~~Limite~~ Derivabilitate

4.1. $f:D\to\mathbb{R}$, $D\subset\mathbb{R}$, $x_0\in D\cap D'$

funcția f este derivabilă în x_0 dacă există (și e finită) limita $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$.

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} := f'(x_0).$$

- Dacă $x_0\in\big((-\infty,x_0)\cap D\big)'$, f are deriv. la stg în x_0 dacă există și e finită.

$$\lim_{\substack{x\to x_0\\ x<x_0}}\frac{f(x)-f(x_0)}{x-x_0} := f'_s(x_0).$$

- $x_0\in\big((x_0,\infty)\cap D\big)'$ — $\lim\limits_{\substack{x\to x_0\\ x>x_0}}\frac{f(x)-f(x_0)}{x-x_0} := f'_d(x_0)$

- $x_0\in\big((-\infty,x_0)\cap D\big)'\cap\big((x_0,\infty)\cap D\big)'$

$$[(-\infty,x_0)\cup(x_0,\infty)]\cap D = \mathbb{R}\setminus\{x_0\}\cap D = \mathbb{R}\cap D\setminus\{x_0\} \Leftrightarrow D\setminus\{x_0\}.$$

x_0

Operații a) $f+g$ der în x_0 $\to (f+g)'(x_0)=f'(x_0)+g'(x_0)$.

b) $f\cdot g$ der în x_0 $\to (f\cdot g)'(x_0)=f'(x_0)\cdot g(x_0)+f(x_0)+g'(x_0)$.

$$\left(\frac{f}{g}\right)'(x_0)=\frac{f'(x_0)\cdot g(x_0)-f(x_0)\cdot g'(x_0)}{g^2(x_0)},\quad g(x)\neq 0.$$

$u:D\to E$, $f:E\to\mathbb{R}$, $y_0=u(x_0)$, $y_0\in E\cap E'$, u deriv în x_0, f deriv în y_0.

$$(f\circ u)'(x_0)=f'(u(x_0))\cdot u'(x_0).$$

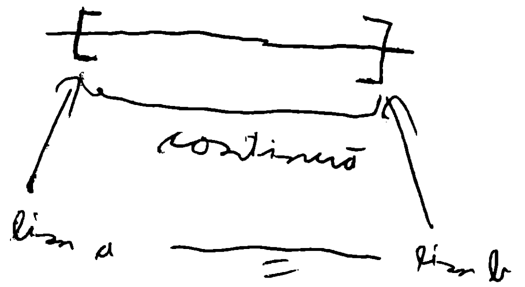
$f:D\to R$, $D\subset R$, $x_0\in D\cap D'$. f derivabilă în x_0 $\Rightarrow$ f continuă în x_0.

Th. Fermat f deriv în pct. de ext. local x_0 și $x_0\in I^0$ $\Rightarrow f'(x_0)=0$.

Th. Rolle: $f:I\to R$, I - interval, $a<b\in I$.

1. f continuă pe $[a,b]$
2. f e deriv pe (a,b) // putem afla deriv în un punct limită.
3. $f(a)=f(b)$.

$\Rightarrow$ Există $c\in(a,b)$ în care derivata se anulează.



Th. Lagrange 1. f cont $[a,b]$
2. f deriv (a,b)

$(\exists)$ cel puțin un $c\in(a,b)$, cu $f(b)-f(a)=f'(c)(b-a)$.

$$f(b)-f(a)=f'(c)(b-a) \Leftrightarrow \frac{f(b)-f(a)}{b-a}=f'(c)$$

$f:D\to R$, $D\subset R$, $x_0\in D\cap D'$

Th. L'Hospital: $f,g:I\to R$, $I\subset R$ interval și $x_0\in I'$

Pres. că:
1. $\boxed{\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x)=0}$;
2. f, g deriv pe $I\setminus\{x_0\}$ $\boxed{f,g \text{ deriv } D\setminus\{x_0\}}$
3. $g'(x)\neq 0$, $\forall x\in I\setminus\{x_0\}$ $\boxed{g'(x)\neq 0}$
4. $(\exists)\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$, $(\exists)\lim_{x\to x_0}\frac{f'(x)}{g'(x)}\in\overline{R}$

$\Rightarrow$ (a) $g(x)\neq 0$, $(\forall)x\in I\setminus\{x_0\}$.

(b) $\frac{f}{g}$ are lim în x_0 și

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}\overset{\frac{0}{0}}{=}\lim_{x\to x_0}\frac{f'(x)}{g'(x)}.$$

$$\lim_{x\to x_0}=\frac{f(x)}{g(x)}\overset{\frac{0}{0}}{=}\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$ — limita există

$g'(x)\neq 0$.

Șiruri de nr. reale

I. Șiruri de nr. reale convergente

$(x_n)_{n\ge 1} \subset \mathbb{R}$ șir de nr. reale (convergent) $\Leftrightarrow (\exists l \in \mathbb{R})(\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n \ge n_\varepsilon)$ $(|x_n - l| < \varepsilon)$

Notare: $x_n \to l$ sau $\lim\limits_{n\to\infty} x_n = l$.

[Orice limită art. fct. convergentă]

există un nr. real numit limită ast. încât oricare ar fi ε pozitiv, există un nog $n_\varepsilon \in \mathbb{N}$ de la care oricare ar fi $n >= n_\varepsilon$, modulul diferenței dintre x_n și l este mai mic decât ε.

$$(\exists l \in \mathbb{R})(\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n \ge n_\varepsilon)(|x_n - l| < \varepsilon).$$

• Șir Cauchy.

$(\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n, m \ge n_\varepsilon)(|x_n - x_m| < \varepsilon)$.

$\Leftrightarrow (\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n \ge n_\varepsilon)(\forall p \ge 1)(|x_{n+p} - x_n| < \varepsilon)$

a) $x_n \to l \Rightarrow |x_n| \to |l|$

b) $|x_n| \to |l| \Leftrightarrow |x_n - l| \to 0$, $x_n \to 0$ $\Rightarrow |x_n| \to 0$.

Cr. Maj

a) $(x_n)_{n\ge1}, (y_n)_{n\ge1} \subset \mathbb{R}$, $l \in \mathbb{R}$ cu $|x_n - l| \le y_n$, $n \ge k$

b) $(x_n)_{n\ge1}, (y_n)_{n\ge1}, (z_n)_{n\ge1} \subset \mathbb{R}$ cu $x_n \le y_n \le z_n$, $n \ge k$, $l \in \mathbb{R}$ și $x_n \to l$, $z_n \to l$, atunci $y_n \to l$.

a) $(x_n)_{n\ge1}, (y_n)_{n\ge1} \subset \mathbb{R}$, $l \in \mathbb{R}$ cu $|x_n - l| \le y_n$, $n \ge k$ și $y_n \to 0$ $\to x_n \to l$.

Șiruri în $\mathbb{R}$ și $\overline{\mathbb{R}}$

Analiză

$(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subset \mathbb{R}$ conv., $x_n \to l_1$, $y_n \to l_2$

$x_n + y_n \to l_1 + l_2$

$x_n y_n \to l_1 l_2$

$\frac{x_n}{y_n} \to \frac{l_1}{l_2}$, $l_2 \neq 0$

$x_n^{y_n} \to l_1^{l_2}$, $x_n > 0$, $n \geq k$.

- **Serii de Nr. Reale**

- $(x_n)_{n\geq 1} \subset \mathbb{R}$
- $x_1+x_2+\dots+x_n+\dots = \boxed{\text{ceva}}$
- $S_1 := x_1,\ S_2 := x_1+x_2;\quad ;\ S_n := x_1+x_2+\dots+x_n,\ (\forall)\, n\geq 1.$

$\longrightarrow (S_n)_{n\geq 1}$ șirul de nr. parțiale asociate lui $(x_n)_{n\geq 1}$.

- Dacă $(S_n)_{n\geq 1}$ are limită $l\in\overline{\mathbb{R}}$, $\boxed{\text{ceva}} = l$.

! $\left((x_n)_{n\geq 1}, (S_n)_{n\geq 1}\right) \overset{not}{=\!=}: \sum\limits_{n\geq 1} x_n$ serie de nr. reale.

$\sum\limits_{n\geq 1} x_n$ ~~conv~~ (C) $\Leftrightarrow (S_n)_{n\geq 1}$ (C)

$\sum\limits_{n\geq 1} x_n$ (D) $\Leftrightarrow (S_n)_{n\geq 1}$ (D)

$-\infty \quad +\infty$ (sau fără lim)

$\left.\begin{array}{l} \sum\limits_{n\geq 1} x_n \ (C) \\ \sum\limits_{n\geq 1} x_n \ (D) \ \text{— are limită } \{\pm\infty\} \end{array}\right]$ are sumă $\sum\limits_{n=1}^{\infty} x_n$

— nu are limită — OSCILANTA

---

a) $\sum\limits_{n\geq 1} \frac{1}{n(n+1)}$; $\quad x_n := \frac{1}{n(n+1)},\ n\geq 1$; $\quad S_n := x_1+x_2+\dots+x_n = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}$

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$\Rightarrow S_n = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n}} - \frac{1}{n+1}$

$S_n = \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1} \longrightarrow 1$

$\sum\limits_{n\geq 1} \frac{1}{n(n+1)}$ (C) și are suma $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

---

b) $\sum\limits_{n\geq 1} \frac{1}{2^n}$

$x_n := \frac{1}{2^n},\ n\geq 1$; $S_n = x_1+x_2+\dots+x_n = \frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}$

Obs că $\frac{1}{2^n} = \frac{1}{2^{n-1}} - \frac{1}{2^n} \Rightarrow \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{2^2} + \dots +$

$\frac{1}{2^n} = x \Leftrightarrow 2x = \frac{1}{2^{n-1}}$

$x = \frac{1}{2^{n-1}} - \frac{1}{2^n} = \frac{2^n - 2^{n-1}}{2^n \cdot 2^{n-1}} = \frac{(2-1)\cancel{2^{n-1}}}{2^n\cdot \cancel{2^{n-1}}}$ V

$+ \frac{1}{2^{n-1}} - \frac{1}{2^n} = \frac{1}{1} - \frac{1}{2^n} =$

$= \frac{2^n - 1}{2^n} \longrightarrow 1$

$a = \frac{1}{2^n}$

$2a = 2 \cdot \frac{1}{2^n} = \frac{2}{2^n} = \frac{1}{2^{n-1}}$

$2a - a = a = \frac{1}{2^{n-1}} - \frac{1}{2^n} = \frac{2^n - 2^{n-1}}{2^n \cdot 2^{n-1}} = \frac{2^{n-1}(2-1)}{2^n \cdot 2^{n-1}} = \frac{1}{2^n}$ V.

