Ecuații diferențiale și cu derivate parțiale

Laborator 10

05.12.2020

Să se rezolve următoarele ecuații:

$\boxed{a)}$ $x''-2x'+5x=e^t(t\cos 2t - t^2\sin 2t)$

T b) $x''+9x=t\sin 3t+\cos 3t$

$\boxed{c)}$ $x''-9x=e^{2t}+te^t-t^2-2$

<u>Rezolvare</u>

a) $x''-2x'+5x=e^t(t\cos 2t - t^2\sin 2t)$

$x''-2x'+5x=0$

$$\left.\begin{array}{l} x=e^{rt}\\ x'=re^{rt}\\ x''=r^2e^{rt}\end{array}\right\} \Leftrightarrow r^2e^{rt}-2re^{rt}+5e^{rt}=0 \;\Big|:e^{rt}$$

$$\left.\begin{array}{l} r^2-2r+5=0\\ \Delta=4-20=-16\end{array}\right\} \Leftrightarrow r_{1,2}=\frac{2\pm 4i}{2}=1\pm 2i \quad \left(\alpha\pm\beta i,\ \begin{array}{l}\alpha=1\\ \beta=2\end{array}\right)$$

• $e^{t}\cos 2t,\ e^{t}\sin 2t$ - sist. fund de sol.

$$\boxed{x_0=C_1e^t\cos 2t + C_2e^t\sin 2t}$$

• $f(t)=e^{\alpha t}(P_1(t)\cos\beta t + P_2(t)\sin\beta t)$, $\alpha=1, \beta=2 \to \alpha\pm\beta i=1\pm 2i$

gr. $P_1=1$, gr. $P_2=2$.

$$x_p=te^t\left[(\lambda_2t^2+\lambda_1t+\lambda_0)\cos 2t+(\beta_2t^2+\beta_1t+\beta_0)\sin 2t\right]$$

$$=e^t\left[(\lambda_2t^3+\lambda_1t^2+\lambda_0t)\cos 2t+(\beta_2t^3+\beta_1t^2+\beta_0t)\sin 2t\right]$$

$$x_p'=e^t\left[(\lambda_2t^3+\lambda_1t^2+\lambda_0t)\cos 2t+(\beta_2t^3+\beta_1t^2+\beta_0t)\sin 2t\right]+$$

$$+e^t\left[(3\lambda_2t^2+2\lambda_1t+\lambda_0)\cos 2t+(\lambda_2t^3+\lambda_1t^2+\lambda_0t)(-2)\sin 2t+\right.$$

$$\left.+(3\beta_2t^2+2\beta_1t+\beta_0)\sin 2t+(\beta_2t^3+\beta_1t^2+\beta_0t)\cdot 2\cos 2t\right]$$

$$=e^t\left[(\lambda_2t^3+\lambda_1t^2+\lambda_0t+3\lambda_2t^2+2\lambda_1t+\lambda_0+2\beta_2t^3+2\beta_1t^2+2\beta_0t)\cos 2t\right.$$

$$\left.+(\beta_2t^3+\beta_1t^2+\beta_0t-2\lambda_2t^3-2\lambda_1t^2-2\lambda_0t+3\beta_2t^2+2\beta_1t+\beta_0)\sin 2t\right]$$

$$\cdot x_p'' = e^t\Big[(\lambda_2 t^3+\lambda_1 t^2+\lambda_0 t+3\lambda_2 t^2+2\lambda_1 t+\lambda_0+2\beta_2 t^3+3\beta_1 t^2+2\beta_0 t)\cos 2t+$$

$$+(\beta_2 t^3+\beta_1 t^2+\beta_0 t-2\lambda_2 t^3-2\lambda_1 t^2-2\lambda_0 t+3\beta_2 t^2+2\beta_1 t+\beta_0)\sin 2t\Big]+$$

$$+e^t\Big[(3\lambda_2 t^2+2\lambda_1 t+\lambda_0+6\lambda_2 t+2\lambda_1+6\beta_2 t^2+4\beta_1 t+2\beta_0)\cos 2t+$$

$$(\lambda_2 t^3+\lambda_1 t^2+\lambda_0 t+3\lambda_2 t^2+2\lambda_1 t+\lambda_0+2\beta_2 t^3+2\beta_1 t^2+2\beta_0 t)\cdot(-2)\sin 2t$$

$$+(3\beta_2 t^2+2\beta_1 t+\beta_0-6\lambda_2 t^2-4\lambda_1 t-2\lambda_0+6\beta_2 t+2\beta_1)\sin 2t+$$

$$(\beta_2 t^3+\beta_1 t^2+\beta_0 t-2\lambda_2 t^3-2\lambda_1 t^2-2\lambda_0 t+3\beta_2 t^2+2\beta_1 t+\beta_0)\cdot 2\cos 2t\Big].$$

