Laborator10

Ecuaţii

Să se rezolve următoarele ecuaţii:

$$
\begin{aligned}
&a)\ x^{II} - 2x^{I} + 5x = e^{t}(t\cos 2t - t^2 \sin 2t)\\
&b)\ x^{II} + 9x = t\sin t + \cos 3t\\
&c)\ x^{II} - 9x = e^{2t} + te^{t} - t^2 - 2
\end{aligned}
$$

Rezolvare

Exerciţiu 1. a) - Video1 Video2

a) $x'' - 2x' + 5x = e^{t}(t\cos 2t - t^2 \sin 2t)$

$x'' - 2x' + 5x = 0$

$x = e^{rt},\ x' = re^{rt},\ x'' = r^2 e^{rt}$

$r^2 e^{rt} - 2re^{rt} + 5e^{rt} = 0 \quad | : e^{rt}$

$r^2 - 2r + 5 = 0$

$\Delta = 4 - 20 = -16$

$r_{1,2} = \frac{2 \pm 4i}{2} = 1 \pm 2i \quad \left(\alpha \pm i\beta,\ \begin{matrix}\alpha = 1\\ \beta = 2\end{matrix}\right)$

$e^{1\cdot t}\cos 2\cdot t,\ e^{1\cdot t}\sin 2t \to S.f.s.$

$\boxed{x_0 = c_1 e^{t}\cos 2t + c_2 e^{t}\sin 2t}$

$f(t) = e^{\alpha t}(P_1(t)\cos \beta t + P_2(t)\sin \beta t),\ \begin{matrix}\alpha = 1\\ \beta = 2\end{matrix} \to \alpha \pm i\beta = 1 \pm 2i$

$gr.\,P_1 = 1,\ gr.\,P_2 = 2$

$x_p = t\cdot e^{t}\left[(\lambda_2 t^2 + \lambda_1 t + \lambda_0)\cos 2t + (\beta_2 t^2 + \beta_1 t + \beta_0)\cdot \sin 2t\right]$

$x_p = e^{t}\left[(\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t)\cos 2t + (\beta_2 t^3 + \beta_1 t^2 + \beta_0 t)\sin 2t\right]$

$$
\begin{aligned}
x_p' = {} & e^{t}\left[(\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t)\cos 2t + (\beta_2 t^3 + \beta_1 t^2 + \beta_0 t)\sin 2t\right]\\
& + e^{t}\left[(3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0)\cos 2t + (\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t)\cdot(-2)\sin 2t\right.\\
& \left. + (3\beta_2 t^2 + 2\beta_1 t + \beta_0)\sin 2t + (\beta_2 t^3 + \beta_1 t^2 + \beta_0 t)\cdot 2\cos 2t\right]
\end{aligned}
$$

$$
\begin{aligned}
x_p' = e^{t}\big[& (\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t + 3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0 + 2\beta_2 t^3 + 2\beta_1 t^2 + 2\beta_0 t)\cos 2t\\
& + (\beta_2 t^3 + \beta_1 t^2 + \beta_0 t - 2\lambda_2 t^3 - 2\lambda_1 t^2 - 2\lambda_0 t + 3\beta_2 t^2 + 2\beta_1 t + \beta_0)\sin 2t\big]
\end{aligned}
$$

$$
\begin{aligned}
x_p'' = {} & e^{t}\big[(\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t + 3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0 + 2\beta_2 t^3 + 2\beta_1 t^2 + 2\beta_0 t)\cos 2t +\\
& (\beta_2 t^3 + \beta_1 t^2 + \beta_0 t - 2\lambda_2 t^3 - 2\lambda_1 t^2 - 2\lambda_0 t + 3\beta_2 t^2 + 2\beta_1 t + \beta_0)\sin 2t\big]\\
& + e^{t}\big[(3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0 + 6\lambda_2 t + 2\lambda_1 + 6\beta_2 t^2 + 4\beta_1 t + 2\beta_0)\cos 2t +\\
& (\lambda_2 t^3 + \lambda_1 t^2 + \lambda_0 t + 3\lambda_2 t^2 + 2\lambda_1 t + \lambda_0 + 2\beta_2 t^3 + 2\beta_1 t^2 + 2\beta_0 t)\cdot(-2)\sin 2t\\
& + (3\beta_2 t^2 + 2\beta_1 t + \beta_0 - 6\lambda_2 t^2 - 4\lambda_1 t - 2\lambda_0 + 6\beta_2 t + 2\beta_1)\sin 2t\\
& + (\beta_2 t^3 + \beta_1 t^2 + \beta_0 t - 2\lambda_2 t^3 - 2\lambda_1 t^2 - 2\lambda_0 t + 3\beta_2 t^2 + 2\beta_1 t + \beta_0)\cdot 2\cos 2t\big]
\end{aligned}
$$

