Laboratorul 11 - TEMĂ Petculescu Mihai - Silviu

Să se rezolve următoarele sisteme de ecuații diferențiale:

1. b) $\begin{cases} x' = y \\ y' = 4x \end{cases}$

$\begin{cases} x = \alpha_1 e^{rt}, & y = \alpha_2 e^{rt} \\ x' = \alpha_1 r e^{rt}, & y' = \alpha_2 r e^{rt} \end{cases} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{rt} \\ \alpha_2 e^{rt} \end{pmatrix}$

$\begin{cases} \alpha_1 r e^{rt} = \alpha_2 e^{rt} \quad |:e^{rt} \\ \alpha_2 r e^{rt} = 4\alpha_1 e^{rt} \quad |:e^{rt} \end{cases} \Leftrightarrow \begin{cases} \alpha_1 r = \alpha_2 \\ \alpha_2 r = 4\alpha_1 \end{cases} \Leftrightarrow \begin{cases} r\alpha_1 - \alpha_2 = 0 \\ 4\alpha_1 - r\alpha_2 = 0 \end{cases}$

$D = \begin{vmatrix} r & -1 \\ 4 & -r \end{vmatrix} = -r^2 + 4 = (2+r)(2-r) \Leftrightarrow \begin{cases} r_1 = -2 \\ r_2 = 2 \end{cases}$

• Pt $r_1 = -2$

$\begin{cases} -2\alpha_1 = \alpha_2 \\ 4\alpha_1 = -2\alpha_2 \end{cases} \Leftrightarrow -2\alpha_1 = \alpha_2, \alpha_2 \in R$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = -2 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \end{pmatrix}$

• Pt $r_2 = 2$

$\begin{cases} 2\alpha_1 = \alpha_2 \\ 4\alpha_1 = 2\alpha_2 \end{cases} \Rightarrow 2\alpha_1 = \alpha_2, \alpha_2 \in R$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = 2 \Rightarrow \alpha_{r_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow X_{r_2} = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$

$X = C_1 X_{r_1} + C_2 X_{r_2} = C_1 \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} C_1 e^{-2t} + C_2 e^{2t} \\ -2C_1 e^{-2t} + 2C_2 e^{2t} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{cases} x = C_1 e^{-2t} + C_2 e^{2t} \\ y = -2C_1 e^{-2t} + 2C_2 e^{2t} \end{cases}$

d) $\begin{cases} x' = -x + 4y \\ y' = y + 2z \\ z' = 3z \end{cases}$

$$\begin{cases} x=\alpha_1 e^{\lambda t}, \; y=\alpha_2 e^{\lambda t}, \; z=\alpha_3 e^{\lambda t} \\ x'=\alpha_1\lambda e^{\lambda t}, \; y'=\alpha_2 \lambda e^{\lambda t}, \; z'=\alpha_3\lambda e^{\lambda t} \end{cases} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{\lambda t} \\ \alpha_2 e^{\lambda t} \\ \alpha_3 e^{\lambda t} \end{pmatrix}$$

$$\begin{cases} \alpha_1\lambda e^{\lambda t} = -\alpha_1 e^{\lambda t} + 4\alpha_2 e^{\lambda t} \;\; |:e^{\lambda t} \\ \alpha_2\lambda e^{\lambda t} = \alpha_2 e^{\lambda t} + 2\alpha_3 e^{\lambda t} \;\; |:e^{\lambda t} \\ \alpha_3\lambda e^{\lambda t} = 3\alpha_3 e^{\lambda t} \;\; |:e^{\lambda t} \end{cases} \Leftrightarrow \begin{cases} \alpha_1\lambda = -\alpha_1 + 4\alpha_2 \\ \alpha_2\lambda = \alpha_2 + 2\alpha_3 \\ \alpha_3\lambda = 3\alpha_3 \end{cases} \Leftrightarrow \begin{cases} (-1-\lambda)\alpha_1 + 4\alpha_2 = 0 \\ (1-\lambda)\alpha_2 + 2\alpha_3 = 0 \\ (3-\lambda)\alpha_3 = 0 \end{cases}$$

$$D = \begin{vmatrix} -1-\lambda & 4 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda)(3-\lambda) = 0 \Leftrightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 1 \\ \lambda_3 = 3 \end{cases}$$

