Gerrymandering for Primaries: Redistricting with Endogenous Candidates Selection[*]

Paola Moscariello[†]

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Abstract

I study optimal redistricting when candidates' policy positions and voters' behavior respond endogenously to district composition. I show that while standard gerrymandering strategies can backfire under these conditions, accounting for this endogeneity makes gerrymandering an even more potent tool for manipulating electoral outcomes. Using an optimal transport representation, I characterize the ideal gerrymandering plan, which strategically creates wedges between moderate and extreme opponents, fostering the emergence of extreme candidates. This approach enables gerrymanderers to exploit shifts in voter behavior following primary elections, turning their rivals' diversity into a liability. The model predicts that optimal gerrymandering can exacerbate polarization in the U.S. House of Representatives by generating significant ideological gaps among elected representatives, even without creating homogeneous districts. Furthermore, I find that while mandated "majority-minority" districts may hinder parties from exploiting endogeneity, "minority opportunity" districts could potentially trigger adverse voting patterns among non-minority voters.

Keywords: Gerrymandering, Optimal Transport, Polarization.

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[†]Department of Economics, Princeton University, pm27@princeton.edu.

1 Introduction

Partisan gerrymandering is the deliberate manipulation of electoral district boundaries to create an unfair advantage for a particular political party. This practice has gained prominence due to the increasing geographical polarization of the American electorate, with Democrats concentrated in urban areas and Republicans in rural regions. This self-sorting phenomenon allows political strategists to predict voters' preferences based on their location with a high degree of accuracy. The advent of sophisticated computer-assisted districting tools has further exacerbated the issue, enabling political parties to craft highly effective redistricting plans. A notable example is the Republican Party's Redistricting Majority Project (REDMAP) in 2012. Despite Republican candidates receiving 1.4 million fewer votes than their Democratic counterparts in House elections, REDMAP's strategic redistricting efforts resulted in a 33-seat majority for the Republicans (Daley, 2020).

This paper studies optimal redistricting when candidates' policy positions and, consequently, voters' behavior, respond to the districts in place. To understand the implications of such an assumption, it is useful to recall the classical model of redistricting. Consider Figure 1. There's a U.S. state with a finite population of voters, two-thirds of which are Democrats (in blue) and one-third are Republicans (in red). A Republican gerrymanderer has to partition voters into three equipopulous districts, so as to maximize the number of districts with a Republican majority. She applies the standard "pack-and-crack" technique. That is, she creates two "cracked" districts, District 1 and 2, consisting of just enough Republicans to have a majority, and one "packed" district, made up of only Democrats. Thanks to gerrymandering, Republicans win two-thirds of districts with just one-third of overall Republican supporters.

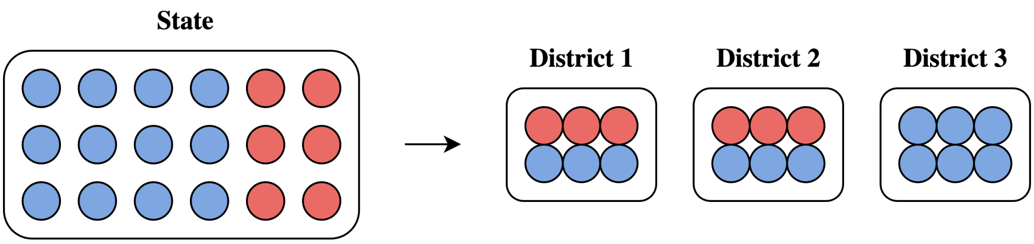


Figure 1: Standard pack-and-crack redistricting.

Now suppose that voters' behavior is endogenous to the redistricting itself. Consider Figure 2. In the state, a small fraction of Democrats and an even smaller fraction of Repub-

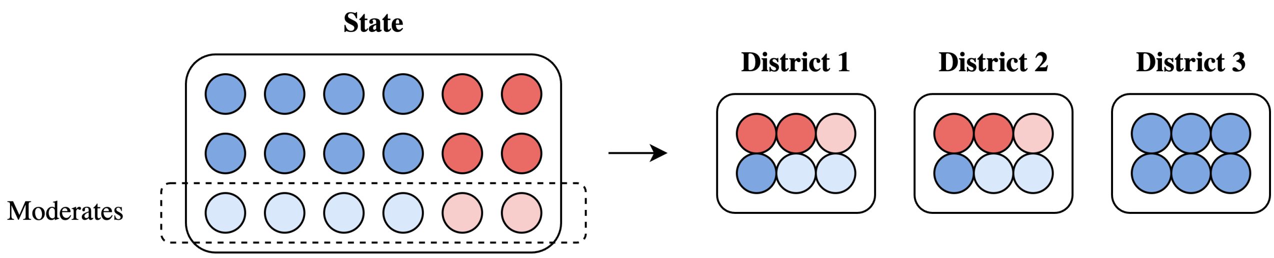


Figure 2: Pack-and-crack backfires.

licans are moderates, represented by less intense shades of blue and red respectively. The majority of voters are partisans (or extremists). Districts use two-stage elections to select representatives. During party primaries, either a moderate or extreme candidate emerges, depending on which voter type constitutes the majority within their party. In general elections, while partisans consistently vote for their party's candidate, moderate voters prefer a moderate candidate from the opposing party over an extreme candidate from their own party. Under these conditions, the standard "pack-and-crack" gerrymandering strategy can backfire. For example, consider District 1 and District 2 in Figure 2. In both districts, there is a majority of partisans among Republicans and a majority of moderates among Democrats. Hence, the Republican candidate caters to partisans, while the Democratic candidate caters to moderates. The only moderate Republican voter prefers voting for the moderate Democratic candidate rather than for a partisan Republican. Consequently, the gerrymandering attempt backfires, and Republicans fail to win any districts.

A real-life equivalent of such "dummymandering" can be found in the case of Oregon's fifth district: after Democrats redrew boundaries in anticipation of 2022 mid-term elections, progressive Jamie McLeod-Skinner unexpectedly defeated seven-term centrist incumbent Kurt Schrader in the Democratic primary. Crucially, McLeod-Skinner's victory hinged on a forty-point advantage in Deschutes County, which was only added to the district through the recent redistricting. This shift allowed Republican Lori Chavez-De Remer to appeal to moderate voters and flip the district for the first time since 1994 (Flaccus, 2022; Glueck, 2022; Scott and Weigel, 2022).

This example demonstrates how ignoring voters' preference intensities during redistricting can inadvertently convert moderate Republican voters into (moderate) Democratic voters, ultimately undermining the intended gerrymandering effects.

Does this mean that gerrymandering is less powerful than previously thought? Quite the opposite. Consider a scenario where a Republican redistricter is aware of and exploits the endogeneity of voter behavior. Figure 3 illustrates such a gerrymandering plan. In District 1, despite having a Democratic majority, the composition is heterogeneous. There are sufficient Democratic partisans so that the Democratic candidate cater to their preferred policy, while the two moderate Democrats align more closely with the Republican candidate, who cater to the single moderate Republican voter in the district. Consequently, moderate Democrats prefer to vote Republican, even though they would have voted Democratic in a district composed solely of moderate Democrats. As a result, Republicans win this district. A similar dynamic plays out in District 2, leading to another Republican victory. District 3 consists of a majority of partisan Republicans, ensuring a straightforward Republican win. By strategically considering the endogeneity of candidates' selection and voters' behavior, the redistricter manages to secure all three districts for the Republican party, despite Republicans constituting only one-third of the overall population.

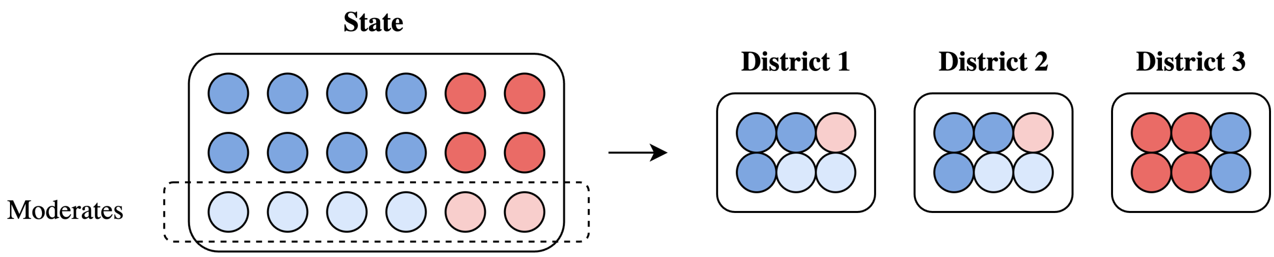


Figure 3: Gerrymandering is even more powerful than previously thought.

This paper aims to formalize the intuitions illustrated so far by incorporating a two-stage district election process into a standard optimal gerrymandering problem. I demonstrate how gerrymandering efforts can be undermined when primary voters favor extreme candidates over centrists. Leveraging an optimal transport (Monge, 1781; Kantorovich, 1942) representation of my problem, I derive a comprehensive characterization of the ideal gerrymandering plan. This characterization illuminates how savvy redistricting parties can capitalize on shifts in voter behavior after primary candidates selection. Their strategy? Crafting district boundaries that drive a wedge between moderates and extremists in the opposing party, effectively turning their rivals' diversity into a liability.

The model. There is a continuum of voters with single-peaked preferences over a unidimensional policy space. Voters with bliss points above an exogenous cutoff are Republicans, while voters below the cutoff are Democrats. A Republican designer partitions voters into equipopulous districts to maximize the number of districts won by Republicans. Given that redistricting in the U.S. takes place roughly every ten years, it is conceivable that gerrymanderers cannot perfectly anticipate future populations' preferences at the time of redistricting. Therefore, I allow the designer to face uncertainty about voters' preferences, parametrized by a one-dimensional aggregate shock.

The model incorporates a two-stage election process at the district level to determine voting behavior. First, in the primary elections, Democratic and Republican voters separately select their party candidates. Then, in the general election, all district voters choose between the two primary winners. Drawing on the work of Owen and Grofman (2006), I model electoral incentives so that primary candidates position themselves to appeal to their respective party medians[1]. In districts lacking Republican or Democratic voters, the corresponding party's candidate defaults to the most moderate position within their ideological spectrum. The general election outcome is decided by the district's median voter. The candidate whose position most closely aligns with the district median emerges victorious.

When devising an optimal plan, the designer needs to take into account three objects for each district: the median and the two party medians. This constitute a novel technical challenge relative to standard models. In fact, when voting behavior is exogenous, the designer only needs to keep track of district medians, and ensure that as many as possible end up being on the Republican spectrum. This reduced complexity transforms the problem into a variation of Bayesian persuasion (Kamenica and Gentzkow, 2011) for medians, which can be solved with recent off-the-shelf tools (Yang and Zentefis, 2024).

Results. My model captures gerrymandering's direct impact on voter incentives, necessitating a departure from standard information design techniques. In the first part of my paper, I reframe the redistricting problem as a matching problem of voter types above and below the population median.

Intuitively, the designer hedges against voter preference uncertainty by creating districts with exactly two distinct voter types in equal proportion. This approach enables two winning strategies. The first is the standard "cracking" strategy, where at least one of the voter

[1] In the Appendix, I provide microfoundations. While such modeling choice appears sensible to me, it is important to note that strict dependence on party medians is not necessary for my results. The driving force behind the model is responsiveness of candidates to party preferences, whatever be the rule. In Section 6, I present an extension where candidates position at party quantiles different than medians.

types is Republican. The second is a hybrid "cracking" and "packing" strategy, where both voter types are Democrats. These districts are "packed" in the sense of containing only Democrats, but "cracked" with respect to within-party preferences. If the gap between the two Democratic voter types is sufficiently large, the more moderate one may prefer voting Republican.

The larger the wedge between voter types, the higher the probability of winning the district. Thus, districts with a significant gap between voter types are "safer" than more homogeneous ones. While the designer would ideally maximize this gap for all districts, she is constrained by the overall distribution of voter types. Essentially, her problem is that of deciding which voter types to allocate to safer, more heterogeneous, districts, and which to less safe, more homogeneous, districts.

Technically, this constitutes a Monge-Kantorovich optimal transport problem, optimizing over joint distributions with marginals representing the distribution of voter types conditional on being above or below the median. The solution to this problem depends on the distribution of uncertainty about voter preferences. Off-the-shelf results from optimal transport literature prove unhelpful, except in particular cases such as when the density of the shock is monotone.

The second part of my paper characterizes the solution under the assumption that the aggregate shock is s-shaped around zero. I demonstrate that there's a unique optimal redistricting plan, and it is a convex combination of two matching strategies. For voters in the distribution's tails ("outside" voters), matching occurs in a "positive assortative manner", allocating extremists in the same district as moderates. Conversely, for voters closer to the median ("inside" voters), matching follows a "negative assortative manner", allocating extremists with extremists and moderates with moderates.

Intuitively, this dual strategy serves distinct purposes. In districts more likely to be won, the designer aims to maintain a roughly constant gap in within-district voter types. This approach allows for equal risk distribution across districts, hence the positive assortative matching. For districts likely to be lost, the designer's goal shifts to damage control. Negative assortative matching allows to achieve this goal by creating districts with significant disparities in safety levels.

Under the additional assumption that the shock is symmetric around zero, I derive the equivalent of a first-order condition for my setup. This condition intuitively states that the optimal plan must maximize the number of districts exhibiting positive assortative matching. While the structure of such solution is independent of the original distribution of voters, comparative statics show that increasing (decreasing) the fraction of Republicans in the population decreases (increases) the number of negative assortative matches.

In Section 6, I extend the model to consider a scenario where candidates' positions are determined by a general quantile of the preference distribution within each party, rather than just the median. For the case of a uniform distribution of voter preferences, I show that the optimal plan involves a three-wise positive assortative matching. This result demonstrates that the strategic creation of wedges between voter types remains crucial even under alternative candidate positioning rules, though the specific implementation of the strategy adapts to the new context.

Political and legal implications. In Section 5, I explore implications of my gerrymandering model that are pertinent to current political and legal debates. First, I examine how optimal gerrymandering strategies can exacerbate polarization in the U.S. House of Representatives, even without creating homogeneous districts. The model predicts a notable ideological gap among elected representatives, with moderate supporters of the redistricting party at one end and extreme opponents at the other. Second, I analyze the legislative implications for "majority-minority" districts, mandated by federal law to ensure minority representation. The analysis suggests these districts may impede both Democrats and Republicans from exploiting their opponents' diversity for electoral gain. However, in "minoirity opportunity" districts where minorities comprise $40-50\%$ of the population, the possibility of "white backlash" becomes tangible. In these districts, the success of minority candidates hinges on white voters' willingness to support them, potentially benefiting Republican candidates as outlined in my model.

Outline of the Paper. The paper is organized as follows: After reviewing relevant literature, Section 2 provides intuitive examples illustrating the model's key mechanisms. Section 3 presents the formal model and shows how it can be reframed as an optimal transport problem. Section 4 characterizes the solution, while Section 5 explores the political and legal implications of gerrymandering. Section 6 examines model extensions, and Section 7 concludes with a comparison to existing literature and a discussion of empirical implications.

1.1 Related Literature

Optimal partisan gerrymandering. This paper primarily relates to the economic theory literature on optimal partisan gerrymandering, starting with Owen and Grofman (1988). While their work focuses on a binary voter type model, subsequent research has emphasized the importance of considering voters' preference intensities in redistricting strategies. Most papers incorporate this nuance by introducing aggregate and/or idiosyncratic uncertainty to voter preferences, typically modeling extreme voters as more partisan and moderate voters as more susceptible to swings. Friedman and Holden (2008) examine a scenario where aggregate uncertainty significantly outweighs idiosyncratic uncertainty. Their model predicts an optimal strategy of allocating extreme supporting voters to the same districts as extreme opposing voters, effectively neutralizing the latter's influence. The authors term this approach "matching slices" (on either side of the preference distribution), which can be interpreted as a form of negative assortative matching. Kolotilin and Wolitzky (2024) develop a more comprehensive model of gerrymandering that does not impose restrictions on the relative magnitudes of aggregate and idiosyncratic uncertainty. Their findings suggest that when idiosyncratic uncertainty dominates, the optimal strategy involves segregating the most extreme opposing voters while matching the remaining voters in a negative assortative manner. Importantly, their empirical analysis indicates that scenarios where idiosyncratic uncertainty outweighs aggregate uncertainty are most relevant in practice.

I contribute to this literature by allowing individual voters' behavior to depend on the districts in place. One implication of this approach is that it results in a different treatment of extreme opposing voters. While previous literature either predicts such extreme voters to be segregated or matched with extreme supporters, my model exploits them to turn moderate voters against their own party. To do so, the optimal plan prescribes to "mis-match slices" (on either side of the preference distribution) resulting in novel political and legal implications, as I discuss in Section 5.

The literature also explores competitive scenarios in gerrymandering. Gul and Pesendorfer (2010) examine competition between two parties, each controlling redistricting in distinct areas. Building on their earlier work, Friedman and Holden (2020) extend their 2008 model to consider competing designers.

