# Ecuații diferențiale și cu derivate parțiale

## Laborator 07

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1. Să se rezolve urm. ec. de ordin n:

$\boxed{a)}$ $x^{IV} = t+2$, $x(0)=1$, $x'(0)=2$, $x''(0)=-1$, $x'''(0)=0$

b) $x'' = t\sin t$, $x(0)=1$, $x'(0)=2$

$\boxed{c)}$ $x''' = \sin t + \cos t$, $x(0)=1$, $x'(0)=2$, $x'''(0)=3$

$\boxed{d)}$ $x'' = \frac{1}{t}$, $x(1)=1$, $x'(1)=2$

e) $x''' = \ln t$, $x(1)=2$, $x'(1)=1$, $x''(1)=0$

Rezolvare:

b) $x'' = t\sin t$, $x(0)=1$, $x'(0)=2$

$x^{(n)} = f(t)$ ; $x'' = f(t)$

• $x' = \int x''\,dt = \int \underbrace{t}_{f}\,\underbrace{\sin t}_{g'}\,dt = -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + c_1$

$f'=1$, $g=-\cos t$

• $x = \int x'\,dt = \int(-t\cos t + \sin t + c_1)\,dt = -\int \underbrace{t}_{f}\,\underbrace{\cos t}_{g'}\,dt + \int \sin t\,dt + \int c_1\,dt$

$f'=1$, $g=\sin t$

$= -(t\sin t - \int \sin t\,dt) + \int \sin t\,dt + c_1\int dt$

$= -t\sin t + 2(-\cos t) + c_1 t + c_2$

• $x(t) = -t\sin t - 2\cos t + c_1 t + c_2$

$\left.\begin{array}{l} x(0) = -2 + c_2 = 1 \Rightarrow c_2 = 3 \\ x'(0) = c_1 = 2 \end{array}\right\} \Rightarrow x_{PC}(t) = -t\sin t = -2\cos t + 2t + 3.$

l) $x''' = \ln t$, $x(1)=2$, $x'(1)=1$, $x''(1)=0$.

$x''' = f(t)$.

• $x'' = \int x''' \, dt = \int 1 \cdot \ln t \, dt = t \ln t - \int \frac{1}{t}\, t \, dt = t\ln t - t + C_1$

$\downarrow \quad \downarrow$

$g' \quad f$

$g = t, \ f' = \frac{1}{t}$

• $x' = \int x'' \, dt = \int (t\ln t - t + C_1)\, dt = \int t \ln t \, dt - \int t\, dt + C_1 \int dt =$

$\downarrow \quad \downarrow$

$g' \quad f$

$g = \frac{t^2}{2}, \ f' = \frac{1}{t}$

$= \frac{t^2}{2}\ln t - \int \frac{1}{t}\cdot \frac{t^2}{2}\, dt - \frac{t^2}{2} + C_1 t = \frac{t^2}{2}\ln t - \frac{1}{2}\cdot\frac{t^2}{2} - \frac{t^2}{2} + C_1 t + C_2.$

• $x' = \frac{t^2}{2}\ln t - \frac{t^2}{4} - \frac{t^2}{2} + C_1 t + C_2 = \frac{t^2}{2}\ln t - \frac{3t^2}{4} + C_1 t + C_2$

• $x = \int x' \, dt = \int \left(\frac{t^2}{2}\ln t - \frac{3}{4}t^2 + C_1 t + C_2\right) dt = \frac{1}{2}\int t^2 \ln t \, dt - \frac{3}{4}\int t^2 \, dt +$

$\downarrow \quad \downarrow$

$g' \quad f$

$g = \frac{t^3}{3}, \ f' = \frac{1}{t}$

$+ C_1 \int t\, dt + C_2 \int dt = \frac{1}{2}\left(\frac{t^3}{3}\ln t - \int \frac{1}{t}\cdot\frac{t^3}{3}\, dt\right) - \frac{3}{4}\cdot\frac{t^3}{3} + C_1 \frac{t^2}{2} + C_2 t =$

$= \frac{t^3}{6}\ln t - \frac{1}{6}\cdot\frac{t^3}{3} - \frac{t^3}{4} + C_1\frac{t^2}{2} + C_2 t + C_3.$