• Dc $\lambda_1 = -1$

$$\begin{cases} 4\alpha_2 = 0 \\ 2\alpha_2 + 2\alpha_3 = 0 \\ 4\alpha_3 = 0 \end{cases} \Rightarrow \alpha_2 = 0, \alpha_3 = 0, \alpha_1 \in R$$

Fie $\alpha_1 = 1 \Rightarrow \alpha_{\lambda_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow X_{\lambda_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot e^{-t} = \begin{pmatrix} e^{-t} \\ 0 \\ 0 \end{pmatrix}$

• Dc $\lambda_2 = 1$

$$\begin{cases} -2\alpha_1 + 4\alpha_2 = 0 \\ 2\alpha_3 = 0 \end{cases} \Rightarrow \alpha_3 = 0, \alpha_1 = +2\alpha_2, \alpha_1, \alpha_2 \in R$$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = 2 \Rightarrow \alpha_{\lambda_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \Rightarrow X_{\lambda_2} = \begin{pmatrix} e^{t} \\ 2e^{t} \\ 0 \end{pmatrix}$

• Dc $\lambda_3 = 3$

$$\begin{cases} -4\alpha_1 + \alpha_2 = 0 \\ -2\alpha_2 + 2\alpha_3 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3, \; \alpha_1 \in R$$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = \alpha_3 = 1 \Rightarrow \alpha_{\lambda_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow X_{\lambda_3} = \begin{pmatrix} e^{3t} \\ e^{3t} \\ e^{3t} \end{pmatrix}$

$$X = C_1 X_{\lambda_1} + C_2 X_{\lambda_2} + C_3 X_{\lambda_3} = C_1 \begin{pmatrix} e^{-t} \\ 0 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} e^{t} \\ 2e^{t} \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} e^{3t} \\ e^{3t} \\ e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} C_1 e^{-t} + C_2 e^{t} + C_3 e^{3t} \\ 0 + 2C_2 e^{t} + C_3 e^{t} \\ 0 + 0 + C_3 e^{t} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = C_1 e^{-t} + C_2 e^{t} + C_3 e^{3t} \\ y = 2C_2 e^{t} + C_3 e^{t} \\ z = C_3 e^{t} \end{cases}$$

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• Să se rezolve următoarele sisteme de ecuații diferențiale:

2. a) $\begin{cases} x' = -y \\ y' = 9x \end{cases}$

$\begin{cases} x = \alpha_1 e^{rt}, & y = \alpha_2 e^{rt} \\ x' = \alpha_1 r e^{rt} & y' = \alpha_2 r e^{rt} \end{cases} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{rt} \\ \alpha_2 e^{rt} \end{pmatrix}$

$\begin{cases} \alpha_1 r e^{rt} = -\alpha_2 e^{rt} \mid : e^{rt} \\ \alpha_2 r e^{rt} = 9\alpha_1 e^{rt} \mid : e^{rt} \end{cases} \Leftrightarrow \begin{cases} \alpha_1 r = -\alpha_2 \\ \alpha_2 r = 9\alpha_1 \end{cases} \Leftrightarrow \begin{cases} -r\alpha_1 - \alpha_2 = 0 \\ 9\alpha_1 - r\alpha_2 = 0 \end{cases}$

$D = \begin{vmatrix} -r & -1 \\ 9 & -r \end{vmatrix} = r^2 + 9 = 0 \Leftrightarrow r_{1,2} = \pm 3i$

• Pt $r = 3i$ $(\gamma = 0, \beta = 3)$

$\begin{cases} -3i\alpha_1 - \alpha_2 = 0 \\ 9\alpha_1 - 3i\alpha_2 = 0 \end{cases} \Leftrightarrow \begin{cases} 3i\alpha_1 = -\alpha_2 \\ 3\alpha_1 = i\alpha_2 \end{cases} \Leftrightarrow \alpha_1 = \frac{i}{3}\alpha_2, \alpha_2 \in R$