Information design. As Kolotilin and Wolitzky (2024) show,[2] the gerrymandering problem can be mapped onto an information design problem. The distribution of voter preferences serves as the "prior," districts function as "posteriors," and a redistricting plan represents a distribution of districts that satisfies a constraint mathematically equivalent to Bayes plausibility. Indeed, in the special case of exogenous policies, my model becomes a variant of Bayesian persuasion (Kamenica and Gentzkow, 2011) for medians, which can be solved with recent off-the-shelf tools (Yang and Zentefis, 2024), as I show in Section 3. My solution sheds light on information design problems where payoffs depend on more intricate aspects of posterior distributions, such as the relative positions of conditional medians, rather than single summary statistics like means or medians.

Optimal transport. To characterize the solution, I leverage a Monge-Kantorovich optimal transport (Monge, 1781; Kantorovich, 1942) representation of my model. Drawing on results from Chiappori et al. (2010) and Santambrogio (2015), I establish the existence and uniqueness of the solution and characterize it for key benchmark cases. My contribution to this literature lies in characterizing the solution to an optimal transport problem where the surplus function is symmetric and s-shaped in a linear function of the arguments.

Applications of optimal transport to economic theory are becoming more and more popular. Boerma et al. (2023) is an example in the context of labor markets with concave mismatch costs. Like in my paper, they find a non-pure solution to their optimal transport problem. In the context of gerrymandering, Kolotilin and Wolitzky (2024) rely on their companion paper, Kolotilin et al. (2023), and show how their problem can be connected to a constrained version of optimal transport, called martingale optimal transport (MOT).

Two-stage elections models. I consider a model of two-stage elections at the district level, similar to Coleman (1971) and Owen and Grofman (2006), in which voters in primary elections are unsure about the position of the median voter in the district.

Other topics in gerrymandering. The broader literature on gerrymandering tackles a variety of different issues. The effects of redistricting on policy choice is considered by Shotts (2002) and Besley and Preston (2007), while the impact of gerrymandering on polarization in the House of Representatives is addressed, for instance, by McCarty et al. (2009). Other

[2] Lagarde and Tomala (2021) and Gomberg, Pancs, and Sharma (2023) also make this point less generally.

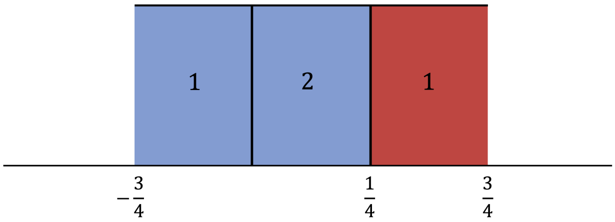


Figure 4: Optimal redistricting with exogenous supporters, matching slices.

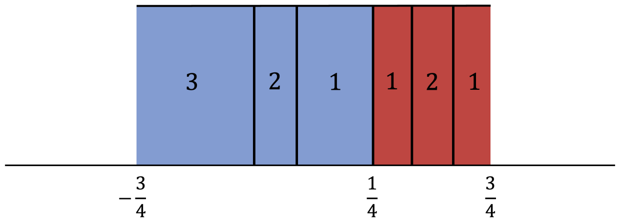


Figure 5: Alternative optimal redistricting with exogenous supporters.

important topics in redistricting that I do not explore in this paper relate to: including geographic constraints on gerrymandering (Puppe and Tasnádi, 2009), accounting for differential voter turnout (Bouton et al., 2023), and measures of electoral maldistricting (Gomberg et al., 2023).

2 Examples

In order to illustrate the mechanisms at the heart of this paper, I provide some stylized examples. Section 3 and Section 4 will be entirely devoted to formalizing the intuition explained in the current section.

2.1 Example 1: Exogenous Policies

Consider a population of voters with single-peaked preferences and ideal points distributed uniformly on the interval $\left[-\frac{3}{4}, \frac{3}{4}\right]$. Call Republicans those voters to the right of $\frac{1}{4}$. Call Democrats those voters to the left of $\frac{1}{4}$. Suppose there is a Republican redistricter who wants to allocate the population into equal-sized districts in order to maximize the number of districts won by Republican candidates. Consider the case of exogenous policies: the Republican and Democratic candidates position themselves at $\frac{3}{4}$ and at $-\frac{1}{4}$, respectively, independently of the district. Then, in any district, the Republican candidate wins if and only if the fraction of voters above $\frac{1}{4}$ is at least $\frac{1}{2}$. The optimal gerrymander will create as many "cracked" districts as possible, comprising half Republicans and half Democrats, and "pack" the remaining Democrats in districts made up of Democratic voters only. In Figure 4, one such possibility is outlined: the redistricter creates $\frac{2}{3}$ of districts of type 1, matching

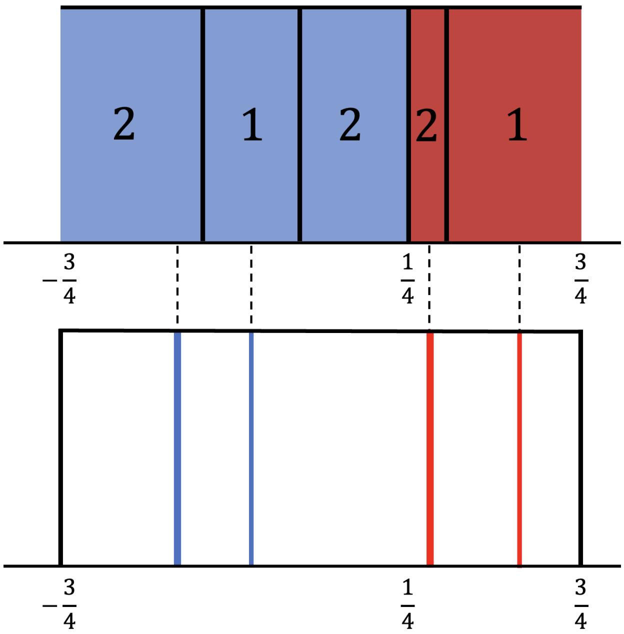


Figure 6: Optimal redistricting with endogenous supporters (top); candidates' positions (bottom).

all the Republicans to half of the Democrats, the most extreme ones, and $\frac{1}{3}$ of districts of type 2, packing moderates. In total, Republicans win $\frac{2}{3}$ of districts. This kind of plan is called "matching slices", because the redistricter neutralizes extreme Democrats by matching them with extreme Republicans, while moderate Democrats are packed together.

Figure 5 depicts another optimal redistricting plan, featuring three types of districts, and where extreme Democrats, rather than moderate Democrats, are packed. If the redistricter can distinguish Republicans from Democrats, but can not tell voters apart within the parties, the plan depicted in Figure 4 and the one in Figure 5 are indistinguishable to her.

2.2 Example 2: Endogenous Policies

Consider the same situation as in the previous example, but imagine that Republican (Democratic) candidates locate themselves at the district median of Republicans (Democrats). The redistricting plan outlined in Figure 4, where "slices" match masses from opposite tails of the distribution, is not optimal anymore. In particular, consider the following profitable deviation: first, take the left boundaries of district type 1 and slide them to the right, switching some moderates in district type 2 with some extremists in district type 1; second, separate moderate Republicans from extreme Republicans into different districts. The outcome of this process is depicted in Figure 6. In particular, the top picture in Figure 6 illustrates the distribution of voters and the redistricting plan, while the bottom picture in Figure 6 shows the canditates' positions in the different district types. In the bottom panel, the relative thickness of a line relative to another represents the relatively higher proportion of candidates with one position relative to another. District type 1 is made up of $\frac{1}{2}$ Republicans, the right tail of the distribution, and $\frac{1}{2}$ Democrats. Republicans win all districts of type 1 by the standard pack-and-crack principle outlined in the previous example. District type 2 is made up by a majority of Democrats, half extremist and half moderate, and some moderate Republicans. However, since the Democratic candidate is an extremist, there are enough moderate Democrats voting for the Republican candidate, a moderate, to have the Republicans win this type of district as well. Hence, with this plan, Republicans win all the districts. Note that this kind of plan requires "mis-matching tails". In particular, districts of type 2 are created by allocating moderate and extreme Democrats to the same district, creating a gap between them and encouraging the emergence of an extreme Democratic candidate.

Now consider the plan depicted in Figure 5, reported in Figure 7, with the addition of candidates' positions. In this case, not only the plan is not optimal anymore, but it backfires: in districts of type 1, the Republican candidate is too extreme, and some moderate Republicans prefer voting for the Democratic candidate. Since the districts are very competitive by design, this is sufficient for Democrats to win the district.

2.3 Example 3: Distribution of Candidates' Positions

Consider the same setup as in the previous examples, but with a slightly different distribution: a mass $\frac{1}{2}$ of voters is distributed uniformly between $-\frac{4}{5}$ and 0 and the remaining mass is distributed uniformly between 0 and $\frac{8}{5}$. Republicans constitute the small mass of voters with types above 1, while all remaining voters are Democrats. Again, in any district, Republican and Democratic candidates locate themselves at their respective party medians. The designer needs to create a redistricting plan in order to maximize the number of elected Republican representatives. In this case, the number of Republicans is even smaller than in the previous examples, and the designer wants to apply the principle of mis-matching tails, presented in the previous example, to win as many districts as possible. That is, she wants to create as many districts as possible so that there's a gap between "moderate" and "extreme" Democrats, encouraging the emergence of extreme Democratic candidates, so that moderate

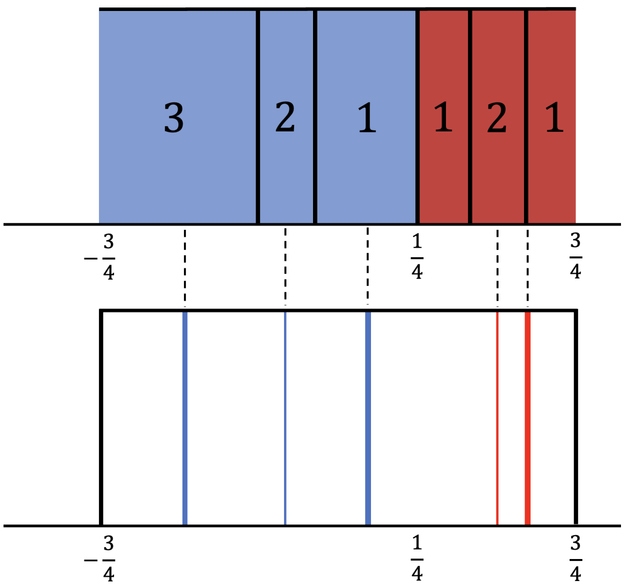


Figure 7: Optimal redistricting with exogenous supporters(top); candidates' positions (bottom).

Republicans have higher chances of winning general elections. She comes up with the redistricting plan in Figure 8. The interesting part of Figure 8 is the bottom picture, depicting candidates' positions: Democratic candidates take positions in $\left[-\frac{4}{5}, -0\right]$, while Republican candidates take positions in $\left[1, \frac{8}{5}\right]$. No candidate arises in the middle of the distribution, the mass in $[0, 1]$. Hence, the winning district representative will be either a Republican, with position $v > 1$, or an "extreme" Democrat, with position $v \leqslant 0$. Unless one party wins all districts, the distribution of district representatives is bimodal. The intuition is straightforward: the party in charge of redistrcting, the Republican party, has a minority of supporters, so it needs to dilute the power of its opponents. Specifically, it needs to dilute the power of its moderate opponents. That's because moderate Democrats constitute the median of the population distribution, the quantile with greatest power in majority-rule-based elections. The optimal plan eviscerates the middle of the distribution into small parts, which are then merged to the lower and upper tails of the distribution.

These examples suggest some take-aways. First, whenever the redistricter can geographically identify extremists and moderates within parties, she can increase the expected number of districts won by one party. The optimal plan pairs extremists to moderates and moderates to extremists, mis-matching tails. Second, when the population is geographically sorted across parties, but not within parties, so that the redistricter can tell Republicans

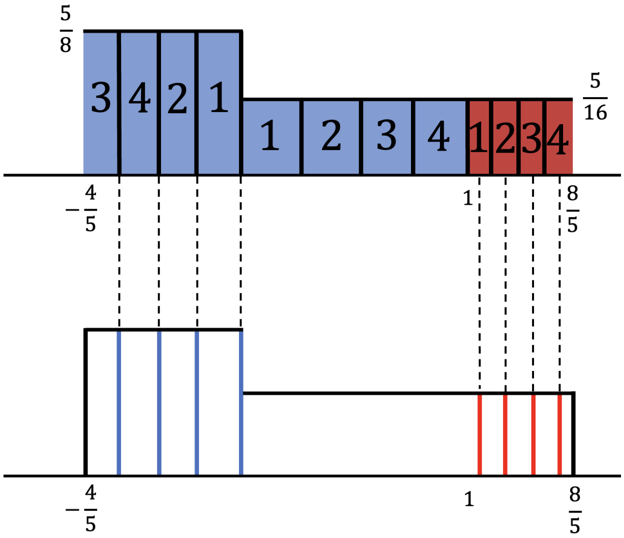


Figure 8: Distribution of voters (top); third quartile of the Democrats, in blue, and first quartile of the Republicans, in red (bottom).

from Democrats, but not extremists from moderates, knife-edge pack-and-crack strategies can backfire. In particular, a cracked district that should elect, say, a Republican, might instead elect a Democrat, if the proposed Republican candidate is too extreme. Third, optimal redistricting entails an endogenous preference for extremism. The designer engineers the election of Democratic extremists who run against Republican moderates, so as to capture as many moderate Democrats as possible. The result is that median candidates are often primaried out in favor of extreme candidates, and the distribution of district representatives has a gap in the midlle.

The rest of the paper formally shows that the intuitions highlighted in these examples hold true generally.

3 The model

3.1 Setup and Statement of the Problem

Voters and Parties. Consider a continuum of voters with single-peaked preferences over uni-dimensional policy space $[\underline{v}, \overline{v}]$, a closed interval on the real line $\mathbb{R}$. A voter's ideal point, sometimes referred to as her type, is denoted by $v \in [\underline{v}, \overline{v}]$, with population distribution $\phi \in \Delta([\underline{v}, \overline{v}])$[3], assumed to have a strictly increasing and continuously differentiable cumulative distribution function (CDF), F, on the interior of $[\underline{v}, \overline{v}]$. I normalize the median of F to be $v^m = 0$. Moreover, I assume voters' preferences to be symmetric around their ideal points[4].

There are two parties, the Democratic and the Republican party. Party affiliation is determined by a threshold $k \in [\underline{v}, \overline{v}]$. In particular, I call Republicans the voters such that $v \geqslant k$ and Democrats the voters such that $v < k$.

Gerrymandering. A Republican designer, or redistricter, is in charge of creating equipopulous districts so as to maximize his party's seat share. She allocates voters among a continuum of districts based on their type v, thus determining the distribution $\pi \in \Delta([\underline{v}, \overline{v}])$, with CDF π, of voter types within a district. A redistricting plan $\mathcal{H} \in \Delta(\Delta([\underline{v}, \overline{v}]))$[5] is, therefore, a distribution over distributions: it specifies the measure of districts with each distribution π of voter types. To satisfy the equipopulous requirement typical of gerrymandering, any redistricting plan must be such that the following budget constraint holds[6]:

$$\int \pi d\mathcal{H}(\pi) = \phi. \tag{BC}$$

For instance, uniform redistricting imposes all districts to be the same, thus $\mathcal{H}(\phi) = 1$, while perfect segregation imposes that each voter type $v \in [\underline{v}, \overline{v}]$ constitutes a district on their own.

Vote Shares. In any district with distribution $\pi \in \text{supp}(\mathcal{H})$[7], two candidates emerge, a Democratic candidate (D) and a Republican candidate (R). Ultimately, and given the sym-

[3]For any complete and separable metric space X, I let $\Delta(X)$ denote the set of Borel probability measures on X, endowed with the weak* topology.

[4]For instance, a voter type v's preference for policy $x \in [\underline{v}, \overline{v}]$ could be represented by utility function $U(x; y, t) = -|x - v|$.

[5]Note that $\Delta([\underline{v}, \overline{v}])$ is metrizable as a complete and separable metric space with the weak* topology.

[6]The integral sign is used to denote the Lebesgue integral as defined in Aliprantis and Border (1994).

[7]Throughout, $\text{supp}(\cdot)$ denotes the support of a probability measure, defined as the set of all points whose every open neighborhood has positive measure.

metry assumption on voters' preferences, each voter chooses the candidate whose policy position is closer to their ideal point. The designer wins a district if and only if she receives a majority of the district vote. Suppose there is an exogenous rule that determines the positions of candidates and how they respond to preferences in the district's population. In particular, suppose that D and R locate at the lowest median of $\pi(\cdot|v<k)$[8](the median of Democratic affiliates) and at the highest median of $\pi(\cdot|v \geqslant k)$ (the median of Republican affiliates)[9], respectively. There are various reasons why primary candidates might cater to their respective party medians rather than converge to the overall district median (e.g. voter myopia). In the Appendix, I show that this can be justified via a model of primary elections, where both voters and candidates are uncertain about the position of the population median (Owen and Grofman, 2006). However, it is important to note that the strict dependence on party medians is not necessary for my results to go through. For instance, the results are robust to any alternative rule that determines the position of candidate D (respectively, R) through a linear combination of the median of Democratic (respectively, Republican) affiliates and the population median.

