# Ecuații diferențiale și cu derivate parțiale
## Laborator 07
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1. Să se rezolve urm. ec. de ordin n:

$\boxed{a)}$ $x^{IV} = t+2$, $x(0)=1$, $x'(0)=2$, $x''(0)=-1$, $x'''(0)=0$

b) $x'' = t\sin t$, $x(0)=1$, $x'(0)=2$

$\boxed{c)}$ $x''' = \sin t + \cos t$, $x(0)=1$, $x'(0)=2$, $x'''(0)=3$

$\boxed{d)}$ $x'' = \frac{1}{t}$, $x(1)=1$, $x'(1)=2$

e) $x''' = \ln t$, $x(1)=2$, $x'(1)=1$, $x''(1)=0$

<u>Rezolvare</u>:

b) $x'' = t\sin t$, $x(0)=1$, $x'(0)=2$

$x^{(n)} = f(t)$ ; $x'' = f(t)$

• $x' = \int x''\,dt = \int \underset{f}{t}\, \underset{g'}{\sin t}\,dt = -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + c_1$

$f'=1$, $g=-\cos t$

• $x = \int x'\,dt = \int(-t\cos t + \sin t + c_1)\,dt = -\int \underset{f}{t}\,\underset{g'}{\cos t}\,dt + \int \sin t\,dt + \int c_1\,dt$

$f'=1$, $g=\sin t$

$= -(t\sin t - \int \sin t\,dt) + \int \sin t\,dt + c_1\int dt$

$= -t\sin t + 2(-\cos t) + c_1 t + c_2$

• $x(t) = -t\sin t - 2\cos t + c_1 t + c_2$

$\left.\begin{array}{l} x(0) = -2 + c_2 = 1 \Rightarrow c_2 = 3 \\ x'(0) = c_1 = 2 \end{array}\right\} \Rightarrow x_{PC}(t) = -t\sin t = -2\cos t + 2t + 3.$

l) $x''' = \ln t$, $x(1)=2$, $x'(1)=1$, $x''(1)=0$.

$x''' = f(t)$.

• $x'' = \int x''' \, dt = \int \underset{g'}{1} \cdot \underset{f}{\ln t} \, dt = t \ln t - \int \frac{1}{t} t \, dt = t\ln t - t + C_1$

$g = t$, $f' = \frac{1}{t}$

• $x' = \int x'' \, dt = \int (t \ln t - t + C_1) \, dt = \int \underset{g'}{t} \underset{f}{\ln t} \, dt - \int t \, dt + C_1 \int dt =$

$g = \frac{t^2}{2}$, $f' = \frac{1}{t}$

$= \frac{t^2}{2} \ln t - \int \frac{1}{t} \cdot \frac{t^2}{2} dt - \frac{t^2}{2} + C_1 t = \frac{t^2}{2} \ln t - \frac{1}{2} \cdot \frac{t^2}{2} - \frac{t^2}{2} + C_1 t + C_2$.

• $x' = \frac{t^2}{2} \ln t - \frac{t^2}{4} - \frac{\overset{2}{\cancel{t^2}}}{2} + C_1 t + C_2 = \frac{t^2}{2} \ln t - \frac{3t^2}{4} + C_1 t + C_2$

• $x = \int x' \, dt = \int \left( \frac{t^2}{2} \ln t - \frac{3}{4} t^2 + C_1 t + C_2 \right) dt = \frac{1}{2} \int \underset{g'}{t^2} \underset{f}{\ln t} \, dt - \frac{3}{4} \int t^2 dt +$

$g = \frac{t^3}{3}$, $f' = \frac{1}{t}$

$+ C_1 \int t \, dt + C_2 \int dt = \frac{1}{2} \left( \frac{t^3}{3} \ln t - \int \frac{1}{t} \cdot \frac{t^3}{3} dt \right) - \frac{3}{4} \cdot \frac{t^3}{3} + C_1 \frac{t^2}{2} + C_2 t =$

$= \frac{t^3}{6} \ln t - \frac{1}{6} \cdot \frac{t^3}{3} - \frac{t^3}{4} + C_1 \frac{t^2}{2} + C_2 t + C_3$.

