Temă – Curs 7 Dotculescu Mihai-Gilvia

1. Să se ortogonalizeze și apoi ortonormeze următoarele sisteme de vectori:

b) $B = \{(1,2,2),(1,1,-5),(3,2,0)\} \subset \mathbb{R}^3/\mathbb{R}$.

c) $B = \{(-1,2,1),(1,-1,2),(1,2,0)\} \subset \mathbb{R}^3/\mathbb{R}$.

Rezolvare:

b) $\vec{x_1} = (1,2,2)$; $\vec{x_2} = (1,1,-5)$; $\vec{x_3} = (3,2,0)$.

$$D = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 2 & -5 & 0 \end{vmatrix} = -30+4-6+10 = -22 \neq 0$$

$\Rightarrow B$ - l.i $\Rightarrow$ se poate ortogonaliza.

Baza ortogonală va fi $B^{\perp} = \{\vec{y_1}, \vec{y_2}, \vec{y_3}\}$

$\vec{y_1} = \vec{x_1} = (1,2,2)$; $\vec{y_2} = \vec{x_2} + \alpha \vec{y_1}$; $\vec{y_3} = \vec{x_3} + \beta_1 \vec{y_1} + \beta_2 \vec{y_2}$

$\vec{y_2} = \vec{x_2} + \alpha \vec{y_1} \mid \langle \vec{y_1} \rangle$.

$\langle \vec{y_2}, \vec{y_1} \rangle = \langle \vec{x_2}, \vec{y_1} \rangle + \alpha \langle \vec{y_1}, \vec{y_1} \rangle$.

$\langle \vec{y_2}, \vec{y_1} \rangle = 0$

$\langle \vec{x_2}, \vec{y_1} \rangle = 1\cdot 1 + 2\cdot 1 + 2(-5) = 1+2-10 = -7$

$\langle \vec{y_1}, \vec{y_1} \rangle = 1\cdot 1 + 2\cdot 2 + 2\cdot 2 = 1+4+4 = 9$.

$\Rightarrow 0 = -7 + 9\alpha \Rightarrow \alpha = \frac{7}{9}$.

$\Rightarrow \vec{y_2} = (1,1,-5) + \frac{7}{9}\cdot(1,2,2) = \left(1+\frac{7}{9}, 1+\frac{14}{9}, -5+\frac{14}{9}\right) = \left(\frac{16}{9}, \frac{23}{9}, \frac{-31}{9}\right)$

- $\vec{y_3} = \vec{x_3} + \beta_1 \vec{y_1} + \beta_2 \vec{y_2} \mid \cdot \langle \vec{y_1} \rangle$

$\langle \vec{y_3}, \vec{y_1} \rangle = \langle \vec{x_3}, \vec{y_1} \rangle + \beta_1 \langle \vec{y_1}, \vec{y_1} \rangle + \beta_2 \langle \vec{y_2}, \vec{y_1} \rangle$.

$\langle \vec{y_3}, \vec{y_1} \rangle = 0$

$\langle \vec{x_3}, \vec{y_1} \rangle = 3\cdot 1 + 2\cdot 2 + 2\cdot 0 = 3+4 = 7$

$\langle \vec{y_2}, \vec{y_1} \rangle = 0$ $\qquad \Rightarrow 0 = 7 + 9\beta_1 \Rightarrow \beta_1 = \frac{-7}{9}$

$\langle \vec{y_1}, \vec{y_1} \rangle = 9$

- $\vec{y_3} = \vec{x_3} + \beta_1 \vec{y_1} + \beta_2 \vec{y_2} \mid \cdot \langle \vec{y_2} \rangle$.

$\langle \vec{y_3}, \vec{y_2} \rangle = \langle \vec{x_3}, \vec{y_2} \rangle + \beta_1 \langle \vec{y_1}, \vec{y_2} \rangle + \beta_2 \langle \vec{y_2}, \vec{y_2} \rangle$.

$\langle \vec{y_3}, \vec{y_2} \rangle = 0$.

