POLS/CS&SS 503:
Advanced Quantitative Political Methodology

LINEAR REGRESSION ESTIMATOR

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Overview

Coeficients of a simple regression

$$Y_i = A + BX_i + E_i$$

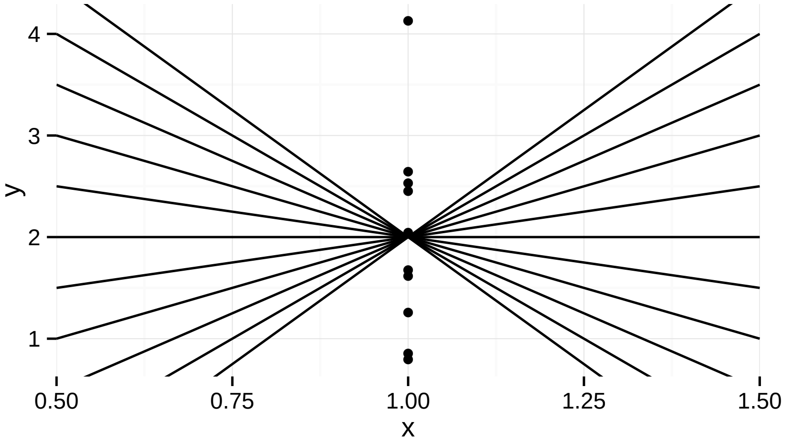
The least squares coefficients are

$$A = \bar{Y} + B\bar{X}$$

$$B = \frac{\sum_i (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_i \left(X_i - \bar{X}\right)^2}$$

- State B in terms of covariance of X and Y and variance?
- State B in terms of correlation of X and Y and standard deviations?
- What values can B take if SD(X) $=$ SD(Y) $= 1$?
- What is $\hat{Y}$ for $X = \bar{X}$?
- What happens to B as V(X) decreases? V(Y) decreases? If V(X) $= 0$

Least squares when V $X = 0$



Least squares coefficients are unidentified if $\mathsf{V}\,x = 0$

- If $\mathsf{V}\,x = 0$ then least squares solution is unidentified
- There is no unique value of A, B that $\arg\min_{A,B} \sum_i E_i^2$

```r
y <- c(1, 2, 3, 4, 5)
x <- 1
ybar <- mean(y)
ybar
```

```
## [1] 3
```

```r
# A = 2, B = 1
sum((y - 2 - 1 * x) ^ 2)
```

```
## [1] 10
```

```r
# A = -7, B = 10
sum((y + 7 - 10 * x) ^ 2)
```

```
## [1] 10
```

Coefficients of a multiple regression

$$\vec{Y} = \boldsymbol{X}\boldsymbol{b} + \boldsymbol{e}$$

- $\boldsymbol{b} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$. *Not that intuitive!*
- Coefficient $\boldsymbol{b}_j$ is

$$\boldsymbol{b}_k = \frac{\mathsf{C}(\boldsymbol{y}, \tilde{\boldsymbol{x}}_j)}{\mathsf{V}(\tilde{\boldsymbol{x}}_k)}$$

- Where $\tilde{\boldsymbol{x}}_j$ are the residuals of $\boldsymbol{x}_j$ on all X_h where $h \neq j$

$$\tilde{X}_{j,i} = X_{j,i} - \tilde{A} - \sum_{h \neq j} \tilde{B}_h X_h$$

Regression example

See `multiple_regression_anatomy.R`

Least Squares coefficients are unidentified if $(X'X)^{-1}$ does not exist

- Common cases in which $(X'X)^{-1}$ does not exist:
    - Number of observations less than $k+1$
    - X_k is constant
    - X_k is a linear function of other variables: $X_k = \sum_{j \neq k} c_j X_j$.
        - dummy variables for all categories of a categorical variable
        - variable multiplied by the constant of another variable

Which of these would be cases of collinearity and why?