• $x(1) = \frac{-1}{18} - \frac{1}{4} + \frac{C_1}{2} + C_2 + C_3 = 2$

$x'(1) = \frac{-3}{4} + C_1 + C_2 = 1$

$x''(1) = -1 + C_1 = 0 \Rightarrow \boxed{C_1 = 1} \Rightarrow C_2 = 1 + \frac{3}{4} - 1 = \frac{3}{4}$

$C_3 = 2 + \frac{1}{18} + \frac{1}{4} - \frac{1}{2} - \frac{3}{4} = 1 + \frac{1}{18} = \frac{19}{18}$

$x_{PC} = \frac{t^3}{6} \ln t - \frac{t^3}{18} - \frac{t^3}{4} + \frac{t^2}{2} + \frac{3}{4} t + \frac{19}{18}$

2. Să se rez. urm. ec. de ordin n:

$\boxed{a)}$ $e^{x''} - (x'')^2 = t+1$

b) $x'' - \sqrt{x''} = t+3$

$\boxed{c)}$ $x'' + \ln x'' = t-5$

<u>Rezolvare:</u>

b) $x'' - \sqrt{x''} = t+3$

$F(t, x'') = 0$

• $x'' = y$

$y - \sqrt{y} = t+3 \Rightarrow t = y - \sqrt{y} - 3 = \varphi(y) \Rightarrow dt = \left(1 - \frac{1}{2\sqrt{y}}\right) dy$

• $x' = \int x'' dt = \int y \cdot \left(1 - \frac{1}{2\sqrt{y}}\right) dy = \int \left(y - y \cdot \frac{1}{2\sqrt{y}}\right) dy = \int y\, dy - \frac{1}{2} \int \sqrt{y}\, dy =$

$= \frac{y^2}{2} - \frac{1}{2} \cdot \int y^{\frac{1}{2}} dy = \frac{y^2}{2} - \frac{1}{2} \cdot \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_1 = \frac{y^2}{2} - \frac{1}{2} \cdot \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + C_1 =$

$= \frac{y^2}{2} - \frac{1}{2} \cdot \frac{2}{3} y^{\frac{3}{2}} + C_1$

• $x = \int x' dt = \int \left(\frac{y^2}{2} - \frac{1}{3} y^{\frac{3}{2}} + C_1\right)\left(1 - \frac{1}{2y^{\frac{1}{2}}}\right) dy = \int \left(\frac{y^2}{2} - \frac{y^2}{2} \cdot \frac{1}{2y^{1/2}} - \frac{1}{3} y^{\frac{3}{2}} + \right.$

$\left. + \frac{1}{3} y^{\frac{3}{2}} \cdot \frac{1}{2y^{1/2}} + C_1 - \frac{C_1}{2y^{1/2}}\right) dy = \frac{1}{2} \int y^2 dy - \frac{1}{4} \int y^{2-\frac{1}{2}} dy - \frac{1}{3} \int y^{\frac{3}{2}} dy$

$+ \frac{1}{6} \int y\, dy + C_1 \int dy - C_1 \int \frac{1}{2\sqrt{y}} dy = \frac{1}{2} \cdot \frac{y^3}{3} - \frac{1}{4} \cdot \frac{y^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{1}{3} \cdot \frac{y^{\frac{3}{2}+1}}{\frac{3}{2}+1}$

$+ \frac{1}{6} \cdot \frac{y^2}{2} + C_1 y - C_1 \sqrt{y} + C_2$

$= \frac{y^3}{6} - \frac{1}{4} \cdot \frac{2}{5} \cdot y^{\frac{5}{2}} - \frac{1}{3} \cdot \frac{2}{5} \cdot y^{\frac{5}{2}} + \frac{1}{12} y^2 + C_1 y - C_1 \sqrt{y} + C_2.$