• $x(1) = \frac{-1}{18} - \frac{1}{4} + \frac{C_1}{2} + C_2 + C_3 = 2$

$x'(1) = \frac{-3}{4} + C_1 + C_2 = 1$

$x''(1) = -1 + C_1 = 0 \Rightarrow \boxed{C_1 = 1} \Rightarrow C_2 = 1 + \frac{3}{4} - 1 = \frac{3}{4}$

$C_3 = 2 + \frac{1}{18} + \frac{1}{4} - \frac{1}{2} - \frac{3}{4} = 1 + \frac{1}{18} = \frac{19}{18}$

$x_{PC} = \frac{t^3}{6}\ln t - \frac{t^3}{18} - \frac{t^3}{4} + \frac{t^2}{2} + \frac{3}{4}t + \frac{19}{18}$

2. Să se rez. urm. ec. de ordin n:

$\boxed{a)}$ $e^{x''} - (x'')^2 = t+1$

b) $x'' - \sqrt{x''} = t+3$

$\boxed{c)}$ $x'' + \ln x'' = t-5$

<u>Rezolvare:</u>

b) $x'' - \sqrt{x''} = t+3$

$F(t, x'') = 0$

• $x'' = y$

$y - \sqrt{y} = t+3 \Rightarrow t = y - \sqrt{y} - 3 = \varphi(y) \Rightarrow dt = \left(1 - \frac{1}{2\sqrt{y}}\right)dy$

• $x' = \int x''\,dt = \int y\cdot\left(1 - \frac{1}{2\sqrt{y}}\right)dy = \int\left(y - y\cdot\frac{1}{2\sqrt{y}}\right)dy = \int y\,dy - \frac{1}{2}\int\sqrt{y}\,dy =$

$= \frac{y^2}{2} - \frac{1}{2}\cdot\int y^{\frac{1}{2}}dy = \frac{y^2}{2} - \frac{1}{2}\cdot\frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_1 = \frac{y^2}{2} - \frac{1}{2}\cdot\frac{y^{\frac{3}{2}}}{\frac{3}{2}} + C_1 =$

$= \frac{y^2}{2} - \frac{1}{2}\cdot\frac{2}{3}y^{\frac{3}{2}} + C_1$

• $x = \int x'\,dt = \int\left(\frac{y^2}{2} - \frac{1}{3}y^{\frac{3}{2}} + C_1\right)\left(1 - \frac{1}{2y^{\frac{1}{2}}}\right)dy = \int\left(\frac{y^2}{2} - \frac{y^2}{2}\cdot\frac{1}{2y^{1/2}} - \frac{1}{3}y^{\frac{3}{2}} +\right.$

$\left.+ \frac{1}{3}y^{\frac{3}{2}}\cdot\frac{1}{2y^{1/2}} + C_1 - \frac{C_1}{2y^{1/2}}\right)dy = \frac{1}{2}\int y^2 dy - \frac{1}{4}\int y^{2-\frac{1}{2}}dy - \frac{1}{3}\int y^{\frac{3}{2}}dy$

$+ \frac{1}{6}\int y\,dy + C_1\int dy - C_1\int\frac{1}{2\sqrt{y}}dy = \frac{1}{2}\cdot\frac{y^3}{3} - \frac{1}{4}\cdot\frac{y^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{1}{3}\cdot\frac{y^{\frac{3}{2}+1}}{\frac{3}{2}+1}$

$+ \frac{1}{6}\cdot\frac{y^2}{2} + C_1 y - C_1\sqrt{y} + C_2$

$= \frac{y^3}{6} - \frac{1}{4}\cdot\frac{2}{5}\cdot y^{\frac{5}{2}} - \frac{1}{3}\cdot\frac{2}{5}\cdot y^{\frac{5}{2}} + \frac{1}{12}y^2 + C_1 y - C_1\sqrt{y} + C_2.$

3. Să se rez. urm. ec. de ordin n:

a) $x''' = \sqrt{1+x''}$, $x(0)=x'(0)=x''(0)=0$

$\boxed{b)}$ $x''+x'\,tg\,t = \sin 2t$

$\boxed{c)}$ $t^2 x'' + 2(x')^2 = 0$, $x(1)=2$, $x'(1)=3$

$\boxed{d)}$ $x'''-x''=t$, $x(1)=1$, $x'(1)=-1$, $x''(1)=2$

e) $t^2 x'' + t x' = 1$

$\boxed{f)}$ $t x''' + x'' = 1+t$

$\boxed{g)}$ $(1+t^2)x'' - 2t x' = 0$, $x(0)=0$, $x'(0)=3$

$\boxed{h)}$ $x^{(5)} + x^{(4)} = 0$.