$S_n = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{8}} + \dots + \cancel{\frac{1}{2^{n-2}}} - \cancel{\frac{1}{2^{n-1}}} + \cancel{\frac{1}{2^{n-1}}} - \frac{1}{2^n}$

$= \frac{1}{1} - \frac{1}{2^n} = 1 - \frac{1}{2^n} = \cancel{\frac{2^n - 1}{2^n}} \to 1 - \frac{1}{\infty} = 1 - 0 = 1.$

$\sum_{n \geq 1} \frac{1}{2^n}$ (C) și are suma $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

---

c) $\sum_{n \geq 1} \ln \frac{n+1}{n}$

$\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n$

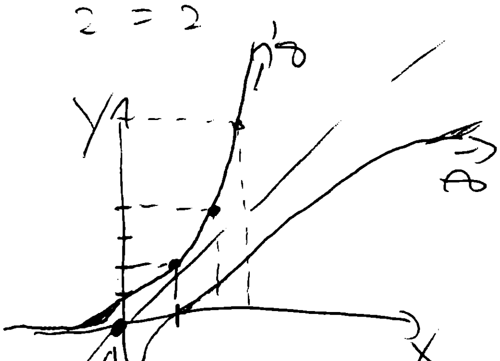
$x_n := \ln \frac{n+1}{n},\ n \geq 1 = \ln(n+1) - \ln(n),\ n \geq 1.$

$S_n := x_1 + x_2 + \dots + x_n = \cancel{\ln 2} - \ln 1 + \ln 3 - \cancel{\ln 2} + \dots + \ln(n+1) - \ln n$

$= (\cancel{\ln 2} - \ln 1) + (\cancel{\ln 3 - \ln 2}) + (\cancel{\ln 4 - \ln 3}) + \dots + (\cancel{\ln(n-2) - \ln(n-3)}) + \cancel{\ln(n-1) - \ln(n-2)}$

$+ \cancel{\ln n - \ln(n-1)} + \ln(n+1) - \cancel{\ln n} = \ln(n+1) - \ln 1 = \ln(n+1) \xrightarrow[\infty]{} \infty$

$2^x = 2^{x=1}$



ln/log $(\infty) \to \infty$

ln/log $(0) \to -\infty$

d) $\sum_{n \geq 1} \frac{1}{\sqrt[3]{n}}$;

$a = \frac{1}{\sqrt[3]{n}}$ $\sqrt{n}$ $a = \frac{1}{\sqrt[n]{n^2}}$

Tetuleanu Mihai - Iulian

Seminar 23.10.2019

1. Arătați că $\frac{(n!)^2}{(2n)!\cdot a^n} \to 0$, $a>1$.

Fie $a_n := \frac{(n!)^2}{(2n)!\cdot a^n}$

Observăm că $a_n>0$, $n\geq 1$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!^2}{(2n+2)!\cdot a^{n+1}} \cdot \frac{a^n\cdot 2n!}{(n!)^2} = \frac{(n+1)!^2}{(n!)^2}\cdot\frac{a^n}{a^{n+1}}\cdot\frac{2n!}{(2n+2)!} =$$

$$= \frac{(n+1)^2}{a\cdot(2n+1)(2n+2)} = \frac{\cancel{n}(1+\frac{1}{n})\cdot\cancel{n}(1+\frac{1}{n})}{a\cdot 2\cancel{n}(1+\frac{1}{2n})\cdot 2\cancel{n}(1+\frac{1}{2n})} \xrightarrow[D.Alc.]{Cr.} \frac{1\cdot 1}{a\cdot 2\cdot 1\cdot 2\cdot 1} = \frac{1}{4a}$$

$$a>1 \Rightarrow \frac{1}{4a}<0 \xrightarrow{Cr.Rap.} a_n \to 0.$$

2. Stabiliți natura șirului:

$x_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}$, $n\geq 1$.

$(\forall)\varepsilon>0$, $(\exists)n_\varepsilon\in N$, $(\forall)n\geq n_\varepsilon$ $\wedge(\forall)p\geq 1$, $|x_{n+p}-x_n|<\varepsilon$ $(\forall)$

$$|x_{n+p}-x_n| = \left|\sum_{K=1}^{n+p}\frac{1}{n} - \sum_{K=1}^{n}\frac{1}{n}\right| = \left|\sum_{K=n+1}^{n+p}\frac{1}{n}\right| = \frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+p} =$$

$$= \frac{1}{n+1}\left(1+\frac{1}{1+\frac{1}{n+1}}+\frac{1}{1+\frac{2}{n+1}}+\ldots+\frac{1}{1+\frac{p-1}{n+1}}\right) \to \frac{1}{\infty}\cdot p = 0 < 1 < \varepsilon$$

$$n_\varepsilon := 1;\ n\geq n_\varepsilon,\ (\forall)p\geq 1 \Rightarrow |x_{n+p}-x_n|<\varepsilon \xrightarrow[def]{} (x_n)_{n\geq 1} \text{ Cauchy}$$

$$\xrightarrow[TH.\ Cauchy]{} (x_n)_{n\geq 1} \text{ convergent}$$

E 4.3.1. Stabiliţi derivabilitatea funcţiilor $f:\mathbb{R}\to\mathbb{R}$.

a) $f(x)=\begin{cases}\cos\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$; b) $f(x)=\begin{cases}x\sin\frac{2}{x^2}, & x\neq 0\\ 0, & x=0\end{cases}$; c) $f(x)=\begin{cases}x^2\cos\frac{1}{2x}, & x\neq 0\\ 0, & x=0\end{cases}$; d) $f(x)=\begin{cases}x^2+x, & x\leq 0\\ \sin x, & x\geq 0\end{cases}$

Rezolvare:

a) Considerăm şirurile $x_n:=\frac{1}{2n\pi}\searrow 0$, $y_n:=\frac{1}{\left(2n\pi+\frac{\pi}{2}\right)}\searrow 0$.

Deoarece $f(x_n)=\cos 2n\pi = 1\to 1$, $f(y_n)=\cos\left(2n\pi+\frac{\pi}{2}\right)=0\to 0$, rezultă că f nu are limită la dreapta în 0 (analog la stânga în 0), deci nu e derivabilă sau continuă în 0.

Evident f derivabilă pe $\mathbb{R}^*$ (provenind din operaţii algebrice şi de compunere cu funcţii derivabile pe $\mathbb{R}^*$) şi $f'(x)=\left(\cos\frac{1}{x}\right)'=-\sin\frac{1}{x}\cdot\left(\frac{-1}{x^2}\right)=\frac{1}{x^2}\sin\frac{1}{x}$.

b) f continuă în 0, deoarece $\exists \lim\limits_{x\to 0} f(x)$ şi $\lim\limits_{x\to 0} f(x)=0=f(0)$.

Deoarece $\left|x\sin\frac{2}{x^2}\right|\leq |x|$, $x\neq 0$, din Cr. Maj va rezulta că limita lui f în 0 şi este egală cu 0. Studiem derivabilitatea lui f în 0: $\frac{f(x)-f(0)}{x-0}=\sin\frac{2}{x^2}$, $x\neq 0$ care nu are limită în 0, deci nu-i derivabilă în 0.

Evident f derivabilă pe $\mathbb{R}^*$ şi $f'(x)=\left(x\sin\frac{2}{x^2}\right)'=\sin\frac{2}{x^2}+x\cdot\left(-\cos\frac{2}{x^2}\right)\cdot\frac{-4}{x^3}=\sin\frac{2}{x^2}+$
$+\frac{4}{x^2}\cos\frac{2}{x^2}$

c) f e continuă în 0, deoarece $(\exists)\lim\limits_{x\to 0} f(x)$ şi $\lim\limits_{x\to 0} f(x)=0=f(0)$.

Într-adevăr, deoarece $\left|x^2\cos\frac{1}{x}\right|\leq x^2$, $x\neq 0$, din Cr. Maj rezultă că există limita lui f în 0 şi este egală cu 0.

Studiem derivabilitatea lui f în 0, avem: $\frac{f(x)-f(0)}{x-0}=\frac{\cancel{x^2}\cos\frac{1}{2x}-0}{x}=x\cos\frac{1}{2x}$
$x\neq 0$, care are limită în 0, cu limita 0.

Evident f este derivabilă pe $\mathbb{R}^*$ şi $f'(x)=\left(x^2\cos\frac{1}{2x}\right)'=2x\cos\frac{1}{2x}+x^2\left(-\sin\frac{1}{2x}\right)\cdot\frac{1}{2}\cdot\frac{-1}{x^2}=$
$=2x\cos\frac{1}{2x}+\frac{1}{2}\sin\frac{1}{2x}$.

d) f continuă în 0, deoarece $(\exists)\lim\limits_{x\to 0} f(x)$ şi $\lim\limits_{x\to 0} f(x)=0=f(0)$.

Evident f este derivabilă pe $\mathbb{R}^*$ şi $f'(x)=2x+1$, $x\leq 0$ şi $f'(x)=\cos x$, $x\geq 0$.

Petrulescu Mihai-Eduard

E 4.3.2. Să se calculeze, utilizând regulile lui l'Hospital, următoarele limite:

a) $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x\sin 2x}$; b) $\lim\limits_{x\to 1}(x-1)\exp\left(\dfrac{1}{x-1}\right)$; c) $\lim\limits_{x\to 1}\dfrac{x^x-x}{\ln x-x+1}$; d) $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{xe^{ax}-2ax}$ $(a\in\mathbb{R})$;

e) $\lim\limits_{x\to 0}\left(\dfrac{\arcsin x}{x}\right)^{x^{-2}}$

Rezolvare:

Se observă că la punctele a), c), d) avem cazul $\frac{0}{0}$, e) avem 1^∞ și b) avem $0\cdot\infty$. Pe ele le vom transforma în cazuri în care putem aplica regulile lui l'Hospital.

a) Fie $f(x)=1-\cos^3 x$, $g(x)=x\sin 2x$, $x\in D=\left(\frac{-\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$. Observăm că sunt îndeplinite condițiile din teorema lui l'Hospital $\left(\frac{0}{0}\right)$:

1) $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0} g(x)=0$; $\lim\limits_{x\to 0} f'(x)=\lim\limits_{x\to 0} g'(x)=0$

2) f și g sunt derivabile pe D și $f'(x)=3\cos^2(x)\sin(x)$, $g'(x)=\sin 2x+2x\cos 2x$.

3) f' și g' sunt derivabile pe D și $f''(x)=3\cos^3 x-6\cos x\sin^2 x$, $g''(x)=4\cos 2x-$
$-4x\sin 2x$

4) $g''(x)\neq 0,\ \forall x\in D=\left(\frac{-\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$.

5) $\lim\limits_{x\to 0}\dfrac{f''(x)}{g''(x)}=\lim\limits_{x\to 0}\dfrac{3\cos^3 x-6\cos x\sin^2 x}{4\cos 2x-4x\sin 2x}=\dfrac{3}{4}$

Conform Th. l'Hospital avem $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x\sin 2x}=\dfrac{3}{4}$.

b) Deoarece $(x-1)\exp\left(\frac{1}{x-1}\right)=\ln(x-1)+\ln\left(e^{\frac{1}{x-1}}\right)=\dfrac{1}{x-1}+\ln(x-1)$, $x\in\mathbb{R}\setminus\{1\}$, $x>0$.

Notăm $f(x)=(x-1)\ln(x-1)+1$, $g(x)=x-1$. Avem:

1) f,g derivabile pe $(0,\infty)\setminus\{1\}$ și $f'(x)=\ln(x-1)+1$, $g'(x)=1$

2) $g'(x)=1\neq 0$, $\forall x\in(0,\infty)\setminus\{1\}$.

3) $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1} g(x)=0$

4) $\lim\limits_{x\to 1}\dfrac{f'(x)}{g'(x)}=\lim\limits_{x\to 1}\dfrac{\cancel{(x-1)}\ln(x-1)+1}{1}=\infty$?

c) Fie $f(x)=x^x-x$, $g(x)=\ln x-x+1$, $x\in(0,\infty)\setminus\{1\}$.

