Ecuații diferențiale și cu derivate parțiale
Laborator 08
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Să se rezolve următoarele ecuații diferențiale liniare de ordin n omogene cu coeficienți constanți.

I. a) $x''' - 3x'' + 2x' = 0$, $x(0)=1$, $x'(0)=2$, $x''(0)=0$

$\boxed{b)}$ $2x''' - 3x'' + x' = 0$, $x(0)=-1$, $x'(0)=2$, $x''(0)=1$

$\boxed{c)}$ $x''' - 7x'' + 14x' - 8x = 0$, $x(0)=1$, $x'(0)=0$, $x''(0)=1$

d) $x'' - 4x' + 3x = 0$, $x(0)=2$, $x'(0)=4$

$\boxed{e)}$ $x''' - x' = 0$

Rezolvare

a) $x''' - 3x'' + 2x' = 0$, $x(0)=1$, $x'(0)=2$, $x''(0)=0$

$$\left.\begin{aligned} x &= e^{rt} \\ x' &= re^{rt} \\ x'' &= r^2e^{rt} \\ x''' &= r^3e^{rt} \end{aligned}\right\} \Rightarrow r^3e^{rt} - 3r^2e^{rt} + 2re^{rt} = 0 \quad |\cdot e^{rt}$$

$r^3 - 3r^2 + 2r = 0$

$r(r^2 - 3r + 2) = 0$

$r_1 = 0$

$r^2 - 3r + 2 = 0$

$\Delta = 9 - 8 = 1 \Rightarrow r_{2,3} = \dfrac{3 \pm 1}{2}$ $\begin{cases} r_2 = 2 \\ r_3 = 1 \end{cases}$

$e^{0\cdot t}, e^{2\cdot t}, e^{1\cdot t}$ - sist. fundam. de sol.

• $x(t) = c_1 + c_2e^{2t} + c_3e^t$; • $x'(t) = c_2 + 2e^{2t} + c_3e^t$

$\boxed{x(0) = c_1 + c_2 + c_3 = 1}$ $x'(0) = \boxed{2c_2 + c_3 = 2}$

• $x''(t) = 2c_2 + 2e^{2t} + c_3e^t$

$x''(0) = \boxed{4c_2 + c_3 = 0}$

$$\begin{cases} C_1 + C_2 + C_3 = 1 \\ 2C_2 + C_3 = 2 \\ 4C_2 + C_3 = 0 \end{cases} \Rightarrow 2C_2 = -2 \Leftrightarrow \boxed{C_2 = -1} \quad \Rightarrow \boxed{C_3 = 4}$$

$$C_1 = 1 - C_2 - C_3 = 1 + 1 - 4 = \boxed{-2 = C_1}$$

$$\underline{x_{PC} = -2 - e^{2t} + 4e^{t}}$$

d) $x'' - 4x' + 3x = 0$, $x(0) = 2$, $x'(0) = 4$

$$\left.\begin{array}{l} x = e^{rt} \\ x' = re^{rt} \\ x'' = r^2 e^{rt} \end{array}\right\} \Rightarrow r^2 e^{rt} - 4re^{rt} + 3e^{rt} = 0 \quad \Big| : e^{rt}$$

$$r^2 - 4r + 3 = 0$$

$$\Delta = 16 - 12 = 4 \Rightarrow r_{1,2} = \frac{4 \pm 2}{2} \quad \begin{cases} r_1 = 3 \\ r_2 = 1 \end{cases}$$

$e^{3 \cdot t}$, $e^{1 \cdot t}$ - sist. fundam. de sol.

