Curs13 - Rezolvare

Exemplu 1

$$\boxed{\begin{cases} x' = \dfrac{x}{x-y} \\ y' = \dfrac{y}{x-y} \end{cases}} \overset{x-y\neq 0}{\iff} \begin{cases} \dfrac{dx}{dt} = \dfrac{x}{x-y} \\ \dfrac{dy}{dt} = \dfrac{y}{x-y} \end{cases} \iff \begin{cases} \dfrac{dx}{x} = \dfrac{dt}{x-y} \\ \dfrac{dy}{y} = \dfrac{dt}{x-y} \end{cases} \iff \frac{dx}{x} = \frac{dt}{x-y} = \frac{dy}{y}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\int \frac{1}{x}dx = \int \frac{1}{y}dx$$

$$\ln|x| = \ln|y| + C_1$$

$$\ln|x| - \ln|y| = C_1$$

$$\ln\left|\frac{x}{y}\right| = C_1$$

$$\boxed{\frac{x}{y} = C_1 \Rightarrow \varphi_1(t,x,y) = \frac{x}{y}}$$

$$\frac{dx}{x} = \frac{dy}{y} = \underbrace{\frac{dx-dy}{x-y} = \frac{dt}{x-y}}$$

$$dx - dy = dt$$

$$d(\underbrace{x-y}_{u}) = dt$$

$$du = dt$$

$$\int du = \int dt$$

$$u = t + C_2$$

$$x - y - t = C_2 \Rightarrow$$

$$\boxed{\varphi_2(t,x,y) = x - y - t}$$

Verif. dacă φ_1, φ_2 sunt funcțional indep.

$$\frac{D(\varphi_1, \varphi_2)}{D(x,y)} \neq 0$$

$$\begin{vmatrix} \dfrac{\partial \varphi_1}{\partial x} & \dfrac{\partial \varphi_1}{\partial y} \\ \dfrac{\partial \varphi_2}{\partial x} & \dfrac{\partial \varphi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{y} & -\dfrac{x}{y^2} \\ 1 & -1 \end{vmatrix} = -\frac{1}{y} + \frac{x}{y^2} = \frac{x-y}{y^2} \neq 0$$

$\Rightarrow \varphi_1, \varphi_2$ indep.

$\Rightarrow$ Sol. implicită a sist. este

$$\begin{cases} \dfrac{x}{y} = C_1 \\ x - y - t = C_2 \end{cases}$$

Exemplu 2 - Video

$$\begin{cases} x' = x \\ y' = y \\ z' = -2xy \end{cases}, \ x, y \neq 0 \iff \begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = y \\ \dfrac{dz}{dt} = -2xy \end{cases} \iff \begin{cases} \dfrac{dx}{x} = dt \\ \dfrac{dy}{y} = dt \\ \dfrac{dz}{-2xy} = dt \end{cases} \iff \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{-2xy} = dt$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\int \frac{1}{x}dx = \int \frac{1}{y}dy$$

$$\ln|x| = \ln|y| + C_1$$

$$\ln|x| - \ln|y| = C_1$$

$$\ln\left|\frac{x}{y}\right| = C_1$$

$$\boxed{\frac{x}{y} = C_1 = \varphi_1(t,x,y,z)}$$

$$\frac{dx}{x} = dt$$

$$\int \frac{1}{x}dx = \int dt$$

$$\ln|x| = t + C_2$$

$$\boxed{\ln|x| - t = C_2 = \varphi_2(t,x,y,z)}$$

$$d(xy) = x\,dy + y\,dx$$

$$\overset{y/}{\frac{dx}{x}} = \overset{x/}{\frac{dy}{y}} = \frac{y\,dx}{xy} = \frac{x\,dy}{xy} = \underbrace{\frac{y\,dx + x\,dy}{2xy} = \frac{dz}{-2xy}}$$

$$\frac{d(xy)}{2xy} = \frac{dz}{-2xy} \quad \Big| \cdot 2xy$$

$$d(xy) = -dz$$

(Not. $xy = u \Rightarrow du = -dz$, $\int du = -\int dz$, $u = -z + c$)

$$\int d(xy) = -\int dz$$

$$xy = -z + C_3$$

$$\boxed{xy + z = C_3 = \varphi_3(t,x,y,z)}$$

$$\frac{D(\psi_1,\psi_2,\psi_3)}{D(x,y,z)} = \begin{vmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_1}{\partial z} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} & \frac{\partial \psi_2}{\partial z} \\ \frac{\partial \psi_3}{\partial x} & \frac{\partial \psi_3}{\partial y} & \frac{\partial \psi_3}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} & 0 \\ \frac{1}{\pm x} & 0 & 0 \\ y & x & 1 \end{vmatrix} = -\frac{-\cancel{x}}{y^2} \cdot \frac{1}{\pm \cancel{x}} = \pm \frac{1}{y^2} \neq 0$$

$\Rightarrow \psi_1, \psi_2, \psi_3$ functional indep.