Fie $\alpha_2 = 1 \Rightarrow \alpha_1 = \frac{i}{3}$

$\alpha_1 = \begin{pmatrix} \frac{i}{3} \\ 1 \end{pmatrix} \Rightarrow X_1 = \begin{pmatrix} \frac{i}{3} \\ 1 \end{pmatrix} \cdot e^{0t}(\cos 3t + i \sin 3t) = \begin{pmatrix} \frac{i}{3}(\cos 3t + i \sin 3t) \\ \cos 3t + i \sin 3t \end{pmatrix} =$

$= \underbrace{\begin{pmatrix} \frac{-1}{3}\sin 3t \\ \cos 3t \end{pmatrix}}_{\tilde{X}_1} + i \underbrace{\begin{pmatrix} \frac{1}{3}\cos 3t \\ \sin 3t \end{pmatrix}}_{\tilde{\tilde{X}}_1}$

$X = C_1 \tilde{X}_1 + C_2 \tilde{\tilde{X}}_1 = C_1 \begin{pmatrix} \frac{-1}{3}\sin 3t \\ \cos 3t \end{pmatrix} + C_2 \begin{pmatrix} \frac{1}{3}\cos 3t \\ \sin 3t \end{pmatrix} = \begin{pmatrix} C_1 \frac{-1}{3}\sin 3t + C_2 \frac{1}{3}\cos 3t \\ C_1 \cos 3t + C_2 \sin 3t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{cases} x = C_1 \frac{-1}{3}\sin 3t + C_2 \frac{1}{3}\cos 3t \\ y = C_1 \cos 3t + C_2 \sin 3t \end{cases}$

c) $\begin{cases} x' = x - 3y \\ y' = 3x + y \end{cases}$

$\begin{cases} x = \alpha_1 e^{rt}, & y = \alpha_2 e^{rt} \\ x' = \alpha_1 r e^{rt}, & y' = \alpha_2 r e^{rt} \end{cases} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{rt} \\ \alpha_2 e^{rt} \end{pmatrix}$

$$\begin{cases} \alpha_1 r e^{rt} = \alpha_1 e^{rt} - 3\alpha_2 e^{rt} \;\Big|: e^{rt} \\ \alpha_2 r e^{rt} = 3\alpha_1 e^{rt} + \alpha_2 e^{rt} \;\Big|: e^{rt} \end{cases} \Leftrightarrow \begin{cases} \alpha_1 r = \alpha_1 - 3\alpha_2 \\ \alpha_2 r = 3\alpha_1 + \alpha_2 \end{cases} \Leftrightarrow \begin{cases} (1-r)\alpha_1 - 3\alpha_2 = 0 \\ 3\alpha_1 + (1-r)\alpha_2 = 0 \end{cases}$$

$$D = \begin{vmatrix} 1-r & -3 \\ 3 & 1-r \end{vmatrix} = (1-r)^2 + 9 = 0 \Leftrightarrow r^2 - 2r + 10 = 0$$

$$D = 4 - 4\cdot 10 = -36$$

$$r_{1,2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

• Fie $r = 1+3i$ $(\mu = 1, \beta = 3)$

$$\begin{cases} -3i\alpha_1 - 3\alpha_2 = 0 \Leftrightarrow \alpha_1 = i\alpha_2,\ \alpha_2 \in \mathbb{R} \\ 3\alpha_1 - 3i\alpha_2 = 0 \end{cases}$$

