Ecuații diferențiale și cu derivate parțiale
Laborator 12
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Să se rezolve următoarele sisteme de ecuații diferențiale:

1. a) $\begin{cases} x' = 2x+z \\ y' = x+y \\ z' = 2y+2z \end{cases}$ $\boxed{b)}$ $\begin{cases} x' = -x-2y \\ y' = 2x+y \\ z' = 3x+2z \end{cases}$ $\boxed{c)}$ $\begin{cases} x' = x-2y+3z \\ y' = y \\ z' = -3x+z \end{cases}$

Rezolvare.

a) $\begin{cases} x' = 2x+z \\ y' = x+y \\ z' = 2y+2z \end{cases}$ $x = \alpha_1 e^{rt}$, $y = \alpha_2 e^{rt}$, $z = \alpha_3 e^{rt}$

$x' = \alpha_1 r e^{rt}$, $y' = \alpha_2 r e^{rt}$, $z' = \alpha_3 r e^{rt}$

$\begin{cases} \alpha_1 r = 2\alpha_1 + \alpha_3 \\ \alpha_2 r = \alpha_1 + \alpha_2 \\ \alpha_3 r = 2\alpha_2 + 2\alpha_3 \end{cases} \Leftrightarrow \begin{cases} (2-r)\alpha_1 + \alpha_3 = 0 \\ \alpha_1 + (1-r)\alpha_2 = 0 \\ 2\alpha_2 + (2-r)\alpha_3 = 0 \end{cases}$ $(*)$

$D = \begin{vmatrix} 2-r & 0 & 1 \\ 1 & 1-r & 0 \\ 0 & 2 & 2-r \end{vmatrix} \overset{L_1+L_2+L_3}{=\!=\!=} \begin{vmatrix} 3-r & 3-r & 3-r \\ 1 & 1-r & 0 \\ 0 & 2 & 2-r \end{vmatrix} = (3-r)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-r & 0 \\ 0 & 2 & 2-r \end{vmatrix} =$

$= (3-r)\left[(1-r)(2-r) + 2 - 2 + r\right] = (3-r)(2 - r - 2r + r^2 + r) = (3-r)(r^2 - 2r + 2) = 0$

$\Rightarrow \begin{cases} 3-r = 0 \Rightarrow r_1 = 3 \\ r^2 - 2r + 2 = 0 \end{cases}$

$\Delta = 4 - 8 = -4 \Rightarrow r_{2,3} = \frac{2 \pm 2i}{2} = 1 \pm i$ $(\mu \pm i\beta,\ \mu = 1,\ \beta = 1)$

• Pt $r_1 = 3$

$(*) \Rightarrow \begin{cases} -\alpha_1 + \alpha_3 = 0 \\ \alpha_1 - 2\alpha_2 = 0 \Rightarrow \alpha_1 = 2\alpha_2 \\ 2\alpha_2 - \alpha_3 = 0 \Rightarrow \alpha_3 = 2\alpha_2,\ \alpha_2 \in \mathbb{R} \end{cases}$

Fie $\alpha_2 = 1 \Rightarrow \alpha_1 = \alpha_3 = 2 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} 2e^{3t} \\ e^{3t} \\ 2e^{3t} \end{pmatrix}$

• Pt $r_2 = 1+i$

$$(*) \Rightarrow \begin{cases} (1-i)\alpha_1 + \alpha_3 = 0 \\ \alpha_1 - i\alpha_2 = 0 \\ 2\alpha_2 + (1-i)\alpha_3 = 0 \end{cases} \Rightarrow \begin{array}{l} \alpha_3 = (i-1)\alpha_1 \\ \alpha_2 = \frac{1}{i}\alpha_1 = -i\alpha_1, \ \alpha_1 \in R \end{array}$$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = -i, \ \alpha_3 = i-1 \Rightarrow \alpha_{n_2} = \begin{pmatrix} 1 \\ -i \\ i-1 \end{pmatrix} \Rightarrow X_{n_2} = \begin{pmatrix} e^t(\cos t + i\sin t) \\ -i\, e^t(\cos t + i \sin t) \\ (i-1)\, e^t(\cos t + i\sin t) \end{pmatrix}$