$$\bullet e^t\Big[(\lambda_2 t^3+\lambda_1 t^2+\lambda_0 t+3\lambda_2 t^2+2\lambda_1 t+\lambda_0+2\beta_2 t^3+2\beta_1 t^2+2\beta_0 t+3\lambda_2 t^2+2\lambda_1 t+$$

$$+\lambda_0+6\lambda_2 t+2\lambda_1+6\beta_2 t^2+4\beta_1 t+2\beta_0+2\beta_2 t^3+2\beta_1 t^2+2\beta_0 t-4\lambda_2 t^3-4\lambda_1 t^2$$

$$-4\lambda_0 t+6\beta_2 t^2+4\beta_1 t+2\beta_0)\underline{\cos 2t}+(\beta_2 t^3+\beta_1 t^2+\beta_0 t-2\lambda_2 t^3-2\lambda_1 t^2-$$

$$-2\lambda_0 t+3\beta_2 t^2+2\beta_1 t+\beta_0-2\lambda_2 t^3-2\lambda_1 t^2-2\lambda_0 t-6\lambda_2 t^2-4\lambda_1 t-2\lambda_0-$$

$$-4\beta_2 t^3-4\beta_1 t^2-4\beta_0 t+3\beta_2 t^2+2\beta_1 t+\beta_0-6\lambda_2 t^2-4\lambda_1 t-2\lambda_0+6\beta_2 t$$

$$+2\beta_1)\sin 2t\Big]$$

$$-2e^t\Big[(\lambda_2 t^3+\lambda_1 t^2+\lambda_0 t+3\lambda_2 t^2+2\lambda_1 t+\lambda_0+2\beta_2 t^3+2\beta_1 t^2+2\beta_0 t)\cos 2t$$

$$+(\beta_2 t^3+\beta_1 t^2+\beta_0 t-2\lambda_2 t^3-2\lambda_1 t^2-2\lambda_0 t+3\beta_2 t^2+2\beta_1 t+\beta_0)\sin 2t\Big]$$

$$+5e^t\Big[(\lambda_2 t^3+\lambda_1 t^2+\lambda_0 t)\cos 2t+(\beta_2 t^3+\beta_1 t^2+\beta_0 t)\sin 2t\Big]=e^t(t\cos 2t-t^2\sin 2t)\,|:e^t$$

$$(-3\lambda_2 t^3+4\beta_2 t^3-3\lambda_1 t^2+6\lambda_2 t^2+4\beta_1 t^2+12\beta_2 t^2+4\lambda_1 t+4\beta_0 t+6\lambda_2 t+$$

$$+8\beta_1 t-3\lambda_0 t+2\lambda_0+2\lambda_1+4\beta_0-2\lambda_2 t^3-2\lambda_1 t^2-2\lambda_0 t-6\lambda_2 t^2-4\lambda_1 t-$$

$$-2\lambda_0-4\beta_2 t^3-4\beta_1 t^2-4\beta_0 t+5\lambda_2 t^3+5\lambda_1 t^2+5\lambda_0 t)\cos 2t$$

$$(-3\beta_2 t^3-4\lambda_2 t^3-3\beta_1 t^2-4\lambda_1 t^2+6\beta_2 t^2-12\lambda_2 t^2-3\beta_0 t-4\lambda_0 t+$$

$$+4\beta_1 t-8\lambda_1 t+6\beta_2 t+2\beta_0-4\lambda_0+2\beta_1-2\beta_2 t^3-2\beta_1 t^2-2\beta_0 t+4\lambda_2 t^3+$$

$$+4\lambda_1 t^2+4\lambda_0 t-6\beta_2 t^2-4\beta_1 t-2\beta_0+5\beta_2 t^3+5\beta_1 t^2+5\beta_0 t)\sin 2t$$

$$=t\cos 2t-2t^2\sin 2t$$

$$\begin{cases} 12\beta_2 t^2 + 6\lambda_2 t + 8\beta_1 t + 2\lambda_1 + \beta_0 = t \\ -12\lambda_2 t^2 - 8\lambda_1 t + 6\beta_2 t - 4\lambda_0 t + 2\beta_1 = -t^2 \end{cases}$$

$$\begin{cases} 12\beta_2 = 0 \Leftrightarrow \boxed{\beta_2 = 0} \\ 6\lambda_2 + 8\beta_1 = 1 \\ 2\lambda_1 + \beta_0 = 0 \Rightarrow \boxed{\beta_0 = 0} \\ -12\lambda_2 = -1 \Rightarrow \boxed{\lambda_2 = \frac{1}{12}} \\ -8\lambda_1 + 6\beta_2 = 0 \Rightarrow \boxed{\lambda_1 = 0} \\ -4\lambda_0 + 2\beta_1 = 0 \end{cases}$$

