Probabilități și Statistică Matematică

Seminar 15.04.2021

1. Se consideră variabila aleatoare discretă $X:\begin{pmatrix} -2 & -1 & 0 & 1 & 2 & 3 \\ \frac{1}{12} & \frac{2}{12} & \frac{2}{12} & \frac{5}{12} & \frac{1}{12} & \frac{1}{12} \end{pmatrix}$

Să se calculeze media, dispersia și deviația standard ale lui X.

- $E(X) = -2\cdot\frac{1}{12} - 1\cdot\frac{2}{12} + 0\cdot\frac{2}{12} + 1\cdot\frac{5}{12} + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{12} = \frac{1}{2}$

- $E(X^2) = 4\cdot\frac{1}{12} + 1\cdot\frac{2}{12} + 0\cdot\frac{2}{12} + 1\cdot\frac{5}{12} + 4\cdot\frac{1}{12} + 9\cdot\frac{1}{12} = 2$

  $\text{var}(X) = E(X^2) - [E(X)]^2 = 2 - \left(\frac{1}{2}\right)^2 = \frac{7}{4}$

- $\sigma(X) = \sqrt{\text{var}(X)} = \frac{\sqrt{7}}{2}$

2. Fie v.a continuă X având densitatea de repartiție $f:\mathbb{R}\to\mathbb{R}$, $f(x) = \begin{cases} x/4, & x\in[1,3] \\ 0, & \text{în rest} \end{cases}$

Să se calculeze media, dispersia și deviația standard ale lui X.

- $E(X) = \int_1^3 x f(x)\,dx = \int_1^3 x\cdot\frac{x}{4}\,dx = \frac{1}{4}\cdot\frac{x^3}{3}\Big|_1^3 = \frac{13}{6}$

- $E(X^2) = \int_1^3 x^2 f(x)\,dx = \int_1^3 x^2\frac{x}{4}\,dx = \frac{1}{4}\cdot\frac{x^4}{4}\Big|_1^3 = 5$

  $\text{var}(X) = E(X^2) - [E(X)]^2 = 5 - \frac{169}{36} = \frac{11}{36}$

- $\sigma(X) = \sqrt{\text{var}(X)} = \frac{\sqrt{11}}{6}$

3. Se consideră vectorul aleator discret (X,Y) cu repartiția dată în tabelul:

$X \backslash Y$	-1	0	1	2	p_i
-1	$1/6$	$1/12$	$1/12$	$1/24$	$9/24$
0	$1/24$	$1/6$	$1/12$	$1/24$	$8/24$
1	$1/24$	$1/24$	$1/6$	$1/24$	$7/24$
q_j	$6/24$	$7/24$	$8/24$	$3/24$	1

a) Să se calculeze covarianța v.a X și Y

b) Să se calculeze coeficientul de corelație al v.a X și Y.

c) Să se calculeze media și matricea de covarianță ale lui $Z=(X,Y)$.

Rezolvare

a) $\text{cov}(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y)$

$E(X) = -1\cdot\frac{9}{24} + 0\cdot\frac{8}{24} + 1\cdot\frac{7}{24} = \frac{-1}{12}$

$$E(Y) = -1\cdot\frac{6}{24} + 0\cdot\frac{7}{24} + 1\cdot\frac{8}{24} + 2\cdot\frac{3}{24} = \frac{1}{3}$$

$$E(X\cdot Y) = -1\cdot\left(-1\cdot\frac{1}{6} + 0\cdot\frac{1}{12} + 1\cdot\frac{1}{12} + 2\cdot\frac{1}{24}\right) + 0\cdot\left(-1\cdot\frac{1}{24} + 0\cdot\frac{1}{6} + 1\cdot\frac{1}{12} + 2\cdot\frac{1}{24}\right) +$$

$$+ 1\cdot\left(-1\cdot\frac{1}{24} + 0\cdot\frac{1}{24} + 1\cdot\frac{1}{6} + 2\cdot\frac{1}{24}\right) = \frac{5}{24}$$

$$cov(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y) = \frac{5}{24} + \frac{1}{36} = \frac{17}{72}$$

b) $$\rho(X,Y) = \frac{cov(X,Y)}{\sigma(X)\cdot\sigma(Y)} = \frac{\frac{17}{72}}{\sqrt{\frac{95}{144}\cdot\frac{35}{36}}} \simeq 0{,}295$$

$$E(X^2) = 1\cdot\frac{9}{24} + 0\cdot\frac{8}{24} + 1\cdot\frac{7}{24} = \frac{16}{24} = \frac{2}{3}$$

$$var(X) = E(X^2) - [E(X)]^2 = \frac{2}{3} - \frac{1}{144} = \frac{95}{144} \Rightarrow \sigma(X) = \sqrt{\frac{95}{144}}$$

$$E(Y^2) = 1\cdot\frac{6}{24} + 0\cdot\frac{7}{24} + 1\cdot\frac{7}{24} + 4\cdot\frac{3}{24} = \frac{26}{24} = \frac{13}{12}$$

$$var(Y) = E(Y^2) - [E(Y)]^2 = \frac{13}{12} - \frac{1}{9} = \frac{35}{36} \Rightarrow \sigma(Y) = \sqrt{\frac{35}{36}}$$

c) $$E(Z) = (E(X), E(Y)) = \left(-\frac{1}{12}, \frac{1}{3}\right)$$

$$cov(Z) = \begin{pmatrix} var(X) & cov(X,Y) \\ cov(X,Y) & var(Y) \end{pmatrix} = \begin{pmatrix} 95/144 & 17/72 \\ 17/72 & 35/36 \end{pmatrix}$$

4. Fie vectorul aleator (X,Y) cu densitatea de probabilitate $f:\mathbb{R}^2\to\mathbb{R}$.

$$f(x,y) = \begin{cases} 1/5\,(x+y+1), & x\in[0,1],\ y\in[0,2] \\ 0, & \text{în rest} \end{cases}$$

a) Să se calculeze covarianța v. a X și Y

b) Să se calculeze coeficientul de corelație la v.a X și Y.