$$X_{n_2} = \begin{pmatrix} \underline{e^t \cos t} + i e^t \sin t \\ -i e^t \cos t + \underline{e^t \sin t} \\ i e^t \cos t - \underline{e^t \sin t} - \underline{e^t \cos t} - i e^t \sin t \end{pmatrix} = \underbrace{\begin{pmatrix} e^t \cos t \\ e^t \sin t \\ -e^t \sin t - e^t \cos t \end{pmatrix}}_{\tilde{X}_{n_2}} + i \underbrace{\begin{pmatrix} e^t \sin t \\ -e^t \cos t \\ e^t \cos t - e^t \sin t \end{pmatrix}}_{\tilde{\tilde{X}}_{n_2}}$$

$$X = c_1 X_{n_1} + c_2 \tilde{X}_{n_2} + c_3 \tilde{\tilde{X}}_{n_2} = c_1 \begin{pmatrix} 2e^{3t} \\ e^{3t} \\ 2e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} e^t \cos t \\ e^t \sin t \\ -e^t \sin t - e^t \cos t \end{pmatrix} + c_3 \begin{pmatrix} e^t \sin t \\ -e^t \cos t \\ e^t \cos t - e^t \sin t \end{pmatrix}$$

$$X = \begin{pmatrix} 2c_1 e^{3t} + c_2 e^t \cos t + c_3 e^t \sin t \\ c_1 e^{3t} + c_2 e^t \sin t - c_3 e^t \cos t \\ 2c_1 e^{3t} - c_2 e^t(\sin t + \cos t) + c_3 e^t(\cos t - \sin t) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x = 2c_1 e^{3t} + c_2 e^t \cos t + c_3 e^t \sin t$$

$$y = c_1 e^{3t} + c_2 e^t \sin t - c_3 e^t \cos t$$

$$z = 2c_1 e^{3t} - c_2 e^t (\sin t + \cos t) + c_3 e^t (\cos t - \sin t)$$

Să se rezolve următoarele sisteme de ecuații diferențiale:

2. a) $\begin{cases} x' = 4x - y - z \\ y' = x + 2y - z \\ z' = x - y + 2z \end{cases}$ $\boxed{b)}$ $\begin{cases} x' = 7x - 5y + 10z \\ y' = 4x - 2y + 8z \\ z' = x - y + 4z \end{cases}$

Rezolvare:

a) $\begin{cases} x' = 4x - y - z \\ y' = x + 2y - z \\ z' = x - y + 2z \end{cases}$ $x = \alpha_1 e^{rt}, \quad y = \alpha_2 e^{rt}, \quad z = \alpha_3 e^{rt}$

$x' = \alpha_1 r e^{rt}, \quad y' = \alpha_2 r e^{rt}, \quad z' = \alpha_3 r e^{rt}$

$$\begin{cases} \alpha_1 r = 4\alpha_1 - \alpha_2 - \alpha_3 \\ \alpha_2 r = \alpha_1 + 2\alpha_2 - \alpha_3 \\ \alpha_3 r = \alpha_1 - \alpha_2 + 2\alpha_3 \end{cases} \Rightarrow \begin{cases} (4-r)\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ \alpha_1 + (2-r)\alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_2 + (2-r)\alpha_3 = 0 \end{cases} \quad (*)$$

$$D = \begin{vmatrix} 4-r & -1 & -1 \\ 1 & 2-r & -1 \\ 1 & -1 & 2-r \end{vmatrix} \overset{C_1+C_2+C_3}{=\!=\!=} \begin{vmatrix} 2-r & -1 & -1 \\ 2-r & 2-r & -1 \\ 2-r & -1 & 2-r \end{vmatrix} = (2-r) \begin{vmatrix} 1 & -1 & -1 \\ 1 & 2-r & -1 \\ 1 & -1 & 2-r \end{vmatrix}$$

