Laborator11 - Rezolvare

Exerciţiu 1. a) - [Video]()

1.a) $\begin{cases} x' = x \\ y' = x + 3y \end{cases}$

$x = \alpha_1 e^{rt}, \quad y = \alpha_2 e^{rt} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{rt} \\ \alpha_2 e^{rt} \end{pmatrix}$

$x' = \alpha_1 r e^{rt}, \quad y' = \alpha_2 r e^{rt}$

$\begin{cases} \alpha_1 r e^{rt} = \alpha_1 e^{rt} & |: e^{rt} \\ \alpha_2 r e^{rt} = \alpha_1 e^{rt} + 3\alpha_2 e^{rt} & |: e^{rt} \end{cases}$

$\begin{cases} \alpha_1 r = \alpha_1 \\ \alpha_2 r = \alpha_1 + 3\alpha_2 \end{cases} \Leftrightarrow \begin{cases} (1-r)\alpha_1 = 0 \\ \alpha_1 + (3-r)\alpha_2 = 0 \end{cases}$

$\Delta = \begin{vmatrix} 1-r & 0 \\ 1 & 3-r \end{vmatrix} = (1-r)(3-r) = 0$

$r_1 = 1$
$r_2 = 3$

Pt. $r_1 = 1$: $\begin{cases} 0 = 0 \\ \alpha_1 + 2\alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = -2\alpha_2, \ \alpha_2 \in \mathbb{R}$

Fie $\alpha_2 = 1 \Rightarrow \alpha_1 = -2 \Rightarrow \alpha_{r_1} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} -2 \cdot e^{1 \cdot t} \\ 1 \cdot e^{1 \cdot t} \end{pmatrix} = \begin{pmatrix} -2e^t \\ e^t \end{pmatrix}$

Pt. $r_2 = 3$: $\begin{cases} -2\alpha_1 = 0 \\ \alpha_1 = 0 \end{cases} \Rightarrow \alpha_1 = 0, \ \alpha_2 \in \mathbb{R}$

Fie $\alpha_2 = 1 \Rightarrow \alpha_{r_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow X_{r_2} = \begin{pmatrix} 0 \cdot e^{3 \cdot t} \\ 1 \cdot e^{3 \cdot t} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{3t} \end{pmatrix}$

$X = C_1 X_{r_1} + C_2 X_{r_2} = C_1 \begin{pmatrix} -2e^t \\ e^t \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ e^{3t} \end{pmatrix} = \begin{pmatrix} -2C_1 e^t \\ C_1 e^t \end{pmatrix} + \begin{pmatrix} 0 \\ C_2 e^{3t} \end{pmatrix}$

$= \begin{pmatrix} -2C_1 e^t \\ C_1 e^t + C_2 e^{3t} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} x = -2C_1 e^t \\ y = C_1 e^t + C_2 e^{3t} \end{cases}$

Exerciţiu 1. c)

1.c) $\begin{cases} x' = 2x \\ y' = x + y \\ z' = x + 2y + 3z \end{cases}$

$x = \alpha_1 e^{rt}, \quad y = \alpha_2 e^{rt}, \quad z = \alpha_3 e^{rt}$

$x' = \alpha_1 r e^{rt}, \quad y' = \alpha_2 r e^{rt}, \quad z' = \alpha_3 r e^{rt}$

$\begin{cases} \alpha_1 r e^{rt} = 2\alpha_1 e^{rt} & |: e^{rt} \\ \alpha_2 r e^{rt} = \alpha_1 e^{rt} + \alpha_2 e^{rt} & |: e^{rt} \\ \alpha_3 r e^{rt} = \alpha_1 e^{rt} + 2\alpha_2 e^{rt} + 3\alpha_3 e^{rt} & |: e^{rt} \end{cases}$

$\begin{cases} \alpha_1 r = 2\alpha_1 \\ \alpha_2 r = \alpha_1 + \alpha_2 \\ \alpha_3 r = \alpha_1 + 2\alpha_2 + 3\alpha_3 \end{cases} \Rightarrow \begin{cases} (2-r)\alpha_1 = 0 \\ \alpha_1 + (1-r)\alpha_2 = 0 \\ \alpha_1 + 2\alpha_2 + (3-r)\alpha_3 = 0 \end{cases}$

