Recapitulare

1. Funcții (Bert Thiend Russell)

Def: $\circ)(X, Y, f)$ – o funcție
$X \neq \emptyset$, $Y \neq \emptyset$ ↳ o lege prin care se asociază oricărui element $x \in X$ un element unic $y \in Y \overset{not}{\equiv} f(x)$

se va nota $f: X \to Y$ (sau $X \xrightarrow{f} Y$)

1) $f: X \to Y$ se numește injectivă $\overset{def}{\iff}$ $(\forall) x, x' \in X, (x \neq x' \Rightarrow f(x) \neq f(x'))$

---

0. Logică matematică

p - predicat, $p \in D$

$v: D \to \{0, 1\}$

$\wedge$ - ȘI, $\vee$ - SAU, $\urcorner$ - NON.
(binari) (unar)

Def: $v(p \wedge q) \overset{def}{=} v(p) \wedge v(q)$ (Manual 9)

$v(p \vee q) \overset{def}{=} v(p) + v(q) - v(p \wedge q)$

Ex: $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$

$(p \wedge q) \vee r = (p \vee r) \wedge (q \vee r)$

Te: $\urcorner(p \vee q) \equiv \urcorner p \wedge \urcorner q$ } Legile lui de Morgan

$\urcorner(p \wedge q) \equiv \urcorner p \cup \urcorner q$

$C(A \cup B) = CA \cap CB$

Ⓟ: $(\forall x)\underbrace{(x \in \mathbb{R} \Rightarrow x^2 \geq 0)}_{p(x)}$

$(\exists x)(x \in \mathbb{R}, x^2 < 0)$

? $\Rightarrow$ $(p \Rightarrow q) \overset{not}{\equiv} \urcorner p \vee q$ (not p sau q)

$\urcorner(\forall x)(p(x)) \equiv (\exists x)\urcorner p(x)$

(d).

f injectivă $\Leftrightarrow (\forall x, x' \in X)\ (f(x) = f(x') \Rightarrow x = x')$.

X - domeniu

Y - codomeniu

$(p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p)$ - principiul red. la absurd.

$$\Updownarrow$$

$$\neg\neg q \vee \neg p$$

f surj

2) $f: X \to Y$, surj $\Leftrightarrow (\forall y \in Y)\ (\exists x \in X)\ (y = f(x))$

3) $f: X \to Y$ bij $\Leftrightarrow$ f inj $\wedge$ f surj

$\Leftrightarrow (\forall y \in Y)\ (\exists!\ x \in X)\ (f(x) = y)$

$$\Downarrow$$

se poate def o nouă fct, $f^{-1}: Y \to X$, $f^{-1}(y) =$ "acel unic $x \in X$ pt care $y = f(x)$".

---

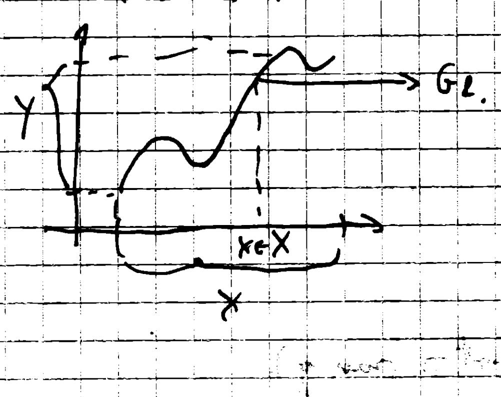
f nu-i inj: $(\exists x, x' \in X)\ (x \neq x' \wedge f(x) = f(x'))$

f nu-i surj: $(\exists y \in Y)\ (\forall x \in X)\ (y \neq f(x))$

f bij $\Leftrightarrow (\forall y \in Y)\ (y = f(x) \Leftrightarrow x = f^{-1}(y) \in Y)$

Def 4) $f: X \to Y$ nu $G_f \overset{\text{not}}{:=} \underbrace{\{(x, f(x)) \mid x \in X\}}_{\in X \times Y}$ - graficul lui f)

Obs: Dacă $X, Y \in \mathbb{R}$, $f: X \to Y$, graficul G_f poate fi vizualizat în rdm, într-un sistem de axe



- f inj: $(\forall y \in Y)$ paralela prin y la Ox "taie" G_f în maxim un pct.

- f surj — " — politic un pct.

- f bij — " — exact un pct.

5) $\underbrace{X \xrightarrow{f} Y \xrightarrow{g} Z}_{g\circ f} \rightsquigarrow g\circ f: X\to Z$, $(g\circ f)(x) \stackrel{def}{=} g(f(x))$ $x\in X$.

Obs: $f:X\to Y$ bijectivă $\Rightarrow f^{-1}\circ f(x)=x$, $(\forall)x\in X$

$f\circ f^{-1}(y)=y$, $(\forall)y\in Y$.

Ex: $f,g:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases}-x, & x<0\\ x^2+1, & x\geq 0\end{cases}$ $\quad g(x)=\begin{cases}x+1, & x<1\\ -x^2, & x\geq 1\end{cases}$

!!! Temă: $g\circ f=?$, $f\circ g=?$

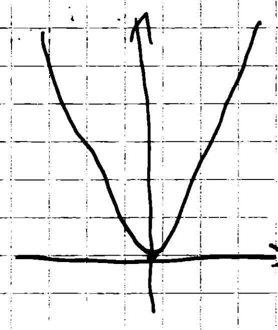
• Funcții elementare

I. Funcția putere și radical

1) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^n$, $n\in\mathbb{R}$

n par

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x^2	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4
x^4		1	$\frac{1}{16}$	0		1	

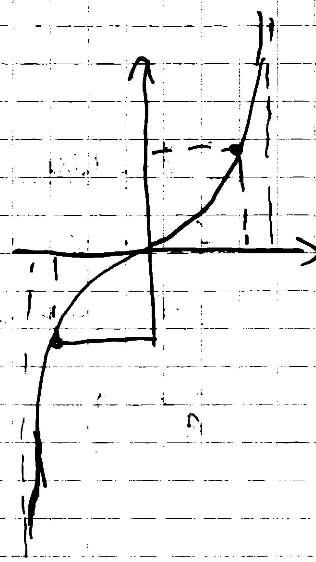


Obs: f nu e inject. nici surj.

$f|_{[0,\infty)}:[0,\infty)\to[0,\infty)$ bijectivă.

n impar.

x	-2	-1	0	$\frac{1}{3}$	$\frac{1}{2}$	1
x^3	-8	-1	0	$\frac{1}{27}$	$\frac{1}{8}$	1
x^5	-32	-1	0		$\frac{1}{32}$	1



f impar: sim. față de origine.

$f:\mathbb{R}\to\mathbb{R}$ funcț. bij.

2. Radical.

Def a) n par $\left(f|_{[0,\infty)}\right)^{-1}:[0,\infty)\to[0,\infty)$ f. rad. de ordin n par.

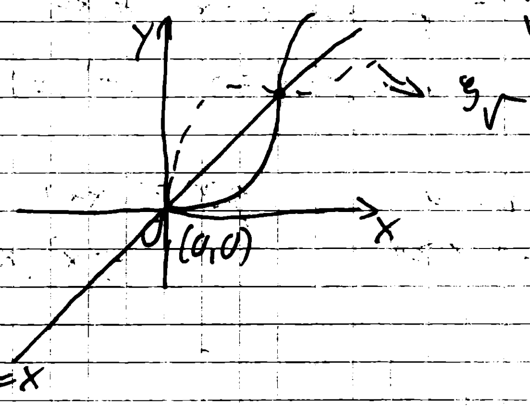
b) n impar $f^{-1}:\mathbb{R}\to\mathbb{R}$, f. rad. de ordin n impar.

Generalizare

$f:\mathbb{R}\to\mathbb{R},\ f(x)=x^n,\ x\in\mathbb{N}^*$

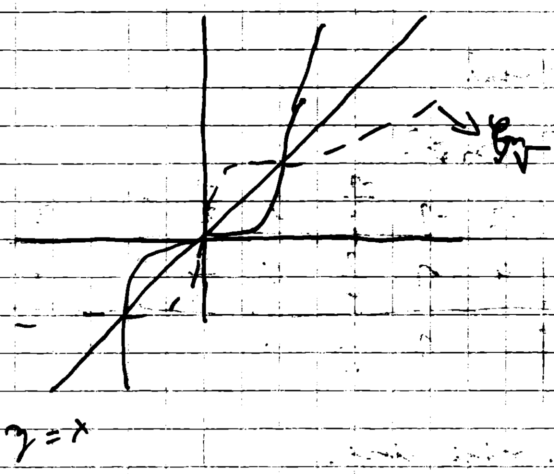
a) n par

$\sqrt[n]{\cdot} \overset{not}{=} \left(f|_{[0,\infty)}\right)^{-1}$



b) n impar

$\sqrt[n]{\cdot} \overset{not}{=} f^{-1}$



II Funcțiile exponențiale și logaritm

Fie $a>0,\ a\neq 1$

$$x\in\mathbb{R} \rightsquigarrow a^x \overset{def}{=} \begin{cases} \underbrace{a\cdot a\cdot\ldots\cdot a}_{n\ ori}, & x=n\in\mathbb{N}^* \quad (i) \\ \frac{1}{a^k}, & x=k\in\mathbb{Z}_- \quad (ii) \\ \sqrt[q]{a^p}, & x=\frac{p}{q}\in\mathbb{Q},\ (p,q)=1,\ p\in\mathbb{Z}^*,\ q\in\mathbb{N}^* \quad (iii) \\ *, & x\in\mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

$\boxed{a^0=1}$

$x = x_0, x_1 x_2 \ldots x_n \ldots$

$a^{x^{(n)}} \leftsquigarrow x^{(n)} := x_0, x_1 \ldots x_n \in \mathbb{Q} \leftarrow$

$a^{y^{(n)}} \leftsquigarrow y^{(n)} := x_0, x_1 \ldots "x_n+1" \in \mathbb{Q}$

$$\Rightarrow \begin{cases} x^{(n)} \le x < y^{(n)} \\ x^{(n)} \nearrow_n,\ y^{(n)} \searrow_n \end{cases}$$

(!) Şirurile $\left(a^{x(n)}\right)_{n\geq 1}$, $\left(a^{y(n)}\right)_{n\geq 1}$ au aceeaşi limită (!!!)

* Fie deci a^x (prin definiţie)

P $a^{x+y} = a^x \cdot a^y$, $x, y \in \mathbb{R}$.

$a^{xy} = (a^x)^y$ — $x, y \in \mathbb{R}$

$(ab)^x = a^x \cdot b^x$

$b > 0, b \neq 1$.

Def: $f:\mathbb{R}\to\mathbb{R}$, $f(x) = a^x$ — fct. exp. de bază a.

Obs: f s. cresc., $a > 1$ ⟹ f inj.
f s. descr. $0 < a < 1$

f surj. ⟹ $f:\mathbb{R}\to(0,\infty)$

$f:\mathbb{R}\to(0,\infty)$, $f(x) = a^x$ bij ... (?)

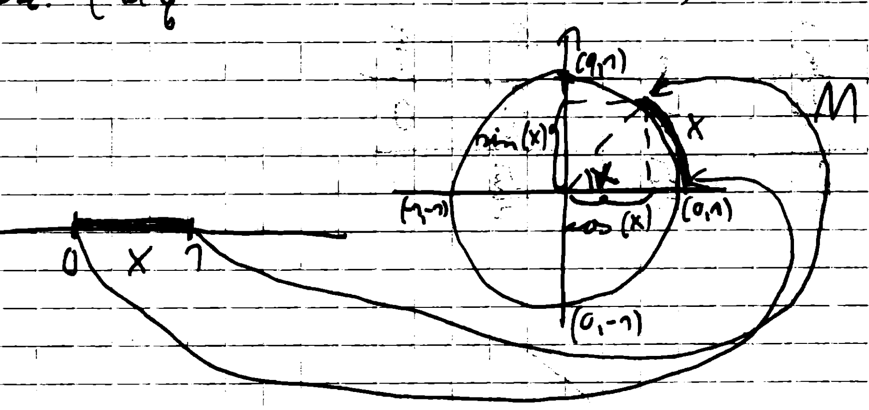
există $f^{-1}:(0,\infty)$

Def: $\log_a b =$

$f: X\to Y$ bij ⟺ $(\forall y \in Y)$ (ec. $y = f(x)$ ⟺ $x = f^{-1}(y) \in X$.

III. Fct. trigonometrice

$(\forall \ell)$ $\left(\frac{L_\ell}{d_\ell} = \text{"mereu acelaşi nr"} \overset{not}{=} \pi\right)$
ec.



Def: $\cos, \sin: \mathbb{R}\to[-1,1]$, $\cos(x) \overset{def}{=}$ "$pr_{Ox} M(x)$" (proiecţie cu semn)

($M:\mathbb{R}\to\mathcal{C}_1$ e fct "înfăşurătoare" a lui $\mathbb{R}$ pe $\mathcal{C}_1$")

$\sin(x) \stackrel{def}{=}$ "$\text{pr}_{Oy} M(x)$" (—n—).

$\underline{P}$: $\cos^2 x + \sin^2 x = 1$, $(\forall)\, x \in \mathbb{R}$.

$\cos(-x) = \cos(x)$ —n—

$\sin(-x) = -\sin x$ —n—

$\cos(x+2\pi) = \cos x$ —n—

$\sin(x+2\pi) = \sin x$ —n—

- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$, $\sin\left(\frac{\pi}{2} - x\right) = \cos x$, $x \in \left[0, \frac{\pi}{2}\right]$

$\cos(\pi - x) = -\cos x$, $\sin(\pi - x) = \sin x$, $x \in [0, \pi]$.

$\cos(2\pi - x) = \cos x$, $\sin(2\pi - x) = -\sin x$, $x \in [0, 2\pi]$.

- $\boxed{\begin{aligned} \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\ \sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \end{aligned}}$

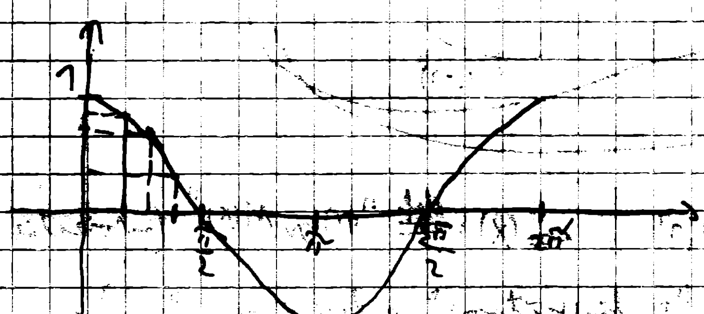
- $\sin\alpha - \sin\beta = \sin\frac{\alpha-\beta}{2} \cos\frac{\alpha+\beta}{2}$.

$x+y=\alpha;\ x-y=\beta \Rightarrow \left\{ x = \frac{\alpha+\beta}{2},\ y = \frac{\alpha-\beta}{2}. \right.$

- $\cos 2x = \cos^2 x - \sin^2 x$

! Știe toate (toate formulele trigonometrice)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1



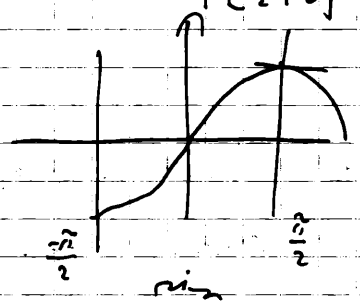
Obs: Fct $\cos\big|_{[0,\pi]} : [0,\pi] \to [-1,1]$ bijectivă.

$\Rightarrow \left(\cos\big|_{[0,\pi]}\right)^{-1} \overset{=}{\underset{def}{}} \arccos.$

$y \in [0,1]$, $\arccos(-y) = \pi - \arccos y$

Temă: G_f sin, arcsin, $G_{\sin}$.

Obs: $\sin\big|_{[-\frac{\pi}{2},\frac{\pi}{2}]} : \underset{bij}{[-\frac{\pi}{2},\frac{\pi}{2}]} \to [-1,1]$, $\left(\sin\big|_{[-\frac{\pi}{2},\frac{\pi}{2}]}\right)^{-1} \overset{=}{\underset{def}{}} \arcsin.$



$y \in [0,1]$, $\arcsin(-y) = -\arcsin y$

Notă: 10% P

10% Rasp (AS).

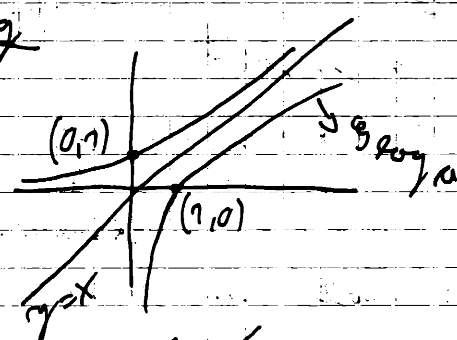
20% Lucrare

10% T

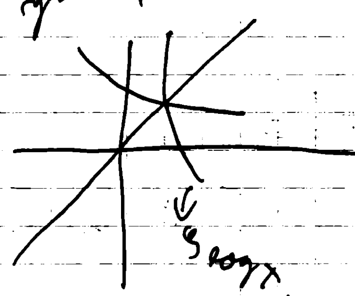
50% Ex fin.

Fct exp și log

$a > 1$



$0 < a < 1$



Carte: Nunică – Analiză matematică. Aplicații. Vol I. (Calcul Diferențial) 50 lei (... Ion Chiriță).

P $\log_a(AB) = \log_a A + \log_a B$

$\log_a \frac{A}{B} = \log_a A - \log_a B$

$\log_a A^{\alpha} = \alpha \log_a A$

$\log_a A = \frac{\log_b A}{\log_b a}$, $b > 0$, $b \neq 1$.

1. Şiruri de nr. reale ($\mathbb{R}$)

Un şir e o fct: $(x : \mathbb{N}^* \to \mathbb{R})$ (şir de nr. reale)

$$\Updownarrow \Leftarrow x(n) \overset{not}{=\!=}: x_n, \ n \in \mathbb{N}^*$$

$$\boxed{(x_n)_{n\in\mathbb{N}^*} \subseteq \mathbb{R}} \Leftrightarrow \boxed{x_1, x_2, \dots, x_n, \dots}$$

$(x_n)_{n\in\mathbb{N}^*}$ – cresc $\Leftrightarrow x_n \le x_{n+1}$, $(\forall) n \in \mathbb{N}^*$

– s. cresc $<$

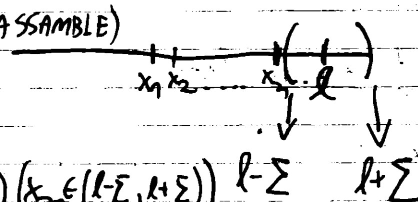
– desc $\ge$

– s. desc. $>$

$(x_n)_{n\in\mathbb{N}^*}$ – mărginit inferior $\Leftrightarrow (\exists \alpha \in \mathbb{R})(\forall n \ge 1)(\alpha \le x_n)$

superior $(\exists \beta \in \mathbb{R})(\forall n \ge 1)(x_n \le \beta)$

Def: I Şiruri convergente (BASIC – ASSAMBLE) (Cauchy)



$(x_n)_{n\ge 1} \subset \mathbb{R}$ s.n. convergent $\overset{def}{\Leftrightarrow}$

$\overset{def}{\Leftrightarrow} (\exists l \in \mathbb{R})(\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N}^*)(\forall n \ge n_\varepsilon)(x_n \in (l-\varepsilon, l+\varepsilon))$

$\Updownarrow$

$|x_n - l| < \varepsilon.$

($\forall$ urmat de $\exists$, litera n primeşte indice de la $\forall$)

$*\to$

Ex1: 1) $x_n = \frac{1}{n}$, $n \ge 1$ $\quad l = 0$

$\varepsilon > 0$ $\quad \left|\frac{1}{n} - 0\right| < \varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n.$

(Epsilon sau ε) $\quad n_\varepsilon = \left[\frac{1}{\varepsilon}\right] + 1.$

$n \ge n_\varepsilon \Rightarrow \frac{1}{\varepsilon} < n \Leftrightarrow |x_n - 0| < \varepsilon \xrightarrow{def.} (x_n)_{n\ge 1}$ convergent

---

*. **Obs1**: l din def de mai sus este unic, $l \overset{not}{=\!=} \lim_{n\to\infty} x_n$ – limita şirului.

Se mai scrie $x_n \xrightarrow[n\to\infty]{} l$ sau $x_n \to l$.

Obs 2: $x_n \to l \Rightarrow |x_n - l| \to 0$. $\quad \left(\begin{array}{l} ||x_n-l|-0| < \varepsilon \\ |x_n-l| < \varepsilon \end{array} \right)$

$l \in \mathbb{R}$

Pt $l = 0$, $x_n \to 0 \Leftrightarrow |x_n| \to 0$

La Ex 1: se poate scrie doar $\frac{1}{n} \to 0$.

Ex 2: $x_n = \frac{1}{2^n}, n \geq 1$; $l = 0$

$\varepsilon > 0$, $|x_n - 0| < \varepsilon \Leftrightarrow \frac{1}{2^n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < 2^n \Leftrightarrow \log_2 \frac{1}{\varepsilon} < n$.

$n_\varepsilon = \left[\log_2 \frac{1}{\varepsilon}\right] + 1$; $n \geq n_\varepsilon \Rightarrow n > \log_2 \frac{1}{\varepsilon} \Leftrightarrow \left|\frac{1}{2^n} - 0\right| < \varepsilon$

$\Rightarrow \frac{1}{2^n} \to 0$.

Din Ex 1: $x_n = \frac{1}{n^k}$, $n \geq 1$

$k \in \mathbb{N}^*$

modificări: $\left|\frac{1}{n^k} - 0\right| < \varepsilon$, $\frac{1}{n^k} < \varepsilon$, $\frac{1}{\sqrt[k]{\varepsilon}} < n$.

Ex 2: $x_n = \frac{1}{a^n}$, $n \geq 1$

$a > 1$

modificări: $\frac{1}{a^n} < \varepsilon$, $\frac{1}{\varepsilon} < a^n$, $\log_a \frac{1}{\varepsilon} < n$, $n_\varepsilon = \left[\log_a \frac{1}{\varepsilon}\right] + 1$.

Ex 3: $x_n = \frac{1}{n!}$, $n \geq 1$. $\Rightarrow 0 < x_n \leq \frac{1}{n}$, $n \geq 1$.

$n! \geq n$

Prop ("Criteriul majorării") Cr-Maj (c)

Fie $(x_n)_{n \geq 1}$, $(y_n)_{n \geq 1} \subset \mathbb{R}$, $l \in \mathbb{R}$ a.î. $|x_n - l| \leq y_n$, $n \geq k$.

Dacă $y_n \to 0 \Rightarrow x_n \to l$.

Dem: Imediată din def

La Ex 3: $0 < x_n = \frac{1}{n!} \leq \frac{1}{n}$, $n \geq 1$

$y_n = \frac{1}{n} \xrightarrow[n]{} 0$ $\quad \xRightarrow{\text{Cr Maj}} x_n \to 0$.

Exp 4: $x_n = \underbrace{(-1)^n \frac{1}{n}}$, $n \geq 1$; $|x_n| = \frac{1}{n} \to 0 \Leftrightarrow x_n \to 0$.

= şir alternant (unul neg, unul poz).

Exp 5: $x_n := \frac{\sin n}{n}$, $n \geq 1$; $|x_n| = \frac{|\sin n|}{n} \leq \frac{1}{n}$, $n \geq 1$ | Tr. Maj $\Rightarrow x_n \to 0$

$y_n = \frac{1}{n} \to 0$ $\quad$ $\frac{1}{n} = y_n$

Exp 6: $x_n = \frac{2n+1}{3n+2}$, $n \geq 1$.

$$\left|x_n - \frac{2}{3}\right| = \left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| = \frac{1}{3(3n+2)} \leq \frac{1}{9n} \leq \frac{1}{n} = y_n$$

$$y_n \to 0 \xRightarrow[\text{Tr. Maj}]{} x_n \to \frac{2}{3}$$

__Prop__ (Operaţii algebrice cu şiruri convergente") (C++)

Fie $(x_n)_{n \geq 1}, (y_n)_{n \geq 1} \subset \mathbb{R}$, $x_n \to l_1 \in \mathbb{R}$, $y_n \to l_2 \in \mathbb{R}$.

Atunci: a) $x_n + y_n \longrightarrow l_1 + l_2 \iff \lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$

b) $x_n \cdot y_n \longrightarrow l_1 \cdot l_2$

c) $\frac{x_n}{y_n} \to \frac{l_1}{l_2}$ at $l_2 \neq 0$. $\quad$ ~~$\left(\frac{1}{l_2}\right) \neq 0$~~

d.) $x_n^{y_n} \to l_1^{l_2}$, $l_1 > 0$. $\quad$ (Inegal. lui Euler)

(de la un rang) $\quad$ Dem: Imediată din Def sau Tr. Maj

$|x_n + y_n - (l_1 + l_2)|$
$|x_n - l_1| + |y_n - l_2|$
$< \varepsilon/2 \quad < \varepsilon/2$

* __Obs 3__: A adăuga / scoate un nr. finit de termeni de la un şir nu-i afectează natura. (convergenta).

~~Dem:~~

__Exp 7__: $x_n = \frac{2n^2 - 3n + 1}{5n^2 + 2n - 1} = \frac{n^2\left(2 - \frac{3}{n} + \frac{1}{n^2}\right)}{n^2\left(5 + \frac{2}{n} - \frac{1}{n^2}\right)} \xrightarrow[n]{\text{Op. alg. şir. con}} \frac{2 - 0 + 0}{5 + 0 - 0} = \frac{2}{5}$.

$\frac{2}{n} \to 0$, $\frac{1}{n^2} \to 0$

Exp 8: $x_n = \frac{2n+1}{n^2 - n + 1} = \frac{n\left(2 + \frac{1}{n}\right)}{n^2\left(1 - \frac{1}{n} + \frac{1}{n^2}\right)} \to \frac{2+0}{\infty(1-0+0)}$??

II. Șiruri cu lim. infinite

Def: $(x_n)_{n\geq 1} \subset \mathbb{R}$ are lim $+\infty$ / $-\infty$ $\overset{def}{\Leftrightarrow}$ $(\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n \geq n_\varepsilon)$

$(x_n \in (\varepsilon, \infty) \Leftrightarrow x_n > \varepsilon)$

$(\text{resp. } x_n < \varepsilon)$.

Se mai scrie $\lim_{n\to\infty} x_n = +\infty$ ($-\infty$) sau $x_n \xrightarrow[n]{} \infty$ $(-\infty)$, $x_n \to \infty$ $(-\infty)$

Ex: 1) $x_n = n^k \to \infty$ $(k \in \mathbb{N}^*)$ (dem $\varepsilon \dots$ OK).

2) $x_n = a^n \to \infty$ $(a > 1)$ $(-n)$

3) $x_n = x! \to \infty$

Prop ("Cr. majorării Reloaded")

Fie $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subset \mathbb{R}$, $x_n \leq y_n$, $n \geq K$.

a) Dacă $y_n \to -\infty \Rightarrow x_n \to -\infty$

b) Dacă $x_n \to \infty \Rightarrow y_n \to \infty$

Dem: imediată din Def.

Ex. 4) $x_n = n + (-1)^n$, $n \geq 1$
$\geq n - 1 \to \infty$ $\xRightarrow{\text{Cr. Maj R}}$ $x_n \to \infty$.

5) $x_n = 4n^2 + 5n + 3 \geq n$, $n \geq 1$
$n \xrightarrow[\text{Ex. 1}]{} \infty$ $\xRightarrow{\text{Cr. maj R}}$ $x_n \to \infty$

6) $x_n = 4n^2 - 5n + 3 \dots$

Prop ("Op. Algebrice cu limite de șiruri")

Fie $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subset \mathbb{R}$, $x_n \to l_1 \in \overline{\mathbb{R}}$, $y_n \to l_2 \in \overline{\mathbb{R}}$.

Atunci: a) $x_n + y_n \to l_1 + l_2$ $\boxed{\text{Atenție } \infty - \infty}$

b) $x_n \cdot y_n \to l_1 \cdot l_2$ $\boxed{0 \cdot \infty}$

c) $\frac{x_n}{y_n} \to \frac{l_1}{l_2}$ $\boxed{\frac{0}{0}, \frac{\infty}{\infty}}$

d) $x_n^{y_n} \to l_1^{l_2}$ $\boxed{1^\infty}$ $\boxed{0^0}$, $\boxed{\infty^0}$

$n - n = 0 \to 0$
$(n+1) - n = 1 \to 1$ | nedeterminări

a) $\infty + \infty = \infty$

b) $\pm\infty \cdot \infty = \pm\infty$

c) $\pm \frac{1}{\infty} = 0.$

d) $0^{\infty} = 0.$

$\ldots\ldots$

$1n^2 - n + 1 = n^2\left(1 - \frac{1}{n} + \frac{1}{n^2}\right) \to \infty(1-0+0)$, $\frac{1}{n} \searrow 0$, $\frac{1}{n^2} \searrow 0$

Ex 1 7 $x_n = \frac{n^2 - n + 1}{2n^2+3n+3} \overset{\frac{\infty}{\infty}}{=} \frac{n^2\left(1-\frac{1}{n}+\frac{1}{n^2}\right)}{n^2\left(2+\frac{3}{n}+\frac{3}{n^2}\right)} \to \frac{1-0+0}{2+0+0} = \frac{1}{2}$

Ex 1 8: $x_n = \frac{2n+1}{n^2-n+1} \overset{\frac{\infty}{\infty}}{=} \frac{n\left(2+\frac{1}{n}\right)}{n^2\left(1-\frac{1}{n}+\frac{1}{n^2}\right)} \Rightarrow \frac{2+0}{\infty(1-0+0)} = 0.$

16.10.2019

Ex1. Să se det. lim. șirurilor:

a) $x_n = \frac{n^2-2n+5}{3n-4n^2+1}$, $n \geq 1$ b) $x_n = \frac{2n-1}{n^2+n+1}$, $n \geq 1$; c)

a) $x_n = \frac{n^2-2n+5}{3n-4n^2+1} \overset{\frac{\infty}{\infty}}{=} \frac{\cancel{n^2}\left(1-\frac{2}{n}+\frac{5}{n^2}\right)}{\cancel{n^2}\left(-4+\frac{3}{n}+\frac{1}{n^2}\right)} \to \frac{1-0+0}{-4+0+0} = \frac{1}{-4}$, $n \geq 1$

b) $x_n = \frac{2n-1}{n^2+n+1} \overset{\frac{\infty}{\infty}}{=} \frac{\cancel{n}\left(2-\frac{1}{n}\right)}{n^{\cancel{2}}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)} \Rightarrow \frac{2-0}{\infty\cdot(1+0+0)} = 0.$, $n \geq 1$

c) $x_n = \frac{-n^2+2n^3+2}{2n^2-n+3} \overset{\frac{\infty}{\infty}}{=} \frac{n^3\left(-\frac{1}{n}+2+\frac{2}{n^3}\right)}{n^2\left(2-\frac{1}{n}+\frac{3}{n^2}\right)} \to \frac{\infty(0+2+0)}{2-0+0} =$, $n \geq 1$

Ob. alg. cu lim. șirului
$x_n \to l_1 \in \overline{\mathbb{R}}$
$y_n \to l_2 \in \overline{\mathbb{R}}$
$x_n + y_n \to l_1 + l_2$ (Ex: $\infty - \infty$)
$x_n \cdot y_n \to l_1 \cdot l_2$ (Ex: $0 \cdot \infty$)
$\frac{x_n}{y_n} \to \frac{l_1}{l_2}$ (Ex. $\frac{0}{0}, \frac{\infty}{\infty}$)
$x_n^{y_n} \to l_1^{l_2}$ (Ex $1^{\infty}, 0^0, \infty^0$)
$(l_1 \geq 0$

$= 0. \cancel{\infty}$

d) $x_n = \sqrt{n^2+n} - n \overset{\infty-\infty}{=} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} \overset{\frac{\infty}{\infty}}{=} \frac{n}{n\sqrt{1+\frac{1}{n}}+n} =$, $n \geq 1$

$= \frac{n}{n\left(1+\sqrt{1+\frac{1}{n}}\right)} \to \frac{1}{1+1} = \frac{1}{2}.$, $\frac{1}{n} \searrow 0$

e) $x_n = \sqrt{n^2+2} - \sqrt{n^2+1} \overset{\infty-\infty}{=} \dfrac{n^2+2-(n^2+1)}{\sqrt{n^2+2}+\sqrt{n^2+1}} = \dfrac{1}{\sqrt{n^2+2}+\sqrt{n^2+1}} \to \dfrac{1}{\infty} = 0.$
$n \geq 1$

f) $x_n = \dfrac{2^n+3^n}{2^{n+1}+3^{n+1}} \overset{\frac{\infty}{\infty}}{=} \cancel{\dfrac{2^n\left(1+\left(\frac{3}{2}\right)^n\right)}{2^n\left(2+3\cdot\left(\frac{3}{2}\right)^n\right)}} \quad \dfrac{\cancel{3^n}\left(1+\left(\frac{2}{3}\right)^n\right)}{\cancel{3^n}\left(3+2\left(\frac{2}{3}\right)^n\right)} \Rightarrow \dfrac{1+0}{3+2\cdot 0} =$
$n \geq 1$

$\left(\frac{2}{3}\right)^n \to 0$

$= \dfrac{1}{3}.$

$\dfrac{1}{a^n} \to 0,\ a>1 \iff \left(\dfrac{1}{q}\right)^n \to 0 \qquad q^n \to 0,\ -1<q<1.$

g) $x_n = \dfrac{2^n-3\cdot 4^n+5^n}{4^{n+1}+5^{n+1}} \overset{\frac{\infty}{\infty}}{=} \dfrac{\cancel{5^n}\left(\left(\frac{2}{5}\right)^n-3\cdot\left(\frac{4}{5}\right)^n+1\right)}{\cancel{5^n}\left(4\left(\frac{4}{5}\right)^n+5\right)} \to \dfrac{0-3\cdot 0+1}{4\cdot 0+5} = \dfrac{1}{5}.$
$n \geq 1$

(Metoda maniola Tavi) ! Burtea & Burtea.

Ex 2 Să se det. lim. șirurilor.

a) $x_n = \dfrac{\sin(n+1)}{n+2},\ n\geq 1$; b) $x_n = \dfrac{n\cos n}{n^2+1},\ n \geq 1$

a) $|x_n| = \dfrac{|\sin(n+1)|}{n+2} \leq \dfrac{1}{n+2} \leq \dfrac{1}{n},\ n\geq 1$; $y_n = \dfrac{1}{n} \to 0$ $\xrightarrow[\text{Cr. Maj}]{} x_n \to 0.$

b) $|x_n| = \dfrac{|n\cos n|}{n^2+1} \leq \dfrac{n}{n^2+1} = \dfrac{n}{n^2\left(1+\frac{1}{n^2}\right)} = \dfrac{1}{n\left(1+\frac{1}{n^2}\right)} \to \dfrac{1}{\infty\cdot(1+0)} = \dfrac{1}{\infty} = 0.$

! Gen. lui f) $x_n = \dfrac{\alpha^n+\beta^n}{\alpha^{n+1}+\beta^{n+1}}.$

b) $|x_n| = \dfrac{n|\cos n|}{n^2+1} \leq \dfrac{n}{n^2+1} \leq \dfrac{n}{n^2} = \dfrac{1}{n} = y_n,\ n \geq 1$; $y_n \to 0$ $\xrightarrow{\text{C.R.Maj}} x_n \to 0.$

• Limite remarcabile

(1) Studiu individual: Se cons. $e_n = \left(1+\dfrac{1}{n}\right)^n,\ n\geq 1$ (șirul lui Euler)

se poate demonstra: $\begin{cases} e_n < e_{n+1}, & (\forall)\, n \geq 1 \\ 2 \leq e_n < 3, & (\forall)\, n\geq 1 \end{cases}$

$\boxed{\text{T. Weierstrass}}$: $(x_n)_{n\geq 1} \subset \mathbb{R}$ mon. / marg. $\Rightarrow (x_n)_{n\geq 1}$ conv.

