Laborator03

Enunţuri

1. Să se rezolve următoarele ecuaţii liniare scalare:

$$a)\ \begin{cases} \frac{dx}{dt} = (t^2+1)\cdot x, & t, x \in R_+ \\ x(0) = 2 \end{cases}$$
$$b)\ \begin{cases} \frac{dx}{dt} = \sqrt{t+1}\cdot x, & t \geq -1 \\ x(0) = 2 & x \in R_+ \end{cases}$$
$$c)\ \begin{cases} \frac{dx}{dt} = t^2 \cdot e^t \cdot x, & t \in R \\ x(0) = 3 & x \in R_+ \end{cases}$$
$$d)\ \begin{cases} \frac{dx}{dt} = \frac{t^2+1}{t-1}\cdot x, & t \in R\setminus\{1\} \\ x(0) = 1 & x \in R_+ \end{cases}$$

2. Să se rezolve următoarele ecuaţii diferenţiale afine:

$$a)\ x' + \frac{1-2t}{t^2}\cdot x = 1$$
$$b)\ \begin{cases} t\cdot x' + x = t\cdot sint, & t > 0 \\ x(\pi) = 2 \end{cases}$$
$$c)\ \begin{cases} x' + 2\cdot t\cdot x + t - e^{-t^2} = 0 \\ x(0) = 1 \end{cases}$$
$$d)\ \begin{cases} \frac{dx}{dt} = x - t^2 \\ x(1) = 2 \end{cases}$$
$$e)\ \begin{cases} x'(t) = \frac{1}{t}\cdot x - 1 \\ x(1) = 4 \end{cases}$$

3. Să se rezolve următoarele ecuaţii reductibile la ecuaţii de tip omogen:

$$
\begin{aligned}
&a)\ \ (t^2 - t \cdot x + x^2)dt + (t \cdot x - 2t^2)dx = 0\\
&b)\ \ x' = \frac{2 \cdot t \cdot x}{3t^2 - x^2}\\
&c)\ \ x' = \frac{t \cdot x + x^2}{t^2}\\
&d)\ \ (t + 2x)dt - tdx = 0\\
&e)\ \ t \cdot x \cdot x' - x^2 + 3t^2 = 0
\end{aligned}
$$

4. Să se rezolve următoarele ecuaţii reductibile la ecuaţii de tip omogen:

$$
\begin{aligned}
&a)\ \ (t - 2x + 5)dt + (2t - x + 4)dx = 0\\
&b)\ \ 2 \cdot (t + 4x - 6)dt = (7t + x - 15)dx\\
&c)\ \ (2t - 4x + 6)dt + (t + x - 3)dx = 0\\
&d)\ \ (3t + 3x - 1)dt + (t + x + 1)dx = 0\\
&e)\ \ (t - 2x + 1)dt + (2t - 4x + 3)dx = 0\\
&f)\ \ (t - x - 1) + x'(x - t + 2) = 0
\end{aligned}
$$

Rezolvare

Exerciţiu 01

c) - Video

1 c) $\begin{cases} \dfrac{dx}{dt} = t^2 e^t x \\ x(0) = 3 \end{cases}$, $t \in \mathbb{R}$, $x \in \mathbb{R}_+$

$\dfrac{dx}{dt} = \underbrace{t^2 e^t}_{A(t)} x$ (ec. liniară)

$\dfrac{dx}{x} = t^2 e^t \, dt$

$\int \frac{1}{x} dx = \int t^2 e^t \, dt$

$\int \underbrace{t^2}_{f} \underbrace{e^t}_{g'} dt = t^2 e^t - \int 2t \cdot e^t \, dt$

$f' = 2t,\ g = e^t$

$= t^2 e^t - 2 \int \underbrace{t}_{f} \underbrace{e^t}_{g'} dt$

$f' = 1,\ g = e^t$

$= t^2 e^t - 2\left(t e^t - \int e^t dt\right)$

$= t^2 e^t - 2t e^t + 2 e^t + C$

$\ln x = t^2 e^t - 2te^t + 2e^t + C$

$x = e^{t^2 e^t - 2te^t + 2e^t + C}$

$x(t) = C \cdot e^{e^t(t^2 - 2t + 2)}$

$x(0) = C \cdot e^{e^0(0 - 0 + 2)} = C \cdot e^2 = 3 \Rightarrow \boxed{C = \frac{3}{e^2}} \Rightarrow x_{pc}(t) = \frac{3}{e^2} \cdot e^{e^t(t^2 - 2t + 2)}$

d)