$\Delta = \begin{vmatrix} 2-r & 0 & 0 \\ 1 & 1-r & 0 \\ 1 & 2 & 3-r \end{vmatrix} = (2-r)(1-r)(3-r) = 0$

$r_1 = 1, \ r_2 = 2, \ r_3 = 3$

Pt. $r_1 = 1$: $\begin{cases} \alpha_1 = 0 \\ \alpha_1 = 0 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 = 0 \end{cases} \Rightarrow \begin{array}{l} \alpha_1 = 0 \\ 2\alpha_2 + 2\alpha_3 = 0 \Rightarrow \alpha_3 = -\alpha_2, \ \alpha_2 \in \mathbb{R} \end{array}$

Fie $\alpha_2 = 1 \Rightarrow \alpha_3 = -1 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{1 \cdot t} = \begin{pmatrix} 0 \\ e^t \\ -e^t \end{pmatrix}$

Pt. $r_2 = 2$: $\begin{cases} 0 = 0 \\ \alpha_1 - \alpha_2 = 0 \\ \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \end{cases} \Rightarrow \begin{array}{l} \alpha_2 = \alpha_1 \\ \alpha_3 = -\alpha_1 - 2\alpha_2 = -3\alpha_1 \end{array}$

Fie $\alpha_1 = 1 \Rightarrow X_{r_2} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \Rightarrow X_{r_2} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} e^{2 \cdot t} = \begin{pmatrix} e^{2t} \\ e^{2t} \\ -3e^{2t} \end{pmatrix}$

Pt. $r_3 = 3$: $\begin{cases} -\alpha_1 = 0 \\ \alpha_1 - 2\alpha_2 = 0 \\ \alpha_1 + 2\alpha_2 = 0 \end{cases}$ $\quad \alpha_1 = 0 \Rightarrow \alpha_2 = 0, \ \alpha_3 \in \mathbb{R}$

Fie $\alpha_3 = 1 \Rightarrow \alpha_{r_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow X_{r_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix}$

$X = C_1 X_{r_1} + C_2 X_{r_2} + C_3 X_{r_3} = C_1 \begin{pmatrix} 0 \\ e^t \\ -e^t \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ e^{2t} \\ -3e^{2t} \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix}$

$X = \begin{pmatrix} C_2 e^{2t} \\ C_1 e^t + C_2 e^{2t} \\ -C_1 e^t - 3C_2 e^{2t} + C_3 e^{3t} \end{pmatrix}$

$x = C_2 e^{2t}$
$y = C_1 e^t + C_2 e^{2t}$
$z = -C_1 e^t - 3C_2 e^{2t} + C_3 e^{3t}$

Exerciţiu 2. b) - Video

2a

$$\begin{cases} x' = 2x - y \\ y' = x + 2y \end{cases}$$

$x = \alpha_1 e^{rt}$, $y = \alpha_2 e^{rt} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{rt} \\ \alpha_2 e^{rt} \end{pmatrix}$

$x' = \alpha_1 r e^{rt}$, $y' = \alpha_2 r e^{rt}$

$$\begin{cases} \alpha_1 r e^{rt} = 2\alpha_1 e^{rt} - \alpha_2 e^{rt} & \mid : e^{rt} \\ \alpha_2 r e^{rt} = \alpha_1 e^{rt} + 2\alpha_2 e^{rt} & \mid : e^{rt} \end{cases}$$

$$\begin{cases} \alpha_1 r = 2\alpha_1 - \alpha_2 \\ \alpha_2 r = \alpha_1 + 2\alpha_2 \end{cases} \Rightarrow \begin{cases} (2-r)\alpha_1 - \alpha_2 = 0 \\ \alpha_1 + (2-r)\alpha_2 = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 2-r & -1 \\ 1 & 2-r \end{vmatrix} = (2-r)^2 + 1 = 0$$

$4 - 4r + r^2 + 1 = 0$

$r^2 - 4r + 5 = 0$

$\Delta_r = 16 - 20 = -4$

$r_{1,2} = \frac{4 \pm 2i}{2} = 2 \pm i \quad \left(\gamma \pm i\beta,\ \gamma = 2,\ \beta = 1\right)$

