

POLS/CS&SS 503:  
Advanced Quantitative Political Methodology

# STATISTICAL INFERENCE FOR REGRESSION

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# Overview

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## Overview of Statistical Inference

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# Statistical Inference

- Population:  $Y$
- Parameters of interest:  $\beta$  from  $Y = \beta X + \epsilon$ .
- Sample:  $y$
- Sample statistics (estimates):  $\mathbf{b}$
- Since samples are random, different samples produce  $\mathbf{b}$ ?
  - How do we use the samples to the population parameters?
  - How do we quantify our uncertainty about that estimate?

# Science is about Uncertainty

- Knowledge is never certain
- Goal: Estimating unknowns and **quantifying the uncertainty** of those estimates
- Estimates without uncertainty are incomplete at best, useless or biased at worst

# The Fundamental Problem of Statistical Inference

- We have methods to calculate the probability of a sample and sample statistics **given** we know the population parameters.
- But we don't know the population parameters, so what do we do?
- Two (three) main methods
  - Frequentist: do not calculate the probability of the parameter
    - Hypothesis testing: Assume a hypothesis and check if data is consistent with it.
    - Confidence intervals: find a plausible range of parameters
  - Bayesian: calculate the probability of the parameter

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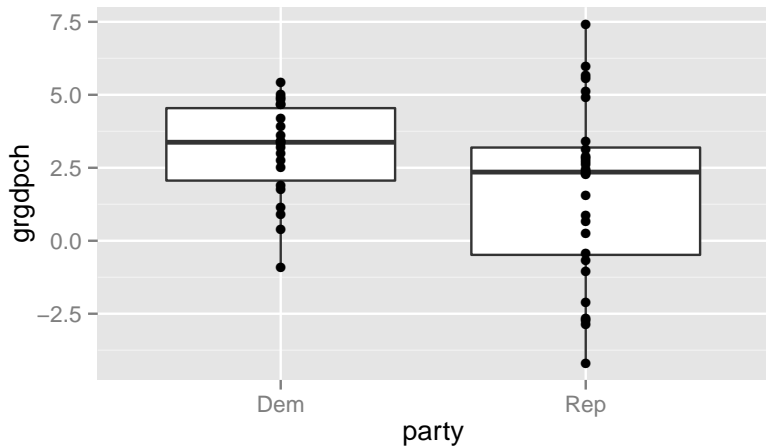
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# Is US Economic Growth Higher Under Democratic Presidents than Republican Presidents?





# Is US Economic Growth Higher Under Democratic Presidents than Republicans?

```
## Source: local data frame [2 x 4]
```

```
##
```

```
##   party      mean      sd length
```

```
## 1   Dem 3.094356 1.672123     22
```

```
## 2   Rep 1.725821 3.014028     28
```

# Sampling Distribution of the Difference in Means

- Want to know  $\mu_D - \mu_R$ ? (Difference in population means)
- What is the sample? What is the population?
- We will be making other dubious assumptions in this example: populations are independent, normal (not important).
- Estimate is  $\bar{x}_D - \bar{x}_R$  (Difference in sample means)
- But the observed sample is random, so how do we characterize the uncertainty in our estimates?

# Sampling Distribution of the Difference in Means

If we knew  $\mu_D, \mu_R, \sigma_R, \sigma_D$ , we could calculate the distribution of  $\bar{x}_D - \bar{x}_R$ .

$$(\bar{x}_D - \bar{x}_R) \sim N\left(\mu_D - \mu_R, \frac{\sigma_R^2}{n_R} + \frac{\sigma_D^2}{n_D}\right)$$

But we don't know the population ...

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# Logic of Significance Tests

- Assume null  $H_0$  and alternative  $H_a$  hypotheses
- Calculate the sampling distribution of the test statistic assuming  $H_0$  is true
- $p$ -value is the probability of data (test statistics) equal or more extreme than the sample
- (optional) At a pre-defined significance level ( $\alpha$ ), reject  $H_0$  if  $p$ -value less than  $\alpha$ , fail to reject if  $p$ -value greater than  $\alpha$ .

