Curs14 - Rezolvare

Exerciţiu 2. c)

2 c) $x''' + 4x'' + 5x' = te^t$ $\quad \alpha = 1$

$x''' + 4x'' + 5x' = 0$

$x = e^{rt},\ x' = re^{rt},\ x'' = r^2e^{rt},\ x''' = r^3e^{rt}$

$r^3 + 4r^2 + 5r = 0$

$r(r^2 + 4r + 5) = 0$

$\boxed{r_1 = 0}$

$\Delta = 16 - 20 = -4 \Rightarrow r_{2,3} = \frac{-4 \pm 2i}{2} = \boxed{-2 \pm i}$

$(\alpha \pm i\beta,\ \alpha = -2,\ \beta = 1)$

$e^{0\cdot t},\ e^{-2\cdot t}\cos 1\cdot t,\ e^{-2\cdot t}\sin 1\cdot t$ – sist. fundam. de sol.

$$\boxed{x_0 = C_1 + C_2e^{-2t}\cos t + C_3e^{-2t}\sin t}$$

$x_p = (A_1t + A_0)e^t$

$x_p' = A_1e^t + (A_1t + A_0)e^t = (A_1 + A_1t + A_0)e^t$

$x_p'' = A_1e^t + (A_1 + A_1t + A_0)e^t = (A_1 + A_1 + A_1t + A_0)e^t$

$= (2A_1 + A_1t + A_0)e^t$

$x_p''' = A_1e^t + (2A_1 + A_1t + A_0)e^t = (3A_1 + A_1t + A_0)e^t$

$3A_1 + A_1t + A_0 + 4(2A_1 + A_1t + A_0) + 5(A_1 + A_1t + A_0) = t$

$3A_1 + A_1t + A_0 + 8A_1 + 4A_1t + 4A_0 + 5A_1 + 5A_1t + 5A_0 = t$

$10A_1t + 16A_1 + 10A_0 = t$

$\begin{cases} 10A_1 = 1 \\ 16A_1 + 10A_0 = 0 \end{cases} \Rightarrow A_1 = \frac{1}{10}$

$A_0 = -\frac{16}{10}\cdot\frac{1}{10} = -\frac{16}{100} = -\frac{4}{25}$

$$\boxed{x_p = \left(\frac{1}{10}t - \frac{4}{25}\right)e^t}$$

$$\boxed{x = x_0 + x_p = C_1 + C_2e^{-2t}\cos t + C_3e^{-2t}\sin t + \left(\frac{1}{10}t - \frac{4}{25}\right)e^t}$$

Exerciţiu 4. a)

4 a) $\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z^2(x-y)}$, $\quad x, y, z > 0,\ x \neq y$

$\frac{dx}{x^2} = \frac{dy}{-y^2}$

$\int \frac{1}{x^2}dx = \int \frac{-1}{y^2}dy$

$-\frac{1}{x} = \frac{1}{y} + C_1$

$-\frac{1}{x} - \frac{1}{y} = C_1$

$\frac{1}{x} + \frac{1}{y} = C_1 = \varphi_1(x,y,z)$

$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dx+dy}{x^2-y^2} = \underbrace{\frac{d(x+y)}{(x-y)(x+y)} = \frac{dz}{z^2(x-y)}}$

$\frac{d(x+y)}{\cancel{(x-y)}(x+y)} = \frac{dz}{z^2\cancel{(x-y)}}$

$\left.\begin{aligned} &\frac{d(x+y)}{x+y} = \frac{dz}{z^2} \\ &\text{Not. } x+y = u \end{aligned}\right\} \Rightarrow \frac{du}{u} = \frac{dz}{z^2} \Leftrightarrow \int \frac{1}{u}du = \int \frac{1}{z^2}dz$

$\ln|u| = -\frac{1}{z} + C_2$

$\ln|u| + \frac{1}{z} = C_2$

$$\boxed{\ln(x+y) + \frac{1}{z} = C_2 = \varphi_2(x,y,z)}$$

Verificăm dc. φ_1, φ_2 sunt indep. $\Leftrightarrow \frac{D(\varphi_1,\varphi_2)}{D(x,y)} \neq 0$

