Temă - Seminar 5

Tetulescu Mihai-Tiberiu

1. Să se afle inversele următorilor operatori, în cazul în care sunt inversabili:

c) $T: R^2 \to R^3$, $T(x_1, x_2) = (x_1 + 3x_2, x_1 - x_2, 2x_1 + x_2)$

$A_T = \begin{pmatrix} 1 & 3 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}$; A_T nu e inversabilă, deoarece nu este pătratică $\Rightarrow T$ nu e inversabil.

d) $T: R^3 \to R^3$, $T(x_1, x_2, x_3) = (x_1 + 2x_3, x_2 - x_3, x_1 + x_2 - 2x_3)$.

$A_T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{pmatrix}$; $\det A_T = -2 + 0 - 0 + 2 + 1 - 0 = -3 \neq 0$
$\Rightarrow A_T$ inversabilă $\Rightarrow T$ inversabil.

	$\vec{v_1}$	$\vec{v_2}$	$\vec{v_3}$	$\vec{e_1}$	$\vec{e_2}$	$\vec{e_3}$
$\vec{e_1}$	$\boxed{1}$	0	2	1	0	0
$\vec{e_2}$	0	1	-1	0	1	0
$\vec{e_3}$	1	1	-2	0	0	1
$\vec{v_1}$	1	0	2	1	0	0
$\vec{e_2}$	0	$\boxed{1}$	-1	0	1	0
$\vec{e_3}$	0	1	-4	-1	0	1
$\vec{v_1}$	1	0	2	1	0	0
$\vec{v_2}$	0	1	-1	0	1	0
$\vec{e_3}$	0	1	$\boxed{-3}$	-1	-1	1
$\vec{v_1}$	1	0	0	1/3	-2/3	2/3
$\vec{v_2}$	0	1	0	1/3	4/3	-1/3
$\vec{v_3}$	0	0	1	1/3	1/3	-1/3

$\} A_T^{-1}$

$\Rightarrow T^{-1}(\vec{x}) = \left(\frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{2}{3}x_3, \frac{1}{3}x_1 + \frac{4}{3}x_2 - \frac{1}{3}x_3, \frac{1}{3}x_1 + \frac{1}{3}x_2 - \frac{1}{3}x_3\right)$

2. Să se determine vectorii și valorile proprii:

c) $T: R^3 \to R^3$, $T(x_1, x_2, x_3) = (2x_1 + 3x_3, 8x_1 + 2x_2 + 2x_3, 3x_1 + 2x_3)$

$\det(A_T - \lambda I) = 0$.

$A_T = \begin{pmatrix} 2 & 0 & 3 \\ 8 & 2 & 2 \\ 3 & 0 & 2 \end{pmatrix}$;

$A_T - \lambda I = \begin{pmatrix} 2 & 0 & 3 \\ 8 & 2 & 2 \\ 3 & 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 0 & 3 \\ 8 & 2-\lambda & 2 \\ 3 & 0 & 2-\lambda \end{pmatrix}$

$\Rightarrow \det(A_T - \lambda I) = (2-\lambda)^3 - 9(2-\lambda) = (2-\lambda)[(2-\lambda)^2 - 9] = 0$

$\Rightarrow \lambda_1 = 2$; $(2-\lambda)^2 = 9 \Rightarrow \begin{cases} 2-\lambda = 3 \\ 2-\lambda = -3 \end{cases} \Leftrightarrow \begin{cases} \lambda_2 = -1. \\ \lambda_3 = 5 \end{cases}$

• Pentru $\lambda_1 = 2 \Rightarrow (A_T - 2I)\vec{\mu} = \vec{0}$

$$\begin{pmatrix} 2 & 0 & 3 \\ 8 & 2 & 2 \\ 3 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3 \\ 8 & 0 & 2 \\ 3 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 3 \\ 8 & 0 & 2 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 3\mu_3 \\ 8\mu_1 + 2\mu_3 \\ 3\mu_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3\mu_3 = 0 \\ 8\mu_1 + 2\mu_3 = 0 \Rightarrow \mu_1 = \mu_3 = 0,\ \mu_2 \in R. \\ 3\mu_1 = 0 \end{cases}$$

• Pentru $\lambda_2 = -1 \Rightarrow (A_T + I)\vec{\mu} = 0.$

$$\begin{pmatrix} 3 & 0 & 3 \\ 8 & 3 & 2 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 3\mu_1 + 3\mu_3 \\ 8\mu_1 + 3\mu_2 + 2\mu_3 \\ 3\mu_1 + 3\mu_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3\mu_1 + 3\mu_3 = 0 \\ 8\mu_1 + 3\mu_2 + 2\mu_3 = 0 \end{cases} \Rightarrow \begin{cases} \mu_1 = -\mu_3 \\ -6\mu_3 + 3\mu_2 = 0 \end{cases} \Leftrightarrow \begin{cases} \mu_1 = -\mu_3 \\ \mu_2 = 2\mu_3 \\ \mu_3 \in R \end{cases}$$

