Laborator14

Exerciţii

1. Să se determine soluţia generală a sistemelor simetrice:

$$a)\ \frac{dx}{x} = -\frac{dy}{2y} = \frac{dz}{-z},\ x \neq 0,\ y \neq 0,\ z \neq 0$$

$$R: \begin{cases} \varphi(x,y,z) = x\sqrt{y} = C_1 \\ \varphi(x,y,z) = xz = C_2 \end{cases}$$

$$b)\ \frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x},\ x \neq y \neq z$$

$$R: \begin{cases} x+y+z = C_1 \\ x^2+y^2+z^2 = C_2 \end{cases}$$

$$c)\ \frac{dx}{xy^2} = \frac{dy}{x^2y} = \frac{dz}{z(x^2+y^2)},\ x \neq 0,\ y \neq 0,\ z \neq 0$$

$$R: \begin{cases} x^2-y^2 = C_1 \\ \frac{xy}{z} = C_2 \end{cases}$$

$$d)\frac{dx}{2y(2-x)} = \frac{dy}{x^2-z^2-y^2-4x} = \frac{dz}{-2yz},\ x > 2,\ y \neq 0,\ z > 0$$

$$R: \begin{cases} \frac{x-2}{y} = C_1 \\ \frac{x^2+y^2+z^2}{z} = C_2 \end{cases}$$

$$e)\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{(y-x)(2x+2y+z)},\ |x| = |y| \neq 0$$

$$R: \begin{cases} xy = C_1 \\ (x+y)(x+y+z) = C_2 \end{cases}$$

$$f)\frac{dx}{2xz} = \frac{dy}{2yz} = \frac{dz}{x^2-y^2},\ |x| \neq |y| \neq 0$$

$$R: \begin{cases} \frac{y}{x} = C_1 \\ 2z^2-x^2+y^2 = C_2 \end{cases}$$

2. Să se rezolve următoarele sisteme cu ajutorul integralelor prime

$$a)\begin{cases} x' = \frac{y}{x-y} \\ y' = \frac{x}{x-y} \end{cases}, \ x \neq y$$

$$R: \begin{cases} x - y + t = C_1 \\ x^2 - y^2 = C_2 \end{cases}$$

$$b)\begin{cases} x' = y \\ y' = -\frac{y^2+1}{x} \end{cases}, \ x \neq 0$$

$$R: \begin{cases} y^2(z^2+1) = C_1 \\ yz + x = C_2 \end{cases}$$

$$c)\begin{cases} x' = 2zy \\ y' = 4xz \\ z' = xy \end{cases}, \ x \neq 0, \ y \neq 0, \ z \neq 0$$

$$R: \begin{cases} \frac{y^2}{2} - 2z^2 = C_1 \\ \frac{x^2}{2} - z^2 = C_2 \\ x^2 + \frac{y^2}{2} - 2z^2 = C_3 \end{cases}$$

$$d)\begin{cases} x' = x \\ y' = y \\ z' = -2xy \end{cases}, \ x \neq 0, \ y \neq 0$$

$$R: \begin{cases} \frac{x}{y} = C_1 \\ ln(x) - t = C_2 \\ xy - z = C_3 \end{cases} \Rightarrow \begin{cases} x = Ce^t \\ y = \frac{C}{C_1}e^t = C_4e^t \\ z = CC_4e^{2t} - C_3 \end{cases}$$

$$e)\begin{cases} x' = y \\ y' = x \\ z' = x - y \end{cases}, \ y \neq x \neq 0$$

$$R: \begin{cases} x^2 - y^2 = C_1 \\ x - y + z = C_2 \\ y - x - e^{-t} = C_3 \end{cases} \Rightarrow x - y = C_2 - z$$

$$f)\begin{cases} x' = y + xy \\ y' = x + yx \\ z' = z^2 - 1 \end{cases}, \ x \neq y, \ |z| > 1$$

$$R: \begin{cases} (x-y)\sqrt{t^2-1} = C_1 \\ ln(x-y) + t = C_2 \\ \frac{1}{2}ln(z^2-1) - t = C_3 \end{cases}$$

$$g) \begin{cases} x' = xy \\ y' = -y^2 \\ z' = -x(1+x^2) \end{cases}, \; x > 0, \; y > 0$$

$$R: \begin{cases} xy = C_1 \\ \frac{1}{y} - t = C_2 \\ \frac{x^3}{3} + \frac{x^5}{5} + xyz = C_3 \end{cases}$$