# Significance Test for Difference in Means

- Null hypothesis:  $H_0 : \mu_D - \mu_R = 0$
- Alternative hypothesis:  $H_a : \mu_D - \mu_R \neq 0$
- The test statistic is

$$t = \frac{\bar{x}_D - \bar{x}_R}{SE(\bar{x}_D - \bar{x}_R)} \quad (1)$$

where

$$SE(\bar{x}_D - \bar{x}_R) = \sqrt{\frac{\sigma_D^2}{n_D} + \frac{\sigma_R^2}{n_R}}$$

- Since we don't know  $\sigma_R^2$  and  $\sigma_D^2$ , use sample variances:  $s_R^2, s_D^2$  as estimators.
- Use  $t$  distributed Student's  $t$  to account for uncertainty from estimating standard deviations. It would be distributed standard Normal if the population standard deviations were known.

# t-distribution

See | <https://jrnold.shinyapps.io/tdist> |

# t-tests for difference of means in R

```
t.test(grgdpch ~ party, # y ~ x
       data = gdp, # dataset
       mu = 0, # H_0: mu_1 - mu_2
       conf.level = 0.95 # confidence level to use for CI
)

##
## Welch Two Sample t-test
##
## data: grgdpch by party
## t = 2.0366, df = 43.679, p-value = 0.04778
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  0.01400377 2.72306582
## sample estimates:
## mean in group Dem mean in group Rep
##          3.094356          1.725821
```



# Significance Tests

- Two approaches:
  - Fisher:  $p$ -value represents the level of evidence against  $H_0$
  - Neyman-Pearson: choose a significance level  $\alpha$  and reject null hypothesis if  $p$ -value is less than  $\alpha$ .
- When making a decision of reject / not reject:
  - Type I error:  $H_0$  true, reject  $H_0$
  - Type II error:  $H_0$  false, fail to reject  $H_0$
- Power:  $1 - \Pr(\text{Type II error})$
- Tests generally focus on Type I error
  - “conservative” – is it really? It prioritizes a hypothesis, and usually the null hypothesis has less evidence than the alternative.
  - much harder to calculate Type II errors

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# Logic of Confidence Intervals

- Find a plausible range of values of the parameter:  $[\bar{x}_{lower}, \bar{x}_{upper}]$
- Only know probability of data given parameter value, so cannot calculate a probability distribution for a parameter value (Bayesian approach)
- Frequentist approach: method to generate intervals which contain the true parameter  $\mu$  in  $C\%$  of the samples.

# What a $100(1 - \alpha)\%$ confidence interval means

**Coverage** A  $100(1 - \alpha)\%$  confidence interval for a parameter  $\theta$ , is an interval generated by a method that generates intervals that include the true parameter  $\theta$  in  $100(1 - \alpha)\%$  of samples.

**Rejection Region** A  $100(1 - \alpha)\%$  confidence interval such that  $H_0 : \theta = \theta'$  cannot be rejected at the  $\alpha$  significance level for all values of  $\theta'$  in the interval, and  $H_0 : \theta = \theta'$  is rejected for all values of  $\theta'$  outside the interval. (not all confidence intervals have this property).

# Confidence levels for difference in means

To get a  $100(1 - \alpha)\%$  confidence interval for a difference of means

$$\bar{x}_D - \bar{x}_R \pm t_{\alpha/2, \nu} \sqrt{\frac{s_D^2}{n_D} + \frac{s_R^2}{n_R}}$$

where  $t_{\alpha/2, \nu}$  is a critical value of the  $t$  distribution such that the tails area of the distribution is  $\alpha$ . The value of  $\nu$  is complicated.

# How to report a confidence interval

- Either of
  - Democratic presidents enjoyed growth rates 1.37 points higher (95% CI: 0.01 to 2.72) than their Republican counterparts.
  - Democrats enjoyed 1.37 points higher growth than Republicans, with a 95 percent confidence interval of 0.01 to 2.72.
- We could calculate any CI we wish: 90 percent, 80 percent, 50 percent, etc.
- The most commonly used are: 90, 95, and 99.

