

Laborator14

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[Exerciții](#)

[Rezolvare](#)

[Exercițiu 1. a\) - Video](#)

[Exercițiu 1. b\)](#)

[Exercițiu 1. c\) - Video](#)

[Exercițiu 1. e\) - Video](#)

[Exercițiu 2. b\)](#)

Exerciții

1. Să se determine soluția generală a sistemelor simetrice:

$$a) \frac{dx}{x} = -\frac{dy}{2y} = \frac{dz}{-z}, \quad x \neq 0, y \neq 0, z \neq 0$$

$$R : \begin{cases} \varphi(x, y, z) = x\sqrt{y} = C_1 \\ \varphi(x, y, z) = xz = C_2 \end{cases}$$

$$b) \frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}, \quad x \neq y \neq z$$

$$R : \begin{cases} x+y+z = C_1 \\ x^2+y^2+z^2 = C_2 \end{cases}$$

$$c) \frac{dx}{xy^2} = \frac{dy}{x^2y} = \frac{dz}{z(x^2+y^2)}, \quad x \neq 0, y \neq 0, z \neq 0$$

$$R : \begin{cases} x^2 - y^2 = C_1 \\ \frac{xy}{z} = C_2 \end{cases}$$

$$d) \frac{dx}{2y(2-x)} = \frac{dy}{x^2 - z^2 - y^2 - 4x} = \frac{dz}{-2yz}, \quad x > 2, y \neq 0, z > 0$$

$$R : \begin{cases} \frac{x-2}{y} = C_1 \\ \frac{x^2+y^2+z^2}{z} = C_2 \end{cases}$$

$$e) \frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{(y-x)(2x+2y+z)}, \quad |x| \neq |y| \neq 0$$

$$R : \begin{cases} xy = C_1 \\ (x+y)(x+y+z) = C_2 \end{cases}$$

$$f) \frac{dx}{2xz} = \frac{dy}{2yz} = \frac{dz}{x^2 - y^2}, \quad |x| \neq |y| \neq 0$$

$$R : \begin{cases} \frac{y}{x} = C_1 \\ 2z^2 - x^2 + y^2 = C_2 \end{cases}$$

2. Să se rezolve următoarele sisteme cu ajutorul integralelor prime

$$a) \begin{cases} x' = \frac{y}{x-y}, & x \neq y \\ y' = \frac{x}{x-y} \end{cases}$$

$$R : \begin{cases} x - y + t = C_1 \\ x^2 - y^2 = C_2 \end{cases}$$

$$b) \begin{cases} x' = y \\ y' = -\frac{y^2+1}{x}, & x \neq 0 \end{cases}$$

$$R : \begin{cases} y^2(z^2 + 1) = C_1 \\ yz + x = C_2 \end{cases}$$

$$c) \begin{cases} x' = 2zy \\ y' = 4xz, & x \neq 0, y \neq 0, z \neq 0 \\ z' = xy \end{cases}$$

$$R : \begin{cases} \frac{y^2}{2} - 2z^2 = C_1 \\ \frac{x^2}{2} - z^2 = C_2 \\ x^2 + \frac{y^2}{2} - 2z^2 = C_3 \end{cases}$$

$$d) \begin{cases} x' = x \\ y' = y, & x \neq 0, y \neq 0 \\ z' = -2xy \end{cases}$$

$$R : \begin{cases} \frac{x}{y} = C_1 \\ \ln(x) - t = C_2 \\ xy - z = C_3 \end{cases} \Rightarrow \begin{cases} x = Ce^t \\ y = \frac{C}{C_1}e^t = C_4e^t \\ z = CC_4e^{2t} - C_3 \end{cases}$$

$$e) \begin{cases} x' = y \\ y' = x, & y \neq x \neq 0 \\ z' = x - y \end{cases}$$

$$R : \begin{cases} x^2 - y^2 = C_1 \\ x - y + z = C_2 \\ y - x - e^{-t} = C_3 \end{cases} \Rightarrow x - y = C_2 - z$$

$$f) \begin{cases} x' = y + xy \\ y' = x + yx, & x \neq y, |z| > 1 \\ z' = z^2 - 1 \end{cases}$$

$$R : \begin{cases} (x - y)\sqrt{t^2 - 1} = C_1 \\ \ln(x - y) + t = C_2 \\ \frac{1}{2}\ln(z^2 - 1) - t = C_3 \end{cases}$$

$$g) \begin{cases} x' = xy \\ y' = -y^2 \\ z' = -x(1+x^2) \end{cases}, x > 0, y > 0$$

$$R : \begin{cases} xy = C_1 \\ \frac{1}{y} - t = C_2 \\ \frac{x^3}{3} + \frac{x^5}{5} + xyz = C_3 \end{cases}$$