Rezolvare

Exerciţiu 1. a) - [Video]

① a) $\frac{dx}{x} = \frac{dy}{-2y} = \frac{dz}{-z}$, $\cancel{x}, y, z \neq 0$.

$\frac{dx}{x} = \frac{dy}{-2y}$

$\int \frac{1}{x} dx = -\frac{1}{2} \int \frac{1}{y} dy$

$\ln|x| = -\frac{1}{2} \ln|y| + C_1 \quad | \cdot 2$

$2\ln|x| = -\ln|y| + C_1$

$\ln|x|^2 + \ln|y| = C_1$

$\ln |x|^2 \cdot |y| = C_1$

$\boxed{x^2 y = C_1 = \varphi_1(x, y, z)}$

$\frac{dx}{x} = \frac{dz}{z}$

$\int \frac{1}{x} dx = -\int \frac{1}{z} dz$

$\ln|x| = -\ln|z| + C_2$

$\ln|x| + \ln|z| = C_2$

$\ln(|x| \cdot |z|) = C_2$

$\boxed{x z = C_2 = \varphi_2(x, y, z)}$

Metoda 1

Fie z variabila indep.

φ_1, φ_2 indep $\Leftrightarrow \frac{D(\varphi_1, \varphi_2)}{D(x, y)} \neq 0 \Leftrightarrow \begin{vmatrix} \frac{\partial \varphi_1}{\partial x} & \frac{\partial \varphi_1}{\partial y} \\ \frac{\partial \varphi_2}{\partial x} & \frac{\partial \varphi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2xy & x^2 \\ z & 0 \end{vmatrix} = -zx^2 \neq 0 \Rightarrow \varphi_1 \varphi_2$ indep.

$\Rightarrow$ Sol. în forma implicită este $\begin{cases} x^2 y = C_1 \\ x z = C_2 \end{cases} \rightarrow x = \frac{C_2}{z} \rightarrow y = \frac{C_1}{x^2} = \frac{C_1}{\frac{C_2^2}{z^2}} = \frac{C_1 z^2}{C_2^2} \Rightarrow$ Sol. în forma explicită $\begin{cases} x = \frac{C_2}{z} \\ y = \frac{C_1 z^2}{C_2^2} \end{cases}$

Metoda 2

φ_1, φ_2 indep. $\Leftrightarrow$ rang $\left(\frac{D(\varphi_1, \varphi_2)}{D(x, y, z)}\right) = 2$

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x} & \frac{\partial \varphi_1}{\partial y} & \frac{\partial \varphi_1}{\partial z} \\ \frac{\partial \varphi_2}{\partial x} & \frac{\partial \varphi_2}{\partial y} & \frac{\partial \varphi_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 2xy & x^2 & 0 \\ z & 0 & x \end{pmatrix}$$

$D_p = \begin{vmatrix} x^2 & 0 \\ 0 & x \end{vmatrix} = x^3 \neq 0 \Rightarrow$ rang $\left(\frac{D(\varphi_1, \varphi_2)}{D(x, y, z)}\right) = 2 \Rightarrow \varphi_1, \varphi_2$ = independente.

