Laborator 14 - TEMĂ

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1. Să se determine soluția generală a sistemelor simetrice:

d) $$\frac{dx}{2y(2-x)} = \frac{dy}{x^2-z^2-y^2-4x} = \frac{dz}{-2yz}, \quad x>2,\ y\neq 0,\ z>0$$

• $$\frac{dx}{2y(2-x)} = \frac{dz}{-2yz} \ \Big|\cdot(-2y) \iff \frac{dx}{x-2} = \frac{dz}{z} \iff \int \frac{1}{x-2}\,dx = \int \frac{1}{z}\,dz \iff$$

$$\iff \ln|x-2| = \ln|z| + C_1 \iff \ln\left|\frac{x-2}{z}\right| = C_1 \iff \boxed{C_1 = \frac{x-2}{z} = \varphi_1(x,y,z)}$$

• $$\frac{{}^{x/}dx}{2y(2-x)} = \frac{{}^{y/}dy}{x^2-z^2-y^2-4x} = \frac{x\,dx}{2xy(2-x)} = \frac{y\,dy}{y(x^2-z^2-y^2-4x)} =$$

$$= \frac{x\,dx + y\,dy}{4xy - 2x^2y + x^2y - z^2y - y^3 - 4xy} = \frac{\frac{1}{2}\,d(x^2+y^2)}{-y^3 - z^2y - x^2y} = \frac{d(x^2+y^2)}{-2y(y^2+z^2+x^2)}$$

$$\frac{d(x^2+y^2)}{-2y(y^2+z^2+x^2)} = \frac{dz}{-2yz} \ \Big|\cdot(-2y) \iff \frac{d(x^2+y^2)}{x^2+y^2+z^2} = \frac{dz}{z} \iff$$

Notăm $x^2+y^2 = u$

$$\iff \frac{du}{u+z^2} = \frac{dz}{z} \iff \int \frac{1}{u+z^2}\,du = \int \frac{1}{z}\,dz \iff \ln|u+z^2| = \ln|z| + C_2 \iff$$

$$\iff C_2 = \ln\left|\frac{u+z^2}{z}\right| \iff \boxed{C_2 = \frac{x^2+y^2+z^2}{z} = \varphi_2(x,y,z)}$$

Verificăm dacă φ_1, φ_2 sunt indep. $\iff \dfrac{D(\varphi_1,\varphi_2)}{D(x,y)} \neq 0$

Fie z – var. indep.

$$\begin{vmatrix} \frac{\partial \varphi_1}{\partial x} & \frac{\partial \varphi_1}{\partial y} \\ \frac{\partial \varphi_2}{\partial x} & \frac{\partial \varphi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{z} & 0 \\ \frac{2x}{z} & \frac{2y}{z} \end{vmatrix} = \frac{2y}{z^2} \neq 0 \Rightarrow \varphi_1, \varphi_2 \text{ indep.}$$

$$\Rightarrow \begin{cases} C_1 = \frac{x-2}{z} \\ C_2 = \frac{x^2+y^2+z^2}{z} \end{cases} \quad \text{(sol. în form. impl.)}$$

$$\Rightarrow \begin{cases} x = C_1 z + 2 \\ y = \sqrt{C_2 z - z^2 - (C_1 z + 2)^2} \end{cases} \quad \text{(sol. în form. expl.)}$$

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1.f) $\dfrac{dx}{2xz} = \dfrac{dy}{2yz} = \dfrac{dz}{x^2-y^2}$, $|x| \neq |y| \neq 0$

• $\dfrac{dx}{2xz} = \dfrac{dy}{2yz} \Big| \cdot 2z \Leftrightarrow \dfrac{dx}{x} = \dfrac{dy}{y} \Leftrightarrow \int \frac{1}{x}\,dx = \int \frac{1}{y}\,dy \Leftrightarrow \ln|x| = \ln|y| + C_1 \Leftrightarrow$

$\Leftrightarrow C_1 = \ln\left|\dfrac{x}{y}\right| \Leftrightarrow \boxed{C_1 = \dfrac{x}{y} = \varphi_1(x,y,z)}$

