

Laborator04

Laborator04

Enunțuri

Rezolvare

Exercițiu 01

c) - Video

Exercițiu 02

a) - Video

c)

Exercițiu 03

a)

b) - Video

Exercițiu 04

a) - Video

c)

Enunțuri

1. Să se rezolve următoarele ecuații de tip Bernoulli.

$$a) \quad x' + \frac{t}{6} \cdot x = \frac{1}{3} \cdot t \cdot x^{-2}$$

$$b) \quad \begin{cases} 3 \cdot t \cdot x^2 \cdot x' + x^3 = 2t \\ x(1) = 2 \end{cases}$$

$$c) \quad \begin{cases} x' = 2 \cdot t \cdot x + t \cdot \sqrt[3]{x} \\ x(0) = 4 \end{cases}$$

$$d) \quad x' + 2 \cdot t \cdot x = 2 \cdot t^3 \cdot x^3$$

2. Să se rezolve următoarele ecuații de tip Riccati.

$$a) \quad x' + x^2 - 2 \cdot x \cdot \sin t + \sin^2 t - \cos t = 0, \quad \rho_0(t) = \sin t$$

$$b) \quad x' = t \cdot x^2 - 2 \cdot t^2 \cdot x + t^3 + 1, \quad \rho_0(t) = t$$

$$c) \quad \begin{cases} x' = -x^2 \cdot \sin t + \frac{2 \cdot \sin t}{\cos^2 t}, \\ x(0) = 2 \end{cases} \quad \rho_0(t) = \frac{1}{\cos t}$$

3. Să se rezolve următoarele ecuații cu diferențiale exacte.

$$a) \quad \frac{t}{x^2} dt + \frac{x^2 - t^2}{x^3} dx = 0$$

$$b) \quad \begin{cases} 2 \cdot t \cdot x dt + (t^2 + x^2) dx = 0 \\ x(1) = 3 \end{cases}$$

$$c) \quad (t^2 + x^2 + 2t) dt + 2 \cdot t \cdot x dx = 0$$

$$d) \quad t dt + x dx = \frac{-t dx - x dt}{t^2 + x^2}$$

4. Să se rezolve următoarele ecuații căutând un factor integrant.

$$a) \quad (x^2 - 2 \cdot t \cdot x) dt + t^2 dx = 0$$

$$b) \quad 2 \cdot t \cdot x dt = (t^2 - x^2) dx$$

$$c) \quad (2 \cdot t \cdot x - t) dt + (x^2 + x + 2t^2) dx = 0$$

Rezolvare

Exercițiu 01

c) - [Video](#)

$$\textcircled{1} C) \begin{cases} x' = 2tx + t\sqrt{x} \\ x(0) = 4 \end{cases}$$

$$x' = 2tx + t\sqrt{x} \quad \left| : x^{1/3} \right. \quad x^{1/3} u = \frac{1}{3}$$

$$\frac{x'}{x^{1/3}} = 2t \frac{x}{x^{1/3}} + t$$

$$x' \cdot x^{-1/3} = 2tx^{2/3} + t = 2tx^{2/3} + t$$

$$u = x^{1/3} \Rightarrow u' = \frac{1}{3} x^{-2/3} x' = \frac{1}{3} x' \cdot x^{-2/3}$$

$$x' \cdot x^{-2/3} = 2tx^{2/3} + t \quad \left| \cdot \frac{2}{3} \right.$$

$$\frac{2}{3} x' x^{-2/3} = \frac{4}{3} tx^{2/3} + \frac{2}{3} t$$

$$u' = \frac{4}{3} tu + \frac{2}{3} t$$

$$\text{Etapa 1} \quad u' = \frac{4}{3} tu \Rightarrow \frac{du}{dt} = \frac{4}{3} tu$$

$$\Leftrightarrow \frac{du}{u} = \frac{4}{3} t dt \Rightarrow \int \frac{1}{u} du = \frac{4}{3} \int t dt$$

$$\Leftrightarrow \ln|u| = \frac{4}{3} \cdot \frac{t^2}{2} + C = \frac{2t^2}{3} + C \Rightarrow u = e^{\frac{2t^2}{3} + C}$$

$$u_0 = C \cdot e^{\frac{2t^2}{3}}$$

$$\text{Etapa 2: } \varphi_0 = C(t) \cdot e^{\frac{2t^2}{3}}$$

$$\left(C(t) \cdot e^{\frac{2t^2}{3}} \right)' = \frac{4}{3} t \cdot C(t) \cdot e^{\frac{2t^2}{3}} + \frac{2}{3} t$$