(!!) $- \; l_n < l_{n+1}$

$(\forall) n \geq 1$ | T.W. $\Longrightarrow (l_n)_{n\geq 1}$ conv.

$- \; 2 \leq l_n < 3$

$- \; l \stackrel{def}{=\!=} \lim_{n\to\infty} l_n$

$\boxed{1'}$ (!!!) $- (\forall)(x_n)_{n\geq 1} \subset \mathbb{R},\ x_n \to \infty \Rightarrow \left(1+\frac{1}{x_n}\right)^{x_n} \to l$

OK

$\boxed{1''}$ $\forall (y_n)_{n\geq 1} \subset \mathbb{R},\ y_n \to 0 \Rightarrow (1+y_n)^{\frac{1}{y_n}} \to l$

(!!!)

$\boxed{1'''}$ — " — $\Rightarrow \frac{\ln(1+y_n)}{y_n} \to 1$

OK

$\boxed{1^{IV}}$ — " — $\Rightarrow \frac{a^{y_n}-1}{y_n} \to \ln a$, $a>0$.

Ex3. Să se det. lim. șirurilor.

a) $x_n = \left(\frac{n+1}{n+2}\right)^n,\ n\geq 1$. $\quad 1^\infty$

b) $x_n = \left(\frac{n^2-n+1}{n^2+n+1}\right)^{\frac{n^2}{n+1}},\ n\geq 1$ $\quad 1^\infty$

c) $x_n = n \ln\left(1+\frac{1}{n+1}\right),\ n\geq 1$ $\quad 0^\infty$

d) $x_n = n^3 \ln\left(\frac{2n+1}{2n+2}\right) \ln\left(\frac{3n^2+1}{3n^2+2}\right),\ n\geq 1$. $\quad 0^\infty$

e) $x_n = n(\sqrt[n]{2}-1),\ n\geq 1$ $\quad 0^\infty$

f) $x_n = n(\sqrt[n]{3}-\sqrt[n]{2}),\ n\geq 1$.

a) $x_n = \left(1+\frac{n+1}{n+2}-1\right)^n = \left(1+\frac{(-1)}{n+2}\right)^n = \left[\underbrace{\left(1+\frac{(-1)}{n+2}\right)^{\frac{n+2}{-1}}}_{\to e \ (1^\infty)}\right]^{n\cdot\frac{-1}{n+2}} =$

$\Rightarrow e^{\lim\limits_{n\to\infty}\frac{-n}{n+2}} = e^{-1}.$

b) $x_n = \left(1+\frac{n^2-n+1}{n^2+n+1}-1\right)^{\frac{n^2}{n+1}} = \left[\left(1+\frac{-2n}{n^2+n+1}\right)^{\frac{n^2+n+1}{-2n}}\right]^{\frac{-2n}{n^2+n+1}\cdot\frac{n^2}{n+1}}$

$\longrightarrow e^{\lim\limits_{n\to\infty}\frac{-2n^3}{(n^2+n+1)(n+1)}} = e^{\frac{-2}{1}} = e^{-2}$

c) $x_n = \frac{\ln\left(1+\frac{1}{n+1}\right)}{\frac{1}{n+1}} \cdot \frac{n}{n+1} \Rightarrow 1\cdot 1 = 1.$

($\ln\left(1+\frac{1}{n+1}\right) \to 1$, $\frac{1}{n+1} \to 1$, $\frac{n}{n+1} \to 1$)

d) $x_n = \frac{\ln\left(\frac{2n+2}{2n+2} - \frac{1}{2n+2}\right)}{\frac{-1}{2n+2}} \cdot \frac{\frac{-1}{2n+2}}{\frac{-1}{2n+2}} \cdot n^3 \cdot \frac{\ln\left(1-\frac{1}{3n^2+2}\right)}{\frac{-1}{3n^2+2}} \cdot \frac{\frac{-1}{3n+2}}{\frac{-1}{3n^2+2}}$

$\Rightarrow \left(\frac{-1}{2n+2}\right)\left(\frac{-1}{3n^2+2}\right)n^3 \to 1\cdot 1\cdot\frac{1}{6} = \frac{1}{6}.$

e) $\to \ln 2$

② <u>Studiu individual</u>: Se poate arăta că: $+|\cos x| \le \left|\frac{\sin x}{x}\right| \le 1, (\forall) x \in \left[0, \frac{\pi}{2}\right]$

2') $- (\forall)\ (x_n)_{n\ge 1},\ x_n \to 0 \Rightarrow \frac{\sin x_n}{x_n} \to 1$

2'') —"— $\Rightarrow \frac{\operatorname{tg} x_n}{x_n} \to 1$

2''') —"— $\Rightarrow \frac{\arcsin x_n}{x_n} \to 1$

) —"— $\Rightarrow \frac{\operatorname{arctg} x_n}{x_n} \to 1$

<u>Ex4</u>. Să se det. limita șirurilor.

a) $x_n = n\sin\frac{2\pi}{n}, n\ge 1$

d) $x_n = n^2\left(\cos\frac{\pi}{n} - \cos\frac{\pi}{n+1}\right), n \ge 1.$

b) $x_n = n^2 \sin\frac{\pi}{n} \arcsin\frac{n}{n^2+1}, n\ge 1$

c) $x_n = \frac{\operatorname{tg}\frac{\pi}{n+1}}{\operatorname{arctg}\frac{\pi n}{n^2+1}}, n \ge 1.$

a) $x_n = \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} \cdot \frac{2\pi}{\cancel{n}} \cdot \cancel{n} \Rightarrow 1 \cdot 2\pi = 2\pi$.

$\downarrow_n 1$

b) $x_n = \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} \cdot \frac{\pi}{\cancel{n}} \cdot n^2 \cdot \frac{\sin\frac{\pi}{n+1}}{\frac{\pi}{n+1}} \cdot \frac{\pi}{n+1} \Rightarrow \pi^2 \cdot 1 \cdot \pi^2 \cdot 1 \cdot 1$.

~~d) $x_n = n^2$~~ (c) $\frac{1}{2}$.

22.10.2019

Şiruri de nr. reale

T_1: $(x_n)_{n\geq 1} \subseteq \mathbb{R}$ convergent $\Rightarrow$ $(x_n)_{n\geq 1}$ mărginit
$\not\Leftarrow$

Dem: Din teorie, prin epsilon ε
Imediată din definiție.

T_2 (Weierstrass) $(x_n)_{n\geq 1} \subseteq \mathbb{R}$ monoton şi mărginit $\Rightarrow$ $(x_n)_{n\geq 1}$ conv.
$\not\Leftarrow$

Dem: (Schiță) $A \subseteq \mathbb{R}$ se numește (majorată) mărginită superior
$\overset{def}{\Leftrightarrow}$ $(\exists)\, \beta \in \mathbb{R}$, $(\forall) x \in A$, $x \leq \beta$
s.n. majorant al lui A

N. Boboc - An. Mat
E.U.B

Axioma (Cantor) $A \subseteq \mathbb{R}$ majorată $\Rightarrow$ $(\exists) M \in \mathbb{R}$ unic
a.î. (1) M majorant pt A $\Rightarrow M \leq M'$
(2) M' majorant pt A

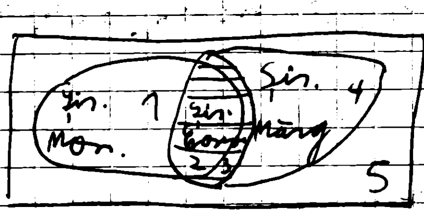
M s.n. majorarea superioară a lui A
$\overset{not}{\|}$
sup A (supremul lui A)

Obs: Analog se definesc not. de mult. minorantă şi infimul
(inf A)

I. $(x_n)_{n\geq 1}$ cresc. și mărg. sup $\xRightarrow{\text{Ax. Cantor}}$ $(\exists)$ sup $A \in \mathbb{R}$.

Se dem. că $(x_n)_{n\geq 1}$ conv. cu $\lim\limits_{n\to\infty} = \sup A$, $A = \{x_n \mid n \geq 1\} \subseteq \mathbb{R}$.

II. $(x_n)_{n\geq 1}$ descresc. și mărg. inf $\Rightarrow$ $(\exists)$ inf $A \in \mathbb{R}$ și se deduce că $(x_n)_{n\geq 1}$ conv. cu $\lim\limits_{n\to\infty} x_n = \inf A$ $\cancel{\nexists}$



1. $x_n = n$
2. $x_n = \frac{1}{n}$
3. $x_n = (-1)^n \frac{1}{n}$
4. $x_n = (-1)^n$
5. $x_n = (-1)^n \cdot n$

Th 3

Def: $(x_n)_{n\geq 1} \subseteq \mathbb{R}$. S.n. subșir al lui x_n fct. $x \circ k : \mathbb{N}^* \to \mathbb{R}$,
$\quad$ $\|$ $\qquad\qquad\qquad\qquad$ $k : \mathbb{N}^* \to \mathbb{N}^*$ s. crescătoare.
$(x : \mathbb{N}^* \to \mathbb{R})$

$$\boxed{(x \circ k)(n) \overset{not}{=} x_{k_n},\ n \in \mathbb{N}^*}$$

~~$x \circ k : \mathbb{N}^* \to \mathbb{R}$~~ $\overset{not}{=} (x_{k_n}) \subseteq \mathbb{R}$, $n \geq 1$

Th 3 $(x_n)_{n\geq 1} \subseteq \mathbb{R}$ are limita $l \in \overline{\mathbb{R}} \Rightarrow \forall (x_{k_n})_{n\geq 1}$ subș. a lui $(x_n)_{n\geq 1}$

$$x_{k_n} \xrightarrow[n]{} l$$

Ex: $x_n = (-1)^n$, $n \geq 1$

$x_{2n} = 1 \xrightarrow[n]{} 1$

$x_{2n+1} = -1 \xrightarrow[n]{} -1$

$\xRightarrow{\text{Th3}}$ $(x_n)_{n\geq 1}$ nu are limită.

PROP. (Criteriul Raportului) : $(x_n)_{n\geq 1} \subseteq (0, \infty)$

(1) $\frac{x_{n+1}}{x_n} \xrightarrow[n]{} l < 1 \Rightarrow x_n \xrightarrow[n]{} 0$

(2) $\frac{x_{n+1}}{x_n} \xrightarrow[n]{} l > 1 \Rightarrow x_n \xrightarrow[n]{} \infty$.

Dem: Se bazează pe Cr. Majorării

Exerciții: Să se studieze natura șirurilor

a) $x_n = \frac{n^k}{a^n}$, $n \geq 1$, $k \in \mathbb{N}^*$; $a>1$; b) $x_n = \frac{a^n}{n!}$, $a>1$, $n \geq 1$.

Soluția 1: b) Observ că $x_n > 0$, $(\forall)$ $a \geq 1$.

(Mon): $\boxed{\frac{x_{n+1}}{x_n} = \frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}} = \frac{a}{n+1}}$ (*) < 1, $(\forall)$ $n \geq n_a = [a]+1$.

$\Downarrow$

$(x_n)_{n \geq n_a}$ strict descresc.

$\xRightarrow{TCW}$

(Mărginit): Evident: $0 < x_n \leq x_{n_a}$, $n \geq n_a$.

$\Rightarrow (x_n)_{n \geq n_a}$ convergent $\xRightarrow[\text{(dată finită)}]{(!)}$ $(x_n)_{n \geq 1}$ convergent $\Rightarrow$

$\Rightarrow (\exists)$ $l = \lim\limits_{n \to \infty} x_n \in \mathbb{R}$

(*) $\Rightarrow x_{n+1} = x_n \cdot \frac{a}{n+1}$, $n \geq 1$ $\xRightarrow[n \to \infty]{}$ $l = l \cdot 0 = 0$.

Soluția 2: $x_n > 0$, $(\forall)$ $n \geq 1$ $\Longrightarrow \frac{x_{n+1}}{x_n} = \ldots = \frac{a}{n+1} \xrightarrow[n]{\text{Cr. Rap.}} 0$; $0<1$ $\Rightarrow$

$\Rightarrow x_n \xrightarrow[n]{} 0$

(!) - Studiază un limbaj de asamblare

- Studiază (din nou) algoritmică.

Def: $(a_n)_{n \geq 1}, (b_n)_{n \geq 1} \subset \mathbb{R}$

$a_n \to 0$, $b_n \to 0$

$(a_n)_{n \geq 1}$ converge mai rapid ca $(b_n)_{n \geq 1}$ (simb $a_n \ll b_n$)

$\overset{def}{\Longleftrightarrow} \frac{a_n}{b_n} \xrightarrow[n]{} 0$.

Exp: a) $\frac{1}{n!} \ll \frac{1}{a^n}$ (vezi soluția anterioară).

b) $\frac{1}{a^n} \ll \frac{1}{n^k}$.

$\Theta(n^2)$ $\quad \frac{n^2}{2^n} \xrightarrow[n]{} 0$

$\Theta(2^n)$

Criteriul Stolz - Cesaro / Lema Stolz - Cesaro.

Exp: Să se studieze natura șirului $x_n = \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{n}$, $n \geq 1$.

? 1883, Cauchy - Nașterea analizei ?

Fie $(a_n)_{n\geq 1}, (b_n)_{n\geq 1} \subseteq \mathbb{R}$ cu proprietățile

a) $(b_n)_{n\geq 1}$ monoton nemărginit.

b) $\frac{a_{n+1}-a_n}{b_{n+1}-b_n} \to l \in \overline{\mathbb{R}}$

Atunci $\frac{a_n}{b_n} \to l$

! STUDIU INDIV.

Th. Weierstrass UPDATE.

$(x_n)_{n\geq 1}$ cresc. și nemărg. (desc.)

$\overset{\text{imp}}{\Longrightarrow} x_n \to \overset{\infty}{-\infty}$.

Dem: foarte tehnică, se omite (Cr. Maj + ε).

Exp: $x_n = \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{n}$, $n \geq 1$

$a_n = 1+\frac{1}{2}+\dots+\frac{1}{n}$; $b_n = n$, $n \geq 1$.

Obs. că (a) $(b_n)_{n\geq 1}$ cresc. și nemărg.

(b) $\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{\left(\cancel{1+\frac{1}{2}+\dots}+\frac{1}{n+1}\right)-\left(\cancel{1+\frac{1}{2}+\dots+\frac{1}{n}}\right)}{\cancel{n}+1-\cancel{n}}$

$= \frac{1}{n+1} \xrightarrow[n]{} 0$.

$\overset{\text{Cr. S-C}}{\Longrightarrow} \frac{a_n}{b_n} = x_n \xrightarrow[n]{} 0$

SE MI NAR

Cor 1: $a_n \to l \in \overline{\mathbb{R}} \Rightarrow \frac{a_1+a_2+\dots+a_n}{n} \to l$ (M. arit tind la l)

Cor 2: $\underset{a_n>0}{a_n} \longrightarrow l \in [0,\infty] \Rightarrow \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} \longrightarrow l$

Cor 3: (D'Alembert) $a_n > 0$, $\frac{a_{n+1}}{a_n} \to l \Longrightarrow \sqrt[n]{a_n} \to l$, $(0^0, \infty^0)$

SEMINAR

23.10.2019.

Ex: Să se studieze natura convergenței șirurilor:

Cr. Rap. a) $x_n = \frac{n^k}{a^n}$, $a>1$, $k \in N^*$, $n \geq 1$ $\left(\frac{\infty}{\infty}\right)$

Cr. S-C b) $x_n = \frac{\ln n}{n}$, $n \geq 1$ $\left(\frac{\infty}{\infty}\right)$

Cr. S-C c) $x_n = \frac{1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}}{\sqrt{n}}$, $n \geq 1$. $\left(\frac{\infty}{\infty}\right)$

Cr. S-C d) $x_n = \frac{1^p+2^p+\dots+n^p}{n^{p+1}}$, $n \geq 1$, $p \in N^*$ $\left(\frac{\infty}{\infty}\right)$

Cr. D.Alb e) $x_n = \sqrt[n]{n!}$, $n \geq 1$ (∞^0)

Cr. D.Alb f) $x_n = \sqrt[n]{n(n+1)\dots(2n)}$, $n \geq 1$ (∞^0)

Cr. D.Alb. g) $x_n = \sqrt[n]{\frac{(n!)^2}{(2n)! \cdot 8^n}}$, $n \geq 1$.

R: Cr. Raportului

$x_n > 0$, $\frac{x_{n+1}}{x_n} \to l$, $\begin{cases} l<1 \Rightarrow x_n \to 0 \\ l>1 \Rightarrow x_n \to \infty \end{cases}$

REC
APL
TUL
ARE

R: Cr. Stolz - Cesaro

$(a_n)_n, (b_n)_n \subseteq \mathbb{R}$ cu prop.

(1) $(b_n)_{n \geq 1}$ monot $\uparrow$ nemărg.

(2) $\frac{a_{n+1}-a_n}{b_{n+1}-b_n} \to l \in \overline{\mathbb{R}}$

$\Rightarrow \frac{a_n}{b_n} \to l$.

Cor 1: $a_n \to l \in \overline{\mathbb{R}} \Rightarrow \frac{a_1+\dots+a_n}{n} \to l$

Cor 2: $a_n > 0$, $a_n \to l \in [0,\infty)$ $\Rightarrow \sqrt[n]{a_1 \dots a_n} \to l$.

Cor 3: (Cr. D. Alembert)

$a_n > 0$, $\frac{a_{n+1}}{a_n} \to l \Rightarrow \sqrt[n]{a_n} \to l$

Aplic Cor 1 pt $(\ln a_n)_{n \geq 1}$, $l \neq 0$.

$b_n = \frac{a_n}{a_{n-1}}$, $n \geq 2$ $\to l$.

$b_1 = a_1$.

$\sqrt[n]{b_1 b_2 \dots b_n} = \sqrt[n]{a_1 \frac{a_2}{a_1} \frac{a_3}{a_2} \dots}$

a) $x_n = \dfrac{n^k}{a^n}$

Obs. că $x_n > 0$; $n \geq 1$.

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)^k}{a^{n+1}} \cdot \frac{a^n}{n^k} = \frac{1}{a}\left(\underbrace{\frac{n+1}{n}}_{\to 1}\right)^k \xrightarrow[n]{} \frac{1}{a}\cdot 1^k = \frac{1}{a}.$$

$$a > 1 \Rightarrow \frac{1}{a} < 1 \xrightarrow{\text{Cr. Rap.}} x_n \to 0.$$

b) Fie $\left.\begin{array}{l} a_n := \ln n \\ b_n := n \end{array}\right\} n \geq 1.$

(1) $(b_n)_{n\geq 1}$ mon. cresc. și nemărg.

(2) $$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{\ln(n+1)-\ln(n)}{n+1-n} = \ln\left(\frac{n+1}{n}\right) \to \ln 1 = 0.$$

$$\xrightarrow{\text{Cr. S-C}} \frac{a_n}{b_n} = x_n \to 0.$$

c) Fie $a_n := 1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$

$b_n := \sqrt{n}$.

(1): $(b_n)_{n\geq 1}$ mon. cresc. și nemărg.

(2): $$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n+1}} - \left(1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}\right)}{\sqrt{n+1}-\sqrt{n}} = \cancel{\frac{\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}}{\sqrt{n+1}-\sqrt{n}}} =$$

$$= \cancel{\frac{\frac{\sqrt{n}-\sqrt{n+1}}{\sqrt{n}\sqrt{n+1}}}{\sqrt{n+1}-\sqrt{n}}} =$$

$$= \frac{\frac{1}{\sqrt{n+1}}}{\sqrt{n+1}-\sqrt{n}} = \frac{\frac{1}{\sqrt{n+1}}}{\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}} = \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}} = \frac{\sqrt{n}}{\sqrt{n}}\left(\frac{\sqrt{1+\frac{1}{n}}+1}{\sqrt{1+\frac{1}{n}}}\right) \to \frac{\sqrt{1+0}+1}{\sqrt{1+0}} =$$

$$= \frac{2}{1} = 2$$

(1) și (2) $\Rightarrow \dfrac{a_n}{b_n} \to 2$.

d) Fie $\left.\begin{array}{l} a_n := 1^p + 2^p + \dots + n^p \\ b_n := n^{p+1} \end{array}\right. , n \geq 1.$

(1) $(b_n)_{n\geq 1}$ mon. cresc. și nemărg. (Evident)

c) $\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{(1^p+2^p+\dots+n^p+(n+1)^p)-(1^p+2^p+\dots+n^p)}{(n+1)^{p+1}-n^{p+1}} = \frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}}$

$= \frac{n^p + C_p^1 n^{p-1} + \dots + 1\cdot C_p^p}{\cancel{n^{p+1}} + C_{p+1}^1 n^p + \dots + \cancel{n^{p+1}}} \to \frac{1}{C_{p+1}^1} = \frac{1}{p+1}$

$\xRightarrow{\text{Th. S-C}} \frac{a_n}{b_n} = x_n \to \frac{1}{p+1}$.

e) $a_n := n!,\ n\geq 1$.

Obs. că $a_n > 0,\ n\geq 1$.

$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} = n+1 \xrightarrow[n]{} \infty \xRightarrow{\text{Th. D'Alb.}} \sqrt[n]{a_n} \to \infty$.

f) $a_n := n(n+1)\dots(2n)$

Obs. că $a_n > 0,\ n\geq 1$.

$\frac{a_{n+1}}{a_n} = \frac{(n+1)(n+2)\dots(2n)(2n+1)(2n+2)}{n(n+1)\dots(2n)} = \frac{(2n+1)(2n+2)}{n} = \frac{4n^2+6n+2}{n}$

$\to \infty$

$\xRightarrow{\text{Th. D'Alb.}} \sqrt[n]{a_n} \to \infty$

g) $a_n := \frac{(n!)^2}{(2n)!\cdot 8^n},\ n\geq 1$

$a_n := \frac{(n!)^2}{(2n)!\cdot a^n},\ n\geq 1,\ a>1$

Temă $\boxed{\frac{(n!)^2}{(2n)!\cdot a^n} \to 0,\ a>1}$! Exerciții

$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!^2}{(2n+2)!\cdot a^{n+1}}}{a_n} = \frac{(n+1)!^2}{(2n+2)!\cdot a^{n+1}} \cdot \frac{(2n)!\cdot a^n}{(n!)^2} = \frac{(n+1)!^2}{n!^2}\cdot\frac{a^n}{a^{n+1}}\cdot\frac{(2n)!}{(2n+2)!} =$

$= (n+1)^2 \cdot \frac{1}{a}\cdot\frac{1}{(2n+1)(2n+2)} = \frac{(n+1)^2}{a(2n+1)(2n+2)} \to \frac{1}{4a} \xRightarrow{\text{Th. Rap.}} a_n \to 0.$

$\xRightarrow{\text{D.Alb.}} \sqrt[n]{a_n} \to \frac{1}{4a}$.

Ex. Std. natura șirurilor.

a) $x_n = \frac{\cos x}{2} + \frac{\cos 2x}{2^2} + \dots + \frac{\cos nx}{2^n}$, $n \geq 1$, $x \in \mathbb{R}$.

b) $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$, $n \geq 1$.

! Șiruri Cauchy

Def: $(x_n)_{n\geq 1} \subseteq \mathbb{R}$ s.n. șir Cauchy $\Leftrightarrow (\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n,m \geq n_\varepsilon)$

$(|x_n - x_m| < \varepsilon)$

$\Leftrightarrow (\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})\begin{pmatrix}\forall n \geq n_\varepsilon \\ \forall p \geq 1\end{pmatrix}(|x_{n+p} - x_n| < \varepsilon)$

Prop 1: $(x_n)_{n\geq 1} \subseteq \mathbb{R}$ convergent $\Rightarrow (x_n)_{n\geq 1}$ Cauchy.

Dem: Imediat din def.

Prop 2: $(x_n)_{n\geq 1} \subseteq \mathbb{R}$ Cauchy $\Rightarrow (x_n)_{n\geq 1}$ mărginit

Def: Imediat din def.

Prop 3: (LEMA LUI CEZARO):

$(x_n)_{n\geq 1} \subseteq \mathbb{R}$ mărginit $\Rightarrow \exists (x_{k_n})_{n\geq 1}$ subșir. conv.

Prop 4: $(x_n)_{n\geq 1}$ Cauchy și admite un subșir convergent $\Rightarrow$

$\Rightarrow (x_n)_{n\geq 1}$ convergent

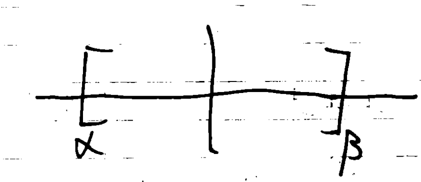
Dem: Imediat din def.

TH (CAUCHY) $(x_n)_{n\geq 1} \subseteq \mathbb{R}$ ~~cauchy~~ conv. $\Leftrightarrow (x_n)_{n\geq 1}$ Cauchy

Dem: „$\Rightarrow$" Prop 1 „$\Leftarrow$" Prop 2+3+4

Schiță (dem Prop 3)

$(\exists)\ \alpha, \beta \in \mathbb{R}$, $\alpha \leq x_n \leq \beta$, $\forall n \geq 1$.



Înjumătățind mereu intervalul $I_0 := [\alpha, \beta]$ a.î. în fiecare interval obținut să avem o inf. de termeni ai șirului se obț:

$(I_n)_{n\geq 1}$, $l(I_n) = \frac{\beta - \alpha}{2^n}$, $n \geq 0$, $I_{n+1} \subset I_n$, $n \geq 0$.

Th. [Șirul de intervale închise, descrescător] $\Leftrightarrow$ Axioma Cantor.

$(I_n)_{n\geq 0}$, $I_n = [\alpha_n, \beta_n]$, $n \geq 0$, $I_{n+1} \subset I_n$, $n \geq 0$.

$\Rightarrow \bigcap_{n\geq 0} I_n \neq \emptyset$

Dacă, în plus, ~~$n\geq 0$~~ $\ell[I_n] \xrightarrow[n]{} 0 \Rightarrow \bigcap_{n\geq 0} I_n = \{c\}$.

În fiecare I_n, aleg $x_{k_n} \in I_n$, $n\geq 0$. Aplic. Th. șir. de int. închise lui

$(I_n)_{n\geq 1} \Rightarrow \bigcap_{n\geq 0} I_n = \{c\}$ $\Big| \Rightarrow |x_{k_n} - c| \leq \frac{\beta - \alpha}{2^n}$, $n \geq 0$. $\Big| \Rightarrow x_{k_n} \to c$.

$x_{k_n} \xrightarrow[n]{} c$ $\quad \frac{\beta-\alpha}{2^n} \xrightarrow[n]{} 0$

Rol a) $\varepsilon > 0$, $|x_{n+p} - x_n| = \left|\sum_{k=1}^{n+p} \frac{\cos kx}{2^k}\right| - \left|\sum_{k=1}^{n} \frac{\cos kx}{2^k}\right| =$

$= \left|\sum_{k=n+1}^{n+p} \frac{\cos kx}{2^k}\right| \leq \sum_{k=n+1}^{n+p} \left|\frac{\cos kx}{2^k}\right| \leq \sum_{k=n+1}^{n+1} \frac{1}{2^k} = \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{n+p}}$

$= \frac{1}{2^{n+1}}\left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{p-1}}\right) = \frac{1}{2^{n+1}} \frac{1-\frac{1}{2^p}}{\frac{1}{2}} < \frac{1}{2^n} < \varepsilon$.

$\Leftrightarrow \frac{1}{\varepsilon} < 2^n \Leftrightarrow \log_2 \frac{1}{\varepsilon} < n$ $\quad p \geq 1$

$n_\varepsilon := [\log_2 \frac{1}{\varepsilon}] + 1$; $n \geq n_\varepsilon \Rightarrow |x_{n+p} - x_n| < \varepsilon$

$\forall p \geq 1$ $\quad \xRightarrow{def} (x_n)_{n\geq 1}$ Cauchy.

$\xRightarrow[\text{TH(CAUCHY)}]{} (x_n)_{n\geq 1}$ convergent.

Structure Computer Organization. Andrew Tanenbaum.

a) $(\forall)\varepsilon > 0$, $(\exists) n_\varepsilon \in \mathbb{N}$, $(\forall) n \geq n_\varepsilon \wedge (\forall) p \geq 1$, $|x_{n+p} - x_n| < \varepsilon$

$|x_{n+p} - x_n| = \left|\sum_{k=1}^{n+p} \frac{\cos kx}{2^k} - \sum_{k=1}^{n} \frac{\cos kx}{2^k}\right| = \left|\sum_{k=n+1}^{n+p} \frac{\cos kx}{2^k}\right| \leq \sum_{k=n+1}^{n+p} \frac{1}{2^k} =$

$$= \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots + \frac{1}{2^{n+p}} = \frac{1}{2^{n+1}}\left(1+\frac{1}{2}+\dots+\frac{1}{2^{p-1}}\right) = \frac{1}{2^{n+1}}\cdot\frac{2^{p}-1}{2^{p-1}} = \frac{1}{2^{n+1}}\cdot\frac{1-\frac{1}{2^{p}}}{\frac{1}{2}} <$$

$$< \frac{1}{2^{n}} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < 2^{n} \Leftrightarrow \log_2\frac{1}{\varepsilon} < n \Rightarrow n_\varepsilon := \left[\log_2\frac{1}{\varepsilon}\right]+1.$$

$n \ge n_\varepsilon$, $(\forall)\, p \ge 1 \Rightarrow |x_{n+p}-x_n| < \varepsilon \xRightarrow[def.]{} (x_n)_{n\ge1}$ Cauchy $\xrightarrow[Cauchy]{TH.} (x_n)_{n\ge1}$ conv.

$$\left[1+\frac{1}{2}+\dots+\frac{1}{2^{n}} = \frac{2^{n+1}-1}{2^{n}}\right]$$

! PARTIAL : 19.11.2019

29.10.2019.

Serii de Nr. Reale.

- Se cons. $(x_n)_{n\ge1} \subset \mathbb{R}$
- ~~Se pune probl. să se dea un sens riguros sumei $x_1+x_2+\dots+x_n+\dots$ = suma~~
- Se rezolvă probl. astfel.

$S_1 := x_1$; $S_2 := x_1+x_2$; ... ; $S_n := x_1+\dots+x_n$ $(\forall)\, n\ge1$.

$\rightarrow (S_n)_{n\ge1}$ s.n. şirul sumelor parţiale (ordonate) asociate lui $(x_n)_{n\ge1}$.

• Dacă $(S_n)_{n\ge1}$ are limită $l\in\overline{\mathbb{R}}$, atunci: $\boxed{\text{suma}} = l$

Def: Fie $(x_n)_{n\ge1} \subseteq \mathbb{R}$. Atunci $((x_n)_{n\ge1}, (S_n)_{n\ge1}) \overset{not}{=:} \sum_{n\ge1} x_n$ s.n. serie de nr. reale (de term. general $(x_n)_{n\ge1}$)

1) $\sum_{n\ge1} x_n$ s.n. serie convergentă $\overset{def}{\Longleftrightarrow}$ $(S_n)_{n\ge1}$ convergent

2) $\sum_{n\ge1} x_n$ s.n. serie divergentă $\overset{def}{\Longleftrightarrow}$ $(S_n)_{n\ge1}$ divergent

!! $(x_n)_{n\ge1}$ diverg $\xLeftrightarrow[def]{}$ $(x_n)_{n\ge1}$ nu e converg.

Există posibilitățile:

a) $\exists \lim\limits_{n\to\infty} S_n \in \{\pm\infty\}$ $\vee$ b) $(S_n)_{n\geq 1}$ nu are limită

În situațiile 1) și 2) a) $\lim\limits_{n\to\infty} S_n \overset{not}{=\!=}: \sum\limits_{n=1}^{\infty} x_n \in \overline{\mathbb{R}}$ $\xrightarrow{\text{s.n. suma seriei}} \sum\limits_{n\geq 1} x_n$

La 2b: seria s.n. oscilantă.

Ex1: Să se det. suma seriilor:

a) $\sum\limits_{n\geq 1} \frac{1}{n(n+1)}$; b) $\sum\limits_{n\geq 1} \frac{1}{2^n}$; c) $\sum\limits_{n\geq 1} \ln\frac{n+1}{n}$, d) $\sum\limits_{n\geq 1} \frac{1}{\sqrt{n}}$.

Sol: a) $x_n := \frac{1}{n(n+1)}$, $n \geq 1$

$$S_n := x_1 + x_2 + \ldots + x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$$

Obs. că $x_n = \frac{1}{n} - \frac{1}{n+1}$

$$S_n = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \ldots + \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} \xrightarrow[n]{} 1.$$

$\Rightarrow \sum\limits_{n\geq 1} \frac{1}{n(n+1)}$ convergentă și are suma $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Avem scrierea: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} + \ldots = 1$.

---

b) $x_n := \frac{1}{2^n}$, $n \geq 1$.

$$S_n := x_1 + x_2 + \ldots + x_n = \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$$

$$\left[1 + q + \ldots + q^n = \frac{1 - q^{n+1}}{1 - q}, q \neq 1\right]$$

$$S_n = \frac{1}{2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}}\right) = \cancel{\frac{1}{2}} \cdot \frac{1 - \frac{1}{2^n}}{\cancel{1 - \frac{1}{2}}} = 1 - \frac{1}{2^n} \xrightarrow[n]{} 1 - 0 = 1.$$

$\Rightarrow \sum\limits_{n\geq 1} \frac{1}{2^n}$ conv. și are suma $\sum\limits_{n=1}^{\infty} \frac{1}{2^n} = 1$

Avem: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} + \ldots = 1$.

c) $x_n := \ln \frac{n+1}{n} = \ln(n+1) - \ln n, \ n \geq 1.$

$S_n := x_1 + x_2 + \ldots + x_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \ldots + (\ln(n+1) - \ln n)$

$= (\cancel{\ln 2} - \ln 1) + (\cancel{\ln 3} - \cancel{\ln 2}) + \ln(4 - \cancel{\ln 3}) + \ldots \cancel{\ln(n-1)} - \cancel{\ln(n-2)} + \cancel{\ln(n)} - \cancel{\ln(n-1)} +$

$+ \ln(n+1) - \cancel{\ln(n)}$

$= \ln(n+1) - \ln 1$

$= \ln(n+1) \xrightarrow[n]{} \infty$

$\Rightarrow \sum_{n\geq 1} \ln \frac{n+1}{n}$ divergentă, cu suma $\sum_{n=1}^{\infty} \ln \frac{n+1}{n} = +\infty$

(au oricâți naturali)

Aveam: $\ln \frac{2}{1} + \ln \frac{3}{2} + \ldots + \ln\left(\frac{n+1}{n}\right) + \ldots = \infty$

! $\boxed{\text{Obs}}$ a) $\sum_{n\geq 1} x_n$ (C) conv $\Rightarrow x_n \xrightarrow[n]{} 0$

$x_n = S_n - S_{n-1}$, $S_n \to 0$, $S_{n-1} \to 0$

! $\boxed{\text{Obs 2}}$ $\sum_{n\geq 1} x_n \sim \sum_{n\geq k} x_n$.

(/ convergența)

! $\boxed{\text{Obs}}$ b) $\sum_{n\geq 1} x_n$ S.T.P. $\Rightarrow S_n \nearrow \Rightarrow \sum_{n\geq 1} x_n$ are sumă și $\sum_{n=1}^{\infty} x_n \in [0, \infty]$.