$\Rightarrow \begin{cases} x^2 y = C_1 \\ x z = C_2 \end{cases}$ (Sol. in formă implicită)

Exerciţiu 1. b)

① a) $\dfrac{dx}{z-y}=\dfrac{dy}{x-z}=\dfrac{dz}{y-x}$, $x\neq y\neq z$

$$\frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{y-x}=\frac{dx+dy+dz}{\cancel{z}-\cancel{y}+\cancel{x}-\cancel{z}+\cancel{y}-\cancel{x}}=\frac{d(x+y+z)}{0}$$

$$\Rightarrow d(x+y+z)=0 \Rightarrow \boxed{x+y+z=C_1=\varphi_1(x,y,z)}$$

$$\frac{x\,dx}{x(z-y)}=\frac{y\,dy}{y(x-z)}=\frac{z\,dz}{z(y-x)}=\frac{x\,dx}{xz-xy}=\frac{y\,dy}{xy-yz}=\frac{z\,dz}{yz-xz}$$

$$=\frac{x\,dx+y\,dy+z\,dz}{\cancel{xz}-\cancel{xy}+\cancel{xy}-\cancel{yz}+\cancel{yz}-\cancel{xz}}=\frac{\frac{1}{2}d(x^2+y^2+z^2)}{0}$$

$$\Rightarrow d(x^2+y^2+z^2)=0 \Rightarrow \boxed{x^2+y^2+z^2=C_2=\varphi_2(x,y,z)}$$

Verif. dacă φ_1, φ_2 sunt indep. $\Big\} \Rightarrow \dfrac{D(\varphi_1,\varphi_2)}{D(x,y)}\neq 0$.
Fie z variabila indep.

$$\frac{D(\varphi_1,\varphi_2)}{D(x,y)}=\begin{vmatrix}\frac{\partial\varphi_1}{\partial x} & \frac{\partial\varphi_1}{\partial y}\\ \frac{\partial\varphi_2}{\partial x} & \frac{\partial\varphi_2}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & 1\\ 2x & 2y\end{vmatrix}=2y-2x=2(y-x)\neq 0$$
$\Rightarrow \varphi_1, \varphi_2$ indep.

$$\boxed{\frac{a}{b}=\frac{c}{d}}=\frac{a+c}{b+d}=\frac{a-c}{b-d}=\frac{\lambda a}{\lambda b}$$

$\dfrac{dx}{0} \Rightarrow dx=0 \Rightarrow x=ct$

$dx\pm dy=d(x\pm y)$

$x\,dx+y\,dy=\frac{1}{2}d(x^2+y^2)$

$y\,dx+x\,dy=d(xy)$

Sol. în formă implicită este
$$\begin{cases}x+y+z=C_1\\ x^2+y^2+z^2=C_2\end{cases}$$

Exerciţiu 1. c) - Video

① c) $\dfrac{dx}{xy^2}=\dfrac{dy}{x^2y}=\dfrac{dz}{z(x^2+y^2)}$, $x,y,z\neq 0$

$\dfrac{dx}{xy^2}=\dfrac{dy}{x^2y}$ $\Big|\cdot xy$

$\dfrac{dx}{y}=\dfrac{dy}{x}$

$x\,dx=y\,dy$

$\int x\,dx=\int y\,dy$

$\dfrac{x^2}{2}=\dfrac{y^2}{2}+C_1$

$\boxed{x^2-y^2=C_1=\varphi_1(x,y,z)}$

$$\frac{{}^{y/}dx}{xy^2}=\frac{{}^{x/}dy}{x^2y}=\frac{y\,dx}{xy^3}=\frac{x\,dy}{x^3y}=\frac{y\,dx+x\,dy}{xy^3+x^3y}=\frac{d(xy)}{xy\underbrace{(y^2+x^2)}}=\frac{dz}{z\underbrace{(x^2+y^2)}}$$

$\dfrac{d(xy)}{xy(y^2+x^2)}=\dfrac{dz}{z(x^2+y^2)}$ $\Big|\cdot(x^2+y^2)$

$\dfrac{d(xy)}{xy}=\dfrac{dz}{z}$

Notăm $xy=u$ $\Rightarrow \dfrac{du}{u}=\dfrac{dz}{z} \Leftrightarrow \int\frac{1}{u}du=\int\frac{1}{z}dz \Leftrightarrow \ln|u|=\ln|z|+C_2$