Rezolvare

a) $x''' = \sqrt{1+x''}$, $x(0)=x'(0)=x''(0)=0$.

$F(t,x'',x''')=0$

$\boxed{x''=y} \Rightarrow x'''=y' \Leftrightarrow y'=\sqrt{1+y} \Leftrightarrow \frac{dy}{dt}=\sqrt{1+y} \Leftrightarrow \frac{dy}{\sqrt{1+y}}=dt$

$2\int \frac{1}{2\sqrt{1+y}}\,dy = \int dt$;

$2\sqrt{1+y} = t + C \;|:2 \Leftrightarrow \sqrt{1+y} = \frac{t}{2} + C \Leftrightarrow 1+y = \left(\frac{t}{2}+C\right)^2$

$$\boxed{y = \left(\frac{t}{2}+C\right)^2 - 1}$$

• $x'' = \left(\frac{t}{2}+C\right)^2 - 1$

• $x' = \int \left[\left(\frac{t}{2}+c\right)^2 - 1\right] dt = \int \left(\frac{t^2}{4} + tc + c^2 - 1\right) dt$

$= \frac{1}{4}\int t^2\,dt - C\int t\,dt + C^2\int dt - \int dt$

$= \frac{1}{4}\cdot\frac{t^3}{3} - C\cdot\frac{t^2}{2} + c^2 t - t + C_1 = \frac{t^3}{12} - \frac{Ct^2}{2} + c^2 t - t + C_1$

• $x = \int \left(\frac{t^3}{12} - \frac{Ct^2}{2} + c^2 t - t + c_1\right) dt = \frac{1}{12}\int \frac{t^3}{1}\,dt - \frac{C}{2}\int t^2\,dt + c^2\int t\,dt -$

$- \int t\,dt + C_1\int dt = \frac{1}{12}\cdot\frac{t^4}{4} - \frac{C}{2}\cdot\frac{t^3}{3} + C^2\cdot\frac{t^2}{2} - \frac{t^2}{2} + C_1 t + C_2$

• $x(t) = \frac{t^4}{48} - \frac{Ct^3}{6} + \frac{C^2 t^2}{2} - \frac{t^2}{2} + C_1 t + C_2$

$$x(0) = \boxed{C_2 = 0}$$

$$x'(0) = \boxed{C_1 = 0}$$

$$x''(0) = c^2 - 1 = 0 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1$$

$$\Rightarrow x_{PC} = \frac{t^4}{48} \pm \frac{t^3}{6} + \frac{t^2}{2} - \frac{t^2}{2}$$

$$x_{PC} = \frac{t^4}{48} \pm \frac{t^3}{6}.$$

2) $t^2 x'' + t x' = 1$

$$F(t, x', x'') = 0$$

$$\boxed{x' = y} \Rightarrow x'' = y'$$

$$t^2 y' + t y = 1 \,|: t^2 \Leftrightarrow y' + \underbrace{\frac{1}{t}}_{A(t)} y = \underbrace{\frac{1}{t^2}}_{B(t)}$$

Etapa 1: $y' + \frac{1}{t} y = 0$

$$\frac{dy}{dt} = \frac{-1}{t} y \Leftrightarrow \frac{dy}{y} = \frac{-1}{t} dt \Leftrightarrow \frac{dy}{y} = \frac{-1}{t} dt \Leftrightarrow \int \frac{1}{y} = -\int \frac{1}{t} dt$$

$$\ln|y| = -\ln t + C \Leftrightarrow \ln|y| = -\ln t + \ln C \Rightarrow \boxed{y_0 = \frac{c}{t}}$$