• $\dfrac{{}^{x/}dx}{2xz} = \dfrac{{}^{y/}dy}{2yz} = \dfrac{x\,dx}{2x^2 z} = \dfrac{y\,dy}{2y^2 z} = \dfrac{x\,dx - y\,dy}{2z(x^2-y^2)} = \dfrac{\frac{1}{2}d(x^2-y^2)}{2z(x^2-y^2)} = \dfrac{dz}{x^2-y^2} \Leftrightarrow$

$\Leftrightarrow \dfrac{d(x^2-y^2)}{4z(x^2-y^2)} = \dfrac{dz}{x^2-y^2} \Big| \cdot (x^2-y^2)$

$\dfrac{d(x^2-y^2)}{4z} = dz \Leftrightarrow d(x^2-y^2) = 4z\,dz \Leftrightarrow \int 1\,d(x^2-y^2) = 4\int z\,dz \Leftrightarrow$

$\Leftrightarrow x^2-y^2 = 2z^2 + C_2 \Leftrightarrow \boxed{C_2 = 2z^2 - x^2 + y^2 = \varphi_2(x,y,z)}$

Verificăm dacă φ_1, φ_2 sunt indep. $\Big|\Leftrightarrow \dfrac{D(\varphi_1,\varphi_2)}{D(x,y)} \neq 0$

Fie z - variabila indep.

$$\begin{vmatrix} \dfrac{\partial \varphi_1}{\partial x} & \dfrac{\partial \varphi_1}{\partial y} \\ \dfrac{\partial \varphi_2}{\partial x} & \dfrac{\partial \varphi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{y} & \dfrac{-x}{y^2} \\ -2x & 2y \end{vmatrix} = 2 - \dfrac{2x^2}{y^2} = \dfrac{2(y^2-x^2)}{y^2} \neq 0$$

$\Rightarrow \begin{cases} C_1 = \dfrac{x}{y} \\ C_2 = 2z^2 - x^2 + y^2 \end{cases}$ (sol. în formă imp.)

$\Rightarrow \begin{cases} x = C_1 y \\ y = C_2 + (C_1 y)^2 - 2z^2 \end{cases}$ (sol în formă exp.)

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2. Să se rezolve următoarele sisteme cu ajutorul integralelor prime.

a) $\begin{cases} x' = \dfrac{y}{x-y}, & x \neq y \\ y' = \dfrac{x}{x-y} \end{cases} \Leftrightarrow \begin{cases} \dfrac{dx}{dt} = \dfrac{y}{x-y} \\ \dfrac{dy}{dt} = \dfrac{x}{x-y} \end{cases} \Leftrightarrow \begin{cases} \dfrac{dx}{y} = \dfrac{dt}{x-y} \\ \dfrac{dy}{x} = \dfrac{dt}{x-y} \end{cases} \Leftrightarrow \dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dt}{x-y}$

• $\dfrac{dx}{y} = \dfrac{dy}{x} \Leftrightarrow x\,dx = y\,dy \Leftrightarrow \int x\,dx = \int y\,dy \Leftrightarrow \dfrac{x^2}{2} = \dfrac{y^2}{2} + C_1 \Rightarrow$

$\Rightarrow \boxed{C_1 = x^2 - y^2 = \varphi_1(t,x,y)}$

• $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dx-dy}{y-x} = \dfrac{dt}{x-y} \Leftrightarrow \dfrac{d(x-y)}{-(x-y)} = \dfrac{dt}{x-y} \quad |\cdot(x-y) \Leftrightarrow d(x-y) = -dt \Leftrightarrow$

$\Leftrightarrow \int d(x-y) = -\int dt \Leftrightarrow x-y = -t + C_2 \Leftrightarrow \boxed{C_2 = x-y+t = \varphi_2(t,x,y)}$

Verif. dacă φ_1, φ_2 sunt indep. $\Leftrightarrow \dfrac{D(\varphi_1,\varphi_2)}{D(x,y)} \neq 0$

$$\begin{vmatrix} \dfrac{\partial \varphi_1}{\partial x} & \dfrac{\partial \varphi_1}{\partial y} \\ \dfrac{\partial \varphi_2}{\partial x} & \dfrac{\partial \varphi_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 1 & -1 \end{vmatrix} = -2x + 2y = 2(y-x) \neq 0$$