$\langle \vec{x_3}, \vec{y_2} \rangle = 3\cdot\frac{16}{9} + 2\cdot\frac{23}{9} + 0\cdot\left(\frac{-31}{9}\right) = \frac{16}{3} + \frac{46}{9} = \frac{94}{9}$

$\langle \vec{y_1}, \vec{y_2} \rangle = 0$

$\langle \vec{y_2}, \vec{y_2} \rangle = \frac{16}{9}\cdot\frac{16}{9} + \frac{23\cdot 23}{9\cdot 9} + \left(\frac{-31}{9}\right)\cdot\left(\frac{-31}{9}\right) = \frac{256}{81} + \frac{529}{81} + \frac{961}{81} = \frac{1746}{81} = \frac{194}{9}$

$\Rightarrow 0 = \frac{94}{9} + \beta_2 \cdot \frac{194}{9} \Rightarrow \beta_2 = \frac{-94}{9}\cdot\frac{9}{194} = \frac{-94}{194} = \frac{-47}{97}$

$\Rightarrow \vec{y_3} = (3,2,0) + \left(\frac{-47}{97}\right)\cdot\left(\frac{16}{9}, \frac{23}{9}, \frac{-31}{9}\right) + \left(\frac{-7}{9}\right)\cdot(1,2,2) =$

$= \left(3 - \frac{47}{97}\cdot\frac{16}{9} - \frac{7}{9}\cdot 1,\ 2 - \frac{47}{97}\cdot\frac{23}{9} - \frac{7}{9}\cdot 2,\ 0 + \frac{47}{97}\cdot\frac{31}{9} - \frac{7}{9}\cdot 2\right) =$

$= \left(\frac{132}{97}, \frac{-1087}{873}, \frac{11}{97}\right)$

$\bullet\ B^{\perp} = \left\{(1,2,2), \left(\frac{16}{9}, \frac{23}{9}, \frac{-31}{9}\right), \left(\frac{132}{97}, \frac{-1087}{873}, \frac{11}{97}\right)\right\}$

Pentru a ortonorma:

$B_n^{\perp} = \left\{\frac{1}{\|\vec{y_1}\|}\cdot\vec{y_1}, \frac{1}{\|\vec{y_2}\|}\cdot\vec{y_2}, \frac{1}{\|\vec{y_3}\|}\cdot\vec{y_3}\right\}$

$\|\vec{y_1}\| = \sqrt{1^2+2^2+2^2} = 3$

$\|\vec{y_2}\| = \sqrt{\left(\frac{16}{9}\right)^2 + \left(\frac{23}{9}\right)^2 + \left(\frac{-31}{9}\right)^2} = \sqrt{\frac{256+529+961}{81}} = \sqrt{\frac{1746}{81}} = \frac{41\sqrt{65}}{9}$

$\|\vec{y_3}\| = \sqrt{\left(\frac{132}{97}\right)^2 + \left(\frac{-1087}{873}\right)^2 + \left(\frac{11}{97}\right)^2}$

2) $B = \{(-1,2,1), (1,-1,2), (1,2,0)\} \subset \mathbb{R}^3/\mathbb{R}$.

$\vec{x_1} = (-1,2,1)$, $\vec{x_2} = (1,-1,2)$, $\vec{x_3} = (1,2,0)$.

$D = \begin{vmatrix} -1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = 4+2+1+4 = 11 \neq 0$

$\Rightarrow B$ - l.i $\Rightarrow$ se poate ortogonaliza.

$B^{\perp} = \{\vec{y_1}, \vec{y_2}, \vec{y_3}\}$

$\vec{y_1} = \vec{x_1} = (-1,2,1)$; $\vec{y_2} = \vec{x_2} + \alpha\vec{y_1}$; $\vec{y_3} = \vec{x_3} + \beta_1\vec{y_1} + \beta_2\vec{y_2}$

$\vec{y_2} = \vec{x_2} + \alpha\vec{y_1} \mid \cdot \langle \vec{y_1} \rangle$

$\langle \vec{y_2}, \vec{y_1} \rangle = \langle \vec{x_2}, \vec{y_1} \rangle + \alpha \langle \vec{y_1}, \vec{y_1} \rangle$

$\langle \vec{y_2}, \vec{y_1} \rangle = 0$

$\langle \vec{x_2}, \vec{y_1} \rangle = 1\cdot(-1) + (-1)2 + 2\cdot 1 = -1.$

$\langle \vec{y_1}, \vec{y_1} \rangle = (-1)\cdot(-1) + 2\cdot 2 + 1\cdot 1 = 6. \Rightarrow 0 = -1 + \alpha\cdot 6 \Rightarrow \alpha = \frac{1}{6}$