$$x(t) = C_1 e^{3t} + C_2 e^{t} \Leftrightarrow x(0) = \boxed{C_1 + C_2 = 2}$$

$$x'(t) = C_1 \cdot 3e^{3t} + C_2 e^{t} \Leftrightarrow x'(0) = \boxed{3C_1 + C_2 = 4}$$

$$\begin{cases} C_1 + C_2 = 4 \\ 3C_1 + C_2 = 4 \end{cases}$$

$$2C_1 = 2 \Rightarrow C_1 = 1 \qquad \Rightarrow \underline{x_{PC} = e^{3t} + e^{t}}$$

$$C_2 = 1$$

II. a) $x'' + 2x' + x = 0$, $x(0) = 1$, $x'(0) = 3$

b) $x^{IV} - 5x'' + 4x = 0$

c) $x''' - 6x'' + 12x' - 8x = 0$

d) $x^{IV} - 2x'' + x = 0$

e) $x^{IV} + 2x''' + x'' = 0$

f) $x^{(6)} - x^{(5)} - 4x^{(4)} + 2x''' + 5x'' - x' - 2x = 0$

g) $x^{(7)} + 3x^{(6)} + 3x^{(5)} + x^{(4)} = 0$.

II a) $x''+2x'+x=0$, $x(0)=1$, $x'(0)=3$

$$\left.\begin{aligned} x&=e^{rt}\\ x'&=re^{rt}\\ x''&=r^2e^{rt}\end{aligned}\right\} \Rightarrow r^2e^{rt}+2re^{rt}+e^{rt}=0 \mid : e^{rt}$$

$$r^2+2r+1=0$$

$$(r+1)^2=0$$

$$\underline{r_1}=\underline{r_2}=\underline{-1}$$

e^{-1t}, te^{-1t} – sist. fundam. de sol

$$x(t)=c_1e^{-t}+c_2te^{-t} \Leftrightarrow x(0)=\boxed{c_1=1}$$

$$x'(t)=c_1(-e^{-t})+c_2(e^{-t}+t(-e^{-t})) \Leftrightarrow x'(0)=\boxed{-c_1+c_2=3}$$

$$c_2=3+c_1=3+1=4 \Rightarrow \boxed{c_2=4}$$

$$\underline{x_{PC}=e^{-t}+4te^{-t}}$$

II b) $x^{(6)}-x^{(5)}-4x^{(4)}+2x'''+5x''-x'-2x=0$

$$\left.\begin{aligned} x&=e^{rt}\\ x'&=re^{rt}\\ x''&=r^2e^{rt}\\ x'''&=r^3e^{rt}\\ x^{(4)}&=r^4e^{rt}\\ x^{(5)}&=r^5e^{rt}\\ x^{(6)}&=r^6e^{rt}\end{aligned}\right\} \Rightarrow r^6e^{rt}-r^5e^{rt}-4r^4e^{rt}+2r^3e^{rt}+5r^2e^{rt}-re^{rt}-2e^{rt}=0 \mid : e^{rt}$$

$$r^6-r^5-4r^4+2r^3+5r^2-r-2=0$$

- $r^6-r^5-4r^4+2r^3+5r^2-r-2 \mid r-1$

$$\begin{aligned}
&r^6-r^5-4r^4+2r^3+5r^2-r-2 \quad\Big|\ \underline{r-1}\\
&\underline{-r^6+r^5}\qquad\qquad\qquad\qquad\quad\ \ r^5-4r^3-2r^2+3r+2\\
&/\quad /\quad -4r^4+2r^3+5r^2-r-2\\
&\qquad\ \ \underline{+4r^4-4r^3}\\
&\qquad\quad /\quad -2r^3+5r^2-r-2\\
&\qquad\qquad\ \ \underline{+2r^3-2r^2}\\
&\qquad\qquad\quad /\quad 3r^2-r-2\\
&\qquad\qquad\qquad\ \underline{-3r^2+3r}\\
&\qquad\qquad\qquad\quad /\quad 2r-2\\
&\qquad\qquad\qquad\qquad\ \underline{-2r+2}\\
&\qquad\qquad\qquad\qquad\quad /\quad /
\end{aligned}$$

$$\begin{aligned}
&r^5-4r^3-2r^2+3r+2 \quad\Big|\ \underline{r-1}\\
&\underline{-r^5+r^4}\qquad\qquad\qquad\quad\ \ r^4+r^3-3r^2-5r-2\\
&/\quad r^4-4r^3-2r^2+3r+2\\
&\ \ \underline{-r^4+r^3}\\
&\quad /\quad -3r^3-2r^2+3r+2\\
&\qquad\ \ \underline{+3r^3-3r^2}\\
&\qquad\quad /\quad -5r^2+3r+2\\
&\qquad\qquad\ \underline{+5r^2-5r}\\
&\qquad\qquad\quad /\quad -2r+2\\
&\qquad\qquad\qquad\ \underline{+2r-2}\\
&\qquad\qquad\qquad\quad /\quad /
\end{aligned}$$