To close the model, I assume that whenever $\text{supp}(\pi) \subseteq [k,\infty)$ (respectively, $\text{supp}(\pi) \subseteq (-\infty,k]$), D (respectively, R) takes position k. This assumption is equivalent to stating that even in a district with a very high Democratic (respectively, Republican) majority, there is always an arbitrarily small fraction of moderate Republicans (respectively, Democrats).

Formally, call $c_{\pi,D}$ and $c_{\pi,R}$ the position taken by D and R in district π. Then:

$$(c_{\pi,D}, c_{\pi,R}) = \begin{cases} (v^m_{\pi,D}, k) & if \ \text{supp}(\pi) \subseteq (-\infty, k] \\ (k, v^m_{\pi,R}) & if \ \text{supp}(\pi) \subseteq [k,\infty) \\ (v^m_{\pi,D}, v^m_{\pi,R}) & otherwise \end{cases},$$

where:

$$v^m_{\pi,D} = \inf_a \left\{a : \ \pi([a,\overline{v}]|v<k) \geqslant \frac{1}{2}\right\} \cap \left\{a : \ \pi([\underline{v},a]|v<k) \geqslant \frac{1}{2}\right\}$$
$$v^m_{\pi,R} = \sup_a \left\{a : \ \pi([a,\overline{v}]|v \geqslant k) \geqslant \frac{1}{2}\right\} \cap \left\{a : \ \pi([\underline{v},a]|v \geqslant k) \geqslant \frac{1}{2}\right\}.$$

Voters choose their district representative by majority rule. Hence, the candidate who wins the elections is the one closer to the district's median. Formally, the position of the district

[8]Throughout, $M(Y|X)$ denotes the conditional probability distribution of Y given X according to measure M.

[9]Without loss of generality, the tie breaking rule is chosen so as to insure the existence of an optimum.

representative, c_π, is:

$$c_\pi = \begin{cases} c_{\pi,D} & if \;\; v_\pi^m < \frac{c_{\pi,R}+c_{\pi,D}}{2} \\ c_{\pi,R} & if \;\; v_\pi^m \geqslant \frac{c_{\pi,R}+c_{\pi,D}}{2} \end{cases}$$

where:

$$v_\pi^m = \sup_a \;\; \left\{a: \; \pi([a,\overline{v}]) \geqslant \frac{1}{2}\right\} \cap \left\{a: \; \pi([\underline{v},a]) \geqslant \frac{1}{2}\right\}.$$

The designer wins district π if the winning candidate is the Republican candidate; that is, if $c_\pi \geqslant k$. Given redistricting plan $\mathcal{H}$, the designer's vote share is:

$$\int \mathbb{1}\left(c_\pi \geqslant k\right) d\mathcal{H}(\pi) = \int \mathbb{1}\left(v_\pi^m - \frac{c_{\pi,R}+c_{\pi,D}}{2} \geqslant 0\right) d\mathcal{H}(\pi).$$

Aggregate Uncertainty. Suppose that, after the designer commits to a plan, but before candidates choose their positions, an aggregate location shock affects all voters[10]. Formally, each voter experiences a common shock $\omega \in \mathbb{R}$, so that her ideal point becomes $v-\omega$. Assume that ω has distribution $\gamma \in \Delta(\mathbb{R})$, with CDF G, assumed to be Lipschitz continuous[11] and strictly increasing on $[2\underline{v}-2\overline{v}, 2\overline{v}-2\underline{v}]$. Any shock ω induces a new preference distribution in each district. For any π, I call $\pi^\omega = \pi(v+\omega)$ such induced distribution.

Redistricter's Problem. The redistricter wishes to maximize the expected seat share won by Republicans. Her problem is:

$$\max_{\mathcal{H}\in\Delta(\Delta([\underline{v},\overline{v}]))} \int\int \mathbb{1}\left(v_{\pi^\omega}^m - \frac{c_{\pi^\omega,D}+c_{\pi^\omega,R}}{2} \geqslant 0\right) d\mathcal{H}(\pi) d\gamma(\omega) \tag{RP}$$

$$s.t. \int \pi d\mathcal{H}(\pi) = \phi.$$

A redistricting plan $\mathcal{H}$ is *feasible* if it satisfies constraint (BC). It is *optimal* if it is a solution to (RP).

[10] A shock to preferences is particularly relevant in my setup, given that redistricting opportunities usually present themselves only every ten years. In any case, my solution encompasses the case in which the shock is arbitrarily small, so as to approximate the solution in the absence of a shock.

[11] For instance, if G is everywhere continuously differentiable, it satisfies the assumptions. I merely require Lipschitz continuity to allow for G to have a set of non-differentiability points of at most Lebesgue measure zero.

It is instructive to compare (RP) to a redistricting problem with exogenous policies, whose solution can be found with off-the-shelf tools from recent literature. Suppose that candidates' positions are fixed: $c_{\pi^\omega,R} = \overline{v}$, $c_{\pi^\omega,D} = \underline{v}$. The redistricter's problem under exogenous policies is:

$$\max_{\mathcal{H}\in\Delta(\Delta([\underline{v},\overline{v}]))} \int\int \mathbb{1}\left(v^m_{\pi^\omega} \geqslant k^\star\right) d\mathcal{H}(\pi) d\gamma(\omega)$$

$$s.t. \int P d\mathcal{H}(\pi) = F. \tag{RPEx}$$

where $k^\star = \frac{\underline{v}+\overline{v}}{2}$. The object of interest to the designer is an element of $\Delta(\Delta([\underline{v},\overline{v}]))$, a distribution over distributions. However, in the case of exogenous policies, the objective function only depends on district medians. Hence, the problem can be mapped to a maximization problem over elements of $\Delta([\underline{v},\overline{v}])$, which are much simpler objects to work with. The set of feasible distributions of medians can be characterized using Theorem 2 in Yang and Zentefis (2024). The transformed problem is:

$$\max_{\chi\in\Delta([\underline{v},\overline{v}])} \int\int \mathbb{1}\left(v-\omega \geqslant k^\star\right) d\chi(v) d\gamma(\omega)$$

$$s.t. \quad \max\{2F(v)-1,0\} \leqslant X(v) \leqslant \min\{2F(v),1\}, \text{ for all } v \in [\underline{v},\overline{v}],$$

where X is the CDF associated with $\chi \in \Delta([\underline{v},\overline{v}])$. Switching the order of integration, it can be rewritten as:

$$\max_{\chi\in\Delta([\underline{v},\overline{v}])} \int G(v-k^\star) d\chi(v)$$

$$s.t. \quad \max\{2F(v)-1,0\} \leqslant X(v) \leqslant \min\{2F(v),1\}, \text{ for all } v \in [\underline{v},\overline{v}].$$

Since the problem is linear in χ, it suffices to focus attention on the extreme points of the set of feasible distributions of medians. Again, Yang and Zentefis (2024) characterize such set in Theorem 1. In my case, since G is increasing, the solution is $\chi^\star = \max\{2F-1,0\}$, which dominates all other feasible distributions of medians, in the first-order-stochastic-dominance sense. We can now recover all solutions to (RPEx) as all those redistricting plans that induce $\chi^\star$. It is easy to see that there are many such plans. The following proposition, already proven with a different argument by Kolotilin and Wolitzky (2024), describes such plans and summarizes the result for exogenous policies.

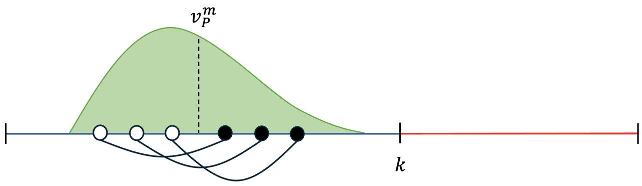


Figure 9: district π (in green) is split at its median, and types above v_π^m (the black circles) are matched to types below v_π^m (the white circles) in a positive assortative manner.

Proposition 0. *A feasible redistricting plan $\mathcal{H} \in \Delta(\Delta([\underline{v}, \overline{v}]))$ is a solution to the redistricting problem with exogenous policies (RPEx) if and only if, for all $\pi \in \operatorname{supp}\mathcal{H}$, except for at most a zero-measure subset, there exist $v_\pi \geqslant 0$ such that $\pi(\{v_\pi\}) = \pi(\{v : v \leqslant 0\}) = \frac{1}{2}$.*

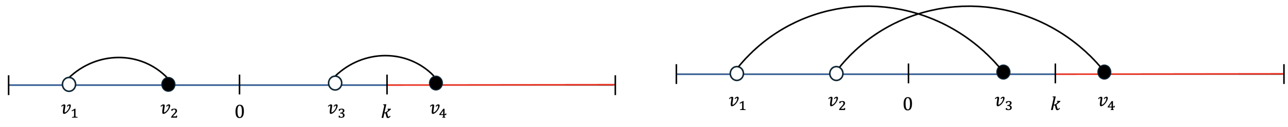
In the case of endogenous policies, it is not possible to reduce the object of interest to a distribution over a uni-dimensional space, because the objective function depends both on district medians and on candidates' positions. Nevertheless, in the next subsection, I show that the problem can still be simplified and solved using a different set of tools. In particular, I show that the object of interest can be reduced to a joint distribution over two uni-dimensional spaces, thus invoking the literature on the so-called optimal transport.

3.2 The Designer's Problem as an Optimal Transport Problem

While (RP) is a challenging problem, it can be simplified and restated as an optimal transport problem. The following result provides necessary conditions for optimality and constitutes the main ingredient for such a transformation.

Proposition 1. *A feasible redistricting plan $\mathcal{H} \in \Delta(\Delta([\underline{v}, \overline{v}]))$ is optimal only if, for all $\pi \in \operatorname{supp}(\mathcal{H})$, except for at most a zero-measure subset, there exist $v' \geqslant 0$ and $v'' \leqslant 0$ such that $\pi(\{v'\}) = \pi(\{v''\}) = \frac{1}{2}$.*

Proposition 1 constrains the set of feasible plans by requiring any candidate for optimality to allow for a continuum of districts, each with at most binary support. Moreover, each district in an optimal plan must place half the mass on a voter type above the median of F and the rest on a voter type below the median of F. Effectively, this proposition states that (RP) is a matching problem of voter types across the median of their population distribution. The proof of Proposition 1 proceeds in two steps, for which I now provide an



(a) Districts $\overline{\pi}$ and $\underline{\pi}$

(b) Alternative matching.

Figure 10: Districts must match voter types across the median of F.

intution.

First, any optimal plan can be emulated with a *pairwise* plan, a plan in which each district has at most a binary support. The intuition is that any district π can be "split" at the median and fragmented into multiple (possibly a continuum of) districts, each of them containing at most two voter types, from above and below the median of π, respectively. The trick consists in matching the highest types above the median of π to the highest types below the median of π, and so on, in a *positive assortative manner*. Figure 9 illustrates this construction. As it turns out, for each such new district $\hat{\pi}$, it holds that $\mathbb{1}(c_{\hat{\pi}^\omega} \geqslant k) \geqslant \mathbb{1}(c_{\pi^\omega} \geqslant k)$, no matter the realization of the shock ω.

Second, pairwise plans are in fact strictly beneficial. While, for some values of $\omega \in \text{supp}(\gamma)$, the designer may be indifferent between π and $\hat{\pi}$, such indifference is broken for a positive-measure subset $\Omega \subseteq \text{supp}(\gamma)$. This intuition becomes clear by looking at Figure 9. As long as $\text{supp}(\pi^\omega)$ is contained in $[\underline{v}, k]$, district π is lost. Nevertheless, there is a ω - threshold below which a positive measure of the $\bullet$ are always closer to k than they are to their $\circ$ matches. Then, such districts are won even if their support is contained in $[\underline{v}, k]$.

Finally, districts must not only be pairwise, but they must also match voter types across the median of F. Figure 10 illustrates this point. Suppose there are two districts, $\overline{\pi}$ and $\underline{\pi}$, as in Figure 10a. District $\overline{\pi}$ matches two voter types $v_4 > v_3$ that are both above zero, while $\underline{\pi}$ matches two voter types $v_2 > v_1$ that are both below zero. Consider the alternative matching shown in Figure 10b. Pairing v_1 with v_3 and v_2 with v_4 is without loss for all values of the location shock, and it is strictly advantageous for a positive-measure subset of such values.

Proposition 1 justifies the definition of a relation between the set redistricting plans and the set of joint distributions over $[\underline{v}, 0] \times [0, \overline{v}]$. The remaining of this subsection is dedicated to formalizing such relation.

Define by $\Delta_2 \subseteq \Delta(\Delta([\underline{v}, \overline{v}]))$ the set of feasible plans $\mathcal{H}$ such that, for all $\pi \in \text{supp}(\mathcal{H})$, other than at most a zero-measure subset, there exists $v' \geqslant 0$ and $v'' \leqslant 0$ with $\pi(\{v'\}) =$

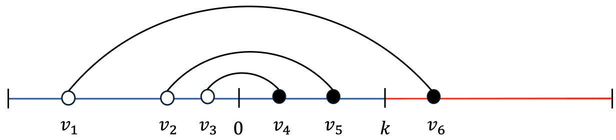


Figure 11: Of the three depicted districts, only the one containing voter types v_3 and v_4 is lost when $\omega = 0$.

$\pi(\{v''\}) = \frac{1}{2}$. Note that districts in Δ_2 placing positive mass on $v' \geqslant 0$ and $v'' \leqslant 0$ can be won in one of two ways. For instance, after the realization of the location shock ω, it can be that $v' - \omega$ is above k, so that the district has at least a majority of Republican affiliates. For such a district, I have $v^m_{\pi^\omega} = c_{\pi^\omega,R} = v'$ and $c_{\pi^\omega,D} = v''$. Alternatively, it can be that $v' - \omega$ is less than k, but still closer to k than it is to $v'' - \omega$, so that the district is split fifty-fifty between "extreme" and "moderate" Democrats, the latter choosing the default moderate Republican candidate. For such a district, I have $v^m_{\pi^\omega} = v'$, $c^\omega_{\pi,R} = k$, and $c^\omega_{\pi,D} = v''$, with $k - v' \leqslant v' - v''$. In Figure 11, three types of districts are shown. When $\omega = 0$, only the district containing voter types v_3 and v_4 is lost.

Now, define $\phi' = \phi(\cdot|v \geqslant 0)$ and $\phi'' = \phi(\cdot|v < 0)$. In words, ϕ' is the distribution of voter types conditional on them being above the population median, while ϕ'' is the distribution of voter types conditional on them being below the population median. I call $T(\phi', \phi'') \subseteq \Delta([\underline{v}, 0] \times [0, \overline{v}])$ the set of joint distributions having marginals ϕ' and ϕ''[12]. Following the literature on optimal transport, I sometimes refer to $T(\phi', \phi'')$ as the set of *transport plans* from ϕ' to ϕ''.

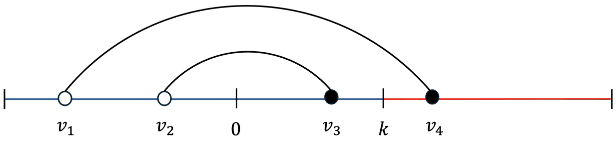
As it turns out, there is a one-to-one map between Δ_2 and $T(\phi', \phi'')$, so that I can rewrite (RP) as:

$$\max_{\tau \in T(\phi',\phi'')} \int\int \mathbb{1}(v' - \omega \geqslant k) + \mathbb{1}(v' - \omega < k)\mathbb{1}\left(v' - \omega - \frac{v'' - \omega}{2} \geqslant k\right) d\tau(v', v'') d\gamma(\omega).$$

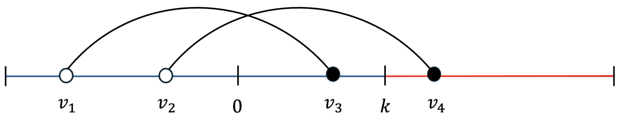
By switching the order of integration and further manipulating the above, I get the optimal transport problem:

$$\max_{\tau \in T(\phi',\phi'')} \int G\left(2v' - v'' - k\right) d\tau(v', v''). \tag{OTP}$$

[12]Formally, $T(\phi', \phi'') = \left\{\tau \in \Delta([\underline{v}, 0] \times [0, \overline{v}]) : \mathrm{proj}_{[\underline{v},0]}\#\tau = \phi'', \mathrm{proj}_{[0,\overline{v}]}\#\tau = \phi'\right\}$, where: $\mathrm{proj}_{[\underline{v},0]}$ and $\mathrm{proj}_{[0,\overline{v}]}$ denote the projection functions of $[\underline{v}, 0] \times [0, \overline{v}]$ on $[\underline{v}, 0]$ and $[0, \overline{v}]$, respectively; $(\mathrm{proj}_{[\underline{v},0]}\#\tau)(A) = \tau(\mathrm{proj}^{-1}_{[\underline{v},0]}(A))$ and $(\mathrm{proj}_{[0,\overline{v}]}\#\tau)(A) = \tau(\mathrm{proj}^{-1}_{[0,\overline{v}]}(A))$ for all A measurable.