3. Să se rez. urm. ec. de ordin n:

a) $x''' = \sqrt{1+x''}$, $x(0)=x'(0)=x''(0)=0$

$\boxed{b)}$ $x''+x'\,tg\,t = \sin 2t$

$\boxed{c)}$ $t^2 x'' + 2(x')^2 = 0$, $x(1)=2$, $x'(1)=3$

$\boxed{d)}$ $x'''-x''=t$, $x(1)=1$, $x'(1)=-1$, $x''(1)=2$

e) $t^2x''+tx'=1$

$\boxed{f)}$ $tx'''+x''=1+t$

$\boxed{g)}$ $(1+t^2)x''-2tx'=0$, $x(0)=0$, $x'(0)=3$

$\boxed{h)}$ $x^{(5)}+x^{(4)}=0$.

Rezolvare

a) $x''' = \sqrt{1+x''}$, $x(0)=x'(0)=x''(0)=0$.

$F(t,x'',x''')=0$

$\boxed{x''=y} \Rightarrow x'''=y' \Leftrightarrow y'=\sqrt{1+y} \Leftrightarrow \frac{dy}{dt}=\sqrt{1+y} \Leftrightarrow \frac{dy}{\sqrt{1+y}} = dt$

$2\int \frac{1}{2\sqrt{1+y}}\,dy = \int dt$;

$2\sqrt{1+y}=t+C \;|:2 \Leftrightarrow \sqrt{1+y}=\frac{t}{2} + C \Leftrightarrow 1+y=\left(\frac{t}{2}+C\right)^2$

$\boxed{y=\left(\frac{t}{2}+C\right)^2-1}$

• $x''=\left(\frac{t}{2}+C\right)^2-1$

• $x'=\int\left[\left(\frac{t}{2}+c\right)^2-1\right]dt = \int\left(\frac{t^2}{4}+tc+c^2-1\right)dt$

$=\frac{1}{4}\int t^2dt - C\int t\,dt + C^2\int dt - \int dt$

$=\frac{1}{4} \cdot \frac{t^3}{3} - C\cdot\frac{t^2}{2} + C^2t - t + C_1 = \frac{t^3}{12} - \frac{Ct^2}{2}+C^2t-t+C_1$

• $x=\int\left(\frac{t^3}{12}-\frac{Ct^2}{2}+C^2t-t+C_1\right)dt = \frac{1}{12}\int\frac{t^3}{1}dt - \frac{C}{2}\int t^2dt + C^2\int t\,dt -$

$-\int t\,dt + C_1\int dt = \frac{1}{12}\cdot\frac{t^4}{4} - \frac{C}{2}\cdot\frac{t^3}{3} + C^2\cdot\frac{t^2}{2} - \frac{t^2}{2}+C_1t+C_2$

• $x(t)=\frac{t^4}{48}-\frac{Ct^3}{6}+\frac{C^2t^2}{2}-\frac{t^2}{2}+C_1t+C_2$

$x(0) = \boxed{C_2 = 0}$

$x'(0) = \boxed{C_1 = 0}$

$x''(0) = c^2 - 1 = 0 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1$

$\Rightarrow x_{PC} = \frac{t^4}{48} \pm \frac{t^3}{6} + \frac{t^2}{2} - \frac{t^2}{2}$

$x_{PC} = \frac{t^4}{48} \pm \frac{t^3}{6}$.

2) $t^2 x'' + t x' = 1$

$F(t, x', x'') = 0$

$\boxed{x' = y} \Rightarrow x'' = y'$

$t^2 y' + t y = 1 \,|: t^2 \Leftrightarrow y' + \underbrace{\frac{1}{t}}_{A(t)} y = \underbrace{\frac{1}{t^2}}_{B(t)}$

Etapa 1: $y' + \frac{1}{t} y = 0$

$\frac{dy}{dt} = \frac{-1}{t} y \Leftrightarrow \frac{dy}{y} = \frac{-1}{t} dt \Leftrightarrow \frac{dy}{y} = \frac{-1}{t} dt \Leftrightarrow \int \frac{1}{y} = -\int \frac{1}{t} dt$