$\Rightarrow \vec{y_2} = (1,-1,2) + \frac{1}{6}(-1,2,1) = \left(1-\frac{1}{6}, -1+\frac{1}{3}, 2+\frac{1}{6}\right) = \left(\frac{5}{6}, \frac{-2}{3}, \frac{13}{6}\right).$

$$\vec{y_3} = \vec{x_3} + \beta_1 \vec{y_1} + \beta_2 \vec{y_2} \;\Big|\cdot \langle \vec{y_1} \rangle$$

$$\langle \vec{y_3}, \vec{y_1} \rangle = \langle \vec{x_3}, \vec{y_1} \rangle + \beta_1 \langle \vec{y_1}, \vec{y_1} \rangle + \beta_2 \langle \vec{y_2}, \vec{y_1} \rangle.$$

$$\langle \vec{y_3}, \vec{y_1} \rangle = 0$$

$$\langle \vec{y_2}, \vec{y_1} \rangle = 0$$

$$\langle \vec{x_3}, \vec{y_1} \rangle = 1\cdot(-1) + 2\cdot 2 + 0\cdot 1 = 3. \Rightarrow 0 = 3 + 6\beta_1 \Rightarrow \beta_1 = \frac{-3}{6} = \frac{-1}{2}$$

$$\vec{y_3} = \vec{x_3} + \beta_1 \vec{y_1} + \beta_2 \vec{y_2} \;\Big|\cdot \langle \vec{y_2} \rangle$$

$$\langle \vec{y_3}, \vec{y_2} \rangle = 0$$

$$\langle \vec{y_1}, \vec{y_2} \rangle = 0$$

$$\langle \vec{y_3}, \vec{y_2} \rangle = \langle \vec{x_3}, \vec{y_2} \rangle + \beta_1 \langle \vec{y_1}, \vec{y_2} \rangle + \beta_2 \langle \vec{y_2}, \vec{y_2} \rangle$$

$$\langle \vec{y_1}, \vec{y_2} \rangle = 0$$

$$\langle \vec{x_3}, \vec{y_2} \rangle = 1\cdot\frac{5}{6} + 2\cdot\left(\frac{-2}{3}\right) + 0\cdot\frac{13}{6} = \frac{5}{6} - \frac{4}{3} = \frac{5-8}{6} = \frac{-3}{6} = \frac{-1}{2}$$

$$\langle \vec{y_2}, \vec{y_2} \rangle = \frac{5}{6}\cdot\frac{5}{6} + \left(\frac{-2}{3}\right)\cdot\left(\frac{-2}{3}\right) + \frac{13}{6}\cdot\frac{13}{6} = \frac{25}{36} + \frac{4}{9} + \frac{169}{36} = \frac{210}{36} = \frac{35}{6}$$

$$\Rightarrow 0 = \frac{-1}{2} + \frac{35}{6}\beta_2 \Rightarrow \frac{35}{6}\beta_2 = \frac{1}{2} \Rightarrow \beta_2 = \frac{1}{2}\cdot\frac{6}{35} = \frac{3}{35}$$

$$\Rightarrow y_3 = (1,2,0) + \left(\frac{-1}{2}\right)(-1,2,1) + \frac{3}{35}\left(\frac{5}{6}, \frac{-2}{3}, \frac{13}{6}\right).$$

$$= \left(1 + \frac{1}{2} + \frac{3}{35}\cdot\frac{5}{6},\; 2 - 1 - \frac{6}{35\cdot 3},\; 0 - \frac{1}{2} + \frac{3}{35}\cdot\frac{13}{6}\right) = \left(\frac{11}{7}, \frac{33}{35}, \frac{-11}{70}\right)$$

$$B^{\perp} = \left\{(-1,2,1), \left(\frac{5}{6}, \frac{-2}{3}, \frac{13}{6}\right), \left(\frac{11}{7}, \frac{33}{35}, \frac{-11}{70}\right)\right\}$$

$$B^{\perp}_{n} = \left\{\frac{\vec{y_1}}{\|\vec{y_1}\|}, \frac{\vec{y_2}}{\|\vec{y_2}\|}, \frac{\vec{y_3}}{\|\vec{y_3}\|}\right\}$$

$$\|\vec{y_1}\| = \sqrt{(-1)^2 + 2^2 + 1^2} = \sqrt{6}$$

$$\|\vec{y_2}\| = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{13}{6}\right)^2} = \sqrt{\frac{210}{36}} = \frac{\sqrt{210}}{6}$$

$$\|\vec{y_3}\| = \sqrt{\left(\frac{11}{7}\right)^2 + \left(\frac{33}{35}\right)^2 + \left(\frac{-11}{70}\right)^2}$$