<u>Temă - Seminar 7</u> Ietculescu Mihai-Lidoiu

1. Să se ortogonalizeze și apoi ortonormeze:

b) $B = \{\underbrace{(-1,2,1)}_{\vec{x_1}}, \underbrace{(3,0,1)}_{\vec{x_2}}, \underbrace{(2,2,0)}_{\vec{x_3}}\} \subset \mathbb{R}^3$

$D = \begin{vmatrix} -1 & 3 & 2 \\ 2 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} = 4+6+2 = 12 \neq 0 \Rightarrow B$ l.i.

$\Rightarrow$ deci se poate ortogonaliza.

$B^{\perp} = \{\vec{y_1}, \vec{y_2}, \vec{y_3}\}$

- $\vec{y_1} = \vec{x_1} = (-1,2,1)$

$\vec{y_2} = \vec{x_2} + \alpha \vec{y_1}$

$\vec{y_3} = \vec{x_3} + \beta_1 \vec{y_1} + \beta_2 \vec{y_2}$

- $\vec{y_2} = \vec{x_2} + \alpha \vec{y_1} \mid \cdot \langle \vec{y_1} \rangle$

$\langle \vec{y_2}, \vec{y_1} \rangle = \langle \vec{x_2}, \vec{y_1} \rangle + \alpha \langle \vec{y_1}, \vec{y_1} \rangle$

$\langle \vec{y_2}, \vec{y_1} \rangle = 0$

$\langle \vec{x_2}, \vec{y_1} \rangle = 3 \cdot (-1) + 0 \cdot 2 + 1 \cdot 1 = -2$

$\langle \vec{y_1}, \vec{y_1} \rangle = (-1)\cdot(-1) + 2 \cdot 2 + 1 \cdot 1 = 6. \Rightarrow 0 = -2 + 6\alpha \Rightarrow \alpha = \frac{1}{3}$

$\Rightarrow \vec{y_2} = (3,0,1) + \frac{1}{3}(-1,2,1) = \left(3 - \frac{1}{3}, 0 + \frac{2}{3}, 1 + \frac{1}{3}\right) = \left(\frac{8}{3}, \frac{2}{3}, \frac{4}{3}\right).$

$\vec{y_3} = \vec{x_3} + \beta_1 \vec{y_1} + \beta_2 \vec{y_2} \mid \cdot \langle \vec{y_1} \rangle$

$\langle \vec{y_3}, \vec{y_1} \rangle = \langle \vec{x_3}, \vec{y_1} \rangle + \beta_1 \langle \vec{y_1}, \vec{y_1} \rangle + \beta_2 \langle \vec{y_2}, \vec{y_1} \rangle.$

$\langle \vec{y_3}, \vec{y_1} \rangle = 0; \ \langle \vec{y_2}, \vec{y_1} \rangle = 0.$

$\langle \vec{x_3}, \vec{y_1} \rangle = 2 \cdot (-1) + 2 \cdot 2 + 0 \cdot 1 = 2 \Rightarrow 0 = 2 + 6\beta_1 \Rightarrow \beta_1 = \frac{-1}{3}.$

- $y_3 = \vec{x_3} + \beta_1 \vec{y_1} + \beta_2 \vec{y_2} \mid \cdot \langle \vec{y_2} \rangle$

$\langle \vec{y_3}, \vec{y_2} \rangle = \langle \vec{x_3}, \vec{y_2} \rangle + \beta_1 \langle \vec{y_1}, \vec{y_2} \rangle + \beta_2 \langle \vec{y_2}, \vec{y_2} \rangle.$

$\langle \vec{y_3}, \vec{y_2} \rangle = 0; \ \langle \vec{y_1}, \vec{y_2} \rangle = 0.$

$\langle \vec{x_3}, \vec{y_2} \rangle = 2 \cdot \frac{8}{3} + 2 \cdot \frac{2}{3} + 0 \cdot \frac{4}{3} = \frac{20}{3}$

$\langle \vec{y_2}, \vec{y_2} \rangle = \frac{8}{3} \cdot \frac{8}{3} + \frac{2}{3} \cdot \frac{2}{3} + \frac{4}{3} \cdot \frac{4}{3} = \frac{64}{9} + \frac{4}{9} + \frac{16}{9} = \frac{84}{9} = \frac{28}{3}$