$$\begin{array}{l|l} \lambda^4+\lambda^3-3\lambda^2-5\lambda-2 & \lambda-2 \\ \hline -\lambda^4+2\lambda^3 & \lambda^3+3\lambda^2+3\lambda+1 \end{array}$$

$/ \quad 3\lambda^3-3\lambda^2-5\lambda-2$

$\quad -3\lambda^3+6\lambda^2$

$\quad / \quad 3\lambda^2-5\lambda-2$

$\quad\quad -3\lambda^2+6\lambda$

$\quad\quad / \quad \lambda-2$

$\quad\quad\quad -\lambda+2$

$\quad\quad\quad / \quad /$

• $(\lambda-1)^2(\lambda-2)(\lambda+1)^3=0$

$\left.\begin{array}{l} \lambda_1=\lambda_2=1 \\ \lambda_3=2 \\ \lambda_4=\lambda_5=\lambda_6=-1 \end{array}\right\} \Rightarrow e^{1t}, te^{1t}, e^{2t}, e^{-1t}, te^{-1t}, t^2e^{-1t}$ – sist. fund. de sol.

$$x(t)=c_1e^t+c_2te^t+c_3e^{2t}+c_4e^{-t}+c_5te^{-t}+c_6t^2e^{-t}$$

III a) $x''+x=0$, $x(0)=3$, $x'(0)=5$

$\boxed{b)}$ $x^{IV}+4x=0$

c) $x^{IV}+8x''+16x=0$

$\boxed{d)}$ $x''+4x'+13x=0$

$\boxed{e)}$ $x''+4x'+5x=0$

I Rezolvare

a) $x''+x=0$, $x(0)=3$, $x'(0)=5$

$\left.\begin{array}{l} x=e^{\lambda t} \\ x'=\lambda e^{\lambda t} \\ x''=\lambda^2e^{\lambda t} \end{array}\right\} \Rightarrow \lambda^2e^{\lambda t}+e^{\lambda t}=0 \;|: e^{\lambda t}$

$\lambda^2+1=0$

$\lambda^2=-1$

$\lambda=\pm i \quad (\alpha\pm i\beta,\ \alpha=0, \beta=1)$

$e^{0t}\cos t\cdot 1,\ e^{0t}\cdot\sin 1t$ – sist. fundam. de sol.

$x(t)=c_1\cos t+c_2\sin t \Rightarrow x(0)=c_1\cos 0+c_2\sin 0=\boxed{c_1=3}$

$x'(t)=c_1(-\sin t)+c_2\cdot\cos t \Rightarrow x'(0)=c_1(-\sin 0)+c_2\cos 0=\boxed{c_2=5}$

$x_{PC}=3\cos t+5\sin t$

III c) $x^{IV}+8x''+16x=0$

$$\left.\begin{array}{l} x=e^{rt} \\ x'=re^{rt} \\ x''=r^2e^{rt} \\ x'''=r^3e^{rt} \\ x^{IV}=r^4e^{rt} \end{array}\right\} \Rightarrow r^4e^{rt}+8r^2e^{rt}+16e^{rt}=0 \mid : e^{rt}$$

$$r^4+8r^2+16=0$$

Not $r^2=p \Rightarrow p^2+8p+16=0$

$$(p+4)^2=0$$

$\Rightarrow p=-4$ (de 2 ori)

$r^2=-4$ (de 2 ori)

$r=\pm 2i$ (de 2 ori) $(\alpha \pm i\beta, \alpha=0, \beta=2)$

$e^{0\cdot t}\cos 2t,\ e^{0t}\sin 2t,\ t e^{0t}\cos 2t,\ t e^{0t}\sin 2t$

– sist. fund. de sol.

