

POLS/CS&SS 503:  
Advanced Quantitative Political Methodology

# MATRIX ALGEBRA, LINEAR REGRESSION

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# Agenda

- Linear regression as finding a “best” line
- Linear regression as the conditional expectation function
- How linear regression relates to the normal distribution

# What is regression?

## Regression

distribution of a **response** (outcome) variable  $Y$  — or summary of that distribution — as a function of **explanatory** variables  $X_1, \dots, X_k$ .

## Ordinary Least Squares

Finds a  $\hat{Y} = \mathbf{X}\mathbf{B}$  that minimizes  $\sum (Y_i - \hat{Y}_i)^2$ . This estimates a linear conditional expectation function  $E(Y|X_1, \dots, X_k)$ .

# OLS Objective Function

One  $X$

Find the line

$$\hat{Y} = A + BX$$

such that

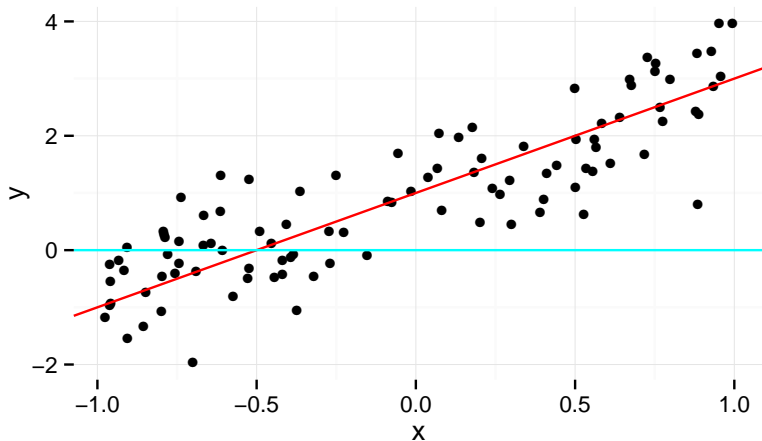
$$A, B = \arg \min_{A, B} S(A, B)$$

where

$$S(A, B) = \sum_i E_i^2 = \sum_i (Y_i - \hat{Y}_i)^2 = \sum_i (Y_i - A - BX_i)^2$$

How do we minimize this?

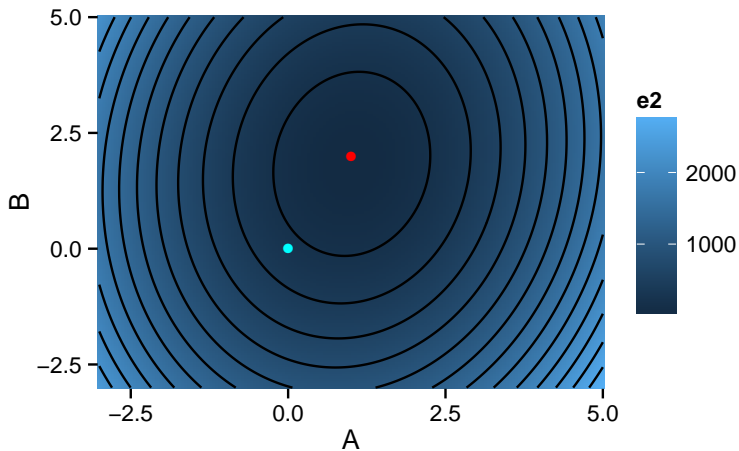
# What does the OLS objective function look like?



Data generated by  $Y_i = 1 + 2X_i + E_i$ . Lines are  $A = 1, B = 2$ , and  $A = 0, B = 0$ .

$\sum E_i^2$  as a function of  $A$  and  $B$

Least squares is the minimum of this function



# Finding the best $A, B$ in Least Squares

One  $X$

To minimize, set partial derivatives equal to 0 and solve:

$$\frac{\partial S(A, B)}{\partial A} = \sum (-1)(2)(Y_i - A - BX_i) = 0$$

$$\frac{\partial S(A, B)}{\partial B} = \sum (-X_i)(2)(Y_i - A - BX_i) = 0$$

Rearrange to get

$$A = \bar{Y} - B\bar{X}$$

$$B = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} = \frac{c(X, Y)}{v(X)}$$

# Implications of the OLS Solution

Least squares  $A$  and  $B$

$$A = \bar{Y} - B\bar{X}$$

$$B = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} = \frac{c(X, Y)}{v(X)}$$

