POLS/CS&SS 503:

Advanced Quantitative Political Methodology

BINARY DEPENDENT VARIABLES

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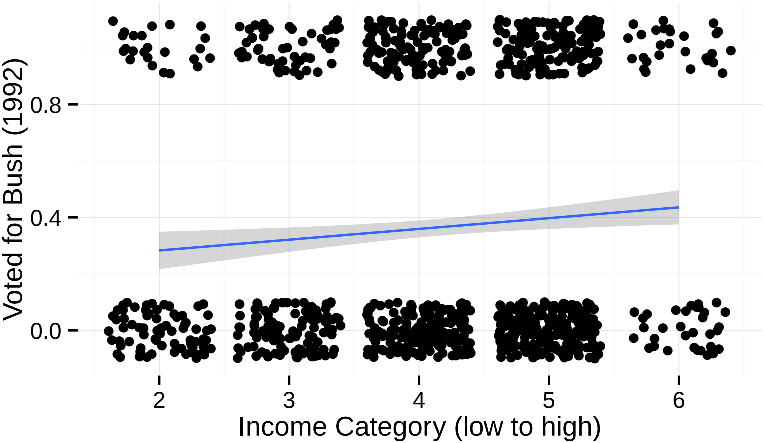
Overview

Example of Linear Probability Model

Vote for Bush in U.S. Presidential Election 1992



Residuals in LPM

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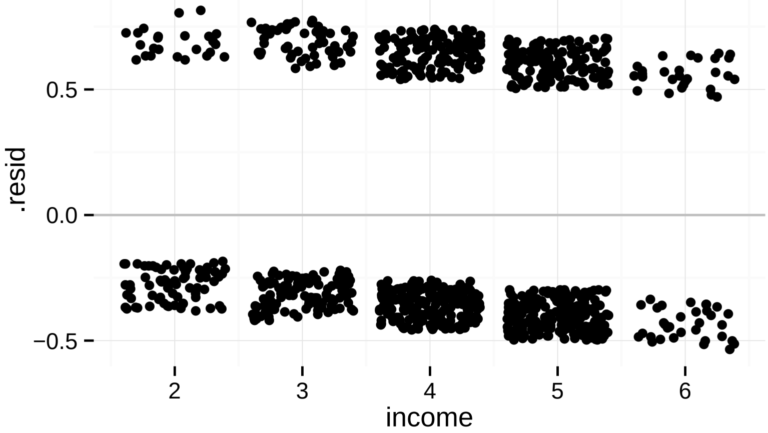
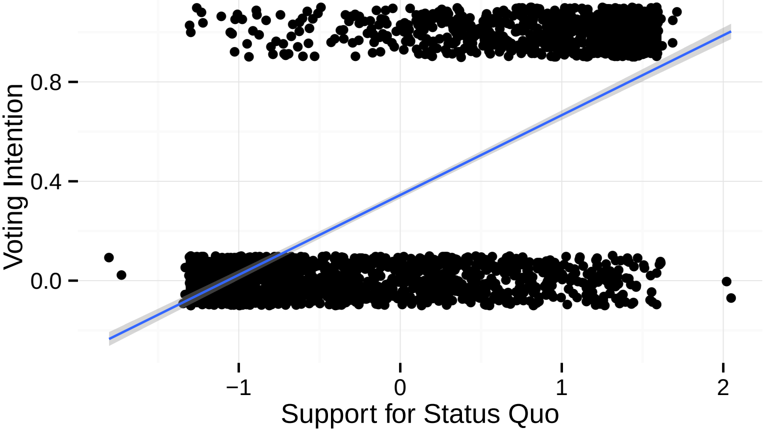