(serie cu termeni poz)

d) $x_n := \frac{1}{\sqrt[n]{n}}$

!!! $\left[\sqrt[n]{n} \xrightarrow[n]{} 1 \text{ cu def cu } \Sigma\right]$

$x_n \xrightarrow[n]{} \frac{1}{1} = 1 \neq 0 \xRightarrow[\text{Obs a}]{} \sum_{n\geq 1} \frac{1}{\sqrt[n]{n}}$ divergentă

Dar $\sum_{n\geq 1} x_n$ S.T.P $\Rightarrow$ are sumă

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} = +\infty.$

Aveam: $1 + \frac{1}{2} + \ldots + \frac{1}{\sqrt[n]{n}} + \ldots = \infty.$

! TH (Cauchy - Criteriul gen. de converg. al seriilor)

$\sum_{n\geq 1} x_n$ convergent $\Leftrightarrow (\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N}) \begin{pmatrix} \forall n \geq n_\varepsilon \\ \forall p \geq 1 \end{pmatrix} \left(|x_{n+1} + \ldots + x_{n+p}| < \varepsilon\right)$

def. $\Updownarrow$ $(S_n)_{n\geq 1}$ conv $\xLeftrightarrow[\text{TH. Cauchy}]{} (S_n)_{n\geq 1}$ Cauchy $\overset{\text{def}}{\Leftrightarrow}$.

Criteriul 1 de convergență a seriilor (de comparație)

Fie $\sum_{n\geq1} x_n$, $\sum_{n\geq1} y_n$ S.T.P. a.î. $x_n \leq y_n$, $n \geq k$.

a) Dacă $\sum_{n\geq1} y_n$ (c), atunci $\sum_{n\geq1} x_n$ (c) — c – convergent.

b) Dacă $\sum_{n\geq1} x_n$ (D), atunci $\sum_{n\geq1} y_n$ (D) — D – divergent

Demonstrație: imediată din Crit. Cauchy (Vezi Obs c)

Ex: (Serii de referință)

(1) Seria geometrică (S.G): $\sum_{n\geq0} q^n =: x_n$

$$S_n = x_0 + x_1 + \dots + x_n = 1 + q + q^2 + \dots + q^n = \begin{cases} \frac{1-q^{n+1}}{1-q}, & q \neq 1. \\ n+1, & q = 1 \to \infty \end{cases}$$

$$\boxed{q^n \to \begin{cases} 0, & q \in (-1,1) \\ \infty, & q > 1 \end{cases}}$$

$$\frac{1-q^{n+1}}{1-q} \xrightarrow[n]{} \begin{cases} \frac{1}{1-q}, & q \in (-1,1) \\ +\infty, & q > 1 \\ \text{nu are limită}, & q \leq -1 \end{cases}$$

! $\sum_{n\geq0} q^n$
- (c) cu suma $\frac{1}{1-q}$, pt $q \in (-1,1)$
- (d) cu suma $+\infty$ pt. $q \geq 1$
- (osc) pt $q \leq -1$
  oscilantă

(2) Seria Armonică (S.A.)

$$\boxed{\sum_{n\ge 1} \frac{1}{n^{\alpha}} \quad \begin{cases} (C) & \text{pt } \alpha > 1 \\ (D) & \text{pt } \alpha \le 1 \end{cases}}$$

Ex. Să se studieze conv. seriilor:

a) $\sum_{n\ge 1} \frac{1}{n+2^n}$; b) $\sum_{n\ge 1} \frac{1}{\sqrt{n^4+2}}$; c) $\sum_{n\ge 1} \sin \frac{1}{n^2}$; d) $\sum_{n\ge 1} \sin \frac{1}{n}$.

a) $x_n := \frac{1}{n+2^n} \le \underbrace{\frac{1}{2^n}}_{=: y_n}$, $n \ge 1$.

$\sum_{n\ge 1} y_n$ (C) (S.G. cu $q = \frac{1}{2}$)

$\Rightarrow \sum_{n\ge 1} x_n$ (C)

b) $x_n := \frac{1}{\sqrt{n^4+2}} \le \frac{1}{\sqrt{n^4}} = \underbrace{\frac{1}{n^2}}_{=: y_n}$, $n \ge 1$

$\sum y_n$ (C) (S.A. $\alpha = 2$)

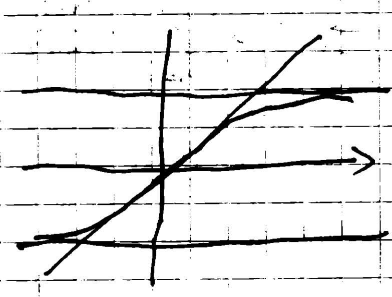
$\xrightarrow{\text{Crit. 1 comp.}} \sum_{n\ge 1} x_n$ (C).

!! Pt. calculul alte ramuri ale Analizei (Analiză Complexă, Teoria Structurii, An. Reală)

c) $x_n := \sin \frac{1}{n^2} < \underbrace{\frac{1}{n^2}}_{=: y_n}$, $n \ge 1$ (grafic sinus)

$\sum_{n\ge 1} y_n$ (C) (S.A. $\alpha = 2$

$\Rightarrow \sum_{n\ge 1} x_n$ (C)



sin;

Criteriul 2 de comp. a seriilor (de comparație)

Fie $\sum_{n\geq 1} x_n$, $\sum_{n\geq 1} y_n$, S.T.P. a.î. $\exists\ \lim_{n\to\infty} \frac{x_n}{y_n} = l \in (0,\infty)$

Atunci $\sum_{n\geq 1} x_n \sim \sum_{n\geq 1} y_n$

au aceeași natură

Dem (schiță) — 0, $l-\varepsilon$, l, $l+\varepsilon$, ∞

$\Rightarrow \exists\ \varepsilon_0 > 0,\ l-\varepsilon_0 > 0 \Rightarrow \exists\ n_0,\ n \geq 0$.

$(l-\varepsilon) y_n < x_n < (l+\varepsilon) y_n$

d) $x_n := \sin\frac{1}{n},\ n\geq 1$

$y_n := \frac{1}{n},\ n\geq 1$

$\frac{x_n}{y_n} = \frac{\sin\frac{1}{n}}{\frac{1}{n}} \to 1 \in (0,\infty)$

$\overset{Cr.2\ comp.}{\Longrightarrow} \sum_{n\geq 1} x_n \sim \sum_{n\geq 1} y_n$ | $\Rightarrow \sum x_n\ (D)$

$\sum y_n\ (D)\ (S.A.\ \alpha = 1)$

30.10.2019.

Seminar

Ex1 Să se det. suma seriilor

a) $\sum_{n\geq 1} \frac{1}{n(n+1)}$; b) $\sum_{n\geq 2} \ln\left(\frac{n^2-1}{n^2}\right)$; c) $\sum_{n\geq 1} \frac{1}{4n^2-1}$; d) $\sum_{n\geq 1} \frac{2n+1}{3n+2}$

e) $\sum_{n\geq 1} (-1)^{n+1} \frac{3^{n-1}}{4^{n+2}}$

REC.: $\sum_{n\geq 1} x_n \overset{not}{:=} ((x_n)_{n\geq 1}, (S_n)_{n\geq 1})$ s.n. serie

$S_n = x_1 + x_2 + \dots + x_n,\ n \geq 1$.

1) $\sum_{n\geq 1} x_n\ (C) \Leftrightarrow (S_n)_{n\geq 1}$ conv.

2) $\sum_{n\geq 1} x_n\ (D) \Leftrightarrow (S_n)_{n\geq 1}$ div. — a) $\exists\ \lim S_n \in \{\pm\infty\}$; b) $\nexists\ \lim S_n$

În situațiile 1) și 2) a) $\sum_{n=1}^{\infty} x_n := \lim_{n\to\infty} S_n$

└ s.n. suma seriei

Obs 1) $\sum_{n\geq1} x_n$ S.T.P $\Rightarrow$ are sumă

2) $\sum_{n\geq1} x_n$ (C) $\Rightarrow$ $x_n \to 0$

---

a) $x_n := \frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{2}{n(n+2)}\right) = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$S_n = x_1+x_2+\dots+x_n = \frac{1}{1\cdot3} + \frac{1}{2\cdot4} + \frac{1}{3\cdot5} + \dots + \frac{1}{n(n+2)} =$

$= \frac{1}{2}\left(\frac{1}{1} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \dots + \cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n-1}} - \frac{1}{n+1} + \cancel{\frac{1}{n}} - \frac{1}{n+2}\right)$

$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) \longrightarrow \frac{1}{2}\cdot\frac{3}{2} = \frac{3}{4}$.

$\Rightarrow \sum_{n\geq1} x_n$ (C) și are suma $\sum_{n=1}^{\infty} x_n = \frac{3}{4}$.

b) $x_n := \ln\left(\frac{n^2-1}{n^2}\right) = \ln(n^2-1) - \ln n^2$ X

$= \ln\left(1 - \frac{1}{n^2}\right)$ X

$= \ln\left(\frac{\frac{n-1}{n}}{\frac{n}{n+1}}\right) = \ln\frac{n-1}{n} - \ln\frac{n}{n+1}$ ✓

$S_n = x_2 + x_3 + \dots + x_n = \ln\frac{1}{2} - \cancel{\ln\frac{2}{3}} + \cancel{\ln\frac{2}{3}} - \cancel{\ln\frac{3}{4}} + \dots + \cancel{\ln\frac{n-2}{n-1}} - \cancel{\ln\frac{n-1}{n}} +$

$+ \cancel{\ln\frac{n-1}{n}} - \ln\frac{n}{n+1}$

$S_n = \ln\frac{1}{2} - \ln\frac{n}{n+1} = \ln\left(\frac{\frac{1}{2}}{\frac{n}{n+1}}\right) = \ln\frac{n+1}{2n} = \ln\frac{1+\frac{1}{n}}{2}$.

$\to \ln\frac{1}{2} - \ln 1 = \ln\frac{1}{2} = -\ln 2$.

$\Rightarrow \sum_{x\geq2} x_n$ (C) și are suma $\sum_{x=2}^{\infty} x_n = -\ln 2$.

c) $x_n := \frac{1}{4n^2-1} = \frac{1}{(2n+1)(2n-1)} =$

$\frac{1}{2n-1} - \frac{1}{2n+1} = \frac{2n+1-2n+1}{(2n+1)(2n-1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$

$S_n = x_1 + x_2 + \dots + x_n$

$= \frac{1}{2}\left(\frac{1}{1} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \dots + \cancel{\frac{1}{2n-3}} + \cancel{\frac{1}{2n-1}} - \cancel{\frac{1}{2n-1}} + \frac{1}{2n+1}\right)$

$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2n+1}\right) \to \frac{1}{2}\cdot 1 = \frac{1}{2}$

$\Rightarrow \sum_{n\ge1} x_n$ (C) şi are suma $\sum_{n\ge1}^{\infty} x_n = \frac{1}{2}$.

~~d)~~ $\cancel{x_n := \frac{2n+1}{3n+2} = \frac{3n+2-(n+1)}{3n+2} = 1 - \frac{n+1}{3n+2} = 1 - \frac{1}{3}\left(\frac{3n+3}{3n+2}\right) = 1 - \frac{1}{3}\left(1 + \frac{1}{3n+2}\right)}$

$\cancel{\frac{1}{3n+1} - \frac{1}{3n+2} =}$

e) $(-1)^{n+1}\frac{3^{n-1}}{4^{n+2}} = \frac{1}{4^3}\cdot\left(-\frac{3}{4}\right)^{n-1}\cdot\cancel{(-1)^2}$

~~d)~~ $\cancel{x_n}$ ~~$\sum_{n\ge1} x_n$ S.T.P ⇒ are suma.~~

$\cancel{x_n := \frac{2n+1}{3n+2} \to \frac{2}{3}}$

d) $x_n := \frac{2n+1}{3n+2} \to \frac{2}{3} \ne 0 \xRightarrow{\text{Obs } 2} \sum_{n\ge1} x_n$ (D) $\Big| \Rightarrow \sum_{n=1}^{\infty} x_n = \infty$.

Dar $\sum x_n$ STP $\xRightarrow[\text{Obs } 1]{}$ are suma.

e) $x_n := (-1)^{n+1}\frac{3^{n-1}}{4^{n+2}} = \frac{1}{4^3}\cdot\left(-\frac{3}{4}\right)^{n-1}, \ n\ge1.$

$$\sum_{n=1}^{\infty} x_n = \frac{1}{4^3}\sum_{n=1}^{\infty}\left(\frac{-3}{4}\right)^{n-1} = \frac{1}{4^3}\cdot\sum_{n=0}^{\infty}\left(\frac{-3}{4}\right)^{n} = \frac{1}{4^3}\cdot\frac{1}{1-\left(-\frac{3}{4}\right)} =$$

$$= \frac{1}{4^3}\cdot\frac{1}{1+\frac{3}{4}} = \frac{1}{4^3}\cdot\frac{4}{7} = \frac{1}{7\cdot4^2}$$

$\frac{1+3+\dots+(2n-1)}{n^2}$; $\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum 1 = \cancel{2}\frac{n(n+1)}{\cancel{2}} - n = n^2$

$\frac{1\cdot3+2\cdot4+\dots+n(n+2)}{n^2}$; $\sum_{k=1}^{n} k(k+2)$.

Ex2. Să se studieze natura seriilor

a) $\sum\limits_{n\ge1} \frac{1}{\sqrt[3]{n^4+2n^2+2}}$; b) $\sum\limits_{n\ge2}\frac{1}{\ln n}$; c) $\sum\limits_{n\ge1} \operatorname{tg}\frac{1}{n}$; d) $\sum\limits_{n\ge1}\operatorname{tg}\frac{1}{n^2}$;

e) $\sum\limits_{n\ge1}\ln\left(1+\frac{1}{n\sqrt{n}}\right)$

REC: Cr 1. comp

$\sum x_n, \sum y_n$ STP, $0\le x_n\le y_n$, $n\ge1$.

a) $\sum y_n$ (C) $\Rightarrow \sum x_n$ (C)

b) $\sum x_n$ (D) $\Rightarrow \sum y_n$ (D)

Cr 2. comp.

$\sum x_n, \sum y_n$ STP.

Dc. $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = l\in(0,\infty) \Rightarrow \sum x_n \sim \sum y_n$

S.G $\sum\limits_{n\ge0} q^n$ — (C) pt $q\in(-1,1)$; (D) cu suma ∞ pt $q\ge1$; osc. pt $q\le -1$

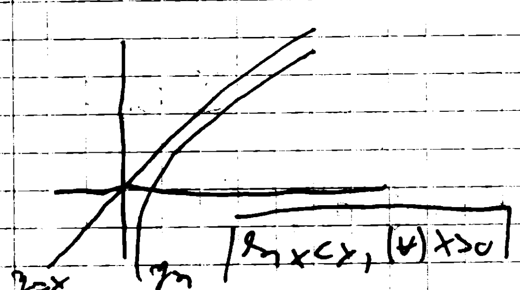
S.A $\sum\limits_{n\ge1}\frac{1}{n^\alpha}$ — (C) pt $\alpha>1$; (D) pt $\alpha\le1$

---

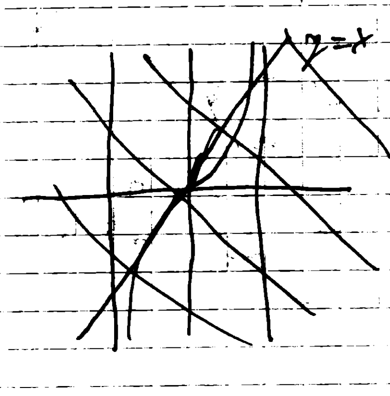
a) $\underset{\substack{\vee\\0}}{x_n} := \frac{1}{\sqrt[3]{n^4+2n^2+2}} \le \frac{1}{\sqrt[3]{n^4}} = \frac{1}{n^{\frac{4}{3}}} =: y_n$ $\overset{\text{Cr 1}}{\underset{\text{comp.}}{\Rightarrow}} \sum x_n$ (C)

Dar $\sum y_n$ (C) (S.A cu $\alpha = \frac{4}{3} > 1$)

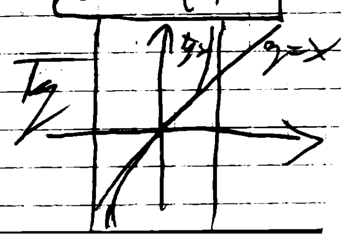
b) ~~$x_n := \frac{1}{\ln n}$~~



$\boxed{\ln x < x,\ (\forall)x>0}$



$\boxed{\operatorname{tg} x > x \quad (\forall) x\in\left(0,\frac{\pi}{2}\right)}$



b) $x_n := \frac{1}{\ln n} > \frac{1}{n}, n \ge 2$
* $\sum_{n\ge 2} \frac{1}{n}$ (D) (S.A cu $\alpha = 1$)

$\xRightarrow{\text{Cr.1 Comp.}} \sum_{n\ge 2} \frac{1}{\ln n}$ (D).

* $\sum_{n\ge 2} \frac{1}{n} \sim \sum_{n\ge 1} \frac{1}{n}$ (au aceeași nat., nu se mod. decât suma)

c) $x_n := \operatorname{tg}\frac{1}{n} > \frac{1}{n}, n \ge 1$.
$\sum_{n\ge 1} \frac{1}{n}$ (D) (S.A. cu $\alpha = 1$)

$\xRightarrow{\text{Cr.1 Comp.}} \sum_{n\ge 1} \operatorname{tg}\frac{1}{n}$ (D)

$\sum_{n\ge 1} x_n$ (D).

d) $x_n := \operatorname{tg}\frac{1}{n^2}$; $y_n := \frac{1}{n^2}$.

//Obs: încercând Cr.1. Comp nu merge, așa că aplicăm criteriul 2.

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\operatorname{tg}\frac{1}{n^2}}{\frac{1}{n^2}} = 1 \in (0,\infty) \xRightarrow[\text{Comp.}]{\text{Cr.2}} \sum x_n \sim \sum y_n$$

Dar $\sum y_n$ (C) (S.A, $\alpha = 2$)

$\Rightarrow \sum x_n$ (C).

Obs: a) $x_n := \frac{1}{\sqrt[3]{n^4+2n^2+2}}$; $y_n := \frac{1}{n^{4/3}}$.

b) Obs: $\boxed{\frac{\ln(1+x_n)}{x_n} \to 1 \text{ dacă } x_n \to 0}$!!!

f) $\sum_{n\ge 1} \operatorname{arctg}\frac{1}{n^2+1}$

e) $x_n := \ln\left(1+\frac{1}{n\sqrt{n}}\right)$; $y_n := \frac{1}{n\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n\sqrt{n}}\right)}{\frac{1}{n\sqrt{n}}} = 1 \in (0,+\infty) \xRightarrow{\text{Cr.2}}$$

$\Rightarrow \sum_{n\ge 1} x_n \sim \sum_{n\ge 1} y_n$

Dar $\sum_{n\ge 1} y_n = \sum_{n\ge 1} \frac{1}{n^{\frac{3}{2}}}$ (C) (S.A cu $\alpha = \frac{3}{2} > 1$)

$\Rightarrow \sum x_n$ (C).

$tg : \underbrace{\mathbb{R} \setminus \{ \frac{\pi}{2} + k\pi \mid k \in \mathbb{Z} \}}_{D_{tg}} \to \mathbb{R}$, $tg\, x :\overset{def}{=} \frac{\sin x}{\cos x}$.

$tg(x+\pi) = tg\, x, \quad \forall x \in D_{tg}$

$tg\big|_{(-\frac{\pi}{2}, \frac{\pi}{2})} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ bijectivă

$arctg := \left(tg\big|_{(-\frac{\pi}{2}, \frac{\pi}{2})} \right)^{-1}$.

?

f) $x_n := arctg \frac{1}{n^2+1}$

$y_n := \frac{1}{n^2}$ etc.

Curs 5.11.2019.

PARTIAL: 19.11.2019 — $12^{00} - 13^{00}$: Mate + Info 12

— $13^{00} - 14^{00}$: Info 11.

Nota partial - pe grup.

• Serii de nr. reale.

- Cr. Cauchy - ansamblu
- Cr. Comp 1 - C
- Cr. Comp 2 - C++
- Cr. Radicalului - C# + Cr. Raportului

S.T.P.

Cr. Radicalului

$\sum x_n$ STP a.î. $\exists\, \lim_{n\to\infty} \sqrt[n]{x_n} = l$

Cr. Raportului

$\sum x_n$ STP a.î. $\exists \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$

Atunci $l < 1 \Rightarrow \sum x_n$ (C)

$l > 1 \Rightarrow \sum x_n$ (D)

$l = 1$ ~~NSPSNCAC~~

(Nu se poate spune nimic dp. caz)

Dem (Schiță) Cr. Rap : Fie $\varepsilon > 0$ cu $l - \varepsilon < l < l + \varepsilon < 1 \Rightarrow (\exists) n_\varepsilon, n \geq n_\varepsilon$,

$x_n < \frac{(l+\varepsilon)^n}{q}$ $\xrightarrow{Cr. Comp 1}$ $\sum x_n$ (C)

$\sum q^n$ (C) etc.

Ex 1: Studiați convergența seriilor.

a) $\sum\limits_{n\ge 1} \frac{(n!)^2}{(2n)!}$; b) $\sum\limits_{n\ge 1} \left(\frac{n^2+n+1}{2n^2+n+3}\right)^n$

a) $x_n := \frac{(n!)^2}{(2n)!}, \; n \ge 1.$

$$\frac{x_{n+1}}{x_n} = \frac{[(n+1)!]^2}{(2n+2)!}\cdot\frac{(2n)!}{(n!)^2} = \frac{(n+1)^2}{(2n+1)(2n+2)} \longrightarrow \frac{1}{4}$$

$\frac{1}{4} < 1$ $\xRightarrow{\text{Cr. Rap.}} \sum x_n \,(C)$

b) $x_n := \left(\frac{n^2+n+1}{2n^2+n+3}\right)^n, \; n \ge 1.$

$$\sqrt[n]{x_n} = \frac{n^2+n+1}{2n^2+n+3} \longrightarrow \frac{1}{2}$$

$\frac{1}{2} < 1$ $\xRightarrow{\text{Cr. Rad}} \sum x_n \,(C).$

Prop: Criteriul condensării (ȘI ALTELE).

$\sum\limits_{n\ge 1} x_n$ S.T.P, $x_n \searrow 0 \Rightarrow \sum\limits_{n\ge 1} x_n \sim \sum\limits_{n\ge 1} 2^n \cdot x_{2^n}.$

Dem:

$\le 2^0 x_{2^0}$ $\quad \le 2^1 x_{2^1}$ $\quad \le 2^2 x_{2^2}$ $\quad \le 2^3 x_{2^3}$ $\quad \le 2^4 \cdot x_{2^4}$

$x_1 + x_2 + x_3 + x_4 + x_5 + \dots + x_7 + x_8 + x_9 + \dots + x_{15} + x_{16} + x_{17} \dots$

$\ge 2^0 \cdot x_2$ $\quad \ge 2^1 \cdot x_4$ $\quad \ge 4 \cdot x_8$ $\quad \ge 8 \cdot x_{16}$

Aplicație: Seria armonică (S.A.)

$$\sum_{n\ge 1} \frac{1}{n^\alpha} \underset{\alpha>0}{\sim} \sum 2^n \cdot \frac{1}{(2^n)^\alpha} = \sum_{n\ge 1} \left(\frac{1}{2^{\alpha-1}}\right)^n$$

(C) dacă $\frac{1}{2^{\alpha-1}} < 1 \iff \alpha > 1$

(D) dacă $\frac{1}{2^{\alpha-1}} \ge 1 \iff \alpha \le 1$

$\alpha \in \mathbb{R}$

Serii de nr. reale arbitrare

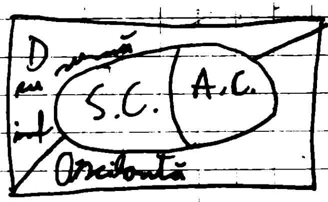
Def: 1) $\sum_{n\ge1} x_n$; $x_n\in\mathbb{R}$, $n\ge1$ s.n. absolut convergentă (A.C.) $\overset{def.}{\Longleftrightarrow}$

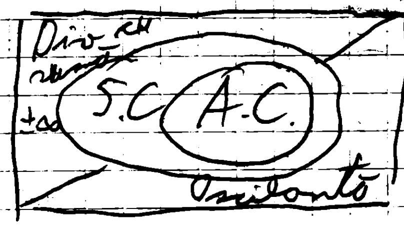
$\Leftrightarrow \sum |x_n|$ (C) (S.T.P.)

(cr. Cauchy și dem.) $|x_{n+1}+\dots+x_{n+p}|\le|x_{n+1}|+\dots+|x_{n+p}|$

Obs: $\sum x_n$ (A.C.) $\underset{\not\Leftarrow}{\Longrightarrow}$ $\sum x_n$ (C)

2) O serie $\sum x_n$ (C) și nu e (A.C.) s.n. semiconvergentă (S.C.)



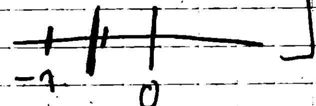


Ex: $\sum_{n\ge1}(-1)^n\cdot\frac{1}{n}$; $\left|\sum_{n\ge1}(-1)^n\cdot\frac{1}{n}\right|=\sum_{n\ge1}\frac{1}{n}$ (D)

$\Rightarrow \sum(-1)^n\frac{1}{n}$ nu e (A.C.)

$\Rightarrow \sum_{n\ge1}(-1)^n\frac{1}{n}$ (S.C.)

$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}$ ... Voi dem. că $\sum(-1)^n\frac{1}{n}$ (C).



Prop (Operații algebrice cu serii)

1) Fie $\sum_{n\ge1}x_n$, $\sum_{n\ge1}y_n$, $x_n, y_n\in\mathbb{R}$, $(\forall)n\ge1$, $\alpha\in\mathbb{R}, \alpha\ne0$.

$$\cancel{1)\sum_{n\ge1} x}\ \sum_{n\ge1}^{\infty}(x_n+y_n)=\sum_{n\ge1}^{\infty}x_n+\sum_{n\ge1}^{\infty}y_n$$

$$\sum_{n\ge1}^{\infty}\alpha x_n=\alpha\sum_{n\ge1}^{\infty}x_n$$

2) $\sum_{n\ge1}x_n$, $\sum_{n\ge1}y_n$ S.T.P. cu suma $+\infty$ $\rightarrow \sum_{n\ge1}(x_n+y_n)$ S.T.P cu suma $+\infty$

Dem: Imed. din def. și op. alg. cu limite de șiruri.

!!!!!

Prop. ("Criteriul Abel")

$\sum_{n\geq 1} x_n$, $x_n = \alpha_n \cdot u_n$, $n \geq 1$, $\alpha_n \searrow 0$, $(u_1 + u_2 + \dots + u_n)_{n\geq 1}$ mărginit

$\Longrightarrow \sum_{n\geq 1}^{\infty} x_n$ (C)

Dem: Netrivială.

Exemplu (Cr. Leibniz)

$\sum_{n\geq 1} x_n$, $x_n = (-1)^n \cdot \alpha_n$, $\alpha_n \searrow 0 \Rightarrow \sum_{n\geq 1} x_n$ (C)

Dem: Din Cr. Abel cu $u_n = (-1)^n$.

Aplicații

1) $\sum_{n\geq 1} (-1)^n \frac{1}{n}$; $\alpha_n := \frac{1}{n} \searrow 0 \xrightarrow{\text{Cr. Leibniz}} \sum_{n\geq 1} (-1)^n \frac{1}{n}$ (C)

2) $\sum_{n\geq 1} \frac{\cos nx}{2^n}$ | $x_n := \frac{\cos nx}{2^n}$, $n \geq 1$, $x \in \mathbb{R}$

TEMA $\boxed{3)}$ $\sum_{n\geq 1} \frac{\cos nx}{n^2}$

$|x_n| = \frac{|\cos nx|}{2^n} \leq \frac{1}{2^n}$, $n \geq 1$, $x \in \mathbb{R}$

$\sum_{n\geq 1} \frac{1}{2^n}$ (C) (S.G. cu $q = \frac{1}{2}$) $\xrightarrow{\text{Cr. Comp.}}$

4) $\sum_{n\geq 1} \frac{\cos nx}{n}$

$\Rightarrow \sum |x_n|$ (C) $\Leftrightarrow \sum x_n$ (A.C.) $\Rightarrow$ (C)

4) $\alpha_n := \frac{1}{n}$, $u_n := \cos nx$, $n \geq 1$.

TEMA

$u_1 + u_2 + \dots + u_n = \cos x + \cos 2x + \dots + \cos nx \; \Big| \cdot 2\sin\frac{x}{2} =$

$= \sin(\dots) - \sin(\dots)$ mărginit (A.C.)

$\xrightarrow{\text{Cr. Abel}} \sum \frac{\cos nx}{n}$ (C).

! Miron Nicolescu: Analiză matematică !

Seminar 6.11.2015.

Ex. Să se studieze natura seriilor

a) $\sum\limits_{n\ge1}\left(\frac{n+1}{n}\right)^n$; b) $\sum\limits_{n\ge1}\frac{1}{2^n+n}$; c) $\sum\limits_{n\ge1}\operatorname{arctg}\frac{1}{n}$, d) $\sum\limits_{n\ge1}\ln\left(1+\frac{1}{\sqrt{n}}\right)$;

e) $\sum\limits_{n\ge1}\left(\frac{3n-2}{4n+1}\right)^n$; f) $\sum\limits_{n\ge1}\frac{(2n)!}{(n!)^2}$, g) $\sum\limits_{n\ge1}\frac{a^n}{n!}$, $a>0$; h) $\sum\limits_{n\ge1}\frac{1}{(\ln n)^n}$;

i) $\sum\limits_{n\ge1}\frac{1\cdot5\cdot\ldots\cdot(4n-3)}{1\cdot6\cdot11\ldots(5n-4)}$; j) $\sum\limits_{n\ge2}\frac{1}{n(\ln n)^{10}}$; k) $\sum\limits_{n\ge1}(-1)^n\frac{1}{n\sqrt{n}}$; l) $\sum\limits_{n\ge1}(-1)^{n-1}\frac{1}{\sqrt[3]{n}}$

m) $\sum\limits_{n\ge1}(-1)^{n-1}\frac{\ln n}{n\sqrt[3]{n}}$; n) $\sum\limits_{n\ge1}\frac{\sin nx}{n^2}$, $x\in\mathbb{R}$; o) $\sum\limits_{n\ge1}\frac{\sin nx}{n}$, $x\in\mathbb{R}$.

REC: Obs: 1) $\sum x_n (C) \overset{\Rightarrow}{\not\Leftarrow} x_n\to0$

2) $\sum x_n$ S.T.P $\Rightarrow$ are sumă

S.T.P.

Cr. Comp 1: $0\le x_n\le y_n$, $n\ge k$.

$\sum y_n(C)\Rightarrow\sum x_n(C)$

$\sum x_n(D)\Rightarrow\sum y_n\ (D)$

Cr. Comp 2: $\sum x_n$, $\sum y_n$ STP a.î. $\lim\frac{x_n}{y_n}=l\in(0,\infty)\Rightarrow\sum x_n\sim\sum y_n$

Cr. Rad: $\sum x_n$ STP a.î. $\lim\sqrt[n]{x_n}=l$

Cr. Rap: $\sum x_n$ STP a.î. $\lim\frac{x_{n+1}}{x_n}=l$.

$l<1\Rightarrow\sum x_n\ (C)$

$l>1\Rightarrow\sum x_n\ (D)$.

a) $x_n:=\left(\frac{n+1}{n}\right)^n\to l\ne0\xRightarrow{\text{Obs }1}\sum x_n\ (D)$.

b) $x_n:=\frac{1}{2^n+n}\le\frac{1}{2^n}=y_n$ | $\xRightarrow{\text{Cr. Comp}}\sum x_n\ (C)$.

$\sum y_n\ (C)(S.G\ q=\frac{1}{2})$

c) $x_n:=\operatorname{arctg}\frac{1}{n}$; $y_n:=\frac{1}{n}$.

$\lim\limits_{n\to\infty}\frac{x_n}{y_n}=\frac{\operatorname{arctg}\frac{1}{n}}{\frac{1}{n}}=1\in(0,\infty)\xRightarrow{\text{Cr. Comp 2}}\sum x_n\sim\sum y_n$

Dar $\sum y_n\ (D)$ (S.A. cu $\alpha=1$) $\Rightarrow\sum x_n\ (D)$

d) $\boxed{\sum \frac{1}{n\sqrt{n}} \text{ (D) din. seriei trecute}}$

Σ $x_n := \ln\left(1+\frac{1}{\sqrt{n}}\right)$; $y_n = \frac{1}{\sqrt{n}}$ (limita tinde la 0, nu la ∞)

$$\frac{x_n}{y_n} = \frac{\ln\left(1+\frac{1}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}} \to 1 \in (0,\infty) \xRightarrow{\text{Cr. Comp. 2}} \sum x_n \sim \sum y_n \Bigg| \Rightarrow \sum x_n \text{ (D)}.$$

$$\sum y_n \text{ (D)}$$

— GREȘIT —

e) $x_n := \left(\frac{3n-2}{4n+1}\right)^n$

$$\sqrt[n]{x_n} = \frac{3n-2}{4n+1} \to \frac{3}{4}$$

$$\frac{3}{4} < 1 \xRightarrow{\text{Cr. Rad.}} \sum x_n \text{ (C)}.$$

f) $x_n := \frac{(2n)!}{(n!)^2}$

$$\frac{x_{n+1}}{x_n} = \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{(n!)^2}{2n!} = \frac{(2n+1)(2n+2)}{(n+1)^2} \to 4 \Bigg\} \xRightarrow{\text{Cr. Rap.}} \sum x_n \text{ (D)}.$$

$$4 > 1$$

d) $\ln\left(1+\frac{1}{\sqrt{n}}\right) \to \ln 2 \neq 0 \to$ divergentă.

d') $\ln\left(1+\frac{1}{n\sqrt{n}}\right)$ la fel ca d, doar că tratăm $y_n := \frac{1}{n\sqrt{n}}$.

$$\sum x_n \sim \sum y_n \sim \sum \frac{1}{n}.$$

g) $x_n := \frac{a^n}{n!}$, $n \geq 1$, $a > 0$.

$$\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \frac{a}{n+1} \to 0 \Bigg| \xRightarrow{\text{Cr. Rap.}} \sum x_n \text{ (C)}.$$

$$0 < 1$$

h) $x_n := \frac{1}{(\ln n)^n}$; $\sqrt[n]{x_n} = \frac{1}{\ln n} \to 0$, $0<1$ $\xRightarrow{\text{Cr. Rad.}} \sum x_n$ (C)

i) $x_n = \frac{1\cdot5\cdot9\cdot\ldots\cdot(4n-3)}{1\cdot6\cdot11\cdot\ldots\cdot(5n-4)}$

$$\frac{x_{n+1}}{x_n} = \frac{1\cdot5\cdot9\cdot\ldots\cdot(4n-3)(4n+1)}{1\cdot6\cdot11\cdot\ldots\cdot(5n-4)(5n+1)}\cdot\frac{1\cdot6\cdot11\cdot\ldots\cdot(5n-4)}{1\cdot5\cdot9\cdot\ldots\cdot(4n-3)} = \frac{4n+1}{5n+1}\to\frac{4}{5}$$

$\frac{4}{5}<1$ $\Rightarrow$

$\xRightarrow{\text{Cr. Rap}} \sum x_n$ (C)

j) $x_n := \frac{1}{n(\ln n)^{10}} \searrow 0$

$\xRightarrow{\text{Cr. Cond}} \sum x_n \sim \sum 2^n \cdot x_{2^n}$

$$\sum 2^n \cdot x_{2^n} = \sum 2^n \cdot \frac{1}{2^n(\ln 2^n)^{10}} = \sum \frac{1}{n^{10}(\ln 2)^{10}} \sim \sum \frac{1}{n^{10}} \text{ (C)}$$

$\Rightarrow \sum x_n$ (C)

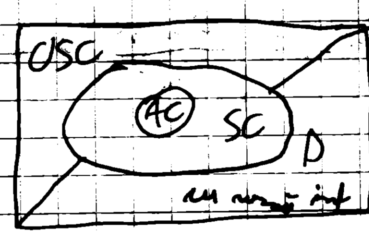
RE.C: $\sum x_n, x_n \in \mathbb{R}$

Def: $\sum x_n$ s.n. (A.C) $\xLeftrightarrow{\text{def}} \sum |x_n|$ (C).