Rezolvare

a) $cov(X,Y) = E(X,Y) - E(X)\cdot E(Y)$

$$\bullet\, E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_0^1 x\cdot\frac{2x+4}{5}\,dx =$$

$$= \frac{1}{5}\left(2\cdot\frac{x^3}{3} + 4\cdot\frac{x^2}{2}\right)\Big|_0^1 = \frac{8}{15}$$

$$f_X(x)\,dx = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{5}\int_0^2 (x+y+1)\,dy = \frac{1}{5}\left(xy + \frac{y^2}{2} + y\right)\Big|_0^2$$

$$= \frac{2x+4}{5},\ x\in[0,1]$$

• $E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^2 y\cdot\frac{2y+3}{10}\,dy =$

$= \frac{1}{10}\left(2\cdot\frac{y^3}{3} + 3\cdot\frac{y^2}{2}\right)\Big|_0^2 = \frac{17}{15}$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{1}{5}(x+y+1)\,dx = \frac{1}{5}\left(\frac{x^2}{2} + yx + x\right)\Big|_0^1 = \frac{2y+3}{10},\ y\in[0,2]$

• $E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \frac{1}{5}\int_0^1 x^2(2x+4)\,dx = \frac{11}{30}$

$var(X) = E(X^2) - [E(X)]^2 = \frac{11}{30} - \frac{64}{225} = \frac{37}{450}$

• $E(Y^2) = \int_{-\infty}^{\infty} y^2\cdot f_Y(y)\,dy = \frac{1}{10}\int_0^2 y^2(2y+3)\,dy = \frac{8}{15}$

$var(Y) = E(Y^2) - [E(Y)]^2 = \frac{8}{5} - \frac{289}{225} = \frac{71}{225}$

• $E(X\cdot Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy = \frac{1}{5}\int_0^1\int_0^2 xy(x+y+1)\,dx\,dy =$

$= \frac{1}{5}\int_0^1 \left(x^2\cdot\frac{y^2}{2} + x\cdot\frac{y^3}{3} + x\cdot\frac{y^2}{2}\right)\Big|_0^2 dx = \frac{1}{5}\int_0^1 \left(2x^2 + \frac{8}{3}x + 2x\right)dx =$

$= \frac{1}{5}\left(2\cdot\frac{x^3}{3} + \frac{8}{3}\cdot\frac{x^2}{2} + x^2\right)\Big|_0^1 = \frac{1}{5}\left(2\cdot\frac{1}{3} + \frac{8}{3}\cdot\frac{1}{2} + 1\right) = \frac{1}{5}(2+1) = \frac{3}{5}.$

$cov(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y) = \frac{3}{5} - \frac{8}{15}\cdot\frac{17}{15} = \frac{-1}{225}.$

b) $\rho(X,Y) = \frac{cov(X,Y)}{\sigma(X)\,\sigma(Y)} = \frac{\frac{-1}{225}}{\sqrt{\frac{37}{450}\cdot\frac{71}{225}}} \simeq 0,02758$

TEMĂ

1. V.a $X: \begin{pmatrix} -1 & 0 & 2 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$

Să se calc $E(X), E(3X), E(4X-2), var(X), \sigma(X)$.

2. Să se calc. val. medie și dispersia v.a care are dens. de prob:

$f(x) = \begin{cases} 1-|1-x|, & x\in(0,2) \\ 0, & \text{altfel} \end{cases}$

Rezolvare

1. $E(X) = -1\cdot\frac{2}{10} + 0\cdot\frac{3}{10} + 2\cdot\frac{5}{10} = \frac{-2+10}{10} = \frac{8}{10} = \frac{4}{5}$

$E(3X) = 3\cdot E(X) = \frac{12}{5}$

$E(4X-2) = (-6)\cdot\frac{2}{10} + (-2)\cdot\frac{3}{10} + 6\cdot\frac{5}{10} = \frac{-12-6+30}{10} = \frac{12}{10} = \frac{6}{5}$

$E(x^2) = 1 \cdot \frac{2}{10} + 0 \cdot \frac{3}{10} + 4 \cdot \frac{5}{10} = \frac{22}{10} = \frac{11}{5}$

$var(X) = E(x^2) - [E(x)]^2 = \frac{11}{5} - \frac{16}{25} = \frac{39}{25}$

$\sigma(X) = \sqrt{var(X)} = \frac{\sqrt{39}}{5}$

2. $E(x) = \int_0^2 x f(x)\, dx = \int_0^2 x(1-(1-x))\, dx = \frac{x^2}{2} - \left|\frac{x^2}{2} - \frac{x^3}{3}\right| \Big|_0^2 =$

$= \frac{4}{2} - \left|\frac{4}{2} - \frac{8}{3}\right| = \frac{12-4}{6} = \frac{4}{3}$

$E(x^2) = \int_0^2 x^2 f(x)\, dx = \int_0^2 x^2 - (x^2 - x^3)\, dx = \frac{x^3}{3} - \left|\frac{x^3}{3} - \frac{x^4}{4}\right| \Big|_0^2 =$

$= \frac{8}{3} - \frac{16}{12} = \frac{5}{3}$

$var(X) = E(x^2) - [E(x)]^2 = \frac{-1}{9}$