Example of Linear Probability Model

Vote Intention in Chilean Plebiscite in 1973



Residuals in LPM

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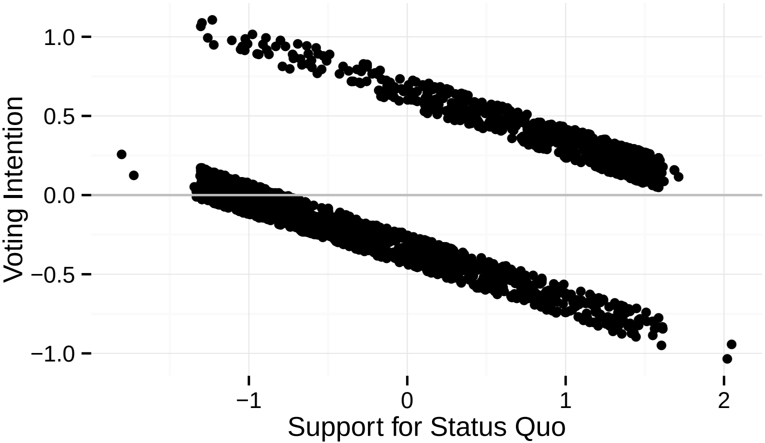


Linear Probability Model

OLS with a binary dependent variable. When $Y_i \in \{0, 1\}$:

$$Y_i = \alpha + \beta X_i + \epsilon_i$$

The expected value is a probability

$$E(Y_i|X_i) = \text{Pr}(Y_i = 1|X_i) = \alpha + \beta X_i$$

Problems with the LPM

- Errors are not normally distributed

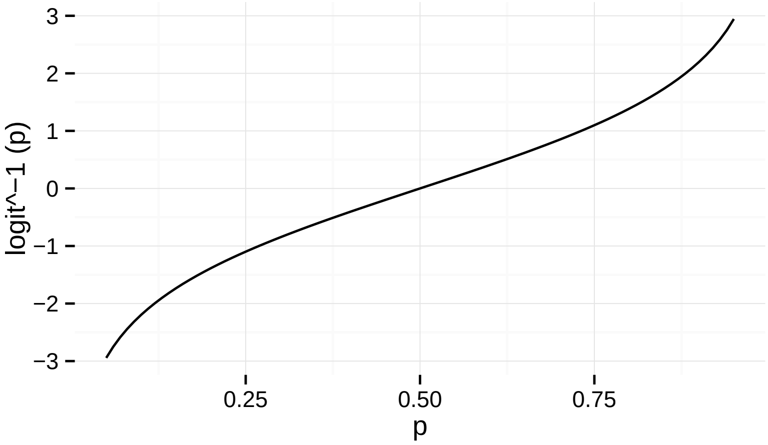
$$\epsilon_i|Y_i = 1 = 1 - E(Y_i|X_i) = 1 - (\alpha - \beta X_i) = 1 - \pi_i$$
$$\epsilon_i|Y_i = 1 = 1 - E(Y_i|X_i) = 1 - (\alpha - \beta X_i) = -\pi_i$$

- Errors have non-constant variance (heteroskedasticity)

$$V(\epsilon_i) = \pi(1 - \pi_i)$$

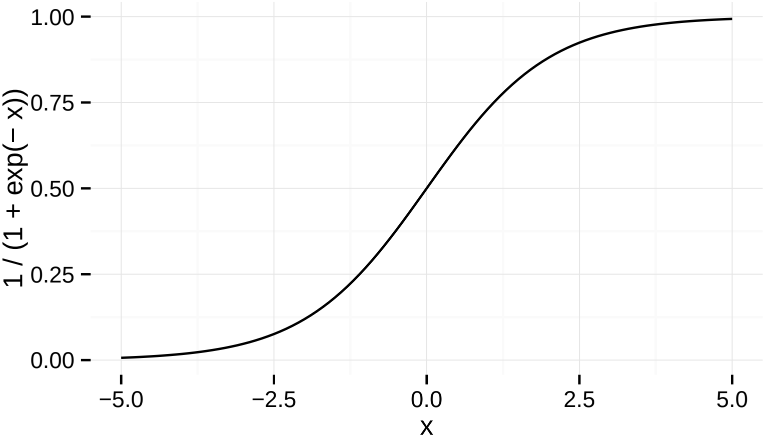
- Expected values $\alpha + \beta X_i$ can extend beyond (0, 1)
- Improper specification leads to bias; heteroskedasticity and errors leads to incorrect standard errors.