Obs: $\sum x_n$ (AC) $\Rightarrow \sum x_n$ (C)
$\not\Leftarrow$

Cr. Abel: $\sum x_n$, $x_n = \alpha_n \cdot u_n$, $\alpha_n \searrow 0$, $(u_1+u_2+\ldots+u_n)_{n\geq1}$ mărg.
$\Rightarrow \sum x_n$ (C)

Cr. Leibniz: $\sum x_n$, $x_n = (-1)^n \cdot \alpha_n$, $\alpha_n \searrow 0 \Rightarrow \sum x_n$ (C).



---

k) $x_n := (-1)^n \cdot \frac{1}{n\sqrt{n}}$

$$\sum_{n\geq1} \left|(-1)^n \frac{1}{n\sqrt{n}}\right| = \sum \frac{1}{n\sqrt{n}} = \sum \frac{1}{n^{\frac{3}{2}}} \text{ (C) (S.A.)}$$

$\alpha = \frac{3}{2} > 1$ $\Rightarrow \sum x_n$ (A.C)

l) $x_n := (-1)^{n-1} \frac{1}{\sqrt[3]{n}}$

$$\sum_{n\geq1} \left|(-1)^{n-1} \frac{1}{\sqrt[3]{n}}\right| = \sum_{n\geq1} \frac{1}{\sqrt[3]{n}} = \sum \frac{1}{n^{\frac{1}{3}}} \text{ ~~(C) (S.A.)~~}$$

~~$\alpha = \frac{1}{3} > 1$~~ ~~$\Rightarrow \sum x_n$~~

$$\sum |x_n| = \sum \frac{1}{\sqrt[3]{n}} = \frac{1}{n^{\frac{1}{3}}} \; (D) \; (S.A. \text{ cu } \alpha = \tfrac{1}{3} < 1) \Rightarrow \sum x_n \text{ nu este } (A.C)$$

$$\alpha_n = \frac{1}{\sqrt[3]{n}} \searrow 0 \xrightarrow{\text{Cr. Leibniz}} \sum (-1)^{n-1} \cdot \alpha_n = \sum x_n \, (C) \Rightarrow \sum x_n \, (S.C)$$

Parțial: Cr. Raport și Cr. Rad.

$$m) \; x_n := (-1)^{n-1} \frac{\sin n}{n\sqrt[3]{n}}$$

$$|x_n| = \left| (-1)^{n-1} \frac{\sin n}{n\sqrt[3]{n}} \right| = \frac{|\sin n|}{n\sqrt[3]{n}} \leq \frac{1}{n\sqrt[3]{n}} = \frac{1}{n^{\frac{4}{3}}} =: y_n, \; n \geq 1$$

$$\sum y_n \, (C) \; (S.A \text{ cu } \alpha = \tfrac{4}{3} > 1 \Rightarrow$$

$$\xrightarrow{\text{Cr. Comp. 1}} \sum |x_n| \, (C) \Leftrightarrow \sum x_n \, (A.C.)$$

$\boxed{\text{Seminar 1} \to (R): 1.2.6, 1.2.7, 1.2.8, 1.2.9, 1.2.13; \; (P): 1.3.4, 1.3.5, 1.3.6, 1.3.7, 1.3.12; \;\; \text{Seminar 2} \to 2.2.1, 2.2.2, 2.2.3, 2.2.4, 2.2.5; \; 2.3.1 \ldots \;\; \boxed{\text{Ex: } 2.3.5}}$

$$n) \; x_n := \frac{\sin nx}{n^2}, \; x \in \mathbb{R}.$$

$$|x_n| = \left| \frac{\sin nx}{n^2} \right| = \frac{|\sin nx|}{n^2} \leq \frac{1}{n^2} =: y_n, \; n \geq 1 \quad \xrightarrow{\text{Cr. Comp. 1}} \sum |x_n| \, (C) \Leftrightarrow$$

$$\sum y_n \, (C) \; (S.A \text{ cu } \alpha = 2 > 1)$$

$$\Leftrightarrow \sum x_n \, (A.C.)$$

$$o) \; x_n := \frac{\sin nx}{n}$$

$$\alpha_n := \frac{1}{n}, \; u_n := \sin nx, \; n \geq 1 \; (x \in \mathbb{R}).$$

$$u_1 + u_2 + \ldots + u_n = \sin x + \sin 2x + \ldots + \sin nx := E_n(x).$$

$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$

$-2\sin\alpha \sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$.

$$-2 E_n(x)\cdot \sin\left(\frac{x}{2}\right) = \cos\frac{3x}{2} - \cos\frac{x}{2} + \cos\frac{5x}{2} - \cos\frac{3x}{2} + \dots + \cos\left(nx+\frac{x}{2}\right) - \cos\left(nx-\frac{x}{2}\right)$$

$$= \cos\left(nx+\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right)$$

$\Rightarrow E_n(x) = \dfrac{-\cos\left(nx+\frac{x}{2}\right) + \cos\frac{x}{2}}{2\sin\frac{x}{2}}$ $\quad$ pt $x \neq 2k\pi,\ k\in\mathbb{Z}$.

$\Rightarrow (E_n(x))_{n\geq 1}$ mărginit pt $\forall x \neq 2k\pi,\ k\in\mathbb{Z}$ $\Big|$ $\overset{\text{Cr. Abel}}{\Rightarrow}$ $\sum \alpha_n u_n$ (C).

$\alpha_n \searrow 0$

TEMA: $\sum\limits_{n\geq 1} \dfrac{\cos nx}{n},\ x\in\mathbb{R}$ $\Big|$ $x = 2k\pi,\ k\in\mathbb{Z} \Rightarrow$ seria devine $\sum$ (l) ...

CAPITOLUL 3. Limite de funcții și continuitate

12.11.2019

I. Cazul: $f: D \subseteq \mathbb{R} \to \mathbb{R}$

Def: 1) $V \subseteq \mathbb{R}$ s.n. vecinătate a lui $\begin{cases} a \in \mathbb{R} \overset{def}{\Longleftrightarrow} (\exists \varepsilon > 0)((a-\varepsilon, a+\varepsilon) \subseteq V) \\ a = +\infty \overset{def}{\Longleftrightarrow} (\exists \varepsilon > 0)((\varepsilon, \infty) \subseteq V) \\ a = -\infty \overset{def}{\Longleftrightarrow} (\exists \varepsilon > 0)((-\infty, -\varepsilon) \subseteq V) \end{cases}$

$a \in \overline{\mathbb{R}}$; $\mathcal{V}(a) \overset{not}{:=} \{V \subseteq \mathbb{R} \mid V \text{ vecinătate a lui } a\}$

2) $D \subseteq \mathbb{R}$; $a \in \overline{\mathbb{R}}$ s.n. punct de acumulare al lui D $\overset{def}{\Longleftrightarrow} (\forall V \in \mathcal{V}(a))((V \setminus \{a\}) \cap D \neq \emptyset)$

$D' \overset{not}{:=} \{a \in \overline{\mathbb{R}} \mid a \text{ punct de acumulare al lui } D\}$.

(!) Dem 2 rânduri

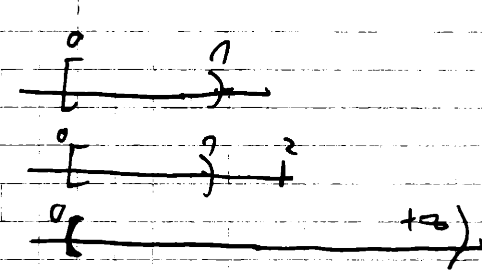
Obs: $a \in \overline{\mathbb{R}}$, $D \subseteq \mathbb{R} \Rightarrow \begin{cases} - a \in \mathbb{R},\ a \in D' \overset{!}{\Longleftrightarrow} (\forall \varepsilon > 0)(((a-\varepsilon, a+\varepsilon) \setminus \{a\}) \cap D \neq \emptyset) \\ - a = +\infty,\ a \in D' \overset{!}{\Longleftrightarrow} (\forall \varepsilon > 0)((\varepsilon, \infty) \cap D \neq \emptyset) \\ - a = -\infty,\ a \in D' \overset{!}{\Longleftrightarrow} (\forall \varepsilon > 0)((-\infty, -\varepsilon) \cap D \neq \emptyset) \end{cases}$

Ex1: Determinați D' pt:

a) $D = [0,1] \Rightarrow D' = [0,1]$.

b) $D = [0,1) \cup \{2\} \Rightarrow D' = [0,1]$

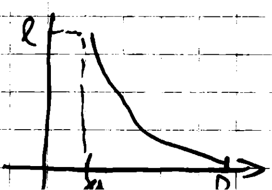
c) $D = (0, \infty) \Rightarrow D' = [0, \infty]$



d) $D = \mathbb{N}^* \Rightarrow D' =$

Def 3: $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $a \in D'$, $l \in \overline{\mathbb{R}}$; f are limita l în a $\overset{def}{\Longleftrightarrow}$

$\Leftrightarrow (\forall V \in \mathcal{V}(l))(\exists U \in \mathcal{V}(a))(\forall x \in (U \setminus \{a\}) \cap D)(f(x) \in V)$.



Se scrie $\lim_{x\to a} f(x) = l$ sau $f(x) \xrightarrow[x\to a]{} l$

Obs: Dacă $a\in\mathbb{R}$, $l\in\mathbb{R}$, atunci $\lim_{x\to a} f(x) = l \Leftrightarrow (\forall \varepsilon>0)$

$$(\exists \delta_\varepsilon>0)\left(x\in(a-\delta_\varepsilon, a+\delta_\varepsilon)\cap D\right)\left(f(x)\in(l-\varepsilon, l+\varepsilon)\right)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$|x-a|<\delta_\varepsilon,\ x\in D,\ x\neq a \qquad\qquad |f(x)-l|<\varepsilon.$$

TEMĂ: scrieți cel. 8 variante caracterizare a limitei, cazul lui $(a\in\mathbb{R}, l=\infty)$ $V(a=\infty, l\in\mathbb{R})$ etc.

(Topologie) // Donald Knuth, Latex

Caracterizarea pe șiruri a limitelor de funcții.

Prop: Fie $f: D\subseteq\mathbb{R}\to\mathbb{R}$, $a\in D'$, $l\in\overline{\mathbb{R}}$. ~~Atunci~~ $\lim_{x\to a} f(x) = l \Leftrightarrow \boxed{\forall (x_n)_{n} \subseteq D,\ x_n\to a,\ x_n\neq a} \Rightarrow$

$\boxed{\Rightarrow f(x_n)\to l}$

Dem: Imediată, din definiție, dar cu 9 cazuri.

Prop. (Cr. Maj. Sim., Ext.)

I. Fie $f: D\subseteq\mathbb{R}\to\mathbb{R}$, $l\in\mathbb{R}$, $a\in D'$, $|f(x)-l|\le g(x)$, $\forall x\in (V\setminus\{a\})\cap D$, $V\in\mathcal{V}(a)$.

Dacă $\lim_{x\to a} g(x) = 0$, atunci $\lim_{x\to a} f(x) = l$.

II. Fie $f, g: D\subseteq\mathbb{R}\to\mathbb{R}$, $a\in D'$, $f(x)\le g(x)$, $\forall x\in (V\setminus\{a\})\cap D$, $V\in\mathcal{V}(a)$.

Dacă — $\lim_{x\to a} f(x) = +\infty$, atunci $\lim_{x\to a} g(x) = +\infty$

— $\lim_{x\to a} g(x) = -\infty$, atunci $\lim_{x\to a} f(x) = -\infty$.

~~Prop~~

Dem: Se utilizează Carac. „pe șiruri" a lim. fct + Cr. Maj. pt. șiruri

Teorema "Operații algebrice cu lim. de funcții"

Fie $f, g: D \subseteq \mathbb{R} \to \mathbb{R}$, $a \in D'$ și ex. $\lim_{x\to a} f(x) = l_1 \in \overline{\mathbb{R}}$, ex. $\lim_{x\to a} g(x) = l_2 \in \overline{\mathbb{R}}$. At.

a) ex: $\lim_{x\to a} (f(x)+g(x)) = l_1 + l_2$ (cu excepția $\boxed{\infty - \infty}$)

b) ex: $\lim_{x\to a} (f(x)\cdot g(x)) = l_1 \cdot l_2$ (cu exc. $\boxed{0\cdot\infty}$)

c) ex: $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{l_1}{l_2}$ (cu exc. $\boxed{\frac{0}{0}}$, $\boxed{\frac{\infty}{\infty}}$)

d) ex: $\lim_{x\to a} f(x)^{g(x)} = l_1^{l_2}$ (cu exc. $\boxed{0^0}$, $\boxed{\infty^0}$, $\boxed{1^\infty}$)

$f(x) > 0$, $\forall x \in (V \setminus \{a\}) \cap D$, $V \in \mathcal{V}(a)$

Dem: Se utilizează Teorema „cu șiruri" a lim. fct + Op. alg. cu limite de șiruri

Limite de fct remarcabile

Utilizând Teorema „cu șiruri" a limitei fct + Lim. rem. de șiruri

$\Rightarrow$ (1) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$; (1') $\lim_{y\to 0} (1+y)^{\frac{1}{y}} = e$; (1'') $\lim_{y\to 0} \frac{\ln(1+y)}{y} = 1$.

(2) $\lim_{x\to 0} \frac{\sin x}{x} = 1$, (2') $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$; (2'') $\lim_{x\to 0} \frac{\operatorname{tg} x}{x} = 1$;

(3) $\lim_{x\to\infty} \frac{x^k}{a^x} = 0$, $a > 1$, $k \in \mathbb{N}^*$ (!) ; (3') $\lim_{x\to\infty} \frac{\ln x}{x^k} = 0$, $k \in \mathbb{N}^*$.

(1''') $\lim_{y\to 0} \frac{a^y - 1}{y} = \ln a$, $a > 0$

(2''') $\lim_{x\to 0} \frac{\operatorname{arctg} x}{x} = 1$

Ex 1) $\lim_{x\to-\infty} \frac{1-x^3+2x^2}{5x+x^2+x^3} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\cancel{x^3}\left(-1+\frac{2}{x}+\frac{1}{x^3}\right)}{\cancel{x^3}\left(1+\frac{1}{x}+\frac{5}{x^2}\right)} = \frac{0-1+0}{1+0+0} = -1$

2) $\lim_{x\to\pm\infty} \frac{1}{x} = 0$.

2) $\lim_{x\to\infty} \sqrt{x^2+1} + x = \lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}-x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}-\cancel{x}} = \frac{1}{\infty+\infty} = \frac{1}{\infty} = 0$

3) $\lim_{x\to\infty} \left(\frac{x+1}{x-1}\right)^{\sqrt{x}} = \left[\lim_{x\to\infty} \left[\underbrace{\left(1+\frac{2}{x-1}\right)^{\frac{x-1}{2}}}_{\to e}\right]\right]^{\frac{2\sqrt{x}}{x-1}} = e^{\lim_{x\to\infty} \frac{2\sqrt{x}}{x-1}} =$

$= e^{\lim_{x\to\infty} \frac{2\cancel{\sqrt{x}}}{\cancel{x}\sqrt{x}\,(1-\frac{1}{x})}} = e^{\frac{2}{\infty(1-0)}} = e^{\frac{2}{\infty}} = e^{0} = 1.$

4) $\lim_{x\to 0} \frac{tg\, 2x}{arcsin\, 3x} = \lim_{x\to 0} \frac{tg\, 2x}{2x} \cdot \frac{3x}{arctg\, 3x} \cdot \frac{2\cancel{x}}{3\cancel{x}} = 1 \cdot \frac{1}{1} \cdot \frac{2}{3} = \frac{2}{3}$

<u>Def</u>: $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $a \in ((-\infty, a) \cap D)'$ (resp. $a \in ((a,\infty) \cap D)'$)

Se spune că: f are <u>lim la stg în a</u> <u>egală $l \in \mathbb{R}$</u> $\overset{def}{\Leftrightarrow}$ $\exists$. $\lim_{x\to a} \left(f|_{(-\infty,a)\cap D}\right)(x) = l$

$\left(\text{Se scrie } \lim_{\substack{x\to a \\ x<a}} f(x) = l\right) := l_s(a)$

f — n —, la stg - n - $\overset{def}{\Leftrightarrow}$ $\exists$. $\lim_{x\to a} \left(f|_{(a,\infty)\cap D}\right)(x) = l.$

$\left(\text{Se scrie } \lim_{\substack{x\to a \\ x>a}} f(x) = l.\right) := l_d(a)$

<u>Prop</u> (Caracterizarea limitei cu limite laterale)

$f: D \subseteq \mathbb{R} \to \mathbb{R}$, $a \in ((-\infty,a)\cap D)' \cap ((a,\infty)\cap D)'$, $l \in \mathbb{R} \Rightarrow$

$\Rightarrow \exists.\ \lim_{x\to a} f(x) = l \Leftrightarrow \exists.\ l_s(a), \exists.\ l_d(a)$ și $l_s(a) = l_d(a) = l$

<u>Dem</u>: Imediată, din def.

Seminar 14.11.2019

Ex: Să se calculeze:

a) $\lim_{x\to\infty} \frac{-3x^4+2x^2-1}{2x^4-x+2}$ $\frac{\infty}{\infty}$ c) $\lim_{x\to -\infty} \frac{\sqrt{9x^2-5x+8}}{2x+1}$ $\frac{\infty}{\infty}$ e) $\lim_{x\to 4} \frac{x-4}{\sqrt{2x+1}-3}$

b) $\lim_{x\to -\infty} \frac{-2x^2+x-1}{3x-2}$ $\frac{\infty}{\infty}$ d) $\lim_{x\to 1} \frac{x^2-x}{x^2-4x+3}$ $\frac{0}{0}$ f) $\lim_{x\to 0} \frac{\sin 5x}{\operatorname{arctg} 4x}$

g) $\lim_{x\to -\frac{2}{3}} \frac{\arcsin(2+3x)}{\operatorname{tg}(9x^2-4)}$ i) $\lim_{x\to 0} \frac{\ln(1+2x)}{\ln(1+5x)}$ k) $\lim_{x\to 0} (1+x\operatorname{arctg} x)^{\frac{1}{\sin^2 x}}$

h) $\lim_{x\to \frac{1}{2}} \frac{3^x-\sqrt{3}}{4x^2-1}$ j) $\lim_{x\to \infty} \left(\frac{3x+1}{4+3x}\right)^{\frac{x^2}{x+2}}$ l) $\lim_{x\to 0} \frac{e^{x^2}-\cos x}{x^2}$.

---

a) $l = \lim_{x\to\infty} \frac{x^4(-3+\frac{2}{x^2}-\frac{1}{x^4})}{x^4(2-\frac{1}{x^3}+\frac{2}{x^4})} = \frac{-3+0-0}{2-0+0} = \frac{-3}{2}$

b) $l = \lim_{x\to -\infty} \frac{x^2(-2+\frac{1}{x}-\frac{1}{x^2})}{x(3-\frac{2}{x})} = \lim_{x\to -\infty} \frac{x(-2+\frac{1}{x}-\frac{1}{x^2})}{3-\frac{2}{x}} = \frac{-\infty(-2+0-0)}{3+0} = \infty$

c) $\cancel{l = \lim_{x\to -\infty} \frac{x\sqrt{9-\frac{5}{x}+\frac{8}{x^2}}}{x(2+\frac{1}{x})} = \frac{\sqrt{9-0+0}}{2+0} = \frac{3}{2}}$

$= \lim_{x\to -\infty} \frac{\sqrt{x^2(9-\frac{5}{x}+\frac{8}{x^2})}}{x(2+\frac{1}{x})} = \lim_{x\to -\infty} \frac{-x\sqrt{9-\frac{5}{x}+\frac{8}{x^2}}}{x(2+\frac{1}{x})} = \frac{-\sqrt{9-0+0}}{2+0} = \frac{-3}{2}$.

d) $l = \lim_{x\to 1} \frac{x(x-1)}{(x-1)(x-3)} = \lim_{x\to 1} \frac{x}{x-3} = \frac{1}{-2} = \frac{-1}{2}$.

e) $l = \frac{(x-4)(\sqrt{2x+1}+3)}{2x+1-9} = \frac{(x-4)(\sqrt{2x+1}+3)}{2x-8} = \lim_{x\to 4} \frac{\cancel{(x-4)}(\sqrt{2x+1}+3)}{2\cancel{(x-4)}} =$

$= \lim_{x\to 4} \frac{\sqrt{2x+1}+3}{2} = \frac{\sqrt{9}+3}{2} = \frac{6}{2} = 3$.

f) $l = \lim_{x\to 0} \frac{\frac{\sin 5x}{5x}}{\frac{\operatorname{arctg} 4x}{4x}} \cdot \frac{5\cancel{x}}{4\cancel{x}} = \frac{5}{4}$.

$$\boxed{\lim_{x\to a} \frac{\sin(u(x))}{u(x)} = 1, \text{ dacă } \lim_{x\to a} u(x) = 0}$$

~~g)~~

g) $l = \lim\limits_{x \to -\frac{2}{3}} \dfrac{\cancel{\arcsin(2+3x)}\ 2+3x}{\dfrac{tg(9x^2-4)}{9x^2-4}} \cdot \dfrac{\cancel{2+3x}}{(3x-2)\cancel{(3x+2)}} = 1 \cdot \dfrac{1}{3\cdot\frac{-2}{3}-2} = \dfrac{-1}{4}$

h) $l = \lim\limits_{x \to \frac{1}{2}} \dfrac{\sqrt{3}\left(3^{x-\frac{1}{2}}-1\right)}{(2x-1)(2x+1)} = \dfrac{\sqrt{3}}{2} \lim\limits_{x \to \frac{1}{2}} \dfrac{3^{x-\frac{1}{2}}-1}{x-\frac{1}{2}} \cdot \dfrac{1}{2x+1} =$

$$\boxed{\lim_{x \to a} \frac{a^{u(x)}-1}{u(x)} = \ln a \text{ și } \lim_{x \to a} u(x) = 0}$$

$= \dfrac{\sqrt{3}}{2} \cdot \ln 3 \cdot \dfrac{1}{2} = \dfrac{\sqrt{3}}{4} \ln 3.$

i) $l = \lim\limits_{x \to 0} \dfrac{\dfrac{\ln(1+2x)}{2x}}{\dfrac{\ln(1+5x)}{5x}} \cdot \dfrac{2\cancel{x}}{5\cancel{x}} = \dfrac{1}{1} \cdot \dfrac{2}{5} = \dfrac{2}{5}.$

$$\boxed{\lim_{x \to a} \frac{\ln(1+u(x))}{u(x)} = 1 \text{ și } \lim_{x \to a} u(x) = 0}$$

j) $l = \lim\limits_{x \to \infty} \left(1 + \dfrac{3x+1}{3x+4} - 1\right)^{\frac{x^2}{x+2}} = \lim\limits_{x \to \infty} \left[\left(1 + \dfrac{-3}{3x+4}\right)^{\frac{3x+4}{-3}}\right]^{\frac{-3}{3x+4} \cdot \frac{x^2}{x+2}} =$

$= e^{\lim\limits_{x \to \infty} \frac{-3x^2}{(x+2)(3x+4)}} = e^{\frac{-3}{3}} = e^{-1} = \dfrac{1}{e}$

k) $l = \lim\limits_{x \to 0} \left[\left(1 + x\, arctg\, x\right)^{\frac{1}{x\, arctg\, x}}\right]^{x\, arctg\, x \cdot \frac{1}{\sin x \cdot \arcsin x}} =$

$= e^{\lim\limits_{x \to 0} \frac{x\, arctg\, x}{\sin x \cdot \arcsin x}} = e^{\lim\limits_{x \to 0} \frac{x}{\sin x} \cdot \frac{arctg\, x}{x} \cdot \frac{x}{\arcsin x}}$, $\dfrac{x}{\sin x} \to \dfrac{1}{1}$, $\dfrac{arctg\, x}{x} \to 1$, $\dfrac{x}{\arcsin x} \to \dfrac{1}{1}$

$= e^{\frac{1}{1} \cdot 1 \cdot \frac{1}{1}} = e^{1}.$

Def: $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $a \in D$, f s.n. continuă în a $\overset{def}{\iff} (\forall V \in \mathcal{V}(f(a)))$ $(\exists U \in \mathcal{V}(a))$ $(\forall x \in (U \cap D))$ $(f(x) \in V)$. $\{$ f cont pe $D_1 \subset D \overset{def}{\iff} (\forall a \in D_1)$ (f cont în a)

Obs: $a \in D \setminus D' \Rightarrow$ f cont în a.

Prop: ("Caract cont cu limită") Fie $f: D \subseteq \mathbb{R} \to \mathbb{R}$, $a \in D \cap D'$. Atunci f cont în a $\iff$ $\exists$. $\lim_{x \to a} f(x) = f(a)$.

Dem: "$\Leftarrow$" Obs. $x \neq a$, $x = a$ ...

"$\Rightarrow$" OK.

Prop: ("Operații alg și de comp. fct cont")

I. $f, g: D \subseteq \mathbb{R} \to \mathbb{R}$, $a \in D$, f, g cont în a. (cont pe D) $\Rightarrow$ $f+g$, $f \cdot g$, $\frac{f}{g}$ cont în a (pe D)

$\hookrightarrow$ ($g(a) \neq 0$ resp. $g \neq 0$ pe D)

II. $D \subseteq \mathbb{R} \xrightarrow{u} E \subseteq \mathbb{R} \xrightarrow{f} \mathbb{R}$, $a \in D$, $b := u(a)$

u cont în a (pe D), f cont în b (pe E) $\Bigg| \Longrightarrow$ $f \circ u$ cont în a (pe D).

Obs (!!!) Toate fct elementare sunt continue pe domeniile lor maxime de definiție.

Ex: $\sin \frac{1}{x}$ ~~...~~ $\underset{\neq 0}{x} \xrightarrow{u} \frac{1}{x} \xrightarrow{f = \sin} \mathbb{R}$.

Exe: Să se studieze continuitatea funcțiilor

a) $f(x) = \begin{cases} \frac{\sin x}{2x}, & x < 0 \\ a, & x = 0 \\ \frac{x^2 + x}{tg(x^2+x)}, & x \in \left(0, \frac{1}{2}\right] \end{cases}$, $f: \left(-\infty, \frac{1}{2}\right] \to \mathbb{R}$. $a \in \mathbb{R}$.

b) $f(x) = \begin{cases} ax + 1, & x \in (-\infty, -1] \\ bx - 1, & x \in (-1, 2) \\ 3, & x = 2 \end{cases}$ $f: (-\infty, 2] \to \mathbb{R}$.

c) $f(x)=\begin{cases} \sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$, $f:\mathbb{R}\to\mathbb{R}$ d) $f(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$, $f:\mathbb{R}\to\mathbb{R}$.

b) $f(x)=\begin{cases} \left|\sin\frac{1}{x}\right|\cdot\frac{1}{1+2^{\frac{1}{x}}}, & x\neq 0 \\ 0, & x=0. \end{cases}$

---

a) f este continuă pe $(-\infty,0)\cup\left[0,\frac{1}{2}\right]$ (provenind din op. algebrice și de comp. cu fct elem.)

$l_s(0)=\lim\limits_{\substack{x\to 0\\ x<0}} f(x)=\lim\limits_{\substack{x\to 0\\ x<0}}\frac{\sin x}{2x}=\frac{1}{2}\cdot 1=\frac{1}{2}$.

$l_d(0)=\lim\limits_{\substack{x\to 0\\ x>0}} f(x)=\lim\limits_{\substack{x\to 0\\ x>0}}\frac{\overbrace{x^2+x}^{u(x)}}{tg\underbrace{(x^2+x)}_{u(x)}}=\frac{1}{1}=1$

$\xrightarrow[\text{în limită}]{\text{Există cont}}$ nu $\exists \lim\limits_{x\to 0} f(x) \Rightarrow$

$\Rightarrow$ f nu e cont în 0, $(\forall)\, a\in\mathbb{R}$.

b) f este continuă pe $(-\infty,-1)\cup(-1,2)$ (provenind din op. alg. și de comp. cu fct elementară)

$l_s(-1)=\lim\limits_{\substack{x\to -1\\ x<-1}} f(x)=\lim\limits_{\substack{x\to -1\\ x<-1}}(ax+1)=-a+1$.

$l_d(-1)=\lim\limits_{\substack{x\to -1\\ x>-1}} f(x)=\lim\limits_{\substack{x\to -1\\ x>-1}}(bx-1)=-b-1$

$\xrightarrow{\text{Există lim. în lim. fct}}$ f are lim în $-1 \Leftrightarrow$

$\Leftrightarrow -a+1=-b-1$.

$\underbrace{\Leftrightarrow a-b=2.}_{(\dots)}$

f va fi cont în $-1 \Leftrightarrow (\dots)\ a-b=2$

$f(2)=3$

$l_s(2)=\lim\limits_{\substack{x\to 2\\ x<2}} f(x)=\lim\limits_{\substack{x\to 2\\ x<2}} bx-1=2b-1$

$\xrightarrow[\text{în limită}]{\text{Există. cont}} 2b-1=3$

$\Leftrightarrow 2b=4$

$\Leftrightarrow b=2$

Din 1 și 2 $\Rightarrow$ f cont pe $(-\infty,2] \Leftrightarrow \begin{cases} a-b=2 \\ b=2 \end{cases} \Leftrightarrow \begin{cases} a=4 \\ b=2. \end{cases}$

c) Se știe că $\nexists$ lim $\sin(x)$, $x \to \infty$ | $x_n := 2n\pi \to \infty$, $\sin x_n = 0 \to 0$; $y_n := 2n\pi + \frac{\pi}{2} \to \infty$, $\sin y_n = 1 \to 1$. $\Rightarrow$

$\overset{\text{Teorema „cu șiruri" a lim.}}{\Longleftarrow}$

Analog $\nexists$ lim $\cos x$ $x \to \infty$ și nici $-\infty$.

$x_n := \frac{1}{2n\pi} \searrow 0$, $f(x_n) = \sin(2n\pi) = 0 \to 0$

$y_n := \frac{1}{2n\pi + \frac{\pi}{2}} \searrow 0$, $f(y_n) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1 \to 1$

$\xrightarrow[\text{a limitei}]{\text{Teorema „cu șiruri"}}$ $\nexists$ lim $\sin\frac{1}{x}$, $x \to 0$, $x > 0$

Analog $\nexists$ lim $\sin\frac{1}{x}$, $x \to 0$, $x < 0$ $\to \nexists$ lim $f(x)$, $x \to 0$

$\xrightarrow[\text{Teorema „cu șiruri" a lim.}]{}$ f nu e cont în 0.

d) $\left|x \sin\frac{1}{x}\right| \leq |x|$, $(\forall)\, x \neq 0$

$\left|x \sin\frac{1}{x}\right| = |f(x)| \searrow 0$; $|x| = g(x)$, $\exists$ lim $g(x) = 0$, $x \to 0$

$\xrightarrow[\text{Teorema cont.-lim}]{\text{Th. Maj.}}$ $\exists$ lim $x \sin\frac{1}{x} = 0$, $x \to 0$. Dar $f(0) = 0$ $\xrightarrow{\text{Teorema...}}$ f cont în 0.

Ex. c, d f cont pe R (provenind din op. alg...)

TEMĂ: l (--) - 3 bilete parțial

- 3.2.4, 3.2.5 (rezolvate)
- 3.3.3, 3.3.4.

Seminar – 20.11.2019 (Derivabilitate – Recapitulare)

Def: $f : D \subseteq \mathbb{R} \to \mathbb{R}$, $a \in D \cap D'$. f se zice că are derivată în $a \overset{def}{\Longleftrightarrow}$

$\Leftrightarrow$ $\exists$ $\lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a} \in \overline{\mathbb{R}}$

$\| not$

$f'(a)$ – s.n. derivata lui f în a.

Dacă în plus, $f'(a) \in \mathbb{R}$, s.n. f e derivabilă în a

f s.n. derivabilă pe $D_1 \subset D \underset{def}{\Longleftrightarrow} (\forall a \in D_1)$ (f este derivabilă în a)

(aplicația $D_1 \ni a \longmapsto f'(a) \in \mathbb{R}$ s.n. derivata lui f)

Obs: f deriv în $a \in D \cap D' \underset{\not\Leftarrow}{\Longrightarrow}$ f cont în a $\left(\underset{x \neq a}{f(x) - f(a)} = \dfrac{f(x)-f(a)}{x-a} \cdot (x-a) \right.$

$\left. \xrightarrow[x \to a]{} \underset{\mathbb{R}}{\underset{\cap}{f'(a)}} \cdot 0 = 0 \right)$

0) $f(x) = c \Rightarrow f'(x) = 0$, $\forall x \in \mathbb{R}$. Se scrie $\boxed{c' = 0}$

Exp: 1) $f(x) = x^n$, $n \in \mathbb{N}^*$, $a \in \mathbb{R}$

$x \neq a \Rightarrow \dfrac{f(x)-f(a)}{x-a} = \dfrac{x^n - a^n}{x-a} = \dfrac{\cancel{(x-a)}(x^{n-1} + x^{n-2} \cdot a + \dots + a^{n-1})}{\cancel{(x-a)}} =$

$\xrightarrow[x \to a]{} n \cdot a^{n-1} \overset{def}{\Longrightarrow} f$ deriv în a și $f'(a) = n \cdot a^{n-1}$

$a \mapsto x \Rightarrow f'(x) = n x^{n-1}$
$\forall x \in \mathbb{R}$

Se scrie prescurtat $\boxed{(x^n)' = n \cdot x^{n-1}, \forall x \in \mathbb{R}}$ (scriere nu foarte riguroasă)

2) $f(x) = \sin x$, $x \in \mathbb{R}$, $a \in \mathbb{R}$.

$x \neq a \Rightarrow \dfrac{f(x)-f(a)}{x-a} = \dfrac{\sin x - \sin a}{x-a} = \dfrac{2 \sin \frac{x-a}{2} \cos \frac{x+a}{2}}{\frac{x-a}{2}} \xrightarrow[x \to a]{} 1 \cdot \cos a = \cos a.$

$\Rightarrow f$ deriv în a și $f'(a) = \cos a$.

$a \mapsto x \Rightarrow f'(x) = \cos x$, $(\forall) x \in \mathbb{R}$.

Se scrie $\boxed{(\sin x)' = \cos x, \forall x \in \mathbb{R}}$

— — — — — —

$\Rightarrow$ așa se obțin majoritatea derivatelor fct. elem.

Prop (Operații algebrice cu fct derivabile – Reguli de derivare)

Fie $f, g : D \subseteq \mathbb{R} \to \mathbb{R}$, derivabile în $a \in D \cap D'$ [derivabile pe $D_1 \subset D \cap D'$]

Atunci $f+g, f\cdot g, \frac{f}{g}$ derivabile în a (resp. pe D_1).
$\hookrightarrow$ ($g(a) \neq 0$ sau $g \neq 0$ pe D_1)

$$(f+g)'(a) = f'(a) + g'(a) \; ; \; (fg)'(a) = f'(a)\cdot g(a) + f(a)\cdot g'(a).$$

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\,g(a) - f(a)\,g'(a)}{g^2(a)}$$

resp.: $(f+g)' = f' + g'$; $(f\cdot g)' = f'g + fg'$; $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$; $(\ldots)$ pe D_1

Prop (Derivata inversei)

Fie $D \subseteq \mathbb{R} \xrightarrow[f \text{ bijecție}]{} E \subseteq \mathbb{R}$, $a \in D$, f derivabilă în a cu $f'(a) \neq 0$.

