#### POLS/CS&SS 503: Advanced Quantitative Political Methodology

#### LINEAR REGRESSION ESTIMATOR

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#### Overview

Regression Coefficient Anatomy

Linear Regression Population Model

Sampling Distribution

Estimators and mean squared error

Gauss-Markov Theorem

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## Coeficients of a simple regression

$$Y_i = A + BX_i + E_i$$

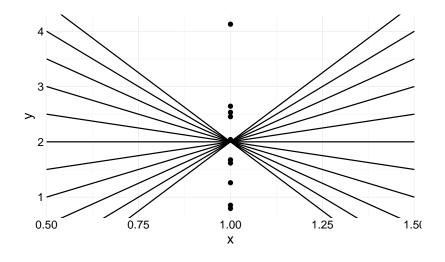
The least squares coefficients are

$$A = \bar{Y} + B\bar{X}$$

$$B = \frac{\sum_{i} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i} (X_i - \bar{X})^2}$$

- State B in terms of covariance of X and Y and variance?
- State B in terms of correlation of X and Y and standard deviations?
- What values can B take if SD(X) = SD(Y) = 1?
- What is  $\hat{Y}$  for  $X = \bar{X}$ ?
- What happens to B as  $\mathrm{V}(X)$  decreases?  $\mathrm{V}(Y)$  decreases? If  $\mathrm{V}(X)=0$

## Least squares when $\operatorname{V} X = 0$



#### Least squares coefficients are unidentified if Vx=0

- If  $\forall x = 0$  then least squares solution is unidentified
- There is no unique value of A,B that  $\arg\min_{A,B}\sum_i E_i^2$

```
y < -c(1, 2, 3, 4, 5)
x <- 1
ybar <- mean(y)
ybar
## [1] 3
\# A = 2, B = 1
sum((y - 2 - 1 * x) ^ 2)
## [1] 10
\# A = -7, B = 10
sum((y + 7 - 10 * x) ^ 2)
## [1] 10
```

## Coefficients of a multiple regression

$$\vec{Y} = Xb + e$$

- $\boldsymbol{b} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$ . Not that intuitive!
- Coefficient  $oldsymbol{b}_j$  is

$$oldsymbol{b}_k = rac{\mathsf{C}(oldsymbol{y}, ilde{oldsymbol{x}}_j)}{\mathsf{V}( ilde{oldsymbol{x}}_k)}$$

• Where  $\tilde{m{x}}_j$  are the residulals of  $m{x}_j$  on all  $X_h$  where h 
eq j

$$\tilde{X}_{j,i} = X_{j,i} - \tilde{A} - \sum_{h \neq j} \tilde{B}_h X_h$$

### Regression example

See  $multiple\_regression\_anatomy.R$ 

# Least Squares coefficients are unidentified if $\left(X'X\right)^{-1}$ does not exist

- Common cases in which  $(X'X)^{-1}$  does not exist:
  - Number of observations less than k+1
  - $X_k$  is constant
  - $X_k$  is a linear function of other variables:  $X_k = \sum_{j \neq k} c_j X_j$ .
    - · dummy variables for all categories of a categorical variable
    - variable multiplied by the constant of another variable

# Which of these would be cases of collinearity and why?

- There is a variable that takes values "white", "black", "hispanic", "asian", "other". You include a dummy variable for each category.
- · GDP, GDP per capita, and population
- Log GDP, log GDP per capita, and log population
- GDP in millions of dollars: GDP in trillions of dollars
- · GDP measured in nominal value; GDP measured in real terms
- · Regression with 3 variables and 4 observations

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## Population model in a simple regression

$$Y_i = \alpha + \beta X_i + \epsilon_i$$

Assumptions for statistical inference

- 1. X is not invariant:  $\mathsf{V}(X) > 0$
- 2. *Linearity*. Average value of error given x is 0.  $E(\epsilon_i) = E(\epsilon_i|x_i) = 0$

$$\mu_i = E(Y_i) = \mathsf{E}(Y|X_i) = \mathsf{E}(\alpha + \beta X_i + \epsilon_i) = \alpha + \beta x_i$$

3. Constant variance Variance of the errors is the same regardless of the value of  $\boldsymbol{X}$ 

$$V(Y|x_i) = E(\epsilon_i^2) = \sigma_{\epsilon}^2$$

- 4. Independence: Observations are sampled independently.  ${\sf Cor}(\epsilon_i,\epsilon_j)=0$  for all  $i\neq j$ .
- 5. Fixed X or X measured without error and independent of the error.
- 6. Errors are normally distributed  $\epsilon_i \sim N\left(0,\sigma_\epsilon^2\right)$

## Population model in a multiple regression

$$Y_i = \alpha + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \dots + \beta_k x_{i,k} + \epsilon_i$$