- $\bar{X}, \bar{Y}$  is in the regression line
- $\sum X_i E_i = 0$

$$\begin{aligned}\sum X_i E_i &= \sum X_i (Y_i - A - B X_i) \\ &= \sum X_i Y_i - A \sum X_i - B \sum X_i^2 = 0\end{aligned}$$

- $\sum \hat{Y}_i E_i = 0$
- Errors  $E$  uncorrelated with  $\hat{Y}$  and  $X$



# OLS Objective Function

## Multiple $X$

Find plane

$$Y = A + B_1X_1 + B_2X_2 + \cdots + B_kX_k$$

such that

$$A, B_1, \dots, B_k = \arg \min_{A, B_1, \dots, B_k} S(A, B_1, \dots, B_k)$$

where

$$\begin{aligned} S(A, B_1, \dots, B_k) &= \sum_i E_i^2 = \sum_i (Y_i - \hat{Y}_i)^2 \\ &= \sum_i (Y_i - A - \sum_{j=1}^k B_j X_{i,j})^2 \end{aligned}$$

How do we minimize this?

# Finding the best $A, B$ in Least Squares Regression

Multiple  $X$

Set partial derivatives equal to 0 and solve system of equations for

$$\frac{\partial S(A, B_1, B_2, \dots, B_k)}{\partial A} = \sum (-1)(2)(Y_i - A - BX_i) = 0$$

$$\frac{\partial S(A, B_1, B_2, \dots, B_k)}{\partial B_1} = \sum (-X_{i,1})(2)(Y_i - A - B_1X_{i,1} - \dots - B_kX_{i,k}) = 0$$

$$\vdots = \vdots$$

$$\frac{\partial S(A, B_1, B_2, \dots, B_k)}{\partial B_k} = \sum (-X_{i,k})(2)(Y_i - A - B_1X_{i,1} - \dots - B_kX_{i,k}) = 0$$

Not as easy ...

# Linear Regression in Matrix Form

Scalar representation

$$Y_i = B_0 + B_1 X_{i,1} + B_2 X_{i,2} + \dots B_k X_{i,k} + E_i$$

Equivalent matrix representation

$$\underset{n \times 1}{\mathbf{y}} = \underset{n \times (k+1)}{\mathbf{X}} \underset{(k+1) \times 1}{\mathbf{b}} + \underset{n \times 1}{\mathbf{e}}$$

or

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_{1,1} & X_{2,1} & \cdots & X_{k,1} \\ 1 & X_{1,2} & X_{2,2} & \cdots & X_{k,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{1,n} & X_{2,n} & \cdots & X_{k,n} \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_k \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix}$$

# Linear Regression in Matrix Form

## Objective Function

The linear regression is

$$\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$$

Want to find the  $\mathbf{b}$  that minimizes the squared errors:

$$\arg \min_{\mathbf{b}} S(\mathbf{b})$$

where

$$\begin{aligned} S(\mathbf{b}) &= \sum E_i^2 = \mathbf{e}'\mathbf{e} \\ &= (\mathbf{y} - \mathbf{X}\mathbf{b})'(\mathbf{y} - \mathbf{X}\mathbf{b}) \end{aligned}$$

Why does  $\mathbf{e}$  need to be transposed?

# Linear Regression in Matrix Form

## Transpose of Sums

$$(A + B)' = A' + B'$$

$$\left( \begin{bmatrix} 10 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} \right)' = ?$$
$$? = ?$$

# Linear Regression in Matrix Form

Transpose of a product

$$(XB)' = B'X'$$

$$\begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = ?$$

$$? = ?$$

# Simplify $e'c$

$$\begin{aligned}e'e &= (\mathbf{y} - \mathbf{X}\mathbf{b})'(\mathbf{y} - \mathbf{X}\mathbf{b}) \\&= (\mathbf{y}' - (\mathbf{X}\mathbf{b})')(\mathbf{y} - \mathbf{X}\mathbf{b}) && \text{distribute the transpose} \\&= (\mathbf{y} - \mathbf{b}'\mathbf{X})(\mathbf{y} - \mathbf{X}\mathbf{b}) && \text{substitute } \mathbf{b}'\mathbf{X}' \text{ for } (\mathbf{X}\mathbf{b})' \\&= \mathbf{y}'\mathbf{y} - \mathbf{b}'\mathbf{X}'\mathbf{y} - \mathbf{y}'\mathbf{X}\mathbf{b} + \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b} && \text{multiply out} \\&= \mathbf{y}'\mathbf{y} - 2\mathbf{b}'\mathbf{X}'\mathbf{y} + \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b} && \text{simplify}\end{aligned}$$