Rezolvare

Exercițiu 1. a) - [Video](#)

① a) $\frac{dx}{x} = \frac{dy}{-2y} = \frac{dz}{-z}, \quad x, y, z \neq 0.$

$$\frac{dx}{x} = \frac{dy}{-2y}$$

$$\int \frac{1}{x} dx = -\frac{1}{2} \int \frac{1}{y} dy$$

$$\ln|x| = -\frac{1}{2} \ln|y| + C_1 \quad | \cdot 2$$

$$2 \ln|x| = -\ln|y| + C_1$$

$$\ln|x|^2 + \ln|y| = C_1$$

$$\ln|x|^2 \cdot \ln|y| = C_1$$

$$\boxed{x^2 y = C_1 = \psi_1(x, y, z)}$$

$$\frac{dx}{x} = \frac{dz}{-z}$$

$$\int \frac{1}{x} dx = -\int \frac{1}{z} dz$$

$$\ln|x| = -\ln|z| + C_2$$

$$\ln|x| + \ln|z| = C_2$$

$$\ln(x \cdot |z|) = C_2$$

$$\boxed{xz = C_2 = \psi_2(x, y, z)}$$

Metoda 1

ψ_1, ψ_2 răndelea liniare

$$\psi_1, \psi_2 \text{ liniare} \Leftrightarrow \frac{\Delta(\psi_1, \psi_2)}{\Delta(x, y)} \neq 0 \Leftrightarrow$$

$$\left| \begin{array}{cc} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} \end{array} \right| = \left| \begin{array}{cc} 2xy & x^2 \\ z & 0 \end{array} \right| = -z x^2 \neq 0 \Rightarrow \psi_1, \psi_2 \text{ liniare}$$

$$\rightarrow \text{Sol. în formă implicită este } \begin{cases} xy = C_1 \\ xz = C_2 \end{cases} \rightarrow \boxed{\begin{cases} x = \frac{C_1}{y} \\ z = \frac{C_2}{x} \end{cases}} \rightarrow \text{Sol. în formă explicită}$$

Metoda 2

$$\psi_1, \psi_2 \text{ liniare} \Leftrightarrow \text{rang} \left(\frac{\Delta(\psi_1, \psi_2)}{\Delta(x, y, z)} \right) = 2$$

$$\left(\begin{array}{ccc} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_1}{\partial z} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} & \frac{\partial \psi_2}{\partial z} \end{array} \right) = \left(\begin{array}{ccc} 2xy & x^2 & 0 \\ z & 0 & x \end{array} \right)$$

$$\Delta_{p_2} \left| \begin{array}{cc} x^2 & 0 \\ 0 & x \end{array} \right| = x^3 \neq 0 \Rightarrow \text{rang} \left(\frac{\Delta(\psi_1, \psi_2)}{\Delta(x, y, z)} \right) = 2 \rightarrow \psi_1, \psi_2 = \text{liniare independente.}$$

$$\rightarrow \begin{cases} xy = C_1 \\ xz = C_2 \end{cases} \quad (\text{Sol. în formă implicită})$$

Exercițiu 1. b)

$$\textcircled{1} \text{ a) } \frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}, \quad x \neq y \neq z$$

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} = \frac{dx+dy+dz}{z-y+x-z+y-x} = \frac{d(x+y+z)}{0}$$

$$\Rightarrow d(x+y+z) = 0 \Rightarrow x+y+z = C_1 = \psi_1(x, y, z)$$

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x} = \frac{x \cdot dx - y \cdot dy}{x-z-yx} = \frac{y \cdot dy - z \cdot dz}{y-xz}$$

$$= \frac{x \cdot dx + y \cdot dy + z \cdot dz}{z^2 - xy + xy - yz + yz - xz} = \frac{1}{2} d(x^2 + y^2 + z^2)$$

$$\Rightarrow d(x^2 + y^2 + z^2) = 0 \Rightarrow x^2 + y^2 + z^2 = C_2 = \psi_2(x, y, z)$$

Verifică dacă ψ_1, ψ_2 sunt indep. $\Rightarrow \frac{\Delta(\psi_1, \psi_2)}{\Delta(x, y)} \neq 0$.
Pp. z - variabila îndep.