Etapa 2: $\varphi_0 = \frac{c(t)}{t}$

$$\left(\frac{c(t)}{t}\right)' + \frac{1}{t} \cdot \frac{C(t)}{t} = \frac{1}{t^2} \Leftrightarrow \frac{c'(t) \cdot t - c(t)}{t^2} + \frac{C(t)}{t^2} = \frac{1}{t^2}$$

$$\frac{c'(t)}{t} - \cancel{\frac{C(t)}{t^2}} + \cancel{\frac{C(t)}{t^2}} = \frac{1}{t^2} \Leftrightarrow \frac{C'(t)}{t} = \frac{1}{t^2} \,|\cdot t \Leftrightarrow C'(t) = \frac{1}{t}$$

$$C(t) = \int \frac{1}{t} dt = \ln t + C_1$$

$$\boxed{\varphi_0 = \frac{\ln t + C_1}{t}} \Leftrightarrow \boxed{y = \varphi_0 + y_0 = \frac{c}{t} + \frac{\ln t}{t}}$$

$$x' = \frac{c}{t} + \frac{\ln t}{t}$$

$$x = \int x' dt = \int \left(\frac{c}{t} + \frac{\ln t}{t}\right) dt = c \cdot \int \frac{1}{t} dt + \int \frac{1}{t} \ln t \, dt$$

$$= c \cdot \ln t + \frac{\ln^2 t}{2} + C_1$$

4. Să se rez. urm. ec. de ordin n:

$\boxed{a)}$ $t^2 x x'' = (x - t x')^2$

$\boxed{b)}$ $t x x'' + t(x')^2 - x x' = 0$

c) $t^2 x x'' + t^2 (x')^2 - 5t x x' + 4x^2 = 0$, $x(1)=1$, $x'(1)=0$.

Rezolvare

c) $t^2 x x'' + t^2 (x')^2 - 5t x x' + 4x^2 = 0$, $x(1)=1$, $x'(1)=0$.

$F(t, x, x', x'') = 0$

$$F(t, mx, mx', mx'') = t^2 \cdot mx \cdot mx'' + t^2 \cdot (mx')^2 - 5t \cdot mx \cdot mx' + 4(mx)^2$$
$$= t^2 m^2 x x'' + t^2 m^2 (x')^2 - 5t m^2 x x' + 4m^2 x^2$$
$$= m^2 (t^2 x x'' + t^2 (x')^2 - 5t x x' + 4x^2)$$
$$= m^2 \cdot F(t, x, x', x'')$$

$\boxed{x' = xy} \Rightarrow x'' = x' \cdot y + x y' = x y^2 + x y'$

$t^2 \cdot x (x y^2 + x y') + t^2 (xy)^2 - 5t x^2 y + 4x^2 = 0$

$t^2 x^2 y^2 + t^2 x^2 y' + t^2 x^2 y^2 - 5t x^2 y + 4x^2 = 0 \,|: x^2$

$\underline{t^2 y^2} + t^2 y' + \underline{t^2 y^2} - 5t y + 4 = 0$

$t^2 y' + 2t^2 y^2 - 5t y + 4 = 0 \,|: t^2$

$y' + 2y^2 - \frac{5}{t} y + \frac{4}{t^2} = 0$

$y' = \underbrace{-2}_{A(t)} y^2 + \underbrace{\frac{5}{t}}_{B(t)} y - \underbrace{\frac{4}{t^2}}_{C(t)}$ (ec. Riccati).

• Verific dacă $\frac{1}{t}$ e sol.

$\frac{-1}{t^2} = -2 \cdot \frac{1}{t^2} + \frac{5}{t} \cdot \frac{1}{t} - \frac{4}{t^2}$ (A)

$\boxed{y = z + \frac{1}{t}} \Rightarrow y' = z' - \frac{1}{t^2}$

$z' - \frac{1}{t^2} = -2\left(z + \frac{1}{t}\right)^2 + \frac{5}{t}\left(z + \frac{1}{t}\right) - \frac{4}{t^2} = -2z^2 - 4\frac{z}{t} - 2\frac{1}{t^2} + \frac{5z}{t} + \frac{5}{t^2} - \frac{4}{t^2}$

$\Leftrightarrow z' = -2z^2 + \frac{z}{t} \Leftrightarrow z' = \underbrace{-2}_{A(t)} z^2 + \underbrace{\frac{1}{t}}_{B(t)} z$ (Ec. Bernoulli cu $\alpha = 2$)

$$\frac{z'}{z^2} = -2 + \frac{1}{t}\cdot\frac{z}{z^2}$$

$$z'\cdot z^{-2} = -2 + \frac{1}{t}\cdot z^{-1}$$

$$\boxed{u = z^{-1}} \Rightarrow u' = -z^{-2}z'$$

$$-z'z^{-2} = 2 - \frac{1}{t}z^{-1} \Rightarrow u' = 2 - \frac{1}{t}u \text{ (ec. afină)}$$