$\Rightarrow 0 = \frac{20}{3} + \frac{84}{9}\beta_2 \Rightarrow \frac{84}{9}\beta_2 = \frac{-20}{3} \Rightarrow \beta_2 = \frac{-20}{3} \cdot \frac{9}{84} = \frac{-60}{84} = \frac{-5}{7}$

- $\vec{y_3} = (2,2,0) + \left(\frac{-1}{3}\right)(-1,2,1) + \left(\frac{-5}{7}\right)\left(\frac{8}{3}, \frac{2}{3}, \frac{4}{3}\right) = \left(2 + \frac{1}{3} - \frac{5}{7} \cdot \frac{8}{3}, 2 - \frac{2}{3} - \frac{5}{7} \cdot \frac{2}{3},\right.$

$\left. 0 - \frac{1}{3} - \frac{5}{7} \cdot \frac{4}{3}\right) = \left(\frac{3}{7}, \frac{6}{7}, \frac{-9}{7}\right)$

$B^{\perp} = \{(-1,2,1), (\frac{8}{3},\frac{2}{3},\frac{4}{3}), (\frac{3}{7},\frac{6}{7},\frac{-9}{7})\}$

$B^{\perp}_1 = \{\frac{\vec{v_1}}{\|\vec{v_1}\|}, \frac{\vec{v_2}}{\|\vec{v_2}\|}, \frac{\vec{v_3}}{\|\vec{v_3}\|}\}$

$\|\vec{v_1}\| = \sqrt{(-1)^2+2^2+1^2} = \sqrt{1+4+1} = \sqrt{6}$;

$\|\vec{v_2}\| = \sqrt{(\frac{8}{3})^2+(\frac{2}{3})^2+(\frac{4}{3})^2} = \sqrt{\frac{64}{9}+\frac{4}{9}+\frac{16}{9}} = \sqrt{\frac{84}{9}} = \frac{2\sqrt{21}}{3}$

$\|\vec{v_3}\| = \sqrt{(\frac{3}{7})^2+(\frac{6}{7})^2+(\frac{-9}{7})^2} = \sqrt{\frac{9}{49}+\frac{36}{49}+\frac{81}{49}} = \sqrt{\frac{126}{49}} = \frac{3\sqrt{14}}{7}$

2. a) Să se verifice dacă vectorii $\vec{AB}$ și $\vec{DC}$ sunt perpendiculari, iar dacă nu sunt să se determine măsura unghiului dintre ei.

b) Să se verifice dacă punctele A, B, C sunt coliniare, iar în caz contrar să se determine aria ΔABC.

c) Să se verifice dacă punctele A, B, C, D sunt coplanare, iar în caz contrar să se afle volumul paralelipipedului determinat de aceste patru puncte.

d) Să se afle înălțimea din A a ΔABC

e) Să se afle înălțimea tetraedrului determinat de punctele: A, B, C, D.

1) $A(1,-1,0)$; $B(3,2,1)$; $C(0,1,1)$; $D(-1,1,2)$;

a) $\vec{AB} = (x_B - x_A, y_B - y_A, z_B - z_A) = (3-1, 2+1, 1-0) = (2,3,1)$.

$\vec{DC} = (x_C - x_D, y_C - y_D, z_C - z_D) = (0+1, 1-1, 1-2) = (1,0,-1)$.

$\langle \vec{AB}, \vec{DC} \rangle = 2\cdot 1 + 3\cdot 0 + 1\cdot(-1) = 2-1 = 1 \neq 0 \Rightarrow \vec{AB} \not\perp \vec{DC}$

$\cos(\vec{AB}, \vec{DC}) = \frac{\langle \vec{AB}, \vec{DC} \rangle}{\|\vec{AB}\| \cdot \|\vec{DC}\|}$

$$\left.\begin{array}{l} \|\vec{AB}\| = \sqrt{2^2+3^2+1^2} = \sqrt{4+9+1} = \sqrt{14} \\ \|\vec{DC}\| = \sqrt{1^2+0^2+(-1)^2} = \sqrt{1+1} = \sqrt{2} \end{array}\right\} \Rightarrow \cos(\vec{AB}, \vec{DC}) = \frac{1}{\sqrt{14}\cdot\sqrt{2}} = \frac{1}{\sqrt{28}} = \frac{\sqrt{28}}{28}$$

b) Verificăm dacă:

$\vec{AB} \times \vec{AC} = 0$

$\vec{AC} = (x_C - x_A, y_C - y_A, z_C - z_A) = (0-1, 1+1, 1-0) = (-1,2,1)$.