Atunci f^{-1} deriv în $b = f(a)$ și $(f^{-1})'(b) = \frac{1}{f'(a)}$

Obs: se pot obține astfel și celelalte derivate ale unor fct elem. care sunt (tg, ctg, $arcsin$, $arccos$, $arctg$, ...

Ex: $arctg \overset{def}{:=} \left(tg\big|_{\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)} \right)^{-1} : \mathbb{R} \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{f = tg / \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)} \mathbb{R}$$

$a \in \qquad b \in \; = tg\, a$.

f deriv în a și $f'(a) = \frac{1}{\cos^2 a} \neq 0$.

$\overset{\text{Prop (Der fct inv)}}{\Longrightarrow}$ $arctg$ deriv în $b = tg\, a$ și $(arctg)'(b) = \frac{1}{f'(a)} = \cos^2 a$.

$$\cos^2 a = \frac{1}{1+tg^2 a} = \frac{1}{1+tg^2(arctg\, b)} = \frac{1}{1+b^2}.$$

Prop (Derivata funcțiilor compuse) $a \in D\cap D' \rightsquigarrow b := u(a) \in E\cap E'$

Fie
$$\begin{tikzcd} D \subseteq \mathbb{R} \arrow[r, "u"] \arrow[rr, bend right, "f\circ u"'] & E \subseteq \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$\left.\begin{array}{l} u \text{ deriv. în } a \text{ (resp. pe } D_1 \subset D\cap D') \\ f \text{ deriv. în } u(a) \text{ (resp. pe } u(D_1)) \end{array}\right| \Rightarrow f\circ u$ deriv. în a (pe D_1) și

$(f\circ u)'(a) = f'(u(a))\cdot u'(a)$ (resp. $(f\circ u)' = f'\circ u \cdot u'$ pe D_1).

$$\left(\ldots \quad \frac{f(u(x))-f(u(a))}{x-a} = \underbrace{\frac{f(u(x))-f(u(a))}{u(x)-u(a)}}_{\xrightarrow{x\to a} f'(u(a))} \cdot \underbrace{\frac{u(x)-u(a)}{x-a}}_{\xrightarrow{x\to a} u'(a)} \right)$$

Obs: Combinând tabel I al derivatelor cu Prop anterioară $\Rightarrow$ tabelul II al derivatelor.

$(\underbrace{\sin}_{f} u)' = \cos u \cdot u'$ ș.a.m.d. ...

Ex. Să se studieze continuitatea funcțiilor.

a) $f(x) = \begin{cases} \sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$; b) $f(x) = \begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$; c) $f(x) = \begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0. \end{cases}$

d) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x>0 \\ 0, & x\leq 0. \end{cases}$

Sol: Știm (vezi sem. trecut) că nu există $\lim_{x\to 0}\sin\frac{1}{x} \Rightarrow f$ nu e cont. în 0. $\Rightarrow f$ nu e deriv. în 0.

Evident f deriv. pe $\mathbb{R}^*$ (provine din op. alg. și de comp. cu fct. deriv.) și $f'(x) = \left(\sin\frac{1}{x}\right)' = \cos\frac{1}{x}\cdot\left(\frac{-1}{x^2}\right) = \frac{-1}{x^2}\cdot\cos\frac{1}{x}$, $(\forall)\, x\neq 0$.

Obs: Singurele fct. elem. care au dom. de derivabilitate diferit de domeniul de definiție sunt:

$\sqrt[n]{x}$, arcsin, arccos

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$, $x>0$ $\left(D=[0,\infty),\ D_{deriv}=(0,\infty)\right)$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $x\in(-1,1)$ $\left(D=[-1,1],\ D_{deriv}=(-1,1)\right)$

$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$, — " — —

$(\sqrt[3]{x})' = \frac{1}{3\sqrt[3]{x^2}}$, $x\neq 0$ $\left(D=\mathbb{R},\ D_{deriv}=\mathbb{R}^*\right)$

b) ~~Știm că $\nexists$. $\lim\limits_{x\to 0} \sin\frac{1}{x}$ $\Rightarrow$ f nu e cont în 0. $\Leftrightarrow$ f nu e deriv în 0.~~

~~$f'(x) = \left(x\sin\frac{1}{x}\right)' = (x)'\cdot\sin\frac{1}{x} + x\cdot\left(\sin\frac{1}{x}\right)' = \sin\frac{1}{x} + x\cos\frac{1}{x}$.~~

Evident f deriv pe $\mathbb{R}^*$ — " — și $f'(x) = \left(x\sin\frac{1}{x}\right)' = (x)'\cdot\sin\frac{1}{x} + x\left(\sin\frac{1}{x}\right)'$

$= \sin\frac{1}{x} + x\cdot\cos\frac{1}{x}\cdot\left(\frac{-1}{x^2}\right) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$ $\forall x\neq 0$.

Știm că $\exists$. $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0 = f(0)$ $\Leftrightarrow$ f cont în 0.

$\frac{f(x)-f(0)}{x-0} = \frac{x\sin\frac{1}{x} - 0}{x-0} = \sin\frac{1}{x}$, nu are lim. în 0 $\Rightarrow$ f nu are derivată în 0.

c) Evident f deriv pe $\mathbb{R}^*$ (...) și $f'(x) = \left(x^2\sin\frac{1}{x}\right)' = (x^2)'\sin\frac{1}{x} + x^2\cdot\left(\sin\frac{1}{x}\right)'$

$= 2x\sin\frac{1}{x} + x^2\cdot\cos\frac{1}{x}\left(\frac{-1}{x^2}\right) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$, $(\forall x\neq 0)$.

Analog cu b) $\exists \lim f(\cdot) = 0 = f(0) \Leftrightarrow$ f cont în origine.

$\frac{f(x)-f(0)}{x-0} = \frac{x^2\sin\frac{1}{x} - 0}{x-0} = x\sin\frac{1}{x} \xrightarrow[x\to 0]{} 0$ (Cr. Maj)

$\Rightarrow$ f deriv și în 0 și $f'(0)=0$.

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

[numim 50 pag notițe ?? - form Lagrange, Rolle, ...].

d) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x>0 \\ 0, & x \leq 0. \end{cases}$

f evident deriv. pe $(-\infty, 0) \cup (0, \infty)$ (provine din fct. elem. care sunt deriv. și din compuneri de fct. elem.).

și $f'(x) = \begin{cases} 0, & x<0 \\ e^{-\frac{1}{x^2}} \cdot \left(\frac{2}{x^3}\right), & x>0 \end{cases} = \begin{cases} 0, & x<0 \\ \frac{2}{x^3} \cdot e^{-\frac{1}{x^2}}, & x>0. \end{cases}$

se poate obs. că $f_s(0) = f_d(0) = 0$
$\Rightarrow$ f cont. în 0

$$\frac{f(x)-f(0)}{x-0} = \frac{e^{-\frac{1}{x^2}} - 0}{x-0} = \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}}$$

Fermat
$\Downarrow$
Rolle
$\Downarrow$
Lagrange
$\Downarrow$
Cauchy
$\Downarrow$
L'Hospital

$$\lim_{\substack{x\to 0\\ x>0}} \frac{f(x)-f(0)}{x-0} = \lim_{\substack{x\to 0\\ x>0}} \frac{e^{-\frac{1}{x^2}} - 0}{x-0} = \lim_{\substack{x\to 0\\ x>0}} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} =$$

$\frac{1}{x} = y$

$$= \lim_{y\to\infty} \frac{y}{e^{y^2}} \overset{l'H}{=} \lim_{y\to\infty} \frac{1}{2y\, e^{y^2}} =$$

$\left.\begin{array}{l} x\to 0 \\ x>0 \end{array}\right| \Rightarrow \frac{1}{x} = y \to \infty$

$= \frac{1}{\infty} = 0 \Rightarrow$ f este deriv. la dr. în 0 și $f'_d(0) = 0$.

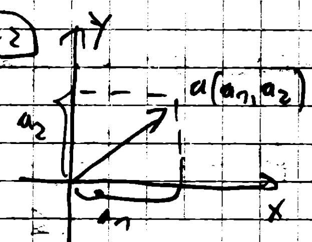
Evident $f'_s(0) = 0$.

$\Downarrow$

f deriv. în 0 și $f'(0) = 0$.

$$f''(a) \overset{def}{=} (f')'(a) \qquad \ldots \quad \frac{\partial^2 f}{\partial x \partial y}(a, b)$$

Curs 26.11.2019



II. Cazul $f: D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

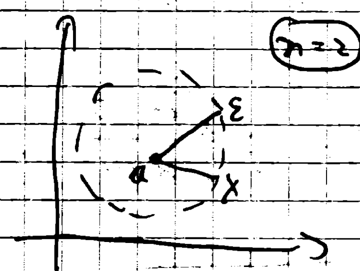
$f(x_1 \dots x_n) = (f_1(x_1 \dots x_n), \dots, f_m(x_1 \dots x_n))$

Pentru $a=(a_1 \dots a_n)\in\mathbb{R}^n$, considerăm $\|a\|_2 \overset{def}{=} \sqrt{a_1^2+\dots+a_n^2}$

↳ s.n. norma lui a (norma euclidiană)

$a\in\mathbb{R}^n$, $\varepsilon>0$ fixați $\rightsquigarrow$ $B(a,\varepsilon) \overset{def}{=} \{x\in\mathbb{R}^n \mid \|x-a\|_2<\varepsilon\}$

s.n. bila de centru a și rază ε



$\|(x_1,x_2)-(a_1,a_2)\|_2<\varepsilon$

$(x_1-a_1)^2+(x_2-a_2)^2<\varepsilon^2$.

(Elem. împr. din fizică)

Def $(x^k)_{k\ge 1}\subset\mathbb{R}^n$, $x^k=(x_1^k,\dots,x_n^k)$, $(\forall)\, k\ge 1$.

$(x^k)_{k\ge 1}$ s.n. convergent (în $\mathbb{R}^n$) $\overset{def}{\iff}$ $(\exists\, l=(l_1 \dots l_n)\in\mathbb{R}^n)$

$(\forall\varepsilon>0)(\exists\, k_\varepsilon\in\mathbb{N})(\forall\, k>k_\varepsilon)(\|x^k-l\|_2<\varepsilon)$.

Obs: l este unic cu proprietatea, și scriem $\lim\limits_{k\to\infty} x^k = l$ (în $\mathbb{R}^n$) sau $x^k \xrightarrow[k]{\mathbb{R}^n} l$

$\Updownarrow$

$\boxed{\|x^k-l\|_2 \xrightarrow[k]{\mathbb{R}} 0}$

Prop. (Teorema pe componente a conv. în $\mathbb{R}^n$).

$$\mathbb{R}^n \ni \underset{(x_1^k,\dots,x_n^k)}{\overset{\|}{x^k}} \xrightarrow[k]{\mathbb{R}^n} \underset{(l_1,\dots l_n)}{\overset{\|}{l\in\mathbb{R}^n}}$$

Dem: Se utilizează inegalitățile

$$|x_i| \le \sqrt{x_i^2+x_{i+1}^2+\dots+x_n^2} \le |x_i|+\dots+|x_n|,\ \forall\, i=\overline{1,n}$$

Ex: $x^k = \left(\frac{1}{k}, \frac{1}{2^k}, \frac{1}{k!}\right) \xrightarrow[k]{\mathbb{R}^3} (0,0,0)$.
($n=3$)

<u>Def 1)</u>: $V \in \mathbb{R}^n$ s.n. <u>vecinătate a lui $a \in \mathbb{R}^n$</u> $\overset{}{\underset{def}{\iff}} (\exists\, \varepsilon > 0)\,(B(a,\varepsilon) \subseteq V)$.

$\mathcal{V}(a) :\overset{not}{=} \{V \subseteq \mathbb{R} \mid V$ vec. a lui $a\}$.

2) $D \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ s.n. punct de acumulare a lui $D \iff \forall\, V \in \mathcal{V}(a)$ $(V \setminus \{a\}) \cap D \neq \emptyset$

$D' :\overset{not}{=} \{a \in \mathbb{R}^n \mid a$ pct. de ac. al lui $D\}$.

<u>Ex</u>: $D = [0,1) \times [0,1) \Rightarrow D' = [0,1] \times [0,1]$.

<u>Obs.</u>: $a \in D' \iff (\forall\, \varepsilon > 0)\,(B(a,\varepsilon) \setminus \{a\}) \neq \emptyset$

Ex: $D = \left([0,1] \cup \{2\}\right) \times [0,1]) \Rightarrow D' = \left([0,1] \cup \{2\}\right) \times [0,1]$.

$D = \left([0,1) \cup \{2\}\right) \times [0,1) \cup \{(3,1)\}. \Rightarrow D' = \left([0,1] \cup \{2\}\right) \times [0,1]$

(! Analiză Complexă Anul 2 – lucruri interesante – Riemann – limită orizont)

<u>Def</u>: $D \subseteq \mathbb{R}^n$, $a \in D'$, $\ell \in \mathbb{R}^n$, $f : D \to \mathbb{R}^n$.
<u>f se spune că are limita ℓ în a</u> $\underset{def}{\iff} (\forall\, V \in \mathcal{V}(\ell))\,(\exists\, U \in \mathcal{V}(a))\,(\forall\, x \in (U \setminus \{a\}) \cap D))$
$(f(x) \in V)$ → (not. $\lim\limits_{x \to a} f(x) = \ell$).

(James Munkres – Topologie)

• <u>Prop</u> ("Variant ε–δ a limitei def. vect.")

Def: $\lim\limits_{x \to a} f(x) = \ell \iff (\forall\, \varepsilon > 0)\,(\exists\, \delta_\varepsilon > 0)\,(\forall\, x \in D \setminus \{a\})$; $(\|x-a\|_2 < \delta_\varepsilon)$ *
$\Updownarrow$
$(x \in B(a, \delta_\varepsilon))$

*$(\|f(x)-\ell\|_2 < \varepsilon)$
$\Updownarrow$
$f(x) \in B(\ell, \varepsilon)$.

Dem: Identică cazului $n=m=1$, $a\in\mathbb{R}$, $l\in\mathbb{R}$ cu $\|\cdot\|_2$ în loc de $|\cdot|$

Prop: ("Caract. cu șiruri a lim. fct. vect.")

Dt. $\lim_{x\to a} f(x)=l \Leftrightarrow \forall (x^k)_k \subset D\setminus\{a\} \Rightarrow \left(f(x^k) \xrightarrow[k]{\mathbb{R}^m} l\right)$

$$\Updownarrow$$

$\left(\forall i=\overline{1,n}\right)\left(x_i^k \xrightarrow[k]{\mathbb{R}} a_i\right) \Leftrightarrow \left(x^k \xrightarrow[k]{\mathbb{R}^n} a\right)$

$\left(\forall j=\overline{1,m}\right)\left(f_j(x^k) \xrightarrow[k]{\mathbb{R}} l_j\right)$

Dem: Identică cazului $n=m=1$, cu $\|\cdot\|_2$ în loc de $|\cdot|$.

Prop: (Criteriul majorării pt lim fct vect").

Fie $f:D\subseteq\mathbb{R}^n \to \mathbb{R}^m$, $g:D\subseteq\mathbb{R}^n\to\mathbb{R}$, $l\in\mathbb{R}^m$, $a\in D'$ a.î. $\exists\, U_0\in\mathcal{V}(a)$ cu

$\|f(x)-l\|_2 \le g(x)$, $\forall x\in(D\setminus\{a\})\cap U_0$ și $\exists \lim_{x\to a} g(x)=0\in\mathbb{R}$.

Atunci $\exists \lim_{x\to a} f(x)=l\in\mathbb{R}^m$

Dem: Imediată din "Caract. cu șiruri a lim fct vect" +

+ "Cr. Maj. pt șiruri de nr. reale")

EXP: Să se studieze existența limitei fcț. în pct. indicate.

a) $f(x,y)=\dfrac{xy}{x^2+y^2}$; $(a,b)=(0,0)$ b) $f(x,y)=\dfrac{x^2y}{x^2+y^2}$, $(a,b)=(0,0)$

Soluție: a) Cons. șirul $(x_n,y_n)=\left(\frac{1}{n},\frac{1}{n}\right)\xrightarrow[n]{}(0,0) \Rightarrow f(x_n,y_n)=\dfrac{\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n^2}}=\dfrac{1}{2}$

$\xrightarrow[n]{}\dfrac{1}{2}$.

$(x_n',y_n')=\left(\frac{1}{n},\frac{-1}{n}\right)\xrightarrow[n]{}(0,0)\Rightarrow f(x_n',y_n')=\dfrac{\frac{-1}{n}\cdot\frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n^2}}=-\dfrac{1}{2}\xrightarrow[n]{}\dfrac{-1}{2}$

$\boxed{* f:\underbrace{\mathbb{R}^2\setminus\{(0,0)\}}_{D}\to\mathbb{R}} \Rightarrow \boxed{D'=\mathbb{R}^2}$

Enunț pe șiruri $\Longrightarrow$ nu $\exists$ $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

a lim fct vect

b) Fie $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0) \Rightarrow f(x_n, y_n) = \dfrac{\frac{1}{n^3}}{\frac{1}{n^2}+\frac{1}{n^2}} = \frac{1}{2n} \xrightarrow[n]{} 0.$

$(x_n', y_n') = \left(\frac{1}{n}, \frac{-1}{n}\right) \to (0,0) \Rightarrow f(x_n', y_n') = \dfrac{\frac{-1}{n^3}}{\frac{1}{n^2}+\frac{1}{n^2}} \Rightarrow \frac{-1}{2n} \xrightarrow[n]{} 0.$

Nu pot spune că $\exists$ $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$, dar, dacă ar exista atunci nu ar fi trebui să fie 0.

$$|f(x,y)-0| = \frac{x^2 |y|}{x^2+y^2} \le \underbrace{|y|}_{=g(x,y)}, \ \forall (x,y)\neq(0,0)$$

$\exists \lim\limits_{(x,y)\to(0,0)} g(x,y) = 0$

$\xrightarrow[\text{fct. vect.}]{\text{Th. Majorării}}$ $\exists \lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$

Def: $f: D \overset{\subseteq R^n}{\longrightarrow} R^m$ s.n. continuă în $a \in D \overset{def}{\iff}$

$\iff (\forall V \in \mathcal{V}_{(f(a))})(\exists U \in \mathcal{V}_{(a)})(\forall x \in (U\setminus\{a\})\cap D)(f(x)\in V).$

Obs: Dacă $a \in D \setminus D'$, atunci f trivial cont în a.

Prop: ("Enunț ε-δ a cont fct. vect") f cont în a $\iff$

$\iff (\forall \varepsilon>0)(\exists \delta_\varepsilon>0)(\forall x \in D, \|x-a\|_2 < \delta_\varepsilon)(\|f(x)-f(a)\|_2 < \varepsilon).$

$\iff \left(\forall (x^k)_{k\ge 1} \subset D,\ x^k \xrightarrow[k]{R^n} a \ \Big|\Rightarrow f(x^k) \xrightarrow[k]{R^m} f(a)\right)$

Prop: (Enunț cont cu lim al fct vect) f cont în $a \in D \cap D' \iff \exists \lim\limits_{x\to a} f(x) = f(a)$.

Prop: (Op. algebrice ~~cu fct. cont~~ și de compunere cu fct vect cont)

I $f: D^{\subseteq R^n} \to R^m$, $g: D^{\subseteq R^n} \to R$ cont în a $\Rightarrow$ $f\cdot g$ cont în a.

II $f, g: D^{\subseteq R^n} \to R^m$ cont în $a \in D \Rightarrow f+g$ cont în a

III $D^{\subseteq R^n} \xrightarrow{f} E^{\subseteq R^m} \xrightarrow{g} R^p$, f cont în $a\in D$, g cont în $f(a)\in E$ $\Rightarrow$ $g\circ f$ cont în a.

Seminar 27.11.2019

Ex 1. Să se studieze existența limitei funcțiilor în punctele indicate.

a) $f(x,y) = \dfrac{\ln(x^2+2y^2+1)}{\sqrt{x^2+2y^2}}$, $(a,b)=(0,0)$ | $D = \mathbb{R}^2 \setminus \{(0,0)\}$, $D' = \mathbb{R}^2$

b) $f(x,y) = \dfrac{x+y}{x-y}$, (a,a), $a \in \mathbb{R}$.

TEMA: c) $f(x,y) = \dfrac{y^2+x}{y^2-x}$, $(a,\sqrt{a})$, $a \geq 0$

Ex 2 Să se studieze continuitatea funcțiilor

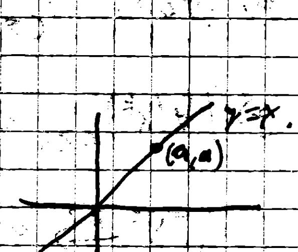
a) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

b) $f(x,y) = \begin{cases} \dfrac{\sin xy}{x}, & x \neq 0 \\ y, & x = 0 \end{cases}$

Ex 1: a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\ln(1+x^2+2y^2)}{\sqrt{x^2+2y^2}} \quad \underset{\substack{(x,y)\to(0,0) \Rightarrow t\to 0 \\ (x,y)\neq(0,0) \Rightarrow t>0}}{\overset{x^2+2y^2 = t}{=}} \quad \lim\limits_{\substack{t\to 0 \\ t>0}} \dfrac{\ln(1+t)}{\sqrt{t}} = \lim\limits_{\substack{t\to 0 \\ t>0}} \dfrac{\ln(1+t)}{t} \cdot \sqrt{t}$

$= 1 \cdot 0 = 0.$

b) $D = \{(x,y) \in \mathbb{R}^2 \mid y \neq x\}$, $D' = \mathbb{R}^2$.



Fie $((x_n,y_n))_{n\geq 1} \subset D \setminus \{(a,a)\}$, $(x_n,y_n) \xrightarrow[n]{} (a,a)$

$\Updownarrow$

$\begin{cases} x_n \to a \\ y_n \to a \end{cases}$, $y_n \neq x_n$

$\Downarrow$

$f(x_n,y_n) = \dfrac{x_n+y_n}{x_n-y_n} \xrightarrow[n]{} \dfrac{2a}{0}$

i) $\boxed{a=0}$ $y=mx$, $m\neq 1$ $\Rightarrow f(x, mx)=\frac{x+mx}{x-mx}=\frac{1+m}{1-m}\xrightarrow[x\to 0]{}\frac{1+m}{1-m}$.
$x\neq 0$

$\exists \lim\limits_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x+y}{x-y}=\frac{1+m}{1-m}$, $(\forall) m\neq 1$. $\Rightarrow$ Nu ex. $\lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x-y}$

ii) $\boxed{a\neq 0}$ Cons. din nou $\underbrace{(x_n,y_n)_{n\geq 1}\subset D\setminus\{(a,a)\}, (x_n,y_n)\to(a,a)}_{\Updownarrow}$

$\begin{cases} x_n\to a \\ y_n\to a\end{cases}$, $x_n\neq y_n$

- implică, în plus, condiția $x_n<y_n \Rightarrow f(x_n,y_n)=\frac{x_n-y_n}{x_n-y_n}\to\frac{2a}{-0}$.

$\Rightarrow \exists. \lim\limits_{\substack{(x,y)\to(a,a)\\ x<y}} \frac{x+y}{x-y}=\begin{cases}-\infty, & a<0\\ +\infty, & a>0\end{cases} = \begin{cases}-\infty, & a>0\\ +\infty, & a<0\end{cases}$

Analog $\exists \lim\limits_{\substack{(x,y)\to(a,a)\\ x>y}} \frac{x+y}{x-y}=\begin{cases}+\infty, & a<0\\ -\infty, & a>0\end{cases}$

$\Rightarrow$ Nu ex $\lim\limits_{(x,y)\to(a,a)} \frac{x+y}{x-y}$.

(// Matlab – pt. a vedea multe resurse • 3DGrapher –

!! TEMA: $\boxed{\text{Ex 1 c}}$

Ex2. a) $(x_n,y_n)=\left(\frac{1}{n},\frac{1}{n}\right)\to(0,0) \Rightarrow f(x_n,y_n)=\frac{\frac{1}{n^2}}{\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}}=\frac{1}{\sqrt{2}n}\xrightarrow[n]{}0$.

Încercând cu alte șiruri care tind la $(0,0)$, $f(\ldots)\to 0$.

$|f(x,y)-0|=\frac{|xy|}{\sqrt{x^2+y^2}}\leq\frac{1}{2}\sqrt{|xy|}=g(x,y)$, $\forall(x,y)\neq(0,0)$ | th. Maj.

(ineg. mediilor pt x^2, y^2) $\exists \lim\limits_{(x,y)\to(0,0)} g(x,y)=0$

$\Rightarrow \exists \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0$ | ambele cont / cu lim fct cont $\Rightarrow f$ cont în $(0,0)$.

Dar $f(0,0)=0$

Evident f este continuă în orice $(x,y) \neq (0,0)$, (provenind din op. alg. și de compunere ale fct cont)

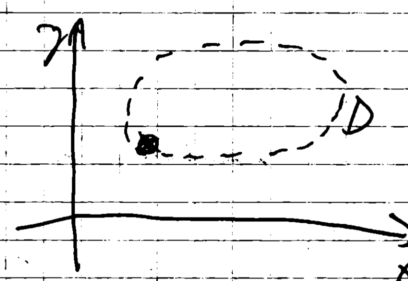
pag 88 ; 3.2.11, 3.2.12 Exerciții rămase.

Derivate parțiale (pt $f = f(x,y)$)

Def: $f : D \xrightarrow{} R$, $D \subseteq R^2$ deschisă

$\Downarrow$

$\boxed{\forall (a,b) \in D,\ \exists \varepsilon > 0,\ B((a,b), \varepsilon) \subseteq D.}$



Se spune că f este derivabilă (parțial) în raport cu x în (a,b) $\overset{def}{\Longleftrightarrow}$

$\Leftrightarrow \exists \lim\limits_{x\to a} \dfrac{f(x,b)-f(a,b)}{x-a} \in R =: \dfrac{\partial f}{\partial x}(a,b)$

$\parallel$

$(x \longmapsto f(x,b))'(a)$

$\cap$

$D_b := \{x \in R \mid (x,b) \in D\}$.

$(a,b) \in D$, D deschisă $\Rightarrow a \in D_b \cap D_b'$

Exp: $\dfrac{\partial f}{\partial x}(1,1) = ?$ pt $f(x,y) = x^2 y$.

$\dfrac{\partial f}{\partial x}(1,1) \overset{V1}{=} \lim\limits_{x\to 1} \dfrac{f(x,1)-f(1,1)}{x-1} = \lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = 1+1 = 2$

$\overset{V2}{=}$ Obs. $(x \longmapsto f(x,1))'(1) = (x \longmapsto x^2)'(1) = 2x \Big|_{x=1} \Rightarrow$

$$\frac{\partial f}{\partial x}(x,y)=2x\cdot y=2xy \iff \frac{\partial f}{\partial x}(x,y)=x^2.$$

Ex. Să se calc. derivatele parțiale ale fct:

a) $f(x,y)=x^2y+2xy^3$; b) $f(x,y)=\frac{xy}{x^2+y^2}$; c) $f(x,y)=x^y$

d) $f(x,y)=\ln(x^2+y^2)$; e) $f(x,y)=x\cdot e^{-(x^2+y^2)}$.

a) $\frac{\partial f}{\partial x}(x,y)=2xy+2y^3$; $\frac{\partial f}{\partial y}(x,y)=2x^2+6xy^2$; $(\forall)(x,y)\in\mathbb{R}^2$.

b) $D=\mathbb{R}^2\setminus\{(0,0)\}$

$$\frac{\partial f}{\partial x}(x,y)=\frac{y(x^2+y^2)-xy\cdot 2x}{(x^2+y^2)^2}=\frac{y^3-xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{x(x^2+y^2)-xy\cdot 2y}{(x^2+y^2)^2}=\frac{x^3-xy^2}{(x^2+y^2)^2},\quad \forall (x,y)\in D.$$

c) $\frac{\partial f}{\partial x}(x,y)=y\,x^{y-1}$ $\qquad D=[[0,\infty)\setminus\{0\}]\times\mathbb{R}$

$\frac{\partial f}{\partial y}(x,y)=x^y\cdot\ln x$, $(\forall)(x,y)\in D$.

d) $D=\mathbb{R}^2\setminus\{(0,0)\}$

$$\frac{\partial f}{\partial x}(x,y)=\frac{1}{x^2+y^2}\cdot 2x=\frac{2x}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{1}{x^2+y^2}\cdot 2y=\frac{2y}{x^2+y^2}$$

$(\forall)(x,y)\in D$.

$(\ln u)'=\frac{1}{u}\cdot u'$

$\ln u(x,y)=\frac{1}{u(x,y)}\cdot\frac{\partial u}{\partial x}(x,y)\ldots$

e) $D = \mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) = 1 \cdot e^{-(x^2+y^2)} + x \cdot e^{-(x^2+y^2)} \cdot \frac{\partial}{\partial x}(-(x^2+y^2))$$

$$= e^{-(x^2+y^2)}(1-2x^2)$$

$(\forall)(x,y) \in D$

$$\frac{\partial f}{\partial y}(x,y) = x \cdot e^{-(x^2+y^2)} \cdot \frac{\partial}{\partial y}(-(x^2+y^2))$$

$$= -2xy\, e^{-(x^2+y^2)}$$

Curs 3.12.2019

„Introducere liniară algebră"

$$\mathbb{R}^n = \underbrace{\mathbb{R}\times\mathbb{R}\times\ldots\times\mathbb{R}}_{n\text{ ori}} = \{(x_1,\ldots,x_n) \mid x_i \in \mathbb{R},\ i=\overline{1,n}\}.$$

$(\mathbb{R},+,\cdot)$ corp com.

(1) se definește „+": $(x_1,\ldots,x_n)+(y_1,\ldots,y_n) \overset{def}{=} (x_1+y_1,\ldots,x_n+y_n)$, $\forall x=(x_1\ldots x_n)\in\boxed{\mathbb{R}}$ $\left(\forall y=(y_1\ldots y_n)\ \ \forall\alpha\in\mathbb{R}\right)$

– „$\cdot\alpha$" $\alpha\cdot(x_1,\ldots,x_n) \overset{def}{=} (\alpha x_1,\ldots,\alpha x_n)$

se poate dem. că $(\mathbb{R}^n,+)$ grup. comut.: $\begin{cases}\alpha(x+y)=\alpha x+\alpha y\\ (\alpha+\beta)\cdot x = \alpha x+\beta x\\ (\alpha\cdot\beta)\cdot x = \alpha\cdot(\beta\cdot x)\\ 1\cdot x = x\end{cases}$

Un V cu două op. "+" și "$\cdot\alpha$" se notează $(V,+,\cdot\alpha)$ și s.n. $\mathbb{R}$-spațiu vect.

(2) $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.n. $\mathbb{R}$-liniară $\Leftrightarrow$ (L_1) $T(x+y)=T(x)+T(y)$, $(\forall)x,\alpha\in\mathbb{R}$.
$n,m\in\mathbb{N}^*$ (L_2) $T(\alpha x)=\alpha T(x)$

$T(1)=:a \Rightarrow T(x)=T(x\cdot 1)=xT(1)=a$

Prop: $T:\mathbb{R}^n\to\mathbb{R}^m$ liniară $\Rightarrow \exists!\ A_T\in M_{m,n}(\mathbb{R})$, $T(x)=A_T\cdot\tilde{x}$ $\left(\tilde{x}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\right)$

Dem: $x=(x_1\ldots x_n)=(x_1,0,\ldots,0)+(0,x_2,\ldots,0)+\ldots+(0,0,\ldots x_n)$
$=\sum_{i=1}^{n} x_i e_i$, $e_i=(0,\ldots,0,\underset{i}{1},0,\ldots,0)$

$\Rightarrow T(x) \overset{1\,și\,2}{=} \sum_{i=1}^{n} x_i T(e_i)$; $A_T :=$

$\alpha_i = \begin{pmatrix}\alpha_{1i}\\ \vdots\end{pmatrix}$ $(\ldots)$

Caz particular: $\boxed{m=2,\ n=1}$ (x_1,x_2) : $T:\mathbb{R}^2\to\mathbb{R}$ liniară $\Rightarrow(\exists)\ \alpha,\beta\in\mathbb{R}$.
$A_T=(\alpha,\beta)$

CAPITOL 5. Diferențiabilitatea fct. vect. cu variabilă vectorială.

• Vom prezenta concepte și rezultate pt fct.:

I. $f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

II. $f: D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Def: I $f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, D derivabilă $\left(\underset{def}{\Leftrightarrow} \forall (a,b) \in D, \exists \varepsilon > 0,\ B(a,b,\varepsilon) \subseteq D\right)$

$(a,b) \in D$.

Se spune că f este derivata parțială în raport cu x (resp. y) în (a,b)

$\Leftrightarrow$ ex. $\lim\limits_{x \to a} \dfrac{f(x,b) - f(a,b)}{x-a} \in \mathbb{R} \overset{not}{=\!=} \dfrac{\partial f}{\partial x}(a,b) \to$ s.n. derivata parțială a lui f în raport cu x în (a,b)

$\left(\text{resp. ex. } \lim\limits_{y \to b} \dfrac{f(a,y) - f(a,b)}{y-a} \in \mathbb{R} \overset{not}{=\!=} \dfrac{\partial f}{\partial y}(a,b) \text{ — " — cu y — " — } \right)$

II. Fie $f: D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $f(x_1, \dots, x_n) = \left(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)\right)$

D derivabilă $\left(\underset{def}{\Leftrightarrow} \forall \underset{(a_1,\dots,a_n)}{\overset{\parallel}{a}} \in D, \exists \varepsilon > 0, B(a,\varepsilon) \subseteq D\right)$, $\underset{(a_1,\dots,a_n)}{\overset{\parallel}{a}} \in D$, $i = \overline{1,n}$

Se spune că f este deriv. parțială în raport cu x_i în a $\underset{}{\overset{def}{\Longleftrightarrow}}$ $\lim \dfrac{(a_1, \dots a_{i-1}, x_i, \dots a_n)}{x_i - a}$

$\in \mathbb{R}^m$ s.n. derivata parț. a lui f în rap. cu x_i în i $\;=\; \dfrac{\partial f}{\partial x_i}(a)$

$f: D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $a \in D'$

$l = (l_1, \dots, l_m) \in \mathbb{R}^m$

ex. $\lim\limits_{x \to a} f(x) = l \Leftrightarrow$

$(\forall j = \overline{1,m})\left(\text{ex. } \lim\limits_{x \to a} f_j(x) = l_j\right)$

Obs. există $\dfrac{\partial f}{\partial x_i}(a) \Leftrightarrow (\forall)\, j = \overline{1,m}$, ex. $\dfrac{\partial f}{\partial x_i}(a)$ și

$\dfrac{\partial f}{\partial x_i}(a) = \left(\dfrac{\partial f_1}{\partial x_1}(a), \dots, \dfrac{\partial f_m}{\partial x_m}(a)\right)$

pt $i = \overline{1,n}$ fixat

Def: I. $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$, $(a,b) \in D$. Atunci f s.n. derivabilă în (a,b)

$$\overset{def}{\Leftrightarrow} \left(\exists T: \mathbb{R}^2 \to \mathbb{R} \text{ liniar}\right)\left(\text{ℓℓ} \lim_{(x,y)\to(a,b)} \frac{f(x,y)-f(a,b)-T(x-a,y-b)}{\|(x,y)-(a,b)\|_2} = 0\right)$$

Obs. 1) T din def de mai sus este unic, $T \underset{not}{=:} df(a,b)$ s.n. diferențiala lui f în (a,b)

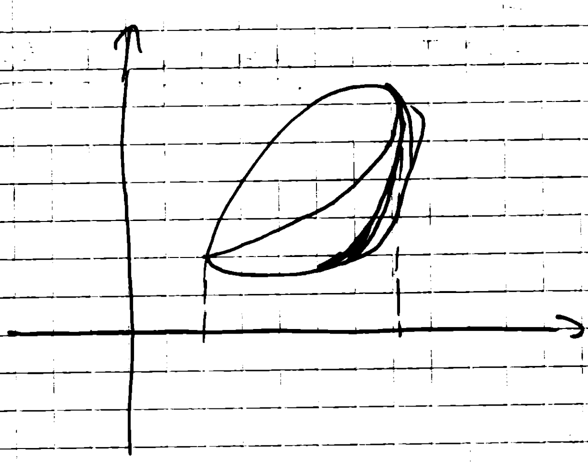
2) f dif în $(a,b) \Rightarrow f(x,y)-f(a,b) \overset{apr.\ egal.}{\simeq} T(x,y)-T(a,b)$

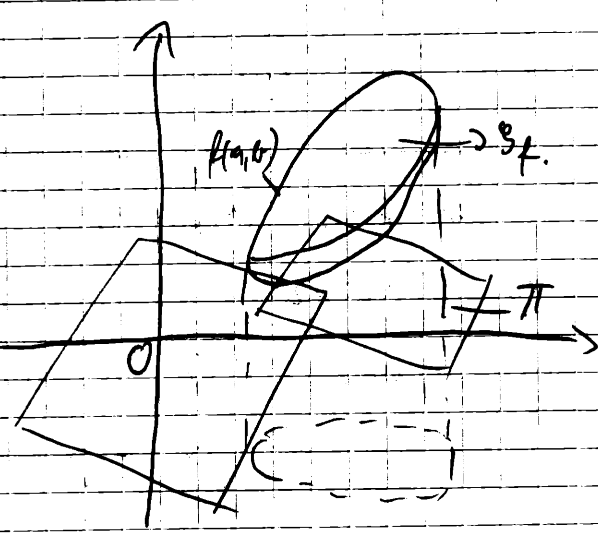
$(\exists)\alpha,\beta \in \mathbb{R}$ $\qquad = \alpha x + \beta y - \alpha a - \beta b$.