$\Rightarrow \begin{cases} C_1 = x^2 - y^2 \\ C_2 = x - y + t \end{cases}$ (sol. în form. impl)

$\Leftrightarrow \begin{cases} x = C_2 + y - t \\ y = (C_2 + y - t)^2 - C_1 \end{cases}$ (sol. în form. expl)

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2. c) $\begin{cases} x' = 2zy, & x \neq 0 \\ y' = 4xz & y \neq 0 \\ z' = xy & z \neq 0 \end{cases} \Leftrightarrow \begin{cases} \frac{dx}{dt} = 2xy \\ \frac{dy}{dt} = 4xz \\ \frac{dz}{dt} = xy \end{cases} \Leftrightarrow \begin{cases} \frac{dx}{2zy} = dt \\ \frac{dy}{4xz} = dt \\ \frac{dz}{xy} = dt \end{cases}$

$\Leftrightarrow \frac{dx}{2zy} = \frac{dy}{4xz} = \frac{dz}{xy} = dt$

• $\frac{dy}{4xz} = \frac{dz}{xy} \, | \cdot x \Leftrightarrow \frac{dy}{4z} = \frac{dz}{y} \Leftrightarrow y\,dy = 4z\,dz \Leftrightarrow \int y\,dy = \int 4z\,dz \Leftrightarrow$

$\Leftrightarrow \frac{y^2}{2} = 2z^2 + C_1 \Leftrightarrow \boxed{C_1 = \frac{y^2}{2} - 2z^2 = \varphi_1(t,x,y,z)}$

• $\frac{dx}{2zy} = \frac{dz}{xy} \, | \cdot y \Leftrightarrow \frac{dx}{2z} = \frac{dz}{x} \Leftrightarrow x\,dx = 2z\,dz \Leftrightarrow \int x\,dx = \int 2z\,dz \Leftrightarrow$

$\Leftrightarrow \frac{x^2}{2} = z^2 + C_2 \Leftrightarrow \boxed{C_2 = \frac{x^2}{2} - z^2 = \varphi_2(t,x,y,z)}$

• $\frac{{}^{x/}dx}{2zy} = \frac{{}^{y/}dy}{4xz} = \frac{x\,dx - y\,dy}{-2xyz} = \frac{dz}{-2xy} \Leftrightarrow x\,dx - y\,dy = z\,dz \Leftrightarrow$

$\Leftrightarrow x^2 + \frac{y^2}{2} = 2z^2 + C_3 \Leftrightarrow \boxed{C_3 = x^2 + \frac{y^2}{2} - 2z^2 = \varphi_3(t,x,y,z)}$

Verif. dacă $\varphi_1, \varphi_2, \varphi_3$ sunt indep. $\Leftrightarrow \frac{D(\varphi_1,\varphi_2,\varphi_3)}{D(x,y,z)} \neq 0$.

$$\begin{vmatrix} \frac{\partial\varphi_1}{\partial x} & \frac{\partial\varphi_1}{\partial y} & \frac{\partial\varphi_1}{\partial z} \\ \frac{\partial\varphi_2}{\partial x} & \frac{\partial\varphi_2}{\partial y} & \frac{\partial\varphi_2}{\partial z} \\ \frac{\partial\varphi_3}{\partial x} & \frac{\partial\varphi_3}{\partial y} & \frac{\partial\varphi_3}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & y & -4z \\ x & 0 & -2z \\ 2x & y & -4z \end{vmatrix} = -4xyz \neq 0$$

$\Rightarrow \begin{cases} C_1 = \frac{y^2}{2} - 2z^2 \\ C_2 = \frac{x^2}{2} - z^2 \\ C_3 = x^2 + \frac{y^2}{2} - 2z^2 \end{cases}$ (sol. în form. imp.)

$\Leftrightarrow \begin{cases} x = \sqrt{2(C_2 + z^2)} \\ y = \sqrt{2(C_1 + 2z^2)}, \quad z = \sqrt{\frac{1}{2}\left(C_3 - x^2 - \frac{y^2}{2}\right)} \end{cases}$ (sol. în form. exp.)

