Laborator12 - Rezolvare

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Exerciţiu 1.a - Video

1 a) $\begin{cases} x' = 2x + z \\ y' = x + y \\ z' = 2y + 2z \end{cases}$

$x = \alpha_1 e^{rt},\ y = \alpha_2 e^{rt},\ z = \alpha_3 e^{rt}$

$x' = \alpha_1 r e^{rt},\ y' = \alpha_2 r e^{rt},\ z' = \alpha_3 r e^{rt}$

$$\begin{cases} \alpha_1 r = 2\alpha_1 + \alpha_3 \\ \alpha_2 r = \alpha_1 + \alpha_2 \\ \alpha_3 r = 2\alpha_2 + 2\alpha_3 \end{cases} \Leftrightarrow \begin{cases} (2-r)\alpha_1 + \alpha_3 = 0 \\ \alpha_1 + (1-r)\alpha_2 = 0 \\ 2\alpha_2 + (2-r)\alpha_3 = 0 \end{cases} \quad (*)$$

$$\Delta = \begin{vmatrix} 2-r & 0 & 1 \\ 1 & 1-r & 0 \\ 0 & 2 & 2-r \end{vmatrix} \underset{=}{L_1+L_2+L_3} \begin{vmatrix} 3-r & 3-r & 3-r \\ 1 & 1-r & 0 \\ 0 & 2 & 2-r \end{vmatrix} = (3-r)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-r & 0 \\ 0 & 2 & 2-r \end{vmatrix} = (3-r)\left[(1-r)(2-r) + \cancel{2} - \cancel{2} + r\right]$$

$$= (3-r)(2 - \cancel{r} - 2r + r^2 + \cancel{r}) = (3-r)(r^2 - 2r + 2) = 0$$

$3 - r = 0 \Rightarrow r_1 = 3$

$r^2 - 2r + 2 = 0$

$\Delta_1 = 4 - 8 = -4 \Rightarrow r_{2,3} = \dfrac{2 \pm 2i}{2} = 1 \pm i \quad (\gamma \pm i\beta,\ \gamma = 1,\ \beta = 1)$

Pt. $r_1 = 3$

$(*) \Rightarrow \begin{cases} -\alpha_1 + \alpha_3 = 0 \\ \alpha_1 - 2\alpha_2 = 0 \\ 2\alpha_2 - \alpha_3 = 0 \end{cases} \begin{matrix} \Rightarrow \alpha_3 = \alpha_1 \\ \Rightarrow \alpha_1 = 2\alpha_2 \\ \Rightarrow \alpha_3 = 2\alpha_2 \end{matrix},\ \alpha_2 \in \mathbb{R}$

Fie $\alpha_2 = 1 \Rightarrow \alpha_1 = \alpha_3 = 2 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} 2e^{3t} \\ e^{3t} \\ 2e^{3t} \end{pmatrix}$

Pt. $r_2 = 1 + i$

$(*) \Rightarrow \begin{cases} (1-i)\alpha_1 + \alpha_3 = 0 \\ \alpha_1 - i\alpha_2 = 0 \\ 2\alpha_2 + (1-i)\alpha_3 = 0 \end{cases} \begin{matrix} \Rightarrow \alpha_3 = (i-1)\alpha_1 \\ \Rightarrow \alpha_2 = \frac{1}{i}\alpha_1 = -i\alpha_1,\ \alpha_1 \in \mathbb{C} \end{matrix}$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = -i,\ \alpha_3 = i - 1 \Rightarrow \alpha_{r_2} = \begin{pmatrix} 1 \\ -i \\ i-1 \end{pmatrix} \Rightarrow X_{r_2} = \begin{pmatrix} e^t(\cos t + i\sin t) \\ -ie^t(\cos t + i\sin t) \\ (i-1)e^t(\cos t + i\sin t) \end{pmatrix}$

$$X_{r_2} = \begin{pmatrix} e^t\cos t + ie^t\sin t \\ -ie^t\cos t + e^t\sin t \\ ie^t\cos t - e^t\sin t - e^t\cos t - ie^t\sin t \end{pmatrix} = \underbrace{\begin{pmatrix} e^t\cos t \\ e^t\sin t \\ -e^t\sin t - e^t\cos t \end{pmatrix}}_{\tilde{X}_{r_2}} + i\underbrace{\begin{pmatrix} e^t\sin t \\ -e^t\cos t \\ e^t\cos t - e^t\sin t \end{pmatrix}}_{\tilde{\tilde{X}}_{r_2}}$$