(a) The district containing voter types $\overline{v}'$, $\underline{v}''$ considerably safer than the one containing types $\overline{v}''$, $\underline{v}'$.



(b) The two districts are of similar safety.

Figure 12: Two alternative configurations of districts.

A transport plan τ in $T(\phi', \phi'')$ is *pure* whenever $\{v', v''\} \in \text{supp}(\tau)$ implies $\{v', \tilde{v}''\}, \{\tilde{v}', v''\} \notin \text{supp}(\tau)$ for $v' \neq \tilde{v}'$ and $v'' \neq \tilde{v}''$. Intuitively, purity requires that no "splitting of masses" occurs across voter types. A pure transport plan is sometimes referred to as a *transport map*.

Define $T^\star \subseteq T(\phi', \phi'')$ as the set of solutions to (OTP) and $\Delta_2^\star \subseteq \Delta_2$ as the set of solutions to (RP). The optimal transport problem (OTP) is *equivalent* to the redistricter's problem (RP) if there exists a bijection from $T^\star$ to $\Delta_2^\star$ mapping each solution to (OTP) to a solution to (RP). The following theorem summarizes the discussion in this subsection.

Theorem 1. *The optimal transport problem (OTP) is equivalent to the redistricter's problem (RP).*

Theorem 1 simplifies the original problem significantly, allowing me to concentrate on a more straightforward problem. In the following section, I will focus on finding a solution to this simplified problem.

4 Characterizing the Optimal Redistricting Plan

In this section, I characterize the solution(s) to (OTP) under different assumptions on G. As it turns out, the optimal redistricting plan depends heavily on the shape of G and, in most cases, on the shape of F.

4.1 Benchmark Cases

I start by describing a few benchmark cases before focusing on the most realistic case of a symmetric, S-shaped shock. Suppose that G is strictly convex on its support. The location shock "shifts" the distribution of preferences. A negative shock shifts it to the right, increasing the fraction of Repiblican voters (hence it is a favorable shock), while a posi-

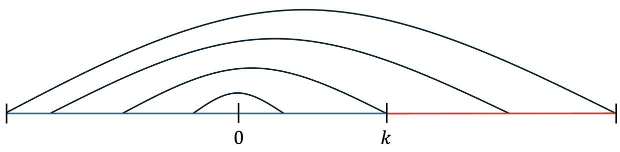


Figure 13: When G is convex, the unique optimal transport plan maps types above 0 to types below 0 in a negative assortative manner.

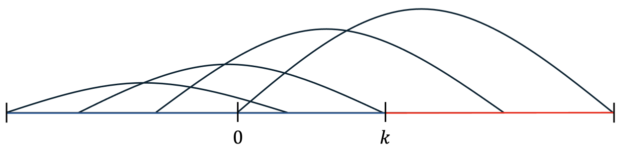


Figure 14: When G is concave, the unique optimal transport plan maps types above 0 to types below 0 in a positive assortative manner.

tive shock shifts the distribution to the left (so it is unfavorable). When G is convex, the marginal benefit of slightly improving a competitive district's safety is outweighed by the cost of marginally reducing the safety of an already secure district. Consider Figure 12. Panel 12a depicts two districts, the one containing voter types v_1, v_4 considerably safer than the one containing types v_2, v_3. One can think of constructing two alternative districts of intermediate safety, as in panel 12b, by matching v_1 to v_3 and v_2 to v_4. Because G is convex, the designer always prefers the matching in 12*a* to the one in 12*b*. In other words, the designer wants to create very safe districts consisting of extremists of both parties, alongside very unsafe districts consisting of more moderate voters. Based on this intuition, it can be shown that there exists a unique, pure solution to OTP and that such solution maps types above zero to types below zero in a *negative assortative manner*[13]. Figure 13 illustrates such map.

Concave Shock. The case of a strictly concave G is analogous to that of a strictly convex G. In this case, the marginal benefit of slightly improving a competitive district's safety exceeds the cost of marginally reducing the safety of an already secure district. Hence, the designer prefers configuration in Figure 12b to the one in Figure 12a and tries to create districts of

[13]Formally, a *negative assortative map* $\tau \in T(\phi', \phi'')$ is such that, for all $(v', v''), (\tilde{v}', \tilde{v}'') \in \text{supp}(\tau)$, $v' > \tilde{v}'$ implies $v'' \leqslant \tilde{v}''$.

similar safety. It can be shown that there exists a unique, pure solution to OTP that maps types above 0 to types below 0 in a *positive assortative manner*[14], as illustrated in Figure 14.

Uniform Shock. For the sake of completeness, I analyze the case when G is affine on its support. It is easy to see that, given the linearity of G, problem OTP does not depend on T, therefore $T^\star = T(\phi', \phi'')$. In other words, any redistricting plan in Δ_2 is optimal.

The following result formalizes the findings for benchmark cases.

Proposition 2. *Consider the following cases:*

- *If G is strictly convex on its support, there exists a unique, pure solution $\tau \in T(\phi', \phi'')$ to OTP and it is a negative assortative map.*
- *If G is strictly concave on its support, there exists a unique, pure solution $\tau \in T(\phi', \phi'')$ to OTP and it is a positive assortative map.*
- *If G is affine on its support, any $\tau \in T(\phi', \phi'')$ is a solution to (OTP).*

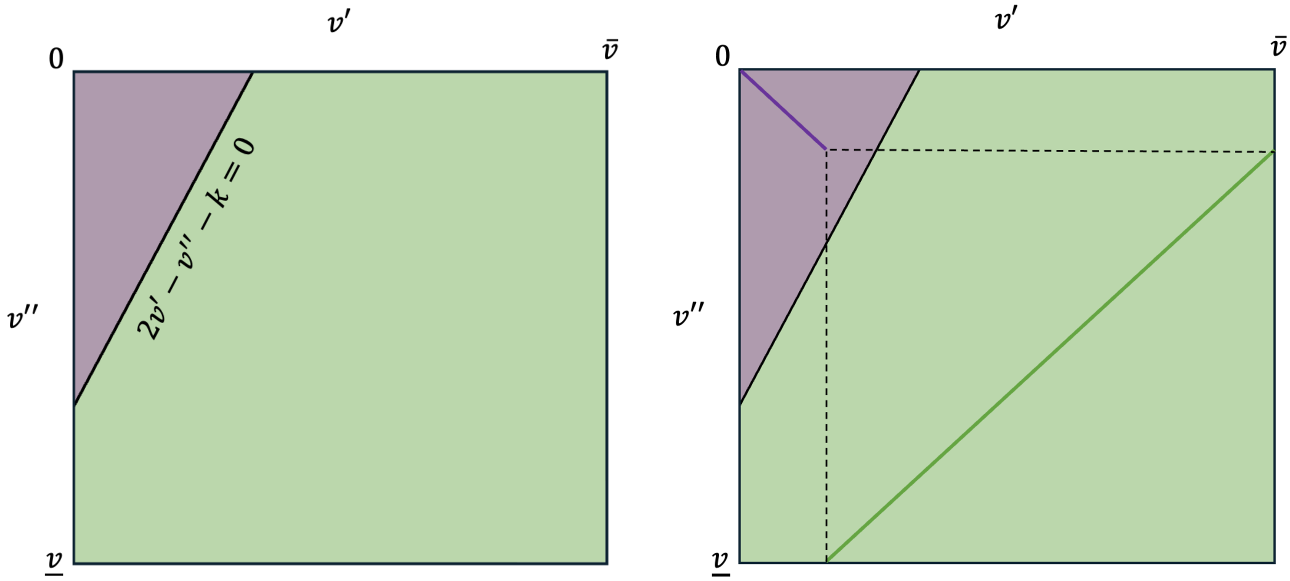
In the next subsections, I will make use of the above benchmark results as building blocks to characterize the solutions to (OTP) when the shock is strictly convex below zero and strictly concave above zero, or S-shaped.

4.2 S-Shaped Shock

Suppose that G that is strictly s-shaped around zero[15]. As it turns out, (OTP) admits a unique solution. Moreover, one can capitalize on the benchmark cases studied in the previous subsection to show that such solution is the convex combination of a positive assortative map and a negative assortative map, each over an appropriate subset of $[0, \overline{v}] \times [\underline{v}, 0]$. Further specifics of G determine which exact subsets host negative or positive assortments, along with additional characteristics of the solution, like purity. Figure 15a depicts $[0, \overline{v}] \times [\underline{v}, 0]$ and partitions it into two subsets. For any (v', v'') belonging to the purple region, $2v' - v'' - k$ is less than zero, while for any (v', v'') belonging to the green region, $2v' - v'' - k$ is greater than zero. Hence, G is convex on the purple subset of $[0, \overline{v}] \times [\underline{v}, 0]$ and concave on the green

[14]Formally, a *positive assortative map* $\tau \in T(\phi', \phi'')$ is such that, for all $(v', v''), (\tilde{v}', \tilde{v}'') \in \text{supp}(\tau), v' > \tilde{v}'$ implies $v'' \geqslant \tilde{v}''$.

[15]Formally, G is strictly convex on $[\underline{v}, 0]$ and strictly concave on $[0, \overline{v}]$.



(a) The purple subset of $[\underline{v}, 0] \times [0, \overline{v}]$ hosts negative assortments, while the green subset hosts positive assortments.

(b) Example of $\tau = \alpha\tau^- + (1-\alpha)\tau^+$ under ϕ uniform.

Figure 15: The unique solution to (OTP) is the convex combination of a negative assortative map and a positive assortative map.

one. Remembering the discussion for benchmark cases, it should not be surprising that any two couples of voter types falling in the green subset of $[0, \overline{v}] \times [\underline{v}, 0]$ must constitute a positive assortment, in order to be part of a solution to (OTP). Similarly, any two couples of voter types falling in the purple subset of $[0, \overline{v}] \times [\underline{v}, 0]$ must constitute a negative assortment. The following result formalizes such intuition.

Proposition 3. *Suppose G is s-shaped around 0. There exist τ^+, $\tau^- \in \Delta([0, \overline{v}] \times [\underline{v}, 0])$ and $\alpha \in [0, 1]$ such that:*

1. *τ^- is a negative assortative map with* $\mathrm{supp}(\tau^-) \subseteq \{(v', v'') \in [0, \overline{v}] \times [\underline{v}, 0] : 2v' - v'' - k \geqslant 0\}$
2. *τ^+ is a positive assortative map with* $\mathrm{supp}(\tau^+) \subseteq \{(v', v'') \in [0, \overline{v}] \times [\underline{v}, 0] : 2v' - v'' - k \leqslant 0\}$
3. *$(1 - \alpha)\tau^- + \alpha\tau^+$ is the unique solution to OTP.*

As an example, Figure 15b depicts the support of a pure candidate for optimality, when ϕ is uniform. In this case, the linearity of F results in a linear mapping of voter types above zero and below zero, with negative slope for matches in the purple region and positive slope for matches in the green region.

While Figure 15b depicts the support of a pure transport plan, the actual solution to (OTP) may very well not be pure, meaning that some voter types might belong to a positive assortative map and a negative assortative map, simultaneously. As I show next, this is indeed the most likely case under a symmetric s-shaped shock, whenever the optimum requires $\alpha < 1$.

4.2.1 Symmetric S-Shaped Shock

I now consider an s-shaped shock that is symmetric around zero. Any normal shock with mean zero falls under this category. For instance, a normal shock with sufficiently small variance is of particular interest because it approximates the solution to (RP) in the absence of an aggregate shock. Proposition 3 justifies the definition of set $T^{\pm}$ as the set of all plans $\tau \in T(\phi', \phi'')$ for which there exist $\tau^{+},\ \tau^{-} \in \Delta([0,\overline{v}] \times [\underline{v}, 0])$ and $\alpha \in [0,1]$ such that:

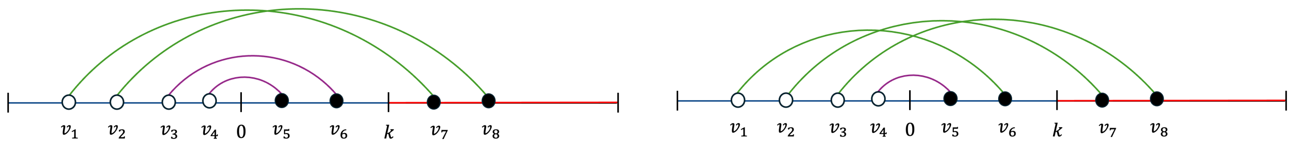
1. τ^{-} is a negative assortative map with $\operatorname{supp}(\tau^{-}) \subseteq \{(v', v'') \in [0,\overline{v}] \times [\underline{v}, 0] : 2v' - v'' - k \geqslant 0\}$
2. τ^{+} is a positive assortative map with $\operatorname{supp}(\tau^{+}) \subseteq \{(v', v'') \in [0,\overline{v}] \times [\underline{v}, 0] : 2v' - v'' - k \leqslant 0\}$
3. $\tau = (1-\alpha)\tau^{-} + \alpha\tau^{+}$.

The symmetry assumption allows me to derive a first-order condition that further characterizes the solution within $T^{\pm}$. The following result states that, at the optimum, either τ is fully positive assortative, or the support of its positive assortative part, τ^{+}, is tangent to the line $2v' - v'' - k = 0$.

Proposition 4. *Suppose G is symmetric and s-shaped around 0. If $\tau = \alpha\tau^{+} + (1 - \alpha\tau^{-}) \in T^{\pm}$ is the solution to (OTP), there exist $v',\ v'' \in \operatorname{supp}(\tau^{+})$ such that:*

$$(1-\alpha)(2v' - v'' - k) = 0.$$

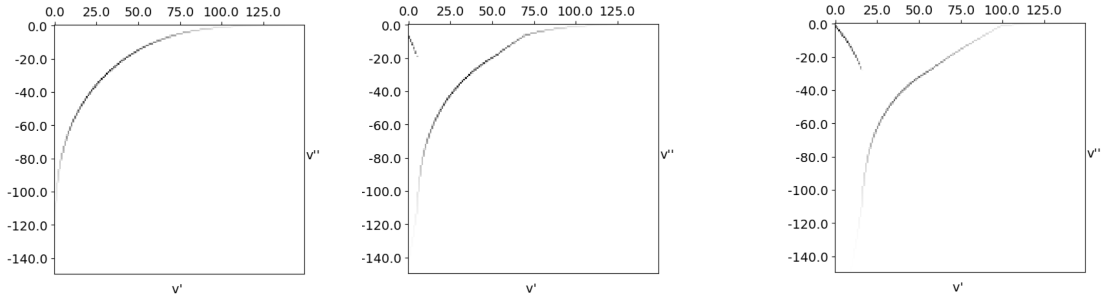
Figure 16 demonstrates the intuition behind Proposition 4. In Figure 16a, we see a configuration consistent with Proposition 3. Here, voter types v_1, v_2, v_7, and v_8 are matched positively assortatively, while v_3, v_4, v_5, and v_6 are matched negatively assortatively. Now, consider breaking the match between v_3 and v_6 and including them in the positive assortative match, as shown in Figure 16b. If this new configuration belongs to $T^{\pm}$, it proves



(a) The purple subset of $[\underline{v}, 0] \times [0, \overline{v}]$ hosts negative assortments, while the green subset hosts positive assortments.

(b) Example of $\tau = \alpha\tau^- + (1-\alpha)\tau^+$ under F uniform.

Figure 16: The unique solution to (OTP) is the convex combination of a negative assortative map and a positive assortative map.



(a) $k = 60$

(b) $k = 75$

(c) $k = 100$

Figure 17: Solution for simulated F normal, G normal, for $k = 60,\ 75,\ 100$

superior to the original arrangement. The key insight is that Figure 16b introduces an additional match where $2v' - v'' - k > 0$, compared to Figure 16a. While this addition reduces the surplus from existing positive matches, the overall benefit outweighs the cost. This occurs because the symmetry of the shock causes G to be steeper around zero than at any point above zero.

Figure 17 shows the simulated solution when both F and G are normal, for different values of k. Note that as k increases, and the fraction of Republicans decreases, more and more voters are matched in a negative assortative manner.

5 Implications

Gerrymandering has drawn intense scrutiny for decades, not only for its blatant manipulation of electoral boundaries but also for its far-reaching impacts on democratic representation and political polarization. In this section, I discuss two implications of my model that speak to the current political and legal debate.

5.1 Gerrymandering and Congress Polarization

The relationship between gerrymandering and partisan polarization in the U.S. House of Representatives is often oversimplified in public discourse. While frequently cited as a primary driver of heightened partisanship and legislative gridlock, the actual dynamics are far more nuanced and multifaceted. Empirical research on gerrymandering's impact remains divided, with scholars like McCarty et al. (2009) questioning its significance, while others identify measurable effects (Kenny et al., 2023).