$\Leftrightarrow (\exists!\, \alpha,\beta \in \mathbb{R})\ (f(x,y) \simeq f(a,b) + \alpha(x-a) + \beta(y-b))$.

$V(x,y) \simeq V(a,b)$

$\Leftrightarrow (\exists!\, \alpha,\beta \in \mathbb{R})\ \left(G_f \underset{(x,y)\simeq(a,b)}{\simeq} G_\pi,\ \pi: z = f(a,b) + \alpha(x-a) + \beta(y-b)\right)$





$f(a_1,\dots,a_n)$ = $\boxed{\begin{array}{l} G_T = G_{df(a,b)} \\ T(x,y) = \alpha x + \beta y \end{array}}$

Def II: Fie $f: D \underset{\text{derivabilă}}{\xrightarrow{\subseteq \mathbb{R}^n}} \mathbb{R}^m$, $a \in D$. Atunci f s.n. derivabilă în a $\Leftrightarrow$

$$\left(\exists T: \mathbb{R}^n \to \mathbb{R}^m \text{ liniară}\right)\left(\text{ℓℓ} \lim_{x\to a} \frac{f(x)-f(a)-T}{\dots} \ ?\right)$$

Teoremă: I. $f: D \xrightarrow{\subset \mathbb{R}^2} \mathbb{R}$, $(a,b) \in D$, f dif în (a,b) $\Rightarrow$ $\begin{cases} f \text{ cont în } (a,b) \\ \exists \frac{\partial f}{\partial x}(a,b), \exists \frac{\partial f}{\partial y}(a,b) \end{cases}$
deschisă

şi $\begin{cases} \frac{\partial f}{\partial x}(a,b) = df(a,b)\underbrace{(1,0)}_{e_1} \\ \frac{\partial f}{\partial y}(a,b) = df(a,b)\underbrace{(0,1)}_{e_2} \end{cases}$

II $f: D \xrightarrow{\subseteq \mathbb{R}^n} \mathbb{R}^m$, $a = (a_1, \dots, a_n) \in D$, f dif în a $\Rightarrow$ $\begin{cases} f \text{ cont în } a \\ \forall i = \overline{1,n}, \exists \frac{\partial f}{\partial x_i} = df(a)(e_i) \end{cases}$
deschisă

Obs. În II dacă f dif în $a \Rightarrow A_{df(a)} = \left(\frac{\partial f_j}{\partial x_i}(a)\right)$, $j = \overline{1,m}$, $i = \overline{1,n}$

$A_T = (\alpha_{ji})$,

$\alpha_{ji} = T(e_i)_j$.

$J_f(a) \to$ s.n. jacobiana lui f în a

Ex: $f(x,y) = (\overset{f_1}{xy}, \overset{f_2}{x^2 - y^2})$.

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$$J_f(x,y) = \begin{array}{cc} \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix} & \begin{matrix} - f_1 \\ - f_2 \end{matrix} \\ \begin{matrix} x \quad\quad & y \end{matrix} & \end{array}$$

Deci $df(a)(v_1, \dots, v_n) = J_f(a)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(a)$, $\forall x \in (v_1, \dots, v_n) \in \mathbb{R}^n$

$\Pi(v_1, \dots, v_n) = v_i \to$ proiecţia pe axa Ox_i. $\boxed{\Pi_i = dx_i}$
(de la $\mathbb{R}^n$ în $\mathbb{R}$) liniară

$d\Pi_i(a) = \Pi_i$

$\Rightarrow d_{f(a)} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\, dx_i$

$T: \mathbb{R}^n \to \mathbb{R}^m$ liniară $\to \forall a \in \mathbb{R}^n$, T dif în a şi $dT(a) = T$.

Caz part. $n=2, m=1$: $d_f(a,b)(u,v) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)$

$= \frac{\partial f}{\partial x}(a,b)u + \frac{\partial f}{\partial y}(a,b)v, \ (\forall)\ u,v \in \mathbb{R}^2$

$$df(a,b) = \frac{\partial f}{\partial x}(a,b)\, dx + \frac{\partial f}{\partial y}(a,b)\, dy.$$

$dx = \pi_1$

$dy = \pi_2$.

Seminar 4.12.2019

BREVIAR:

- $f: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $a=(a_1 \dots a_n) \in D$, f dif. în a $\Rightarrow$ $(\forall i = \overline{1,n}, \exists \frac{\partial f}{\partial x_i}(a))$
  derivabilă, $f=(f_1, \dots, f_m)$

$J_f(a) := \left(\frac{\partial f_j}{\partial x_i}(a)\right)_{\substack{j=\overline{1,m} \\ i=\overline{1,n}}}$ → m. jacobiană lui f în a

$df(a)(v) = J_f(a) \cdot \tilde{v}$, $v = (v_1 \dots v_n)$

$$df(a)(v_1 \dots v_n) = \gamma\left(\frac{\partial f_j}{\partial x_1}(a) \dots \frac{\partial f_j}{\partial x_n}(a)\right)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Ex. Calculați J_f și df pentru:

a) $f(x,y) = \left(x^2 - y^2, \frac{x}{y}\right)$;

b) $f(x,y) = (xy, 2x-y, x^2y - 2xy^3)$;

c) $f(x,y,z) = (x^2+y^2+z^2, xy+yz+zx)$

d) $f(x,y,z) = \left(\frac{xz}{y}, \ln(x^2+y^2), ze^{xyz}\right)$

Dt $n=2, m=1$; $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$, $(a,b) \in D$,
f dif. în (a,b)

$$J_f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right); \ df(a,b)(u,v) = J_f(a,b) \cdot \begin{pmatrix} u \\ \vdots \\ v \end{pmatrix} = \frac{\partial f}{\partial x}(a,b)u + \frac{\partial f}{\partial y}(a,b)v.$$

$\forall (u,v) \in \mathbb{R}^2$

$$df(a,b) = \frac{\partial f}{\partial x}(a,b)\,dx + \frac{\partial f}{\partial y}(a,b)\,dy; \quad u = \pi_1(u,v) = dx(u,v).$$

---

a) $D = \mathbb{R} \times \mathbb{R}^*$ (deschisă)

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y), & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y), & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix} \begin{matrix} \to f_1 \\ \to f_2 \end{matrix} = \begin{pmatrix} 2x & -2y \\ \frac{1}{y} & \frac{-x}{y^2} \end{pmatrix}$$

$\frac{\partial}{\partial x}$ $\qquad$ $\frac{\partial x}{\partial y}$

$$d_f(x,y)(u,v) = J_f(x,y)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2xu - 2yv \\ \frac{u}{y} - \frac{xv}{y^2} \end{pmatrix}, \quad \forall (u,v) \in \mathbb{R}^2, \ \forall (x,y) \in D.$$

$$d f(x,y) = d f(x,y)\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 2x\,dx - 2y\,dy \\ \frac{dx}{y} - \frac{x\,dy}{x^2} \end{pmatrix}, \ \forall (x,y) \in D$$

~~Immone~~ b): $D = \mathbb{R} \times \mathbb{R}$;

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y), & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y), & \frac{\partial f_2}{\partial y}(x,y) \\ \frac{\partial f_3}{\partial x}(x,y), & \frac{\partial f_3}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} y & x \\ 2 & -1 \\ 2xy - 2y^3, & x^2 - 6xy^2 \end{pmatrix}$$ ✓

$$d_f(x,y)(u,v) = J_f(x,y)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cancel{2y + 2x}\ uy + vx \\ 2u - v \\ u(2xy - 2y^3) + v(x^2 - 6xy^2) \end{pmatrix} \quad \forall (u,v) \in \mathbb{R}^2, \ \forall (x,y) \in D = \mathbb{R}^2$$ ✓

$$d_f(x,y) = J_f(x,y)\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} dx \cdot y + dy \cdot x \\ 2dx - dy \\ dx(2xy - 2y^3) + dy(x^2 - 6xy^2) \end{pmatrix} \quad \forall (x,y) \in \mathbb{R}^2$$ ✓

c) $D = \mathbb{R}^3$

$$J_f(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y,z), & \frac{\partial f_1}{\partial y}(x,y,z), & \frac{\partial f_1}{\partial z}(x,y,z) \\ \frac{\partial f_2}{\partial x}(x,y,z), & \frac{\partial f_2}{\partial y}(x,y,z), & \frac{\partial f_3}{\partial z}(x,y,z) \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \end{pmatrix}$$

~~$d_f(x,y) = J_f(x,y)($~~

~~$d(x,y)(u,v) = J_f(x,y)$~~

$$d_f(x,y,z)(u,v,w) = J_f(x,y,z)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2xu + 2yv + 2zw \\ (y+z)u + (x+z)v + (y+x)w \end{pmatrix}$$ ✓

$(\forall)\ (u,v,w) \in \mathbb{R}^3$

$(\forall)\ (x,y,z) \in \mathbb{R}^3$

$$d_f(x,y,z) = J_f(x,y,z)\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} 2x\,dx + 2y\,dy + 2z\,dz \\ (y+z)dx + (x+z)dy + (y+x)dz \end{pmatrix}, \ (\forall)(x,y,z)\in\mathbb{R}^3$$

✓

Partial

$$x_n = \sin\frac{1}{n\sqrt[3]{n}} \ ; \ y_n = \frac{1}{n\sqrt[3]{n}}$$

Nota: 9.

$$\frac{x_n}{y_n} = \frac{\sin\frac{1}{n\sqrt[3]{n}}}{\frac{1}{n\sqrt[3]{n}}} \to 1 \overset{\text{Cr. Comp. 2}}{\Rightarrow} \sum_{n\geq 1} x_n \sim \sum_{n\geq 1} y_n$$

Trebuie să arăt că y_n (C) (Cr. Comp. 1).

d) $D = \mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R}$ $\quad (\mathbb{R}\times\mathbb{R}^*\times\mathbb{R})$

$$J_f(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y,z), & \frac{\partial f_1}{\partial y}(x,y,z), & \frac{\partial f_1}{\partial z}(x,y,z) \\ \frac{\partial f_2}{\partial x}(x,y,z), & \frac{\partial f_2}{\partial y}(x,y,z), & \frac{\partial f_2}{\partial z}(x,y,z) \\ \frac{\partial f_3}{\partial x}(x,y,z), & \frac{\partial f_3}{\partial y}(x,y,z), & \frac{\partial f_3}{\partial z}(x,y,z) \end{pmatrix} = \begin{pmatrix} \frac{z}{y}, & \frac{-xz}{y^2}, & \frac{x}{y} \\ \frac{2x}{x^2+y^2}, & \frac{2y}{x^2+y^2}, & 0 \\ ze^{xyz}yz, & ze^{xyz}xz, & e^{xyz}(1+xyz) \end{pmatrix}$$

$$\frac{\partial}{\partial z}\left(ze^{xyz}\right) = e^{xyz} + z\cdot e^{xyz}xy = e^{xyz}(1+xyz)$$

$$d_f(x,y,z)(u,v,w) = J_f(x,y,z)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{z}{y}u + \frac{-xz}{y^2}v + \frac{x}{y}w \\ \frac{2x}{x^2+y^2}u + \frac{2y}{x^2+y^2}v + 0 \\ ze^{xyz}yz\cdot u + ze^{xyz}xz\cdot v + e^{xyz}(1+xyz)\cdot w \end{pmatrix}$$

$\forall (x,y,z)\in D$

$\forall (u,v,w)\in\mathbb{R}^3$

$$d_f(x,y,z) = J_f(x,y,z)\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \frac{z}{y}dx + \frac{-xz}{y^2}dy + \frac{x}{y}dz \\ \frac{2x}{x^2+y^2}dx + \frac{2y}{x^2+y^2}dy \\ ze^{xyz}yz\cdot dx + ze^{xyz}xz\cdot dy + e^{xyz}(1+xyz)\cdot dz \end{pmatrix}$$

$\forall x,y,z\in\mathbb{R}^3$

Def: $f: D \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}$, $(a,b) \in D$ a.î. există $V \in \mathcal{V}(a,b)$, a.î. $\boxed{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} : V \cap D \to \mathbb{R}}$

deschisă

Derivatele parțiale de ordin 2 ale lui f în (a,b) sunt (dacă există)

$$\frac{\partial^2 f}{\partial x^2}(a,b) \overset{not}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(a,b) \qquad \frac{\partial^2 f}{\partial x \partial y}(a,b) \overset{not}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a,b)$$

;

$$\frac{\partial^2 f}{\partial y \partial x}(a,b) \overset{not}{=} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(a,b) \qquad \frac{\partial^2 f}{\partial y^2}(a,b) \overset{not}{=} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(a,b)$$

Ex: Să se scrie d.p. de ordin 2 ale funcțiilor:

a) $f(x,y) = x^3 + y^3 - xy$

b) $f(x,y,z) = \ln(x^2+y^2+z^2)$

a) $\frac{\partial f}{\partial x}(x,y) = 3x^2 - y$; $\frac{\partial f}{\partial y}(x,y) = 3y^2 - x$. $\}$ $(\forall)(x,y) \in D = \mathbb{R}^2$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 6x \left(= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y)\right) \checkmark$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = -1 \left(= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)\right)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = 6y \checkmark$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = -1$$

$$H_f(a,b) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b), & \frac{\partial^2 f}{\partial y \partial x}(a,b) \\ \frac{\partial^2 f}{\partial x \partial y}(a,b), & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix}$$

$\to$ s.n. hessiana lui f în (a,b)

$$H_f(x,y) = \begin{pmatrix} 6x & -1 \\ -1 & 6y \end{pmatrix}$$

b) $D = \mathbb{R}^3 \setminus \{(0,0,0)\}$

$\frac{\partial f}{\partial x}(x,y,z) = \frac{1}{x^2+y^2+z^2} \cdot 2x$; $\frac{\partial f}{\partial z}(x,y,z) = \frac{1}{x^2+y^2+z^2} \cdot 2z$;

$\frac{\partial f}{\partial y}(x,y,z) = \frac{1}{x^2+y^2+z^2} \cdot 2y$,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y,z) = \left(\frac{1}{x^2+y^2+z^2}\right)' \cdot 2x + 2 \cdot \frac{1}{x^2+y^2+z^2}$$

$$= (-1) \cdot \frac{1}{(x^2+y^2+z^2)^2} \cdot 2x \cdot 2x + \frac{2}{x^2+y^2+z^2}$$

$$= \frac{-4x^2}{(x^2+y^2+z^2)^2} + \frac{2}{x^2+y^2+z^2}$$

$$= \frac{2}{x^2+y^2+z^2}\left(\frac{-2x^2}{x^2+y^2+z^2} + 1\right)$$

$$= \frac{-2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^2} \to \frac{2(y^2+z^2-x^2)}{x^2+y^2+z^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y,z) = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2+z^2}\right) = 2x \cdot \frac{-1}{x^2+y^2+z^2} \cdot 2y = \frac{-4xy}{x^2+y^2+z^2}$$

— — — —

Prin analogie (datorită simetriei funcției) $\Rightarrow$

$$\cancel{H_f(x,y,z) = \frac{1}{x^2+y^2+z^2} \cdot \left(2(y^2+z^2-x^2),\ 2(x^2+z^2-y^2),\ 2(x^2+y^2-z^2) \right)}$$

$$H_f(x,y,z) = \frac{1}{x^2+y^2+z^2}\begin{pmatrix} 2(y^2+z^2-x^2) & -4xy & -4xz \\ -4xy & 2(x^2+z^2-y^2) & -4yz \\ -4xz & -4yz & 2(x^2+y^2-z^2) \end{pmatrix}$$

Ex:

Prop: (Criteriul de diferențiabilitate)

I f: $D \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}$ (deschisă), (a,b) atunci f diferențiabilă în (a,b) $\Leftrightarrow$ 1) $\exists$ $\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \in \mathbb{R}$

2) $\exists$ $\lim\limits_{(x,y)\to(a,b)} \frac{f(x,y)-f(a,b)-\frac{\partial f}{\partial x}(a,b)(x-a)-\frac{\partial f}{\partial y}(a,b)(y-b)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$

Dem: Imediată din def dif + Th. ("Dif $\Rightarrow$ Cont + D.P")

II. Fie $f: D \xrightarrow{\subseteq \mathbb{R}^n} \mathbb{R}^m$ (deschisă), $a \in D$. Atunci f dif în a $\Leftrightarrow$ 1) $\forall i=\overline{1,n}$, $\exists$ $\frac{\partial f}{\partial x_i}(a) \in \mathbb{R}$

2)

2) $\exists$ $\lim\limits_{x \to a} \frac{f(x)-f(a)-\sum\limits_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i-a_i)}{\|x-a\|} = 0 \in \mathbb{R}^m$.

Dem: — " —

Prop: (Criteriul de dif "cu derivate parțiale")

I. Fie $f: D \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}$ (deschisă), $(a,b) \in D$ a.î. $\exists$ $V \in \mathcal{V}((a,b))$, $\exists$ $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}: V \cap D \to \mathbb{R}$

cu prop. că $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ continue în (a,b). $\Rightarrow$

$\Rightarrow$ Atunci f dif în (a,b).

II Fie $f: D \xrightarrow{\subseteq \mathbb{R}^n} \mathbb{R}^m$ (deschisă), $a \in D$ a.î. $\exists V \in \mathcal{V}(a)$ $\exists$ $\frac{\partial f}{\partial x_i}: V \cap D \to \mathbb{R}^m$, $i=\overline{1,n}$ și

sunt cont în a. Atunci f dif în a.

Dem: Tehnică, se omite.

Corolar I: $f: D \overset{\subseteq \mathbb{R}^2}{\underset{\text{deschisă}}{\longrightarrow}} \mathbb{R}$, $(a,b) \in D$, $f \in C^1(D)$, "f este de clasă C^1 pe D" $\Leftrightarrow$

$\Leftrightarrow$ $\exists$ $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}: D \to \mathbb{R}$ și sunt cont. pe D $\Rightarrow$ f dif pe D.

II: $f: D \overset{\subseteq \mathbb{R}^n}{\underset{\text{deschisă}}{\longrightarrow}} \mathbb{R}^m$, $a \in D$, $f \in C^1(D) \Leftrightarrow \forall i = \overline{1,n}$, $\exists$ $\frac{\partial f}{\partial x_i}: D \to \mathbb{R}$ și sunt cont pe D

$\Rightarrow$ f dif pe D.

Ex I: Fie $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. Să se studieze dif sa pe $\mathbb{R}^2$.

Deoarece nu $\exists$ $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ (vezi seminar...) $\Rightarrow$ f nu e cont în $(0,0)$.

$\Rightarrow$ f nu e dif în $(0,0)$.

Dar $\frac{\partial f}{\partial x}(x,y) = \frac{y(x^2+y^2) - 2x(xy)}{(x^2+y^2)^2} = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$, $\frac{\partial f}{\partial y}(x,y) = \frac{x^3 - y^2 x}{(x^2+y^2)^2}$.

Evident $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (raport de fct. pol, ...etc.) $\Leftrightarrow$

$\Leftrightarrow$ $f \in C^1(\mathbb{R}^2 \setminus \{(0,0)\}) \xRightarrow{\text{Cor}(\dots)} f$ dif pe $\mathbb{R}^2 \setminus \{(0,0)\}$

Def I Fie $f: D \overset{\subseteq \mathbb{R}^2}{\underset{\text{deschisă}}{\longrightarrow}} \mathbb{R}$, $(a,b) \in D$ a.î. $\exists$ $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}: V \cap D \to \mathbb{R}$, $V \in \mathcal{V}((a,b))$.

Derivatele parțiale de ordin 2 ale lui f în (a,b) sunt (dacă există)

$\frac{\partial^2 f}{\partial x^2}(a,b) \overset{not}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(a,b)$, $\frac{\partial^2 f}{\partial y \partial x}(a,b) \overset{not}{=} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(a,b)$.

$\frac{\partial^2 f}{\partial x \partial y}(a,b) \overset{not}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a,b)$, $\frac{\partial^2 f}{\partial^2 y}(a,b) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(a,b)$.

II. Fie $f: D \overset{\subseteq \mathbb{R}^n}{\underset{\text{deschisă}}{\longrightarrow}} \mathbb{R}^m$ a.î. $\exists$ $\frac{\partial f}{\partial x_i}: V \cap D \to \mathbb{R}^m$, $V \in \mathcal{V}(a)$, $i = \overline{1,n}$. $\forall j = \overline{1,n}$ se scrie

$\frac{\partial^2 f}{\partial x_j \partial x_i}(a) \overset{not}{=} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(a) \to$ s.n. derivata parțială de ordin 2 a lui f în a în raport cu $x_j - x_i$.

Teoremă (Schwarz I): I. $f : D \underset{\text{deschisă}}{\subseteq} \mathbb{R}^2 \to \mathbb{R}$, $(a,b) \in D$ a.î. $(\exists) V \in \mathcal{V}(a,b)$, a.î. $\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x} : V \cap D \to \mathbb{R}$ şi sunt cont. în (a,b).

Atunci $\frac{\partial^2 f}{\partial y \partial x}(a,b) = \frac{\partial^2 f}{\partial x \partial y}(a,b)$.

II. $f : D \underset{\text{deschisă}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$, $a \in D$ a.î. $\exists V \in \mathcal{V}(a)$, a.î. $\frac{\partial^2 f}{\partial x_j \partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j} : V \cap D \to \mathbb{R}^m$, $i, j = \overline{1,n}, i \neq j$

şi sunt cont. în a. Atunci $\frac{\partial^2 f}{\partial x_j \partial x_i}(a) = \frac{\partial^2 f}{\partial x_i \partial x_j}(a)$.

Corolar I. $f : D \underset{\text{deschisă}}{\subseteq} \mathbb{R}^2 \to \mathbb{R}$, $f \in C^2(D)$ ($\Leftrightarrow$ "f este de clasă C^2 pe D" $\overset{\text{def}}{\Leftrightarrow}$ există derivatele parţiale de ordin 2 pe D şi sunt cont. pe D) $\Rightarrow$

$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ pe D.

II. $f : D \underset{\text{deschisă}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$, $f \in C^2(D)$ ($\overset{\text{def}}{\Leftrightarrow}$ are d.p. de ordin 2 pe D şi sunt cont. pe D)

$\Rightarrow \forall i, j = \overline{1,n}, i \neq j$, $\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ pe D

Obs: Pt II. şi $j = i$ se notează $\frac{\partial^2 f}{\partial x_i^2}(a) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right)(a)$.

!! **Derivate parţiale de fct. compuse**

I. $D \underset{\text{deschisă}}{\subseteq} \mathbb{R}^2 \xrightarrow{\psi} E \underset{\text{deschisă}}{\subseteq} \mathbb{R}^2 \xrightarrow{\varphi} \mathbb{R}$, $(a,b) \in D$, $\psi(a,b) \in E$

$f := \varphi \circ \psi$

$\psi(x,y) = (u(x,y), v(x,y)) = (u, v)$

$f(x,y) = (\varphi \circ \psi)(x,y) = \varphi(\psi(x,y)) = \varphi(u(x,y), v(x,y))$.

II. $D \xrightarrow[\text{deschisă}]{\psi} E \xrightarrow[\text{deschisă}]{\varphi} \mathbb{R}^n$, $D \subset \mathbb{R}^n$, $E \subset \mathbb{R}^m$

$a=(a_1, \dots, a_n) \longmapsto \psi(a)$

$f := \varphi \circ \psi$

$\psi(x_1, \dots, x_n) = (u_1(x_1, \dots, x_n), \dots, u_m(x_1, \dots, x_n))$

$f(x_1, \dots, x_n) = (\varphi \circ \psi)(x_1, \dots, x_n) = \varphi(\psi(x_1, \dots, x_n)) = \varphi(u_1(x_1, \dots, x_n), \dots,$

$u_m(x_1, \dots, x_n))$

<u>Teoremă</u> ("de derivare a fct. compuse") // În condițiile de mai sus avem.

I dacă ψ dif în (a,b) iar φ dif în $\psi(a,b)$, atunci $f = \varphi \circ \psi$ dif în (a,b)

și $d(\varphi \circ \psi)(a,b) = \underbrace{d\varphi(\psi(a,b))}_{\mathcal{L}(\mathbb{R}^2;\mathbb{R})} \circ \underbrace{d\psi(a,b)}_{\mathcal{L}(\mathbb{R}^2;\mathbb{R})} \in \mathcal{L}(\mathbb{R}^2;\mathbb{R}^2)$.

$\Updownarrow$

$$J_f(a,b) = J_{\varphi \circ \psi}(a,b) = J_\varphi(\psi(a,b)) \cdot J_\psi(a,b).$$

$$\left(\frac{\partial f}{\partial x}(a,b) \quad \frac{\partial f}{\partial y}(a,b)\right) = \left(\frac{\partial \varphi}{\partial u}\left(\frac{\psi(a,b)}{u(a,b)\ v(a,b)}\right) \quad \frac{\partial \varphi}{\partial v}(\psi(a,b))\right) \cdot$$

$$\cdot \begin{pmatrix} \frac{\partial u}{\partial x}(a,b) & \frac{\partial u}{\partial y}(a,b) \\ \frac{\partial v}{\partial x}(a,b) & \frac{\partial v}{\partial y}(a,b) \end{pmatrix} = \frac{\partial f}{\partial x_{(y)}}(a,b) = \frac{\partial \varphi}{\partial u}(u(a,b), v(a,b)) \cdot \frac{\partial u}{\partial x_{(y)}}(a,b) +$$

$$+ \frac{\partial \varphi}{\partial v}(u(a,b), v(a,b)) \cdot \frac{\partial v}{\partial x_{(y)}}(a,b)$$

II. Dacă ψ dif în a, iar φ dif în $\psi(a)$, atunci $f = \varphi \circ \psi$ dif în a și $df(a) = d(\varphi \circ \psi)$

$(a) = d(\varphi(\psi(a))) \circ d\psi(a) \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$

$\Updownarrow$

$$M_{mn}(\mathbb{R}) \ni J_f(a) = J_{\varphi \circ \psi}(a) = \underbrace{J_\varphi(\psi(a))}_{\in M_{mm}(\mathbb{R})} \cdot J_\psi(a) \in M_{mn}(\mathbb{R})$$

$$\left(\frac{\partial f_k}{\partial x_i}(a)\right)_{\substack{k=\overline{1,m} \\ i=\overline{1,n}}} = \left(\frac{\partial \varphi_k}{\partial u_j}(\psi(a))\right)_{\substack{k=\overline{1,m} \\ j=\overline{1,m}}} \cdot \left(\frac{\partial u_j}{\partial x_i}(a)\right)_{\substack{j=\overline{1,m} \\ i=\overline{1,n}}}$$

$$\boxed{\Rightarrow \frac{\partial f_k}{\partial x_i}(a) = \sum_{j=1}^{m} \frac{\partial \varphi_k}{\partial u_j}(\psi(a)) \cdot \frac{\partial u_j}{\partial x_i}(a), \quad \forall\, k=\overline{1,p},\ i=\overline{1,n}}$$

Ex) Să se calc. der. de ordin 1 pt:

a) $f(x,y) = \varphi(x^2-y^2, xy)$, $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi \in C^1(\mathbb{R}^2)$

b) $f(x,y,z) = \varphi(xy, yz-x, xyz)$, $\varphi : \mathbb{R}^3 \to \mathbb{R}$, $\varphi \in C^1(\mathbb{R}^3)$.

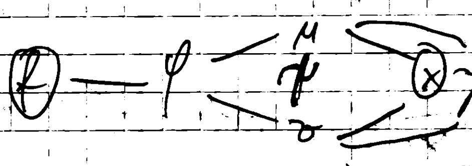
a) $$\frac{\partial f}{\partial x}(x,y) = \frac{\partial \varphi}{\partial u}(x^2-y^2, xy) \cdot \frac{\partial u}{\partial x}(x,y) + \frac{\partial \varphi}{\partial v}(x^2-y^2, xy) \cdot \frac{\partial v}{\partial x}(x,y) =$$

$$= \frac{\partial \varphi}{\partial u}(x^2-y^2, xy) \cdot 2x + \frac{\partial \varphi}{\partial v}(x^2-y^2, xy) \cdot y$$

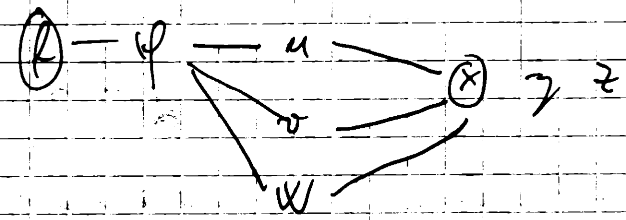
$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial \varphi}{\partial u}(x^2-y^2, xy) \cdot \frac{\partial u}{\partial y}(x,y) + \frac{\partial \varphi}{\partial v}(x^2-y^2, xy) \cdot \frac{\partial v}{\partial y}(x,y) =$$

$$= \frac{\partial \varphi}{\partial u}(x^2-y^2, xy) \cdot (-2y) + \frac{\partial \varphi}{\partial v}(x^2-y^2, xy) \cdot x$$

$\forall\, x,y \in \mathbb{R}^2$



$$\frac{\partial f}{\partial x(y)} = \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial x(y)} + \frac{\partial \varphi}{\partial v} \cdot \frac{\partial v}{\partial x(y)}$$

b) 

$$\frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial \varphi}{\partial w} \cdot \frac{\partial w}{\partial x}$$

Seminar - 11.12.2019

1) $f(x) = \varphi(u(x)) \Rightarrow f'(x) = \varphi'(u(x)) \cdot u'(x)$.

2) $f(x,y) = \varphi(u(x,y)) \Rightarrow \dfrac{\partial f}{\partial x_{(y)}}(x,y) = \varphi'(u(x,y)) \cdot \dfrac{\partial u}{\partial x_{(y)}}(x,y)$.

3) $f(x,y) = \varphi(u(x,y), v(x,y)) \Rightarrow \dfrac{\partial f}{\partial x}(x,y) = \dfrac{\partial \varphi}{\partial u}(u(x,y), v(x,y)) \cdot \dfrac{\partial u}{\partial x}(x,y) +$

$+ \dfrac{\partial \varphi}{\partial v}(u(x,y), v(x,y)) \cdot \dfrac{\partial v}{\partial x_{(y)}} \cdot (x,y)$.

Ex. Să se scrie d.p. de ordinul I pentru:

a) $f(x,y) = \varphi(\overset{u}{x^2 y}, \overset{v}{\tfrac{x}{y}})$, $\varphi: \mathbb{R}^2 \to \mathbb{R}$, $\varphi \in C^1(\mathbb{R}^2)$

b) $f(x,y) = \varphi\left(\frac{x}{y}, \frac{y}{x}, x^2 - y^2\right)$, $\varphi: \mathbb{R}^3 \to \mathbb{R}$, $\varphi \in C^1(\mathbb{R}^3)$

c) $f(x,y,z) = \varphi(xyz, xy + yz - zx)$, $\varphi: \mathbb{R}^2 \to \mathbb{R}$, $\varphi \in C^1(\mathbb{R}^2)$

d) $\varphi(x,y,z) = u\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, $u: \mathbb{R}^3 \to \mathbb{R}$, $u \in C^1(\mathbb{R}^3)$

e) $u(\rho, \theta) = f(\rho\cos\theta, \rho\sin\theta)$, $f: \mathbb{R}^2 \to \mathbb{R}$, $f \in C^1(\mathbb{R}^2)$.

a) $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{\partial \varphi}{\partial x^2 y}\left(x^2 y, \frac{x}{y}\right) \cdot \dfrac{\partial x^2 y}{\partial x}(x,y) + \dfrac{\partial \varphi}{\partial \frac{x}{y}}\left(x^2 y, \frac{x}{y}\right) \cdot \dfrac{\partial \frac{x}{y}}{\partial y}$

$$\frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial u} \cdot 2xy + \frac{\partial \varphi}{\partial v} \cdot \left(\frac{+1}{y}\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial \varphi}{\partial u} \cdot x^2 + \frac{\partial \varphi}{\partial v} \cdot \left(\frac{-x}{y^2}\right).$$

b) $f \quad \varphi \quad \begin{matrix} u \\ v \\ w \end{matrix} \quad x,y$

$$\frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial u} \cdot \frac{1}{y} + \frac{\partial \varphi}{\partial v} \cdot \frac{-y}{x^2} + \frac{\partial \varphi}{\partial w} \cdot 2x$$

$$\frac{\partial f}{\partial y} = \frac{\partial \varphi}{\partial u} \cdot \frac{-x}{y^2} + \frac{\partial \varphi}{\partial v} \cdot \frac{1}{x} + \frac{\partial \varphi}{\partial w} \cdot (-2y)$$

c) $\dfrac{\partial f}{\partial x} = \dfrac{\partial \varphi}{\partial u} \cdot yz + \dfrac{\partial \varphi}{\partial v} \cdot (y-z)$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial \varphi}{\partial u} \cdot xz + \dfrac{\partial \varphi}{\partial v} (x+z)$

$\dfrac{\partial f}{\partial z} = \dfrac{\partial \varphi}{\partial u} \cdot xy + \dfrac{\partial \varphi}{\partial v} (y-x)$.

d) $\dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial a} \cdot \dfrac{1}{y} + \dfrac{\partial u}{\partial b} \cdot 0 + \dfrac{\partial u}{\partial c} \cdot \dfrac{-1}{x^2} =$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial a} \cdot \dfrac{-x}{y^2} + \dfrac{\partial u}{\partial b} \cdot \dfrac{1}{z} + \dfrac{\partial u}{\partial c} \cdot 0$

$\dfrac{\partial f}{\partial z} = \dfrac{\partial u}{\partial a} \cdot 0 + \dfrac{\partial u}{\partial b} \cdot \dfrac{-y}{z^2} + \dfrac{\partial u}{\partial c} \cdot 0$

Humanitas: Fizica Teoretică

---

e) $\dfrac{\partial u}{\partial \rho} = \dfrac{\partial f}{\partial u} \cdot \overset{\cos\theta}{\cancel{(\cos\theta + \sin\theta \cdot \rho)}} + \dfrac{\partial f}{\partial v} \overset{\sin\theta}{\cancel{(\sin\theta)}}$

$\dfrac{\partial u}{\partial \theta} = \dfrac{\partial f}{\partial u} \cdot (-\rho \sin\theta) + \dfrac{\partial f}{\partial v} (+\rho \cos\theta)$

$\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \left(\dfrac{\partial \varphi}{\partial u} \cdot \dfrac{\partial u}{\partial x} + \dfrac{\partial \varphi}{\partial v} \cdot \dfrac{\partial v}{\partial x}\right)$ | Lucru suplimentar.