$$C'(t) \cdot e^{\frac{2t^2}{3}} + C(t) \cdot e^{\frac{2t^2}{3}} \cdot \frac{4}{3} t = \frac{4}{3} t \cdot C(t) \cdot e^{\frac{2t^2}{3}} + \frac{2}{3} t$$

$$C'(t) \cdot e^{\frac{2t^2}{3}} = \frac{2}{3} t \quad \left| : e^{\frac{2t^2}{3}} \right.$$

$$C'(t) = \frac{2}{3} t e^{-\frac{2t^2}{3}} \Rightarrow C(t) = \frac{2}{3} \int \frac{1}{3} t e^{-\frac{2t^2}{3}} dt$$

$$C(t) = -\frac{1}{2} \cdot e^{-\frac{2t^2}{3}} + C_1$$

$$u = -\frac{1}{3} t$$

$$\varphi_0 = \left(-\frac{1}{2} e^{-\frac{2t^2}{3}} + C_1 \right) \cdot e^{\frac{2t^2}{3}}$$

$$\varphi_0 = -\frac{1}{2} + C_1 e^{\frac{2t^2}{3}}$$

$$u = u_0 + \varphi_0 = C \cdot e^{\frac{2t^2}{3}} - \frac{1}{2} + C_1 e^{\frac{2t^2}{3}}$$

$$u = C \cdot e^{\frac{2t^2}{3}} - \frac{1}{2}$$

$$u = x^{1/3} \Rightarrow u' = x^{-2/3} x' = \sqrt{u^3}$$

$$x = \left(C e^{\frac{2t^2}{3}} - \frac{1}{2} \right)^3$$

$$C e^{\frac{2t^2}{3}} - \frac{1}{2} > 0$$

$$x(0) = \sqrt{\left(C - \frac{1}{2} \right)^3} = 4$$

$$\left(C - \frac{1}{2} \right)^3 = 16$$

$$C - \frac{1}{2} = \sqrt[3]{16}$$

$$C = \frac{1}{2} + \sqrt[3]{16}$$

$$x_{pc} = \sqrt{\left[\left(\frac{1}{2} + \sqrt[3]{16} \right) \cdot e^{\frac{2t^2}{3}} - \frac{1}{2} \right]^3}$$

3

Exercițiu 02

a) - [Video](#)

② a) Laborator 4.

$$x' + \underbrace{x^2}_{A(t)} - \underbrace{2x \sin t}_{B(t)} + \underbrace{\sin^2 t - \cos t}_{f(t)} = 0, \quad \varphi_0(t) = \sin t$$

$$x = y + \varphi_0 \Rightarrow x = y + \sin t \Rightarrow x' = y' + \cos t$$

$$y' + \cos t + (y + \sin t)^2 - 2 \sin t (y + \sin t) + \sin^2 t - \cos t = 0$$

$$y' + y^2 + 2y \sin t + \sin^2 t - 2y \sin t - 2 \sin^2 t + \sin^2 t = 0$$

$$y' + y^2 = 0$$

$$\frac{dy}{dt} = -y^2$$

$$\frac{dy}{-y^2} = dt \Rightarrow \int -\frac{1}{y^2} dy = \int dt \Leftrightarrow \frac{1}{y} = t + C \Rightarrow y = \frac{1}{t+C}, \quad t+C \neq 0$$

$$\Rightarrow x = \frac{1}{t+C} + \sin t$$

c)

② c)

$$x' = -x^2 \frac{\sin t}{\cos^2 t} + \frac{2 \sin t}{\cos^2 t}, \quad \varphi_0(t) = \frac{1}{\cos t}$$

$$x(0) = 2 \quad A(t) \quad f(t)$$

$$x = y + \varphi_0 \Rightarrow x = y + \frac{1}{\cos t} \Rightarrow x' = y' + \frac{\sin t}{\cos^2 t}$$

$$y' + \frac{\sin t}{\cos^2 t} = -\left(y + \frac{1}{\cos t}\right)^2 \cdot \sin t + \frac{2 \sin t}{\cos^2 t}$$

$$= -y^2 \sin t - 2y \frac{1}{\cos t} \cdot \sin t - \frac{\sin t}{\cos^2 t} + \frac{2 \sin t}{\cos^2 t}$$

