Temă - Curs 4 Detculescu Mihai-Liviu

1. Verificați dacă $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x_1, x_2, x_3) = (x_1 - x_2, 2x_2 + x_3, x_1 + x_2 - x_3)$ e o aplicație liniară.

• $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}), \ \forall \vec{x}, \vec{y} \in \mathbb{R}^3$

$$T(\vec{x} + \vec{y}) = T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$$
$$= (x_1 + y_1 - x_2 - y_2, 2(x_2 + y_2) + x_3 + y_3, x_1 + y_1 + x_2 + y_2 - x_3 - y_3)$$
$$= (x_1 - x_2, 2x_2 + x_3, x_1 + x_2 - x_3) + (y_1 - y_2, 2y_2 + y_3, y_1 + y_2 - y_3)$$
$$= T(\vec{x}) + T(\vec{y}) \quad (1)$$

• $T(\alpha \vec{x}) = \alpha T(\vec{x}), \ \forall \alpha \in \mathbb{R}, \vec{x} \in \mathbb{R}^3$

$$T(\alpha \vec{x}) = T(\alpha(x_1, x_2, x_3)) = T(\alpha x_1, \alpha x_2, \alpha x_3)$$
$$= (\alpha x_1 - \alpha x_2, 2\alpha x_2 + \alpha x_3, \alpha x_1 + \alpha x_2 - \alpha x_3)$$
$$= \alpha(x_1 - x_2, 2x_2 + x_3, x_1 + x_2 - x_3) = \alpha \cdot T(\vec{x}) \quad (2)$$

Din (1) și (2) $\Rightarrow$ T e operator liniar.

2. Să se verifice dacă urm. aplicații sunt injective și surjective:

a) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x_1, x_2) = (x_1 + x_2, x_1 - 2x_2, 2x_1 + 3x_2)$

$\operatorname{Ker} T = \{\vec{x} \in \mathbb{R}^3 \mid T(\vec{x}) = \vec{0}\}$

$T(\vec{x}) = \vec{0}$; $T(x_1, x_2) = \vec{0}$

$(x_1 + x_2, x_1 - 2x_2, 2x_1 + 3x_2) = (0, 0, 0)$

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 - 2x_2 = 0 \\ 2x_1 + 3x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_2 \\ -3x_2 = 0 \\ x_2 = 0 \end{cases} \Rightarrow x_2 = x_1 = 0 \Rightarrow \operatorname{Ker} T = \{\vec{0}\} \Rightarrow T \text{ inj}$$

$\operatorname{Im} T = \{\vec{w} \in \mathbb{R}^3 \mid \exists \vec{x} \in \mathbb{R}^2 \text{ a.î. } T(\vec{x}) = \vec{w}\}$

$T(\vec{x}) = \vec{w}$; $T(x_1, x_2) = (w_1, w_2, w_3)$

$$\begin{cases} x_1 + x_2 = w_1 \\ x_1 - 2x_2 = w_2 \\ 2x_1 + 3x_2 = w_3 \end{cases} ; \quad A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 3 \end{pmatrix} ; \quad D_1 = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3 \neq 0;$$

$$D_c = \begin{vmatrix} 1 & 1 & w_1 \\ 1 & -2 & w_2 \\ 2 & 3 & w_3 \end{vmatrix} = 0 \Leftrightarrow -2w_3 + 3w_1 + 2w_2 + 4w_1 - 3w_2 - w_3 = 0$$
$$\Leftrightarrow 7w_1 - w_2 + 2w_3 = 0$$
$$w_2 = 7w_1 + 2w_3.$$

$Im\,T = \{(w_1, 7w_1 + 2w_3, w_3) \mid w_1, w_3 \in R\} \Rightarrow Im\,T \neq R^3 \Rightarrow T$ nu e surj.

b) $T: R^2 \to R^2$, $T(x_1, x_2) = (2x_1 - x_2, 3x_1 + x_2)$

$Ker\,T = \{\vec{x} \in R^2 \mid T(\vec{x}) = \vec{0}\}$

$T(\vec{x}) = \vec{0}$

$$\begin{cases} 2x_1 - x_2 = 0 \\ 3x_1 + x_2 = 0 \end{cases} \quad "+"$$

$5x_1 = 0 \Rightarrow x_1 = 0 \Rightarrow x_2 = 0 \Rightarrow Ker\,T = \{\vec{0}\} \Rightarrow T$ inj.