1 d) $\begin{cases} \dfrac{dx}{dt} = \dfrac{t^2+1}{t-1}\cdot x, & t\in\mathbb{R}\setminus\{1\},\ x\in\mathbb{R}_+ \\ x(0)=1 \end{cases}$

$$\frac{dx}{dt} = \underbrace{\frac{t^2+1}{t-1}}_{A(t)}\cdot x \quad \text{(ec. lin. scalară)}$$

$$\frac{dx}{x} = \frac{t^2+1}{t-1}\,dt$$

$$\int \frac{1}{x}\,dx = \int \frac{t^2+1}{t-1}\,dt$$

$$\int \frac{t^2+1}{t-1}\,dt = \int\left(t+1+\frac{2}{t-1}\right)dt$$

$$\begin{array}{l|l} t^2+1 & t-1 \to g \\ \hline -t^2+t & t+1 \to c \\ \quad\ t+1 & \\ \ -t+1 & \\ \quad\ \ 2 \to r & \end{array} \qquad \frac{f}{g} = c + \frac{r}{g}$$

$$\ln x = \frac{t^2}{2}+t+2\ln|t-1|+C$$

$$x = e^{t^2/2+t+2\ln|t-1|+C} = e^{\frac{t^2}{2}+t}\cdot e^{\ln(t-1)^2}\cdot e^{C}$$

$$x(t) = C\cdot(t-1)^2\cdot e^{\frac{t^2}{2}+t}$$

$$x(0) = C\cdot(0-1)^2\cdot e^0 = \boxed{C=1} \Rightarrow x_{PC}(t) = (t-1)^2\cdot e^{\frac{t^2}{2}+2}$$

Exerciţiu 02

b) - Video

2 b) $\begin{cases} t x' + x = t\sin t, & t>0 \\ x(\pi)=2 \end{cases}$

$$t\cdot\frac{dx}{dt}+x = t\sin t \;\Big|\, :t$$

$$\frac{dx}{dt}+\underbrace{\frac{1}{t}}_{A(t)}x = \underbrace{\sin t}_{B(t)} \quad \text{(ec. afină)}$$

Etapa 1 $\quad \frac{dx}{dt}+\frac{1}{t}x=0$

$$\frac{dx}{dt} = -\frac{1}{t}x$$

$$\frac{dx}{x} = -\frac{1}{t}\,dt \Leftrightarrow \int\frac{1}{x}\,dx = \int\frac{1}{t}\,dt$$

$$\ln|x| = -\ln t + C$$

$$\ln|x| = -\ln t + \ln C = \ln\frac{C}{t}$$

$$\boxed{x_0 = \frac{C}{t}}$$

Etapa 2 $\quad \varphi_0 = \frac{C(t)}{t}$

$$\left(\frac{C(t)}{t}\right)' + \frac{1}{t}\cdot\frac{C(t)}{t} = \sin t$$

$$\frac{C'(t)\cdot t - C(t)}{t^2} + \frac{C(t)}{t^2} = \sin t$$

$$\frac{C'(t)}{t} - \cancel{\frac{C(t)}{t^2}} + \cancel{\frac{C(t)}{t^2}} = \sin t$$

$$C'(t) = \underbrace{t\sin t}_{f(t)}$$

$$C(t) = \int \underbrace{t}_{f}\,\underbrace{\sin t}_{g'}\,dt = -t\cos t + \int\cos t\,dt = -t\cos t + \sin t + C_1$$

$$f'=1,\quad g=-\cos t$$

$$\Rightarrow \boxed{\varphi_0 = \frac{-t\cos t+\sin t+C_1}{t}}$$

$$\Rightarrow x(t) = x_0+\varphi_0 = \frac{-t\cos t+\sin t+C}{t}$$

$$x(\pi) = \frac{\pi+C}{\pi} = 2 \Rightarrow \boxed{C=\pi}$$

$$x_{PC} = \frac{-t\cos t+\sin t+\pi}{t}$$

c)