#### Assumptions for statistical inference

- 1.  $\it X$  is not invariant and no  $\it X$  is a perfect linear function of the others.
- 2. Linearity.  $E(\epsilon_i) = 0$
- 3. Constant variance  $V(\epsilon_i) = \sigma_\epsilon^2$
- 4. Independence Observations are sampled independently.  $Cor(\epsilon_i, \epsilon_j) = 0$  for all  $i \neq j$ .
- 5. Fixed X or X measured without error and independent of the error
- 6. Normality Errors are normally distributed  $\epsilon_{i} \sim N\left(0,\sigma_{\epsilon}^{2}\right)$

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#### **Definitions**

population The observations of interest. May be theoretical sample The data you have.

parameter A function of the population distribution statistic A function of the sample sampling distribution The distribution of a statistic calculated from the distribution of samples of a given size drawn from a population.

 ${\tt See \ Sampling\_Distributions.Rmd}$ 

## Sampling Distribution of Simple Regression Coefficients

The sampling distributions of A, B given  $Y_i = \alpha + \beta X_i + \epsilon_i$ 

expected values (linearity)

$$E(A) = \alpha$$
$$E(B) = \beta$$

variances (linearity, constant variance, independence)

$$\begin{aligned} \mathsf{V}(A) &= \frac{\sigma_{\epsilon}^2}{n} \cdot \frac{\sum x_i^2}{\sum (x_i - \bar{x})^2} \\ \mathsf{V}(B) &= \frac{\sigma_{\epsilon}^2}{\sum (x_i - \bar{x})^2} = \frac{\sigma_{\epsilon}^2}{n \, \mathsf{V}(x)} \end{aligned}$$

normal distribution (normal errors)

$$\begin{split} A \sim N(\mathrm{E}(A), \mathrm{V}(A)) \\ B \sim N(\mathrm{E}(B), \mathrm{V}(B)) \end{split}$$

# Coefficient sampling distributions in multiple regression

The sampling distributions of  $B_k$  given  $Y_i = \alpha + \sum \beta_i X_{i,i} + \epsilon_i$ 

- Expected value:  $\mathsf{E}(B_K) = \beta_k$
- · Variance:

$$V(B_{j}) = \frac{1}{(1 - R_{j}^{2})} \frac{\sigma_{\epsilon}^{2}}{\sum (x_{i,j} - \bar{x}_{j})^{2}}$$
$$= \frac{\sigma_{\epsilon}^{2}}{\sum_{i} (x_{i,j} - \hat{x}_{i,j})^{2}}$$

Where  $R_j^2$  is  $R^2$  from regression of  $X_j$  on other X, and  $\hat{x}_{ij}$  are fitted values from that regression.

- Normally distributed if errors are normally distributed or as  $n \to \infty$ .
- $oldsymbol{\cdot}$   $oldsymbol{b}$  is multivariate normally distributed

$$\boldsymbol{b} \sim N\left(\beta, \sigma_{\epsilon}^2(X'X)^{-1}\right)$$

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## Let's define some things

```
statistic Function of a sample, e.g. Sample mean \bar{x}=\frac{1}{n}\sum x_i parameter Function of the population distribution, e.g. Expected value \mu of the normal distribution. estimator Method to use a sample statistic (estimate) to infer a population parameter (estimand)
```

## How to determine if an estimator is good?

- Is  $\hat{\beta} = (X'X)^{-1}X'y$  a good estimator for  $\beta$ ?
- Would another estimator be better?
- First, need criteria to by which to judge estimators

## What makes an estimator good?

- Bias
- Variance
- Efficiency (mean squared error)
- Consistency

#### Bias and Variance

#### Bias

On average how far off is the estimator?

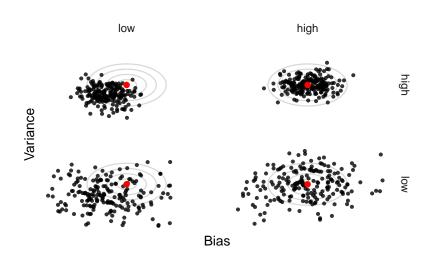
$$\mathrm{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta}) - \beta$$

#### Variance

Does the estimator give similar results in different samples?

$$V(\hat{\beta}) = E\left(\left(\beta - E(\hat{\beta})\right)^2\right)$$

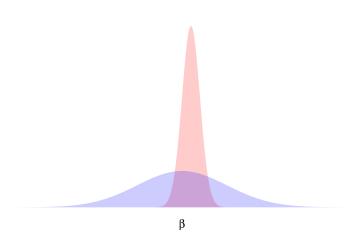
#### Bias and Variance Visualized



## What makes an estimator good?

- Unbiased methods may still miss the truth by a large amount, just direction not systematic
- Unbiased estimates can be horrible: random draw from numbers 0-24 for time of day
- Biased estimates are not necessarily terrible: a clock that's 2 minutes fast