Pt. $r = 2 + i$

$$\begin{cases} -i\alpha_1 - \alpha_2 = 0 \\ \alpha_1 - i\alpha_2 = 0 \end{cases} \Rightarrow \alpha_2 = i\alpha_1,\ \alpha_1 \in \mathbb{R}$$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = i \Rightarrow \alpha_r = \begin{pmatrix} 1 \\ i \end{pmatrix} \Rightarrow X_r = \begin{pmatrix} 1 \cdot e^{2t}(\cos t + i \sin t) \\ i \cdot e^{2t}(\cos t + i \sin t) \end{pmatrix}$

$$X_r = \begin{pmatrix} e^{2t}\cos t + i e^{2t} \sin t \\ i e^{2t} \cos t - e^{2t} \sin t \end{pmatrix} = \underbrace{\begin{pmatrix} e^{2t}\cos t \\ -e^{2t}\sin t \end{pmatrix}}_{\tilde{X}_r} + i \underbrace{\begin{pmatrix} e^{2t}\sin t \\ e^{2t}\cos t \end{pmatrix}}_{\approx\atop X_r}$$

$$X = C_1 \tilde{X}_r + C_2 \overset{\approx}{X}_r = C_1 \begin{pmatrix} e^{2t}\cos t \\ -e^{2t}\sin t \end{pmatrix} + C_2 \begin{pmatrix} e^{2t}\sin t \\ e^{2t}\cos t \end{pmatrix}$$

$$= \begin{pmatrix} C_1 e^{2t}\cos t + C_2 e^{2t}\sin t \\ -C_1 e^{2t}\sin t + C_2 e^{2t}\cos t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$x = C_1 e^{2t}\cos t + C_2 e^{2t} \sin t$

$y = -C_1 e^{2t} \sin t + C_2 e^{2t} \cos t$

Exerciţiu 3. a) - Video

3a

$$\begin{cases} x' = 2x + z \\ y' = x + y \\ z' = 2y + 2z \end{cases}$$

$x = \alpha_1 e^{rt}$, $y = \alpha_2 e^{rt}$, $z = \alpha_3 e^{rt}$

$x' = \alpha_1 r e^{rt}$, $y' = \alpha_2 r e^{rt}$, $z' = \alpha_3 r e^{rt}$

$$\begin{cases} \alpha_1 r e^{rt} = 2\alpha_1 e^{rt} + \alpha_3 e^{rt} & \mid : e^{rt} \\ \alpha_2 r e^{rt} = \alpha_1 e^{rt} + \alpha_2 e^{rt} & \mid : e^{rt} \\ \alpha_3 r e^{rt} = 2\alpha_2 e^{rt} + 2\alpha_3 e^{rt} & \mid : e^{rt} \end{cases}$$

$$\begin{cases} \alpha_1 r = 2\alpha_1 + \alpha_3 \\ \alpha_2 r = \alpha_1 + \alpha_2 \\ \alpha_3 r = 2\alpha_2 + 2\alpha_3 \end{cases} \Leftrightarrow \begin{cases} (2-r)\alpha_1 + \alpha_3 = 0 \\ \alpha_1 + (1-r)\alpha_2 = 0 \\ 2\alpha_2 + (2-r)\alpha_3 = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 2-r & 0 & 1 \\ 1 & 1-r & 0 \\ 0 & 2 & 2-r \end{vmatrix} \xrightarrow{L_1 + L_2 + L_3} \begin{vmatrix} 3-r & 3-r & 3-r \\ 1 & 1-r & 0 \\ 0 & 2 & 2-r \end{vmatrix} = (3-r) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-r & 0 \\ 0 & 2 & 2-r \end{vmatrix} =$$

$$= (3-r)\left[(1-r)(2-r) + \cancel{2} - \cancel{2} + r\right] = (3-r)(2 - \cancel{r} - 2r + r^2 + \cancel{r}) = (3-r)(r^2 - 2r + 2) = 0$$