$$\overset{C_1+C_2}{\underset{C_1+C_3}{=\!=\!=}} (2-r) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 3-r & 0 \\ 1 & 0 & 3-r \end{vmatrix} = (2-r)(3-r)^2 = 0$$

$$\begin{cases} 2 - r = 0 \Rightarrow r_1 = 2 \\ (3-r)^2 = 0 \Rightarrow r_2 = r_3 = 3 \end{cases}$$

• Pt $r_1 = 2$

$(*) \Rightarrow \begin{cases} 2\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_3 = 0 \Rightarrow \alpha_1 = \alpha_3, \quad \alpha_1 \in R \\ \alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2 \end{cases}$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = \alpha_3 = 1 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}$

• Pt $r_2 = r_3 = 3$

$(*) \Rightarrow \begin{cases} \alpha_1 - \alpha_2 - \alpha_3 = 0 \Rightarrow \alpha_1 = \alpha_2 + \alpha_3, \quad \alpha_2, \alpha_3 \in R \\ \alpha_1 - \alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_2 - \alpha_3 = 0 \end{cases}$

$$(\alpha_2+\alpha_3, \alpha_2, \alpha_3) = (\alpha_2, \alpha_2, 0) + (\alpha_3, 0, \alpha_3) = \alpha_2 \underbrace{(1,1,0)}_{\alpha_{12}} + \alpha_3 \underbrace{(1,0,1)}_{\alpha_{13}} \Rightarrow$$

$$\Rightarrow X_{12} = \begin{pmatrix} e^{3t} \\ e^{3t} \\ 0 \end{pmatrix}, \quad X_{13} = \begin{pmatrix} e^{3t} \\ 0 \\ e^{3t} \end{pmatrix}$$

$$X = C_1 X_{11} + C_2 X_{12} + C_3 X_{13} = C_1 \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ e^{3t} \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} e^{3t} \\ 0 \\ e^{3t} \end{pmatrix}$$

$$X = \begin{pmatrix} C_1 e^{2t} + C_2 e^{3t} + C_3 e^{3t} \\ C_1 e^{2t} + C_2 e^{3t} \\ C_1 e^{2t} + C_3 e^{3t} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} x = C_1 e^{2t} + C_2 e^{3t} + C_3 e^{3t} \\ y = C_1 e^{2t} + C_2 e^{3t} \\ z = C_1 e^{2t} + C_3 e^{3t} \end{cases}$$

Să se rezolve următoarele sisteme de ecuații diferențiale:

3. a) $\begin{cases} x' = y \\ y' = -x + 2y \end{cases}$ b) $\begin{cases} x' = 2x + y \\ y' = 2y + 4z \\ z' = x - z \end{cases}$ $\boxed{c)}$ $\begin{cases} x' = x + y - 2z \\ y' = 4x + y \\ z' = 2x + y - z \end{cases}$

$\boxed{d)}$ $\begin{cases} x' = 2x - y - z \\ y' = 2x - y - 2z \\ z' = -x + y + 2z \end{cases}$ $\boxed{e)}$ $\begin{cases} x' = x - y \\ y' = -y - z \\ z' = -z \end{cases}$

Rezolvare

a) $\begin{cases} x' = y \\ y' = -x + 2y \end{cases}$ $x = \alpha_1 e^{rt}, \quad y = \alpha_2 e^{rt}$

$x' = \alpha_1 r e^{rt}, \quad y' = \alpha_2 r e^{rt}$

$\begin{cases} \alpha_1 r = \alpha_2 \\ \alpha_2 r = -\alpha_1 + 2\alpha_2 \end{cases} (*') \Rightarrow \begin{cases} -r\alpha_1 + \alpha_2 = 0 \\ -\alpha_1 + (2-r)\alpha_2 = 0 \end{cases} (*)$

$$D = \begin{vmatrix} -r & 1 \\ -1 & 2-r \end{vmatrix} = -r(2-r) + 1 = r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1$$