$\ln|u|-\ln|z|=C_2$

$\ln\left|\dfrac{u}{z}\right|=C_2$

$\dfrac{u}{z}=C_2$

$\boxed{\dfrac{xy}{z}=C_2=\varphi_2(x,y,z)}$

Verif. dc. φ_1,φ_2 sunt indep. $\Big| \Leftrightarrow \dfrac{D(\varphi_1,\varphi_2)}{D(x,y)}\neq 0$.
Pp. z – variabilă indep.

$$\begin{vmatrix}\frac{\partial\varphi_1}{\partial x} & \frac{\partial\varphi_1}{\partial y}\\ \frac{\partial\varphi_2}{\partial x} & \frac{\partial\varphi_2}{\partial y}\end{vmatrix}=\begin{vmatrix}2x & -2y\\ \frac{y}{z} & \frac{x}{z}\end{vmatrix}=\frac{2x^2}{z}+\frac{2y^2}{z}=\frac{2}{z}(x^2+y^2)\neq 0 \rightarrow \varphi_1,\varphi_2 \text{ indep.}$$

Exerciţiu 1. e) - Video

① e) $\dfrac{dx}{x(x+y)}=\dfrac{dy}{-y(x+y)}=\dfrac{dz}{(y-x)(2x+2y+z)}$, $|x|+|y|\neq 0$

$\dfrac{dx}{x(x+y)}=\dfrac{dy}{-y(x+y)}$ $\Big|\cdot(x+y)$

$\dfrac{dx}{x}=-\dfrac{dy}{y}$

$\int\frac{1}{x}dx=-\int\frac{1}{y}dy$

$\ln|x|=-\ln|y|+C_1$

$\ln|x|+\ln|y|=C_1$

$\ln|xy|=C_1$

$\boxed{xy=C_1=\varphi_1(x,y,z)}$

$$\frac{dx}{x(x+y)}=\frac{dy}{-y(x+y)}=\frac{dx+dy}{x(x+y)-y(x+y)}=\frac{d(x+y)}{(x+y)\underbrace{(x-y)}}=\frac{dz}{\underbrace{(y-x)}(2x+2y+z)}$$

$\dfrac{d(x+y)}{(x+y)(x-y)}=\dfrac{dz}{-(x-y)[2(x+y)+z]}$ $\Big|\cdot(x-y)$

$\dfrac{d(x+y)}{x+y}=\dfrac{dz}{-[2(x+y)+z]}$, Notăm $x+y=u$ $\Rightarrow \dfrac{du}{u}=\dfrac{dz}{-2u-z}$

$\dfrac{du}{dz}=\dfrac{u}{-2u-z}=\dfrac{1}{-2-\frac{z}{u}}$

Fie z variabila indep.

$\dfrac{du}{dz}=\dfrac{1}{-2-\frac{1}{\frac{u}{z}}}$, $\dfrac{u}{z}=v \Rightarrow u=vz \Rightarrow u'=v'z+v$

$v'z+v=\dfrac{1}{-2-\frac{1}{v}} \Rightarrow v'z=\dfrac{v}{-2v-1}-v=\dfrac{v+2v^2+v}{-2v-1}=\dfrac{2v^2+2v}{-2v-1}$

$\dfrac{dv}{dz}\cdot z=\dfrac{2(v^2+v)}{-(2v+1)} \Rightarrow \dfrac{2v+1}{v^2+v}dv=-\dfrac{2}{z}dz$

$$\int \frac{2v+1}{v^2+v}\,dv = -2\int \frac{1}{z}\,dz$$

$$\ln|v^2+v| = -2\ln|z| + C_2$$

$$\ln|v^2+v| + \ln|z|^2 = C_2$$

$$\ln|z^2(v^2+v)| = C_2$$

$$z^2(v^2+v) = C_2$$

$$z^2\left(\frac{u^2}{z^2}+\frac{u}{z}\right) = C_2$$

$$\cancel{z^2}\cdot\frac{u^2+zu}{\cancel{z^2}} = C_2$$

$$(x+y)^2 + z(x+y) = C_2$$

$$\boxed{(x+y)(x+y+z) = C_2 = \varphi_2(x,y,z)}$$

z – var. indep.