$$8\beta_1 = 1 - 6\lambda_2 = 1 - 6\cdot\frac{1}{12} = 1 - \frac{1}{2} = \frac{1}{2} \Leftrightarrow \boxed{\beta_1 = \frac{1}{16}}$$

$$4\lambda_0 = 2\beta_1 = 2\cdot\frac{1}{16} = \frac{1}{8} \Leftrightarrow \boxed{\lambda_0 = \frac{1}{32}}$$

$$\boxed{x_p = te^t\left[\left(\frac{1}{12}t^2 + \frac{1}{32}\right)\cos 2t + \frac{1}{16}t\sin 2t\right]}$$

$$x = x_0 + x_p = c_1 e^t \cos 2t + c_2 e^t \sin 2t + te^{2t}\left[\left(\frac{1}{12}t^2 + \frac{1}{32}\right)\cos 2t + \frac{1}{16}t\sin 2t\right]$$

c) $x'' - 9x = \underbrace{e^{2t}}_{f_1} + \underbrace{te^t}_{f_2} \underbrace{- t^2 - 2}_{f_3}$

• $x'' - 9x = 0$

$$\left.\begin{aligned} x &= e^{rt} \\ x' &= re^{rt} \\ x'' &= r^2 e^{rt} \end{aligned}\right\} \Leftrightarrow r^2 e^{rt} - 9e^{rt} = 0 \,|: e^{rt}|.$$

$r^2 - 9 = 0 \Leftrightarrow r^2 = 9 \Leftrightarrow r_{1,2} = \pm 3$

• e^{3t}, e^{-3t} – sist. fund. de sol.

$$\boxed{x_0 = c_1 e^{3t} + c_2 e^{-3t}}$$

• $x_{p_1} \overset{\alpha=2}{=\!=} e^{2t}A$

$x_{p_1}' = 2Ae^{2t}$; $x_{p_1}'' = 4Ae^{2t}$

$4Ae^{2t} - 9Ae^{2t} = e^{2t} \,|: e^{2t}$

$4A - 9A = 1 \Leftrightarrow -5A = 1 \Rightarrow A = \frac{-1}{5} \Rightarrow \boxed{x_{p_1} = \frac{-1}{5}e^{2t}}$

• $x_{p_2} \overset{\alpha=2}{=\!=} e^t(\lambda_1 t + \lambda_0)$

$x_{p_2}' = e^t(\lambda_1 t + \lambda_0) + e^t\lambda_1 = e^t(\lambda_1 t + \lambda_0 + \lambda_1)$

$x_{p_2}'' = e^t(\lambda_1 t + \lambda_0 + \lambda_1) + e^t(\lambda_1) = e^t(\lambda_1 t + \lambda_0 + 2\lambda_1)$

$e^t(\lambda_1 t + \lambda_0 + 2\lambda_1) - 9e^t(\lambda_1 t + \lambda_0) = te^t \,|: e^t$

$\lambda_1 t + \lambda_0 + 2\lambda_1 - 9\lambda_1 t - 9\lambda_0 = t$

$$\begin{cases} -8\lambda_1 = 7 & \Rightarrow \lambda_1 = \frac{-7}{8} \\ 2\lambda_1 - 8\lambda_0 = 0 & 8\lambda_0 = 2\lambda_1 = 2 \cdot \frac{-7}{8} = \frac{-7}{4} \Rightarrow \lambda_0 = \frac{-7}{32} \end{cases}$$

$$\boxed{x_{p_2} = e^t\left(-\frac{7}{8}t - \frac{7}{32}\right)}$$

- $x_{p_3} = \lambda_2 t^2 + \lambda_1 t + \lambda_0$

$x_{p_3}' = 2\lambda_2 t + \lambda_1$; $x_{p_3}'' = 2\lambda_2$

$2\lambda_2 - 9(\lambda_2 t^2 + \lambda_1 t + \lambda_0) = -t^2 - 2 \Leftrightarrow 2\lambda_2 - 9\lambda_2 t^2 - 9\lambda_1 t - 9\lambda_0 = -t^2 - 2$

$$\begin{cases} -9\lambda_2 = -1 \\ -9\lambda_1 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda_2 = \frac{1}{9} \\ \lambda_1 = 0 \end{cases}$$

$2\lambda_2 - 9\lambda_0 = -2 \Rightarrow 9\lambda_0 = 2 + 2\lambda_2 = 2 + \frac{2}{9} = \frac{20}{9} \Rightarrow \lambda_0 = \frac{20}{81}$

$$\boxed{x_{p_3} = \frac{1}{9}t^2 + \frac{20}{81}}$$

- $x = x_0 + x_{p_1} + x_{p_2} + x_{p_3}$

$x = c_1 e^{3t} + c_2 e^{-3t} - \frac{1}{5}e^{2t} + e^t\left(\frac{-7}{8}t - \frac{7}{32}\right) + \frac{1}{9}t^2 + \frac{20}{81}$.