$$y' = \underbrace{-\sin t \cdot y^2}_{B(t)} - \underbrace{2 \frac{\sin t}{\cos t} \cdot y}_{A(t)}, \quad \alpha = 2$$

$$: y^2$$

$$\frac{y'}{y^2} = -\sin t - 2 \frac{\sin t}{\cos t} \cdot \frac{1}{y}$$

$$y' \cdot y^{-2} = -\sin t - 2 \frac{\sin t}{\cos t} y^{-1}$$

$$u = y^{-1} \Rightarrow u' = -1 \cdot y^{-2} \cdot y'$$

$$-y' y^{-1} = \sin t + 2 \frac{\sin t}{\cos t} y^{-1}$$

$$u' = \sin t + \frac{2 \sin t}{\cos t} u$$

$$u = \frac{1}{y} \Rightarrow y = \frac{1}{u} = \frac{1}{\frac{-\cos t}{3} + \frac{C}{\cos^2 t}}$$

$$x_{\text{h}} = y + \frac{1}{\cos t} = \frac{1}{\frac{C}{\cos^2 t} - \frac{\cos t}{3}} + \frac{1}{\cos t}$$

$$x(0) = \frac{1}{C - \frac{1}{3}} + 1 = 2 \Rightarrow \frac{1}{C - \frac{1}{3}} = 1 \Rightarrow C - \frac{1}{3} = 1 \Rightarrow C = 1 + \frac{1}{3} = \frac{4}{3}$$

$$x_{\text{pC}} = \frac{1}{\frac{4}{\cos^2 t} - \frac{\cos t}{3}} + \frac{1}{\cos t}$$

Exercițiu 03

a)

3.a. Seria 4.

$$\underbrace{\frac{t}{x^2} dt}_{P(t,x)} + \underbrace{\frac{x^2 - t^2}{x^3} dx}_{Q(t,x)} = 0$$

$$\left. \begin{aligned} \frac{\partial P}{\partial x} &= \frac{-t \cdot 2x}{x^4} = -\frac{2t}{x^3} \\ \frac{\partial Q}{\partial t} &= \left(\frac{1}{x} - \frac{t^2}{x^3} \right)_t = -\frac{2t}{x^3} \end{aligned} \right\} \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial t}$$

$$F(t, x) = C, \quad F(t, x) = \int_0^t P(z, x_0) dz + \int_{x_0}^x Q(t, \sigma) d\sigma$$

$$= \int_0^t \frac{z}{x_0^2} dz + \int_{x_0}^x \frac{\sigma^2 - t^2}{\sigma^3} d\sigma = \frac{1}{x_0^2} \cdot \frac{z^2}{2} \Big|_0^t + \left(\frac{1}{x_0} \cdot \frac{1}{\sigma} - \frac{t^2}{\sigma^2} \right) \Big|_{x_0}^x$$

$$= \frac{t^2}{2x_0^2} + \ln|\sigma| \Big|_{x_0}^x - t^2 \int_{x_0}^x \sigma^{-3} d\sigma = \frac{t^2}{2x_0^2} + \ln|x| - \ln|x_0| - t^2 \left(\frac{\sigma^{-2}}{-2} \right) \Big|_{x_0}^x$$

$$F(t, x) = \frac{t^2}{2x_0^2} + \ln|x| - \ln|x_0| + \frac{t^2}{2x^2} \cdot \frac{t^2}{x_0^2}$$

$$\ln|x| + \frac{t^2}{2x^2} = C \quad \rightarrow +C$$

b) - [Video](#)

3) b) $\begin{cases} 2tx \, dt + (t^2 + x^2) \, dx = 0 \\ x(1) = 3 \end{cases}$

$$\underbrace{2tx \, dt}_{P(t,x)} + \underbrace{(t^2 + x^2) \, dx}_{Q(t,x)} = 0$$

$$\left. \begin{aligned} \frac{\partial P}{\partial x} &= 2t \\ \frac{\partial Q}{\partial t} &= 2t \end{aligned} \right\} \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial t}$$

$$F(t, x) = C$$

$$\int_0^t P(z, 0) dz + \int_0^x Q(t, \sigma) d\sigma =$$

$$= \int_0^t 2 \cdot z \cdot 0 \, dz + \int_0^x (t^2 + \sigma^2) d\sigma =$$

$$= t^2 \sigma \Big|_0^x + \frac{\sigma^3}{3} \Big|_0^x = t^2 x + \frac{x^3}{3}$$