Fie $\alpha_2 = 1 \Rightarrow \alpha_1 = i \Rightarrow \alpha_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \Rightarrow X_r = \begin{pmatrix} i \\ 1 \end{pmatrix} \cdot e^t(\cos 3t + i \sin 3t) =$

$$= \begin{pmatrix} i e^t(\cos 3t + i\sin 3t) \\ e^t(\cos 3t + i \sin 3t) \end{pmatrix} = \underbrace{\begin{pmatrix} -e^t \sin 3t \\ e^t \cos 3t \end{pmatrix}}_{\tilde{X}_r} + i \underbrace{\begin{pmatrix} e^t \cos 3t \\ e^t \sin 3t \end{pmatrix}}_{\tilde{\tilde{X}}_r}$$

$$X = C_1 \tilde{X}_r + C_2 \tilde{\tilde{X}}_r = C_1 \begin{pmatrix} -e^t \sin 3t \\ e^t \cos 3t \end{pmatrix} + C_2 \begin{pmatrix} e^t \cos 3t \\ e^t \sin 3t \end{pmatrix} = \begin{pmatrix} C_1\cdot(-e^t)\sin 3t + C_2 e^t \cos 3t \\ C_1 e^t \cos 3t + C_2 e^t \sin 3t \end{pmatrix} =$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} x = C_1(-e^t)\sin 3t + C_2 e^t \cos 3t \\ y = C_1 e^t \cos 3t + C_2 e^t \sin 3t \end{cases}$$

d) $\begin{cases} x' = -x - 4y \\ y' = 4x - y \end{cases}$

$$\begin{cases} x = \alpha_1 e^{rt}, & y = \alpha_2 e^{rt} \\ x' = \alpha_1 r e^{rt}, & y' = \alpha_2 r e^{rt} \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{rt} \\ \alpha_2 e^{rt} \end{pmatrix}$$

$$\begin{cases} \alpha_1 r e^{rt} = -\alpha_1 e^{rt} - 4\alpha_2 e^{rt} \;\Big|: e^{rt} \\ \alpha_2 r e^{rt} = 4\alpha_1 e^{rt} - \alpha_2 e^{rt} \;\Big|: e^{rt} \end{cases} \Leftrightarrow \begin{cases} \alpha_1 r = -\alpha_1 - 4\alpha_2 \\ \alpha_2 r = 4\alpha_1 - \alpha_2 \end{cases} \Leftrightarrow \begin{cases} (-1-r)\alpha_1 - 4\alpha_2 = 0 \\ 4\alpha_1 + (-1-r)\alpha_2 = 0 \end{cases}$$

$$D = \begin{vmatrix} -1-r & -4 \\ 4 & -1-r \end{vmatrix} = (-1-r)(-1-r) + 16 = r^2 + 2r + 17 = 0$$

$$D = 4 - 4\cdot 17 = -64$$

$$r_{1,2} = \frac{-2 \pm 8i}{2} = -1 \pm 4i$$

• Fie $r = -1 + 4i$

$$\begin{cases} -4i\alpha_1 - 4\alpha_2 = 0 \\ 4\alpha_1 - 4i\alpha_2 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = -i\alpha_1 \\ \alpha_1 = i\alpha_2 \end{cases} \Leftrightarrow \alpha_1 = i\alpha_2,\ \alpha_2 \in \mathbb{R}$$

Fie $\alpha_1 = 1 \Rightarrow \alpha_2 = -i \Rightarrow \alpha_r = \begin{pmatrix} 1 \\ -i \end{pmatrix} \Rightarrow$

$$\Rightarrow X_1 = \begin{pmatrix} 1\cdot e^{-t}(\cos 4t + i\sin 4t) \\ -i\cdot e^{-t}(\cos 4t + i\sin 4t) \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-t}\cos 4t \\ e^{-t}\sin 4t \end{pmatrix}}_{\tilde{X}_1} + i\underbrace{\begin{pmatrix} e^{-t}\sin 4t \\ -e^{-t}\cos 4t \end{pmatrix}}_{\tilde{\tilde{X}}_1}$$

$$X = C_1\tilde{X}_1 + C_2\tilde{\tilde{X}}_1 = C_1\begin{pmatrix} e^{-t}\cos 4t \\ e^{-t}\sin 4t \end{pmatrix} + C_2\begin{pmatrix} e^{-t}\sin 4t \\ -e^{-t}\cos 4t \end{pmatrix}$$

$$= \begin{pmatrix} C_1 e^{-t}\cos 4t + C_2 e^{-t}\sin 4t \\ C_1 e^{-t}\sin 4t - C_2 e^{-t}\cos 4t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = C_1 e^{-t}\cos 4t + C_2 e^{-t}\sin 4t \\ y = C_1 e^{-t}\sin 4t - C_2 e^{-t}\cos 4t \end{cases}$$