$r_1 = 3$, $r^2 - 2r + 2 = 0 \Rightarrow r_{2,3} = 1 \pm i$

Pt. $r_1 = 3$

$$\begin{cases} -\alpha_1 + \alpha_3 = 0 & \Rightarrow \alpha_3 = \alpha_1 \\ \alpha_1 - 2\alpha_2 = 0 & \Rightarrow \alpha_2 = \frac{\alpha_1}{2} \\ 2\alpha_2 - \alpha_3 = 0 \end{cases}$$

Fie $\alpha_1 = 2 \Rightarrow \alpha_2 = 1,\ \alpha_3 = 2 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} 2 \cdot e^{3 \cdot t} \\ 1 \cdot e^{3 \cdot t} \\ 2 \cdot e^{3 \cdot t} \end{pmatrix} = \begin{pmatrix} 2e^{3t} \\ e^{3t} \\ 2e^{3t} \end{pmatrix}$

Pt. $r_2 = 1 + i$, $\gamma = 1, \beta = 1$

$$\begin{cases} (1-i)\alpha_1 + \alpha_3 = 0 \\ \alpha_1 - i\alpha_2 = 0 & \Rightarrow \alpha_1 = i\alpha_2 \\ 2\alpha_2 + (1-i)\alpha_3 = 0 & \alpha_3 = \frac{\overset{1+i}{-2}}{1-i}\alpha_2 = \frac{-\cancel{2}(1+i)}{\cancel{2}}\alpha_2 = -(1+i)\alpha_2 \end{cases}$$

Fie $\alpha_2 = 1 \Rightarrow \alpha_1 = i,\ \alpha_3 = -1-i$

$\alpha_{r_2} = \begin{pmatrix} i \\ 1 \\ -1-i \end{pmatrix} \Rightarrow X_{r_2} = \begin{pmatrix} i \\ 1 \\ -1-i \end{pmatrix} \cdot e^{1 \cdot t}\left(\cos 1 \cdot t + i \sin 1 \cdot t\right) = \begin{pmatrix} ie^t(\cos t + i\sin t) \\ e^t(\cos t + i \sin t) \\ (-1-i) \cdot e^t(\cos t + i \sin t) \end{pmatrix}$

$= \begin{pmatrix} ie^t\cos t - \underline{e^t \sin t} \\ \underline{e^t \cos t} + ie^t \sin t \\ \underline{-e^t \cos t} - ie^t \sin t - ie^t \cos t + \underline{e^t \sin t} \end{pmatrix} = \underbrace{\begin{pmatrix} -e^t \sin t \\ e^t \cos t \\ -e^t\cos t + e^t \sin t \end{pmatrix}}_{\tilde{X}_{r_2}} + i \underbrace{\begin{pmatrix} e^t \cos t \\ e^t \sin t \\ -e^t \sin t - e^t \cos t \end{pmatrix}}_{\tilde{\tilde{X}}_{r_2}}$

$X = c_1 X_{r_1} + c_2 \tilde{X}_{r_2} + c_3 \tilde{\tilde{X}}_{r_2}$

$= c_1 \cdot \begin{pmatrix} 2e^{3t} \\ e^{3t} \\ 2e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} -e^t \sin t \\ e^t \cos t \\ -e^t \cos t + e^t \sin t \end{pmatrix} + c_3 \begin{pmatrix} e^t \cos t \\ e^t \sin t \\ -e^t \sin t - e^t \cos t \end{pmatrix} = \begin{pmatrix} 2c_1 e^{3t} - c_2 e^t \sin t + c_3 e^t \cos t \\ c_1 e^{3t} + c_2 e^t \cos t + c_3 e^t \sin t \\ 2c_1 e^{3t} + c_2(-e^t \cos t + e^t \sin t) + c_3(-e^t \sin t - e^t \cos t) \end{pmatrix}$

$$\begin{cases} x = 2c_1 e^{3t} - c_2 e^t \sin t + c_3 e^t \cos t \\ y = c_1 e^{3t} + c_2 e^t \cos t + c_3 e^t \sin t \\ z = 2c_1 e^{3t} - c_2 e^t(\cos t - \sin t) - c_3 e^t(\sin t + \cos t) \end{cases}$$