Ecuații diferențiale și cu derivate parțiale
Laborator 02
14.10.2020

1. Să se rezolve următoarele ecuații diferențiale direct integrabile:

a) $\begin{cases} x'(t)=t^2+2, & x,t\in\mathbb{R} \\ x(1)=1 \end{cases}$

$\boxed{b)}$ $x'(t)=\frac{1}{t^2-1}$, $t\in(-1,1), x\in\mathbb{R}$

$\boxed{c)}$ $\begin{cases} x'(t)=\frac{1}{1+t^2}, & t,x\in\mathbb{R} \\ x(-1)=-2 \end{cases}$

$\boxed{d)}$ $\begin{cases} x'(t)=\sin t+4t^2 \\ x\left(\frac{\pi}{3}\right)=\frac{-1}{3} \end{cases}$

e) $\begin{cases} x'(t)=\frac{\sqrt{\ln t}}{t}, & t>1, x\in\mathbb{R} \\ x(e)=\frac{5}{3} \end{cases}$

Rezolvare:

a) $x'(t)=\underbrace{t^2+2}_{f(t)}$ (ec. direct integrabilă)

$$x(t)=\int(t^2+2)\,dt=\int t^2\,dt+\int 2\,dt=\frac{t^3}{3}+2t+\mathcal{C}$$

$$x(1)=1=\frac{1}{3}+2+\mathcal{C} \Rightarrow \mathcal{C}=1-\frac{1}{3}-2=\frac{-4}{3} \Rightarrow C=\frac{-4}{3}$$

$$x_{PC}(t)=\frac{t^3}{3}+2t-\frac{4}{3}.$$

e) $x'(t)=\underbrace{\frac{\sqrt{\ln t}}{t}}_{f t}$ (ec. direct integrabilă)

$$x(t)=\int\frac{\sqrt{\ln t}}{t}\,dt=\int\sqrt{\ln t}\cdot\frac{1}{t}\,dt=\int\sqrt{u}\cdot u'\,dt=\int u^{\frac{1}{2}}\cdot u'\,dt$$

$u(t)=\ln t$

$u'(t)=\frac{1}{t}$

$$=\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+\mathcal{C}$$

$$=\frac{2}{3}\cdot(\ln t)^{\frac{3}{2}}+\mathcal{C}$$

$$x(e)=\frac{2}{3}\sqrt{(\ln e)^3}+\mathcal{C}=\frac{2}{3}+C \Rightarrow C=1$$

$$x_{PC}(t)=\frac{2}{3}\sqrt{(\ln t)^3}+1.$$

2. Să se rezolve următoarele ecuații diferențiale cu variabile separate:

$\boxed{a)}$ $\frac{1}{1+t^2}\,dt+\frac{1}{x}\,dx=0$, $x>0, t\in\mathbb{R}$

$\boxed{b)}$ $dx+\frac{1}{t^2-9}\,dt=0$, $t>3, x\in\mathbb{R}$

c) $\sqrt{x}\,dx=\sqrt{t}\,dt$, $t>0, x>0$

$x(1)=1$

$\boxed{d)}$ $\sin t\,dt-\cos x\,dx=0$, $x\in[0,\pi]$

$x(0)=\frac{\pi}{2}$

e) $\frac{x}{1-x^2}\,dx=\frac{1}{1-t}\,dt$, $t<1, x\in(0,1)$

$x(0)=\frac{1}{2}$

Rezolvare

c) $\begin{cases} \sqrt{x}\,dx = \sqrt{t}\,dt, & t>0, x>0 \\ x(1)=1 \end{cases}$

$\sqrt{x}\,dx = \sqrt{t}\,dt$ (ecuație cu variabile separate)

$$\int \sqrt{x}\,dx = \int \sqrt{t}\,dt \Rightarrow \int x^{\frac{1}{2}}dx = \int t^{\frac{1}{2}}dt \Leftrightarrow \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \mathcal{C} \Rightarrow x^{\frac{3}{2}} = t^{\frac{3}{2}} + \mathcal{C}$$

$$\Leftrightarrow x_{(t)} = \sqrt[3]{\left(t^{\frac{3}{2}}+\mathcal{C}\right)^2}$$