$f \leftarrow \dfrac{\partial \varphi}{\partial u}$ $\begin{matrix} u \\ v \end{matrix}$ $x \; y$

Ex. Să se studieze diferențiabilitatea funcțiilor

a) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

c) $f(x,y) = \begin{cases} \dfrac{x^3 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

b) $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

d) $f(x,y) = \sqrt{x^2+y^2}$ (TEMĂ)
(cont. nu are der. în $(0,0)$).

Cr. de dif

$f: D \underset{\text{deschisă}}{\subseteq \mathbb{R}^2} \to \mathbb{R}$, $(a,b)\in D$, f dif în $(a,b) \iff \begin{cases} 1)\ \exists\ \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \in \mathbb{R} \\ 2)\ \exists \end{cases}$

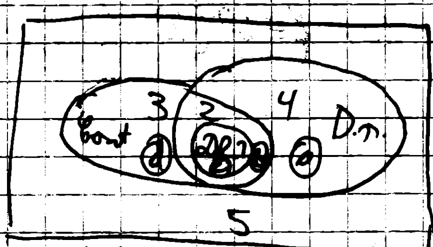
$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-f(a,b)-\frac{\partial f}{\partial x}(a,b)(x-a)-\frac{\partial f}{\partial y}(a,b)(y-b)}{\sqrt{(x-a)^2+(y-b)^2}}$$

Cr. de dif cu d.p.

$f: D \underset{\text{deschisă}}{\subseteq \mathbb{R}^2} \to \mathbb{R}$, $(a,b)\in D$, $\exists\ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} : D\cap V \to \mathbb{R}$ cont în $(a,b) \Rightarrow f$ dif în (a,b) $(V\in \mathcal{V}(a,b))$

Cor

$f\in C^1(D)$, D deschisă $\Rightarrow f$ dif în orice punct din D $\left(\underset{\text{def}}{\iff} f \text{ dif pe } D\right)$.



a) Știm că nu $\exists \lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \Rightarrow f$ nu e cont în $(0,0) \Rightarrow f$ nu e dif în $(0,0)$.

Avem $\frac{\partial f}{\partial x}(x,y) = \ldots = \frac{y^3-yx^2}{(x^2+y^2)^2}$, $\frac{\partial f}{\partial y} = \frac{x^3-xy^2}{(x^2+y^2)^2}$, $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$\Longrightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ cont pe $\mathbb{R}^2\setminus\{(0,0)\}$ (raport de polinoame) $\iff f\in C^1(\underbrace{\mathbb{R}^2\setminus\{(0,0)\}}_{\text{deschisă}})$.

$\xrightarrow{\text{Cor (Cr. dif cu d.p.)}} f$ dif pe $\mathbb{R}^2\setminus\{(0,0)\}$

b) Știm că $\exists \lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0 = f(0,0) \Rightarrow f$ cont în $(0,0)$.

$\hookrightarrow$ (Cr. Maj $\left|\frac{x^2y}{x^2+y^2}\right| \le |y|$)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0}\frac{\frac{x^2\cdot 0}{x^2+0}-0}{x-0} = \lim_{x\to 0} 0 = 0.$$

Analog $\frac{\partial f}{\partial y}(0,0) = \lim \ldots = \lim\limits_{y\to 0} 0 = 0$.

$$\frac{f(x,y)-f(0,0)-\frac{\partial f}{\partial x}(0,0)(x-0)-\frac{\partial f}{\partial y}(0,0)(y-0)}{\sqrt{(x-0)^2+(y-0)^2}} = \frac{\frac{x^2y}{x^2+y^2}-0-0-0}{\sqrt{x^2+y^2}} = \frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}} =: g(x,y)$$

$$\left.\begin{array}{l} y := mx \\ x \neq 0 \end{array}\right| \Rightarrow g(x,mx) = \frac{x^2\cdot mx}{(x^2+m^2x^2)^{\frac{3}{2}}} = \frac{mx^3}{|x|^3\cdot\sqrt{(1+m^2)^3}} \text{ nu are lim în } 0.$$

$$\Downarrow$$

$$\text{nu } \exists \lim_{(x,y)\to(0,0)} g(x,y) \xRightarrow[\text{T. de dif}]{} f \text{ nu e dif în } (0,0).$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{2xy(x^2+y^2)-x^2y\cdot 2x}{(x^2+y^2)^2} = \frac{2xy^3}{(x^2+y^2)^2}, \ (\forall)(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}.$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x^2(x^2+y^2)-x^2\cdot y\cdot 2y}{(x^2+y^2)^2} = \frac{x^4-x^2y^2}{(x^2+y^2)^2}$$

Evident $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ cont pe $\mathbb{R}^2\setminus\{(0,0)\} \Leftrightarrow f\in C^1(\mathbb{R}^2\setminus\{(0,0)\})$

$\xRightarrow{\text{T.c (T. dif cu d.p.)}}$ f dif pe $\mathbb{R}^2\setminus\{(0,0)\}$.

c) Analog cu b) $\exists \lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0) \Rightarrow f$ cont în $(0,0)$.

Analog cu b) $\exists\ \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.

$$\frac{f(x,y)-f(0,0)-\frac{\partial f}{\partial x}(0,0)(x-0)-\frac{\partial f}{\partial y}(0,0)(y-0)}{\sqrt{(x-0)^2+(y-0)^2}} = \frac{x^3y}{(x^2+y^2)^{\frac{3}{2}}} =: g(x,y).$$

$$|g(x,y)| = \underbrace{\left(\frac{x^2}{x^2+y^2}\right)^{\frac{3}{2}}}_{\leq 1}\cdot|y| \leq |y|, \ \forall (x,y)\neq(0,0)$$
$$\exists \lim_{(x,y)\to(0,0)} |y| = 0$$

$\xRightarrow[\text{lim pt}]{\text{T. Maj}} \exists \lim_{(x,y)\to(0,0)} g(x,y) = 0$.

$\xRightarrow{\text{T. de dif}}$ f dif în $(0,0)$.

Analog cu a) şi b), f dif pe $\mathbb{R}^2\setminus\{(0,0)\}$

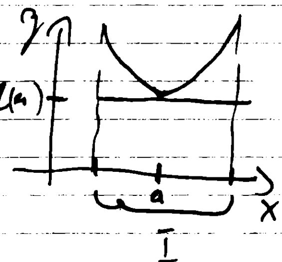
<u>Obs.</u> Analog cu b) (sau a)) $\exists\ \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.

17.12.2019 **Puncte de extrem pentru funcții R de mai multe variabile**

Th (Fermat) (1-dim) Fie $f: \underset{\text{interval}}{I \subseteq \mathbb{R}} \to \mathbb{R}$ a.î. $\begin{cases} a \in \overset{\circ}{I} \text{ (a pct. interior lui } I) \\ a \text{ pct de extrem local} \\ f \text{ derivabilă în } a. \end{cases}$

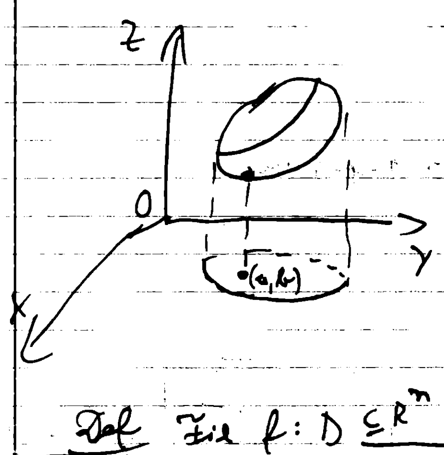
→ Atunci $f'(a) = 0$.

Dem: Imediată, din def.



Th (Fermat) (2-dim) Fie $f: \underset{\text{deschisă}}{D \subseteq \mathbb{R}^2} \to \mathbb{R}$ cu $\begin{cases} (a,b) \in D \\ (a,b) \text{ pct. de extrem local pt } f \\ f \text{ dif în } (a,b). \end{cases}$

Atunci $df(a,b) \equiv 0$ $\left(\Leftrightarrow \begin{cases} \frac{\partial f}{\partial x}(a,b) = 0 \\ \frac{\partial f}{\partial y}(a,b) = 0 \end{cases} \right)$



Def Fie $f: \underset{\text{deschisă}}{D \subseteq \mathbb{R}^n} \to \mathbb{R}$, $a = (a_1, \dots, a_n) \in D$

- a s.n. pct de minim local pt f $\overset{def}{\Longleftrightarrow} (\exists V \in \mathcal{V}(a))(f(a) \leq f(x), \forall x \in V \cap D)$
- a s.n. pct de maxim local pt f $\overset{def}{\Longleftrightarrow} (\exists V \in \mathcal{V}(a))(f(a) \geq f(x), \forall x \in V \cap D)$
- a s.n. pct. critic pt f $\overset{def}{\Longleftrightarrow} (\forall i = \overline{1,n})(\frac{\partial f}{\partial x_i}(x) = 0)$.

Obs: Th. Fermat afirmă că orice pct de extrem local al unei funcții diferențiabile este punct critic, reciproca nefiind adevărată (un pct. critic care nu e de extrem, e.x. punct șa.

Exp: $f(x)=x^3$, $f:\mathbb{R}\to\mathbb{R}$, $x=0$ → punct șa

Th (Sylvester): Fie $f: \underset{\text{deschisă}}{D} \xrightarrow{C^2} \mathbb{R}$, $f\in C^2(\mathbb{R})$, $(a,b)\in D$ pct. critic.

$$\rightsquigarrow H_f(a,b): \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial y \partial x}(a,b) \\ \frac{\partial^2 f}{\partial x \partial y}(a,b) & \frac{\partial^2 f}{\partial^2 y}(a,b) \end{pmatrix}$$

$$D_1 = \left|\frac{\partial^2 f}{\partial^2 x}(a,b)\right|, \quad D_2 = \det H_f(a,b)$$

1) Dacă $D_1, D_2 > 0 \Rightarrow (a,b)$ minim local (strict) pt f.

2) Dacă $D_1 < 0, D_2 > 0 \Rightarrow (a,b)$ — " — maxim — " —

3) Dacă $D_2 < 0 \Rightarrow (a,b)$ pct. șa

Dem: Dificilă, utilizează form. lui Taylor cu rest Lagrange + un rezultat de algebră liniară.

$$\boxed{\begin{array}{l} !\ \text{extrem} \;!=\; \text{pct. de extrem} \;!=\; \text{pct. de ext. al graficului} \\ \quad \| \\ \text{val. fct în} \\ \text{pct. extrem.} \end{array}}$$

Algoritm de studiu al pct. de extrem pt fct. de $\mathbb{R}$ mai multe variabile

I. Det. pct. critice ale funcției

II. Scrierea hessianei (H_f) în pct. critice.

III. Decizia (Th. Sylvester) care din pct critice sunt de extrem (...)

Ex 1 a) $f(x,y) = x^3 + y^3 + 3xy$

b) $f(x,y,z) = x^2 + y^2 + z^2 - xy + x - 2z$.

Să se det. pct. de extrem ale funcțiilor.

Sol(a) I

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 3x^2 + 3y = 0 \\[2ex] \dfrac{\partial f}{\partial y}(x,y) = 3y^2 + 3x = 0 \end{cases} \Leftrightarrow \begin{cases} y = -x^2 \\ x = -y^2 \end{cases} \Leftrightarrow \begin{cases} y = -x^2 \\ x = -x^4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^4 + x = 0 \\ y = -x^2 \end{cases} \Leftrightarrow \begin{cases} x(x^3+1) = 0 \\ y = -x^2 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0,\ x_2 = \sqrt[3]{-1} = -1 \\ y_1 = 0,\ y_2 = -1 \end{cases}$$

$x^4 + x = x(x+1)(x^2 - x + 1)$ // fără sol. complexe

$S = \{(0,0), (-1,-1)\}$.

II. $\dfrac{\partial^2 f}{\partial x^2}(x,y) = 6x$, $\dfrac{\partial^2 f}{\partial y \partial x}(x,y) = 3 = \dfrac{\partial^2 f}{\partial x \partial y}(x,y)$, $\dfrac{\partial^2 f}{\partial y^2} = 6y$.

$Hf(x,y) = \begin{pmatrix} 6x & 3 \\ 3 & 6y \end{pmatrix}$, $Hf(0,0) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$, $Hf(-1,-1) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$

III. $\underline{(0,0)}$ $D_1 = 0$, $D_2 = -9 < 0$ $\Rightarrow (0,0)$ pct. șa

$\underline{(-1,-1)}$ $D_1 = -6 < 0$, $D_2 = 27 > 0$ $\Rightarrow (-1,-1)$ pct de maxim local pt f.

b) I.

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y,z) = 2x - y + 1 = 0 \\[2ex] \dfrac{\partial f}{\partial y}(x,y,z) = 2y - x = 0 \\[2ex] \dfrac{\partial f}{\partial z}(x,y,z) = 2z - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{-2}{3} \\[2ex] y = \dfrac{-1}{3} \\[2ex] z = 1 \end{cases} \qquad S = \left\{\left(\frac{-2}{3}, \frac{-1}{3}, 1\right)\right\}.$$

II. $Hf(x,y,z) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = Hf\left(\frac{-2}{3}, \frac{-1}{3}, 1\right)$.

$D_1 = 2 > 0$

$D_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$

$D_3 = \det H_f\left(\frac{-2}{3}, \frac{-1}{3}, 1\right) = 8 + 0 + 0 - 0 - 0 - 2 = 6 > 0$

$\Rightarrow \left(\frac{-2}{3}, \frac{-1}{3}, 1\right)$ pct de minim local strict.

<u>Seminar 18.12.2019</u>

<u>Ex. Să se determine pct. de extrem local ale funcțiilor</u>

a) $f(x,y) = x^3 + 4xy - 2x^2 - 12x - 2y^2 + 10y$

b) $f(x,y) = x^3 + x^2 y + xy^2 + 1$

c) $f(x,y) = x^4 + y^4 - 2(x-y)^2$

d) $f(x,y) = x^3 + 3xy^2 - 15x - 12y + 1$

$(x,y) \in \mathbb{R}^2$

e) $f(x,y,z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$, $x,y,z > 0$.

a) <u>I. Determinarea pct. critice.</u>

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial x}(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 + 4y - 4x - 12 = 0 \\ 4x - 4y + 10 = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 - 12 = 0 \\ 4x - 4y = -10 \end{cases}$$

$$\Rightarrow \begin{cases} x = 2 \\ y = \frac{9}{2} \end{cases} \lor \begin{cases} x = -2 \\ y = \frac{1}{2} \end{cases}.$$

$S = \left\{\left(2, \frac{9}{2}\right), \left(-2, \frac{1}{2}\right)\right\}$.

II. $H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial y \partial x}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 6x-4 & 4 \\ 4 & -4 \end{pmatrix}$

III. $Hf\left(2,\frac{9}{2}\right) = \begin{pmatrix} 8 & 4 \\ 4 & -4 \end{pmatrix}$

$D_1 = 8 > 0$

$D_2 = \begin{vmatrix} 8 & 4 \\ 4 & -4 \end{vmatrix} = -32 - 16 < 0$

$\Rightarrow \left(2,\frac{9}{2}\right)$ nu e punct de extrem (punct şa).

$Hf\left(-2,\frac{1}{2}\right) = \begin{pmatrix} -16 & 4 \\ 4 & -4 \end{pmatrix}$

$D_1 = -16 < 0$

$D_2 = 64 - 16 > 0$

$\Rightarrow \left(-2,\frac{1}{2}\right)$ punct de maxim local strict pt f.

b) I. $\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 + 2xy + y^2 = 0 \\ x^2 + 2xy = 0 \end{cases} \Leftrightarrow \begin{cases} 2x^2 + y^2 = 0 \\ x^2 + 2xy = 0. \end{cases}$ $\Rightarrow \begin{array}{l} x = 0 \\ y = 0 \end{array}$

$\Leftrightarrow S = \{(0,0)\}$.

II. $Hf(x,y) = \begin{pmatrix} 6x + 2y & 2x + 2y \\ 2x + 2y & 2x \end{pmatrix}$

III $Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$D_1 = 0$

$D_2 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$

$\Rightarrow$ nu putem decide cu Th. Sylvester natura lui $(0,0)$.

$y = x \Rightarrow f(x, x) = 3x^3 + 1$
$\underset{\vdots}{\|}$
$\bar{f}(x)$

Ştim că fct. x^3 are punct şa în 0. $\Rightarrow$ şi fct. $\bar{f}$ are punct şa în 0.

$(0,0)$ este punct şa pentru f.

c) $f(x,y)=x^4+y^4-2(x-y)^2$

$$\begin{cases} \dfrac{\partial f}{\partial x}=4x^3-2(2x-2y)=0 \\ \qquad =4x^3-4x+4y \\ \dfrac{\partial f}{\partial y}=4y^3-2(-2x+2y)=0 \end{cases} \Leftrightarrow \begin{cases} x^3=x-y \\ y^3=y-x \end{cases} \Leftrightarrow \begin{cases} x^3+y^3=0 \\ y^3=y-x \end{cases} \Leftrightarrow$$

$$\begin{cases} y=-x \\ y^3=y-x \end{cases} \Leftrightarrow \begin{cases} y=-x \\ y^3=2y \end{cases} \Leftrightarrow \begin{cases} y=-x \\ y(y-\sqrt{2})(y+\sqrt{2}) \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=-\sqrt{2} \\ y=\sqrt{2} \end{cases} \vee \begin{cases} x=\sqrt{2} \\ y=-\sqrt{2} \end{cases}$$

$S=\{(0,0),(-\sqrt{2},\sqrt{2}),(\sqrt{2},-\sqrt{2})\}$

$$H_f(x,y)=\begin{pmatrix} 12x^2-4 & 4 \\ 4 & 12y^2-4 \end{pmatrix}$$

$H_f(0,0)=\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}$, $D_1=-4<0$, $D_2=0$

$H_f(-\sqrt{2},\sqrt{2})=H_f(\sqrt{2},-\sqrt{2})=\begin{pmatrix} 20 & 4 \\ 4 & 20 \end{pmatrix}$; $D_1=20>0$, $D_2>0$ $\Rightarrow (-\sqrt{2},\sqrt{2}),(\sqrt{2},-\sqrt{2})$ pct de minim local pt f.

$\to y=x \Rightarrow f(x,x)=2x^4=:\varphi(x)\geq 0=\varphi(0)$ → 0 pct de minim pt $\varphi=f|_{y=x}$ (1)

$y=-x \Rightarrow f(x,-x)=+2x^4-8x^2=:\psi(x)$

$\psi'(x)=8x^3-16x$

$\psi'(x)=0 \Leftrightarrow 8x(x^2-2)$

x		$-\sqrt{2}$		0		$\sqrt{2}$	
ψ'	$-$	0	$+$	0	$-$	0	$+$
ψ	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

$\Rightarrow$ 0 pct de maxim local pt $\psi=f|_{y=-x}$ (2)

(1) (2) $\Rightarrow (0,0)$ nu e pct de extrem pt f.

! Th. Catastrofelor - Studiu pct. critice.

d) $f(x,y) = x^3 + 3xy^2 - 15x - 12y + 1$

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 3x^2 + 3y^2 - 15 = 0 \\ \dfrac{\partial f}{\partial y}(x,y) = 6xy - 12 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 + y^2 - 5 = 0 \\ xy - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 + y^2 = 5 \\ xy = 2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (x+y)^2 = 9 \\ xy = 2 \end{cases} \Leftrightarrow \begin{cases} x+y = 3 \\ xy = 2 \end{cases} \vee \begin{cases} x+y = -3 \\ xy = 2 \end{cases}$$

$t^2 - 3t + 2 = 0$ $\qquad\qquad$ $t^2 + 3t + 2 = 0$

$t_{1,2} = 1;\ 2$ $\qquad\qquad$ $t_{1,2} = -1;\ -2$

$$\Leftrightarrow \begin{cases} x = 1 \\ y = 2 \end{cases} \vee \begin{cases} x = 2 \\ y = 1 \end{cases} \vee \begin{cases} x = -1 \\ y = -2 \end{cases} \vee \begin{cases} x = -2 \\ y = -1 \end{cases}$$

$S = \{(1,2), (2,1), (-1,-2), (-2,-1)\}$

! S_2: 4.3.23, 4.3.24, 4.3.26 → 4.3.33 S_4
4.3.39 → 4.3.40 S_3 ([illegible])
→ Refacere Partial [illegible] 14

$$H_f(x,y) = \begin{pmatrix} 6x & 6y \\ 6y & 6x \end{pmatrix}$$

$$H_f(1,2) = \begin{pmatrix} 6 & 12 \\ 12 & 6 \end{pmatrix} \quad D_1 = 6 > 0 \ (\cancel{36 - 144 < 0}) \quad D_2 = 36 - 144 < 0$$ (1,2) pct şa.

$$H_f(2,1) = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix} \quad D_1 = 12 > 0 \quad D_2 = 144 - 36 > 0$$ (2,1) pct minim

$$H_f(-1,-2) = \begin{pmatrix} -6 & -12 \\ -12 & -6 \end{pmatrix} \quad D_1 = -6 < 0 \quad D_2 = 36 - 144 < 0$$ (-1,-2) pct şa

$$H_f(-2,-1) = \begin{pmatrix} -12 & -6 \\ -6 & -12 \end{pmatrix} \quad D_1 = -12 < 0 \quad D_2 = 144 - 36 > 0$$ (-2,-1) pct maxim

l) $f(x,y,z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}, \quad x,y,z > 0.$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 1 - \frac{y^2}{4x^2} \cancel{+ \dots} = 0 \\ \frac{\partial f}{\partial y}(x,y,z) = \frac{2y}{4x} + \frac{-z^2}{y^2} = 0 \\ \frac{\partial f}{\partial z}(x,y,z) = \frac{2z}{y} + \frac{-2}{z^2} = 0 \end{cases} \Leftrightarrow \left[\begin{cases} \cancel{\frac{4}{4}} + \frac{y^2}{4} - \frac{1}{x^2} = 0 \\ \frac{y}{2x} - \frac{z^2}{y^2} = 0 \\ \frac{2z}{y} - \frac{2}{z^2} = 0 \end{cases}\right]$$

$$\Leftrightarrow \begin{cases} (4+y^2)\cdot x^2 = 4 \\ y^3 = 2z^2 x \\ \cancel{2}z^3 = \cancel{2}y \end{cases} \Leftrightarrow \begin{cases} 4x^2 + x^2y^2 = 4 \\ z^{\cancel{6}4} = 2\cancel{z^2}x \\ z^3 = y \end{cases} \Leftrightarrow \begin{cases} 4x^2 + x^2y^2 = 4 \\ z^4 = 2x \\ z^3 = y \end{cases}$$

$$\Leftrightarrow \begin{cases} z^8 + \frac{z^8}{4} \cdot z^6 = 4 \\ z^4 = 2x \\ z^3 = y \end{cases} \Leftrightarrow \begin{cases} 4\cdot z^8 + z^{14} = 16 \\ z^4 = 2x \\ z^3 = y \end{cases}$$

$$\Leftrightarrow \begin{cases} z^8(4 + z^6) = 16 \\ z^4 = 2x \\ z^3 = y \end{cases}$$

$$\begin{cases} 1 + \frac{y^2}{4x^2} = 0 \Leftrightarrow y^2 = -4 \\ \quad z^4 = 2x \\ \quad z^3 = y \end{cases} \Leftrightarrow \begin{cases} z^6 = -4 \end{cases}$$

$$\begin{cases} 1 - \frac{y^2}{4x^2} = 0 \\ y^3 - 2z^2x = 0 \\ 2z^3 - 2y = 0 \end{cases} \Leftrightarrow \begin{cases} 4x^2 = y^2 \\ y^3 = 2z^2x \\ y = z^3 \end{cases} \Leftrightarrow \begin{cases} 4x^2 = y^2 \\ 2xz = 1 \\ 2x = \frac{1}{8x^3} \end{cases} \Leftrightarrow \begin{cases} 16x^4 = 1 \\ y = 2x \\ z = \frac{1}{2x}. \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{1}{2} \\ y = 1 \\ z = 1 \end{cases} \qquad S = \{(\tfrac{1}{2}, 1, 1)\}.$$

$$H_f = \begin{pmatrix} \frac{y^2}{2x^3} & \frac{-y}{2x} & 0 \\ \frac{-y}{2x} & \frac{1}{2x} + \frac{2z^2}{y^3} & \frac{-2z}{y^2} \\ 0 & \frac{-2z}{y^2} & \frac{2}{y} + \frac{4}{z^3} \end{pmatrix}$$

$$H_f(\tfrac{1}{2}, 1, 1) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 6 \end{pmatrix}$$

$D_1 = 1 > 0$

$D_2 = \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = 3 - 1 = 2 > 0$

$D_3 > 0$

$\Rightarrow (\frac{1}{2}, 1, 1)$ pct. de minim.

7.01.2020.

5. Şiruri şi serii de funcţii.

• Şiruri de funcţii: - Se consideră $(f_n)_{n\geq 1}$, $f_n : A \xrightarrow{\subseteq \mathbb{R}} \mathbb{R}$, $n \geq 1$.

- Se pune problema găsirii mulţimii $B := \{x \in A \mid (f_n(x))_n \subseteq \mathbb{R}$ şi convergent$\}$ $\left(\text{Fie } \lim_{n\to\infty} f_n(x) \underset{not}{=:} f(x)\right)$

Def: Fie $(f_n)_{n\geq 1}$, $f_n : A \xrightarrow{\subseteq \mathbb{R}} \mathbb{R}$, $n \geq 1$, $B \subseteq A$, $f : B \to \mathbb{R}$. Se spune că f_n converge simplu la f la B $\underset{def}{\iff}$ $(\forall x \in B)\left(f_n(x) \xrightarrow[n\to\infty]{} f(x)\right)$ Se scrie $f_n \xrightarrow[B]{s} f$.

$\Updownarrow$

$$(\forall x \in B)(\forall \varepsilon > 0)(\exists n_{\varepsilon, x} \in \mathbb{N})(\forall n \geq n_{\varepsilon, x})\left(|f_n(x) - f(x)| < \varepsilon\right)$$

f_n conv. uniform la f pe B $\underset{def}{\iff}$ $(\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n \geq n_\varepsilon)(\forall x \in B)\left(|f_n(x) - f(x)| < \varepsilon\right)$.

(!)

$$\iff (\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n \geq n_\varepsilon)\Big(\underbrace{\sup_{x\in B} |f_n(x) - f(x)|}_{a_n} \leq \varepsilon\Big)$$

$$\iff a_n := \sup_{x \in B} |f_n(x) - f(x)| \xrightarrow[n]{} 0.$$

Se scrie $f_n \xrightarrow[B]{u} f$

Ex 1: Să se studieze conv. simplă şi uniformă a şirurilor de fct:

a) $f_n(x) = \frac{1}{n} \operatorname{arctg} nx$, $x \in \mathbb{R}$ $\quad n \geq 1$

b) $f_n(x) = \frac{x}{x+n}$, $x \in [0, \infty)$

---

a) $x > 0 \Rightarrow \operatorname{arctg} nx \xrightarrow[n]{} \frac{\pi}{2} \Rightarrow f_n(x) = \frac{1}{n} \operatorname{arctg} x \xrightarrow[n]{} 0$.

$x < 0 \Rightarrow \operatorname{arctg} nx \xrightarrow[n]{} -\frac{\pi}{2} \Rightarrow f_n(x) = \dots \xrightarrow[n]{} 0$.

$x=0 \Rightarrow f_n(0)=0 \xrightarrow[n]{} 0.$

$\Rightarrow f_n \xrightarrow[A=B=\mathbb{R}]{s} 0 \overset{\text{not}}{=:} f.$

Obs. not: $|f_n(x)-0| = \frac{1}{n}|\mathrm{arctg}\, nx| \le \frac{\pi}{2n}, \forall x\in\mathbb{R}, \forall n\ge 1.$

$\Rightarrow \sup_{x\in\mathbb{R}} |f_n(x)-0| \le \frac{\pi}{2n}, \forall n\ge 1.$ $\overset{\text{T.M.U.}}{\Longrightarrow} \sup_{x\in\mathbb{R}} |f_n(x)-0| \xrightarrow[n]{} 0$

$\lim \frac{\pi}{2n} \to 0$ $\Updownarrow$

$f_n \xrightarrow[\mathbb{R}]{u} 0$

---

b) $f_n(x) = \frac{x}{x+n} \xrightarrow[n]{} 0 \;\forall x\in[0,\infty) \Leftrightarrow f_n \xrightarrow[{[0,\infty)}]{s} 0$

$[0,\infty) = A = B$, $0 = f$

$|f_n(x)-0| =: g_n(x) = \frac{x}{n+x} = f_n(x), \forall x\in[0,\infty), \forall n\ge 1.$

$g_n'(x) = \frac{x+n-x}{(x+n)^2} = \frac{n}{(x+n)^2} > 0, \forall x\ge 0 \Rightarrow \sup_{x\in[0,\infty)} g_n(x) = 1 \xrightarrow[n]{} 1 \ne 0 \overset{\text{def}}{\Longleftrightarrow} f_n \not\xrightarrow[{[0,\infty)}]{u} 0$

x	0		δ		$+\infty$
g_n'		$+ + + + +$		$+ + +$	
g_n	0	$\nearrow$	$g_n(\delta) = \frac{\delta}{\delta+n}$	$\nearrow$	1

? $f_n \xrightarrow[{[0,\delta]}]{u} 0 \Rightarrow \sup_{x\in[0,\delta]} g_n(x) = \frac{\delta}{\delta+n} \to 0, \forall \delta>0.$

$\Rightarrow f_n \xrightarrow[{[0,\delta]}]{u} 0, \forall \delta > 0.$

Th ("de transfer de continuitate") - TTC.

$f_n: A\subseteq\mathbb{R} \to \mathbb{R}$, $f_n \xrightarrow[B]{u} f$, $B\subseteq A$, f_n cont în $a\in B$ (pe B) $\Rightarrow f$ cont în a (pe B).

T.T.D.: $f_n: I \subseteq \mathbb{R} \to \mathbb{R}$ (int.), $f_n \xrightarrow[I]{s} f$, f_n deriv pe I, $f_n' \xrightarrow[I]{u} g \Rightarrow f$ deriv pe I și $f' = g$.
(th. de derivabilitate)

T.T.I. : $f_n : I=[a,b] \to \mathbb{R}$, $n\geq 1$. $f_n \xrightarrow[{[a,b]}]{u} f$ $\quad (f:[a,b]\to\mathbb{R})$.

(de integrabilitate)

f_n integr. (R) pe $[a,b]$ $\forall n\geq 1$ $\Rightarrow$ f integr. (R) pe $[a,b]$

$$\text{şi } \underbrace{\int_a^b f(x)\,dx}_{\lim f_n} = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

Ex: $f_n(x)=x^n$, $x\in[0,1]$.

$x\in[0,1) \Rightarrow f_n(x)=x^n \xrightarrow[n]{} 0$

$x=1 \Rightarrow f_n(1)=1 \xrightarrow[n]{} 1$

$\Rightarrow f_n \xrightarrow[{[0,1]}]{S} f$, $f(x)=\begin{cases} 0, & x\in[0,1) \\ 1, & x=1.\end{cases}$

Pp: $f_n \xrightarrow[{[0,1]}]{u} f$ (Presupunem)

f_n cont pe $[0,1]$, $\forall n\geq 1$

$\xRightarrow{T.T.C.}$ f cont pe $[0,1]$ ↯ (f disc. în 1)

Serii de funcţii – se cons. $(f_n)_{n\geq 1}$, $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$, $n\geq 1$.

(S.F.) – se pune prob. det. mulţ $B:=\{x\in A \mid \sum_{n\geq 1} f_n(x) \text{ conv}\}$.

Def: Fie $(f_n)_{n\geq 1}$, $f_n : A\subseteq\mathbb{R} \to \mathbb{R}$, $n\geq 1$, $B\subseteq A$, $S:B\to\mathbb{R}$. Se spune că

~~seria de fct~~ $\sum_{n\geq 1} f_n \stackrel{def}{=} \left((f_n)_{n\geq 1}, (S_n)_{n\geq 1}\right)$ s.n. seria de funcţii

$(S_n = f_1+f_2+\dots+f_n$, $n\geq 1$.

s.n. şirul sumelor. fct (...) asoc. lui $(f_n)_{n\geq 1}$.

– $\sum_{n\geq 1} f_n$ este simplu conv. la S pe B $\iff$ $(\forall x\in B)\left(\sum_{n\geq 1} f_n(x)\ (C)\right.$ şi are suma

$\left.\sum_{n\geq 1}^{\infty} f_n(x) = S(x)\right)$ se scrie $\sum_{n\geq 1} f_n \xrightarrow[B]{S} S$. $\quad (\stackrel{\text{lim.}}{\Longleftarrow})$

– $\sum f_n$ – n. integrabilă – n – . $\underset{def}{\iff}$ $S_n \xrightarrow[B]{u} S$ se scrie $\sum_{n\geq 1} f_n \xrightarrow[B]{u} S$.

Obs: Deoarece nu avem procedeu de serii numerice nici la acă calculată suma, nu se poate det fct S din def. ant. Se va utiliza termin $\sum f_n$ este simplu conv pe B.

• Dacă notăm $s(x)=\sum_{n=1}^{\infty} f_n(x), x\in B$ ni serie, $S=\sum_{n=1}^{\infty} f_n$ și scrie.

voi putea scrie $\sum_{n\geq 1} f_n \xrightarrow[B]{S(x)} \sum_{n=1}^{\infty} f_n$; chiar dacă fct. $\sum_{n=1}^{\infty} f_n$ nu se poate

determina efectiv, fiind o fct. „virtuală".

Th (Weierstrass - de conv. simpl. a S.F)

Fie $(f_n)_{n\geq 1}$, $f_n: A^{\subseteq \mathbb{R}} \to \mathbb{R}$, $n\geq 1$, $B\subseteq A$, $(a_n)_{n\geq 1} \in \mathbb{R}$ a.î. $|f_n(x)|\leq a_n$,

$\forall x\in B, \forall n\geq K$ (K un rang....).

Dacă $\sum_{n\geq 1} a_n$ este converg., at. $\sum_{n\geq 1} f_n \xrightarrow[B]{u}$

Exp: $\sum_{n\geq 1} \underbrace{\frac{1}{n^2}\operatorname{arctg} nx}_{f_n(x)} \xrightarrow[\mathbb{R}]{u}$. Introducem.