$$t^2 x + \frac{x^3}{3} = C \quad (\text{sol. în formă implicită})$$

$$1^2 \cdot 3 + \frac{3^3}{3} = C$$

$$\Rightarrow C = 3 + 9 = 12$$

$$PC: \quad t^2 x + \frac{x^3}{3} = 12$$

Exercițiu 04

a) - [Video](#)

$$(1) a) \underbrace{(x^2 - 2tx)}_{P(t,x)} dt + \underbrace{t^2}_{Q(t,x)} dx = 0 \quad | \mu(t)$$

$$\left. \begin{aligned} \frac{\partial P}{\partial x}(t,x) &= 2x - 2t \\ \frac{\partial Q}{\partial t}(t,x) &= 2t \end{aligned} \right\} \Rightarrow \frac{\partial P}{\partial x} \neq \frac{\partial Q}{\partial t}$$

Find $\mu = \mu(t)$

$$\underbrace{\mu(t) \cdot (x^2 - 2tx)}_{P^*(t,x)} dt + \underbrace{\mu(t) \cdot t^2}_{Q^*(t,x)} dx = 0$$

$$\frac{\partial P^*}{\partial x}(t,x) = \mu'(t) \cdot (x^2 - 2tx) + \mu(t) \cdot (2x - 2t)$$

$$\frac{\partial Q^*}{\partial t}(t,x) = \mu(t) \cdot 2t$$

$$\mu'(t)(x^2 - 2tx) + \mu(t)(2x - 2t) = \mu(t) \cdot 2t$$

$$\begin{aligned} \mu'(t) \cdot (x^2 - 2tx) &= \mu(t) \cdot (2t - 2x + 2t) \\ \mu'(t) \cdot x(x - 2t) &= \mu(t) \cdot (4t - 2x) \\ \mu'(t) \cdot x(x - 2t) &= -2 \mu(t) \cdot (x - 2t) \\ \frac{d\mu}{dx} \cdot x &= -2\mu \end{aligned}$$

$$\frac{d\mu}{\mu} = -\frac{2}{x} dx \Rightarrow \int \frac{1}{\mu} = -2 \int \frac{1}{x} dx$$

$$\ln|\mu| = -2 \ln|x| + C$$

$$\ln|\mu| = -\ln x^2 + \ln C$$

$$\mu = \frac{C}{x^2} \quad \text{pp. } C=1$$

$$(x^2 - 2tx) \cdot \frac{C}{x^2} dt + t^2 \cdot \frac{C}{x^2} dx = 0$$

$$F(t,x) = C$$

$$F(t,x) = \int_0^t P(\tau, x_0) d\tau + \int_{x_0}^x Q(t, \sigma) d\sigma$$

$$= \int_0^t \frac{x_0^2 - 2\tau x_0}{x_0^2} d\tau + \int_{x_0}^x t^2 \cdot \frac{1}{\sigma^2} d\sigma$$

$$= \int_0^t \left(1 - \frac{2\tau}{x_0} \right) d\tau + t^2 \int_{x_0}^x \frac{1}{\sigma^2} d\sigma$$

$$= \tau \Big|_0^t - \frac{\tau^2}{x_0} \Big|_0^t + t^2 \cdot \left(-\frac{1}{\sigma} \right) \Big|_{x_0}^x$$

$$= t - \frac{t^2}{x_0} + \frac{t^2}{x} + C$$

$$t - \frac{t^2}{x} = C$$

$$(1) b)$$

$$\underbrace{(x^2 - 2tx)}_{P(t,x)} dt + \underbrace{t^2}_{Q(t,x)} dx = 0 \quad | \mu(t)$$

$$\left. \begin{aligned} \frac{\partial P}{\partial x}(t,x) &= 2x - 2t \\ \frac{\partial Q}{\partial t}(t,x) &= 2t \end{aligned} \right\} \Rightarrow \frac{\partial P}{\partial x} \neq \frac{\partial Q}{\partial t}$$

pp. $\mu = \mu(t)$

$$\underbrace{\mu(t) \cdot (x^2 - 2tx)}_{P^*(t,x)} dt + \underbrace{\mu(t) \cdot t^2}_{Q^*(t,x)} dx = 0$$

$$\frac{\partial P^*}{\partial x}(t,x) = \mu(t) \cdot (2x - 2t)$$

$$\frac{\partial Q^*}{\partial t}(t,x) = \mu'(t) \cdot t^2 + \mu(t) \cdot 2t$$

$$\begin{aligned} \mu'(t) \cdot t^2 + \mu(t) \cdot 2t &= \mu(t) \cdot (2x - 2t) \\ \mu'(t) \cdot t^2 &= \mu(t) \cdot (2x - 2t - 2t) \\ \mu'(t) \cdot t^2 &= \mu(t) \cdot (2x - 4t) \quad \text{[mu e keine]} \end{aligned}$$