Fie z - variabila indep.

$$\begin{vmatrix} \frac{\partial\varphi_1}{\partial x} & \frac{\partial\varphi_1}{\partial y} \\ \frac{\partial\varphi_2}{\partial x} & \frac{\partial\varphi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{1}{x^2} & -\frac{1}{y^2} \\ \frac{1}{x+y} & \frac{1}{x+y} \end{vmatrix} = \frac{1}{x+y}\begin{vmatrix} -\frac{1}{x^2} & -\frac{1}{y^2} \\ 1 & 1 \end{vmatrix} = \frac{1}{x+y}\left(-\frac{1}{x^2} + \frac{1}{y^2}\right) = \frac{1}{x+y}\cdot\frac{x^2-y^2}{x^2y^2} = \frac{(x-y)\cancel{(x+y)}}{\cancel{(x+y)}x^2y^2} = \frac{x-y}{x^2y^2} \neq 0 \Rightarrow \varphi_1, \varphi_2 \text{ indep.}$$

Sol. implicită:

$$\begin{cases} \frac{1}{x} + \frac{1}{y} = c_1 \\ \ln(x+y) + \frac{1}{z} = c_2 \end{cases}$$

Exerciţiu 4. b)

4 b) $\begin{cases} x' = y + xy \\ y' = x + yx \\ z' = z^2 - 1 \end{cases} \Leftrightarrow \begin{cases} \frac{dx}{dt} = y + xy \\ \frac{dy}{dt} = x + yx \\ \frac{dz}{dt} = z^2 - 1 \end{cases} \Leftrightarrow \begin{cases} \frac{dx}{y+xy} = dt \\ \frac{dy}{x+yx} = dt \\ \frac{dz}{z^2-1} = dt \end{cases} \Leftrightarrow \frac{dx}{y+xy} = \frac{dy}{x+yx} = \frac{dz}{z^2-1} = dt$

$$\frac{dx}{y+xy} = \frac{dy}{x+yx} = \frac{dx - dy}{y + \cancel{xy} - x - \cancel{yx}} = \frac{d(x-y)}{y-x} = \frac{dz}{z^2-1}$$

$$\left.\begin{array}{l} \frac{d(x-y)}{-(x-y)} = \frac{dz}{z^2-1} \\ x - y = u \end{array}\right\} \Rightarrow \frac{du}{-u} = \frac{dz}{z^2-1}$$

$$-\int \frac{1}{u} du = \int \frac{1}{z^2-1} dz$$

$$-\ln|u| = \frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| + c_1 \quad |\cdot 2$$

$$2\ln|u| = -\ln\left|\frac{z-1}{z+1}\right| + c_1$$

$$\ln|u|^2 + \ln\left|\frac{z-1}{z+1}\right| = c_1$$

$$\ln\left|u^2\left(\frac{z-1}{z+1}\right)\right| = c_2 \Rightarrow \boxed{(x-y)^2 \cdot \frac{z-1}{z+1} = c_1 = \varphi_1(t,x,y,z)}$$

$$\frac{dz}{z^2-1} = dt$$

$$\int \frac{1}{z^2-1} \cancel{dt} = \int dt$$

$$\frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| = t + c_2$$

$$\boxed{\frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| - t = c_2 = \varphi_2(t,x,y,z)}$$

$$\left.\begin{array}{l} \frac{d(x-y)}{-(x-y)} = dt \\ x - y = u \end{array}\right\} \Rightarrow \frac{du}{-u} = dt$$

$$-\int \frac{1}{u} du = \int dt$$

$$-\ln|u| = t + c_3$$

$$\boxed{\ln|x-y| + t = c_3 = \varphi_3(t,x,y,z)}$$

Verif. dc. $\varphi_1, \varphi_2, \varphi_3$ sunt indep. $\Leftrightarrow \frac{D(\varphi_1,\varphi_2,\varphi_3)}{D(x,y,z)} \neq 0$