$x(t)=C_1\cos 2t+C_2\sin 2t+C_3 t\cos 2t+C_4 t\sin 2t$

IV a) $x^{IV}+2x'''+4x''-2x'-5x=0$

$\boxed{b)}$ $x'''-3x''+9x'+13x=0$

c) $x^{(7)}-x^{(6)}+x^{(5)}-x^{(4)}=0$

$\boxed{d)}$ $x'''-5x''+17x'-13x=0$

$\boxed{e)}$ $x^{V}+4x^{IV}+3x'''-6x'-2x=0$

Rezolvare

a) $x^{IV}+2x'''+4x''-2x'-5x=0$

$$\left.\begin{array}{l} x=e^{rt} \\ x'=re^{rt} \\ x''=r^2e^{rt} \\ x'''=r^3e^{rt} \\ x^{IV}=r^4e^{rt} \end{array}\right\} \Rightarrow r^4e^{rt}+2r^3e^{rt}+4r^2e^{rt}-2re^{rt}-5e^{rt}=0 \mid : e^{rt}$$

$$r^4+2r^3+4r^2-2r-5=0.$$

$$\begin{array}{l|l} n^4+2n^3+4n^2-2n-5 & n-1 \\ -n^4+n^3 & \\ \hline \quad 3n^3+4n^2-2n-5 & \\ \quad -3n^3+3n^2 & \\ \hline \quad\quad 7n^2-2n-5 & \\ \quad\quad -7n^2+7n & \\ \hline \quad\quad\quad 5n-5 & \\ \quad\quad\quad -5n+5 & \\ \hline \quad\quad\quad / \; / & \end{array}$$

$$\begin{array}{l|l} n^3+3n^2+7n+5 & n+1 \\ \cline{2-2} -n^3-n^2 & n^2+2n+5 \\ \hline \quad 2n^2+7n+5 & \\ \quad -2n^2-2n & \\ \hline \quad\quad 5n+5 & \\ \quad\quad -5n-5 & \\ \hline \quad\quad / \; / & \end{array}$$

• $(n-1)(n+1)(n^2+2n+5)=0$

$n_1=1$

$n_2=-1$

$n^2+2n+5=0$

$D=4-20=-16$

$n_{1,2}=\frac{-2\pm 4i}{2}=-1\pm 2i$

$(\alpha\pm i\beta,\ \alpha=-1, \beta=2)$

• $e^{1t}, e^{-1t}, e^{-1t}\cos 2t, e^{-1t}\sin 2t$ - sist. fund de sol.

$x(t)=c_1e^{t}+c_2e^{-t}+C_3e^{-t}\cos t+C_4e^{-t}\sin 2t.$

e) $x^{(7)}-x^{(6)}+x^{(5)}-x^{(4)}=0$

$x=e^{nt}$

$x'=ne^{nt}$

$x''=n^2e^{nt}$

$x'''=n^3e^{nt}$

$x^{IV}=n^4e^{nt}$

$x^{(5)}=n^5e^{nt}$

$x^{(6)}=n^6e^{nt}$

$x^{(7)}=n^7e^{nt}$

$\Rightarrow n^7e^{nt}-n^6e^{nt}+n^5e^{nt}-n^4e^{nt}=0 \quad |:e^{nt}$

$n^7-n^6+n^5-n^4=0$

$n^4(\underbrace{n^3-n^2}+\underbrace{n-1})=0$

$n^4(n-1)(n^2+1)=0$

$n_1=n_2=n_3=n_4=0$

$n-1=0\Rightarrow n_5=1$

$n^2+1=0\Rightarrow n^2=-1\Rightarrow n_{6,7}=\pm 1 \quad (\alpha\pm i\beta, \alpha=0, \beta=1)$

$e^{0t}, te^{0t}, t^2e^{0t}, t^3e^{0t}, e^{1t}, e^{0t}\cos 1t, e^{0t}\sin 1t$

- sist. fund de sol.

$x(t)=C_1+C_2t+C_3t^2+C_4t^3+C_5e^{t}+C_6\cos t+C_7\sin t$