- There is a variable that takes values “white”, “black”, “hispanic”, “asian”, “other”. You include a dummy variable for each category.
- GDP, GDP per capita, and population
- Log GDP, log GDP per capita, and log population
- GDP in millions of dollars; GDP in trillions of dollars
- GDP measured in nominal value; GDP measured in real terms
- Regression with 3 variables and 4 observations

Population model in a simple regression

$$Y_i = \alpha + \beta X_i + \epsilon_i$$

Assumptions for statistical inference

1. *X is not invariant:* $\mathsf{V}(X) > 0$
2. *Linearity*. Average value of error given x is 0. $E(\epsilon_i) = E(\epsilon_i | x_i) = 0$

$$\mu_i = E(Y_i) = \mathsf{E}(Y | X_i) = \mathsf{E}(\alpha + \beta X_i + \epsilon_i) = \alpha + \beta x_i$$

3. *Constant variance* Variance of the errors is the same regardless of the value of X
$$\mathsf{V}(Y | x_i) = E(\epsilon_i^2) = \sigma_\epsilon^2$$
4. *Independence:* Observations are sampled independently. $\mathsf{Cor}(\epsilon_i, \epsilon_j) = 0$ for all $i \neq j$.
5. Fixed X or X measured without error and independent of the error.
6. Errors are normally distributed $\epsilon_i \sim N\left(0, \sigma_\epsilon^2\right)$

Population model in a multiple regression

$$Y_i = \alpha + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_k x_{i,k} + \epsilon_i$$

Assumptions for statistical inference

1. *X is not invariant* and no X is a perfect linear function of the others.
2. *Linearity*. $\mathrm{E}(\epsilon_i) = 0$
3. *Constant variance* $V(\epsilon_i) = \sigma^2_\epsilon$
4. *Independence Observations are sampled independently*. $\mathrm{Cor}(\epsilon_i, \epsilon_j) = 0$ for all $i \neq j$.
5. *Fixed X or X measured without error and independent of the error*
6. *Normality* Errors are normally distributed $\epsilon_i \sim N\left(0, \sigma^2_\epsilon\right)$

Definitions

population The observations of interest. May be theoretical

sample The data you have.

parameter A function of the population distribution

statistic A function of the sample

sampling distribution The distribution of a statistic calculated from the distribution of samples of a given size drawn from a population.

See `Sampling_Distributions.Rmd`

Sampling Distribution of Simple Regression Coefficients

The sampling distributions of A, B given $Y_i = \alpha + \beta X_i + \epsilon_i$

- expected values (linearity)

$$\mathsf{E}(A) = \alpha$$
$$\mathsf{E}(B) = \beta$$

- variances (linearity, constant variance, independence)

$$\mathsf{V}(A) = \frac{\sigma_\epsilon^2}{n} \cdot \frac{\sum x_i^2}{\sum (x_i - \bar{x})^2}$$
$$\mathsf{V}(B) = \frac{\sigma_\epsilon^2}{\sum (x_i - \bar{x})^2} = \frac{\sigma_\epsilon^2}{n\,\mathsf{V}(x)}$$

- normal distribution (normal errors)

$$A \sim N(\mathsf{E}(A), \mathsf{V}(A))$$
$$B \sim N(\mathsf{E}(B), \mathsf{V}(B))$$

Coefficient sampling distributions in multiple regression

The sampling distributions of B_k given $Y_i = \alpha + \sum \beta_j X_{j,i} + \epsilon_i$

- Expected value: $\mathsf{E}(B_K) = \beta_k$
- Variance:

$$\mathsf{V}(B_j) = \frac{1}{(1 - R_j^2)} \frac{\sigma_\epsilon^2}{\sum (x_{i,j} - \bar{x}_j)^2}$$
$$= \frac{\sigma_\epsilon^2}{\sum_i (x_{i,j} - \hat{x}_{i,j})^2}$$

  Where R_j^2 is R^2 from regression of X_j on other X, and $\hat{x}_{ij}$ are fitted values from that regression.