$$\begin{vmatrix} 2(x-y)\frac{z-1}{z+1} & -2(x-y)\frac{z-1}{z+1} & (x-y)^2 \cdot \frac{2}{(z+1)^2} \\ 0 & 0 & \frac{z-1}{(z+1)^3} \\ \frac{1}{x-y} & \frac{-1}{x-y} & 0 \end{vmatrix} = -2\cancel{(x-y)} \cdot \frac{z-1}{z+1} \cdot \frac{z-1}{(z+1)^3} \cdot \frac{1}{\cancel{x-y}} + 2\cancel{(x-y)}\frac{z-1}{z+1} \cdot \frac{z-1}{(z+1)^3} \cdot \frac{1}{\cancel{x-y}}$$

$$= -\frac{2(z-1)^2}{(z+1)^4} + \frac{2(z-1)^2}{(z+1)^4} = 0 \Rightarrow \varphi_1, \varphi_2, \varphi_3 \text{ nu sunt indep.}$$

$$\frac{dx}{y+xy} = \frac{dy}{x+xy} = \frac{dz}{z^2-1} = dt$$

$$\frac{dx}{y+xy} = \frac{dy}{x+xy} \Leftrightarrow \frac{dz}{y(1+x)} = \frac{dy}{x(1+y)}$$

$$\frac{x\,dx}{1+x} = \frac{y\,dy}{1+y}$$

$$\int \frac{x+1-1}{1+x}\,dx = \int \frac{y+1-1}{1+y}\,dy$$

$$\int\left(1 - \frac{1}{1+x}\right)dx = \int\left(1 - \frac{1}{1+y}\right)dy$$

$$x - \ln|1+x| = y - \ln|1+y| + C_3$$

$$\boxed{x - y - \ln|1+x| + \ln|1+y| = C_3 = \psi_4(t,x,y,z)}$$

$$\psi_1(t,x,y,z) = (x-y)^2 \cdot \frac{z-1}{z+1}$$

$$\psi_2(t,x,y,z) = \frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| - t$$

$$\psi_3(t,x,y,z) = \ln|x-y| + t$$

$$\psi_4(t,x,y,z) = x - y - \ln|1+x| + \ln|1+y|$$

Verif. ψ_2, ψ_3, ψ_4 sunt indep.

$$\begin{vmatrix} 0 & 0 & \frac{z-1}{(z+1)^3} \\ \frac{1}{x-y} & \frac{-1}{x-y} & 0 \\ 1-\frac{1}{1+x} & -1+\frac{1}{1+y} & 0 \end{vmatrix} = \frac{z-1}{(z+1)^3} \cdot \begin{vmatrix} \frac{1}{x-y} & \frac{-1}{x-y} \\ \frac{1+x-1}{1+x} & \frac{-1-y+1}{1+y} \end{vmatrix} = \frac{1}{x-y} \cdot \frac{z-1}{(z+1)^3} \begin{vmatrix} 1 & -1 \\ \frac{x}{1+x} & \frac{-y}{1+y} \end{vmatrix}$$

$$= \frac{1}{x-y} \cdot \frac{z-1}{(z+1)^3}\left(\frac{-y}{1+y} + \frac{x}{1+x}\right)$$

$$= \frac{1}{x-y} \cdot \frac{z-1}{(z+1)^3} \cdot \frac{-y-\cancel{xy}+x+\cancel{xy}}{(1+x)(1+y)}$$

$$= \frac{1}{\cancel{x-y}} \cdot \frac{z-1}{(z+1)^3} \cdot \frac{\cancel{x-y}}{(1+x)(1+y)}$$

$$= \frac{z-1}{(z+1)^3(1+x)(1+y)} \neq 0.$$

Sol. implicită!

$$\begin{cases} \frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| - t = C_2 \\ \ln|x-y| + t = C_3 \\ x - y - \ln(1+x) + \ln(1+y) = C_4 \end{cases}$$