$|f_n(x)| = \frac{1}{n^2}|\operatorname{arctg} nx| < \underbrace{\frac{\pi}{2n^2}}_{=a_n}$, $\forall n\geq 1$, $\forall x\in\mathbb{R}$.
$\quad < \frac{\pi}{2}$

T.W. pt. f.s.f. $\Longrightarrow \sum_{n\geq 1} f_n \xrightarrow[\mathbb{R}]{u}$

Dar $\sum_{n\geq 1} a_n \sim \sum_{n\geq 1}\frac{1}{n^2}$ (C) (S.A. $\alpha=2$).

T.T.C. pt. s.f. : $f_n: A\to\mathbb{R}$, $n\geq 1$, $B\subseteq A$; $\sum_{n\geq 1} f_n \xrightarrow[B]{u}$ $\Rightarrow \sum_{n=1}^{\infty} f_n$ cont. pe B.
$\quad$ ↳ cont. pe B

T.T.D. pt. s.f $\equiv$ DTT

(derivare th. cu ...)

↳ $f_n: I^{\subseteq \mathbb{R}} \to \mathbb{R}$, $n\geq 1$; $\sum_{n\geq 1} f_n' \xrightarrow[I]{u}$ $\Rightarrow$
$\quad$ int. $\quad$ ↳ deriv. pe I.

$\sum_{n\geq 1} f_n \xrightarrow[I]{u}$

$\Rightarrow \sum_{n=1}^{\infty} f_n$ (deriv. pe I) și $\left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'$

T.T.I. pt. s.f $\equiv$ ITT.

(integrare ....)

$f_n:[a,b]\to\mathbb{R}$, $n\geq 1$; $\sum_{n\geq 1} f_n \xrightarrow[{[a,b]}]{u}$
$\quad$ ↳ int. pe [a,b]

$\Rightarrow \sum_{n=1}^{\infty} f_n$ ... pe [a,b] și $\int_a^b (\ldots\ldots)$

Laborator - 8.01.2020

Ex. Să se studieze convergența simplă & uniformă a șirurilor de funcții

a) $f_n(x) = \frac{1}{x+n}$, $x \in [0,+\infty)$, $n \geq 1$

b) $f_n(x) = nx \cdot e^{-nx^2}$, $x \in \mathbb{R}$, $n \geq 1$.

T. d) $f_n(x) = \frac{\ln(1+n^4x^2)}{n}$, $x \in [0,1]$, $n \geq 1$.

c) $f_n(x) = \operatorname{arctg} nx$, $x \in \mathbb{R}$, $n \geq 1$.

RECAP.

$f_n : A \xrightarrow{SR} \mathbb{R}$, $n \geq 1$, $B \subseteq A$, $f : B \to A$.

$f_n \xrightarrow[B]{s} f \Leftrightarrow (\forall x \in B)\left(f_n(x) \xrightarrow[n]{} f(x)\right)$

$f_n \xrightarrow[B]{u} f \Leftrightarrow (\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n \geq n_\varepsilon, \forall x \in B)\left(|f_n(x) - f(x)| < \varepsilon\right)$

$\Leftrightarrow \sup_{x \in B} |f_n(x) - f(x)| \xrightarrow[n]{} 0$
!

a) $\left(f_n(x) = \frac{1}{x+n} \xrightarrow[n\to\infty]{} 0, \forall x \in [0,\infty)\right) \Leftrightarrow f_n \xrightarrow[{[0,\infty)}]{s} 0 \; (=f)$

$0 \leq \underbrace{|f_n(x) - 0|}_{g_n(x)} = \frac{1}{n+x} \leq \frac{1}{n}$, $x \in [0,\infty)$, $n \geq 1$ $\Bigg| \Rightarrow \sup_{x \in [0,\infty)} g_n(x) \leq \frac{1}{n}$, $n \geq 1$; $\frac{1}{n} \xrightarrow[n]{} 0$. $\Bigg| \Rightarrow$

$\xrightarrow[\text{Cleștelui}]{\text{Cr. Maj}} \sup_{x \in [0,\infty)} g_n(x) \xrightarrow[n]{} 0 \Leftrightarrow f_n \xrightarrow[{[0,\infty)}]{u} 0$

b) $x=0 \Rightarrow f_n(0)=0 \xrightarrow[n]{} 0$

$x\neq 0 \Rightarrow f_n(x)=\dfrac{nx}{\left(e^{x^2}\right)^n} \xrightarrow[n]{} 0,\ \forall x\neq 0$ $\Bigg| \Rightarrow f_n \xrightarrow[\mathbb{R}]{S} 0.$

$\Downarrow$

$x^2>0 \Leftrightarrow e^{x^2}>1$

$$g_n(x)=|f_n(x)-0|=\frac{n|x|}{e^{nx^2}}=\begin{cases}\dfrac{nx}{e^{nx^2}}, & x\geq 0\\[2ex] \dfrac{-nx}{e^{nx^2}}, & x<0.\end{cases}$$

$$g_n'(x)=\begin{cases}\dfrac{n\cdot e^{nx^2}-nx\cdot e^{nx^2}\cdot 2nx}{e^{2nx^2}}=\dfrac{(1-2nx^2)n}{e^{nx^2}}, & x>0.\\[3ex] \quad -\ \text{n}\ - \qquad\qquad\qquad =\dfrac{-(1-2nx^2)n}{e^{nx^2}}, & x<0.\end{cases}$$

$g_n'(x)=0 \Leftrightarrow 1-2nx^2=0 \Leftrightarrow x=\pm\sqrt{\dfrac{1}{2n}}$

x	$-\infty$		$\frac{-1}{\sqrt{2n}}$		0		$\frac{1}{\sqrt{2n}}$		∞
g_n'		$+$	0	$-$	$?$	$+$	0	$-$	
g_n		$\nearrow$	$\sqrt{\frac{n}{2e}}$	$\searrow$		$\nearrow$	$\sqrt{\frac{n}{2e}}$	$\searrow$	

posibil max

$g_n\left(\pm\dfrac{1}{\sqrt{2n}}\right)=\sqrt{\dfrac{n}{2e}}$

$\sup\limits_{x\in\mathbb{R}} g_n(x)=\sqrt{\dfrac{n}{2e}}$

Deci $\sup\limits_{x\in\mathbb{R}} g_n(x)\xrightarrow[n]{}\infty \Rightarrow f_n \not\xrightarrow[\mathbb{R}]{u} 0.$

? $f_n \xrightarrow[[\delta,\infty)]{u} 0$? $\delta>0$

1) $\boxed{\sup_{x\in[\delta,\infty)} g_n(x) = g_n(\delta) = \frac{n\delta}{e^{n\delta^2}} \xrightarrow[n]{} 0, \quad \forall \delta>0}$ $\Rightarrow$ $\boxed{f_n \xrightarrow[[\delta,\infty)]{u} 0, \forall \delta>0}$

c) $f_n(x) = \operatorname{arctg}(nx)$, $x\in\mathbb{R}, n\geq 1$.

$$\left.\begin{array}{l} x>0 \Rightarrow f_n(x) \xrightarrow[n]{} \frac{\pi}{2} \\ x<0 \Rightarrow f_n(x) \xrightarrow[n]{} \frac{-\pi}{2} \\ x=0 \Rightarrow f_n(0)=0 \xrightarrow[n]{} 0 \end{array}\right\} \Rightarrow f_n \xrightarrow[\mathbb{R}]{s} f,\ f(x)=\begin{cases} \frac{\pi}{2}, & x>0 \\ 0, & x=0 \\ \frac{-\pi}{2}, & x<0. \end{cases}$$

(contradictie)

$\Updownarrow$

Deoarece f_n cont pe $\mathbb{R}$, $\forall n\geq 1$, Pp. $f_n \xrightarrow[\mathbb{R}]{u} f \xRightarrow{T.T.C} f$ cont pe $\mathbb{R}$ $\oint$

(f disc în 0) $\Rightarrow$

$\Rightarrow f_n \not\xrightarrow[\mathbb{R}]{u} f$.

? $\boxed{\delta>0,\ f_n \xrightarrow[[\delta,\infty)]{u} \frac{\pi}{2}}$?

$g_n(x) := |f_n(x) - f(x)|$

$\sup_{x\in[\delta,\infty)} g_n(x) = ?$

$$g_n(x) = \begin{cases} \frac{\pi}{2} - \operatorname{arctg} nx, & x>0 \\ \operatorname{arctg} nx - \frac{\pi}{2}, & x<0 \\ 0, & 0=x \end{cases}$$

~~$g_n'(x) = \begin{cases} \frac{-n}{nx^2+1}, & x>0 \\ \frac{n}{nx^2+1}, & x<0 \end{cases}$~~

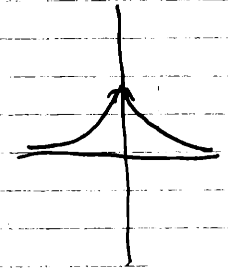
~~$\frac{-n}{nx^2+1} = 0$~~

~~X~~	~~$\frac{-n}{nx^2+1}$~~	~~0~~
~~g'~~		~~0~~
~~g_n~~		

$$g_n(x) = |f_n(x) - f(x)| = \begin{cases} \frac{\pi}{2} - \operatorname{arctg} nx, & x>0 \\ 0, & x=0 \\ \frac{\pi}{2} + \operatorname{arctg} nx, & x<0 \end{cases}$$

$$g_n'(x) = \begin{cases} \frac{-n}{1+(nx)^2}, & x>0 \\ \frac{n}{1+(nx)^2}, & x<0 \end{cases}$$

x	$-\infty$	0	δ $+\infty$
g_n'	$+ + + +$	|	$- - - -$
g_n	$\nearrow$	$\frac{\pi}{2}$ | $\frac{\pi}{2}$	$\searrow$



$$\Rightarrow \begin{cases} \sup_{x\in R} g_n(x) = \frac{\pi}{2} \xrightarrow[n]{} \frac{\pi}{2} \neq 0 \Rightarrow f_n \overset{u}{\underset{R}{\not\to}} 0 \\ \sup_{x\in[\delta,\infty)} g_n(x) = g_n(\delta) = \frac{\pi}{2} - \operatorname{arctg} n\delta \xrightarrow[n]{} 0, \ \forall \delta>0 \Leftrightarrow f_n \xrightarrow[{[\delta,\infty)}]{u} \frac{\pi}{2} \end{cases}$$

<u>Ex 2</u> Să se arate convergența, simplă și uniformă a seriilor de funcții.

a) $\sum_{n\ge1} \frac{\sin nx}{\sqrt[4]{n^5+x^2}}, x\in R$; b) $\sum_{n\ge1} \operatorname{arctg}\frac{2x}{x^2+n^3}, x\in R$; c) $\sum_{n\ge0} \frac{x^n}{n!}, x\in R$.

RECAP.

$f_n : A \subseteq R \to R, n\ge1, B\subseteq A$

$\sum_{n\ge1} f_n :\overset{def}{=} \left((f_n)_{n\ge1}, (S_n)_{n\ge1}\right)$ $(S_n := f_1+f_2+\dots+f_n, n\ge1)$

$\sum_{n\ge1} f_n \xrightarrow[B]{S} :\overset{def}{=} S_n \xrightarrow[B]{S} \Leftrightarrow (\forall x\in B)\left(\sum_{n\ge1} f_n(x)\ (C)\right)$

$\sum_{n\ge1} f_n \xrightarrow[B]{u} \Leftrightarrow S_n \xrightarrow[B]{u}$

$! \ f_n \xrightarrow[B]{u} f \Rightarrow f_n \xrightarrow[B]{S} f$

$\not\Leftarrow$

<u>Th. Weierstrass (conv. unif. a s.f.)</u>

$\sum_{n\ge1} f_n$, $f_n : A \subseteq R \to R, n\ge1$

$B\subseteq A$, $(a_n)_{n\ge1} \subset R$.

$$\left.\begin{array}{l} |f_n(x)| \le a_n, n\ge k, \forall x\in B \\ \sum_{n\ge1} a_n\ (C) \end{array}\right| \Rightarrow \sum_{n\ge1} f_n \xrightarrow[B]{u}$$

Sol: a) $f_n(x) := \dfrac{\sin nx}{\sqrt[4]{n^5+x^2}}$, $n \geq 1$, $x \in \mathbb{R}$.

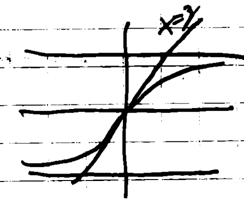
$|f_n(x)| = \dfrac{|\sin nx|}{\sqrt[4]{n^5+x^2}} \leq \dfrac{1}{\sqrt[4]{n^5}} = \underbrace{\dfrac{1}{n^{5/4}}}_{=:a_n}$, $n \geq 1$, $x \in \mathbb{R}$

$\sum_{n\geq 1} a_n$ (C) (S.A. $\alpha = \frac{5}{4} > 1$)

$\overset{T.W.}{\Longrightarrow} \sum_{n\geq 1} f_n \overset{u}{\underset{\mathbb{R}}{\longrightarrow}}$

b) $f_n(x) := \operatorname{arctg} \dfrac{2x}{x^2+n^3}$

(*) $\operatorname{arctg} t \leq |t|$, $\forall t \in \mathbb{R}$



$|f_n(x)| = \left|\operatorname{arctg} \dfrac{2x}{x^2+n^3}\right| \underset{(*)}{\leq} \left|\dfrac{2x}{x^2+n^3}\right| \underset{(**)}{\leq}$

$\leq \underbrace{\dfrac{1}{n^{\frac{3}{2}}}}_{a_n}$, $\forall x \in \mathbb{R}$, $\forall n \geq 1$

$\sum_{n\geq 1} a_n$ (C) (S.A. $\alpha = \frac{3}{2} > 1$)

$\overset{T.W.}{\Longrightarrow} \sum_{n\geq 1} f_n \overset{u}{\underset{\mathbb{R}}{\longrightarrow}}$

$\dfrac{a+b}{2} \geq \sqrt{ab}$, $a, b \geq 0$ $\Rightarrow \dfrac{x^2+n^3}{2} \geq \sqrt{n^3x^2} = |x| \cdot n^{\frac{3}{2}} \Leftrightarrow$

$\Leftrightarrow \dfrac{2|x|}{x^2+n^3} \leq \dfrac{1}{n^{\frac{3}{2}}}$, $\forall x \in \mathbb{R}$, $n \geq 1$ (**)

---

c) $f_n(x) := \dfrac{x^n}{n!}$, $x \in \mathbb{R}$, $n \geq 1$.

$\xrightarrow[n]{} 0 \Rightarrow f_n \overset{s}{\underset{n}{\longrightarrow}} 0$.

$a > 1$, $\dfrac{a^n}{n!} \xrightarrow[n]{} 0$.

Obs: $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \ldots \dfrac{x^n}{n!} + \ldots$

$|f_n(x)| = \dfrac{|x|^n}{n!} \underset{x \in [-\delta, \delta]}{\leq} \dfrac{\delta^n}{n!} =: a_n$ (1)

$\sum \dfrac{\delta^n}{n!}$ (C) (2) (TEMA. Cr. Rap. D'Al. Serii)

(1)(2) $\overset{T.W.}{\Longrightarrow} \sum_{n\geq 0} \dfrac{x^n}{n!} \overset{u}{\underset{[-\delta,\delta]}{\longrightarrow}}$, $\forall \delta > 0$

Cum $\sum_{n\geq 0} f_n'(x) = \sum_{n\geq 0} f_n(x) \overset{u}{\underset{[-\delta,\delta]}{\longrightarrow}}$

$\overset{D.T.T.}{\Longrightarrow} \sum_{n=0}^{\infty} f_n$ deriv. pe $[-\delta, \delta]$ și $\forall \delta > 0$

$\Downarrow$

deriv. pe $\mathbb{R}$

$$\left(\sum_{n=0}^{\infty} f_n\right)' = \sum_{n=0}^{\infty} f_n' \text{ pe } \mathbb{R}.$$

$$\Updownarrow$$

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)' = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x + c,\ \forall x \in \mathbb{R} \longrightarrow \text{pt } x=0 \Rightarrow 1 = 1 + c \Rightarrow c = 0.$$

$$\boxed{\text{În final } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ \forall x \in \mathbb{R}}$$

DATA VIITOARE:

$$\boxed{\begin{aligned} \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned} \quad x \in \mathbb{R}}$$

$$\boxed{e^{ix} = \cos x + i \sin x,\quad x \in \mathbb{R}}$$

e^z, $z \in \mathbb{C}$???

$\sum (-1)^n u_n$, $u_n \searrow 0$

- Transformarea Abel.
- ~~Proiect grafic~~
- Scopul [illegible]: 98% → 80%.

Curs. 14.07.2020

Serii de puteri

• Def: O serie de funcții de forma $\sum_{n\geq 0} a_n \cdot x^n$, $x\in\mathbb{R}$, s.n. serie de puteri (s.

$\quad\quad = f_n$

$B := \{x\in\mathbb{R} \mid \sum_{n\geq 0} a_n x^n$ convergentă$\}$ → s.n. mulț. de convergență a s.p.

$0 \in B$, $\quad a_0 + a_1x^1 + \ldots + a_nx^n + \ldots$

Th. Abel Pt s.p. $\sum_{n\geq 0} a_n x^n$, $\exists!\, R\in[0,\infty)$ a.î. dacă $R=0 \Rightarrow B=\{0\}$.

dacă $R\neq 0$ atunci

raza de conv. a s.p.

(1) $\sum_{n\geq 0} a_n x^n$ (C) pt $(-R,R)$ și (D) pe $\mathbb{R}\setminus(-R,R)$.

(2) $\sum_{n\geq 0} a_n x^n \xrightarrow[{[-r,r]}]{u} \quad \forall\, 0<r<R$.

Obs: pt $R=\infty \Rightarrow B=(-\infty,\infty)=\mathbb{R}$.

Pt. $0<R<+\infty \Rightarrow (-R,R)\subseteq B\subseteq[-R,R]$.

Dem: Simplu, se bazează pe Cr. Rap. pt serii numerice + Th. W. sau c.f.

Th (Hadamard) Pt s.p. $\sum_{n\geq 0} a_n x^n$, $R = \begin{cases} 0, & \omega=\infty \\ \infty, & \omega=0 \\ \frac{1}{\omega}, & 0<\omega<+\infty \end{cases}$ $\Leftrightarrow$ $\boxed{\text{"}R=\frac{1}{\omega}\text{"}}$, $\omega := \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$

Dem: Utilizează Cr. Rad pt serii numerice, etc.

Obs: Raza de convergență a seriei derivatelor seria s.p. este aceeași cu raza seriei inițiale (OK, ușor).

$\Rightarrow$ pt $R\neq 0$ puterea seriei ~~pt~~ ~~seriei~~ $\boxed{\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1},}$

$\boxed{\forall x\in[-r,r],\ \forall\, 0<r<R \Leftrightarrow \forall x\in(-R,R)}$ (*)

2) Se mai poate calcula ω din Th. (Hadamard) astfel:

$$\omega = \overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right|.$$

Ex1 Să se det. mulțimea de convergență și eventual suma seriilor de puteri.

a) $\sum_{n\geq 1} (-1)^n \frac{1}{n\cdot 3^n} x^n$, $x\in\mathbb{R}$; b) $\sum_{n\geq 0} n\cdot x^n$, $x\in\mathbb{R}$.

Sol: $a_n := (-1)^n \frac{1}{n\cdot 3^n}$, $n\geq 1$; $\omega = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{\frac{1}{(n+1)3^{(n+1)}}}{\frac{1}{n3^n}} =$

$= \lim \frac{n}{3(n+1)} = \frac{1}{3}.$

$\xRightarrow{Th. H.} R = \frac{1}{\omega} = 3 \xRightarrow{Th. Abel} (-3,3) \subseteq B \subseteq [-3,3].$

$x=3 \Rightarrow$ seria devine $\sum_{n\geq 1} (-1)^n \frac{1}{n}$ (C) (Cr. Leibniz) $\Rightarrow 3\in B$.

$x=-3 \Rightarrow$ — " — $\sum_{n\geq 1}\frac{1}{n}$ (D) (S.A. $\alpha=1$) $\Rightarrow -3\notin B$.

$\Rightarrow B = (-3,3]$.

b) $a_n = n$, $n\geq 0$, $W = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = 1 \xRightarrow{Th H} R = \frac{1}{W} = 1$

$\xRightarrow{Th. Abel} (-1,1)\subseteq B \subseteq [-1,1]$

$x=1 \Rightarrow \sum n$ (D)

$x=-1 \Rightarrow \sum (-1)^n \cdot n$ (D)

$\Rightarrow B = (-1,1)$.

$$\sum_{n=0}^{\infty} n x^n = x\cdot \sum_{n=1}^{\infty} \underbrace{n x^{n-1}}_{(x^n)'} \xRightarrow{*} \left(\sum_{n=1}^{\infty} x^n\right)' = \left(\frac{1}{1-x} - 1\right)' = \frac{-1}{(1-x)^2}$$

(Cr. algebrice cu lim. de șiruri).

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \ x\in(-1,1) \quad \text{(Suma Geometrică)}$$

$*$

$$\boxed{\sum_{n=0}^{\infty} n x^{n} = \frac{1}{(1-x)^2},\ x\in(-1,1)}\ !!!$$

Th. Taylor $f: I \subseteq \mathbb{R} \to \mathbb{R}$, $f \in C^{n+1}(I) \Rightarrow \forall a\in I,\ \exists \xi \in I$ a.î.
interval. ξ nu

! $$f(x) = \underbrace{f(a) + \frac{1}{1!} f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \dots + \frac{1}{n!} f^{(n)}(a)(x-a)^n}_{T_n(x)} + \underbrace{\frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-a)^{n+1}}_{R_n}\quad \forall x\in I.$$

Dem: tehnică, netrivială.

s.n. formula lui Taylor cu rest Lagrange.

$T_n(x)$ - s.n. polinomul Taylor de grad n asoc. lui f în a.

R_n

s.n. restul Lagrange de ? ord. din n+1 form. lui f Taylor ?

Th. ("de dezvoltabilitate a fct în serii de puteri")
$f: I \subseteq \mathbb{R}$ int, $f\in C^{\infty}(I)$, $a\in I \rightsquigarrow$ seria s.n. $\sum_{n\geq 0} \frac{1}{n!} f^{(n)}(a)(x-a)^n$.
cu raza de conv. R.

Ip. $R\neq 0$ și $|R_{n+1}(x)| \xrightarrow[n\to\infty]{} 0$, $\forall x\in I\cap(a-R, a+R)$.

$$\Longrightarrow \boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n,\ \forall x\in I\cap(a-R,a+R)}$$

Dem. Imediat din th. Taylor.

Laborator – 15.01.2020

Ex 1. Să se determine mulțimea de convergență a seriilor de puteri (s.p.).

a) $\sum_{n\geq 0} n! \cdot x^n$; b) $\sum_{n\geq 1} (-1)^n \frac{1}{n^2 \cdot 2^n} \cdot x^n$; c) $\sum_{n\geq 0} \frac{x^n}{(2n+1)!}$

T

Ex 2 — n — și suma s.p.:

a) $\sum_{n\geq 0} n(n+1) \cdot x^n$; b) $\sum_{n\geq 0} \frac{x^{2n+1}}{2n+1}$; c) $\sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{2n+1}$.

Temă

REC

Th. (Abel): Fie o s.p. $\sum_{n\geq 0} a_n x^n$, $x\in\mathbb{R}$ $\exists!\, R \in [0,\infty]$ a.î.

"$a_0 + a_1 x + \dots + a_n x^n + \dots$" — rază de convergență.

$B := \{x\in\mathbb{R} \mid \sum_{n\geq 0} a_n x^n \ (C)\}$.

a) $(-R,R) \subseteq B \subseteq [-R,R]$ pt $R \neq 0$.

b) $\sum a_n x^n \xrightarrow[{[-r,r]}]{u} $, $0<r<R$ pt $R\neq 0$.

(Obs $R = 0 \rightsquigarrow B = \{0\}$)

Th. (H) Fie s.p. $\sum_{n\geq 0} a_n x^n$, "$R = \frac{1}{\omega}$" unde $\omega = \overline{\lim} \sqrt[n]{|a_n|}$ sau

$\omega = \lim \left|\frac{a_{n+1}}{a_n}\right|$.

Obs. Pt o s.p. raza de conv. a seriei derivatelor coincide cu

— a seriei inițiale. $\Rightarrow$ $\left(\text{pt } R\neq 0 \Rightarrow \left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} (a_n x^n)' \right)$

$\forall x \in (-R,R) \Leftrightarrow \forall x\in[-r,r],\ \forall\, 0<r<R$.

$\int_a^b \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n\geq 0} \int_a^b a_n x^n\, dx$.

$\forall [a,b] \subset (-R,R)$, $n=0$.

Sol: a) $a_n := n! \Rightarrow \omega = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} n = \infty \xRightarrow{Th.(H)} \boxed{R=0}$

$\xRightarrow{Th. Abel} B = \{0\}$.

c) $a_n := \frac{1}{(2n+1)!}, n \geq 0 \Rightarrow \omega = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{(2n+3)!}}{\frac{1}{(2n+1)!}} = 0 \xRightarrow{Th.}$

$\Rightarrow R = \infty \xRightarrow{Th. Abel} B = (-\infty, \infty) = \mathbb{R}$.

b) Indicație: $R = 2 (\ldots)$

$x = 2 \Rightarrow$ seria devine $\sum_{n\geq 1} (-1)^n \frac{1}{n^2}$ (C) (Cr. Leibniz)

$x = -2 \Rightarrow$ — " — $\sum_{n\geq 1} \frac{1}{n^2}$ (C) (S.A. $\alpha = 2$)

$\Rightarrow B = [-2, 2]$

Ex 2 b) $a_n :\overset{not}{=} \begin{cases} \frac{1}{2k+1}, & n = 2k+1 \\ 0, & n = 2k \end{cases}$

$\overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| = 0$? ; $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = 0 =$?

$\sum_{n\geq 0} \frac{x^{2n+1}}{2n+1} = : x \sum \frac{(x^2)^n}{2n+1}$.

$y := x^2 \Rightarrow \sum \frac{x^{2n}}{2n+1} = \sum_{n\geq 0} \frac{y^n}{2n+1}$

$a_n' := \frac{1}{2n+1} \Rightarrow \overline{\lim_{n\to\infty}} \left|\frac{a'_{n+1}}{a'_n}\right| = \lim_{n\to\infty} \frac{a'_{n+1}}{a'_n} = 1 = \omega'$

$\frac{2n+1}{2n+3}$

$\xRightarrow{Th. H} R' = 1 \xRightarrow{Th. Abel} (-1, 1) \subseteq B' \subseteq [-1, 1]$.

(a_n', ω_n', R' sunt asociate seriei $\sum_{n\geq 0} \frac{y^n}{2n+1}$)

⇒ notând R şi B ~~seguenin~~ m. de conv a seriei a seriei iniţiale.

$(-1,1) \subseteq B \subseteq [-1,1]$ $(R=1)$

$$\left.\begin{array}{l} x=1 \Rightarrow \sum\limits_{n\geq 0} \dfrac{1}{2n+1} \\ x=-1 \Rightarrow \sum\limits_{n\geq 0} \dfrac{-1}{2n+1} \end{array}\right\} D \left(\sim \sum \frac{1}{n} \ (D) \ (S.A. \ \alpha=1)\right)$$

$\Rightarrow B = (-1,1).$

Obs ⇒ Raza de conv a n. derivabilităţilor rămâne $1 \Rightarrow \left(\sum\limits_{n=0}^{\infty} \dfrac{x^{2n+1}}{2n+1}\right)' =$

$$= \sum_{n=0}^{\infty}\left(\frac{x^{2n+1}}{2n+1}\right)' = \sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} (x^2)^n = \frac{1}{1-x^2},\ x\in(-1,1). \Rightarrow \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \underset{x\in(-1,1)}{=}$$

$$= \int \frac{1}{1-x^2}\,dx = -\ln\left|\frac{x-1}{x+1}\right| + C$$
$x\in(-1,1).$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},\ x\in(-1,1)}$$

Notă: Ideile speciale sunt depinde de bun simţ.

$$\boxed{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = \ln\sqrt{\frac{1+x}{1-x}},\ x\in(-1,1)}$$

a) $a_n := n(n+1) \Rightarrow \omega = \ldots = 1 \Rightarrow R = 1 \Rightarrow (-1,1) \subseteq B \subseteq [-1,1]$

$x=1 \Rightarrow$ Seria devine ~~$\sum$~~ $\sum n(n+1)$ (D)

$x=-1 \Rightarrow$ — " — $\sum (-1)^n n(n+1)$ (D).

$\Rightarrow B = (-1,1).$

Obs. $\Rightarrow \sum\limits_{n=0}^{\infty} n(n+1)x^n = x\sum\limits_{n=0}^{\infty} \underbrace{n(n+1)\cdot x^{n-1}}_{(x^{n+1})''} \overset{\text{DTT}}{\underset{\text{de termeni}}{=}} x\left(\sum\limits_{n=0}^{\infty} x^{n+1}\right)'' =$

$$= x\left(\frac{1}{1-x} - 1\right)'' = x\left(\frac{1}{(1-x)^2}\right) = \frac{2x}{(1-x)^3}.$$

Th. Abel I. Fie s.p. $\sum_{n\geq 0} a_n x^n$ cu raza $R\neq 0$ de conv.

dacă $R\in B \Rightarrow \sum_{n=0}^{\infty} a_n x^n$ conv în R

$(-R\in B)$ $(-R)$.

Ex 3. Să se dezvolte în serii de puteri următoarele funcții în jurul punctelor indicate:

a) $f(x)=e^x$, $x\in\mathbb{R}$, $a=0$. Calc. $\frac{1}{\sqrt{e}}$ cu 3 zec. exacte

b) $f(x)=\cos x$, $x\in\mathbb{R}$, $a=0$. (Tema)

c) $f(x)=\operatorname{arctg}$, $x\in\mathbb{R}$, $a=0$ Calc. π cu 3 zec. exacte.

d) $f(x)=\frac{1}{4x-1}$, $x\in\mathbb{R}\setminus\{\frac{1}{4}\}$ $a=0$, $a=1$

e) $f(x)=\frac{1}{x^2-3x+2}$, $x\in\mathbb{R}\setminus\{1,2\}$ $a=0$, $a=1$ $\left(\frac{1}{x-2}-\frac{1}{x-1}\right)$

T.

Th. (Taylor) $f\in C^{n+1}(I)$, $a\in I \Rightarrow \exists\, \xi\in I$, $f(x)=\left(f(a)+\frac{1}{1!}f'(a)(x-a)+\dots\right.$
$\forall x\in I \Rightarrow \exists\, \xi$ între a și x. int

$\left.+\frac{1}{n!}f^{(n)}(a)(x-a)^n\right)+\left(\frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-a)^{n+1}\right)$ $\forall x\in I$.

$\underset{T_n(x)}{\parallel}$ $R_{n+1}(x)$

Th. (de dezvoltare în s.p. a fct)

$f\in C^\infty(\underset{int}{I})$, $a\in I$, R raza de conv. a s.p. $\sum_{n\geq 0}\frac{1}{n!}f^{(n)}(a)(x-a)^n$

$|R_{n+1}(x)|\xrightarrow[n]{} 0 \Rightarrow f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{n}(a)(x-a)^n$

$\forall x\in I\cap(a-R,a+R)$

~~Th. Abel~~

Sol: a) $I=\mathbb{R}$, $a=0$, $f(x)=e^x$, $f\in C^\infty(\mathbb{R})$ $(f^{(n)}(x)=e^x)$
Ex3 $\forall x\in \mathbb{R}$.

Formăm seria $\sum_{n\geq 0} \frac{1}{n!}\cdot e^0 (x-0)^n = \sum_{n\geq 0} \frac{x^n}{n!}$

$a_n := \frac{1}{n!} \Rightarrow \omega = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{1}{n+1} = 0$.

$\Rightarrow R=\infty \Rightarrow B=(-\infty,\infty)=\mathbb{R}$.

$R_{n+1}(x) = \frac{1}{(n+1)!} e^{\xi} (x-0)^{n+1} = \frac{1}{(n+1)!} x^{n+1} \cdot e^{\xi}, \forall x\in \mathbb{R}$.

$\xRightarrow[\text{Ex. rez.}]{} |R_{n+1}(x)| \xrightarrow[n]{} 0, \forall x\in\mathbb{R} \xRightarrow{\text{Th. (de dezv...)}} \boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \forall x\in\mathbb{R}}$

$\Rightarrow \frac{1}{\sqrt{e}} = e^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\cdot n!}$ $\Big|$ $\Rightarrow e^{-\frac{1}{2}} \simeq 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{16\cdot 24}$

Avem nevoie de f. Taylor cu rest Lagrange pt $f(x)=e^x$, $a=0$, $x=\frac{-1}{2}$.

Obț. $\frac{1}{\sqrt{e}} = e^0 + \frac{1}{1!} e^0\left(\frac{-1}{2}\right) + \dots + \frac{1}{n!}e^0\left(\frac{-1}{2}\right)^n + \frac{1}{(n+1)!}\cdot e^{\xi}\left(\frac{-1}{2}\right)^{n+1}$.

$\left|R_{n+1}\left(\frac{-1}{2}\right)\right| < \frac{1}{10^3} \Leftrightarrow \left|\frac{1}{(n+1)!} e^{\xi} \frac{(-1)^{n+1}}{2^{n+1}}\right| <$

d) Sol: $\frac{1}{4x-1} = -\frac{1}{1-4x} = -\sum_{n=0}^{\infty} (4x)^n = -\sum_{n=0}^{\infty} 4^n\cdot x^n$

$(a=0)$

pt $|4x|<1 \Leftrightarrow x\in\left(\frac{-1}{4},\frac{1}{4}\right)$.

$(a=1)$ $\frac{1}{4x-1} = \frac{1}{4x-4+3} = \frac{1}{3+4(x-1)} = \frac{1}{3}\cdot\frac{1}{1-\left[\frac{-4}{3}(x-1)\right]} =$

pt $\left|\frac{-4}{3}(x-1)\right| < 1 \Leftrightarrow |x-1| < \frac{3}{4}$

$= \frac{1}{3}\sum_{n=0}^{\infty}\left[\frac{-4}{3}(x-1)\right]^n = \sum_{n=0}^{\infty} (-1)^n \frac{4^n}{3^{n+1}}(x-1)^n$ $\Leftrightarrow x\in\left(\frac{1}{4},\frac{7}{4}\right)$.

c) $(\operatorname{arctg} x)' = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n} \Rightarrow$

pt. $|-x^2| < 1 \iff x \in (-1,1)$.

$\Rightarrow \int_0^x \operatorname{arctg} t \, dt = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt \overset{\text{IT.T}}{=\!=} \sum_{n=0}^{\infty} \left(\int_0^x (-1)^n t^{2n} dt \right)$

$\forall x \in (-1,1)$.

$= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} \Big|_0^x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \forall x \in (-1,1)$

$x = 1 \Rightarrow$ seria devine $\sum (-1)^n \frac{1}{2n+1}$ (C) (Cr. Leibniz) $\overset{\text{Th. II Abel}}{\Rightarrow}$

$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ cont. în 1 $\Rightarrow \underset{\frac{\pi}{4}}{\overset{}{\underset{\|}{\operatorname{arctg} 1}}} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$.

* $\sum_{n \geq 0} (-1)^n \cdot u_n \searrow$

$\Rightarrow |S - S_n| < u_{n+1}, \forall n \geq 1$

S (suma seriei)

(*) $\Rightarrow$ Testez $\frac{1}{2n+1} < \frac{1}{10^3}$ devine când $n = 498$ ☹

$$\boxed{\sum_{n=1}^{\infty} (-1)^{n-1} u_n \overset{u_n \searrow 0}{=} \sum_{n=2} (-1)^{n-1} \frac{u_n - u_{n-1}}{2}}$$

Transformarea Euler

Se aplică Tr. Euler de 3 ori pt ca seria să devină rapid conv.

* Programul care să calculeze pi cu 3 zec. exacte.