←

c)

4) c) Sem 4

$$\underbrace{(2tx - t)}_{P(t,x)} dt + \underbrace{(x^2 + x + 2t^2)}_{Q(t,x)} dx = 0$$

$$\left. \begin{aligned} \frac{\partial P}{\partial x}(t,x) &= 2t \\ \frac{\partial Q}{\partial t}(t,x) &= 4t \end{aligned} \right\} \frac{\partial P}{\partial x} \neq \frac{\partial Q}{\partial t}$$

Pp. $\mu = \mu(t)$

$$\underbrace{\mu(t)(2tx - t)}_{P^*(t,x)} dt + \underbrace{\mu(t)(x^2 + x + 2t^2)}_{Q^*(t,x)} dx = 0$$

$$\frac{\partial P^*}{\partial t} = \mu'(t) \cdot 2t \quad ; \quad \frac{\partial Q^*}{\partial t} = \mu'(t)(x^2 + x + 2t^2) + \mu(t) \cdot 4t$$

$$\left. \begin{aligned} \mu'(t)(x^2 + x + 2t^2) + 4t \cdot \mu(t) &= 2t \mu(t) \\ \mu'(t)(x^2 + x + 2t^2) &= -2t \mu(t) \end{aligned} \right\} \text{ (nur in time!)}$$

Pp. $\mu = \mu(t)$

$$\underbrace{\mu(t)(2tx - t)}_{P^*(t,x)} dt + \underbrace{\mu(t)(x^2 + x + 2t^2)}_{Q^*(t,x)} dx = 0$$

$$\frac{\partial P^*}{\partial x} = \mu'(t)(2tx - t) + \mu(t) \cdot 2t$$

$$\frac{\partial Q^*}{\partial t} = \mu(t) \cdot 4t$$

$$\left. \begin{aligned} \mu'(t)(2tx - t) + 2t \mu(t) &= 4t \mu(t) \\ \mu'(t) \cdot t(2x - 1) &= 2t \mu(t) \end{aligned} \right\}$$

$$\mu'(t)(2x - 1) = 2 \mu(t)$$

$$\frac{d\mu}{dx}(2x - 1) = 2 \mu$$

$$\frac{d\mu}{\mu} = \frac{2}{2x - 1} dx$$

$$\int \frac{1}{\mu} d\mu = \int \frac{2}{2x - 1} dx$$

$$\ln|\mu| = \ln|2x - 1| + C$$

$$\ln|\mu| = \ln C |2x - 1|$$

$$\mu = C(2x - 1)$$

Für $C = 1 \Rightarrow \mu(x) = 2x - 1$

$$\underbrace{(2x - 1)(2tx - t)}_{P^*(t,x)} dt + \underbrace{(2x - 1)(x^2 + x + 2t^2)}_{Q^*(t,x)} dx = 0$$

$$F(t, x) = C \cdot t$$

$$F(t, x) = \int_0^t P(\tau, 0) d\tau + \int_0^x Q(\tau, t) d\tau$$

$$= \int_0^t (-1)(-\tau) d\tau + \int_0^x (2\tau - 1)(\tau^2 + \tau + 2t^2) d\tau$$

$$= \int_0^t \tau d\tau + \int_0^x (2\tau^3 + 2\tau^2 + 4\tau t^2 - \tau^2 - \tau - 2t^2) d\tau$$

$$= \frac{\tau^2}{2} \Big|_0^t + \left(\frac{2}{4} \tau^4 + 2 \cdot \frac{\tau^3}{3} + 4t^2 \cdot \frac{\tau^2}{2} - \frac{\tau^3}{3} - \frac{\tau^2}{2} - 2t^2 \tau \right) \Big|_0^x$$

$$= \frac{t^2}{2} + \frac{x^2}{2} + \frac{2x^3}{3} + 2t^2 x^2 - \frac{x^3}{3}$$