1) $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1} g(x)=0$, $\lim\limits_{x\to 1} f'(x)=\lim\limits_{x\to 1} g'(x)=0$

2) f,g, derivabile pe $(0,\infty)\setminus\{1\}$ și $f'(x)=x^x(\ln(x)+1)-1$, $g'(x)=\frac{1}{x}-1$, f',g' derivabile pe D cu $f''(x)=x^x(\ln x+1)^2$, $g''(x)=\dfrac{-1}{x^2}$.

3) $g''(x)=-1\neq 0$

4) $\lim\limits_{x\to 1}\dfrac{f''(x)}{g''(x)}=\dfrac{4}{-1}=-4$.

Petculescu Mihai-Eduard

4.3.23. Să se calculeze jacobianul J_f într-un punct curent pentru funcțiile:

i) $f:\mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x^2y - y, x^3 - y^4)$

ii) $f:\mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (xe^{x+y}, x^2y^3, \ln(2+x^2+y^2))$

iii) $f:(0,\infty)\times\mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y,z) = \left(y^2 - xe^{z^2} - y\ln x, \frac{x}{zy}\right)$

iv) $f:\mathbb{R}^3\setminus\{(0,0,0)\} \to \mathbb{R}^3$, $f(x,y,z) = \left(\frac{1}{\sqrt{x^2+y^2+z^2}}, xyz, xy - zx + 2yz\right)$.

Să se scrie expresia diferențialei df într-un punct curent.

Rezolvare:

i) $J_f(x,y) = \begin{pmatrix} 2xy, & x^2-1 \\ 3x^2, & -4y^3 \end{pmatrix}$; $df(x,y) = J_f(x,y)\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 2xy\,dx + (x^2-1)\,dy \\ 3x^2\,dx - 4y^3\,dy \end{pmatrix}$

ii) $J_f(x,y) = \begin{pmatrix} e^{x+y} + xe^{x+y}, & xe^{x+y} \\ 2xy^3, & 3x^2y^2 \\ \frac{2x}{2+x^2+y^2}, & \frac{2y}{2+x^2+y^2} \end{pmatrix}$; $df(x,y) = J_f(x,y)\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} (x+1)e^{x+y}\,dx + xe^{x+y}\,dy \\ 2xy^3\,dx + 3x^2y^2\,dy \\ \frac{2x}{2+x^2+y^2}\,dx + \frac{2y}{2+x^2+y^2}\,dy \end{pmatrix}$

iii) $J_f(x,y,z) = \begin{pmatrix} -e^{z^2} - \frac{y}{x}, & 2y - \ln x, & -xe^{z^2}\cdot 2z \\ \frac{1}{zy}, & \frac{-x}{zy^2}, & \frac{-x}{yz^2} \end{pmatrix}$; $df(x,y,z) = J_f(x,y,z)\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} =$

$$= \begin{pmatrix} \left(-e^{z^2} - \frac{y}{x}\right)dx + (2y - \ln x)\,dy + (-xe^{z^2}\cdot 2z)\,dz \\ \frac{dx}{zy} - \frac{x\,dy}{zy^2} - \frac{x\,dz}{yz^2} \end{pmatrix}$$

iv) $J_f(x,y,z) = \begin{pmatrix} \frac{-2x}{|x^2+y^2+z^2|}, & \frac{-2y}{x^2+y^2+z^2}, & \frac{-2z}{x^2+y^2+z^2} \\ yz, & xz, & xy \\ y-z, & x+2z, & -x+2y \end{pmatrix}$

$df(x,y,z) = J_f(x,y,z)\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \frac{-2x\,dx}{x^2+y^2+z^2} + \frac{-2y\,dy}{x^2+y^2+z^2} + \frac{-2z\,dz}{x^2+y^2+z^2} \\ yz\,dx + xz\,dy + xy\,dz \\ (y-z)\,dx + (x+2z)\,dy + (-x+2y)\,dz \end{pmatrix}$

Petrubescu Mihai-Liviu

4.3.24 Să se calculeze hessiana H_f într-un punct curent pentru funcțiile:

i) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 y + y^4$

ii) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = xye^x + x^2$

iii) $f:(0,\infty)\times\mathbb{R}^2 \to \mathbb{R}$, $f(x,y,z) = xy + ye^z + z\ln x$

iv) $f:\mathbb{R}^3\setminus\{(0,0,0)\} \to \mathbb{R}$, $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

Să se scrie expresia diferențialei a II-a d^2f într-un punct curent.

Rezolvare:

i) $\dfrac{\partial f}{\partial x} = 3x^2y$; $\dfrac{\partial f}{\partial y} = x^3 + 4y^3$

$$H_f = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x,y), & \dfrac{\partial^2 f}{\partial y\partial x}(x,y) \\ \dfrac{\partial^2 f}{\partial x\partial y}(x,y), & \dfrac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 6xy, & 3x^2 \\ 3x^2, & 12y^2 \end{pmatrix} ; \ df^2(x,y) = \begin{pmatrix} 6xy\,dx + 3x^2\,dy \\ 3x^2\,dx + 12y^2\,dy \end{pmatrix}$$

ii) $\dfrac{\partial f}{\partial x}(x,y) = ye^x + xye^x + 2x$; $\dfrac{\partial f}{\partial y}(x,y) = xe^x$

$$H_f = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x,y), & \dfrac{\partial^2 f}{\partial y\partial x}(x,y) \\ \dfrac{\partial^2 f}{\partial x\partial y}(x,y), & \dfrac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 2ye^x + xye^x + 2, & e^x + xe^x \\ e^x + xe^x, & 0 \end{pmatrix}$$

$$df^2(x,y) = \begin{pmatrix} (2ye^x + xye^x + 2)\,dx + (e^x + xe^x)\,dy \\ (x+1)e^x\,dx \end{pmatrix}$$

iii) $\dfrac{\partial f}{\partial x}(x,y,z) = y + \dfrac{z}{x}$; $\dfrac{\partial f}{\partial y}(x,y,z) = x + e^z$; $\dfrac{\partial f}{\partial z}(x,y,z) = ye^z + \ln x$

$$H_f = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x,y,z), & \dfrac{\partial^2 f}{\partial y\partial x}(x,y,z), & \dfrac{\partial^2 f}{\partial z\partial x}(x,y,z) \\ \dfrac{\partial^2 f}{\partial x\partial y}(x,y,z), & \dfrac{\partial^2 f}{\partial y^2}(x,y,z), & \dfrac{\partial^2 f}{\partial z\partial y}(x,y,z) \\ \dfrac{\partial^2 f}{\partial x\partial z}(x,y,z), & \dfrac{\partial^2 f}{\partial y\partial z}(x,y,z), & \dfrac{\partial^2 f}{\partial z^2}(x,y,z) \end{pmatrix} = \begin{pmatrix} \dfrac{-z}{x^2}, & 1, & \dfrac{1}{x} \\ 1, & 0, & e^z \\ \dfrac{1}{x}, & e^z, & ye^z \end{pmatrix}$$

$$d^2f(x,y,z) = \begin{pmatrix} \dfrac{-z}{x^2}\,dx + dy + \dfrac{1}{x}\,dz \\ dx + e^z\,dz \\ \dfrac{dx}{x} + e^z\,dy + ye^z\,dz \end{pmatrix}$$

Ieteșanu Mihai-Zitorin

iv) $\frac{\partial f}{\partial x}(x,y,z) = \frac{-2x}{x^2+y^2+z^2}$, $\frac{\partial f}{\partial y}(x,y,z) = \frac{-2y}{x^2+y^2+z^2}$, $\frac{\partial f}{\partial z}(x,y,z) = \frac{-2z}{x^2+y^2+z^2}$

$$H_f(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y,z), & \frac{\partial^2 f}{\partial y \partial x}(x,y,z), & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y}(x,y,z), & \frac{\partial^2 f}{\partial y^2}(x,y,z), & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z}(x,y,z), & \frac{\partial^2 f}{\partial y \partial z}(x,y,z), & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} \frac{-2}{x^2+y^2+z^2} + \frac{4x^2}{(x^2+y^2+z^2)^2}, & \frac{(-2x)(-2y)}{(x^2+y^2+z^2)^2}, & \frac{(-2x)(-2z)}{(x^2+y^2+z^2)^2} \\ \frac{(-2x)(-2y)}{(x^2+y^2+z^2)^2}, & \frac{-2}{x^2+y^2+z^2} + \frac{4y^2}{(x^2+y^2+z^2)^2}, & \frac{(-2y)(-2z)}{(x^2+y^2+z^2)^2} \\ \frac{(-2x)(-2z)}{(x^2+y^2+z^2)^2}, & \frac{(-2y)(-2z)}{(x^2+y^2+z^2)^2}, & \frac{-2}{x^2+y^2+z^2} + \frac{4z^2}{(x^2+y^2+z^2)^2} \end{pmatrix}$$

$$; \; d_f^2(x,y,z) = \begin{pmatrix} \left(\frac{-2}{x^2+y^2+z^2} + \frac{4x^2}{(x^2+y^2+z^2)^2}\right)dx + \frac{4xy}{(x^2+y^2+z^2)^2}dy + \frac{4xz}{(x^2+y^2+z^2)^2}dz \\ \frac{4xy\,dx}{(x^2+y^2+z^2)^2} + \left(\frac{-2}{x^2+y^2+z^2} + \frac{4y^2}{(x^2+y^2+z^2)^2}\right)dy + \frac{4yz\,dz}{(x^2+y^2+z^2)^2} \\ \frac{4xz\,dx}{(x^2+y^2+z^2)^2} + \frac{4yz\,dy}{(x^2+y^2+z^2)^2} + \left(\frac{-2}{x^2+y^2+z^2} + \frac{4z^2}{(x^2+y^2+z^2)^2}\right)dz \end{pmatrix}$$

4.3.26. Să se calculeze derivatele parțiale de ordinul I ale funcției

$$f(x,y) = xy \cdot \varphi\left(\frac{x}{\sqrt{x^2+y^2}}\right), \quad (x,y) \neq (0,0), \text{ unde } \varphi : \mathbb{R} \to \mathbb{R} \text{ derivabilă pe } \mathbb{R}.$$

Rezolvare:

$$f(x,y) = xy \cdot \frac{x}{\sqrt{x^2+y^2}} = \frac{x^2 y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial x} = \frac{yx^3 + 2y^3x}{(x^2+y^2)^{\frac{3}{2}}} ; \quad \frac{\partial f}{\partial y} = \frac{x^4}{(y^2+x^2)^{\frac{3}{2}}}$$

Petrulescu Mihai-Silviu

Diferențiabilitate.

$7418 -$
7337
$--87$

A. funcții reale de variabilă reală.

$f: D \to R$, $D \subset R$, $x_0 \in D \cap D'$.

Def: a) f derivabilă în x_0, dacă $(\exists)$ $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} := f'(x_0)$.

b) $x_0 \in \left((-\infty, x_0) \cap D\right)'$, f are deriv. la stg. în x_0 dacă $\exists$ și e finită $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x)-f(x_0)}{x-x_0}$.

$x_0 \in \left((x_0, \infty) \cap D\right)'$, f are dr — — $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} \frac{f(x)-f(x_0)}{x-x_0} := f_d'(x_0)$.

Prop. $f: D \to R$, $D \subset R$, $x_0 \in \left((-\infty, x_0) \cap D\right)' \cap \left((x_0, \infty) \cap D\right)'$, f deriv în x_0 $\Leftrightarrow$ f deriv și la stg și la dr.