$$X = C_1X_{r_1} + C_2\tilde{X}_{r_2} + C_3\tilde{\tilde{X}}_{r_2} = C_1\begin{pmatrix} 2e^{3t} \\ e^{3t} \\ 2e^{3t} \end{pmatrix} + C_2\begin{pmatrix} e^t\cos t \\ e^t\sin t \\ -e^t\sin t - e^t\cos t \end{pmatrix} + C_3\begin{pmatrix} e^t\sin t \\ -e^t\cos t \\ e^t\cos t - e^t\sin t \end{pmatrix}$$

$$X = \begin{pmatrix} 2C_1 e^{3t} + C_2 e^t \cos t + C_3 e^t \sin t \\ C_1 e^{3t} + C_2 e^t \sin t - C_3 e^t \cos t \\ 2C_1 e^{3t} - C_2 e^t (\sin t + \cos t) + e^t (\cos t - \sin t) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} x = 2C_1 e^{3t} + C_2 e^t \cos t + C_3 e^t \sin t \\ y = C_1 e^{3t} + C_2 e^t \sin t - C_3 e^t \cos t \\ z = 2C_1 e^{3t} - C_2 e^t (\sin t + \cos t) + e^t (\cos t - \sin t) \end{cases}$$

Exerciţiu 2.a - Video

$$② \ a) \begin{cases} x' = 4x - y - z \\ y' = x + 2y - z \\ z' = x - y + 2z \end{cases}$$

$$x = \alpha_1 e^{rt}, \ y = \alpha_2 e^{rt}, \ z = \alpha_3 e^{rt}$$

$$x' = \alpha_1 r e^{rt}, \ y' = \alpha_2 r e^{rt}, \ z' = \alpha_3 r e^{rt}$$

$$\begin{cases} \alpha_1 r = 4\alpha_1 - \alpha_2 - \alpha_3 \\ \alpha_2 r = \alpha_1 + 2\alpha_2 - \alpha_3 \\ \alpha_3 r = \alpha_1 - \alpha_2 + 2\alpha_3 \end{cases} \Leftrightarrow \begin{cases} (4-r)\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ \alpha_1 + (2-r)\alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_2 + (2-r)\alpha_3 = 0 \end{cases} \quad (*)$$

$$\Delta = \begin{vmatrix} 4-r & -1 & -1 \\ 1 & 2-r & -1 \\ 1 & -1 & 2-r \end{vmatrix} \overset{C_1+C_2+C_3}{=\!=\!=} \begin{vmatrix} 2-r & -1 & -1 \\ 2-r & 2-r & -1 \\ 2-r & -1 & 2-r \end{vmatrix} = (2-r) \begin{vmatrix} 1 & -1 & -1 \\ 1 & 2-r & -1 \\ 1 & -1 & 2-r \end{vmatrix} \underset{C_1+C_3}{\overset{C_1+C_2}{=\!=\!=}} (2-r) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 3-r & 0 \\ 1 & 0 & 3-r \end{vmatrix}$$

$$= (2-r)(3-r)^2 = 0$$

$$2 - r = 0 \Rightarrow r_1 = 2$$

$$(3-r)^2 = 0 \Rightarrow r_2 = r_3 = 3$$

Pt. $r_1 = 2$

$$(*) \Rightarrow \begin{cases} 2\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_3 = 0 \quad \Rightarrow \alpha_3 = \alpha_1 \\ \alpha_1 - \alpha_2 = 0 \quad \Rightarrow \alpha_2 = \alpha_1 \end{cases} \Rightarrow \alpha_1 \in \mathbb{R}$$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = \alpha_3 = 1 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}$

Pt. $r_2 = r_3 = 3$

$$(*) \Rightarrow \begin{cases} \alpha_1 - \alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_2 - \alpha_3 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 + \alpha_3, \ \alpha_2, \alpha_3 \in \mathbb{R}$$

$$(\alpha_2 + \alpha_3, \alpha_2, \alpha_3) = (\alpha_2, \alpha_2, 0) + (\alpha_3, 0, \alpha_3) = \alpha_2 (1, 1, 0) + \alpha_3 (1, 0, 1) \Rightarrow \alpha_{r_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ \alpha_{r_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$X_{r_2} = \begin{pmatrix} e^{3t} \\ e^{3t} \\ 0 \end{pmatrix}, \ X_{r_3} = \begin{pmatrix} e^{3t} \\ 0 \\ e^{3t} \end{pmatrix}$$

$$X = C_1 X_{r_1} + C_2 X_{r_2} + C_3 X_{r_3} = C_1 \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ e^{3t} \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} e^{3t} \\ 0 \\ e^{3t} \end{pmatrix} = \begin{pmatrix} C_1 e^{2t} + C_2 e^{3t} + C_3 e^{3t} \\ C_1 e^{2t} + C_2 e^{3t} \\ C_1 e^{2t} + C_3 e^{3t} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} x = C_1 e^{2t} + C_2 e^{3t} + C_3 e^{3t} \\ y = C_1 e^{2t} + C_2 e^{3t} \\ z = C_1 e^{2t} + C_3 e^{3t} \end{cases}$$

