POLS/CS&SS 503: Advanced Quantitative Political Methodology

MATRIX ALGEBRA, LINEAR REGRESSION

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Agenda

- · Linear regression as finding a "best" line
- Linear regression as the conditional expectation function
- How linear regression relates to the normal distribution

What is regression?

Regression

distribution of a **response** (outcome) variable Y — or summary of that distribution — as a function of **explanatory** variables X_1, \dots, X_k .

Ordinary Least Squares

Finds a $\hat{Y}=XB$ that minimizes $\sum (Y_i-\hat{Y}_i)^2$). This estimates a linear conditional expectation function $E(Y|X_1,\ldots,X_k)$.

OLS Objective Function

One X

Find the line

$$\hat{Y} = A + BX$$

such that

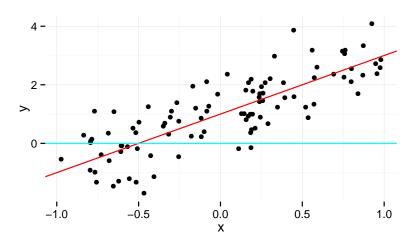
$$A, B = \underset{A,B}{\operatorname{arg\,min}} S(A, B)$$

where

$$S(A, B) = \sum_{i} E_{i}^{2} = \sum_{i} (Y_{i} - \hat{Y}_{i})^{2} = \sum_{i} (Y_{i} - A - BX_{i})^{2}$$

How do we minimize this?

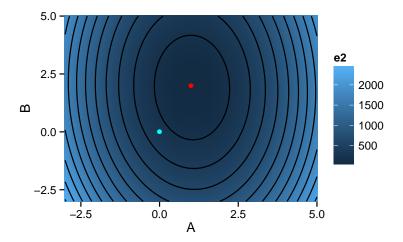
What does the OLS objective function look like?



Data generated by $Y_i=1+2X_i+E_i$. Lines are A=1,B=2, and A=0,B=0.

$\sum E_i^2$ as a function of A and B

Least squares is the minimum of this function



Finding the best A, B in Least Squares One X

To minimize, set partial derivatives equal to 0 and solve:

$$\frac{\partial S(A,B)}{\partial A} = \sum (-1)(2)(Y_i - A - BX_i) = 0$$
$$\frac{\partial S(A,B)}{\partial B} = \sum (-X_i)(2)(Y_i - A - BX_i) = 0$$

Rearrange to get

$$A = \bar{Y} - B\bar{X}$$

$$B = \frac{\sum (X - \bar{X})(Y - \bar{Y})}{\sum (X_i - \bar{X})^2} = \frac{C(X, Y)}{V(X)}$$

Implications of the OLS Solution

Least squares *A* and *B*

$$A = \bar{Y} - B\bar{X}$$

$$B = \frac{\sum (X - \bar{X})(Y - \bar{Y})}{\sum (X_i - \bar{X})^2} = \frac{C(X, Y)}{V(X)}$$

- \bar{X}, \bar{Y} is in the regression line
- $\sum X_i E_i = 0$

$$\sum X_i E_i = \sum X_i (Y_i - A - BX_i)$$
$$= \sum X_i Y_i - A \sum X_i - B \sum X_i = 0$$

- $\sum \hat{Y}_i E_i = 0$
- Errors E uncorrelated with \hat{Y} and X

OLS Objective Function

Multiple X

Find plane

$$Y = A + B_1 X_1 + B_2 X_2 + \dots + B_k X_k$$

such that

$$A, B_1, \dots, B_k = \underset{A, B_1, \dots, B_k}{\operatorname{arg \, min}} S(A, B_1, \dots, B_k)$$

where

$$\begin{split} S(A,B_1,\ldots,B_k) &= \sum_i E_i^2 = \sum_i (Y_i - \hat{Y}_i)^2 \\ &= \sum_i (Y_i - A - \sum_{i=1}^k B_i X_{i,j}) \end{split}$$

How do we minimize this?

Finding the best A, B in Least Squares Regression Multiple X

Set partial derivatives equal to 0 and solve system of equations for

$$\begin{split} \frac{\partial S(A, B_1, B_2, \dots, B_k)}{\partial A} &= \sum (-1)(2)(Y_i - A - BX_i) = 0\\ \frac{\partial S(A, B_1, B_2, \dots, B_k)}{\partial B_1} &= \sum (-X_{i,1})(2)(Y_i - A - B_1X_{i,1} - \dots - B_2X_{i,k}) = 0\\ &\vdots = \vdots\\ \frac{\partial S(A, B_1, B_2, \dots, B_k)}{\partial B_k} &= \sum (-X_{i,k})(2)(Y_i - A - B_1X_{i,1} - \dots - B_2X_{i,k}) = 0 \end{split}$$

Not as easy ...