P 4.1.3. $f, g: D \to R$, $D \subset R$, $x_0 \in D \cap D'$, f, g deriv în x_0

$u: D \to E$

f $E \to D$

Petculescu Mihai-Silviu

4.2.1. $f: \mathbb{R} \to \mathbb{R}$.

a) $f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

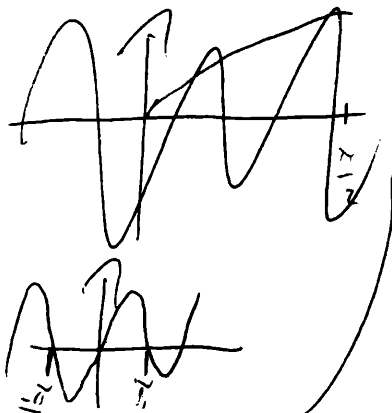
b) $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

a) $x_n = \frac{1}{2n\pi} \to 0$. $\quad f(x_n) = \sin 2n\pi \to 0$.

$y_n = \frac{1}{2n\pi + \frac{\pi}{2}} \to 0$. $\quad f(y_n) = \sin\left(2n\pi + \frac{\pi}{2}\right) = \sin\left(\pi\left(2n + \frac{1}{2}\right)\right) \to 1$

f derivabilă pe $\mathbb{R}^*$

$f'(x) = \left(\sin \frac{1}{x}\right)' = \cos \frac{1}{x} \cdot \left(\frac{1}{x}\right)' = \cos \frac{1}{x} \cdot \frac{-1}{x^2}$



b) $\lim_{x \to 0} f(x) = 0 = f(0)$

[illegible] Mihai-[illegible]

4.2.2. a) $f(x) = tg\, x - x$

$g(x) = x - \sin x$

a) $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$.

b) f, g deriv în D . cu $f'(x) = \frac{1}{\cos^2 x} - 1$, $g'(x) = 1 - \cos x$.

c) $g'(x) \neq 0$, $\forall x \in D$

d) $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\frac{1}{\cos^2 x} - 1}{1 - \cos x} = \lim_{x\to 0} \frac{1 - \cos^2 x}{(1-\cos x)\cos^2 x} =$

$= \lim_{x\to 0} \frac{1+\cos x}{\cos^2 x} = \frac{1+1}{1} = 2$.

l.Hopital
$\Rightarrow \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 2$.

b) $f(x) = e^{x^2} - x\sin x - \cos x$

$g(x) = x^2$, $x \in \mathbb{R}^*$.

$f'(x) = 2x \cdot e^{x^2} - (\cancel{\sin x} + x\cdot\cos x) + \cancel{\sin x} = 2x\cdot e^{x^2} - x\cos x \xrightarrow[0]{} 0$

$g'(x) = 2x \xrightarrow[0]{} 0$

$f''(x) = (2e^{x^2} + 2x\cdot e^{x^2}\cdot 2x) - (\cos x - x\sin x)$.

$= 2e^{x^2} + 4x^2\cdot e^{x^2} - \cos x + x\sin x \xrightarrow[0]{} 2\cdot 1 + 4\cdot 0 - 1 + 0 = 1$

$g''(x) = 2 \xrightarrow[0]{} 2$.

$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \frac{1}{2}$

l.H.
$\Rightarrow \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \frac{1}{2}$.

c) $f(x) = 3^{\sin^2 x} - 3$.

$g(x) = x - \frac{\pi}{2}$

~~$f'(x) = 3^{\sin^2 x} \cdot \ln(\sin^2 x) \cdot = 3^{\sin^2 x} \cdot 2\ln\sin x$~~

$f'(x) = 3^{\sin^2 x} \cdot \ln(a) = \ln(3) \cdot 3^{\sin^2 x} \cdot (\sin^2 x)' =$

$= \ln(3) \cdot 3^{\sin^2 x} \cdot 2\sin x \cdot \cos x$

$g'(x) = 1$. $\quad \lim_{x\to \frac{\pi}{2}} \frac{-2\ln(3)\cdot 3^1 \cdot 0}{1} = 0$.

Ietuşbeanu Mihai-Silviu

d) $f(x) = \ln(\sin 2x)$
$g(x) = \ln(\sin 3x)$ $, x \in (0, \frac{\pi}{6})$

l) $\lim\limits_{x\to 0} x^{tg x}$.

$f'(x) = \cancel{\ln(\sin 2x) \quad (\sin 2x)' = \ln(\sin 2x) \cdot 2 \sin 2x}$.

$\ln(\sin 2x) = \frac{1}{\sin 2x} \cdot \cos 2x \cdot 2 = \frac{2\cos 2x}{\sin 2x}$

$\ln(\sin 3x) = \frac{1}{\sin 3x} \cdot \cos 3x \cdot 3$.

b) $g'(x) \neq 0, \forall x \in (0, \frac{\pi}{6})$.

c) $\lim\limits_{x\to 0} |f(x)| = \lim\limits_{x\to 0} |g(x)| = \infty$.

d) $\lim\limits_{x\to 0} \frac{f'(x)}{g'(x)} = \lim \frac{\frac{2\cos 2x}{\sin 2x}}{\frac{3\cos 3x}{\sin 3x}} = \frac{2\cos 2x \cdot \sin 3x}{3\cos 3x \cdot \sin 2x} = \cos\left(\frac{2x}{3x}\right) \cdot \frac{2}{3} \sin\left(\frac{3x}{2x}\right) \; 1$.

lim. fund.

l) $\lim\limits_{x\to 0} x^{tg x} \xrightarrow{0^0}$

$x^{tg x} = e^{tg x \cdot \ln x} \quad |\ln \Leftrightarrow \ln(x^{tg x}) = tg x \ln x$

$\cancel{x^{tg x} = tg x \cdot x^{tg x - 1} \cdot (tg x)'}$

$x^{tg x} = \ln(x^{tg x}) = tg x \cdot \ln x \Rightarrow e^{tg x \cdot \ln x}$.

$e^{\frac{\ln x}{\frac{1}{tg x}}} \Leftrightarrow \lim\limits_{x\to 0} x^{tg x} = e^{\left(\lim\limits_{x\to 0} \frac{\ln x}{\frac{1}{tg x}}\right)}$

$f(x) = \ln x; \; g(x) = \frac{1}{tg x}$

$f'(x) = \frac{1}{x} \xrightarrow{0} \infty$

$g'(x) = \frac{1}{tg x} = \cancel{(tg x)^{-1} = (-1) \cdot tg x \cdot \frac{1}{ctg x}} = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}$

$\frac{f'(x)}{g'(x)} = \frac{\frac{1}{x}}{\frac{-1}{\sin^2 x}} = \frac{\sin^2 x}{-x} = \frac{\sin x}{x} \cdot (-\sin x) = 1 \cdot 0 = 0$.

$e^0 = 1$

a) $x_n = \frac{1}{n}\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right), n \geq 1.$

$a_n := \left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)$

$b_n = n$, monoton... (evident $(b_n)_n$ s. cresc. și $\lim b_n \to \infty$)

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{1+\frac{1}{2}+\dots+\frac{1}{n}+\frac{1}{n+1}-1-\frac{1}{2}-\dots-\frac{1}{n}}{n+1-n} = \frac{\frac{1}{n+1}}{1} = \frac{1}{n+1} \to 0.$$

S.C. $\Rightarrow (x_n)_n = \left(\frac{a_n}{b_n}\right)_n$ are limită și $\lim_{n\to\infty} x_n = 0$

b) $x_n := \frac{1^n+2^n+\dots+n^n}{n^{n+1}}$

$a_n := 1^n+2^n+\dots+n^n$

$b_n := n^{n+1}$

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{(n+1)^n}{(n+1)^{n+1}-n^{n+1}} = \frac{\cancel{(n+1)^n}}{(n+1)^{\cancel{n+1}}\left(1-\left(\frac{n}{n+1}\right)^{n+1}\right)} =$$

$$\Rightarrow \frac{1}{(\infty)(1-0)} = 0.$$

$$= \frac{n^n + C_n^1 n^{n-1}+\dots}{\cancel{n^{n+1}}+C_{n+1}^1 n^n + C_{n+1}^2 n^{n-1}\cdot 1^2+\dots-\cancel{n^{n+1}}} = \frac{n^n + n n^{n-1}+\dots}{(n+1)n^n + C_{n+1}^2 n^n+\dots+1} =$$

$$\to \frac{1}{n+1}$$

a) $x_n := \frac{\ln n}{n^k}, n \geq 1.$ ($k \in \mathbb{N}^*$ fixat)

$a_n := \ln n$

$b_n := n^k$, evident n^k s. cresc. și $\lim b_n \to \infty$

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{\ln(n+1)-\ln n}{(n+1)^k-n^k} = \frac{\ln\left(\frac{n+1}{n}\right)}{(n+1)^k\left(1-\left(\frac{n}{n+1}\right)^k\right)} = \frac{\ln\left(1+\frac{1}{n}\right)}{\cancel{n^k}+kn^{k-1}+\dots-\cancel{n^k}} =$$

$$= \underbrace{\ln\left(1+\frac{1}{n}\right)}_{\to 0}\cdot\underbrace{\frac{1}{\underbrace{kn^{k-1}+\dots}_{\to\infty}}}_{\to 0} \to 0.$$

	i	j	k
$2 =$	1	0	0
$3 =$	0	1	0
$4 =$	2	0	0
$5 =$	0	0	1
$6 =$	1	1	0
$8 =$	3	0	0
$9 =$	0	2	0
$10 =$	1	0	1
$12 =$	2	1	0

$k = \dots$ ($5^k \neq 0$)

1	0	0	0
2	1	0	0
3	0	1	0
4	2	0	0
5	0	0	1
6	1	1	0
7	3	0	0
8	0	2	0
9	1	0	1
10	2	1	0

d) $x_n = \sqrt[n]{\frac{(2n)!}{(n!)^2}}, n \geq 1.$

$a_n = \sqrt{\cancel{}} \frac{(2n)!}{(n!)^2}$

D'Alembert

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{[(n+1)!]^2} \cdot \frac{(n!)^2}{(2n)!} = \frac{(2n+1)(2n+2)}{(n+1)^2} \to \frac{4}{1} = 4$$

e) $x_n := \frac{\sqrt[n]{n!}}{n}, n \geq 1.$

$$x_n = \sqrt[n]{\frac{n!}{n^n}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\cancel{!}}{(n+1)^{n+1}} \cdot \frac{n^n}{\cancel{n!}} = \left(\frac{n}{n+1}\right)^n \cdot \cancel{(n+1)}$$

$$\left(\frac{n}{n+1}\right)^n = \left(1 + \frac{n}{n+1} - 1\right)^n = \left[\left(1 + \frac{(-1)}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-1}{n+1} \cdot n} =$$

$$\Rightarrow e^{\lim\limits_{n\to\infty} \frac{-n}{n+2}} = e^{\frac{-1}{1}} = e^{-1}$$

$\cancel{\frac{a_{n+1}}{a_n}}$ $\cancel{\frac{a_{n+1}}{a_n}}$ $\frac{(n!)^2}{(2n)! \cdot a^n};$ $a_n = (n!)^2$

$b_n = (2n)! \cdot a^n.$

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{(n+1)!^2 - (n)!^2}{(2n+2)! \cdot a^{n+1} - (2n)! \cdot a^n}$$

$6^2 = 3^2 \cdot 2^2 \Leftrightarrow$

$$\frac{(n!)^2}{(2n)! \cdot a^n} = \frac{(n!)}{\underbrace{(n+1)(n+2)\ldots(2n)}_{(n^n \ldots)} \cdot a^n} \to 0.$$

$(n^n \ldots) \cdot a^n$

Petrescu Mihai-Silviu

1.2.1. a) $\lim\limits_{n\to\infty} \frac{4n+3}{5n+7} = \frac{4}{5}$.