- Normally distributed if errors are normally distributed or as $n \to \infty$.
- $\boldsymbol{b}$ is multivariate normally distributed

$$\boldsymbol{b} \sim N\left(\beta, \sigma_\epsilon^2 (X'X)^{-1}\right)$$

Let's define some things

statistic Function of a sample, e.g. Sample mean $\bar{x} = \frac{1}{n} \sum x_i$

parameter Function of the population distribution, e.g. Expected value μ of the normal distribution.

estimator Method to use a sample statistic (estimate) to infer a population parameter (estimand)

How to determine if an estimator is good?

- Is $\hat{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$ a good estimator for β?
- Would another estimator be better?
- First, need criteria to by which to judge estimators

What makes an estimator good?

- Bias
- Variance
- Efficiency (mean squared error)
- Consistency

Bias and Variance

Bias

On average how far off is the estimator?

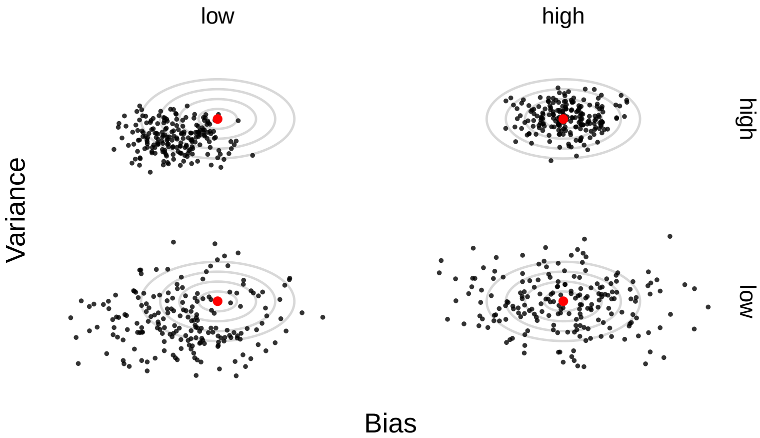
$$\text{bias}(\hat{\beta}) = \mathsf{E}(\hat{\beta}) - \beta$$

Variance

Does the estimator give similar results in different samples?

$$\mathsf{V}(\hat{\beta}) = \mathsf{E}\left(\left(\beta - \mathsf{E}(\hat{\beta})\right)^2 \right)$$

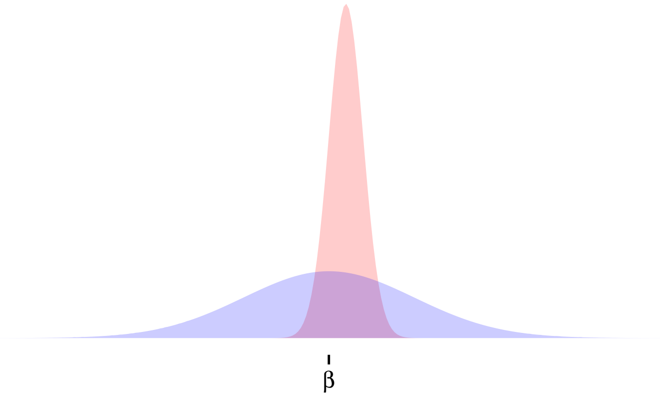
Bias and Variance Visualized



What makes an estimator good?

- Unbiased methods may still miss the truth by a large amount, just direction not systematic
- Unbiased estimates can be horrible: random draw from numbers 0–24 for time of day
- Biased estimates are not necessarily terrible: a clock that's 2 minutes fast

You may prefer a biased, low variance estimator to an unbiased, high variance estimator



Mean Squared Error (MSE)

- MSE is

$$MSE(\hat{\beta}) = \mathsf{E}\left((\hat{\beta} - \beta)^2\right)$$

- MSE trades off bias and variance

$$\begin{aligned} MSE(\hat{\beta}) &= \mathsf{E}((\hat{\beta} - \mathsf{E}(\hat{\beta}))^2) + \mathsf{E}(\mathsf{E}(\hat{\beta}) - \beta))^2 \\ &= \mathsf{V}(\hat{\beta}) + \left(\mathsf{bias}(\hat{\beta}, \beta)\right)^2 \end{aligned}$$