Scalar representation

$$Y_i = B_0 + B_1 X_{i,1} + B_2 X_{i,2} + \dots B_k X_{i,k} + E_i$$

Equivalent matrix representation

$$y = X b + e n \times (k+1) (k+1) \times 1 + e n \times 1$$

or

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_{1,1} & X_{2,1} & \dots & X_{k,1} \\ 1 & X_{1,2} & X_{2,2} & \dots & X_{k,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{1,n} & X_{2,n} & \dots & X_{k,n} \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_k \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix}$$

Objective Function

The linear regression is

$$y = Xb + e$$

Want to find the b that minimizes the squared errors:

$$\underset{b}{\operatorname{arg\,min}} S(b)$$

where

$$S(b) = \sum_{i} E_i^2 = e'e$$

= $(y - Xb)'(y - Xb)$

Why does e need to be transposed?

Transpose of Sums

$$(A + B)' = A' + B'$$

$$\left(\begin{bmatrix} 10 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}\right)' = ?$$

$$? = ?$$

Transpose of a product

$$(XB)' = B'X'$$

$$\begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = ?$$

$$? = ?$$

Simplify *e*′*c*

$$\begin{split} e'e &= (y-Xb)'(y-Xb) \\ &= (y'-(Xb)')(y-Xb) & \text{distribute the transpose} \\ &= (y-b'X)(y-Xb) & \text{substitute } b'X' \text{ for } (Xb)' \\ &= y'y-b'X'y-y'Xb+b'X'Xb & \text{multiply out} \\ &= y'y-2b'X'y+b'X'Xb & \text{simplify} \end{split}$$

- To minimize need to calculate derivative of e'e with respect to b.
- · Need two know two things
 - derivative of scalar with respect to vector (2b'X'y)
 - derivative of quadratic form (b'X'Xb)

What is the derivative of scalar with respect to vector

- Need to take derivative of e'e with respect to b to find b that min the sum of squared.
- · A derivative of a scalar with respect to a vector

$$y = a'x = a_1x_1 + a_2x_2 + \dots + a_nx_n$$
$$\frac{\partial y}{\partial x} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}'$$
$$\frac{\partial y}{\partial x} = a$$

Derivative of a quadratic form

- Equivalent to x^2 is inner product x'x
- Vector analogue of ax^2 is x'Xx, where A is $n \times n$ matrix

$$\frac{\partial ax^2}{\partial x} = 2ax$$
$$\frac{\partial x' Ax}{\partial x} = 2Ax$$

OLS in Matrix Form

Minimizing the objective function

1. Take partial derivative of S(b):

$$\frac{\partial S(b)}{b} = \frac{\partial}{b}(y'y - 2b'X'y + b'X'Xb)$$
$$= 0 - (2y'X) + 2(X'X)b$$

2. Set to 0, and solve for *b*:

$$X'Xb = X'y$$
$$b = (X'X)^{-1}X'y$$

What $(X'X)^{-1}$ implies

- For b to be defined $(X'X)^{-1}$ needs to exist
- X'X must be full rank
- rank of X'X is the same as the rank of X
- The rank of X is between n and k+1, means that $n \ge k+1$ (obs > variables)
- k + 1 columns of X must be linearly independent?
 - · Can you have a full set of dummies?
 - Can you include a variable that is always equal to 3?

Takeaways

- Linear regression is the A, B_1, \dots, B_k that solve $\arg\min_{A, B_1, \dots, B_k} \sum E_i^2$
- Solving for linear regression coefficients is relatively easy; linear equations; there's an explicit solution. No iteration required.

Linear Regression and CEF

Linear Regression and Normal Distribution

Interpretation

CEF justification for linear regression justification

- Conditional Expectation Function is $E(Y_i|X_i=x)$ for all x
- The CEF is the Min Mean Squared Error (MMSE) predictor of Y_i given X_i
- If the population CEF is linear, then the least squares population regression is the CEF
- If the population CEF is not linear, then the least squares line is the MMSE linear estimate of the CEF.
- See Angrist and Pischke, Ch 3.1

Linear Regression and CEF

Linear Regression and Normal Distribution

Interpretation

But I thought linear regression had to do with the normal distribution?

· Linear regression often presented as

$$y_i = X_i \beta + \epsilon_i$$
 $\epsilon_i \sim N(0, \sigma^2)$

- Why? We haven't had to assume normal distributions before now.
- · Helps with statistical inference results.
- However, the CLT handles asymptotic sampling distribution of parameters

Linear Regression and CEF

Linear Regression and Normal Distribution

Interpretation

Interpreting Regression Coefficients β

How the average outcome variable differs, on average:

predictive between **groups of units** that differ by 1 in the relevant explanatory variable while being identical in all other explanatory variables the same

counterfactual in the **same individual** when chaning the relevant explanatory variable 1 unit while holding all other explanatory variables the same

See Gelman and Hill, p. 34; Fox, p. 81

References

- Some slides derived from Christopher Adolph Linear Regression in Matrix Form / Propoerties & Assumptions of Linear Regression. Used with permission.
- · Material included from
 - Fox Ch 2, 5, 9.1-9.2
 - · Angrist and Pischke, Chapter 3.1
 - · Gelman and Hil, Chapter 2