Un șir $(x_n)_{n\ge1}$ e convergent $\Leftrightarrow (\exists\, l \in \mathbb{R})(\forall\, \varepsilon > 0)(\exists\, n_\varepsilon \in \mathbb{N}^*)(\forall\, n \ge n_\varepsilon)$
$(|x_n - l| < \varepsilon)$

$x_n := \frac{4n+3}{5n+7}$ are limita $l = \frac{4}{5}$

$\Leftrightarrow |x_n - l| < \varepsilon \Leftrightarrow \left|\frac{4n+3}{5n+7} - \frac{4}{5}\right| < \varepsilon \Leftrightarrow \left|\frac{20n+15-20n-28}{5(5n+7)}\right| < \varepsilon \Leftrightarrow$

$\Leftrightarrow \left|\frac{-13}{5(5n+7)}\right| < \varepsilon \Leftrightarrow \frac{13}{5(5n+7)} < \varepsilon \Leftrightarrow \frac{25n+35}{13} > \frac{1}{\varepsilon} \Leftrightarrow 25n > \frac{13}{\varepsilon} - 35$

$n > \frac{\frac{13}{\varepsilon}}{25} - \frac{35}{25} = \frac{13}{25\varepsilon} - \frac{5}{7}$

~~6~~
Fie $a_n := \frac{4n+3}{5n+7}$, $n \ge 1$. Avem

$\lim\limits_{n\to\infty} \frac{4n+3}{5n+7} = \frac{4}{5} \Leftrightarrow (\forall\, \varepsilon > 0)(\exists\, n_\varepsilon \in \mathbb{N}^*)(\forall\, n \ge n_\varepsilon)\left(\left|a_n - \frac{4}{5}\right| < \varepsilon\right)$

Dar $\left|a_n - \frac{4}{5}\right| < \varepsilon \Leftrightarrow \ldots \Leftrightarrow n > \frac{13}{25\varepsilon} - \frac{5}{7}$

Cons. $n_\varepsilon = \left[\frac{13}{25\varepsilon} - \frac{7}{5}\right] + 1$ avem că $(\forall)\, n \ge n_\varepsilon$, $n > \frac{13}{25\varepsilon} - \frac{7}{5} \Leftrightarrow$

a) $x_n = \left(\frac{n+1}{n+2}\right)^n, n \geq 1$

$$\left(1+\frac{n+1}{n+2}-1\right)^n = \left(\frac{-1}{n+2}+1\right)^n = \left[\left(1+\frac{-1}{n+2}\right)^{\frac{n+2}{-1}}\right]^{\frac{-1}{n+2}\cdot n} = e^{\frac{-n}{n+2}} \Rightarrow e^{-1} = \frac{1}{e}.$$

b) $$\left(1+\frac{n^2-n+1}{n^2+n+1}-1\right)^{\frac{n^2}{n+1}} = \left[\left(1+\frac{-2n}{n^2+n+1}\right)^{\frac{n^2+n+1}{-2n}}\right]^{\frac{n^2}{n+1}\cdot\frac{-2n}{n^2+n+1}} = e^{-2}$$

c) $$x_n = \frac{\ln\left(1+\frac{1}{n+1}\right)}{\frac{1}{n+1}} = 1$$

$\frac{1}{n+1} \to 0$

$$x_n = \frac{\ln\left(\frac{2n+2}{2n+2}-\frac{1}{2n+2}\right)}{\frac{-1}{2n+2}} = 1 \cdot \frac{\cancel{-1}}{\frac{\cancel{2n+2}}{2n+2}} = \frac{-1}{0}$$

$$x_n = n^3 \ln\left(\frac{2n+1}{2n+2}\right)\ln\left(\frac{3n^2+1}{3n^2+2}\right)$$

$$\frac{8}{\frac{4}{2}} = 4 = \frac{16}{4}$$

$$= n^3 \cdot \underbrace{\frac{\ln\left(1+\frac{-1}{2n+2}\right)}{\frac{-1}{2n+2}}}_{1} \cdot \frac{-1}{2n+2} \cdot \underbrace{\frac{\ln\left(1+\frac{-1}{3n^2+2}\right)}{\frac{-1}{3n^2+2}}}_{1} \cdot \frac{-1}{3n^2+2}$$

$$= \frac{n^3}{6n^3+\dots} \cdot 1 \cdot 1 \to \frac{1}{6}.$$

$$x = \frac{1}{\frac{1}{x}}$$

e) $$n\left(\sqrt[n]{2}-1\right) = \frac{n}{\sqrt[n]{2}+1} = \frac{\infty}{2^{\frac{1}{\infty}}+1} = \frac{\infty}{2} = \infty$$

$$n\left(\sqrt[n]{2}-1\right) = n\left(2^{\frac{1}{n}}-1\right) = n \cdot 2^{\frac{1}{n}} - n$$

$$n\left(\sqrt[n]{2}-1\right) = \frac{\left(2^{\frac{1}{n}}-1\right)}{\frac{1}{n}} = \frac{\left(1+2^{\frac{1}{n}}-2\right)}{\frac{1}{n}} =$$

$$= \frac{\quad}{\frac{-1}{x^2}}$$

1.2.1.

e) $x_n := \left(1+\frac{1}{n}\right)^n$ e convergent (⟺ are limită)

Ineg. lui Bernoulli: $(1+t)^n > 1+nt$, $(\forall)\, t \in (-1,0)\cup(0,\infty)$, $(\forall)\, n \in \mathbb{N}^*$

y_n

Cr. Rap: $(x_n)_{n\ge 1} \subset (0,\infty)$, $\frac{x_{n+1}}{x_n} \to l < 1 \Rightarrow x_n \to 0$.

$\cancel{\frac{x_n}{x_{n+1}} = \left(\frac{n+1}{n}\right)^n \cdot}$ $\qquad y_n := \left(1+\frac{1}{n}\right)^{n+1}$

$$\frac{y_n}{y_{n+1}} = \left(\frac{n+1}{n}\right)^{n+1} \cdot \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+2}} \Leftrightarrow \frac{1}{\left(\frac{n+2}{n+1}\right)^{n+2}} \Leftrightarrow \frac{1}{\left(\frac{n+2}{n+1}\right)^{n+1}} \cdot \frac{n+2}{n+1}$$

$$= \left(\frac{n+1}{n}\right)^{n+1} \cdot \frac{(n+1)^{n+1}}{(n+2)^{n+1}} \cdot \frac{(n+1)}{(n+2)} = \left(\frac{(n+1)^2}{n^2+2n}\right)^{n+1} \cdot \frac{n+1}{n+2} =$$

$$= \left(1+\frac{1}{n^2+2n}\right)^{n+1} \cdot \frac{n+1}{n+2} > \left(1+\frac{n+1}{n^2+2n}\right) \cdot \frac{n+1}{n+2} = \frac{[n(n+2)+n+1]\cdot n+1}{n(n+2)^2} =$$

$$= \frac{n^2(n+2)+n^2+n+1}{n(n+2)^2} = \frac{n(n+2)^2+1}{n(n+2)^2} > 1.$$

$(y_n)_n$ e s. desc. | ⟹ e conv.
$(y_n)_n$ mărg. inf. |
SRP

$\frac{x_{n+1}}{x_n} = \dots > \dots = 1.$

$(x_n)_n$ s. cresc. $\quad x_n < y_n \le y_1 = 4$

$$0 < y_n - x_n = \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{n} \le \frac{y_n}{n} < \frac{y_1}{n} \to 0$$

$$\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \left[1+\frac{1}{n} - 1\right] = \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{n}$$

1.22.

a) $x_n = \dfrac{\sin(n+1)}{n+1}, \quad n \geq 1$

$|x_n| = \left|\dfrac{\sin(n+1)}{n+1}\right| \leq \dfrac{1}{n+1} = y_n, \quad n \geq 1$ $\left|\ \xRightarrow{\text{Cr. Maj}} x_n \to 0.\right.$

$y_n \to 0$

$\dfrac{1}{n+1}$

$\left(\dfrac{2}{2n+2}\right) < \dfrac{2}{2n+1}$

b) $x_n := \dfrac{1+\cos n^2}{2n+1}, \quad n \geq 1$

$|x_n| = \left|\dfrac{1+\cos n^2}{2n+1}\right| \leq \dfrac{2}{2n+1} =: y_n, \quad n \geq 1$ $\left|\ \xRightarrow{\text{Cr. Maj}} x_n \to 0.\right.$

$y_n \to 0$

c) $x_n = \dfrac{n \cos n}{n^2+2}, \quad n \geq 1$

$|x_n| = \left|\dfrac{n \cos n}{n^2+2}\right| \leq \dfrac{n}{n^2+2} =: y_n \quad \geq 1$ $\left|\ \xRightarrow{\text{Cr. Maj}} x_n \to 0.\right.$

$y_n := \dfrac{n}{n^2+2} = \dfrac{1}{n\left(1+\frac{2}{n^2}\right)} \to 0$

1.23. a) $(a_n)_n$ şir de nr. reale monoton şi nem., at. $\lim\limits_{n\to\infty} \dfrac{1}{a_n} \to 0$

Fie $(a_n)_n$ cresc. şi nem. Din faptul că $(a_n)_n$ nem. $\Rightarrow (\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N}^*)\left(a_{n_\varepsilon} > \dfrac{1}{\varepsilon}\right)$

$\left[(\forall n \geq n_\varepsilon)(|x_n - l| < \varepsilon), (\exists l \in \mathbb{R})\right]$

(De la un rang, şirul are doar term. poz.)

$\Rightarrow a_{n_\varepsilon} > 0.$

Din (a_n) ~~nem~~ cresc $\Rightarrow (\forall n \geq n_\varepsilon)(a_n > a_{n_\varepsilon})$

$\xRightarrow{(1) şi (2)} (\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N}^*)(\forall n \geq n_\varepsilon)\left(a_n > \dfrac{1}{\varepsilon} \Leftrightarrow \dfrac{1}{a_n} < \varepsilon\right) \Leftrightarrow \lim\limits_{n\to\infty} \dfrac{1}{a_n} = 0.$

1.2.1. a) Fie $x_n := \frac{4n+3}{5n+7}, n \geq 1$.

Avem $\lim\limits_{x\to\infty} \frac{4n+3}{5n+7} = \frac{4}{5} \Leftrightarrow (\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N}^*)(\forall n \geq n_\varepsilon)\left(\left|x_n - \frac{4}{5}\right| \leq \varepsilon\right)$

Dar $\left|x_n - \frac{4}{5}\right| = \left|\frac{4n+3}{5n+7} - \frac{4}{5}\right| = \left|\frac{20n+15-20n-28}{5(5n+7)}\right| = \left|\frac{-13}{5(5n+7)}\right| < \varepsilon$

$\frac{13}{5(5n+7)} < \varepsilon \Leftrightarrow \frac{5(5n+7)}{13} > \frac{1}{\varepsilon} \Leftrightarrow 5(5n+7) > \frac{13}{\varepsilon} \Leftrightarrow 5n+7 > \frac{13}{5\varepsilon}$

$5n > \frac{13}{5\varepsilon} - 7 \Leftrightarrow n > \frac{13}{25\varepsilon} - \frac{7}{5}$.