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• Să se rezolve următoarele sisteme de ecuații diferențiale:

3. b) $\begin{cases} x' = -x - 2y \\ y' = 2x + y \\ z' = 3x + 2z \end{cases}$

$\begin{cases} x = \alpha_1 e^{rt}, \; y = \alpha_2 e^{rt}, \; z = \alpha_3 e^{rt} \\ x' = \alpha_1 r e^{rt}, \; y' = \alpha_2 r e^{rt}, \; z' = \alpha_3 r e^{rt} \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{rt} \\ \alpha_2 e^{rt} \\ \alpha_3 e^{rt} \end{pmatrix}$

$\begin{cases} \alpha_1 r e^{rt} = -\alpha_1 e^{rt} - 2\alpha_2 e^{rt} \;|: e^{rt} \\ \alpha_2 r e^{rt} = 2\alpha_1 e^{rt} + \alpha_2 e^{rt} \;|: e^{rt} \\ \alpha_3 r e^{rt} = 3\alpha_1 e^{rt} + 2\alpha_3 e^{rt} \;|: e^{rt} \end{cases} \Leftrightarrow \begin{cases} \alpha_1 r = -\alpha_1 - 2\alpha_2 \\ \alpha_2 r = 2\alpha_1 + \alpha_2 \\ \alpha_3 r = 3\alpha_1 + 2\alpha_3 \end{cases} \Leftrightarrow \begin{cases} (-1-r)\alpha_1 - 2\alpha_2 = 0 \\ 2\alpha_1 + (1-r)\alpha_2 = 0 \\ 3\alpha_1 + (2-r)\alpha_3 = 0 \end{cases}$

$D = \begin{vmatrix} -1-r & -2 & 0 \\ 2 & 1-r & 0 \\ 3 & 0 & 2-r \end{vmatrix} = (1-r)(-1-r)(2-r) - (2-r)(-2)(2)$

$= (2-r)(r^2+3) = 0 \Rightarrow \begin{cases} r_1 = 2 \\ r_{2,3} = \pm\sqrt{3}\, i \end{cases}$

• Fie $r_1 = 2$

$\begin{cases} -3\alpha_1 - 2\alpha_2 = 0 \\ 2\alpha_1 - \alpha_2 = 0 \\ 3\alpha_1 = 0 \end{cases} \Rightarrow \alpha_1 = 0, \alpha_2 = 0, \alpha_3 \in R$

Fie $\alpha_3 = 1 \Rightarrow \alpha_{r_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} 0 \\ 0 \\ e^{2t} \end{pmatrix}$

• Fie $r_2 = \sqrt{3}\, i$, $\mu = 0$, $\beta = \sqrt{3}$

$\begin{cases} (-1-\sqrt{3}i)\alpha_1 = 2\alpha_2 \\ 2\alpha_1 = (-1+\sqrt{3}i)\alpha_2 \\ 3\alpha_1 = (-2+\sqrt{3}i)\alpha_3 \end{cases} \Leftrightarrow \alpha_1 = \frac{-1+\sqrt{3}i}{2}\alpha_2, \; \alpha_3 = \frac{3}{-2+\sqrt{3}i}\alpha_1, \; \alpha_2 \in R$