Existing theoretical models, predominantly based on exogenous candidate regimes, have yielded conflicting predictions depending on assumptions about district composition. Models forecasting homogeneous districts, which effectively segregate extreme opponents (Gul and Pesendorfer, 2010; Kolotilin and Wolitzky, 2024), predict a distinct "gap" in the ideological distribution of elected representatives. This gap manifests as a polarized landscape, with moderate supporters of the redistricting party at one end and extreme opponents at the other. In contrast, models predicting district heterogeneity (Friedman and Holden, 2008) anticipate a more continuous spectrum of political representation, without such a pronounced ideological chasm.

By incorporating endogenous candidate emergence, my model predicts the formation of a significant gap in the distribution of district representatives without relying on the creation of homogeneous districts. Let $Q^{\omega}_{\mathcal{H}}$ denote the distribution of district representatives given a redistricting plan $\mathcal{H}$ and a shock realization ω. Formally:

Proposition 5. *For any optimal plan $\mathcal{H}$ and shock realization $\omega < k$, $Q^{\omega}_{\mathcal{H}}((-\omega, k)) = 0$.*

Consistent with existing literature, optimal plans create numerous right-leaning districts that elect moderate Republican candidates. However, they differ from previous models in their treatment of opposition voters. Instead of isolating Democratic voters in homogeneous districts, these plans strategically distribute them across heterogeneous districts, each containing a carefully calibrated mix of moderate and extreme Democratic voters. This nuance yields two critical implications. First, in these heterogeneous "packed" districts, extreme Democratic candidates consistently prevail as the consolidated bloc of extreme Democratic voters outweighs moderate Democrats. Second, these districts effectively disenfranchise moderate Democratic voters, who find themselves without representatives reflecting their political stances.

The analysis in this paper bridges competing theories. It predicts both discontinuities in the distribution of representatives, characteristic of voter-segregating models, and within-district polarization, typically associated with models yielding continuous representative distributions. While existing literature has primarily examined voter segregation as a driver of polarization, this work explores an alternative mechanism: strategic distribution of heterogeneous opposition voters. This approach offers a new perspective on the relationship between redistricting strategies and political polarization, that is worth exploring in future research.

5.2 Legislative implications for "majority-minority" districts

American federal legislation, including the 1965 Voting Rights Act, mandates that electoral district lines cannot be drawn in such a manner as to improperly dilute minorities' voting power. Since the 1986 Supreme Court decision in Thornburg v. Gingles, such laws have been interpreted as actively requiring the creation of districts where racial and ethnic minorities have the concrete opportunity to elect their own representatives. As a consequence, in the 118th Congress there are 26 congressional districts where Black people constitute a strict majority, and 37 are majority Hispanic or Latino (Klein, 2023). Such districts are called "majority-minority".

While such legislation aims to increase minority representation in Congress, it has sparked debate over its broader political implications. Some argue these districts inadvertently segregate Democratic voters, potentially mirroring aspects of an optimal Republican redistricting strategy. The impact of such legislation on partisan outcomes remains complex and unresolved.

As for the case of Congress polarization, the economic theory literature is divided. One school of thought predicts that segregating extreme opponents is optimal, suggesting majority-minority districts benefit Republicans by forcing Democratic gerrymanderers to deviate from their optimal strategy. The opposing view argues for more heterogeneous districts, implying that majority-minority districts disrupt Republicans' desired optimum. In my model, imposing a clear homogeneous majority could prevent both Democrats and Republicans from exploiting the endogeneity of electoral incentives.

In some districts, racial or ethnic minorities may constitute a plurality rather than a majority. These areas, known as "minority opportunity" or "non-majority minority" districts, provide these groups the chance to elect their preferred representatives through coalitions with White voters or other minority groups.

Some political analysts argue that Black candidates can win in constituencies where Black voters comprise less than $40-50\%$ of the population. They contend that creating Black-majority districts is unnecessary and may inadvertently limit potential Black political gains (Canon, 2022).

Recent trends, particularly in the South, show people of color winning more seats in majority-white districts (Lublin et al., 2020). However, the success of this strategy largely depends on White Democrats' willingness to support candidates of color. If they are reluctant to do so, a phenomenon known as "White backlash" may occur, potentially benefiting Republican candidates as described in this paper.

6 Extension: Quantile Redistricting

In the previous sections, I relied on party candidates whose positioning rule is a function of party medians. The implication is that the designer can win a district even if there are no supporters inhabiting it. I now consider a scenario where candidates' positions are determined by a quantile, not necessarily the median, of the distribution of preferences within each party. Let $\frac{1}{2} < q < 1$ be the quantile parameter. Under this setting, the Democratic candidate locates at the q-quantile of the preference distribution conditional on being affiliated with the Democratic party, while the Republican candidate locates at the $(1-q)$-quantile of the preference distribution conditional on being affiliated with the Republican party. In this setting, as it will become clear soon, the designer always needs a non-zero measure of supporters in order to win a district. Therefore, it becomes much harder to characterize the solution in general[16]. Thus, for illustrative purposes, I assume F to be uniform on $[\underline{v}, \overline{v}]$.

Call $c^q_{\pi,D}$ and $c^q_{\pi,R}$ the positions taken by D and R in district π. Then:

$$(c^q_{\pi,D}, c^q_{\pi,R}) = \begin{cases} (v^q_{\pi,D}, k) & if \ \operatorname{supp}(\pi) \subseteq (-\infty, k] \\ (k, v^q_{\pi,R}) & if \ \operatorname{supp}(\pi) \subseteq [k, \infty) \\ (v^q_{\pi,D}, v^q_{\pi,R}) & otherwise \end{cases}$$

where:

$$v^q_{\pi,D} = \inf_a \quad \{a : \ \pi([a, \overline{v}] | v < k) \geqslant q\} \cap \{a : \ \pi([\underline{v}, a] | v < k) \geqslant q\}$$
$$v^q_{\pi,R} = \sup_a \quad \{a : \ \pi([a, \overline{v}] | v \geqslant k) \geqslant 1-q\} \cap \{a : \ \pi([\underline{v}, a] | v \geqslant k) \geqslant 1-q\}.$$

Every other detail of the model is the same as in Section 3. The following proposition characterizes the optimal measure of districts won by the designer for each realization of the shock ω, denoted by $V^\omega(q)$.

Proposition 6. *There are two cases:*

1. *Suppose $F(k+\omega) \leqslant \frac{1}{2q}$. Then $V^\omega(q) = 1$.*

2. *Suppose $F(k+\omega) > \frac{1}{2q}$. Then $V^\omega(q) = \frac{2q}{2q-1}(1 - F(k+\omega))$.*

[16]For starters, the connection to optimal transport requires considering joint distributions with three fixed marginals, rather than just two. Hence, the optimal transport problem itself becomes less tractable.

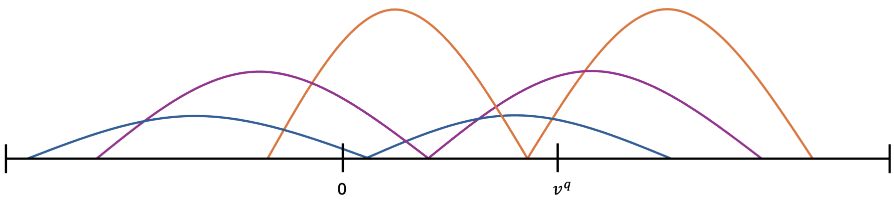


Figure 18: Three-wise positive assortative matching with probability masses of $\frac{1}{2}$, $\frac{1}{2q}$, and $\frac{2q}{2q-1}$.

In the first case, where the fraction of Democratic voters is less than or equal to $\frac{1}{2q}$, all districts are won by the Republican party. In the second case, where the fraction of Democratic voters is greater than $\frac{1}{2q}$, the designer wins $\frac{2q}{2q-1}(1-F(k+\omega))$ district, which is strictly more than $2(1-F(k+\omega))$, the districts she would win under exogenous policies. The intuition for the optimal plan is similar to the one in the previous sections. In each district, the redistricter needs to create a wedge between moderate and extreme Democrats, encouraging the emergence of an extreme Democratic candidate and counting on moderate Democrats to vote for a moderate Republican candidate. The following result characterizes optimal redistricting plans:

Proposition 7. *Define $v^q = F^{-1}\left(\frac{1}{2q}\right)$. A feasible redistricting plan $\mathcal{H}$ is optimal if and only if, for all $\pi \in supp\ (\mathcal{H})$, there exists $v'_\pi \geqslant v^q$ such that:*

1. $\pi(\{v'_\pi\}) = 1 - \pi(\{v : v < v^q\}) = \frac{2q}{2q-1}$, *and*
2. $|v^q_{\pi,D} - v^m_\pi| \geqslant |v^m_\pi - v'_\pi|$.

This proposition states that an optimal redistricting plan under quantile redistricting must satisfy two conditions for each district π in the support of the plan. First, the mass of voters with type exactly equal to v'_π should be $\frac{2q}{2q-1}$, and the mass of voters with type below v_q should be $1 - \frac{2q}{2q-1}$. Second, the distance between $v^q_{P,D}$, which is the lowest median of the district, and the highest median of the district should be greater than or equal to the distance between the highest median and v'_π. As it turns out, a type of redistricting plan called "three-wise positive assortative" plan complies with the requirements of Proposition 7. Figure 18 illustrates such a plan. Each district contains exactly three voter types, one type below 0, one type between 0 and v^q, and another type above v^q, with probability masses of $\frac{1}{2}$, $\frac{1}{2q}$, and $\frac{2q}{2q-1}$, respectively. Given shock ω, the designer wins district π if and only if

$v'_\pi - \omega \geqslant k$. Note that the intuition for the distribution of district representatives works exactly as in Proposition 5. Even under quantile redistricting, the winning candidate is either a Republican with position above k, or a Democrat with position below $-\omega$, so that the distribution of district representatives has a gap in $(-\omega, k)$.

7 Conclusion

In this paper, I have explored the interaction between partisan gerrymandering and policy positioning of candidates at the district level. By incorporating the endogenous response of candidates to the redistricting plan, I have provided a more comprehensive understanding of the consequences of partisan gerrymandering on electoral outcomes and political polarization. The key findings can be summarized as follows:

1. The optimal reredistricting plan differs from the standard "pack-and-crack" approach. Instead of concentrating opposition voters into a few districts and spreading the remaining opposition voters across multiple districts, the optimal plan allocates moderate and extreme opponents to the same districts. This strategy encourages the emergence of extreme opponent candidates, as the primaries are likely to select candidates closer to the extremes of the preference distribution.

2. The designer can exploit the endogeneity of electoral incentives to achieve better outcomes compared to a scenario with exogenous policies. By strategically drawing district boundaries, the designer can influence the positioning of candidates in each district, creating more favorable conditions for the Republican party.

3. Under aggregate uncertainty, the shape of the shock distribution plays a crucial role in determining the optimal redistricting strategy. I use tools from the literature on Optiaml Transportation to show that: when the shock distribution is concave, the designer prefers to smooth the risk across districts by employing positive assortative matching between voters from the upper and lower medians; in contrast, when the shock distribution is convex, the designer opts for negative assortative matching, creating districts that are either highly favorable or highly unfavorable to the Republican party.

4. The optimal reredistricting plan leads to a polarized distribution of district representatives. With moderate and extreme opponents allocated to the same districts, the pri-

maries tend to select candidates who are further away from the center of the preference distribution. This results in a Congress with a higher proportion of extreme representatives, potentially exacerbating political polarization and making consensus-building more challenging.

5. I also investigate the case of quantile redistricting, where candidate positions are determined by a general quantile of the preference distribution within each party. When the distribution of preference is uniform, I show that the optimal plan involves three-wise positive assortative matching.

This study contributes to the literature by providing a novel perspective on the interplay between partisan gerrymandering and candidate positioning. By considering the endogenous response of candidates to the redistricting plan, I have highlighted the potential unintended consequences of partisan gerrymandering on political polarization and the distribution of district representatives. The findings of this paper have important implications for policymakers and scholars interested in electoral reform and the promotion of fair and competitive elections. The results suggest that efforts to mitigate the adverse effects of partisan gerrymandering should not only focus on the shape of district boundaries but also consider the endogenous response of candidates and the resulting impact on political polarization. Future research could extend this work by considering alternative models of candidate positioning, incorporating the role of campaign financing, or examining the long-term dynamics of redistricting and political polarization. Additionally, empirical studies could test the predictions of this model using data on reredistricting plans, candidate positions, and electoral outcomes across different jurisdictions and time periods.

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A A Model of Two-Stage District Elections

In this appendix, I develop a model of probabilistic voting at the district level, providing a foundation for the dependence of district candidates' positions on the conditional medians of their respective vote base. While equilibrium policy divergence can be supported under a variety of models, the dependence of such policies on conditional medians is typical of two-stage election mechanisms. I propose one such model in which, like in many others, two forces work in opposing directions to determine equilibrium policies. One such force, a centripetal force, drives positions towards the district median, fueled by the concern voters have with the ability of their first-stage candidate to win general elections. Another force, a centrifugal force, moves positions away from each other, driven by first-stage voters' uncertainty about the exact position of the district median. Similarly to what was suggested, among others, by Owen and Grofman (2006), equilibrium policies tend to be driven towards party medians.

Consider a continuum of voters in a district. Each voter i has policy preferences given by single-peaked utility $u(\cdot, v_i)$, with $v_i \in [\underline{v}, \overline{v}]$ being the voter's unique ideal point. Further, assume that the distribution π of v_i admits a unique median, conditional median given $v_i \geqslant k$, and conditional median given $v_i < k$, denoted by v_π^m, $v_{\pi,R}^m$, and $v_{\pi,D}^m$.

Two-stage elections are held in the district. In the first, primaries stage, there are two Republican and two Democratic candidates. Voters with $v_i \geqslant k$ elect one of the Republican candidates, while voters with $v_i < k$ elect one of the Democratic candidates, by simple majority, with ties broken uniformly at random. In the second stage, general elections, all voters elect a district representative among the two first-stage winners, by simple majority, with ties broken uniformly at random.

Before the first-stage elections, candidates simultaneously announce a policy position and commit to it. They receive a payoff of 1 if they win the second-stage elections and 0 otherwise.

Suppose that voters and candidates do not know the position of the overall district median v_π^m, and believe it is distributed according to $H \in \Delta([\underline{v}, \overline{v}])$. In addition, candidates know the position of the conditional medians $v_{\pi,R}^m$ and $v_{\pi,D}^m$.

Given voters' single-peaked preferences, the winner of the general elections is the candidate closer to the district median. In the first stage, voters best respond to the anticipated position of the opposing nominee by voting for the candidate maximizing their expected utility. More formally, given the position of the opposing nominee y, a voter with ideal point t votes for the candidate whose position maximizes the following:

$$U(x;y,t) = u(x,t)p(x,y) + u(y,t)(1 - p(x,y)),$$

where $p(x,y)$ is the probability of x winning against y in general elections. As already stated, equilibrium policies in such a model are driven by party medians, but their exact value depends on the functional form of u and H. The literare analyses the model various such specifics. I suggest a *linear setting*, where voter preferences are linear and uncertainty is uniform. I prove the following.

Proposition 8. *Suppose $u(x,v_i) = -|x - v_i|$ and H is uniform on $[\underline{v},\overline{v}]$. There exists a unique Nash equilibrium where the Republican and Democratic candidates set positions $c_{\pi,R} = v^m_{\pi,R}$ and $c_{\pi,D} = v^m_{\pi,D}$, respectively.*

The advantage of the linear setting is that voters' expected utilities in the primaries turn out to be single-peaked, with maximum reached at each voter's ideal point. Then, it can be argued that any candidate positioning at the party median will win first-stage elections against any other candidate at a different position, irrespective of the opposing party's behavior. Alternatively, Owen and Grofman (2006) consider a model where $u(x,v_i) = e^{-\alpha|x-v_i|}$, for some parameter $\alpha > 0$, and uncertainty is normal around the median, with standard deviation σ. In their model, voters' expected utility during primary elections is not necessarily single-peaked. Hence, they need to explicitly rule out a particular strategic behavior in which some first-stage voters anticipate that they will prefer the opponents' candidate at general elections and purposefully sabotage their own primaries by voting for an extremist[17]. Under this credible assumption, they prove the following:

Proposition 9. (Owen and Grofman, 2006). If $\sigma \geqslant \frac{\max\left\{1-e^{1-\alpha(v^m_\pi - v^m_{\pi,D})}, 1-e^{\alpha(v^m_{\pi,R} - v^m_\pi)}\right\}}{\alpha\sqrt{2\pi}}$, there exists a unique Nash equilibrium where the Republican and Democratic candidates set positions $c_{\pi,R} = v^m_{\pi,R}$ and $c_{\pi,D} = v^m_{\pi,D}$, respectively.

[17] Under uniform uncertainty, extreme events have a sufficiently high probability of happening, so that voters are deterred from this kind of "political gamble". Indeed, if they sabotage primaries by voting for an extremist, they run the risk of such candidate winning general elections as well.