Pentru $\lambda_3 = 5 \Rightarrow (A_T - 5I)\vec{\mu} = 0$

$$\begin{pmatrix} 2 & 0 & 3 \\ 8 & 2 & 2 \\ 3 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 3 \\ 8 & -3 & 2 \\ 3 & 0 & -3 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 0 & 3 \\ 8 & -3 & 2 \\ 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -3\mu_1 + 3\mu_3 \\ 8\mu_1 - 3\mu_2 + 2\mu_3 \\ 3\mu_1 - 3\mu_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -3\mu_1 + 3\mu_3 = 0 \Rightarrow \mu_1 = \mu_3 \\ 8\mu_1 - 3\mu_2 + 2\mu_3 = 0 \Rightarrow 8\mu_3 - 3\mu_2 + 2\mu_3 = 0 \Rightarrow \mu_2 = \frac{10}{3}\mu_3,\ \mu_3 \in R. \\ 3\mu_1 - 3\mu_2 = 0 \end{cases}$$

Fie $\mu_3 = 1 \Rightarrow$ un vector propriu corespunzător val. proprii $\lambda_3 = 5$ este

$$\vec{\mu}_{\lambda_3} = \begin{pmatrix} 1 \\ 10/3 \\ 1 \end{pmatrix}$$

d) $T: R^3 \to R^3,\ T(x_1, x_2, x_3) = (x_1 + 5x_3, -2x_1 + x_2 - x_3, 5x_1 + x_3)$

$\det(A_T - \lambda I) = 0$

$$A_T - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 5 \\ -2 & 1-\lambda & -1 \\ 5 & 0 & 1-\lambda \end{pmatrix}$$

$$A_T = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -1 \\ 5 & 0 & 1 \end{pmatrix}$$

$\det(A_T - \lambda I) = 0 \Leftrightarrow (1-\lambda)^3 - 25(1-\lambda) = 0$

$\Leftrightarrow (1-\lambda)\left[(1-\lambda)^2 - 25\right] = 0 \Rightarrow 1-\lambda = 0 \Leftrightarrow \lambda_1 = 1$

$(1-\lambda)^2 = 25 \Rightarrow \begin{cases} 1-\lambda = 5 \\ 1-\lambda = -5 \end{cases} \Rightarrow \begin{cases} \lambda_2 = -4 \\ \lambda_3 = 6 \end{cases}$

- Pentru $\lambda_1 = 1 \Rightarrow (A_T - I)\vec{u} = \vec{0}$

$$\begin{pmatrix} 0 & 0 & 5 \\ -2 & 0 & -1 \\ 5 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 5u_3 \\ -2u_1 - u_3 \\ 5u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 5u_3 = 0 \\ -2u_1 - u_3 = 0 \\ 5u_1 = 0 \end{cases} \Rightarrow \begin{cases} u_1 = u_3 = 0 \\ u_2 \in R \end{cases}$$

Fie $u_2 = 1 \Rightarrow$ Un vector propriu coresp. val. proprii $\lambda_1 = 1$ este $\vec{u}_{\lambda_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

- Pentru $\lambda_2 = -4 \Rightarrow (A_T + 4I)\vec{u} = \vec{0}$

$$\begin{pmatrix} 5 & 0 & 5 \\ -2 & 5 & -1 \\ 5 & 0 & 5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 5u_1 + 5u_3 \\ -2u_1 + 5u_2 - u_3 \\ 5u_1 + 5u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \begin{cases} 5u_1 + 5u_3 = 0 \Rightarrow u_1 = -u_3 \\ -2u_1 + 5u_2 - u_3 = 0 \Rightarrow 2u_3 + 5u_2 - u_3 = 0 \end{cases}$

$5u_2 = -u_3 \Leftrightarrow u_2 = \frac{-u_3}{5}, u_3 \in R.$

Fie $u_3 = 1 \Rightarrow$ Un vector propriu coresp. val. proprii $\lambda_2 = -4$ este $\vec{u}_{\lambda_2} = \begin{pmatrix} 1 \\ -1/5 \\ 1 \end{pmatrix}$

- Pentru $\lambda_3 = 6 \Rightarrow (A_T - 6I)\vec{u} = \vec{0}$

$$\begin{pmatrix} -5 & 0 & 5 \\ -2 & -5 & -1 \\ 5 & 0 & -5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -5u_1 + 5u_3 \\ -2u_1 - 5u_2 - u_3 \\ 5u_1 - 5u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \begin{cases} -5u_1 + 5u_3 = 0 \\ -2u_1 - 5u_2 - u_3 = 0 \\ 5u_1 - 5u_3 = 0 \end{cases}$ $\Rightarrow u_1 = u_3$

$-2u_3 - 5u_2 - u_3 = 0 \Leftrightarrow -5u_3 = 3_3$

$\Rightarrow u_2 = \frac{-3u_3}{5}, u_3 \in R.$

Fie $u_3 = 1 \Rightarrow$ Un vector propriu coresp. val. proprii $\lambda_3 = 6$ este $\vec{u}_{\lambda_3} = \begin{pmatrix} 1 \\ -3/5 \\ 1 \end{pmatrix}$