$$\frac{\Delta(\psi_1, \psi_2)}{\Delta(x, y)} \rightarrow \begin{vmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = 2y - 2x = 2(y - x) \neq 0$$

$$\frac{a}{a+d} = \frac{c}{c+d} \Rightarrow \frac{a+c}{a+d} = \frac{a-c}{c+d} = \frac{na}{nc}$$

$$\frac{dx}{0} \Rightarrow dx = 0 \Rightarrow x = ct$$

$$dx + dy = d(x + y)$$

$$xdx + ydy = \frac{1}{2} d(x^2 + y^2)$$

$$ydx + xdy = d(xy)$$

Sol. în formă implicită este

$$\begin{cases} x+y+z = C_1 \\ x^2+y^2+z^2 = C_2 \end{cases}$$

Exercițiu 1. c) - [Video](#)

$$\textcircled{1} \text{ c) } \frac{dx}{xy^2} = \frac{dy}{x^2y} = \frac{dz}{z(x^2+y^2)}, \quad x, y, z \neq 0$$

$$\frac{dx}{xy^2} = \frac{dy}{x^2y} \quad | \cdot xy$$

$$\frac{dx}{y} = \frac{dy}{x}$$

$$x \cdot dx = y \cdot dy$$

$$x^2 \cdot dx = y^2 \cdot dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + C_1$$

$$x^2 - y^2 = C_1 = \psi_1(x, y, z)$$

$$\frac{dx}{xy^2} = \frac{y}{x^2y} = \frac{dx}{x^3y} = \frac{y \cdot dy}{x^3y} = \frac{y \cdot dy + x \cdot dy}{x^3y + x^3y} = \frac{d(xy)}{xy(y^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} = \frac{dz}{2(x^2 + y^2)}$$

$$\frac{d(xy)}{xy(y^2 + x^2)} = \frac{dz}{2(x^2 + y^2)}$$

$$\frac{d(xy)}{xy} = \frac{dz}{z}$$

$$\text{Notăm } xy = u \Rightarrow \frac{du}{u} = \frac{dz}{z} \Rightarrow \int \frac{1}{u} du = \int \frac{1}{z} dz \Rightarrow \ln|u| = \ln|z| + C_2$$

$$\ln|u| - \ln|z| = C_2$$

$$\ln|\frac{u}{z}| = C_2$$

$$\frac{u}{z} = C_2$$

$$\frac{xy}{z} = C_2 = \psi_2(x, y, z)$$

Verifică dacă ψ_1, ψ_2 sunt indep. $\Rightarrow \frac{\Delta(\psi_1, \psi_2)}{\Delta(x, y)} \neq 0$.
Pp. z - variabila îndep.

$$\begin{vmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ \frac{y}{z} & \frac{x}{z} \end{vmatrix} = \frac{2x^2}{z} + \frac{2y^2}{z} = \frac{2}{z}(x^2 + y^2) \neq 0 \rightarrow \psi_1, \psi_2 \text{ îndep.}$$

Exercițiu 1. e) - [Video](#)

$$\textcircled{1} \text{ e) } \frac{dx}{x(x+y)} = \frac{dy}{y(x+y)} = \frac{dz}{(y-x)(2x+2y+z)}, \quad |x| \neq |y| \neq 0$$

$$\frac{dx}{x(x+y)} = \frac{dy}{y(x+y)} \quad | \cdot (x+y)$$

$$\frac{dx}{x} = -\frac{dy}{y}$$

$$\int \frac{1}{x} dx = -\int \frac{1}{y} dy$$

$$\ln|x| = -\ln|y| + C_1$$

$$\ln|u| + \ln|v| = C_1$$

$$\ln|uv| = C_1$$

$$uv = C_1 = \psi_1(x, y, z)$$

$$\frac{dx}{x(x+y)} = \frac{dy}{y(x+y)} = \frac{dx+dy}{x(x+y)-y(x+y)} = \frac{d(x+y)}{(x+y)(x-y)} = \frac{dz}{(y-x)(2x+2y+z)}$$

$$\frac{d(x+y)}{(x+y)(x-y)} = \frac{dz}{-(x-y)[2(x+y)+z]} \quad | \cdot (x-y)$$

$$\frac{d(x+y)}{x+y} = \frac{dz}{-2(x+y)+z}, \quad \text{Notăm } x+y = u \Rightarrow \frac{du}{u} = \frac{dz}{-2u-z}$$

$$\frac{du}{dz} = \frac{u}{-2u-z} = \frac{1}{-2-\frac{z}{u}}$$

$$\text{Trac 2 variabila îndep.} \quad \frac{du}{dz} = \frac{1}{-2-\frac{z}{u}} \quad , \quad \frac{u}{z} = v \Rightarrow u = vz \Rightarrow u^1 = v^1 z + v^0$$