$$e^t\Big[(\lambda_2 t^3+\lambda_1 t^2+\lambda_0 t+3\lambda_2 t^2+2\lambda_1 t+\lambda_0+2\beta_2 t^3+2\beta_1 t^2+2\beta_0 t+3\lambda_2 t^2+2\lambda_1 t+\lambda_0+6\lambda_2 t+2\lambda_1+6\beta_2 t^2$$
$$+4\beta_1 t+2\beta_0+2\beta_2 t^3+2\beta_1 t^2+2\beta_0 t-4\lambda_2 t^3-4\lambda_1 t^2-4\lambda_0 t+6\beta_2 t^2+4\beta_1 t+2\beta_0)\cos 2t$$
$$+(\beta_2 t^3+\beta_1 t^2+\beta_0 t-2\lambda_2 t^3-2\lambda_1 t^2-2\lambda_0 t+3\beta_2 t^2+2\beta_1 t+\beta_0-2\lambda_2 t^3-2\lambda_1 t^2-2\lambda_0 t-6\lambda_2 t^2-4\lambda_1 t$$
$$-2\lambda_0-4\beta_2 t^3-4\beta_1 t^2-4\beta_0 t+3\beta_2 t^2+2\beta_1 t+\beta_0-6\lambda_2 t^2-4\lambda_1 t-2\lambda_0+6\beta_2 t+2\beta_1)\sin 2t\Big]$$
$$-2e^t\Big[(\lambda_2 t^3+\lambda_1 t^2+\lambda_0 t+3\lambda_2 t^2+2\lambda_1 t+\lambda_0+2\beta_2 t^3+2\beta_1 t^2+2\beta_0 t)\cos 2t$$
$$+(\beta_2 t^3+\beta_1 t^2+\beta_0 t-2\lambda_2 t^3-2\lambda_1 t-2\lambda_0 t+3\beta_2 t^2+2\beta_1 t+\beta_0)\sin 2t\Big]$$
$$+5e^t\Big[(\lambda_2 t^3+\lambda_1 t^2+\lambda_0)\cos 2t+(\beta_2 t^3+\beta_1 t^2+\beta_0 t)\sin 2t\Big]=e^t(t\cos 2t-t^2\sin 2t)\qquad |:e^t$$

$$[-8\lambda_2 t^3+4\beta_2 t^3-3\lambda_1 t^2+6\lambda_1 t^2+4\beta_1 t^2+12\beta_2 t^2+4\lambda_1 t+4\beta_0 t+6\lambda_2 t+8\beta_1 t-3\lambda_0 t+2\lambda_0+2\lambda_1+4\beta_0-2\lambda_2 t^3$$
$$-2\lambda_1 t^2-2\lambda_0 t-6\lambda_2 t^2-4\lambda_1 t-2\lambda_0-4\beta_2 t^3-4\beta_1 t^2-4\beta_0 t+5\lambda_2 t^3+5\lambda_1 t^2+5\lambda_0 t]\cos 2t$$
$$+(-3\beta_2 t^3-4\lambda_2 t^3-4\lambda_1 t^2+6\beta_2 t^2-12\lambda_2 t^2-3\beta_1 t^2-3\beta_0 t-4\lambda_0 t+4\beta_1 t-8\lambda_1 t+6\beta_2 t+2\beta_0-4\lambda_0+2\beta_1$$
$$-2\beta_2 t^3-2\beta_1 t^2-2\beta_0 t+4\lambda_2 t^3+4\lambda_1 t^2+4\lambda_0 t-6\beta_2 t^2-4\beta_1 t-2\beta_0+5\beta_2 t^3+5\beta_1 t^2+5\beta_0 t)\sin 2t=$$
$$t\cos 2t-t^2\sin 2t$$

$$12\beta_2 t^2+6\lambda_2 t+8\beta_1 t+2\lambda_1+4\beta_0=t$$
$$-12\lambda_2 t^2-8\lambda_1 t-4\lambda_0+2\beta_1=-t^2$$

$12\beta_2=0 \quad\to\ \beta_2=0$

$6\lambda_2+8\beta_1=1 \quad\to\ 8\beta_1=1-6\cdot\frac{1}{12}=1-\frac12=\frac12\Rightarrow\beta_1=\frac{1}{16}$

