \Rightarrow x = ut \Rightarrow x' = u't + u$$

$$u't + u = 1 + u + u^2$$

$$u't = 1 + u^2$$

$$\frac{du}{dt} \cdot t = 1 + u^2$$

$$\frac{du}{u^2 + 1} = \frac{1}{t} dt$$

$$\int \frac{1}{u^2 + 1} du = \int \frac{1}{t} dt$$

$$\arctg u = \ln |t| + C$$

$$\arctg \frac{x}{t} = \ln |t| + C \quad (\text{sol. în formă implicită})$$

$$\frac{x}{t} = \operatorname{tg}(\ln |t| + C)$$

$$x = t \operatorname{tg}(\ln |t| + C) \quad (\text{sol. generală în formă explicită})$$