Exerciţiu 3.a - Video

3 a) $\begin{cases} x' = y \\ y' = -x + 2y \end{cases}$

$x = \alpha_1 e^{rt}, \quad y = \alpha_2 e^{rt}$

$x' = \alpha_1 r e^{rt}, \quad y' = \alpha_2 r e^{rt}$

$\begin{cases} \alpha_1 r = \alpha_2 \\ \alpha_2 r = -\alpha_1 + 2\alpha_2 \end{cases} \Leftrightarrow \begin{cases} -\alpha_1 r + \alpha_2 = 0 \\ -\alpha_1 + (2-r)\alpha_2 = 0 \end{cases} \quad (*)$

$\Delta = \begin{vmatrix} -r & 1 \\ -1 & 2-r \end{vmatrix} = -r(2-r) + 1 = -2r + r^2 + 1 = r^2 - 2r + 1 = (r-1)^2 = 0$

$r_1 = r_2 = 1$

$(*) \Rightarrow \begin{cases} -\alpha_1 + \alpha_2 = 0 \\ -\alpha_1 + \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = 1 \Rightarrow \alpha_{r_{1,2}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$ un singur vector l.i

Pt $r = 1$ considerăm $x = (A_1 t + A_0)e^t, \quad y = (B_1 t + B_0)e^t$

$x' = A_1 e^t + (A_1 t + A_0)e^t, \quad y' = B_1 e^t + (B_1 t + B_0)e^t$

$\begin{cases} A_1 + A_1 t + A_0 = B_1 t + B_0 \\ B_1 + B_1 t + B_0 = -A_1 t - A_0 + 2B_1 t + 2B_0 \end{cases}$

$\begin{cases} \boxed{A_1 = B_1} \\ A_1 + A_0 = B_0 \Rightarrow \boxed{A_1 = B_0 - A_0}, \quad B_0, A_0 \in \mathbb{R} \\ B_1 = -A_1 + 2B_1 \\ B_1 + B_0 = -A_0 + 2B_0 \end{cases}$

Fie $B_0 = 1, A_0 = 0 \Rightarrow A_1 = B_1 = 1$

$X_{r_{1,2}} = \begin{pmatrix} te^t \\ (t+1)e^t \end{pmatrix} = \begin{pmatrix} te^t \\ te^t + e^t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ e^t \end{pmatrix}}_{X_{r_1}} + \underbrace{\begin{pmatrix} te^t \\ te^t \end{pmatrix}}_{X_{r_2}}$

$X = C_1 X_{r_1} + C_2 X_{r_2} = C_1 \begin{pmatrix} 0 \\ e^t \end{pmatrix} + C_2 \begin{pmatrix} te^t \\ te^t \end{pmatrix}$

$X = \begin{pmatrix} C_2 t e^t \\ C_1 e^t + C_2 t e^t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{cases} x = C_2 t e^t \\ y = C_1 e^t + C_2 t e^t \end{cases}$

Exerciţiu 3.b

③ a) $\begin{cases} x' = 2x + y \\ y' = 2y + 4z \\ z' = x - z \end{cases}$

$x = \alpha_1 e^{rt}, \quad y = \alpha_2 e^{rt}, \quad z = \alpha_3 e^{rt}$

$x' = \alpha_1 r e^{rt}, \quad y' = \alpha_2 r e^{rt}, \quad z' = \alpha_3 r e^{rt}$

$\begin{cases} \alpha_1 r = 2\alpha_1 + \alpha_2 \\ \alpha_2 r = 2\alpha_2 + 4\alpha_3 \\ \alpha_3 r = \alpha_1 - \alpha_3 \end{cases} \Leftrightarrow \begin{cases} (2-r)\alpha_1 + \alpha_2 = 0 \\ (2-r)\alpha_2 + 4\alpha_3 = 0 \\ \alpha_1 + (-1-r)\alpha_3 = 0 \end{cases} \quad (*)$

$\Delta = \begin{vmatrix} 2-r & 1 & 0 \\ 0 & 2-r & 4 \\ 1 & 0 & -1-r \end{vmatrix} \underset{=}{\overset{L_1+L_2+L_3}{=}} \begin{vmatrix} 3-r & 3-r & 3-r \\ 0 & 2-r & 4 \\ 1 & 0 & -1-r \end{vmatrix} = (3-r)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2-r & 4 \\ 1 & 0 & -1-r \end{vmatrix} = (3-r)\left[(2-r)(-1-r) + 4 - 2 + r\right]$