You may prefer a biased, low variance estimator to an unbiased, high variance estimator



## Mean Squared Error (MSE)

MSE is

$$MSE(\hat{\beta}) = E\left((\hat{\beta} - \beta)^2\right)$$

· MSE trades off bias and variance

$$\begin{split} MSE(\hat{\beta}) &= \mathrm{E}((\hat{\beta} - \mathrm{E}(\hat{\beta}))^2) + \mathrm{E}(\mathrm{E}(\hat{\beta}) - \beta))^2 \\ &= \mathrm{V}(\hat{\beta}) + \left(\mathrm{bias}(\hat{\beta}, \beta)\right)^2 \end{split}$$

- root mean squared error (RMSE)  $\sqrt{\text{MSE}}$ : on average how far is an estimate from the truth
- An efficient estimator has the smallest MSE
- · What is the MSE of an unbiased estimator?

$$MSE(\hat{\beta}) = \mathsf{V}(\hat{\beta}) + \left(\mathsf{bias}(\hat{\beta},\beta)\right)^2 = \mathsf{V}(\hat{\beta}) + 0 = \mathsf{V}(\hat{\beta})$$

## MSE Example

- Suppose population parameter  $\beta=1$
- Consider two estimators  $\hat{eta}_1$  and  $\hat{eta}_2$ .
  - $\hat{\beta}_1 \sim N(1, 1^2)$
  - $\hat{\beta}_2 \sim N(0.5, 0.5^2)$
- · What are the bias, variance, and MSE of each estimator?

## Consistency

 A consistent estimator converges to the parameter value as the number of observations grows

$$\mathsf{E}(\hat{\beta} - \beta) \to 0$$
 as  $n \to \infty$ 

- A concern of econometricians
- May not be as much a concern in finite, small sample sizes
- We will mainly be concerned with efficiency, secondarily with bias, rarely with consistency

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## LS assumptions and consequences of violations

	Assumption		Consequence of violation
1	No perfect collinearity	$\operatorname{rank}({m X}) = k, k < n$	Coefficients unidentified
2	$oldsymbol{X}$ is exogenous	$E(\boldsymbol{X}\epsilon)=0$	Biased, even as $n  o \infty$
3	Disturbances have mean 0	$E(\epsilon) = 0$	Biased, even as $n  o \infty$
4	No serial correlation	$E(\epsilon_i \epsilon_j) = 0, i \neq j$	Unbiased but ineff. Wrong se.
5	Homoskedastic errors	$E(\epsilon'\epsilon') = \sigma^2 \boldsymbol{I}$	Unbiased but ineff. Wrong se.
_6	Normal errors	$\epsilon \sim N(0, \sigma^2)$	se wrong unless $n \to \infty$

Assumptions stronger from top to bottom, 4 and 5 could be combined

#### Unbiasedness of LS

- Only need assumptions 1-3 (no collinearity,  $oldsymbol{X}$  exogenous,  $\mathsf{E}(\epsilon)=0$
- · Start with

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\beta + \epsilon)$$
$$= \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\epsilon$$

· Take the expectation

$$\begin{split} \mathbf{E}(\hat{\boldsymbol{\beta}}) &= \mathbf{E}(\boldsymbol{\beta}) + \mathbf{E}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\epsilon}) \\ &= \mathbf{E}(\boldsymbol{\beta}) + \boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\,\mathbf{E}(\boldsymbol{\epsilon}) \\ &= \mathbf{E}(\boldsymbol{\beta}) \end{split}$$

• Since  $E(\hat{\beta}) = E(\beta)$ , LS is unbiased.

#### Gauss-Markov

- If make assumptions 1–5: LS is the best linear unbiased estimator (BLUE)
- · LS estimator is **linear** because  $\hat{eta} = oldsymbol{M} oldsymbol{y}$ , where  $oldsymbol{M} = (oldsymbol{X}'oldsymbol{X})^{-1}oldsymbol{X}'$
- best is best mean squared error (MSE).
- If LS is unbiased, then its mean squared error is the same as its ...?
- Could exist other non-linear unbiased estimators with smaller MSE,
   e.g. Robust regression when population has fat tailed errors
- If errors are Gaussian, LS is Minimum Variance Unbiased (MVU).
- MVU = for  $\mathit{all}$  estimators that are unbiased.  $\hat{\beta}$  has smallest variance (and MSE).

#### References

- Some slides derived from Christopher Adolph Linear Regression in Matrix Form / Propoerties & Assumptions of Linear Regression. Used with permission.
  - <a href="http://faculty.washington.edu/cadolph/503/topic3.pw.pdf">http://faculty.washington.edu/cadolph/503/topic3.pw.pdf</a>
- Material included from
  - Fox Ch 6, 9.3
  - Angrist and Pischke, Chapter 3

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