Fie $\alpha_2 = 1 \Rightarrow \alpha_1 = \frac{-1+\sqrt{3}i}{2}$, $\alpha_3 = \frac{3}{-2+\sqrt{3}i} \Rightarrow \alpha_{r_2} = \begin{pmatrix} \frac{-1+\sqrt{3}i}{2} \\ 1 \\ \frac{3}{-2+\sqrt{3}i} \end{pmatrix}$

$$\bullet\; X_{r_2} = \begin{pmatrix} \frac{-1+\sqrt{3}i}{2} \\ 1 \\ \frac{3}{-2+\sqrt{3}i} \end{pmatrix} \cdot e^{0t}(\cos\sqrt{3}t + i\sin\sqrt{3}t) = \underbrace{\begin{pmatrix} \frac{-1+\sqrt{3}}{2}\cdot(-1)\cdot\sin\sqrt{3}t \\ \cos\sqrt{3}t \\ \frac{3}{-2+\sqrt{3}}\cdot(-1)\sin\sqrt{3}t \end{pmatrix}}_{\tilde{X}_{r_2}} +$$

$$+\, i \overbrace{\begin{pmatrix} \frac{-1+\sqrt{3}}{2}\cos\sqrt{3}t \\ \sin\sqrt{3}t \\ \frac{3}{-2+\sqrt{3}}\cos\sqrt{3}t \end{pmatrix}}^{\tilde{\tilde{X}}_{r_2}}$$

$$X = C_1 X_{r_1} + C_2 \tilde{X}_{r_2} + C_3 \tilde{\tilde{X}}_{r_3} = C_1 \begin{pmatrix} 0 \\ 0 \\ e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} \frac{1-\sqrt{3}}{2} \sin\sqrt{3}t \\ \cos\sqrt{3}t \\ \frac{3}{2-\sqrt{3}} \sin\sqrt{3}t \end{pmatrix} + C_3 \begin{pmatrix} \frac{-1+\sqrt{3}}{2} \cos\sqrt{3}t \\ \sin\sqrt{3}t \\ \frac{3}{-2+\sqrt{3}} \cos\sqrt{3}t \end{pmatrix} =$$

$$= \begin{pmatrix} C_2 \frac{1-\sqrt{3}}{2} \sin\sqrt{3}t + C_3 \frac{-1+\sqrt{3}}{2} \cos\sqrt{3}t \\ C_2 \cos\sqrt{3}t + C_3 \sin\sqrt{3}t \\ C_1 e^{2t} + C_2 \frac{3}{2-\sqrt{3}} \sin\sqrt{3}t + C_3 \frac{3}{-2+\sqrt{3}} \cos\sqrt{3}t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} x = C_2 \frac{1-\sqrt{3}}{2} \sin\sqrt{3}t + C_3 \frac{-1+\sqrt{3}}{2} \cos\sqrt{3}t \\ y = C_2 \cos\sqrt{3}t + C_3 \sin\sqrt{3}t \\ z = C_1 e^{2t} + C_2 \frac{3}{2-\sqrt{3}} \sin\sqrt{3}t + C_3 \frac{3}{-2+\sqrt{3}} \cos\sqrt{3}t \end{cases}$$

c) $\begin{cases} x' = x - 2y + 3z \\ y' = y \\ z' = -3x + z \end{cases}$

$\begin{cases} x = \alpha_1 e^{rt}, & y = \alpha_2 e^{rt}, & z = \alpha_3 e^{rt} \\ x' = \alpha_1 r e^{rt}, & y' = \alpha_2 r e^{rt}, & z' = \alpha_3 r e^{rt} \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{rt} \\ \alpha_2 e^{rt} \\ \alpha_3 e^{rt} \end{pmatrix}$