Etapa 1 $\quad u' = \frac{-1}{t}u \Leftrightarrow \frac{du}{dt} = \frac{-1}{t}u \Leftrightarrow \frac{1}{u}du = \frac{-1}{t}dt \Leftrightarrow \int\frac{1}{u}du = \int\frac{-1}{t}dt \Leftrightarrow$

$$\Leftrightarrow \ln|u| = -\ln|t| + C \Rightarrow \boxed{u_0 = Ct^{-1}}$$

Etapa 2: $\varphi_0 = C(t)t^{-1}$

$$C'(t)t^{-1} + \cancel{C(t)(-t^{-2})} + \cancel{\frac{1}{t}\cdot C(t)\cdot t^{-1}} = 2$$

$$C'(t)t^{-1} = 2 \Leftrightarrow C(t) = \int 2t\,dt = t^2 + C_1$$

$$\varphi_0 = (t^2 + C_1)t^{-1} = t + C_1t^{-1}$$

$$u = u_0 + \varphi_0 = Ct^{-1} + t + C_1t^{-1} = t + Ct^{-1}$$

- $u = \frac{1}{z} \Leftrightarrow z = \frac{1}{u} = \frac{1}{t + Ct^{-1}}$

- $y = z + \frac{1}{t} \Leftrightarrow y = \frac{1}{t+Ct^{-1}} + \frac{1}{t}$

- $y = \frac{x'}{x} \Leftrightarrow \frac{x'}{x} = \frac{1}{t+Ct^{-1}} + \frac{1}{t} \Leftrightarrow \frac{dx}{x} = \left(\frac{1}{t+Ct^{-1}} + \frac{1}{t}\right)dt \Leftrightarrow$

$$\Leftrightarrow \int\frac{dx}{x}\cancel{dx} = \int\left(\frac{1}{t+Ct^{-1}} + \frac{1}{t}\right)dt$$

$$\ln|x| = \frac{\ln|t^2+C|}{2} + \ln|t| + C_1$$

$$x = C_1t\,(|t^2+C|)^{\frac{1}{2}} \Leftrightarrow x' = \cancel{\frac{C_1(2t^2+C)}{\sqrt{t^2+C}}} = \frac{C_1(|t^2+C|)^{\frac{1}{2}}}{1+Ct^{-2}} + C_1(|t^2+C|)^{\frac{1}{2}}$$

- $x(1) = \cancel{C_1\cdot C^{\frac{1}{2}} = 1}$
- $x'(1) = \cancel{\frac{C_1\cdot(2+C)}{\sqrt{1+C}}} = 0 \ \cancel{\Leftrightarrow C_1(2+C) = 0 \Leftrightarrow}$ $\begin{cases} C_1 = 1 \\ C = -2 \end{cases}$

- $x_{PC} = t\,(|t^2-2|)^{\frac{1}{2}}$

5. Să se rez. urm. ec. de ordin n:

a) $x'' + x^2 = 0$

$\boxed{b)}$ $x'' + x \cdot x' = 0$

$\boxed{c)}$ $x \cdot x''' + 3x' \cdot x'' = 0$

<u>Rezolvare</u>

a) $x'' + x^2 = 0$

$F(x, x', x'') = 0$

- $\boxed{y = \frac{dx}{dt}} = x' \Rightarrow x'' = y' = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = y \cdot \frac{dy}{dx}$

- $y \frac{dy}{dx} + x^2 = 0 \Leftrightarrow y \frac{dy}{dx} = -x^2 \Leftrightarrow y\,dy = -x^2 dx \Leftrightarrow \int y\,dy = -\int x^2 dx \Leftrightarrow$

$\Leftrightarrow \frac{y^2}{2} = \frac{-x^3}{3} + C \Leftrightarrow y^2 = \frac{-2x^3}{3} + c \Leftrightarrow y = \pm\sqrt{\frac{-2x^3}{3} + C}, \quad \frac{-2x^3}{3} + 3 \geq 0$

$y = \sqrt{\frac{-2x^3}{3} + c} = \frac{dx}{dt} \Leftrightarrow \frac{dx}{\sqrt{\frac{-2x^3}{3} + c}} = dt \Leftrightarrow \int \frac{1}{\sqrt{\frac{-2}{3}x^3 + c}} dx = \int dt$

- $t + C_1 = \int \frac{1}{\sqrt{-\frac{2}{3}x^3 + c}} dx$