$$v^1 z + v^0 = \frac{1}{-2-\frac{z}{u}} \Rightarrow v^1 z = \frac{u}{-2u-z} \stackrel{-z}{=} \frac{u + zv^1 + v^0}{-2u-z} = \frac{2v^1 z + 2v^0}{-2u-z}$$

$$\frac{dv}{dz} \cdot z = \frac{2(v^2 + v^0)}{-(2v^1 + 1)} \Rightarrow \frac{2v+1}{v^2+v} dv = -\frac{2}{z} dz$$

$$\begin{aligned} \int \frac{2x+1}{x^2+y^2} dx &= -2 \int \frac{1}{z} dz \\ \ln|x^2+y^2| &= -2 \ln|z| + C_2 \\ \ln|x^2+y^2| + \ln|z|^2 &= C_2 \\ \ln|x^2+y^2| = C_2 &\\ z^2(x^2+y^2) = C_2 &\\ z^2 \left(\frac{u^2}{x^2} + \frac{y^2}{x^2} \right) = C_2 &\\ z^2 \cdot \frac{u^2 + y^2}{x^2} = C_2 &\\ (x+y)^2 + z(x+y) = C_2 &\\ (x+y)(x+y+z) = C_2 = \psi_2(x, y, z) &\end{aligned}$$

z - var. îndep.

ψ_1, ψ_2 îndep $\Leftrightarrow \frac{\Delta(\psi_1, \psi_2)}{\Delta(x, y)} \neq 0$

$$\begin{vmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 2x+2y+z & 2x+2y+z \end{vmatrix} = (2x+2y+z) \begin{vmatrix} y & x \\ 1 & 1 \end{vmatrix} = (2x+2y+z)(y-x) \neq 0$$

$$(x+y)(x+y+z) = x+y+z + (x+y)$$
 $\Rightarrow \psi_1, \psi_2$ îndep.
 Sol. implicită:
 $\begin{cases} xy = C_1 \\ (x+y)(x+y+z) = C_2 \end{cases}$

Exercițiu 2. b)

(2) u) $\begin{cases} x' = y \\ y' = -\frac{y^2+1}{x} \end{cases}, x \neq 0 \Leftrightarrow \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{y^2+1}{x} \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{y^2+1}{x} \end{cases} \Rightarrow \frac{dx}{y} = dt \quad \Rightarrow \frac{dx}{y} = \frac{-x dy}{y^2+1} = dt$

$$\frac{dx}{y} = \frac{-x dy}{y^2+1}$$

$$\frac{dx}{x} = -\frac{y dy}{y^2+1}$$

$$\frac{1}{x} dx = -\frac{1}{2} \frac{2y}{y^2+1} dy$$

$$\ln|x| = -\frac{1}{2} \ln(y^2+1) + C_1$$

$$2 \ln|x| = -\ln(y^2+1) + C_1$$

$$\ln x^2 + \ln(y^2+1) = C_1$$

$$\ln x^2(y^2+1) = C_1$$

$$x^2(y^2+1) = C_1 = \psi_1(x, y)$$

$\frac{d(x)}{y} = \frac{-x dy}{y^2+1} = \frac{y dx}{y^2+1} = \frac{x dy}{-y^2-1} = \frac{y dx + x dy}{y^2-y^2-1} = \frac{d(xy)}{-1} = dt$

$$\frac{d(xy)}{-1} = dt \Rightarrow d(xy) = -dt \quad \Rightarrow du = -dt \quad \Rightarrow \begin{cases} du = -dt \\ u = -t + C_2 \\ u+t = C_2 \end{cases}$$

Notăm $xy = u$

Vezi. de. ψ_1, ψ_2 sunt îndep.

$\Leftrightarrow \frac{\Delta(\psi_1, \psi_2)}{\Delta(x, y)} \neq 0 \quad \left(\begin{array}{c} 2x(y^2+1) \\ y \\ y \end{array} \right) \quad \begin{array}{c} 2y x^2 \\ x \\ x \end{array} \quad \begin{array}{l} = 2x^2(y^2+1) - 2y^2 x^2 \\ = 2x^2 y^2 + 2x^2 - 3y^2 x^2 = 2x^2 \end{array} \neq 0$

$\Rightarrow \psi_1, \psi_2$ îndep.

Sol. implicită: $\begin{cases} x^2(y^2+1) = C_1 \\ xy + t = C_2 \end{cases}$