• Pt $r_1 = r_2 = 1$

$(*) \begin{cases} -\alpha_1 + \alpha_2 = 0 \\ -\alpha_1 + \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 \Rightarrow$ o singură sol. l.i

Alegem $x = (A_1 t + A_0) e^t, \quad y = (B_1 t + B_0) e^t$

$x' = A_1 e^t + (A_1 t + A_0) e^t, \quad y' = B_1 e^t + (B_1 t + B_0) e^t$

$(*) \begin{cases} A_1 + A_1 t + A_0 = B_1 t + B_0 \\ B_1 + B_1 t + B_0 = -A_1 t - A_0 + 2B_1 t + 2B_0 \end{cases}$

$\begin{cases} \boxed{A_1 = B_1} \\ A_1 + A_0 = B_0 \Rightarrow \boxed{A_1 = B_0 - A_0} \\ B_1 = -A_1 + 2B_1 \\ B_1 + B_0 = -A_0 + 2B_0 \end{cases}$

• Fie $B_0 = 1, A_0 = 0 \Rightarrow A_1 = B_1 = 1$

$$X_{\lambda_{1,2}} = \begin{pmatrix} te^t \\ (t+1)e^t \end{pmatrix} = \begin{pmatrix} te^t \\ te^t + e^t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ e^t \end{pmatrix}}_{X_{\lambda_1}} + \underbrace{\begin{pmatrix} te^t \\ te^t \end{pmatrix}}_{X_{\lambda_2}}$$

$$X = C_1 X_{\lambda_1} + C_2 X_{\lambda_2} = C_1 \begin{pmatrix} 0 \\ e^t \end{pmatrix} + C_2 \begin{pmatrix} te^t \\ te^t \end{pmatrix} = \begin{pmatrix} C_2 te^t \\ C_1 e^t + C_2 te^t \end{pmatrix}$$

$$\begin{cases} x = C_2 te^t \\ y = C_1 e^t + C_2 te^t \end{cases}$$

b) $\begin{cases} x' = 2x + y \\ y' = 2y + 4z \\ z' = x - z \end{cases}$ $\quad x = \alpha_1 e^{\lambda t}, \quad y = \alpha_2 e^{\lambda t}, \quad z = \alpha_3 e^{\lambda t}$

$\quad x' = \alpha_1 \lambda e^{\lambda t}, \quad y' = \alpha_2 \lambda e^{\lambda t}, \quad z = \alpha_3 \lambda e^{\lambda t}$

$$\begin{cases} \alpha_1 \lambda = 2\alpha_1 + \alpha_2 \\ \alpha_2 \lambda = 2\alpha_2 + 4\alpha_3 \\ \alpha_3 \lambda = \alpha_1 - \alpha_3 \end{cases} \Rightarrow \begin{cases} (2-\lambda)\alpha_1 + \alpha_2 = 0 \\ (2-\lambda)\alpha_2 + 4\alpha_3 = 0 \\ \alpha_1 + (-1-\lambda)\alpha_3 = 0 \end{cases} \quad (*)$$

$$D = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 4 \\ 1 & 0 & -1-\lambda \end{vmatrix} \xlongequal{L_1+L_2+L_3} \begin{vmatrix} 3-\lambda & 3-\lambda & 3-\lambda \\ 0 & 2-\lambda & 4 \\ 1 & 0 & -1-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2-\lambda & 4 \\ 1 & 0 & -1-\lambda \end{vmatrix} =$$

$$= (3-\lambda)\left[(2-\lambda)(-1-\lambda) + 4 - 2 + \lambda\right] = (3-\lambda)\lambda^2 = 0 \Rightarrow \begin{cases} 3-\lambda = 0 \Rightarrow \lambda_1 = 3 \\ \lambda^2 = 0 \Rightarrow \lambda_2 = \lambda_3 = 0 \end{cases}$$