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# Fun Stuff

- <https://xkcd.com/882/>
- <https://xkcd.com/1478/>



# Confidence Intervals vs. Significance Tests

- Problems with **both**
  - Simply commitment to a certain error rate, given **assumptions**. Does not account for **model uncertainty**.
  - “File drawer problem”, “fishing”: even if it makes sense on an individual test, multiple testing within a research project + selecting on significant results can result in biases.
- Problems with significance tests that CI overcome
  - tests are “weak” - only show one result
  - confidence intervals focus more on substantive significance (parameter values);  $p$ -values ignore all substantive significance.

# Statistical and Substantive Significance

- $p$ -values are a function of estimated effect size ( $B$ ) but also the sample size
- $p$ -values only show statistical significance, not substantive significance.
- Confidence intervals can be more useful for displaying substantive significance

# Confidence Intervals vs. Significance Tests

- Confidence intervals often misinterpreted
- Definition of confidence interval is awkward and not exactly what we want, so often interpreted as probability interval
- But which is clearer?
  - Compared to Republicans, the effect of Democratic presidents on the economy is significantly positive at the 0.05 level.
  - Democratic presidents enjoyed 1.37 points higher growth than Republicans, with a 95 percent confidence interval of 0.01 to 2.72.

# Bayesian vs. Frequentist Statistics

- Confidence intervals and significance tests do not calculate the probability of hypotheses (parameters)
- Bayesian statistics attempts to do so, but
  - requires prior probability of the hypotheses
  - computationally, mathematically more difficult

# Conditional Probability

$$p(A|B) = \frac{p(A \& B)}{P(B)}$$

- What if  $A$  and  $B$  are independent?  $P(A|B) = P(A)$
- What is the sampling distribution?

# Bayes Rule

$$\begin{aligned} p(A|B) &= \frac{p(B|A)p(A)}{\sum_{A'} p(B|A')p(A')} \\ &= \frac{p(B|A)p(A)}{\sum_{A'} p(B|A')p(A')} \\ &\propto p(B|A)p(A) \end{aligned}$$

# Inference and Bayes Rule

Want to find the probability of a hypothesis  $H$  given the data  $D$ :

$$\begin{aligned} p(H|D) &= \frac{p(H|D)p(D)}{\sum_{H'} p(D|H')p(H')} \\ &= \frac{p(D|H)p(H)}{p(D)} \\ &\propto p(D|H)p(H) \end{aligned}$$

- $p(D|H)$  is the likelihood (related to the sampling distribution).
- Where does  $p(H)$  come from?

# Bayesian and Frequentist Statistics

In many research questions we are interested in the probability of the hypothesis  $H$ , given the data  $D$ :  $p(H|D)$ .

**Frequentist** Assume a hypothesis, e.g. null hypothesis  $H_0$ , and calculate the probability of the data:  $p(D|H_0)$

**Bayesian** Assume a prior distribution  $p(H)$  and calculate the probability of the hypothesis given the data:

$$p(H|D) \propto p(D|H)p(H)$$

**My Claim:** Even if using frequentist tests  $p(D|H)$ , a paper assigns prior probabilities to hypotheses, e.g. lit review, logical arguments, etc. to make a Bayesian argument,  $p(D|H)p(H)$ .