$= (3-r)\left(-\cancel{2} - \cancel{2r} + \cancel{r} + r^2 + \cancel{4} - \cancel{2} + \cancel{r}\right) = r^2(3-r) = 0.$

$3 - r = 0 \Rightarrow r_1 = 3$

$r^2 = 0 \Rightarrow r_2 = r_3 = 0$

Pt. $r_1 = 3$

$(*) \Rightarrow \begin{cases} -\alpha_1 + \alpha_2 = 0 \\ -\alpha_2 + 4\alpha_3 = 0 \\ \alpha_1 - 4\alpha_3 = 0 \end{cases}$ $\quad \alpha_1 = \alpha_2 \Rightarrow \alpha_2 = 4\alpha_3 \Rightarrow \alpha_1 = 4\alpha_3, \quad \alpha_3 \in \mathbb{R}$

Fie $\alpha_3 = 1 \Rightarrow \alpha_1 = \alpha_2 = 4 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$

$\Rightarrow X_{r_1} = \begin{pmatrix} 4e^{3t} \\ 4e^{3t} \\ e^{3t} \end{pmatrix}$

Pt. $r_2 = r_3 = 0$

$(*) \Rightarrow \begin{cases} 2\alpha_1 + \alpha_2 = 0 \\ 2\alpha_2 + 4\alpha_3 = 0 \\ \alpha_1 - \alpha_3 = 0 \end{cases}$ $\Rightarrow \alpha_2 = -2\alpha_1, \quad \alpha_1 \in \mathbb{R}.$ $\quad \Rightarrow \alpha_1 = \alpha_3$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = -2, \alpha_3 = 1 \rightarrow \alpha_{r_{2,3}} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow$ un singur vector l.i.

Pt $r = 0$ considerăm $x = (A_1 t + A_0)e^{0\cdot t} = A_1 t + A_0$, $y = (B_1 t + B_0)\cdot e^{0\cdot t} = B_1 t + B_0$, $z = (C_1 t + C_0)e^{0\cdot t} = C_1 t + C_0$

$x' = A_1, \quad y' = B_1, \quad z' = C_1$

$\begin{cases} A_1 = 2A_1 t + 2A_0 + B_1 t + B_0 \\ B_1 = 2B_1 t + 2B_0 + 4C_1 t + 4C_0 \\ C_1 = A_1 t + A_0 - C_1 t - C_0 \end{cases}$

$\begin{cases} 2A_1 + B_1 = 0 \\ 2A_0 + B_0 = A_1 \\ 2B_1 + 4C_1 = 0 \\ 2B_0 + 4C_0 = B_1 \\ A_1 - C_1 = 0 \\ A_0 - C_0 = C_1 \end{cases}$

$\begin{cases} 2A_1 + B_1 = 0 \\ 2B_1 + 4C_1 = 0 \\ A_1 - C_1 = 0 \end{cases}$ $\quad B_1 = -2A \quad \Rightarrow A_1 = C_1 \Rightarrow C_1 = A_1 \quad \Rightarrow A_1 \in \mathbb{R}$

Fie $A_1 = 1 \Rightarrow B_1 = -2, C_1 = 1$

$\begin{cases} 2A_0 + B_0 = 1 \\ 2B_0 + 4C_0 = -2 \\ A_0 - C_0 = 1 \end{cases}$ $\quad \Rightarrow B_0 = 1 - 2A_0 \quad \Rightarrow C_0 = A_0 - 1, \quad A_0 \in \mathbb{R}$

Fie $A_0 = 1 \Rightarrow B_0 = -1, C_0 = 0$

$X_{r_{2,3}} = \begin{pmatrix} t + 1 \\ -2t - 1 \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{X_{r_2}} + \underbrace{\begin{pmatrix} t \\ -2t \\ t \end{pmatrix}}_{X_{r_3}}$

$X = C_1 X_{r_1} + C_2 X_{r_2} + C_3 X_{r_3} = C_1 \begin{pmatrix} 4e^{3t} \\ 4e^{3t} \\ e^{3t} \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} t \\ -2t \\ t \end{pmatrix}$

$= \begin{pmatrix} 4C_1 e^{3t} + C_2 + tC_3 \\ 4C_1 e^{3t} - C_2 - 2C_3 t \\ C_1 e^{3t} + tC_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{cases} x = 4C_1 e^{3t} + C_2 + tC_3 \\ y = 4C_1 e^{3t} - C_2 - 2tC_3 \\ z = C_1 e^{3t} + tC_3 \end{cases}$