- Pt $\lambda_1 = 3$

$$(*) \Rightarrow \begin{cases} -\alpha_1 + \alpha_2 = 0 \Leftrightarrow \alpha_1 = \alpha_2 \\ -\alpha_1 + 4\alpha_3 = 0 \\ \alpha_1 - 4\alpha_3 = 0 \Rightarrow \alpha_1 = 4\alpha_3 \end{cases} \Leftrightarrow \alpha_2 = 4\alpha_3, \ \alpha_3 \in R$$

Fie $\alpha_3 = 1 \Rightarrow \alpha_1 = \alpha_2 = 4 \Rightarrow \alpha_{\lambda_1} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} \Rightarrow X_{\lambda_1} = \begin{pmatrix} 4e^{3t} \\ 4e^{3t} \\ e^{3t} \end{pmatrix}$

- Pt $\lambda_2 = \lambda_3 = 0$

$$(*) \Rightarrow \begin{cases} 2\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_2 = \alpha_1, \ \alpha_1 \in R \\ 2\alpha_1 + 4\alpha_3 = 0 \\ \alpha_1 - \alpha_3 = 0 \Rightarrow \alpha_3 = \alpha_1 \end{cases}$$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = \alpha_3 = 1 \Rightarrow \alpha_{\lambda_{2,3}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Pt $\lambda=0$ căutăm $x=(A_1t+A_0)e^{0t}=A_1t+A_0$, $y=(B_1t+B_0)e^{0t}=B_1t+B_0$,

$z=(C_1t+C_0)e^{0t}=C_1t+C_0$, $x'=A_1$, $y'=B_1$, $z'=C_1$

$$\begin{cases} A_1 = 2A_1t+2A_0+B_1t+B_0 \\ B_1 = 2B_1t+2B_0+4C_1t+4C_0 \\ C_1 = A_1t+A_0-C_1t-C_0 \end{cases}$$

$$\begin{cases} 2A_1+B_1=0 \\ 2A_0+B_0=A_1 \\ 2B_1+4C_1=0 \\ 2B_0+4C_0=B_1 \\ A_1-C_1=0 \\ A_0-C_0=C_1 \end{cases} \Leftrightarrow \begin{cases} 2A_1+B_1=0 \\ B_1+2C_1=0 \\ A_1-C_1=0 \end{cases} \Rightarrow \begin{matrix} B_1=-2C_1,\ C_1\in R \\ A_1=C_1 \end{matrix}$$

Fie $C_1=1 \Rightarrow A_1=1, B_1=-2$

$$\Leftrightarrow \begin{cases} 2A_0+B_0=1 \\ 2B_0+4C_0=-2 \\ A_0-C_0=1 \end{cases} \Rightarrow \begin{matrix} 2A_0+B_0=1 \Rightarrow B_0=-2C_0-1 \\ A_0=1+C_0 \end{matrix}$$

$$X_{\lambda_{2,3}} = \begin{pmatrix} t+1 \\ -2t-1 \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{X_{\lambda_2}} + \underbrace{\begin{pmatrix} t \\ -2t \\ t \end{pmatrix}}_{X_{\lambda_3}}$$

$$X = C_1X_1+C_2X_2+C_3X_3 = C_1\begin{pmatrix} 4e^{3t} \\ 4e^{3t} \\ e^{3t} \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_3\begin{pmatrix} t \\ -2t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 4C_1e^{3t}+C_2+tC_3 \\ 4C_1e^{3t}+(-C_2)-2tC_3 \\ C_1e^{3t}+tC_3 \end{pmatrix}$$

$$\begin{cases} x = 4C_1e^{3t}+C_2+tC_3 \\ y = 4C_1e^{3t}-C_2-2tC_3 \\ z = C_1e^{3t}+tC_3 \end{cases}$$