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ -1 & 2 & 1 \end{vmatrix} = 3\vec{i} + 4\vec{k} - \vec{j} + 3\vec{k} - 2\vec{i} - 2\vec{j} = \vec{i} + 7\vec{k} - 3\vec{j} \neq 0.$$

$\Rightarrow A, B, C$ nu sunt coliniare.

$$A_{ABC} = \frac{\|\vec{AB} \times \vec{AC}\|}{2} = \frac{\sqrt{1^2+7^2+(-3)^2}}{2} = \frac{\sqrt{1+49+9}}{2} = \frac{\sqrt{59}}{2}$$

c) Calculăm produsul mixt al $\vec{AB}, \vec{AC}, \vec{AD}$

$\vec{AB} = (2,3,1)$

$\vec{AC} = (-1,2,1)$

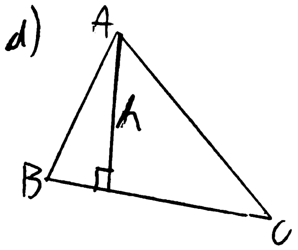
$\vec{AD} = (x_B - x_A, y_D - y_A, z_D - z_A) = (-1-1, 1+1, 2-0) = (-2,2,2)$.

$(\vec{AB}, \vec{AC}, \vec{AD}) = \begin{vmatrix} 2 & -1 & -2 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 8-6-2+4-4+6 = 6 \neq 0$.

$\Rightarrow A, B, C, D$ nu sunt coplanare.

$V = |(\vec{AB}, \vec{AC}, \vec{AD})| = |6| = 6$.

d)



Calculăm A_{ABC} în 2 moduri:

$A = \frac{BC \cdot h}{2}$

$A = \frac{\|\vec{AB} \times \vec{AC}\|}{2} = \frac{\sqrt{59}}{2}$ (din b))

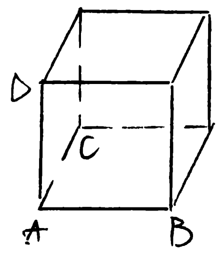
$\vec{BC} = (0-3, 1-2, 1-1) = (-3,-1,0)$.

$BC = \sqrt{(-3)^2 + (-1)^2 + 0^2} = \sqrt{9+1} = \sqrt{10}$

$\Rightarrow A = \frac{BC \cdot h}{2} \Leftrightarrow \frac{\sqrt{59}}{2} = \frac{\sqrt{10} \cdot h}{2} \Rightarrow \sqrt{59} = \sqrt{10} h \Rightarrow h = \frac{\sqrt{59}}{\sqrt{10}} = \frac{\sqrt{10} \cdot \sqrt{59}}{10}$

e) Calculăm volumul în 2 moduri:

$\Rightarrow V_t = \frac{V_p}{6}$; $V_t = \frac{A_b \cdot h}{3}$



$A_b = A_{ABC} = \frac{\sqrt{59}}{2}$

$(\vec{AB}, \vec{AC}, \vec{AD}) = 6$ (din c))

$V_t = \frac{|6|}{6} = 1 \Leftrightarrow 1 = \frac{\frac{\sqrt{59}}{2} \cdot h}{3} \Rightarrow \frac{\sqrt{59}}{2} h = 3$; $h = \frac{6}{\sqrt{59}} = \frac{6\sqrt{59}}{59}$.

2) $A(0,1,1)$, $B(1,1,3)$, $C(-1,0,2)$, $D(2,1,0)$.

a) $\vec{AB} = (1-0, 1-1, 3-1) = (1,0,2)$.

$\vec{DC} = (-1-2, 0-1, 2-0) = (-3,-1,2)$.