- root mean squared error (RMSE) $\sqrt{\mathsf{MSE}}$: on average how far is an estimate from the truth
- An **efficient** estimator has the smallest MSE
- What is the MSE of an unbiased estimator?

$$MSE(\hat{\beta}) = \mathsf{V}(\hat{\beta}) + \left(\mathsf{bias}(\hat{\beta}, \beta)\right)^2 = \mathsf{V}(\hat{\beta}) + 0 = \mathsf{V}(\hat{\beta})$$

MSE Example

- Suppose population parameter $\beta = 1$
- Consider two estimators $\hat{\beta}_1$ and $\hat{\beta}_2$.
    - $\hat{\beta}_1 \sim N(1, 1^2)$
    - $\hat{\beta}_2 \sim N(0.5, 0.5^2)$
- What are the bias, variance, and MSE of each estimator?

Consistency

- A consistent estimator converges to the parameter value as the number of observations grows

$$\mathsf{E}(\hat{\beta} - \beta) \to 0 \qquad\qquad \text{as } n \to \infty$$

- A concern of econometricians
- May not be as much a concern in finite, small sample sizes
- We will mainly be concerned with efficiency, secondarily with bias, rarely with consistency

LS assumptions and consequences of violations

	Assumption		Consequence of violation
1	No perfect collinearity	rank$(\boldsymbol{X}) = k$, $k < n$	Coefficients unidentified
2	$\boldsymbol{X}$ is exogenous	E$(\boldsymbol{X}\epsilon) = 0$	Biased, even as $n \to \infty$
3	Disturbances have mean 0	E$(\epsilon) = 0$	Biased, even as $n \to \infty$
4	No serial correlation	E$(\epsilon_i \epsilon_j) = 0$, $i \neq j$	Unbiased but ineff. Wrong se.
5	Homoskedastic errors	E$(\epsilon' \epsilon') = \sigma^2 \boldsymbol{I}$	Unbiased but ineff. Wrong se.
6	Normal errors	$\epsilon \sim N(0, \sigma^2)$	se wrong unless $n \to \infty$

Assumptions stronger from top to bottom, 4 and 5 could be combined

Unbiasedness of LS

- Only need assumptions 1-3 (no collinearity, $\boldsymbol{X}$ exogenous, $\mathsf{E}(\epsilon) = 0$
- Start with

$$\begin{aligned}\hat{\beta} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'(\boldsymbol{X}\beta + \epsilon) \\ &= \beta + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\epsilon\end{aligned}$$

- Take the expectation

$$\begin{aligned}\mathsf{E}(\hat{\beta}) &= \mathsf{E}(\beta) + \mathsf{E}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\epsilon) \\ &= \mathsf{E}(\beta) + \boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\,\mathsf{E}(\epsilon) \\ &= \mathsf{E}(\beta)\end{aligned}$$

- Since $\mathsf{E}(\hat{\beta}) = \mathsf{E}(\beta)$, LS is unbiased.

Gauss-Markov

- If make assumptions 1–5: LS is the best linear unbiased estimator (BLUE)
- LS estimator is **linear** because $\hat{\beta} = \boldsymbol{My}$, where $\boldsymbol{M} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$
- **best** is best mean squared error (MSE).
- If LS is unbiased, then its mean squared error is the same as its …?
- Could exist other non-linear unbiased estimators with smaller MSE, e.g. Robust regression when population has fat tailed errors
- If errors are Gaussian, LS is Minimum Variance Unbiased (MVU).
- MVU = for *all* estimators that are unbiased. $\hat{\beta}$ has smallest variance (and MSE).

References

- Some slides derived from Christopher Adolph *Linear Regression in Matrix Form / Propoerties & Assumptions of Linear Regression*. Used with permission. <http://faculty.washington.edu/cadolph/503/topic3.pw.pdf>
- Material included from
  - Fox Ch 6, 9.3
  - Angrist and Pischke, Chapter 3

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