Luăm $n_\varepsilon = \left[\frac{13}{25\varepsilon} - \frac{7}{5}\right] + 1$. $\left(m_\varepsilon = 1,\ \frac{13}{25\varepsilon} - \frac{7}{5} < 0\right)$ avem.

$(\forall)\, n \geq n_\varepsilon,\ n > \frac{13}{25\varepsilon} - \frac{7}{5} \Leftrightarrow \left|x_n - \frac{7}{5}\right| < \varepsilon$.

Deci $(\forall \varepsilon > 0)\left(\exists n_\varepsilon \in \mathbb{N}\ \left(n_\varepsilon = \left[\frac{13}{25\varepsilon} - \frac{7}{5}\right] + 1\right)\right)(\forall n \geq n_\varepsilon)\left(\left|x_n - \frac{4}{5}\right| < \varepsilon\right)$.

de unde $\lim\limits_{x\to\infty} \frac{4n+3}{5n+7} = \frac{4}{5}$.

b) Să se arate că şirul $a_n = \frac{2^n + (-2)^n}{2^n}$ nu are limită.

$a_n = \frac{2^n}{2^n} + \frac{(-1)^n \cancel{2^n}}{\cancel{2^n}} = 1 + (-1)^n$

$a_{2k} = 2 \Rightarrow \lim\limits_{x\to\infty} a_{2k} = 2$

$a_{2k+1} = 0 \Rightarrow \lim\limits_{x\to\infty} a_{2k+1} = 0$

Deci $(a_n)_n$ nu are limită, deoarece are 2 subşiruri cu limite distincte.

Petrulescu Mihai-Robert

c) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

$x_n := \sqrt[n]{n} = n^{\frac{1}{n}}$

Notăm $a_n = \sqrt[n]{n} - 1$, $a_n > 0$, $(\forall) n \geq 2$.

$\sqrt[n]{n} = 1 + a_n \Leftrightarrow n = (1+a_n)^n = 1 + C_n^1 a_n + C_n^2 a_n^2 + \dots + C_n^n a_n^n >$

$> C_n^2 a_n^2$, $(\forall) n \geq 2$

$0 < a_n < \sqrt{\frac{n}{C_n^2}} = \sqrt{\frac{2n}{(n-1)n}} = \sqrt{\frac{2}{n-1}}$

$\frac{n!}{(n-2)!\cdot 2!} = \frac{(n-1)n}{2}$ $\qquad$ $C_n^2 a_n^2 = \frac{(n-1)n}{2}\cdot\left(\sqrt[n]{n^2} - 2\sqrt[n]{n} + 1\right)$

$\sqrt[n]{n} - 1 < \frac{2}{n-1}$

$n = (1+a_n)^n > C_n^2 a_n^2$, $(\forall) n \geq 2$.

$2 = (1 + \sqrt[2]{2} - 1)^2 = 2$ ✓.

$C_n^2 a_n^2 < n$

~~Arătăm~~ $\sqrt[n]{n} \to 1 \overset{\text{not}}{\Rightarrow} a_n = \sqrt[n]{n} - 1$. $(|x_n - l| < \varepsilon)$.

Evident $\sqrt[n]{n} = a_n + 1 \Rightarrow n = (a_n+1)^n \overset{B.N.}{=} C_n^0 a_n^0 + C_n^1 a_n + C_n^2 a_n^2 + \dots$

$\geq C_n^2 a_n^2$ $(\forall n >= 2)$.

$\boxed{n > C_n^2 a_n^2} \Leftrightarrow \frac{n}{C_n^2} > a_n^2 \Leftrightarrow \boxed{0 < a_n^2 < \frac{n}{C_n^2}}$

$a_n^2, n=2 \Rightarrow (\sqrt{2}-1)^2 = 2 - 2\sqrt{2} + 1 = 3 - 2\sqrt{2} > 0$.

$0 < a_n < \sqrt{\frac{n}{C_n^2}} = \sqrt{\frac{n}{\frac{(n-1)n}{2}}} = \sqrt{\frac{2\cancel{n}}{\cancel{n}(n-1)}} = \sqrt{\frac{2}{n-1}} \to 0$

Cr. Maj.

$\Rightarrow \lim_n \sqrt{a_n} \to 0$

Cr. Cleștelui

$\lim_n \sqrt[n]{n} \to 1$.

123

b) $\lim_{n\to\infty} a_n = 0$, (b_n) mărg.

$(b_n)_n$ mărg $\Rightarrow \exists M>0$, $|b_n|<M$, $(\forall n\in\mathbb{N}^*)$

Dt $(a_n)_n$, $(\forall \varepsilon>0)(\exists n_\varepsilon\in\mathbb{N}^*)(\forall n>n_\varepsilon)\left(|a_n|<\frac{\varepsilon}{M}\right)$ (a_n conv la 0)

$|a_n-0|<\underset{\sim}{\varepsilon}\cdot\text{const}$.

$(\forall\varepsilon>0)(\exists n_\varepsilon\in\mathbb{N}^*)(\forall n\geq n_\varepsilon)\left(|a_nb_n|<M\cdot\frac{\varepsilon}{M}=\varepsilon\right)$.

$(a_nb_n)_n\to 0$.

a) (a_n) SRP, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=l$; cu $l\in[0,1)$

~~Cr. Raport~~ $\frac{a_{n+1}}{a_n}=l<1$, $(\forall l\in[0,1))\xrightarrow{\text{Cr. Rap}}$ ~~lim~~ $a_n\underset{n}{\to}0$.

$0\leq l<q<1$, $(-1,q)$ vecinătate a lui l.

$(\exists n_0\in\mathbb{N}^*)(\forall n\geq n_0)\left(\frac{a_{n+1}}{a_n}<q\right)$.

$\frac{n^k}{q^{+n}}$; $n^k\cdot q^n$; $q\in(-1,1)$ $\left\{\frac{1}{2},\frac{1}{3},\dots\frac{1}{k}\right\}$

$q^n\to 0$, $(\forall q\in(-1,1))$

$n^k\to$

$|q|<1 \Leftrightarrow |q|=\frac{1}{1+a}$, $a>0$

$n\geq k+1$.

$$<\frac{n^k}{a^{k+1}\cdot C_n^{k+1}}=\frac{n^k}{a^{k+1}\frac{n(n-1)\dots(n-k)}{k!}}$$

$$0\leq n^k|q|^n=\frac{n^k}{(1+a)^n}=\frac{n^k}{\cancel{C_n^0}\,1+C_n^1a+\dots+C_n^ka^k\underbrace{+\dots+a^nC_n^n}}$$

Detușleanu Mihai-Tiberiu

$1+3+5+\ldots+(2k-1)$, $k\geq 1$

$S_{2k}=1+2+\ldots+2k=\frac{2k(2k+1)}{2}$

$2+4+\ldots+2k=$

$1+3+5+\ldots+(2k-1)=$

$=1+2+3+\ldots+(2k-1)+2k-2-4-\ldots-2k$

$=\frac{2k(2k+1)}{2}-2\left(\frac{k(k+1)}{2}\right)$

$=k(2k+1)-k(k+1)$

$=k\cdot k=k^2$

$$\sum_{k=1}^{n}(2k-1)=2\sum_{k=1}^{n}k-n=2\cdot\frac{n(n+1)}{2}=n(n+1)-n=n^2$$

---

e) $x_n=\sqrt{n^2+1}-n$, $n\geq 1$.

$=n\sqrt{1+\frac{1}{n^2}}-n=n\left(\sqrt{1+\frac{1}{n^2}}-1\right)\rightarrow$.

$\sqrt{}+n$

$$\sqrt{n^2+1}-n=\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}=\frac{1}{\sqrt{n^2+1}+n}\rightarrow\frac{1}{\infty+\infty}=\frac{1}{\infty}=0.$$

---

f) $$x_n=\frac{\sqrt{n^2+n}-n}{\sqrt{n^2+1}-n}=\frac{\frac{n}{\sqrt{n^2+n}+n}}{\frac{1}{\sqrt{n^2+1}+n}}\rightarrow\frac{\frac{\infty}{\infty}}{\frac{1}{\infty}}=$$

$$=\frac{n}{\cancel{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}\cdot\cancel{n}\left(1+\sqrt{1+\frac{1}{n^2}}\right)=\frac{\infty}{\cancel{2}}\cdot\cancel{2}=\infty$$

g) $x_n=\sqrt[3]{n^2+1}-\sqrt[3]{n^2-1}$, $n\geq 1$.

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$$x_n=\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right)\left(n^2\cancel{+1}+\sqrt[3]{(n^2+1)(n^2-1)}+n^2\cancel{-1}\right)$$

$$=\frac{\cancel{n^2}+1-\cancel{n^2}+1}{\sqrt{n^2+1}+\sqrt{n^2-1}}\left(2n^2+\sqrt[3]{n^4-1}\right)$$

$$=\frac{4n^2+\sqrt[3]{n^4-1}}{n\sqrt{1+\frac{1}{n^2}}+n\sqrt{1-\frac{1}{n^2}}}\rightarrow\infty$$

$$\frac{n^2+1+1-n^2}{\sqrt[3]{(n^2+1)^2}+\sqrt[3]{(n^2+1)(n^2-1)}+\sqrt[3]{(n^2-1)^2}}=\frac{2}{n^{\frac{4}{3}}\left(\sqrt[3]{\left(1+\frac{1}{n^2}\right)^2}+\sqrt[3]{\left(1+\frac{1}{n^2}\right)\left(1-\frac{1}{n^2}\right)}+\dots\right)}$$

$$\Rightarrow \frac{2}{\infty\cdot\left(\sqrt[3]{1}+\dots\right)}=0.$$

1.2.7

a) $$a_n=\frac{3^n+5^n}{3^{n+1}+5^{n+1}}=\frac{3^{n+1}}{3^{n+1}+5^{n+1}}\cdot\frac{1}{3}+\frac{5^{n+1}}{3^{n+1}+5^{n+1}}\cdot\frac{1}{5}.$$

$$=\frac{1}{5}\cdot\frac{5^{n+1}+5\cdot3^n}{3^{n+1}+5^{n+1}}=\frac{1}{5}\cdot\frac{\cancel{5^{n+1}}\left(1+\underbrace{\left(\frac{3}{5}\right)^n}_{\to 0}\right)}{\cancel{5^{n+1}}\left(\underbrace{\left(\frac{3}{5}\right)^{n+1}}_{\to 0}+1\right)}\to\frac{1}{5}\cdot\frac{1+0}{1+0}=\frac{1}{5}$$

$$a_n=\frac{\alpha^n+\beta^n}{\alpha^{n+1}+\beta^{n+1}}$$

~~Dacă~~ $\alpha>\beta \Leftrightarrow a_n\to \cancel{\frac{\beta}{\alpha}}\ \frac{1}{\alpha}$

$$\frac{\alpha^n+\beta^n}{\alpha^{n+1}+\beta^{n+1}}=\frac{\alpha^n\left(1+\left(\frac{\beta}{\alpha}\right)^n\right)}{\alpha^{n+1}\left(1+\left(\frac{\beta}{\alpha}\right)^{n+1}\right)}\to\frac{1}{\alpha}\cdot\frac{1+0}{1+0}$$