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# Statistical Inference for OLS

- Individual Coefficients:
  - Significance tests:  $\beta_k$
  - Confidence intervals
- Multiple coefficients:
  - Significance test
    - F-test on all slopes:  $H_0 : \beta_1 = \dots = \beta_k = 0$
    - F-test on subset of slopes:  $H_0 : \beta_1 = \dots = \beta_k = 0$
    - F-test on linear combinations of slopes: e.g  $H_0 : \beta_1 - \beta_2 = 0$ .
  - Confidence regions

# Statistical Inference for Individual Coefficients for Simple Regression

If all assumptions of Gauss-Markov hold, variance of sample coefficient  $B$  is

$$V(B) \sim N \left( \beta, \frac{\sigma_{\epsilon}^2}{\sum (x_i - \bar{x})^2} \right)$$

If  $\epsilon$  not normal, approximate as  $n \rightarrow \infty$

- Test statistic for  $H_0 : \beta = \beta_0$  distributed Student's  $t$  with  $n - k - 1$  df.

$$t = \frac{B - \beta_0}{\text{SE}(B)}$$

- Confidence interval is

$$B \pm t_{\alpha/2, n-k-1} \text{SE}(B)$$

- Where  $\text{SE}(B)$  is  $V(B)$  with  $\hat{\sigma}_{\epsilon}^2$  as an estimate of  $\sigma_{\epsilon}$ .

# Estimate of $\sigma_\epsilon^2$

Estimate  $\sigma_\epsilon^2$  from the regression squared errors:

$$\hat{\sigma}_\epsilon^2 = \frac{\sum E_i^2}{n - k - 1}$$

- where  $n - k - 1 = (\text{observations}) - (\text{variables}) - (\text{intercept})$  is the degrees of freedom.
- Similar to mean squared error, but to estimate population divide by degrees of freedom.

# Statistical Inference for Individual Coefficients for Multiple Regression

If multiple variables, and Gauss-Markov assumptions hold, then

$$\mathbf{b} \sim MVN \left( \boldsymbol{\beta}, \sigma_{\epsilon}^2 (\mathbf{X}'\mathbf{X})^{-1} \right)$$

- analogous to the bivariate version
  - calculates all standard errors simultaneously
  - covariances:  $B_i$  and  $B_j$  can be correlated

# Statistical Inference for Individual Coefficients for Multiple Regression

Standard error for a single coefficient

$$SE(B_j) = \sqrt{\hat{\sigma}_\epsilon (X'X)^{-1}_{jj}} = \frac{1}{\sqrt{1 - R_j^2}} \frac{\hat{\sigma}_\epsilon}{\sqrt{\sum (x_{ij} - \bar{x}_j)^2}}$$

- Test statistic for  $H_0 : \beta = \beta_0$  distributed Student's t with  $n - k - 1$  df.

$$t = \frac{\beta - \beta_0}{se}$$

- Confidence interval is

$$B \pm t_{\alpha/2, n-k-1} se$$

- Where  $\hat{\sigma}_\epsilon = \frac{\sum_i E_i^2}{n-k-1}$

# Confidence Interval

## General Definition

In repeated samples,  $C\%$  of samples have a  $C\%$  confidence interval that contains the population (true) parameter  $\theta$ .

- Not a statement about a sample interval, statement about the method
- Each confidence interval either contains  $\theta$  or not, there is no probability. Parameters are fixed, only samples are random.

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Overlapping confidence intervals does not mean  
difference is not statistically significant

See [https:](https://www.cscu.cornell.edu/news/statnews/Stnews73insert.pdf)

[//www.cscu.cornell.edu/news/statnews/Stnews73insert.pdf](https://www.cscu.cornell.edu/news/statnews/Stnews73insert.pdf)

# Significant and Not significant are not statistically significant

- Common example:
  - Regression with several dummy variables
  - Coefficient of dummy variable of category A ( $\beta_A$ ) is significant at 5% level, dummy variable of category B ( $\beta_B$ ) is not significant at the 5% level.
  - Common (wrong) interpretation: A and B are different
  - Correct procedures:
    - Significance test with  $H_0 : \beta_A = \beta_B$
    - calculate confidence interval of  $\beta_A - \beta_B$ .

# References

- Some slides derived from Christopher Adolph *Inference and Interpretation of Linear Regression*. Used with permission. <http://faculty.washington.edu/cadolph/503/topic4.pw.pdf>
- Material included from
  - Fox Ch 6, 9.3