Sol. implicită:

$$\begin{cases} \frac{x}{y} = c_1 & \Rightarrow y = \frac{x}{c_1} = \frac{c_4 e^t}{c_1} \\ \ln|x| - t = c_2 & \rightarrow \ln|x| = t + c_2 \Rightarrow |x| = e^{t+c_2} = e^t \cdot e^{c_2} \Rightarrow \boxed{x = c_4 e^t} \\ xy + z = c_3 & \rightarrow z = c_3 - xy = c_3 - c_4 e^t \cdot \frac{c_4 e^t}{c_1} \end{cases}$$

Sol. explicită:

$$\begin{cases} x = c_4 e^t \\ y = \frac{c_4 e^t}{c_1} \\ z = c_3 - \frac{c_4^2 e^{2t}}{c_1} \end{cases}$$

Exemplu 3

$$\frac{dx}{xy} = \frac{dy}{-y^2} = \frac{z\,dz}{-x(1+x^2)}$$

Trebuie să găsim două integrale prime independente.

$$\frac{dx}{xy} = \frac{dy}{-y^2} \quad \Big| \cdot y$$

$$\frac{dx}{x} = \frac{dy}{-y} \quad \text{(ec cu variabile separate)}$$

$$\int \frac{1}{x}\,dx = -\int \frac{1}{y}\,dy$$

$$\ln|x| = -\ln|y| + C_1$$

$$\ln|x| + \ln|y| = C_1 \Rightarrow \ln|xy| = C_1 \Rightarrow \boxed{xy = C_1 = \varphi_1(x,y,z)}$$

$$\frac{dx}{xy} = \frac{z\,dz}{-x(1+x^2)} \quad \Big| \cdot (x)$$

$$\frac{dx}{y} = \frac{z\,dz}{1+x^2}$$

Aflăm pe y din $xy = C_1 \Rightarrow y = \frac{C_1}{x}$

Înlocuim în relația de mai sus $\Rightarrow \frac{dx}{\frac{C_1}{x}} = \frac{z\,dz}{1+x^2}$

$$\frac{x\,dx}{C_1} = \frac{z\,dz}{1+x^2} \quad \text{(ec cu variabile separabile)}$$

$$x(1+x^2)\,dx = C_1 z\,dz$$

$$\int x(1+x^2)\,dx = C_1 \int z\,dz$$

$$\int (x+x^3)\,dx = C_1 \cdot \frac{z^2}{2} + C_2 \iff \frac{x^2}{2} + \frac{x^4}{4} = C_1 \frac{z^2}{2} + C_2 \quad \Big| \cdot 4$$

$$2x^2 + x^4 = 2C_1 z^2 + C_2$$

Înlocuim pe C_1 cu xy

$$\Rightarrow \boxed{2x^2 + x^4 - 2xyz^2 = C_2 = \varphi_2(x,y,z)}$$

Verificam dacă φ_1 și φ_2 sunt independente

$\Leftrightarrow \dfrac{D(\varphi_1,\varphi_2)}{D(x,y)} \neq 0$ (considerăm z variabila independentă)

$$\begin{vmatrix} \dfrac{\partial \varphi_1}{\partial x} & \dfrac{\partial \varphi_1}{\partial y} \\ \dfrac{\partial \varphi_2}{\partial x} & \dfrac{\partial \varphi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 4x+4x^3-2yz^2 & -2xz^2 \end{vmatrix} = -2xyz^2 - x(4x+4x^3-2yz^2)$$

$$= \cancel{-2xyz^2} - 4x^2 - 4x^4 + \cancel{2xyz^2}$$

$$= -4x^2(1+x^2) \neq 0.$$

(deoarece x nu poate fi 0, că altfel nu mai avem ecuații)

Formare Sistem Simetric

$$\begin{cases} x_1' = f_1(t, x_1, x_2 \dots x_n) \\ x_2' = f_2(t, x_1, x_2 \dots x_n) \\ \vdots \\ x_n' = f_n(t, x_1, x_2, \dots x_n) \end{cases} \Leftrightarrow \begin{cases} \frac{dx_1}{dt} = f_1(t, x_1, x_2, \dots x_n) \\ \frac{dx_2}{dt} = f_2(t, x_1, x_2, \dots x_n) \\ \dots \\ \frac{dx_n}{dt} = f_n(t, x_1, x_2, \dots x_n) \end{cases} \Leftrightarrow \begin{cases} \frac{dx_1}{f_1} = dt \\ \frac{dx_2}{f_2} = dt \\ \dots \\ \frac{dx_n}{f_n} = dt \end{cases}$$

$$\Rightarrow \frac{dx_1}{f_1} = \frac{dx_2}{f_2} = \dots = \frac{dx_n}{f_n} = dt$$