Logit Function



$\text{logit}(x) = \log\left(\frac{p}{1-p}\right)$

Inverse Logit (Logistic) Function



$\text{logit}^{-1}(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}$

Logit and Logistic Function

Logit Function

Log-odds: Goes from $(0, 1)$ to $-(\infty, -\infty)$

$$\text{logit}(p) = \log\left(\frac{p}{1-p}\right) = \log(p) - \log(1-p)$$

Logistic or Inverse Logit Function

Goes from $-(\infty, -\infty)$ to $(0, 1)$

$$\text{logit}^{-1}(x) = \frac{1}{1+\exp(-x)} = \frac{\exp(x)}{\exp(x)+1}$$

Logit Model

$$
\begin{aligned}
\text{Pr}(Y_i = 1) &= f(X_i\beta) \\
&= \frac{1}{1 + \text{exp}(-(\alpha + X_i\beta))} \\
&= \text{logit}^{-1}(\alpha + X_i\beta)
\end{aligned}
$$

- Model $\text{Pr}(Y_i = 1)$
- $X_i\beta$ is a linear predictor
- Not OLS anymore; parameters estimated by MLE
- f is a function that maps $(-\infty, +\infty)$ to $(0, 1)$

Logit Model

Alternative specification:

$$\begin{aligned} \text{Pr}(Y_i = 1) &= \pi_i \\ \text{logit}(\pi_i) &= \alpha + X_i\beta \end{aligned}$$

Log-odds of the probability of Y is a linear function

```r
summary(glm(voterep ~ income, data = nes_sample,
            family = binomial(link = "logit")))
```

```
##
## Call:
## glm(formula = voterep ~ income, family = binomial(link = "logit"),
##     data = nes_sample)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -1.0738  -1.0066  -0.8793   1.3584   1.5838
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept) -1.25311    0.27045  -4.633  3.6e-06 ***
## income       0.16741    0.06276   2.668  0.00764 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 1311.4  on 999  degrees of freedom
## Residual deviance: 1304.1  on 998  degrees of freedom
## AIC: 1308.1
##
## Number of Fisher Scoring iterations: 4
```

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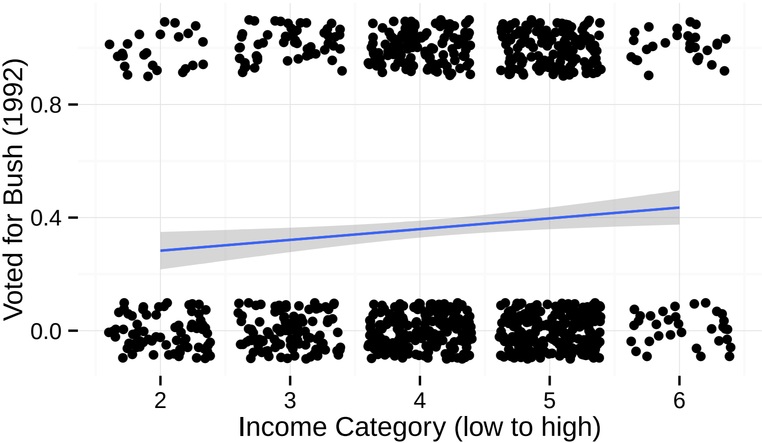
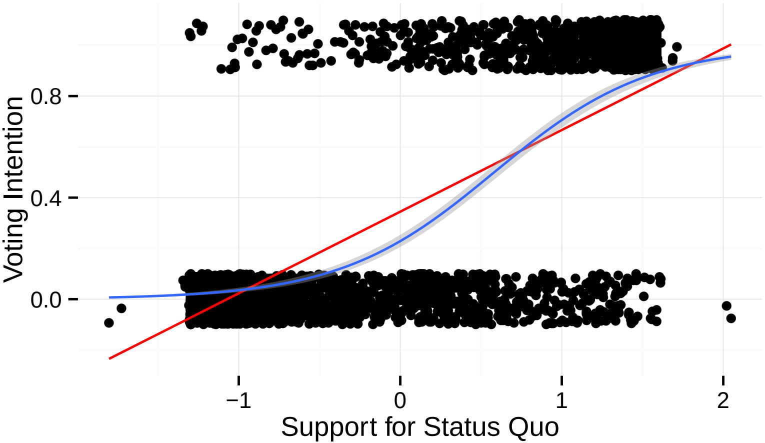


```r
summary(glm(vote_yes ~ statusquo, data = Chile,
            family = binomial(link = "logit")))
```

```
##
## Call:
## glm(formula = vote_yes ~ statusquo, family = binomial(link = "logit"),
##     data = Chile)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -2.4942  -0.4747  -0.2290   0.5747   2.8140
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept) -1.21597    0.06955  -17.48   <2e-16 ***
## statusquo    2.08971    0.07805   26.78   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 3242.0  on 2518  degrees of freedom
## Residual deviance: 1874.9  on 2517  degrees of freedom
##   (181 observations deleted due to missingness)
## AIC: 1878.9
##
## Number of Fisher Scoring iterations: 5
```