$\begin{cases} \alpha_1 r e^{rt} = \alpha_1 e^{rt} - 2\alpha_2 e^{rt} + 3\alpha_3 e^{rt} \mid : e^{rt} \\ \alpha_2 r e^{rt} = \alpha_2 e^{rt} \mid : e^{rt} \\ \alpha_3 r e^{rt} = -3\alpha_1 e^{rt} + \alpha_3 e^{rt} \mid : e^{rt} \end{cases} \Leftrightarrow \begin{cases} \alpha_1 r = \alpha_1 - 2\alpha_2 + 3\alpha_3 \\ \alpha_2 r = \alpha_2 \\ \alpha_3 r = -3\alpha_1 + \alpha_3 \end{cases} \Leftrightarrow \begin{cases} (1-r)\alpha_1 - 2\alpha_2 + 3\alpha_3 = 0 \\ (1-r)\alpha_2 = 0 \\ -3\alpha_1 + (1-r)\alpha_3 = 0 \end{cases}$

$$D = \begin{vmatrix} 1-r & -2 & 3 \\ 0 & 1-r & 0 \\ -3 & 0 & 1-r \end{vmatrix} = (1-r)^3 - 3(-3)(1-r) = (1-r)(10 - 2r + r^2)$$

$D = 4 - 4 \cdot 10 = -36$

$\Rightarrow \begin{cases} r_1 = 1 \\ r_{2,3} = 1 \pm 3i \end{cases}$

• Pt $r_1 = 1$

$\begin{cases} -2\alpha_2 + 3\alpha_3 = 0 \\ -3\alpha_1 = 0 \end{cases} \Rightarrow \alpha_1 = 0, \alpha_2 = \frac{3}{2}\alpha_3, \alpha_3 \in \mathbb{R}$

Fie $\alpha_3 = 1 \Rightarrow \alpha_2 = \frac{3}{2} \Rightarrow \alpha_{r_1} = \begin{pmatrix} 0 \\ \frac{3}{2} \\ 1 \end{pmatrix} \Rightarrow X_{r_1} = \begin{pmatrix} 0 \\ \frac{3}{2} e^t \\ e^t \end{pmatrix}$

• Pt $r_2 = 1 + 3i$ $(\mu = 1, \beta = 3)$

$\begin{cases} -3i\alpha_1 - 2\alpha_2 + 3\alpha_3 = 0 \\ -3i\alpha_2 = 0 \\ -3\alpha_1 - 3i\alpha_3 = 0 \end{cases} \Rightarrow \alpha_2 = 0, \alpha_3 = i\alpha_1, \alpha_1 \in \mathbb{R}$

Fie $\alpha_1 = 1 \Rightarrow \alpha_3 = i \Rightarrow \alpha_{r_2} = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \Rightarrow$

$$\Rightarrow X_{r_2} = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \cdot e^t(\cos 3t + i \sin 3t) = \begin{pmatrix} e^t(\cos 3t + i\sin 3t) \\ 0 \\ ie^t(\cos 3t + i \sin 3t) \end{pmatrix} = \underbrace{\begin{pmatrix} e^t \cos 3t \\ 0 \\ -e^t \sin 3t \end{pmatrix}}_{\tilde{X}_{r_2}} +$$

$$+ i \overbrace{\begin{pmatrix} e^t \sin 3t \\ 0 \\ e^t \cos 3t \end{pmatrix}}^{\tilde{\tilde{X}}_{r_2}}$$

$$X = C_1 X_{n_1} + C_2 \tilde{X}_{n_2} + C_3 \tilde{\tilde{X}}_{n_2} = C_1 \begin{pmatrix} 0 \\ \frac{3}{2} e^t \\ e^t \end{pmatrix} + C_2 \begin{pmatrix} e^t \cos 3t \\ 0 \\ -e^t \sin 3t \end{pmatrix} + C_3 \begin{pmatrix} e^t \sin 3t \\ 0 \\ e^t \cos 3t \end{pmatrix} =$$

$$= \begin{pmatrix} C_2 e^t \cos 3t + C_3 e^t \sin 3t \\ C_1 \frac{3}{2} e^t \\ C_1 e^t + C_2 (-e^t) \sin 3t + C_3 e^t \cos 3t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} x = C_2 e^t \cos 3t + C_2 e^t \sin 3t \\ y = C_1 \frac{3}{2} e^t \\ z = C_1 e^t + C_2 (-e^t) \sin 3t + C_3 e^t \cos 3t \end{cases}$$