$$x(1) = \sqrt[3]{(1+\mathcal{C})^2} = 1 \Rightarrow (1+\mathcal{C})^2 = 1 \Leftrightarrow 1+\mathcal{C} = \pm 1 \; \begin{cases} C=0 \to x_{PC}(t) = \sqrt[3]{\left(t^{\frac{3}{2}}\right)^2} = t \\ C=-2 \to x_{PC}(t) = \sqrt[3]{\left(t^{\frac{3}{2}}-2\right)^2} \end{cases}$$

e) $\frac{x}{1+x^2}dx = \frac{1}{1-t}dt$ (ecuație cu variabile separate)

$$\frac{-1}{2}\cdot\int \frac{-2x}{1-x^2}dx = -\int \frac{-1}{1-t}dt \xrightarrow{\int \frac{u'}{u}dx = \ln|u|} \frac{-1}{2}\ln|1-x^2| = -\ln|1-t| + \mathcal{C}$$

$u(x) = 1-x^2$ $\quad v(t) = 1-t$

$u'(x) = -2x$ $\quad v'(t) = -1$

$$\ln(1-x^2) = 2\ln(1-t) + \ln\mathcal{C}$$

$$\ln(1-x^2) = \ln(1-t^2)\cdot\mathcal{C}$$

$$1-x^2 = C(1-t)^2$$

$$x(t) = \sqrt{1-C(1-t)^2},\quad 1-C(1-t)^2 > 0$$

$$x(0) = \sqrt{1-C} = \frac{1}{2} \Rightarrow 1-C = \frac{1}{4} \Leftrightarrow C = \frac{3}{4}$$

$$x_{PC}(t) = \sqrt{1-\frac{3}{4}(1-t)^2}$$

3. Să se rezolve următoarele ecuații diferențiale cu variabile separabile:

$\boxed{a}$ $(t+1)\cdot x'(t) = 2x-3$

b) $(t^2-1)\cdot x'(t) + 2tx^2 = 0$

c) $x'(t) = \frac{-t}{\sqrt{1+t^2}}\cdot\frac{\sqrt{1+x^2}}{x}$, $\quad x<0, t\in R$

$\boxed{d}$ $\frac{dx}{dt} = \frac{t}{1+t}(1-x)$, $t>-1, x>1$

$x(0)=5$

Rezolvare

b) $(t^2-1)\cdot x'(t) + 2tx^2 = 0$

$$(t^2-1)\frac{dx}{dt} + 2tx^2 = 0 \Leftrightarrow (t^2-1)\frac{dx}{dt} = -2tx^2 \text{ (ecuație cu var. separabile)}$$

$$\frac{-1}{x^2}dx = \frac{2t}{t^2-1}dt;\; \int\frac{-1}{x^2}dx = \int\frac{2t}{t^2-1}dt \Leftrightarrow \frac{1}{x} = \ln|t^2-1| + \mathcal{C} \Leftrightarrow x_{(t)} = \frac{1}{\ln|t^2-1|+\mathcal{C}}$$

$u(t) = t^2-1$

$u'(t) = 2t$

$\ln|t^2-1| + \mathcal{C} \neq 0.$

c) $x'(t) = \frac{-t}{\sqrt{1+t^2}}\cdot\frac{\sqrt{1+x^2}}{x}$, $\quad x<0, t\in R$

$$\frac{dx}{dt} = \frac{-t}{\sqrt{1+t^2}}\cdot\frac{\sqrt{1+x^2}}{x};\quad \frac{x}{\sqrt{1+x^2}}dx = \frac{-t}{\sqrt{1+t^2}}dt$$

$$\int\frac{2x}{2\sqrt{1+x^2}}dx = -\int\frac{2t}{2\sqrt{1+t^2}}dt \xrightarrow{\int\frac{u'}{2\sqrt{u}}dx} \sqrt{1+x^2} = -\sqrt{1+t^2}+\mathcal{C};\quad x_{(t)} = -\sqrt{1-\left(\mathcal{C}-\sqrt{1+t^2}\right)^2},$$

$u(x) = 1+x^2;\; u'(x) = 2x$

$$1+x^2 = \left(\mathcal{C}-\sqrt{1+t^2}\right)^2;\quad 1-\left(\mathcal{C}-\sqrt{1+t^2}\right)^2 > 0$$