Example of Linear Probability Model

Vote Intention in Chilean Plebiscite in 1973



Logit Coefficients are Less Transparent

In linear regression, $\partial Y / \partial X_j = \beta_j$

$$\frac{\partial Y}{\partial X_j} = \frac{\partial}{\partial X_j}(\alpha + \beta_1 X_1 + \ldots \beta_k X_k) = \beta_j$$

In logistic regression, $\partial Y / \partial X_j = \beta_j$

$$\frac{\partial \, \mathsf{Pr}(Y_i = 1)}{\partial X_j} = \frac{\partial}{\partial X_j} \frac{1}{1 + \exp(\alpha + \beta X_i)} = \mathsf{Pr}(Y = 1 | X_i) \, \mathsf{Pr}(Y = 0 | X_i) \beta_j$$

or

$$\frac{\partial logit(Y_i = 1)}{\partial X_j} = \frac{\partial}{\partial X_j} X_i \beta_j = \beta_j$$

- Unlike OLS, the partial derivative depends on value of X_i

The LPM Strikes Back

- LPM has renewed popular among econometricians, causal inference folks -
- See the debate here
- OLS is still Min MSE linear approx of Conditional Expectation Function
- It is biased if functional form is wrong; but so it logit / probit. And the functional form is **always** wrong
- If you care about **average marginal effects** OLS does well

$$\text{Avg. Marginal Effect} = \frac{1}{n}x \sum_i \frac{\partial Y}{\partial x_j}|_{X_i}$$

- Angrist and Pischke recommend LPM with heteroskedasticity consistent errors

Comparing Average Marginal Effects of Logit and LPM

1992 U.S. Election Example

```
mod <- glm(voterep ~ income, data = nes_sample,
           family = binomial(link = "logit"))
mod_aug <- augment(mod, type.predict = "response")
mean(mod_aug$.fitted * (1 - mod_aug$.fitted) * coef(mod)[2])

## [1] 0.03847867

lm(voterep ~ income, data = nes_sample)

##
## Call:
## lm(formula = voterep ~ income, data = nes_sample)
##
## Coefficients:
## (Intercept)       income
##     0.20679      0.03811
```

Comparing Average Marginal Effects of Logit and LPM

Chile Plebiscite Example

```
mod <- glm(vote_yes ~ statusquo, data = Chile,
           family = binomial(link = "logit"))
mod_aug <- augment(mod, type.predict = "response")
mean(mod_aug$.fitted * (1 - mod_aug$.fitted) * coef(mod)[2])

## [1] 0.2436621

lm(vote_yes ~ statusquo, data = Chile)

##
## Call:
## lm(formula = vote_yes ~ statusquo, data = Chile)
##
## Coefficients:
## (Intercept)    statusquo
##      0.3447       0.3215
```

Linear Probability Model

Logit Models

LPM vs. Logit

References

References

- Fox, Ch. 14
- Gelman and Hill, Ch 5. This should have most material you need.