φ_1, φ_2 indep $\Leftrightarrow \dfrac{D(\varphi_1,\varphi_2)}{D(x,y)} \neq 0$

$$\begin{vmatrix} \frac{\partial \varphi_1}{\partial x} & \frac{\partial \varphi_1}{\partial y} \\ \frac{\partial \varphi_2}{\partial x} & \frac{\partial \varphi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 2x+2y+z & 2x+2y+z \end{vmatrix} = (2x+2y+z)\begin{vmatrix} y & x \\ 1 & 1 \end{vmatrix}$$

$$= \underbrace{(2x+2y+z)}_{\neq 0}\underbrace{(y-x)}_{\neq 0}$$

$$\left[(x+y)(x+y+z)\right]'_x = x+y+z+(x+y)$$

$\Rightarrow \varphi_1, \varphi_2$ – indep.

Sol. implicită:

$$\begin{cases} xy = C_1 \\ (x+y)(x+y+z) = C_2 \end{cases}$$

Exerciţiu 2. b)

2 b)
$$\begin{cases} x' = y \\ y' = -\dfrac{y^2+1}{x} \end{cases},\ x \neq 0 \Leftrightarrow \begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\dfrac{y^2+1}{x} \end{cases} \Leftrightarrow \begin{cases} \dfrac{dx}{y} = dt \\ \dfrac{-x}{y^2+1}dy = dt \end{cases} \Leftrightarrow \frac{dx}{y} = \frac{-x\,dy}{y^2+1} = dt$$

$$\frac{dx}{y} = \frac{-x\,dy}{y^2+1}$$

$$\frac{dx}{x} = -\frac{y\,dy}{y^2+1}$$

$$\int \frac{1}{x}\,dx = -\frac{1}{2}\int \frac{2y}{y^2+1}\,dy$$

$$\ln|x| = -\frac{1}{2}\ln|y^2+1| + C_1$$

$$2\ln|x| = -\ln|y^2+1| + C_1$$

$$\ln x^2 + \ln(y^2+1) = C_1$$

$$\ln x^2(y^2+1) = C_1$$

$$\boxed{x^2(y^2+1) = C_1 = \varphi_1(t,x,y)}$$

$$\frac{dx}{y} = \frac{-x\,dy}{y^2+1} = \frac{y\,dx}{y^2} = \frac{x\,dy}{-y^2-1} = \frac{y\,dx + x\,dy}{\cancel{y^2}-\cancel{y^2}-1} = \frac{d(xy)}{-1} = dt$$

$$\frac{d(xy)}{-1} = dt \Leftrightarrow d(xy) = -dt$$

Notăm $\boxed{xy = u}$

$$\Rightarrow du = -dt \Leftrightarrow \int du = -\int dt$$

$$u = -t + C_2$$

$$u + t = C_2$$

$$\boxed{xy + t = C_2 = \varphi_2(t,x,y)}$$

Verif. de. φ_1, φ_2 sunt indep.

$$\Leftrightarrow \frac{D(\varphi_1,\varphi_2)}{D(x,y)} \neq 0 \Leftrightarrow \begin{vmatrix} 2x(y^2+1) & 2yx^2 \\ y & x \end{vmatrix} = 2x^2(y^2+1) - 2y^2x^2$$

$$= \cancel{2x^2y^2} + 2x^2 - \cancel{2y^2x^2} = \underbrace{2x^2}_{\neq 0}$$

$\Rightarrow \varphi_1, \varphi_2$ indep.

Sol. implicită:
$$\begin{cases} x^2(y^2+1) = C_1 \\ xy + t = C_2 \end{cases}$$