B Proofs

B.1 Proofs of Section 3

Proof of Proposition 0. Since the designer wins district $\pi \in \Delta([\underline{v},\overline{v}])$ whenever $v^m_\pi \geqslant k^\star + \omega$, a redistricting plan $\mathcal{H} \in \Delta(\Delta([\underline{v},\overline{v}]))$ can be described by a distribution $\chi \in \Delta([\underline{v},\overline{v}])$ over $v = v^m_\pi$. Using Theorem 2 in Yang and Zentefis (2024), (RPEx) can be stated as:

$$\max_{\chi\in\Delta([\underline{v},\overline{v}])} \int\int \mathbb{1}\left(v-\omega \geqslant k^\star\right) d\chi(v) d\gamma(\omega)$$

$$s.t.\ \ \max\{2F(v)-1,0\} \leqslant X(v) \leqslant \min\{2F(v),1\},\ \text{for all } v \in [\underline{v},\overline{v}].$$

Switching the order of integration, it can be rewritten as:

$$\max_{\chi\in\Delta([\underline{v},\overline{v}])} \int G(v-k^\star) d\chi(v)$$

$$s.t.\ \ \max\{2F(v)-1,0\} \leqslant X(v) \leqslant \min\{2F(v),1\},\ \text{for all } v \in [\underline{v},\overline{v}].$$

By definition of first order stochastic dominance, and since G is strictly increasing, I have:

$$\int G(v-k^\star) d\chi(v) < \int G(v-k^\star) d\max\{2F(v)-1,0\}$$

for any $\chi > \max\{2F-1,0\}$. Hence, the optimal distribution of medians is $\chi^\star = \max\{2F-1,0\}$.

■

Proof of Proposition 1. Consider the following definition:

Definition 1. *A redistricting plan $\mathcal{H}$ is pairwise if $|\operatorname{supp}\pi| \leqslant 2$ for all $\pi \in \operatorname{supp}(\mathcal{H})$.*

The proof of this proposition relies on the following lemma, which states that for any redistricting plan, there exists a pairwise plan that achieves the same value, for each realization of the shock $\omega \in \operatorname{supp}(\gamma)$.

Lemma 1. *For any feasible plan $\mathcal{H}$, there exists a feasible pairwise plan $\hat{\mathcal{H}}$ such that, for all $\omega \in \operatorname{supp}(\gamma)$:*

$$\int \mathbb{1}\left(c_{\hat{\pi}^\omega} \geqslant k\right) d\hat{\mathcal{H}}(\hat{\pi}) \geqslant \int \mathbb{1}\left(c_{\pi^\omega} \geqslant k\right) d\mathcal{H}(\pi)$$

Proof. Take any plan $\mathcal{H} \in \Delta(\Delta([\underline{v},\overline{v}]))$ such that $\int \pi d\mathcal{H}(\pi) = \phi$. First, for any $\pi \in \text{supp}(\mathcal{H})$, construct a measure $\hat{\mathcal{P}}_\pi \in \Delta(\Delta([\underline{v},\overline{v}]))$ such that $\int \hat{\pi} d\hat{\mathcal{P}}_\pi(\hat{\pi}) = \pi$, and, for all $\hat{\pi} \in \text{supp}(\hat{\mathcal{P}}_\pi)$, $\text{supp}(\hat{\pi}) = \{v'_{\hat{\pi}}, v''_{\hat{\pi}}\}$, for some $v'_{\hat{\pi}}, v''_{\hat{\pi}} \in \text{supp}(\pi)$ with:

$$v'_{\hat{\pi}} \geqslant v^m_\pi,$$
$$v''_{\hat{\pi}} \leqslant v^m_\pi,$$
$$\hat{\pi}(\{v'_{\hat{\pi}}\}) = \hat{\pi}(\{v''_{\hat{\pi}}\}).$$

Moreover, for any $\hat{\pi}$, $\hat{\rho} \in \text{supp}(\hat{\mathcal{P}}_\pi)$, let $v'_{\hat{\pi}} > v'_{\hat{\rho}} \implies v''_{\hat{\pi}} \geqslant v''_{\hat{\rho}}$.

Second, construct alternative plan $\hat{\mathcal{H}} \in \Delta(\Delta([\underline{v},\overline{v}]))$ such that, for any measurable set $A \subseteq \Delta([\underline{v},\overline{v}])$, $\hat{\mathcal{H}}(A) = \int \hat{\mathcal{P}}_\pi(A) d\mathcal{H}(\pi)$. By construction, $\hat{\mathcal{H}}$ is feasible and pairwise. I now show that:

$$\int \mathbb{1}\left(c_{\hat{\pi}^\omega} \geqslant k\right) d\hat{\mathcal{H}}(\hat{\pi}) \geqslant \int \mathbb{1}\left(c_{\pi^\omega} \geqslant k\right) d\mathcal{H}(\pi).$$

Specifically, I show that, for any $\pi \in \text{supp}(\mathcal{H})$, $\hat{\pi} \in \text{supp}(\hat{\mathcal{P}}_\pi)$, $\omega \in \text{supp}(\gamma)$, I have that $\mathbb{1}\left(c_{\hat{\pi}^\omega} \geqslant k\right) \geqslant \mathbb{1}\left(c_{\pi^\omega} \geqslant k\right)$.

Consider the following three cases:

1. If $v^m_{\pi^\omega} \geqslant k$, then $c_{\hat{\pi}^\omega,R} = v^m_{\hat{\pi}^\omega} \geqslant k$, which means that $\mathbb{1}\left(c_{\hat{\pi}^\omega} \geqslant k\right) = 1 \geqslant \mathbb{1}\left(c_{\pi^\omega} \geqslant k\right)$.

2. If $v^m_{\pi^\omega} < k$ and $\text{supp}(\hat{\pi}^\omega) = \{v^m_{\pi^\omega}\}$, it must be that, for all $\hat{\rho} \in \text{supp}(\hat{\mathcal{P}}_\pi)$:

$$v''_{\hat{\rho}} < v''_{\hat{\pi}} = v^m_\pi \implies v'_{\hat{\rho}} \leqslant v'_{\hat{\pi}} = v^m_\pi,$$

   which means that $v''_{\hat{\rho}} < v^m_\pi \implies v'_{\hat{\rho}} = v^m_\pi$. Then, it must be that $v^m_{\pi^\omega} = c_{\pi^\omega,D}$ and thus $\mathbb{1}\left(c_{\hat{\pi}^\omega} \geqslant k\right) = \mathbb{1}\left(c_{\pi^\omega} \geqslant k\right) = 0$.

3. If $v^m_{\pi^\omega} < k$, $|\text{supp}(\hat{\pi}^\omega)| = 2$, and $v^m_{\hat{\pi}^\omega} = c_{\hat{\pi}^\omega,R} \geqslant k$, then $\mathbb{1}\left(c_{\hat{\pi}^\omega} \geqslant k\right) = 1 \geqslant \mathbb{1}\left(c_{\pi^\omega} \geqslant k\right)$.

4. If $v^m_{\pi^\omega} < k$, $|\text{supp}(\hat{\pi}^\omega)| = 2$, and $v^m_{\hat{\pi}^\omega} < k$, it must be that $c_{\hat{\pi}^\omega,D} \leqslant c_{\pi^\omega,D}$, $c_{\hat{\pi}^\omega,R} = k \leqslant c_{\pi^\omega,R}$, and $v^m_{\hat{\pi}^\omega} \geqslant v^m_{\pi^\omega}$, so that $\mathbb{1}\left(c_{\hat{\pi}^\omega} \geqslant k\right) \geqslant \mathbb{1}\left(c_{\pi^\omega} \geqslant k\right)$.

□

Consider plan $\mathcal{H} \in \Delta(\Delta([\underline{v},\overline{v}]))$ such that $\int d\pi \mathcal{H}(\pi) = \phi$. The proof proceeds in two steps.

1. First, I show that if $\mathcal{H}$ is optimal it must be that, for all $\pi \in \text{supp}(\mathcal{H})$ (except at most for a zero-measure subset), either $|\text{supp}(\pi)| = 1$, or there exist $v' \neq v''$ such that $\pi(\{v'\}) = \pi(\{v''\}) = \frac{1}{2}$.

Consider the pairwise feasible plan $\hat{\mathcal{H}}$ constructed in Lemma 1. By construction, $\hat{\mathcal{H}}$ is such that for all $\hat{\pi} \in \text{supp}(\hat{\mathcal{H}})$, either $|\text{supp}(\hat{\pi})| = 1$, or there exist $v' \neq v''$ such that $\hat{\pi}(\{v'\}) = \hat{\pi}(\{v''\}) = \frac{1}{2}$. Morevoer, $\hat{\mathcal{H}}$ is such that $\int \mathbb{1}(c_{\hat{\pi}^\omega} \geqslant 0)\, d\hat{\mathcal{H}}(\hat{\pi}) \geqslant \int \mathbb{1}(c_{\hat{\pi}^\omega} \geqslant 0)\, d\mathcal{H}(\hat{\pi})$ for all $\omega \in \text{supp}(\gamma)$.

Suppose that, for a positive-measure subset $S \subseteq \text{supp}(\mathcal{H})$, either $|\text{supp}(\pi)| > 2$ or $\text{supp}(\pi) = \{v', v''\}$ with $v' \neq v''$ and $\pi(\{v'\}) \neq \pi(\{v''\})$. I now show that there exists measurable $\Omega \subseteq \text{supp}(\gamma)$ such that $\gamma(\Omega) > 0$ and:

$$\int \mathbb{1}(c_{\hat{\pi}^\omega} \geqslant k)\, d\hat{\mathcal{H}}(\hat{\pi}) > \int \mathbb{1}(c_{\pi^\omega} \geqslant k)\, d\mathcal{H}(\pi),$$

for all $\omega \in \Omega$.

Take $\pi \in S$ and suppose $\text{supp}(\pi) \subseteq (-\infty, 0]$. Take the measure $\hat{\mathcal{P}}_\pi$ constructed in the proof of Lemma 1. Excluding the cases $|\text{supp}(\pi)| = 1$ and $|\text{supp}(\pi)| = \{v', v''\}$ with $v' \neq v''$, $\pi(\{v'\}) = \pi(\{v''\})$, it must be that $\hat{\mathcal{P}}_\pi\left(\hat{\pi} : v^m_{\hat{\pi}} - c_{\hat{\pi},D} > v^m_\pi - c_{\pi,D}\right) > 0$. For any $\hat{\pi}$ such that $v^m_{\hat{\pi}} - c_{\hat{\pi},D} > v^m_\pi - c_{\pi,D}$, there exist $\underline{\omega} < \overline{\omega}$ such that $2v^m_{\hat{\pi}^\omega} - c_{\hat{\pi}^\omega,D} > 0 > 2v^m_{\pi^\omega} - c_{\pi^\omega,D}$ for all $\omega \in (\underline{\omega}, \overline{\omega})$. Indeed, $2v^m_{\hat{\pi}^\omega} - c_{\hat{\pi}^\omega,D} = 2v^m_{\hat{\pi}} - c_{\hat{\pi},D} - \omega > 0$ for all $\omega \in (v^m_{\hat{\pi}}, 2v^m_{\hat{\pi}} - c_{\hat{\pi},D})$ and $2v^m_{\pi^\omega} - c_{\pi^\omega,D} = 2v^m_\pi - c_{\pi,D} - \omega < 0$ for all $\omega \in (2v_\pi - c_{\pi,D}, +\infty)$. Since $v^m_{\hat{\pi}} - c_{\hat{\pi},D} > v^m_\pi - c_{\pi,D}$, it suffices to take $\underline{\omega} = 2v^m_\pi - c_{\pi,D}$ and $\overline{\omega} = 2v^m_{\hat{\pi}} - c_{\hat{\pi},D}$.
Now, suppose $\text{supp}(\pi) \nsubseteq (-\infty, 0]$. There exists $\omega^\star$ such that, for all $\omega \geqslant \omega^\star$, $\text{supp}(\pi^\omega) \subseteq (-\infty, 0]$. Then, the reasoning for $\text{supp}(\pi) \subseteq (-\infty, 0]$ applies.

2. I showed that if $\mathcal{H}$ is optimal it must be that, for all $\pi \in \text{supp}(\mathcal{H})$, either $|\text{supp}(\pi)| = 1$, or there exist v', v'' such that $\pi(\{v'\}) = \pi(\{v''\}) = \frac{1}{2}$. Now I show that it must be that $v' \geqslant v^m$ and $v'' \leqslant v^m$ for all $\pi \in \text{supp}(\mathcal{H})$ (except for at most a zero-measure subset). Suppose there exist $\overline{S} \subseteq \text{supp}(\mathcal{H})$ such that $\mathcal{H}(\overline{S}) > 0$ and for all $\overline{\pi} \in \overline{S}$, $\text{supp}(\overline{\pi}) = \{\overline{v}'_{\overline{\pi}}, \overline{v}''_{\overline{\pi}}\}$ with $\overline{v}'_{\overline{\pi}} \geqslant \overline{v}''_{\overline{\pi}} \geqslant v^m$. Since $\int \pi d\mathcal{H}(\pi) = \phi$, there must exist $\underline{S} \subseteq \text{supp}(\mathcal{H})$ such that $\mathcal{H}(\underline{S}) = \mathcal{H}(\overline{S}) > 0$ and for all $\underline{\pi} \in \underline{S}$, $\text{supp}(\underline{\pi}) = \{\underline{v}'_{\underline{\pi}}, \underline{v}''_{\underline{\pi}}\}$ with $\underline{v}''_{\underline{\pi}} \leqslant \underline{v}'_{\underline{\pi}} \leqslant v^m$. Consider the measurable set $\hat{S} \subseteq \Delta([\underline{v}, \overline{v}])$ and suppose that, for all $\hat{\pi}' \in \hat{S}$, there exist $\hat{\pi}'' \in \hat{S}$, $\overline{\pi} \in \overline{S}$, and $\underline{\pi} \in \underline{S}$ such that $\hat{\pi}'(\overline{v}'_{\overline{\pi}}) = \hat{\pi}'(\underline{v}'_{\underline{\pi}}) = \frac{1}{2}$, $\hat{\pi}''(\overline{v}''_{\overline{\pi}}) = \hat{\pi}''(\underline{v}''_{\underline{\pi}}) = \frac{1}{2}$. Consider the alternative plan $\hat{\mathcal{H}}$, identical to $\mathcal{H}$ but such that $\hat{\mathcal{H}}(\hat{S}) = \mathcal{H}(\overline{S}) + \mathcal{H}(\underline{S})$ and $\hat{\mathcal{H}}(\overline{S}) = \hat{\mathcal{H}}(\underline{S}) = 0$. Similarly to the first part of this proof, there exist $\Omega \subseteq \text{supp}(\gamma)$ such that $G(\Omega) > 0$ and:

$$\int \mathbb{1}(c_{\pi^\omega} \geqslant k)\, d\hat{\mathcal{H}}(\pi) > \int \mathbb{1}(c_{\pi^\omega} \geqslant k)\, d\mathcal{H}(\pi),$$

for all $\omega \in \Omega$.

■

Proof of Theorem 1. The proof of this result rests on the following lemma.

Lemma 2. *There exists a bijection $t : T(\phi', \phi'') \to \Delta_2$.*

Proof. Call $\Delta_1 \subseteq \Delta([\underline{v}, \overline{v}])$ the set of district distributions $\pi \in \Delta([\underline{v}, \overline{v}])$ such that there exist $v'_\pi \geqslant 0$, $v''_\pi \leqslant 0$ with $\pi(\{v'_\pi\}) = \pi(\{v''_\pi\}) = \frac{1}{2}$. Define the function $s : \Delta_1 \to [\underline{v}, 0] \times [0, \overline{v}]$ such that, for each $\pi \in \Delta_1$, $s(\pi) = (v'_\pi, v''_\pi)$. Note that s is a bijection.

Define the function $t : T(\phi', \phi'') \to \Delta(\Delta_1)$ such that, for all $\tau \in T(\phi', \phi'')$, for all measurable $B \subseteq \Delta_1$, $t(\tau)(B) = \tau(s(B))$. It is easy to see that t is well defined.

First, I show that $t(T(\phi', \phi'')) = \Delta_2$. Indeed, take any $\tau \in T(\phi', \phi'')$. I need to show that $t(\tau)$ is feasible. That is, I show that, for all measurable $A \subseteq [\underline{v}, \overline{v}]$:

$$\int_{\Delta_1} \pi(A) dt(\tau)(\pi) = \phi(A).$$

Note that $\int_{\Delta_1} \pi(A) dt(\tau)(\pi) = \frac{1}{2} \int_{\Delta_1} \pi(A|v \leqslant 0) dt(\tau)(\pi) + \frac{1}{2} \int_{\Delta_1} \pi(A|v \geqslant 0) dt(\tau)(\pi)$. Hence, I show that $\int_{\Delta_1} \pi(A|v \geqslant 0) dt(\tau)(\pi) = \phi'(A)$. The reasoning for $\int_{\Delta_1} \pi(A|v \leqslant 0) dt(\tau)(\pi) = \phi''(A)$ is analogous. By definition of t I have:

$$\int_{\Delta_1} \pi(A|v \leqslant 0) dt(\tau)(\pi) = \int_{\Delta_1} \pi(A|v \leqslant 0) d\tau(s(\pi)).$$

With a change of variable I get:

$$\int_{\Delta_1} \pi(A|v \leqslant 0) d\tau(s(\pi)) = \int_{[0,\overline{v}] \times [\underline{v}, 0]} \mathbb{1}(v' \in A) d\tau(v', v'').$$

Finally:

$$\int_{[0,\overline{v}] \times [\underline{v}, 0]} \mathbb{1}(v' \in A) d\tau(v', v'') = \left(\text{proj}_{[0,\overline{v}]} \# \tau\right)(A) = \phi'(A),$$

where the last equality follows from the definition of transport plan.