② c) $\begin{cases} x' + 2tx + t - e^{-t^2} = 0 \\ x(0) = 1 \end{cases}$

$\frac{dx}{dt} + \underbrace{2t}_{A(t)} x = \underbrace{e^{-t^2} - t}_{B(t)}$

Etapa 1

$\frac{dx}{dt} + 2tx = 0$

$\frac{dx}{dt} = -2tx$

$\frac{dx}{x} = -2t\,dt$

$\int \frac{1}{x}\,dx = -\int 2t\,dt$

$\ln|x| = -t^2 + \mathcal{C}$

$x = e^{-t^2+C} \Rightarrow \boxed{x_0 = C \cdot e^{-t^2}}$

Etapa 2

$\varphi_0 = C(t) \cdot e^{-t^2}$

$\left(C(t) \cdot e^{-t^2}\right)' + 2t \cdot C(t)\, e^{-t^2} = e^{-t^2} - t$

$C'(t) \cdot e^{-t^2} + \cancel{C(t) \cdot e^{-t^2} \cdot (-2t)} + \cancel{2t\, C(t)\, e^{-t^2}} = e^{-t^2} - t$

$C'(t) = 1 - t e^{t^2}$

$C(t) = \int \left(1 - t e^{t^2}\right) dt = \int 1\,dt - \frac{1}{2}\int 2t\, e^{t^2}\,dt$ $\quad u(t) = t^2,\ u'(t) = 2t$

$= t - \frac{1}{2} e^{t^2} + C_1$

$\Rightarrow \varphi_0 = \left(t - \frac{1}{2} e^{t^2} + C_1\right) \cdot e^{-t^2} = \boxed{t e^{-t^2} - \frac{1}{2} + C_1 e^{-t^2}}$

$x(t) = x_0 + \varphi_0 = t e^{-t^2} - \frac{1}{2} + C e^{-t^2}$

$x(0) = -\frac{1}{2} + C = 1 \Rightarrow C = 1 + \frac{1}{2} = \frac{3}{2}$

$x_{PC} = t e^{-t^2} - \frac{1}{2} + \frac{3}{2} e^{-t^2}$

Exerciţiu 03

b) - Video

③ b) $x' = \frac{2tx}{3t^2 - x^2}$

$x' = \frac{\frac{2tx}{t^2}}{\frac{3t^2}{t^2} - \frac{x^2}{t^2}} = \frac{2 \cdot \frac{x}{t}}{3 - \left(\frac{x}{t}\right)^2} \Rightarrow \boxed{f\left(\frac{x}{t}\right) \text{ (ec. de tip omogen)}}$

$\boxed{u = \frac{x}{t}} \Rightarrow x = u \cdot t \Rightarrow x' = u' \cdot t + u$

$u' t + u = \frac{2u}{3 - u^2}$

$u' t = \frac{2u}{3-u^2} - u = \frac{2u - 3u + u^3}{3 - u^2} = \frac{u^3 - u}{3 - u^2}$

$\frac{du}{dt} \cdot t = \frac{u^3 - u}{3 - u^2}$

$\frac{3 - u^2}{u^3 - u}\,du = \frac{1}{t}\,dt$

$\int \frac{3-u^2}{u^3-u}\,du = \int \frac{1}{t}\,dt$

$\int \frac{3-u^2}{u^3-u}\,du = \int \frac{3-u^2}{u(u^2-1)}\,du = \int \frac{3-u^2}{u(u-1)(u+1)}\,du$

$\frac{3-u^2}{u(u-1)(u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1} \quad \Big| \cdot u \quad \Big| \cdot (u-1) \quad \Big| (u+1)$

$\frac{3-u^2}{(u-1)(u+1)} = A + \frac{Bu}{u-1} + \frac{Cu}{u+1} \quad (u = 0 \Rightarrow A = -3)$

$\frac{3-u^2}{u(u+1)} = \frac{A(u-1)}{u} + B + \frac{C(u-1)}{u+1} \quad (u = 1 \Rightarrow B = 1)$

$\frac{3-u^2}{u(u-1)} = \frac{A(u+1)}{u} + \frac{B(u+1)}{u-1} + C \quad (u = -1 \Rightarrow C = 1)$

$\int \frac{3-u^2}{u^3-u}\,du = \int \left(\frac{-3}{u} + \frac{1}{u-1} + \frac{1}{u+1}\right) du = -3\ln|u| + \ln|u-1| + \ln|u+1|$

$\Rightarrow \ln\left|\frac{u^2-1}{u^3}\right| = \ln|t| + \mathcal{C} = \ln|t| + \ln C \Rightarrow \frac{u^2-1}{u^3} = C \cdot |t|$

sol. în formă implicită $\boxed{\frac{\left(\frac{x}{t}\right)^2 - 1}{\left(\frac{x}{t}\right)^3} = C \cdot |t|}$

e)