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2. e) $\begin{cases} x' = y \\ y' = x \\ z' = x - y \end{cases}$, $x \neq y \neq 0 \Leftrightarrow \begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = x \\ \dfrac{dz}{dt} = x - y \end{cases} \Leftrightarrow \begin{cases} \dfrac{dx}{y} = dt \\ \dfrac{dy}{x} = dt \\ \dfrac{dz}{x-y} = dt \end{cases} \Leftrightarrow \dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{x-y} = dt$

$\bullet\ \dfrac{dx}{y} = \dfrac{dy}{x} \Leftrightarrow x\,dx = y\,dy \Leftrightarrow \int x\,dx = \int y\,dy \Leftrightarrow x^2 = y^2 + C_1 \Leftrightarrow$

$\Leftrightarrow \boxed{x^2 - y^2 = C_1 = \varphi_1(t, x, y, z)}$

$\bullet\ \dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dx - dy}{y - x} = \dfrac{d(x-y)}{-(x-y)} = \dfrac{dz}{x-y} \Leftrightarrow d(x-y) = -dz \Leftrightarrow \int d(x-y) = -\int dz \Leftrightarrow$

$\Leftrightarrow x - y = -z + C_2 \Leftrightarrow \boxed{C_2 = x - y + z = \varphi_2(t, x, y, z)}$

$\bullet\ \dfrac{d(x-y)}{-(x-y)} = dt \Leftrightarrow \dfrac{du}{-u} = dt \Leftrightarrow -\int \frac{1}{u}\,du = \int dt \Leftrightarrow -\ln|u| = t + C_3 \Leftrightarrow$

Notăm $u = x - y$ $\qquad \Leftrightarrow C_3 = \ln|x-y| + t \Leftrightarrow \boxed{C_3 = x - y - e^{t} = \varphi_3(t, x, y, z)}$

Verif. dacă $\varphi_1, \varphi_2, \varphi_3$ sunt indep. $\Leftrightarrow \dfrac{D(\varphi_1, \varphi_2, \varphi_3)}{D(x, y, z)} \neq 0$

$$\begin{vmatrix} \frac{\partial \varphi_1}{\partial x} & \frac{\partial \varphi_1}{\partial y} & \frac{\partial \varphi_1}{\partial z} \\ \frac{\partial \varphi_2}{\partial x} & \frac{\partial \varphi_2}{\partial y} & \frac{\partial \varphi_2}{\partial z} \\ \frac{\partial \varphi_3}{\partial x} & \frac{\partial \varphi_3}{\partial y} & \frac{\partial \varphi_3}{\partial z} \end{vmatrix} = \begin{vmatrix} 2x & -2y & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = -2y + 2x = 2(x-y) \neq 0.$$

$\Rightarrow \begin{cases} C_1 = x^2 - y^2 \\ C_2 = x - y + z \\ C_3 = x - y - e^{t} \end{cases}$ (sol. în form. imp)

$\Leftrightarrow x - y = C_2 - z$

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2\. f) $\begin{cases} x' = y + xy \\ y' = x + xy \\ z' = z^2 - 1 \end{cases}$, $x \neq y$, $|z| > 1$ $\Leftrightarrow \begin{cases} \dfrac{dx}{dt} = y + xy \\ \dfrac{dy}{dt} = x + xy \\ \dfrac{dz}{dt} = z^2 - 1 \end{cases} \Leftrightarrow \begin{cases} \dfrac{dx}{y+xy} = dt \\ \dfrac{dy}{x+xy} = dt \\ \dfrac{dz}{z^2-1} = dt \end{cases}$

$$\Leftrightarrow \frac{dx}{y+xy} = \frac{dy}{x+xy} = \frac{dz}{z^2-1} = dt$$

- $\dfrac{dx}{y+xy} = \dfrac{dy}{x+xy} = \dfrac{dx-dy}{-(x-y)} = \dfrac{d(x-y)}{-(x-y)} = dt \Leftrightarrow \dfrac{du}{-u} = dt \Leftrightarrow -\int \frac{1}{u}\,du = \int dt \Leftrightarrow$

Notăm $u = x - y$

$$\Leftrightarrow \boxed{C_1 = \ln|x-y| + t = \varphi_1(t,x,y,z)}$$

- $\dfrac{dz}{z^2-1} = dt \Leftrightarrow \boxed{\frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| - t = C_3 = \varphi_2(t,x,y,z)}$