Second, I show that $\pi \neq \hat{\pi}$ implies $t(\pi) \neq t(\hat{\pi})$. Since $\pi \neq \hat{\pi}$, there exists measurable $B \subseteq [\underline{v}, 0] \times [0, \overline{v}]$ such that $\pi(B) \neq \hat{\pi}(B)$. Then:

$$t(\pi)(B) = \pi(s(B)) \neq \hat{\pi}(s(B)) = t(\hat{\pi})(B)$$

where the second inequality follows from the fact that s is a bijection. □

I now show that $\tau^\star \in T^\star$ if and only if $\mathcal{H}^\star = t(\tau^\star) \in \Delta_2^\star$.

For any $\mathcal{H} \in \Delta_2$, I have:

$$\int\int \mathbb{1}(c_{\pi^\omega} \geqslant k) d\mathcal{H}(\pi) d\gamma(\omega)$$

$$=$$

$$\int\int \mathbb{1}(v'_\pi - \omega \geqslant k) + \mathbb{1}(v'_\pi - \omega < k)\mathbb{1}\left(v'_\pi - \omega - \frac{v''_\pi - \omega + k}{2} \geqslant k\right) d\mathcal{H}(\pi) d\gamma(\omega).$$

By switching the order of integration and further manipulating, I get:

$$\int\int \mathbb{1}(v'_\pi - \omega \geqslant k) + \mathbb{1}(v'_\pi - \omega < k)\mathbb{1}\left(v'_\pi - \omega - \frac{v''_\pi - \omega + k}{2} \geqslant k\right) d\gamma(\omega) d\mathcal{H}(\pi)$$

$$=$$

$$\int\int \mathbb{1}(v'_\pi - \omega \geqslant k) d\gamma(\omega) + \int \mathbb{1}(v'_\pi - \omega < k)\mathbb{1}\left(v'_\pi - \omega - \frac{v''_\pi - \omega + k}{2} \geqslant k\right) d\gamma(\omega) d\mathcal{H}(\pi)$$

$$=$$

$$\int G(v'_\pi - k) + G\left(2v'_\pi - v''_\pi - k\right) - G(v'_\pi - k) d\mathcal{H}(\pi)$$

$$=$$

$$\int G\left(2v'_\pi - v''_\pi - k\right) d\mathcal{H}(\pi).$$

Using Proposition 1, $\mathcal{H}^\star \in \Delta_2$ is optimal if and only if:

$$\int G\left(2v'_\pi - v''_\pi - k\right) d\mathcal{H}(\pi) \leqslant \int G\left(2v'_\pi - v''_\pi - k\right) d\mathcal{H}^\star(P) \text{ for all} \mathcal{H} \in \Delta_2.$$

With a change of variable:

$$\int G\left(2v'_\pi - v''_\pi - k\right) dt(\tau)(P) = \int G\left(2v'_\pi - v''_\pi - k\right) d\tau(s(P)) = \int G\left(2v'_\pi - v''_\pi - k\right) d\tau(v'_\pi, v''_\pi)$$

$$\leqslant$$

$$\int G\left(2v'_\pi - v''_\pi - k\right) dt(\tau^\star)(\pi) = \int G\left(2v'_\pi - v''_\pi - k\right) d\tau^\star(s(\pi)) = \int G\left(2v'_\pi - v''_\pi - k\right) d\tau^\star(v'_\pi, v''_\pi),$$

for all $\tau \in T(\phi', \phi'')$ and for $\tau^\star = t^{-1}(\mathcal{H}^\star)$. Hence, $\mathcal{H}^\star$ is optimal if and only if $\tau^\star$ is optimal.

■

B.2 Proofs of Section 4

In this subsection, I borrow results in Santambrogio (2015) and Chiappori et al. (2010) as building blocks to characterize the solution to (OTP). In particular, Lemma 3, Lemma 4, and

Lemma 5 are adapted for my context from the above references. Given a function $f: X \to \mathbb{R}$ locally Lipschitz, I define its *superderdifferential* at $x_0 \in X$, $\partial f(x_0)$, to consist of the set of real numbers β such that:

$$f(x) \leqslant f(x_0) + \beta(x - x_0) + o(|x - x_0|) \text{ as } x \to x_0,$$

with the error term being allowed to depend on the x_0. Note that if the function is differentiable at x_0, I have that $\partial f(x_0) = \{f'(x_0)\}$. I provide the following definitions.

Definition 2. *G satisfies the twist condition whenever G is locally Lipschitz and $\partial G(a) \cap \partial G(b) = \varnothing$ for all $a \neq b$.*

Definition 3. *G satisfies the sub-twist condition whenever G is locally Lipschitz and, for all $a \in \operatorname{supp}(\gamma)$:*

$$|\{b \in \operatorname{supp}(\gamma) : b \neq a,\ \partial G(a) \cap \partial G(b) \neq \varnothing\}| \leqslant 1.$$

Definition 4. *For any $\tau \in T(\phi', \phi'')$, $\operatorname{supp}(\tau) \subseteq [0, \overline{v}] \times [\underline{v}, 0]$ is cyclically monotone (CM) if, for every $n \in \mathbb{N}$, every permutation σ, and every finite family of points $(v_1', v_1''), \ldots, (v_n', v_n'') \in \operatorname{supp}(\tau)$:*

$$\sum_{i=1}^{k} G(2v_i' - v_i'' - k) \geqslant \sum_{i=1}^{k} G(2v_i' - v_{\sigma(i)}'' - k).$$

Then, the following lemmas hold.

Lemma 3. *If G satisfies the twist condition, there exists a unique, pure solution to (OTP).*

Proof. By Theorem 2 in Chiappori et al. (2010). □

Lemma 4. *If G satisfies the sub-twist condition, there exists a unique solution to (OTP).*

Proof. By Theorem 3 in Chiappori et al. (2010). □

Lemma 5. *If $\tau \in T(\phi', \phi'')$ is a solution to (OTP), then $\operatorname{supp}(\tau)$ is CM.*

Proof. By assumption, G is continuous. Hence, the result holds by Theorem 1.38 in Santambrogio (2015). □

I am now ready to prove the results in Section 4.

Proof of Proposition 2. Suppose G is strictly convex. Then, for all x, either $\partial G(x) = \varnothing$, or $\inf\ \partial G(x') > \sup\ \partial G(x)$ for all $x' > x$. Then, G satisfies the twist condition, and, by Lemma 3, there exists a unique, pure solution $\tau^\star$ to (OTP). Take any $(v', v''), (\tilde{v}', \tilde{v}'') \in \text{supp}(\tau^\star)$, such that $v' > \tilde{v}'$. By Lemma 5, it must be that:

$$G(2v' - v'' - k) + G(2\tilde{v}' - \tilde{v}'' - k) \geqslant G(2v' - \tilde{v}'' - k) + G(2\tilde{v}' - v'' - k).$$

Since G is strictly convex, the function $\Phi(v', v'') = G(2v' - v'' - k)$ is strictly submodular. Suppose that $v'' > \tilde{v}''$. By submodularity:

$$G(2v' - v'' - k) + G(2\tilde{v}' - \tilde{v}'' - k) < G(2v' - \tilde{v}'' - k) + G(2\tilde{v}' - v'' - k).$$

Hence, it must be $v'' \leqslant \tilde{v}''$.

Suppose G is strictly concave. Then, for all x, $\sup\ \partial G(x) < \inf\ \partial G(x')$ for all $x' > x$. Then, G satisfies the twist condition, and, by Lemma 3, there exists a unique, pure solution $\tau^\star$ to (OTP). Take any $(v', v''), (\tilde{v}', \tilde{v}'') \in \text{supp}(\tau^\star)$, such that $v' > \tilde{v}'$. By Lemma 5, it must be that:

$$G(2v' - v'' - k) + G(2\tilde{v}' - \tilde{v}'' - k) \geqslant G(2v' - \tilde{v}'' - k) + G(2\tilde{v}' - v'' - k).$$

Since G is strictly concave, the function $\Phi(v', v'') = G(2v' - v'' - k)$ is strictly supermodular. Suppose that $v'' < \tilde{v}''$. By supermodularity:

$$G(2v' - v'' - k) + G(2\tilde{v}' - \tilde{v}'' - k) < G(2v' - \tilde{v}'' - k) + G(2\tilde{v}' - v'' - k).$$

Hence, it must be $v'' \geqslant \tilde{v}''$.

Now, suppose G is affine. Then, for any $\tau \in T(\phi', \phi'')$:

$$\begin{aligned}\int G(2v' - v'' - k)\,d\tau(v', v'') = G\left(\int (2v' - v'' - k)\,d\tau(v', v'')\right) = \\ = G\left(\int 2v'\,d\phi' - \int v''\,d\phi'' - k\right),\end{aligned}$$

by definition of $T(\phi', \phi'')$. Since the objective function is constant over $T(\phi', \phi'')$, I have that $T^\star = T(\phi', \phi'')$.

■

Proof of Proposition 3. First, since G is strictly convex below 0 and strictly concave above 0, it satisfies the sub-twist condition, even if it does not necessarily satisfy the twist condition. Hence, by Lemma 4, a solution τ to (OTP) exists and is unique.

Take any $(v', v''), (\tilde{v}', \tilde{v}'') \in \text{supp}(\tau)$, such that $2v' - v'' - k < 0$ and $2\tilde{v}' - \tilde{v}'' - k < 0$. Suppose that $v' > \tilde{v}'$. By Lemma 5, it must be that:

$$G(2v' - v'' - k) + G(2\tilde{v}' - \tilde{v}'' - k) \geqslant G(2v' - \tilde{v}'' - k) + G(2\tilde{v}' - v'' - k).$$

Suppose that $v'' > \tilde{v}''$. Since G is strictly convex below 0, I have:

$$G(2v' - v'' - k) + G(2\tilde{v}' - \tilde{v}'' - k) < G(2v' - \tilde{v}'' - k) + G(2\tilde{v}' - v'' - k).$$

Hence, it must be $v'' \leqslant \tilde{v}''$.

Take any $(v', v''), (\tilde{v}', \tilde{v}'') \in \text{supp}(\tau)$, such that $2v' - v'' - k \geqslant 0$ and $2\tilde{v}' - \tilde{v}'' - k \geqslant 0$. Suppose that $v' > \tilde{v}'$. By Lemma 5, it must be that:

$$G(2v' - v'' - k) + G(2\tilde{v}' - \tilde{v}'' - k) \geqslant G(2v' - \tilde{v}'' - k) + G(2\tilde{v}' - v'' - k).$$

Suppose that $v'' < \tilde{v}''$. Since G is strictly concave above 0, I have:

$$G(2v' - v'' - k) + G(2\tilde{v}' - \tilde{v}'' - k) < G(2v' - \tilde{v}'' - k) + G(2\tilde{v}' - v'' - k).$$

Hence, it must be $v'' \geqslant \tilde{v}''$.

Suppose $\tau(\{v', v'' : 2v' - v'' - k \geqslant 0\}) > 0$ and $\tau(\{v', v'' : 2v' - v'' - k < 0\}) > 0$. Define the measure τ^+ to be $\tau^+(\cdot) = \tau(\cdot | \{v', v'' : 2v' - v'' - k \geqslant 0\})$, and the measure τ^- to be $\tau^-(\cdot) = \tau(\cdot | \{v', v'' : 2v' - v'' - k < 0\})$. Note that τ^+ is positive assortative, while τ^- is negative assortative. Moreover:

$$\tau(\cdot) = \tau\left(\{v', v'' : 2v' - v'' - k \geqslant 0\}\right)\tau^+(\cdot) + \tau\left(\{v', v'' : 2v' - v'' - k < 0\}\right)\tau^-(\cdot).$$

Suppose $\tau(\{v', v'' : 2v' - v'' - k \geqslant 0\}) = 0$. Then define $\tau^-(\cdot) = \tau(\cdot)$ and note that it is negative assortative.

Suppose $\tau(\{v', v'' : 2v' - v'' - k < 0\}) = 0$. Then define $\tau^+(\cdot) = \tau(\cdot)$ and note that it is positive assortative.

■

Proof of Proposition 4. Consider plan $\tau = \alpha\tau^+ + (1 - \alpha)\tau^-$, in $T^\pm$. Define $\Gamma^+ = \text{supp}(\tau^+)$, $\Gamma^- = \text{supp}(\tau^-)$, $\mu^+ = \text{proj}_{[0,\overline{v}]}\#\tau^+$, $\mu^- = \text{proj}_{[0,\overline{v}]}\#\tau^-$, $\nu^+ = \text{proj}_{[\underline{v},0]}\#\tau^+$, $\nu^- = \text{proj}_{[\underline{v},0]}\#\tau^-$. Consider any set $\hat{\Gamma} \subseteq \Gamma^-$ and small $0 < \epsilon \leqslant 1$. Construct the measure $\hat{\tau}^- = \tau^-(\cdot | \hat{\Gamma})$, with marginals $\hat{\mu}^- = \text{proj}_{[0,\overline{v}]}\#\hat{\tau}^-$ and $\hat{\nu}^- = \text{proj}_{[\underline{v},0]}\#\hat{\tau}^-$. Define:

$$\tilde{\tau}^- = \frac{\tau^- - \epsilon\tau^-(\hat{\Gamma})\hat{\tau}^-}{1 - \epsilon\tau^-(\hat{\Gamma})},$$

with marginals $\tilde{\mu}^- = \frac{\mu^- - \epsilon\tau^-(\hat{\Gamma})\hat{\mu}^-}{1-\epsilon\tau^-(\hat{\Gamma})}$ and $\tilde{\nu}^- = \frac{\nu^- - \epsilon\tau^-(\hat{\Gamma})\hat{\nu}^-}{1-\epsilon\tau^-(\hat{\Gamma})}$. Then, construct $\tilde{\tau}^+$ to be positive assortative with marginals $\tilde{\mu}^+ = \frac{\mu^+ + \frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})\hat{\mu}^-}{1+\frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})}$ and $\tilde{\nu}^+ = \frac{\nu^+ + \frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})\hat{\nu}^-}{1+\frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})}$. Finally, consider the measure:

$$\tilde{\tau} = \alpha\left(1 + \frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})\right)\tilde{\tau}^+ + (1-\alpha)(1-\epsilon\tau^-(\hat{\Gamma}))\tilde{\tau}^-.$$

The proof relies on the following lemma.

Lemma 6. *If $\tilde{\tau}$ is in $T^{\pm}$, then:*

$$\int G(2v' - v'' - k)d\tilde{\tau} > \int G(2v' - v'' - k)d\tau.$$

Proof. By substitution:

$$\int G(2v' - v'' - k)d\tilde{\tau} =$$

$$= \alpha\left(1 + \frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})\right)\int G(2v' - v'' - k)d\tilde{\tau}^+ + (1-\alpha)(1-\epsilon\tau^-(\hat{\Gamma}))\int G(2v' - v'' - k)d\tilde{\tau}^-,$$

where:

$$+(1-\alpha)(1-\epsilon\tau^-(\hat{\Gamma}))\int G(2v' - v'' - k)d\tilde{\tau}^- =$$

$$= (1-\alpha)\int G(2v' - v'' - k)d\tau^- - (1-\alpha)\epsilon\tau^-(\hat{\Gamma})\int G(2v' - v'' - k)d\hat{\tau}^-.$$

Hence, the inequality in the statement of the present lemma holds if and only if:

$$\alpha\left(1 + \frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})\right)\int G(2v' - v'' - k)d\tilde{\tau}^+ - (1-\alpha)\epsilon\tau^-(\hat{\Gamma})\int G(2v' - v'' - k)d\hat{\tau}^- >$$

$$> \alpha\int G(2v' - v'' - k)d\tau^+.$$

Now, define $\hat{\tau}^+ = \tilde{\tau}^+(\cdot|\operatorname{supp}(\hat{\mu}^-) \times [\underline{v}, 0])$, with marginals $\hat{\mu}^+ = \operatorname{proj}_{[0,\overline{v}]}\#\hat{\tau}^+ = \hat{\mu}^-$ and $\hat{\nu}^+ = \operatorname{proj}_{[\underline{v},0]}\#\hat{\tau}^+$. Morevoer, define $\overline{\tau}^+ = \tilde{\tau}^+(\cdot|\operatorname{supp}(\mu^+) \times [\underline{v}, 0])$, with marginals $\bar{\mu}^+ = \operatorname{proj}_{[0,\overline{v}]}\#\overline{\tau}^+ = \mu^+$ and $\overline{\nu}^+ = \operatorname{proj}_{[\underline{v},0]}\#\overline{\tau}^+$. Note that:

$$\tilde{\tau}^+ = \frac{\overline{\tau}^+ + \frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})\hat{\tau}^+}{1 + \frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})}.$$

By substituting in the inequality:

$$(1-\alpha)\epsilon\tau^-(\hat{\Gamma})\int G(2v' - v'' - k)d\hat{\tau}^+ - (1-\alpha)\epsilon\tau^-(\hat{\Gamma})\int G(2v' - v'' - k)d\hat{\tau}^- >$$

$$> \alpha\int G(2v' - v'' - k)d\tau^+ - \alpha\int G(2v' - v'' - k)d\overline{\tau}^+.$$

By assumption $\tilde{\tau}$ is in $T^{\pm}$, so that $\int G(2v' - v'' - k)d\hat{\tau}^+ > 0$.