$\alpha<\beta \Leftrightarrow a_n\to\frac{1}{\beta}$

$\alpha=\beta \Leftrightarrow \frac{2\alpha^n}{2\alpha^{n+1}}=\frac{2}{2\alpha}=\frac{1}{\alpha}\to\frac{1}{\alpha}.$

$$x_n=\cancel{n}\cdot\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\cdot\frac{2\pi}{\cancel{n}}\Rightarrow 1\cdot 2\pi=2\pi.$$

$$x_n=n\cdot\frac{tg(\dots)}{\frac{n\pi+1}{2n^2}}\cdot\frac{n\pi+1}{2n^2}\to\frac{\pi}{2}.$$

c) $$a_n=\frac{2^n+3\cdot4^n+5^n}{4^{n+1}+5^{n+1}}$$

$$=\frac{5^n\left(1+3\cdot\left(\frac{4}{5}\right)^n+\left(\frac{2}{5}\right)^n\right)}{5^{n+1}\left(1+\left(\frac{4}{5}\right)^{n+1}\right)}=$$

$$n^2\,\frac{\sin(\frac{\pi}{3n})}{\frac{\pi}{3n}}\cdot\left(\frac{\pi}{3n}\right)\cdot\frac{\arcsin(\dots)}{\left(\frac{2}{3n}\right)}\cdot\left(\frac{2}{3n}\right)$$

$$=\frac{2\pi\cdot n^2}{3n\cdot3n}\to\frac{2\pi}{9}=\frac{\pi}{9}$$

~~4.2.75.~~

$$\frac{(\ln n)^{10}}{n\sqrt[3]{n}} = \lim_{n\to\infty} \frac{(\ln n)^{10}}{n} \cdot \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} \to 0$$

$$\lim_{n\to\infty} \frac{[\ln(n)]^{10}}{n} = \lim_{n\to\infty} \ln(n)^9 \cdot \lim_{n\to\infty} \frac{\ln(n)}{n} \to 1 \text{ (L.Fund).}$$

$$\cancel{\lim_{n\to\infty}\left(1+\frac{\ln n}{n}-1\right)}$$

$$\lim \frac{\ln n}{n}, \quad a_n = \ln n \; ; \; b_n = n$$

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{\ln(n+1)-\ln(n)}{n+1-n} = \frac{\ln\left(\frac{n+1}{n}\right)}{1} = \ln\left(1+\frac{1}{n}\right) \to \ln(1)=0.$$

$$\lim_{n\to\infty} \ln(n)^9$$

$$a_n = (\ln n)^{10} \; ; \; b_n = n$$

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{\ln(n+1)^{10}-\ln(n)^{10}}{n+1-n} = \frac{\ln\left(\frac{n+1}{n}\right)^{10}}{1} = \ln\left(1+\frac{1}{n}\right)^{10} \to \ln(1)=0.$$

Ietcuban Mihai-Silviu

Obs $\sum_{n\geq 1} x_n$ (C) $\Rightarrow x_n \to 0$

$$\sum_{n\geq k} x_n \,(C) \sim \sum_{n\geq 1} x_n \,(C)$$

b) $\sum_{n\geq 1} x_n$ S.T.P. $\Rightarrow \sum_{n\geq 1} x_n$ are sumă și $\sum_{n=1}^{\infty} x_n \in [0,\infty]$

$\sqrt[n]{n} \to 1.$

$$n^{\frac{1}{n}} = \infty^{\frac{1}{\infty}} = \infty^{0}$$

$$x_n = n^3(\dots - \dots) = n^3 \cdot (-2) \cdot \sin\left(\frac{\frac{\pi}{n}+\frac{\pi}{n+1}}{2}\right)\sin\left(\frac{\frac{\pi}{n}-\frac{\pi}{n+1}}{2}\right)$$

$$\cos A - \cos B = -2\sin\frac{A-B}{2}\sin\frac{A+B}{2}.$$

$$\frac{\pi}{n} + \frac{\pi}{n+1} = \frac{2n\pi + \pi}{n(n+1)}$$

$$= n^3 \cdot (-2) \cdot \sin\frac{\pi(2n+1)}{n(n+1)} \cdot \sin\frac{\pi}{n(n+1)} = -2n^3 \cdot \frac{\sin(\dots)}{\frac{\pi(2n+1)}{n(n+1)}} \cdot \frac{\pi(2n+1)}{n(n+1)} \cdot \frac{\sin(\dots)}{\frac{\pi}{n(n+1)}} \cdot \frac{\pi}{n(n+1)}$$

$$= -2n^3 \cdot \frac{\pi(2n+1)}{n^2+n} \cdot \frac{\pi}{n^2+n}(\dots) \to -\infty$$

$$= -2\cancel{n^3} \cdot \frac{\sin\left(\frac{\pi(2n+1)}{2n(n+1)}\right)}{\frac{\pi(2n+1)}{2n(n+1)}} \cdot \frac{\sin\frac{\pi}{2n(n+1)}}{\frac{\pi}{2n(n+1)}} \cdot \frac{\pi^2(2n+1)n}{4\cancel{n^2}\cdot(n+1)^2}$$

$$\to -2 \cdot 1 \cdot 1 \cdot \frac{2\pi^2}{4} = \frac{-4\pi^2}{4} = -\pi^2.$$

1.2.9

a) $x_n := \left(\frac{n^2+n+1}{n^2-n+1}\right)^{\frac{n^2}{n+1}} = \left(1 + \frac{n^2+n+1}{n^2-n+1} - 1\right)^{\frac{n^2}{n+1}} = \left(1 + \frac{2n}{n^2-n+1}\right)^{\frac{n^2}{n+1}} =$

$\cancel{\frac{n^2+n+1}{n^2-n+1} = (n-1)}$

$$= \left[\left(1 + \frac{2n}{n^2-n+1}\right)^{\frac{n^2-n+1}{2n}}\right]^{\frac{2n}{n^2-n+1}\cdot\frac{n^2}{n+1}} = e^{\lim\limits_{x\to\infty}\frac{2n^3}{(n^2-n+1)(n+1)}}$$

$$= e^{\frac{2}{1}} = e^2$$

b) $x_n := n\ln\left(\frac{n+2}{n+1}\right) = 1 + n\ln\left(\frac{n+2}{n+1}\right) - 1$

$x_n \to \infty, \quad \left(1 + \frac{1}{x_n}\right)^{x_n} \to e$

$y_n \to 0, \quad (1+y_n)^{\frac{1}{y_n}} \to e$

$y_n \to 0, \quad \frac{\ln(1+y_n)}{y_n} \to 1$

$y_n \to 0 \quad \frac{(1+y_n)^r - 1}{y_n} \to r$

$y_n \to 0, \frac{a^{y_n}-1}{y_n} = \ln a.$

Petrulescu Mihai-Silviu

$x_n \to \infty,\ \left(1+\frac{1}{x_n}\right)^{x_n} \to e$

$y_n \to 0,\ (1+y_n)^{\frac{1}{y_n}} \to e$

$y_n \to 0,\ \frac{\ln(1+y_n)}{y_n} \to 1$

$y_n \to 0;\ \frac{(1+y_n)^{n}-1}{y_n} \to n$

$y_n \to 0,\ \frac{a^{y_n}-1}{y_n} \to \ln a$

b) $x_n := n \ln\left(\frac{n+2}{n+1}\right)$

$$x_n = n\ln\left(1+\frac{1}{n+1}\right) = n \cdot \frac{\ln\left(1+\frac{1}{n+1}\right)}{\frac{1}{n+1}} \cdot \frac{1}{n+1}$$

$$\to \frac{n}{n+1} \cdot \underbrace{\frac{\ln\left(1+\frac{1}{n+1}\right)}{\frac{1}{n+1}}}_{\to 1} = 1.$$

c) $x_n := n(\sqrt[n]{3}-\sqrt[n]{2})$

$$x_n := n \cdot \left(3^{\frac{1}{n}} - 2^{\frac{1}{n}}\right) = n \cdot 3^{\frac{1}{n}} - n \cdot 2^{\frac{1}{n}} = \left(\cancel{n} \cdot \frac{3^{\frac{1}{n}}}{\frac{1}{n}} \cdot \cancel{\frac{1}{n}}\right) - \left(\cancel{n} \cdot \frac{2^{\frac{1}{n}}}{\frac{1}{n}} \cdot \cancel{\frac{1}{n}}\right) =$$

$$= \frac{3^{\frac{1}{n}}-1}{\frac{1}{n}} + \cancel{\frac{1}{\frac{1}{n}}} - \left(\frac{2^{\frac{1}{n}}-1}{\frac{1}{n}} + \cancel{\frac{1}{\frac{1}{n}}}\right) = \ln 3 - \ln 2 = \ln\frac{3}{2}.$$

d) $x_n := n\left[1-\left(\frac{n^3+2}{n^3+1}\right)^3\right]$, $n \geq 1$.

$$x_n := n\left[1-\left(\frac{1}{n^3+1}+1\right)^3\right] = n \cdot \left(1+\frac{1}{n^3+1}\right)$$

$$\cancel{x_n := n - n\left(\frac{n^3+2}{n^3+1}\right)^3 = n - n\left(1+\frac{1}{n^3+1}\right) - 1}$$

$$x_n := n\left[1 - \frac{\left(1+\frac{1}{n^3+1}\right)^3-1}{\frac{1}{n^3+1}} + \frac{1}{\frac{1}{n^3+1}}\right] = n\left[n^3+2 - \frac{\left(1+\frac{1}{n^3+1}\right)^3-1}{\frac{1}{n^3+1}}\right] =$$

$$\to \infty(\infty^3+2-3) = \infty(\infty^3-1) = \infty.$$

$$n\left[1-\left(1+\frac{1}{n^3+1}\right)^3\right] = n - n \cdot \frac{\left(1+\frac{1}{n^3+1}\right)^3}{\frac{1}{n^3+1}} \cdot \frac{1}{n^3+1} = n - \frac{\left(1+\frac{1}{n^3+1}\right)^3}{\frac{1}{n^3+1}} \cdot \frac{n}{n^3+1}$$

$$= n - \frac{\left(1+\frac{1}{n^3+1}\right)^3-1}{\frac{1}{n^3+1}} \cdot \frac{n}{n^3+1} + \left(\frac{\left(1+\frac{1}{n^3+1}\right)^3-1}{\frac{1}{n^3+1}} + \frac{1}{\frac{1}{n^3+1}}\right) \cdot \frac{n}{n^3+1} = -(\dots)$$

$x_n = \frac{1}{n^k}$

$\frac{1}{n^k} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n^k \Leftrightarrow \sqrt[k]{\frac{1}{\varepsilon}} < n \Leftrightarrow \frac{1}{\sqrt[k]{\varepsilon}} < n$

$n_\varepsilon = \left[\frac{1}{\sqrt[k]{\varepsilon}}\right] + 1.$

$x_n = \frac{1}{n!} \Rightarrow 0 < x_n \leq \frac{1}{n}$

Cr. Majorării: $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}, \exists l \in \mathbb{R}$ a.î. $|x_n - l| \leq \overbrace{y_n}^{n\geq k}, y_n \to 0 \Rightarrow x_n \to l$