$Im\,T = \{\vec{w} \in R^2 \mid \exists\, \vec{x} \in R^2$ a.î. $T(\vec{x}) = \vec{w}\}$

$T(\vec{x}) = \vec{w}$

$$\begin{cases} 2x_1 - x_2 = w_1 \\ 3x_1 + x_2 = w_2 \end{cases} \qquad D = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 + 3 = 5 \neq 0 \Rightarrow \exists\, x_1, x_2\ \forall\, w_1, w_2$$

$\Rightarrow Im\,T = R^2 \Rightarrow T$ surj.

c) $T: R^3 \to R^3$, $T(x_1, x_2, x_3) = (x_1 + x_2, x_1 + 2x_2 + x_3, x_3)$

$Ker\,T = \{\vec{x} \in R^3 \mid T(\vec{x}) = \vec{0}\}$

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + 2x_2 + x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 + 2x_2 = 0 \end{cases} \quad "-"$$

$-x_2 = 0 \Leftrightarrow x_2 = 0 \Rightarrow x_1 = 0 \Rightarrow Ker\,T = \{\vec{0}\} \Rightarrow T$ inj.

$Im\,T = \{\vec{w} \in R^3 \mid \exists\, \vec{x} \in R^3$ a.î. $T(\vec{x}) = \vec{w}\}$

$$\begin{cases} x_1 + x_2 = w_1 \\ x_1 + 2x_2 + x_3 = w_2 \\ x_3 = w_3 \end{cases} ; \quad D = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2 - 1 = 1 \neq 0$$

$\Rightarrow \exists\, x_1, x_2, x_3,\ \forall\, w_1, w_2, w_3 \Rightarrow Im\,T = R^3 \Rightarrow T$ surj.

d) $T: R^3 \to R^3$, $T(x_1, x_2, x_3) = (2x_1 - x_3, x_2, 2x_1 + x_2 - x_3)$

$Ker\,T = \{\vec{x} \in R^3 \mid T(\vec{x}) = \vec{0}\}$

$$\begin{cases} 2x_1 - x_3 = 0 \\ x_2 = 0 \\ 2x_1 + x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} 2x_1 - x_3 = 0 \\ 2x_1 - x_3 = 0 \end{cases}$$

$$D = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & -1 \end{vmatrix} = -2 + 2 = 0 \ ; \quad D_p = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 \neq 0.$$

$\Rightarrow$ sist. comp. nedeterminat.

$$\begin{cases} 2x_{\cancel{1}} = x_3 \Rightarrow x_1 = \frac{x_3}{2} \\ x_2 = 0 \end{cases}$$

Notăm $x_3 = \alpha \Rightarrow \operatorname{Ker} T = \{(\frac{\alpha}{2}, 0, \alpha) \mid \alpha \in \mathbb{R}\} \Rightarrow \operatorname{Ker} T \neq \{\vec{0}\} \Rightarrow$ T nu e inj.

$\operatorname{Im} T = \{\vec{w} \in \mathbb{R}^3 \mid \exists \vec{x} \in \mathbb{R}^3$ a.î. $T(\vec{x}) = \vec{w}\}$

$$\begin{cases} 2x_1 - x_3 = \vec{w_1} \\ x_2 = \vec{w_2} \\ 2x_1 + x_2 - x_3 = \vec{w_3} \end{cases}$$

$D = 0,\ D_1 = 2 \neq 0.$

$$D_c = \begin{vmatrix} 2 & 0 & w_1 \\ 0 & 1 & w_2 \\ 2 & 1 & w_3 \end{vmatrix} = 0 \Leftrightarrow 2w_3 - 2w_1 - 2w_2 = 0.$$

$\operatorname{Im} T = \{(w_1, w_2, w_1 + w_2) \mid w_1, w_2 \in \mathbb{R}\}$

$\Rightarrow \operatorname{Im} T \neq \mathbb{R}^3 \Rightarrow$ T nu e surj.