To show the inequality holds, I proceed in three steps:

1. Because G is s-shaped and symmetric around 0, it must be that:

$$\frac{\int G(2v'-v''-k)d\hat{\tau}^+ - \int G(2v'-v''-k)d\hat{\tau}^-}{\int(2v'-v''-k)d\hat{\tau}^+ - \int(2v'-v''-k)d\hat{\tau}^-} >$$
$$> \frac{\int G(2v'-v''-k)d\tau^+ - \int G(2v'-v''-k)d\overline{\tau}^+}{\int(2v'-v''-k)d\tau^+ - \int(2v'-v''-k)d\overline{\tau}^+}$$

2. I have that:

$$\int(2v'-v''-k)d\hat{\tau}^+ - \int(2v'-v''-k)d\hat{\tau}^- =$$
$$= -\int v''d\hat{v}^+ + \int v''d\hat{v}^-,$$

and:

$$\int(2v'-v''-k)d\tau^+ - \int(2v'-v''-k)d\overline{\tau}^+ =$$
$$= -\int v''dv^+ + \int v''d\overline{v}^+.$$

Note that it must be that:

$$v^+ + \frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})\hat{v}^- = \overline{v}^+ + \frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})\hat{v}^+,$$

so that:

$$\int(2v'-v''-k)d\tau^+ - \int(2v'-v''-k)d\overline{\tau}^+ =$$
$$= \frac{1-\alpha}{\alpha}\epsilon\tau^-(\hat{\Gamma})\int(2v'-v''-k)d\hat{\pi}^+ - \int(2v'-v''-k)d\hat{\tau}^-.$$

3. Putting together 1. and 2. delivers the desired inequality.

□

Suppose $(1-\alpha)\neq 0$ and $2v'-v''-k>0$ for all $(v',v'')\in \text{supp}(\tau^+)$. It suffices to show that there exist $\hat{\Gamma}$ and ϵ so that $\tilde{\tau}$ is in $T^{\pm}$. Consider $v^\star = \sup_{\text{supp}(\Gamma^-)}(v',-v'')$. Call $B_\delta(v^\star)\subseteq \Gamma^-$ a neighborhood of $v^\star$ in Γ^- of radius δ.

Because ϕ admits a continuous density, and $2v'-v''-k>0$ for all $v',v''\in\text{supp}(\tau^+)$, there exist δ and ϵ, such that, for $\hat{\Gamma}=B_\delta(v^\star)$, $2v'-v''-k\geqslant 0$ for all $v',v''\in\text{supp}(\hat{\tau}^+)$ and for all $v',v''\in\text{supp}(\overline{\tau}^+)$, so that $\tilde{\tau}$ is in $T^{\pm}$.

■

B.3 Proofs of Section 5

Proof of Proposition 5. By Proposition 1, any district π in an optimal plan $\mathcal{H}$ must be such that $\pi(\{v'\}) = \pi(\{v''\}) = \frac{1}{2}$ for $v' \geqslant 0$ and $v'' \leqslant 0$. Fix shock realization ω. Consider the following cases:

- $v' - \omega \geqslant k$. Then, $c_\pi = v' - \omega \geqslant k$.
- $v' - \omega < k$ and $v' - \omega - \frac{v'' - \omega + k}{2} \geqslant k$. Then $c_\pi = k$.
- $v' - \omega < k$ and $v' - \omega - \frac{v'' - \omega + k}{2} < k$. Then, $c_\pi = v'' - \omega \leqslant -\omega$.

Hence, $Q^\omega_{\mathcal{H}}((-\omega, k)) = 0$.

■

B.4 Proofs of Section 6

Proof of Proposition 6. First, note that a necessary, but not sufficient, condition for the designer to win district π when the realized shock is ω, is that $\pi^\omega(\{v : v \geqslant k\}) \geqslant \frac{2q-1}{2q}$. Suppose not. Then $\pi^\omega(\{v : v < k\}) > 1 - \frac{2q-1}{2q} = \frac{1}{2q}$. The equilibrium position of the Democratic candidate is $v^q_{\pi^\omega,D}$, the leftmost q-quantile of π^ω conditional on $v < k$. Then, it must be that $\pi^\omega(\{v : v \leqslant v^q_{\pi^\omega,D}\}) > \frac{1}{2}$, which means that the district is won by the Democrats. Now, for any redistricting plan $\mathcal{H}$ and any realization of the schok ω, define by $\chi^\omega \in \Delta([0,1])$ the distribution over $x^\omega = \pi^\omega(\{v : v \geqslant k\})$, that is the distribution of Republican voters across districts. Then, an upper bound on the designer utility is:

$$\int \mathbb{1}\left(x^\omega \geqslant \frac{2q-1}{2q}\right) d\chi^\omega(x^\omega) \leqslant \int \frac{2q}{2q-1} x^\omega d\chi^\omega(x^\omega) = \min\left\{1, \frac{2q}{2q-1}(1 - F(k+\omega))\right\},$$

where the equality holds by the law of iterated expectations. To finish the proof, it suffice to show that the above upper bound can always be achieved. Because F is uniform, it is easy to see that is achieved by the redistricting plan that matches any $v' \in [v^q, \overline{v}]$ to a $v'' = v^q - \frac{\overline{v} - v'}{2q-1} \in [0, v^q]$ and to a $v''' = -\frac{q}{2q-1}(\overline{v} - v) \in [\underline{v}, 0]$, with respective weights $\frac{2q-1}{2q}$, $\frac{1}{2q}$, and $\frac{1}{2}$.

■

Proof of Proposition 7. For any redistricting plan $\mathcal{H}$, call $H(\omega)$ the measure of districts won by the designer when the aggregate shock takes realization ω. By Proposition 6, at any realized shock ω, the designer can win at most measure $\min\left\{1, \frac{2q}{2q-1}(1 - F(k+\omega))\right\}$ of districts, so $H(\omega) \leqslant \min\left\{1, \frac{2q}{2q-1}(1-F(k+\omega))\right\}$ at any ω. This implies that any feasible H must satisfy $H(\omega) \leqslant H^\star(\omega)$, where:

$$H^\star(\omega) = \begin{cases} 1 & if \;\; \omega \leqslant v^q - k \\ \frac{2q}{2q-1}(1-F(k+\omega)) & if \;\; \omega > v^q - k \end{cases}$$

The designer expected utility for any feasible H is then:

$$\int H(\omega) d\gamma(\omega) \leqslant \int H^\star(\omega) d\gamma(\omega),$$

with strict inequality if $H(\omega) \neq H^\star(\omega)$ for any ω. If $H^\star$ is attainable at every ω, $\mathcal{H}$ is optimal if and only if it induces $H^\star$. $H^\star$ is always attainable, as shown by the proof of Proposition 6. This means that $\mathcal{H}$-almost all the districts π the designer wins if and only if the shock is at most ω must satisfy $\pi(\{v : v = \omega\}) = 1 - \pi(\{v : v < v^q\}) = \frac{2q-1}{2q}$. However, since to win a district the designer needs at least $\frac{1}{2}$ of voters to vote for the Republican candidate and $\frac{2q-1}{q} \leqslant \frac{1}{2}$, some voters with $v < v^q$ must vote for R. Precisely $\frac{1}{2} - \frac{2q-1}{2q}$ additional voters must prefer $v = \omega$, the Republican candidate, to $v^q_{\pi^\omega,D}$, the Democratic candidate, with $v^q_{\pi^\omega,D}$ being the q - quantile of π^ω conditional on $v < k$. But given that $P(\{v : v < k\}) = \frac{1}{2q}$, the Democratic candidate sits at a median of π. For him not to win the district it must be that:

$$|v^q_{\pi^\omega,D} - v^m_{\pi^\omega}| \geqslant |v^m_{\pi^\omega} - \omega|.$$

■

B.5 Proofs of Appendix A

Proof of Proposition 8. Consider a Democratic voter with ideal point t (the reasoning is similar for a Republican voter). Given position y of the Republican candidate, the expected utility of electing a first-stage Democratic candidate with position x is:

$$U(x;y,t) = u(x,t)p(x,y) + u(y,t)(1-p(x,y)),$$

where $p(x,y)$ is the probability of the Democratic candidate winning against the Republican candidate in the general elections. The proof of this result relies on the following lemma.

Lemma 7. *$U(x;y,t)$ is strictly increasing for $x < t$, strictly decreasing for $x > t$, and achieves its maximum at $x = t$.*

Proof. First, suppose $y > t$. Consider three cases:

Case $x > y$. First, note that $U(x;y,t) \leqslant U(y;y,t)$. Moreover, $U(x;y,t)$ is decreasing:

$$U(x;y,t) = -(x-t)\left(1 - \frac{\frac{x+y}{2} - \underline{v}}{\overline{v} - \underline{v}}\right) - (y-t)\left(\frac{\frac{x+y}{2} - \underline{v}}{\overline{v} - \underline{v}}\right),$$

and by taking the first order derivative:

$$U'(x;y,t) = -\left(1 - \frac{\frac{x+y}{2} - \underline{v}}{\overline{v} - \underline{v}}\right) + \frac{x-t}{2(\overline{v} - \underline{v})} - \frac{y-t}{2(\overline{v} - \underline{v})} =$$

$$= -1 + \frac{x+y-2\underline{v}}{2(\overline{v} - \underline{v})} + \frac{x-y}{2(\overline{v} - \underline{v})} = -1 + \frac{x-2\underline{v}}{2(\overline{v} - \underline{v})} < 0$$

Case $t < x < y$. $U(x;y,t)$ is decreasing:

$$U(x;y,t) = -(x-t)\left(\frac{\frac{x+y}{2} - \underline{v}}{\overline{v} - \underline{v}}\right) - (y-t)\left(1 - \frac{\frac{x+y}{2} - \underline{v}}{\overline{v} - \underline{v}}\right),$$

and by taking the first order derivative:

$$U'(x;y,t) = -\left(\frac{\frac{x+y}{2} - \underline{v}}{\overline{v} - \underline{v}}\right) - \frac{x-t}{2(\overline{v} - \underline{v})} + \frac{y-t}{2(\overline{v} - \underline{v})} =$$

$$= -\frac{x+y-2\underline{v}}{2(\overline{v} - \underline{v})} + \frac{y-x}{2(\overline{v} - \underline{v})} = \frac{-2x+2\underline{v}}{2(\overline{v} - \underline{v})} < 0$$

Case $x < t$. $U(x;y,t)$ is increasing:

$$U(x;y,t) = -(t-x)\left(\frac{\frac{x+y}{2} - \underline{v}}{\overline{v} - \underline{v}}\right) - (y-t)\left(1 - \frac{\frac{x+y}{2} - \underline{v}}{\overline{v} - \underline{v}}\right),$$

and by taking the first order derivative:

$$U'(x;y,t) = \left(\frac{\frac{x+y}{2} - \underline{v}}{\overline{v} - \underline{v}}\right) + \frac{x-t}{2(\overline{v} - \underline{v})} + \frac{y-t}{2(\overline{v} - \underline{v})} =$$

$$= \frac{x+y-2\underline{v}}{2(\overline{v} - \underline{v})} + \frac{y+x-2t}{2(\overline{v} - \underline{v})} = \frac{2(x+y-t-\underline{v})}{2(\overline{v} - \underline{v})} > 0$$

Second, suppose $y < t$. Consider three cases:

Case $x < y$. First, note that $U(x;y,t) < U(y;y,t)$. Moreover, $U(x;y,t)$ is increasing:

$$U(x;y,t) = -(t-x)\left(\frac{\frac{x+y}{2}-\underline{v}}{\overline{v}-\underline{v}}\right) - (t-y)\left(1-\frac{\frac{x+y}{2}-\underline{v}}{\overline{v}-\underline{v}}\right),$$

and by taking the first order derivative:

$$U'(x;y,t) = \left(\frac{\frac{x+y}{2}-\underline{v}}{\overline{v}-\underline{v}}\right) + \frac{x-t}{2(\overline{v}-\underline{v})} - \frac{y-t}{2(\overline{v}-\underline{v})} =$$

$$= \frac{x+y-2\underline{v}}{2(\overline{v}-\underline{v})} + \frac{x-y}{2(\overline{v}-\underline{v})} = \frac{x-2\underline{v}}{2(\overline{v}-\underline{v})} > 0$$

Case $y < x < t$. $U(x;y,t)$ is increasing:

$$U(x;y,t) = -(t-x)\left(1-\frac{\frac{x+y}{2}-\underline{v}}{\overline{v}-\underline{v}}\right) - (t-y)\left(\frac{\frac{x+y}{2}-\underline{v}}{\overline{v}-\underline{v}}\right),$$

and by taking the first order derivative:

$$U'(x;y,t) = \left(1-\frac{\frac{x+y}{2}-\underline{v}}{\overline{v}-\underline{v}}\right) - \frac{x-t}{2(\overline{v}-\underline{v})} + \frac{y-t}{2(\overline{v}-\underline{v})} =$$

$$= 1 - \frac{x+y-2\underline{v}}{2(\overline{v}-\underline{v})} + \frac{y-x}{2(\overline{v}-\underline{v})} = 1 - \frac{-2x+2\underline{v}}{2(\overline{v}-\underline{v})} > 0$$

Case $x > t$. $U(x;y,t)$ is decreasing:

$$U(x;y,t) = -(x-t)\left(1-\frac{\frac{x+y}{2}-\underline{v}}{\overline{v}-\underline{v}}\right) - (t-y)\left(\frac{\frac{x+y}{2}-\underline{v}}{\overline{v}-\underline{v}}\right),$$

and by taking the first order derivative:

$$U'(x;y,t) = -\left(1-\frac{\frac{x+y}{2}-\underline{v}}{\overline{v}-\underline{v}}\right) + \frac{x-t}{2(\overline{v}-\underline{v})} + \frac{y-t}{2(\overline{v}-\underline{v})} =$$

$$= -1 + \frac{x+y-2\underline{v}}{2(\overline{v}-\underline{v})} + \frac{y+x-2t}{2(\overline{v}-\underline{v})} = -1 + \frac{2(x+y-t-\underline{v})}{2(\overline{v}-\underline{v})} < 0.$$

Finally, note that $U(x;y,t)$ is continuous, so it is reaches its maximum at $x = t$, it is strictly decreasing for $x > t$, and strictly increasing for $x < t$.

□

Given Lemma 7, a Democratic candidate at $v^m_{\pi,D}$ will win first-stage elections against any candidate at a different position (and similarly for Republicans). Indeed, Democratic voters to the right of $v^m_{\pi,D}$, accounting for half of all Democratic voters, prefer a candidate at $v^m_{\pi,D}$ to any other candidate to the left of $v^m_{\pi,D}$. Moreover, Democratic voters to the left of $v^m_{\pi,D}$, prefer a candidate at $v^m_{\pi,D}$ to any other candidate to the right of $v^m_{\pi,D}$. Importantly, this reasoning is independent of the Republican candidate's position y. Since positioning at $v^m_{\pi,D}$ ($v^m_{\pi,R}$, respectively) gives a Democratic (Republican) first-stage candidate a positive probability of winning second-stage elections, there exists an equilibrium where both Democratic candidates set at $v^m_{\pi,D}$ and both Republican candidates set at $v^m_{\pi,R}$.

Now, I show there can not be any other equilibrium. First, any situation where first-stage candidates do not tie can not be an equilibrium, because the losing candidate can move to $v^m_{\pi,D}$ ($v^m_{\pi,R}$) and have positive probability of winning second-statge elections. Second, any situation where first-stage candidates do not set the same position can not be an equilibrium. To see this, note that, in order to have different positions and tie at the same time, one of the Democratic (Republican) first-stage candidate needs to choose a position to the left (right) of $v^m_{\pi,D}$ ($v^m_{\pi,R}$). However, any such position is outside of the support of the median distribution, since candidates know the position of the conditional medians, and is therefore dominated by $v^m_{\pi,D}$ ($v^m_{\pi,R}$). Finally, suppose the two first-stage candidates set at the same position, different from $v^m_{\pi,D}$ ($v^m_{\pi,R}$). Then, there exists small $\epsilon > 0$ such that one of the candidates has a profitable deviation by moving closer to $v^m_{\pi,D}$ ($v^m_{\pi,R}$) by ϵ.

■