$\langle \vec{AB}, \vec{DC} \rangle = 1 \cdot (-3) + 0 \cdot (-1) + 2 \cdot 2 = 1 \neq 0 \Rightarrow \vec{AB} \perp \vec{DC}$.

$\cos(\vec{AB}, \vec{DC}) = \frac{\langle \vec{AB}, \vec{DC} \rangle}{\|\vec{AB}\| \cdot \|\vec{DC}\|}$

$\left.\begin{array}{l} \|\vec{AB}\| = \sqrt{1^2+0^2+2^2} = \sqrt{5} \\ \|\vec{DC}\| = \sqrt{(-3)^2+(-1)^2+2^2} = \sqrt{9+1+4} = \sqrt{14} \end{array}\right\} \Rightarrow \cos(\vec{AB}, \vec{DC}) = \frac{1}{\sqrt{5} \cdot \sqrt{14}} = \frac{1}{\sqrt{70}} = \frac{\sqrt{70}}{70}$.

b) $\vec{AC} = (-1-0,\ 0-1,\ 2-1) = (-1,-1,1)$.

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ -1 & -1 & 1 \end{vmatrix} = -\vec{k} - 2\vec{j} + 2\vec{i} - \vec{j} = 2\vec{i} - 3\vec{j} - \vec{k} \neq 0.$$

$\Rightarrow A, B, C$ nu sunt coliniare.

$$A_{ABC} = \frac{\|\vec{AB} \times \vec{AC}\|}{2} = \frac{\sqrt{2^2+(-3)^2+(-1)^2}}{2} = \frac{\sqrt{4+9+1}}{2} = \frac{\sqrt{14}}{2}.$$

c) $\vec{AD} = (2-0,\ 1-1,\ 0-1) = (2,0,-1)$.

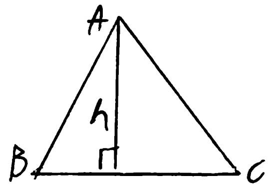
$\vec{AB} = (1,0,2)$, $\vec{AC} = (-1,-1,1)$.

$$(\vec{AB}, \vec{AC}, \vec{AD}) = \begin{vmatrix} 1 & -1 & 2 \\ 0 & -1 & 0 \\ 2 & 1 & -1 \end{vmatrix} = 1 + 4 = 5 \neq 0.$$

$\Rightarrow A, B, C, D$ nu sunt coplanare.

$V = |(\vec{AB}, \vec{AC}, \vec{AD})| = |5| = 5$.

d) Calculăm A_{ABC} în 2 moduri:

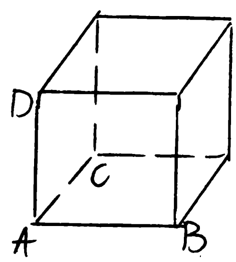


$$A = \frac{BC \cdot h}{2} = \frac{\|\vec{AB} \times \vec{AC}\|}{2} = \frac{\sqrt{14}}{2} \text{ (din b))}$$

$\vec{BC} = (-1-1,\ 0-1,\ 2-3) = (-2,-1,-1) \Rightarrow BC = \sqrt{4+1+1} = \sqrt{6} \Rightarrow \frac{\sqrt{14}}{2} = \frac{\sqrt{6}\,h}{2} \Rightarrow$

$\Rightarrow \sqrt{14} = \sqrt{6}\,h \Leftrightarrow h = \frac{\sqrt{14}}{\sqrt{6}} = \frac{\sqrt{14} \cdot \sqrt{6}}{6}$

e) Calculăm volumul în 2 moduri:



$V_T = \frac{V_P}{6} = \frac{A_b \cdot h}{3}$; $V_P = |5| = 5$ (din c))

$A_b = A_{ABC} = \frac{\sqrt{14}}{2}$

$$\Rightarrow V_T = \frac{5}{6} \Leftrightarrow \frac{5}{6} = \frac{\frac{\sqrt{14}}{2} \cdot h}{3} \Leftrightarrow 15 = \frac{6\sqrt{14}\,h}{2}$$

$$\Rightarrow 15 = 3\sqrt{14}\,h \Rightarrow h = \frac{15}{3\sqrt{14}} = \frac{5}{\sqrt{14}} = \frac{5\sqrt{14}}{14}$$