$\ln|y| = -\ln t + C \Leftrightarrow \ln|y| = -\ln t + \ln C \Rightarrow \boxed{y_0 = \frac{c}{t}}$

Etapa 2: $\varphi_0 = \frac{c(t)}{t}$

$\left(\frac{c(t)}{t}\right)' + \frac{1}{t} \cdot \frac{c(t)}{t} = \frac{1}{t^2} \Leftrightarrow \frac{c'(t) \cdot t - c(t)}{t^2} + \frac{c(t)}{t^2} = \frac{1}{t^2}$

$\frac{c'(t)}{t} - \cancel{\frac{c(t)}{t^2}} + \cancel{\frac{c(t)}{t^2}} = \frac{1}{t^2} \Leftrightarrow \frac{c'(t)}{t} = \frac{1}{t^2} \,|\cdot t \Leftrightarrow c'(t) = \frac{1}{t}$

$c(t) = \int \frac{1}{t} dt = \ln t + C_1$

$\boxed{\varphi_0 = \frac{\ln t + C_1}{t}} \Leftrightarrow \boxed{y = \varphi_0 + y_0 = \frac{c}{t} + \frac{\ln t}{t}}$

$x' = \frac{c}{t} + \frac{\ln t}{t}$

$x = \int x' dt = \int \left(\frac{c}{t} + \frac{\ln t}{t}\right) dt = c \cdot \int \frac{1}{t} dt + \int \frac{1}{t} \ln t \, dt$

$= c \cdot \ln t + \frac{\ln^2 t}{2} + C_1$

4. Să se rez. urm. ec. de ordin n:

$\boxed{a)}$ $t^2 x x'' = (x - t x')^2$

$\boxed{b)}$ $t x x'' + t(x')^2 - x x' = 0$

c) $t^2 x x'' + t^2 (x')^2 - 5t x x' + 4x^2 = 0$, $x(1)=1$, $x'(1)=0$.

<u>Rezolvare</u>

c) $t^2 x x'' + t^2 (x')^2 - 5t x x' + 4x^2 = 0$, $x(1)=1$, $x'(1)=0$.

$F(t, x, x', x'') = 0$

$$F(t, mx, mx', mx'') = t^2 \cdot mx \cdot mx'' + t^2 \cdot (mx')^2 - 5t \cdot mx \cdot mx' + 4(mx)^2$$

$$= t^2 m^2 x x'' + t^2 m^2 (x')^2 - 5t m^2 x x' + 4 m^2 x^2$$

$$= m^2 \left(t^2 x x'' + t^2 (x')^2 - 5t x x' + 4x^2\right)$$

$$= m^2 \cdot F(t, x, x', x'')$$

$\boxed{x' = x y} \Rightarrow x'' = x' \cdot y + x y' = x y^2 + x y'$

$t^2 \cdot x (x y^2 + x y') + t^2 (x y)^2 - 5t x^2 y + 4x^2 = 0$

$t^2 x^2 y^2 + t^2 x^2 y' + t^2 x^2 y^2 - 5t x^2 y + 4x^2 = 0 \mid : x^2$

$\underline{t^2 y^2} + t^2 y' + \underline{t^2 y^2} - 5t y + 4 = 0$

$t^2 y' + 2t^2 y^2 - 5t y + 4 = 0 \mid : t^2$

$y' + 2y^2 - \frac{5}{t} y + \frac{4}{t^2} = 0$

$y' = \underbrace{-2}_{A(t)} y^2 + \underbrace{\frac{5}{t}}_{B(t)} y \underbrace{- \frac{4}{t^2}}_{C(t)}$ (ec. Riccati).

• Verific dacă $\frac{1}{t}$ e sol.

