Seminar06 - Rezolvare

1 a) $x' = r x^2 \left(1 - \frac{x}{k}\right)$

$x(0) = x_0$

$\frac{dx}{dt} = r x^2 \left(1 - \frac{x}{k}\right)$

$\frac{dx}{x^2\left(1 - \frac{x}{k}\right)} = r\, dt$

$\int \frac{1}{x^2\left(1-\frac{x}{k}\right)} dx = r \int dt$

$\int \frac{1}{x^2 \cdot \frac{k-x}{k}} dx = r t + C$

$k \int \frac{1}{x^2(k-x)} dx = r t + C$

$\int \frac{1}{x^2(k-x)} dx$

$\frac{1}{x^2(k-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{k-x} \quad \Big| \cdot x^2 \quad \Big| \cdot k-x$

$\frac{1}{k-x} = Ax + B + \frac{Cx^2}{k-x} \quad (x = 0 \Rightarrow B = \frac{1}{k})$

$\frac{1}{x^2} = \frac{A(k-x)}{x} + \frac{B(k-x)}{x^2} + C \quad \left| x = k \Rightarrow C = \frac{1}{k^2} \right|$

$\frac{1}{x^2(k-x)} = \frac{Akx - Ax^2 + Bk - Bx + Cx^2}{x^2(k-x)} = \frac{(C-A)x^2 + (Ak-B)x + Bk}{x^2(k-x)}$

$C - A = 0$
$Ak - B = 0 \Rightarrow A = \frac{1}{k^2}$
$Bk = 1$

$\Rightarrow \int \frac{1}{x^2(k-x)} dx = \int \left(\frac{\frac{1}{k^2}}{x} + \frac{\frac{1}{k}}{x^2} + \frac{\frac{1}{k^2}}{k-x} \right) dx$

$= \frac{1}{k^2} \int \frac{1}{x} dx + \frac{1}{k} \int \frac{1}{x^2} dx + \frac{1}{k^2} \int \frac{1}{k-x} dx$

$= \frac{1}{k^2} \ln|x| - \frac{1}{kx} + \frac{1}{k^2} \ln(k-x)$

$\frac{1}{k^2} \ln|x| - \frac{1}{kx} + \frac{1}{k^2} \ln|k-x| = rt + C \quad \Big| \cdot k^2$

$\ln|x| - \frac{k}{x} + \ln(k-x) = k^2 r t + C$

$\ln x(k-x) - \frac{k}{x} = k^2 r t + C$

$x(0) = x_0$

$\ln x_0(k - x_0) - \frac{k}{x_0} = C$

$\Rightarrow x_{pc} = \ln(x(k-x)) - \frac{k}{x} = k^2 r t + \ln x_0 (k - x_0) - \frac{k}{x_0}$

① b) $x' = rx\left(1-\frac{x^2}{k}\right)$
$x(0) = x_0$

$$\frac{dx}{dt} = rx\left(1-\frac{x^2}{k}\right) = rx\cdot\frac{k-x^2}{k} = \frac{r}{k}x(k-x^2)$$

$$\frac{dx}{x(k-x^2)} = \frac{r}{k}dt$$

$$\int\frac{1}{x(k-x^2)}dx = \frac{r}{k}\int dt$$

$$\int\frac{1}{x(k-x^2)}dx$$

$$\frac{1}{x(k-x^2)} = \frac{1}{x(\sqrt{k}-x)(\sqrt{k}+x)}$$

$$= \frac{A}{x} + \frac{B}{\sqrt{k}-x} + \frac{C}{\sqrt{k}+x} \quad \Big| x \quad \Big| \sqrt{k}-x$$

$$\frac{1}{k-x^2} = A + \frac{Bx}{\sqrt{k}-x} + \frac{Cx}{\sqrt{k}+x} \quad \left(x=0,\ A=\frac{1}{k}\right)$$

$$\frac{1}{x(\sqrt{k}+x)} = \frac{A(\sqrt{k}+x)}{x} + B + \frac{C(\sqrt{k}-x)}{\sqrt{k}+x} \quad \left(x=\sqrt{k} \Rightarrow B = 1/2k\right)$$

$$\frac{1}{x(k-x^2)} = \frac{A}{x} + \frac{B}{\sqrt{k}-x} + \frac{C}{\sqrt{k}+x} \quad \Big| \cdot \sqrt{k}+x$$

$$\frac{1}{x(\sqrt{k}-x)} = \frac{A(\sqrt{k}+x)}{x} + \frac{B(\sqrt{k}+x)}{\sqrt{k}-x} + C \quad \left(x=-\sqrt{k} \Rightarrow C = -\frac{1}{2k}\right)$$

$$\int\frac{1}{x(k-x^2)}dx = \int\left(\frac{\frac{1}{k}}{x} + \frac{\frac{1}{2k}}{\sqrt{k}-x} - \frac{\frac{1}{2k}}{\sqrt{k}+x}\right)dx$$

$$= \frac{1}{k}\int\frac{1}{x}dx - \frac{1}{2k}\int\frac{-1}{\sqrt{k}-x}dx - \frac{1}{2k}\int\frac{1}{\sqrt{k}+x}dx$$

$$= \frac{1}{k}\ln x - \frac{1}{2k}\ln|\sqrt{k}-x| - \frac{1}{2k}\ln|\sqrt{k}+x|$$

$$= \frac{1}{2k}\left(\ln x^2 - \ln|\sqrt{k}-x| - \ln|\sqrt{k}+x|\right) = \frac{1}{2k}\ln\frac{x^2}{k-x^2}$$

$$\Rightarrow \frac{1}{2k}\ln\frac{x^2}{k-x^2} = \frac{r}{k}t + C \ \Big| \cdot k \Rightarrow \frac{1}{2}\ln\frac{x^2}{k-x^2} = rt + C$$