- To minimize need to calculate derivative of  $e'e$  with respect to  $\mathbf{b}$ .
- Need to know two things
  - derivative of scalar with respect to vector ( $2\mathbf{b}'\mathbf{X}'\mathbf{y}$ )
  - derivative of quadratic form ( $\mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b}$ )

# What is the derivative of scalar with respect to vector

- Need to take derivative of  $\mathbf{e}'\mathbf{e}$  with respect to  $\mathbf{b}$  to find  $\mathbf{b}$  that min the sum of squared.
- A derivative of a scalar with respect to a vector

$$y = \mathbf{a}'\mathbf{x} = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

$$\frac{\partial y}{\partial \mathbf{x}} = [a_1 \quad a_2 \quad \cdots \quad a_n]'$$

$$\frac{\partial y}{\partial \mathbf{x}} = \mathbf{a}$$



# Derivative of a quadratic form

- Equivalent to  $x^2$  is inner product  $\mathbf{x}'\mathbf{x}$
- Vector analogue of  $ax^2$  is  $\mathbf{x}'\mathbf{A}\mathbf{x}$ , where  $\mathbf{A}$  is  $n \times n$  matrix

$$\frac{\partial ax^2}{\partial x} = 2ax$$
$$\frac{\partial \mathbf{x}'\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$$

# OLS in Matrix Form

## Minimizing the objective function

1. Take partial derivative of  $S(\mathbf{b})$ :

$$\begin{aligned}\frac{\partial S(\mathbf{b})}{\partial \mathbf{b}} &= \frac{\partial}{\partial \mathbf{b}}(\mathbf{y}'\mathbf{y} - 2\mathbf{b}'\mathbf{X}'\mathbf{y} + \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b}) \\ &= 0 - (2\mathbf{y}'\mathbf{X}) + 2(\mathbf{X}'\mathbf{X})\mathbf{b}\end{aligned}$$

2. Set to 0, and solve for  $\mathbf{b}$ :

$$\begin{aligned}\mathbf{X}'\mathbf{X}\mathbf{b} &= \mathbf{X}'\mathbf{y} \\ \mathbf{b} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}\end{aligned}$$

# What $(X'X)^{-1}$ implies

- For  $\mathbf{b}$  to be defined  $(X'X)^{-1}$  needs to exist
- $X'X$  must be full rank
- rank of  $X'X$  is the same as the rank of  $X$
- The rank of  $X$  is between  $n$  and  $k + 1$ , means that  $n \geq k + 1$  (obs > variables)
- $k + 1$  columns of  $X$  must be linearly independent?
  - Can you have a full set of dummies?
  - Can you include a variable that is always equal to 3?

# Takeaways

- Linear regression is the  $A, B_1, \dots, B_k$  that solve  $\arg \min_{A, B_1, \dots, B_k} \sum E_i^2$
- Solving for linear regression coefficients is relatively **easy**; linear equations; there's an explicit solution. No iteration required.

## Linear Regression and CEF

Linear Regression and Normal Distribution

Interpretation

# CEF justification for linear regression justification

- Conditional Expectation Function is  $E(Y_i | X_i = x)$  for all  $x$
- The CEF is the Min Mean Squared Error (MMSE) predictor of  $Y_i$  given  $X_i$
- If the population CEF is linear, then the least squares population regression is the CEF
- If the population CEF is not linear, then the least squares line is the MMSE linear estimate of the CEF.
- See Angrist and Pischke, Ch 3.1

Linear Regression and CEF

Linear Regression and Normal Distribution

Interpretation

# But I thought linear regression had to do with the normal distribution?

- Linear regression often presented as

$$y_i = X_i\beta + \epsilon_i \qquad \epsilon_i \sim N(0, \sigma^2)$$

- Why? We haven't had to assume normal distributions before now.
- Helps with statistical inference results.
- However, the CLT handles asymptotic sampling distribution of parameters



Linear Regression and CEF

Linear Regression and Normal Distribution

Interpretation

# Interpreting Regression Coefficients $\beta$

How the average outcome variable differs, on average:

**predictive** between **groups of units** that differ by 1 in the relevant explanatory variable while being identical in all other explanatory variables the same

**counterfactual** in the **same individual** when changing the relevant explanatory variable 1 unit while holding all other explanatory variables the same

See Gelman and Hill, p. 34; Fox, p. 81

# References

- Some slides derived from Christopher Adolph *Linear Regression in Matrix Form / Properties & Assumptions of Linear Regression*. Used with permission.
- Material included from
  - Fox Ch 2, 5, 9.1–9.2
  - Angrist and Pischke, Chapter 3.1
  - Gelman and Hil, Chapter 2