$\frac{-1}{t^2} = -2 \cdot \frac{1}{t^2} + \frac{5}{t} \cdot \frac{1}{t} - \frac{4}{t^2}$ (A)

$\boxed{y = z + \frac{1}{t}} \Rightarrow y' = z' - \frac{1}{t^2}$

$z' - \frac{1}{t^2} = -2\left(z + \frac{1}{t}\right)^2 + \frac{5}{t}\left(z + \frac{1}{t}\right) - \frac{4}{t^2} = -2z^2 - 4\frac{z}{t} - 2\frac{1}{t^2} + \frac{5z}{t} + \frac{5}{t^2} - \frac{4}{t^2}$

$\Leftrightarrow z' = -2z^2 + \frac{z}{t} \Leftrightarrow z' = \underbrace{-2}_{A(t)} z^2 + \underbrace{\frac{1}{t}}_{B(t)} z$ (Ec. Bernoulli cu $\alpha = 2$)

$$\frac{z'}{z^2} = -2 + \frac{1}{t}\cdot\frac{z}{z^2}$$

$$z'\cdot z^{-2} = -2 + \frac{1}{t}\cdot z^{-1}$$

$$\boxed{u = z^{-1}} \Rightarrow u' = -z^{-2}z'$$

$$-z'z^{-2} = 2 - \frac{1}{t}z^{-1} \Rightarrow u' = 2 - \frac{1}{t}u \text{ (ec. afină)}$$

Etapa 1 $\quad u' = \frac{-1}{t}u \Leftrightarrow \frac{du}{dt} = \frac{-1}{t}u \Leftrightarrow \frac{1}{u}du = \frac{-1}{t}dt \Leftrightarrow \int\frac{1}{u}du = \int\frac{-1}{t}dt \Leftrightarrow$

$$\Leftrightarrow \ln|u| = -\ln|t| + C \Rightarrow \boxed{u_0 = Ct^{-1}}$$

Etapa 2: $\varphi_0 = C(t)t^{-1}$

$$C'(t)t^{-1} + \cancel{C(t)(-t^{-2})} + \cancel{\frac{1}{t}\cdot C(t)\cdot t^{-1}} = 2$$

$$C'(t)t^{-1} = 2 \Leftrightarrow C(t) = \int 2t\,dt = t^2 + C_1$$

$$\varphi_0 = (t^2 + C_1)t^{-1} = t + C_1t^{-1}$$

$$u = u_0 + \varphi_0 = Ct^{-1} + t + C_1t^{-1} = t + Ct^{-1}$$

• $u = \frac{1}{z} \Leftrightarrow z = \frac{1}{u} = \frac{1}{t + Ct^{-1}}$

• $y = z + \frac{1}{t} \Leftrightarrow y = \frac{1}{t + Ct^{-1}} + \frac{1}{t}$

• $y = \frac{x'}{x} \Leftrightarrow \frac{x'}{x} = \frac{1}{t + Ct^{-1}} + \frac{1}{t} \Leftrightarrow \frac{dx}{x} = \left(\frac{1}{t + Ct^{-1}} + \frac{1}{t}\right)dt \Leftrightarrow$

$$\Leftrightarrow \int\frac{dx}{x}\cancel{dx} = \int\left(\frac{1}{t + Ct^{-1}} + \frac{1}{t}\right)dt$$

$$\ln|x| = \frac{\ln|t^2 + C|}{2} + \ln|t| + C_1$$

$$x = C_1t\,(|t^2 + C|)^{\frac{1}{2}} \Leftrightarrow x' = \cancel{\frac{C_1(2t^2 + C)}{\sqrt{t^2 + C}}} = \frac{C_1(|t^2 + C|)^{\frac{1}{2}}}{1 + Ct^{-2}} + C_1(|t^2 + C|)^{\frac{1}{2}}$$

• $x(1) = \cancel{C_1\cdot C^{\frac{1}{2}} = 1}$

• $x'(1) = \cancel{\frac{C_1\cdot(2 + C)}{\sqrt{1 + C}}} = 0 \cancel{\Leftrightarrow C_1(2 + C) = 0 \Leftrightarrow}$ $\begin{cases} C_1 = 1 \\ C = -2 \end{cases}$

• $x_{PC} = t\,(|t^2 - 2|)^{\frac{1}{2}}$