$2\lambda_1+4\beta_0=0 \quad\Rightarrow\beta_0=0$

$-12\lambda_2=-1 \quad\Rightarrow\lambda_2=\frac{1}{12}$

$-8\lambda_1=0 \quad\to\ \lambda_1=0$

$-4\lambda_0+2\beta_1=0 \quad\to\ 4\lambda_0=2\beta_1=2\cdot\frac{1}{16}=\frac18\Rightarrow\lambda_0=\frac{1}{32}$

$$\boxed{x_p=te^t\Big[\Big(\frac{1}{12}t^2+\frac{1}{32}\Big)\cos 2t+\frac{1}{16}t\sin 2t\Big]}$$

$$x=x_0+x_p=c_1e^t\cos 2t+c_2e^t\sin 2t+te^t\Big[\Big(\frac{1}{12}t^2+\frac{1}{32}\Big)\cos 2t+\frac{1}{16}t\sin 2t\Big]$$

Exerciţiu 1. c) - [Video]

c) $x''-9x=\underbrace{e^{2t}}_{f_1}+\underbrace{te^t}_{f_2}\underbrace{-t^2-2}_{f_3}$

$x''-9x=0$

$x=e^{rt},\ x'=re^{rt},\ x''=r^2e^{rt}$

$r^2e^{rt}-9e^{rt}=0\quad |:e^{rt}$

$r^2-9=0\Rightarrow r^2=9$

$r_{1,2}=\pm3$

$e^{3t},\ e^{-3t}\to$ S.f.s

$\boxed{x_0=c_1e^{3t}+c_2e^{-3t}}$

$x_{p_3}=\lambda_2t^2+\lambda_1t+\lambda_0$

$x'_{p_3}=2\lambda_2t+\lambda_1$

$x''_{p_3}=2\lambda_2$

$2\lambda_2-9(\lambda_2t^2+\lambda_1t+\lambda_0)=-t^2-2$

$2\lambda_2-9\lambda_2t^2-9\lambda_1t-9\lambda_0=-t^2-2$

$-9\lambda_2=-1 \quad\Rightarrow\lambda_2=\frac19$

$-9\lambda_1=0 \quad\Rightarrow\lambda_1=0$

$2\lambda_2-9\lambda_0=-2 \quad\Rightarrow 9\lambda_0=2+2\lambda_2=2+2\cdot\frac19=\frac{20}{9}\Rightarrow\lambda_0=\frac{20}{81}$

$x_{p_1}\overset{\alpha=2}{=}e^{2t}\cdot A$

$x'_{p_1}=2Ae^{2t}$

$x''_{p_1}=4Ae^{2t}$

$4Ae^{2t}-9Ae^{2t}=e^{2t}\quad |:e^{2t}$

$-5A=1\Rightarrow A=-\frac15$

$\boxed{x_{p_1}=-\frac15e^{2t}}$

$x_{p_2}\overset{\alpha=1}{=}e^t(\lambda_1t+\lambda_0)$

$x'_{p_2}=e^t(\lambda_1t+\lambda_0)+e^t\cdot\lambda_1=e^t(\lambda_1t+\lambda_0+\lambda_1)$

$x''_{p_2}=e^t(\lambda_1t+\lambda_0+\lambda_1)+e^t\cdot\lambda_1=e^t(\lambda_1t+\lambda_0+\lambda_1+\lambda_1)$

$e^t(\lambda_1t+\lambda_0+2\lambda_1)-9e^t(\lambda_1t+\lambda_0)=te^t\quad |:e^t$

$\lambda_1t+\lambda_0+2\lambda_1-9\lambda_1t-9\lambda_0=t$

$-8\lambda_1=1 \quad\Rightarrow\lambda_1=-\frac18$

$2\lambda_1-8\lambda_0=0 \quad\Rightarrow 8\lambda_0=2\lambda_1=2\cdot\left(-\frac18\right)=-\frac14\Rightarrow\lambda_0=-\frac{1}{32}$

$\boxed{x_{p_2}=e^t\left(-\frac18t-\frac{1}{32}\right)}$

$\boxed{x_{p_3}=\frac19t^2+\frac{20}{81}}$

$x=x_0+x_{p_1}+x_{p_2}+x_{p_3}$

$\quad=c_1e^{3t}+c_2e^{-3t}-\frac15e^{2t}+e^t\left(\frac18t-\frac{1}{32}\right)+\frac19t^2+\frac{20}{81}$

[Maple]