③ e) $t x x' - x^2 + 3t^2 = 0 \quad |: tx$

$$x' - \frac{x^{\cancel{2}}}{t\cancel{x}} + \frac{3t^{\cancel{2}}}{\cancel{t}x} = 0$$

$$x' - \frac{x}{t} + 3\frac{t}{x} = 0$$

$$x' = \underbrace{\frac{x}{t} - \frac{3}{\frac{x}{t}}}_{f\left(\frac{x}{t}\right)} \quad \text{– (ec. de tip omogen)}$$

$$\boxed{u = \frac{x}{t}} \Rightarrow x = u\cdot t \Rightarrow x' = u't + u$$

$$u't + \cancel{u} = \cancel{u} - \frac{3}{u}$$

$$\frac{du}{dt}\, t = -\frac{3}{u} \quad \text{(ec. cu var. separabile)}$$

$$u\,du = \frac{-3}{t}\,dt$$

$$\int u\,du = -3\int \frac{1}{t}\,dt$$

$$\frac{u^2}{2} = -3\ln|t| + C$$

$$u^2 = -6\ln|t| + C$$

$$u = \pm\sqrt{-6\ln|t| + C} \ ; \quad -6\ln|t| + C \geq 0$$

$$\frac{x}{t} = \pm\sqrt{-6\ln|t| + C}$$

$$x_{1,2} = \pm t\sqrt{-6\ln|t| + C}.$$

Exerciţiu 04

a)

④ a) $(t - 2x + 5)\,dt + (2t - x + 4)\,dx = 0$

$$(2t - x + 4)\,dx = (-t + 2x - 5)\,dt$$

$$\frac{dx}{dt} = \frac{-t + 2x - 5}{2t - x + 4} = f\left(\frac{a_1 t + b_1 x + c_1}{a_2 t + b_2 x + c_2}\right)$$

$$\Delta = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1 - 4 = -3 \neq 0$$

$$\begin{cases} -t + 2x = 5 \\ 2t - x = -4 \quad |\cdot 2 \end{cases} \Leftrightarrow \begin{cases} -t + 2x = 5 \\ 4t - 2x = -8 \end{cases}$$

$$3t = -3$$

$$\boxed{t_0 = -1} \qquad \boxed{x_0 = 2}$$

$$\boxed{\begin{aligned} x &= u + 2 \\ t &= z - 1 \end{aligned}}$$

$$\frac{du}{dz} = \frac{-(z-1) + 2(u+2) - 5}{2(z-1) - (u+2) + 4} = \frac{-z + \cancel{1} + 2u + \cancel{4} - \cancel{5}}{2z - \cancel{2} - u - \cancel{2} + \cancel{4}}$$

$$= \frac{-z + 2u}{2z - u}^{\,:z} = \frac{-1 + 2\frac{u}{z}}{2 - \frac{u}{z}} = g\left(\frac{u}{z}\right)$$

$$\boxed{v = \frac{u}{z}} \Rightarrow u = v\cdot z \Rightarrow u' = v'z + v$$

$$v'z + v = \frac{-1 + 2v}{2 - v} \Rightarrow v'z = \frac{-1 + 2v}{2 - v} - {}^{2-v)}v$$

$$v'z = \frac{-1 + \cancel{2v} - \cancel{2v} + v^2}{2 - v} = \frac{v^2 - 1}{2 - v}$$

$$\frac{dv}{dz} = \frac{v^2 - 1}{2 - v}$$

$$\frac{2 - v}{v^2 - 1}\,dv = dz \Rightarrow \int \frac{2 - v}{v^2 - 1}\,dv = \int dz$$

$$\int \frac{2-v}{v^2-1}\,dv = \int \frac{2}{v^2-1}\,dv - \frac{1}{2}\int \frac{2v}{v^2-1}\,dv$$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}}\ln\left|\frac{v-1}{v+1}\right| - \frac{1}{2}\ln|v^2-1|$$

$$\Rightarrow \ln\left|\frac{v-1}{v+1}\right| - \frac{1}{2}\ln|v^2-1| = t + C \quad \Big/ \cdot 2$$

$$2\ln\left|\frac{v-1}{v+1}\right| - \ln|v^2-1| = 2t + C$$

$$\ln\left|\frac{v-1}{v+1}\cdot\frac{1}{v^2-1}\right| = 2t + C$$

$$\frac{1}{|v+1|^2} = e^{2t+C} = C\cdot e^{2t}$$

$$\frac{1}{\left(\frac{u}{t}+1\right)^2} = C\cdot e^{2t}$$

$$\frac{1}{\left(\frac{x-2}{t+1}+1\right)^2} = C e^{2t}$$

(sol. în formă implicită)