$(x_n)_{n\geq 1}, (y_n)_{n\geq 1}, \exists l \in \mathbb{R}$ a.î. $|x_n - l| \leq y_n, \forall n \geq k$ $\overset{\text{Dacă}}{\text{și}}$ $y_n \to 0 \Longrightarrow x_n \to l$

$$\left.\begin{array}{l} x_n = \frac{1}{n!} \Leftrightarrow 0 \leq x_n \leq \frac{1}{n}, \ n \geq 1 \\ y = \frac{1}{n} \to 0 \end{array}\right| \overset{\text{Cr. Maj}}{\Longrightarrow} x_n \to 0.$$

Ex 4: $x := \underbrace{(-1)^n \frac{1}{n}}_{\text{șir alternant}}, n \geq 1$

$$\left.\begin{array}{l} |x_n| = \left|(-1)^n \frac{1}{n}\right| = \left|\frac{1}{n}\right| \to 0 \\ x_n \to 0 \Leftrightarrow |x_n| \to 0 \end{array}\right| \Rightarrow x_n \to 0.$$

$x_n := \frac{\sin n}{n}$; $-1 \leq \sin n \leq 1$

$$\left.\begin{array}{l} |x_n| = \left|\frac{\sin n}{n}\right| \leq \frac{1}{n}, n \geq 1 \\ y_n := \frac{1}{n} \to 0 \end{array}\right| \overset{\text{Cr. Maj}}{\Rightarrow} x_n \to 0.$$

$x_n := \frac{2n+1}{3n+2}, n \geq 1.$

$\left|x_n - \frac{2}{3}\right| = \left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| = \frac{3(2n+1) - 2(3n+2)}{3(3n+2)} = \frac{6n+3-6n-4}{3(3n+2)} = \left|\frac{-1}{3(3n+2)}\right| \leq \frac{1}{9n}$

$$\left.\begin{array}{l} \leq \frac{1}{n} \\ y_n := \frac{1}{n} \to 0 \end{array}\right| \overset{\text{Cr. Maj}}{\Longrightarrow} x_n \to \frac{2}{3}.$$

Cr. Maj: d) $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subset \mathbb{R}, l \in \mathbb{R}$ cu $|x_n - l| \leq y_n, n \geq k$ și $y_n \to 0 \Rightarrow x_n \to l$.

Cr. Rap: $(x_n)_{n\geq 1} \subset (0,\infty)$ cu $\frac{x_{n+1}}{x_n} \to l < 1$, $x_n \to 0$

Th. lui Weierstrass: $(x_n)_{n\geq 1} \subset \mathbb{R}$ șir mon și mărg $\Rightarrow (x_n)_{n\geq 1}$ (C).

Th. lui Cauchy $(x_n)_{n\geq 1} \subset \mathbb{R}$

$(x_n)_{n\geq 1}$ (C) $\Leftrightarrow$ șir Cauchy

Ce e un șir: o funcție $f: \mathbb{N}^* \to \mathbb{R}$ notat $(x_n)_{n\in\mathbb{N}^*} \subseteq \mathbb{R}$, $(x_n)_{n\geq 1}$.

mărginire inferioară: $(\exists \alpha \in \mathbb{R})(\forall n \geq 1)(\alpha \leq x_n)$ ✓

sup $(\exists \alpha \in \mathbb{R})(\forall n \geq 1)(x_n \leq \alpha)$

șir conv: $(x_n)_{n\geq 1} \subset \mathbb{R}$ șir conv $\Leftrightarrow (\exists l \in \mathbb{R})(\forall \varepsilon > 0)$ ~~$(\exists n_\varepsilon \geq 1)$~~ $(\exists n_\varepsilon \in \mathbb{N}^*)(\forall n \geq n_\varepsilon)$
$(|x_n - l| < \varepsilon)$

$x_n = \frac{1}{n}, n \geq 1$

$(\exists l \in \mathbb{R})(\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N}^*)(\forall n \geq n_\varepsilon)(|x_n - l| < \varepsilon) \Leftrightarrow \left|\frac{1}{n} - l\right| < \varepsilon \Big|_{l=0} \Leftrightarrow \frac{1}{n} < \varepsilon \Leftrightarrow$

$\Leftrightarrow \frac{1}{\varepsilon} < n \Leftrightarrow n_\varepsilon = \left[\frac{1}{\varepsilon}\right] + 1$

$n \geq n_\varepsilon \Rightarrow \frac{1}{\varepsilon} < n \Leftrightarrow |x_n - 0| < \varepsilon \underset{def}{\Rightarrow} (x_n)_{n\geq 1}$ conv.

b) $x_n \to l \Leftrightarrow |x_n - l| \to 0$
$l = 0$, $|x_n| \to 0$ $\Big| \Rightarrow |x_n| \to 0$

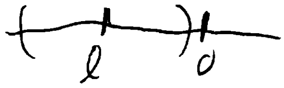
Ex: $x_n = \frac{1}{2^n}, n \geq 1$, $l > 0$.

$|x_n - l| < \varepsilon \Leftrightarrow \frac{1}{2^n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < 2^n \Leftrightarrow \log_2\left(\frac{1}{\varepsilon}\right) < n$

$\Leftrightarrow$ ~~$\log_2 \varepsilon < n$~~ $n_\varepsilon = \left[\log_2 \frac{1}{\varepsilon}\right] + 1$, $n \geq n_\varepsilon \Rightarrow n \geq \log_2 \frac{1}{\varepsilon} \Leftrightarrow \left|\frac{1}{2^n} - 0\right| < \varepsilon$.

Cr. Maj, $(x_n)_{n\ge 1}, (y_n)_{n\ge 1}$, $\exists\, l \in \mathbb{R}$ a.î. $|x_n - l| \le y_n$, $n \ge k$
Dacă $y_n \to 0$ | $\Rightarrow x_n \to l$.

Obs: Adăugarea/scoaterea unui nr. finit de termeni din șir nu-i afectează convergența.



Ex 7: $x = \frac{2n^2-3n+1}{5n^2+2n-1} = \frac{\cancel{n^2}\left(2-\frac{3}{n}+\frac{1}{n^2}\right)}{\cancel{n^2}\left(5+\frac{2}{n}-\frac{1}{n^2}\right)} \xrightarrow{\text{th. alg. cu șir. conv}} \frac{2-0+0}{5+0-0} = \frac{2}{5}$.

Ex 8. $x = \frac{2n-1}{n^2-n+1} = \frac{n\left(2-\frac{1}{n}\right)}{n^2\left(1-\frac{1}{n}+\frac{1}{n^2}\right)} \longrightarrow \frac{2-0}{\infty(1-0+0)} = \frac{2}{\infty} = 0$.

Cr. Maj. Reload $(x_n)_{n\ge 1}, (y_n)_{n\ge 1} \subset \mathbb{R}$, $x_n \le y_n$, $n \ge k$

1) $y_n \to -\infty \Rightarrow x_n \to -\infty$

2) $x_n \to \infty \Rightarrow y_n \to \infty$

$x_n := n + (-1)^n$, $n \ge 1$

$x_n \ge n - 1 =: y_n$
$y_n \to \infty$ | $\Rightarrow x_n \to \infty$

$4n^2+5n+3 \ge n$
$n \to \infty$ | $\Rightarrow x_n \to \infty$

$\frac{n^2-n+1}{2n^2+3n+3} \overset{\infty/\infty}{\Rightarrow} \frac{1}{2}$

$\frac{2n+1}{n^2-n+1} \overset{\infty/\infty}{=} \frac{2+0}{\infty(1-0+0)} = 0$.

---

a) $x_n := \frac{n^2-2n+5}{3n-4n^2+1} \overset{\infty/\infty}{=} \frac{n^2\left(1-\frac{2}{n}+\frac{5}{n^2}\right)}{n^2\left(-4+\frac{3}{n}+\frac{1}{n^2}\right)} \to \frac{1-0+0}{-4+0+0} = \frac{-1}{4}$

b) $x_n := \frac{2n-1}{n^2+n+1} \to 0$

c) $x_n = \frac{-n^2+2n^3+2}{2n^2-n+3} = \infty$

d) $x_n = \sqrt{n^2+n} - n \overset{\infty-\infty}{=} n\sqrt{1+\frac{1}{n}} - n$

$= \frac{\cancel{n^2}+n-\cancel{n^2}}{\sqrt{n^2+n}+n} = \frac{n}{\cancel{n}\left(\sqrt{1+\frac{1}{n}}+1\right)} \to \frac{1}{2}$

Tetulescu Mihai-Iulian

e) $x_n = \sqrt{n^2+2} - \sqrt{n^2+1} = \frac{n^2+2-n^2-1}{\sqrt{n^2+2}+\sqrt{n^2+1}} = \frac{1}{\sqrt{n^2+2}+\sqrt{n^2+1}} \to \frac{1}{\infty} = 0.$

f) $x_n = \frac{2^n+3^n}{2^{n+1}+3^{n+1}} = \frac{3^n\left(\left(\frac{2}{3}\right)^n+1\right)}{3^{n+1}\left(\left(\frac{2}{3}\right)^{n+1}+1\right)} = \frac{1\left(1+\left(\frac{2}{3}\right)^n\right)}{3\left(1+\left(\frac{2}{3}\right)^{n+1}\right)} \to \frac{1(1+0)}{3(1+0)} = \frac{1}{3}.$

g) $\frac{2^n-3\cdot 4^n+5^n}{4^{n+1}+5^{n+1}} = \frac{5^n\left(1-3\cdot\left(\frac{4}{5}\right)^n+\left(\frac{2}{5}\right)^n\right)}{5^{n+1}\left(1+\left(\frac{4}{5}\right)^{n+1}\right)} \Rightarrow \frac{1(1-0+0)}{5(1+0)} = \frac{1}{5}$

$x_n = \frac{\sin(n+1)}{n+2}$

$|x_n| = \left|\frac{\sin(n+1)}{n+2}\right| \leq \frac{1}{n+2} < \frac{1}{n} =: y_n \quad \Big| \xrightarrow{\text{Cr. Maj. Rel.}} x_n \to 0.$

$y_n = \frac{1}{n} \to 0$

$x_n = \frac{n\cos n}{n^2+1}$

$|x_n| = \left|\frac{n\cos n}{n^2+1}\right| \leq \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} =: y_n \quad \Big| \xrightarrow{\text{Cr. Maj. Rel}} x_n \to 0.$

$y_n = \frac{1}{n} \to 0$

$x_n = \left(1+\frac{1}{n}\right)^n \leq$

$1+\frac{1}{n} = \frac{n+1}{n}$

$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} \Leftrightarrow \left(\frac{n+1}{n}\right)^n < \left(\frac{n+2}{n+1}\right)^{n+1}$

$(n+1)^{2n+1} < n^n\cdot(n+2)^{n+1}$

$\frac{a^{y_n}-1}{y_n} \to \ln a$

$\frac{a^{y_n}-1}{y_n} \to \ln a.$

Th. Weierstrass:

~~$\ln(1+y_n$~~

$\left(1+\frac{1}{x_n}\right)^{x_n} \Longrightarrow e$

$(1+y_n)^{\frac{1}{y_n}} = e \;\Big|\ln$

$(1+y_n)^{\frac{1}{y_n}} \to e$, $\frac{1}{y_n} \to \infty$

$\frac{1}{y_n}\ln(1+y_n) = \log_e e = 1 \Rightarrow$