POLS/CS&SS 503:
Advanced Quantitative Political Methodology

LINEAR REGRESSION ESTIMATOR

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Overview

Regression Coefficient Anatomy

Linear Regression Population Model

Sampling Distribution

Estimators and mean squared error

Gauss-Markov Theorem

Coeficients of a simple regression

$$Y_i = A + BX_i + E_i$$

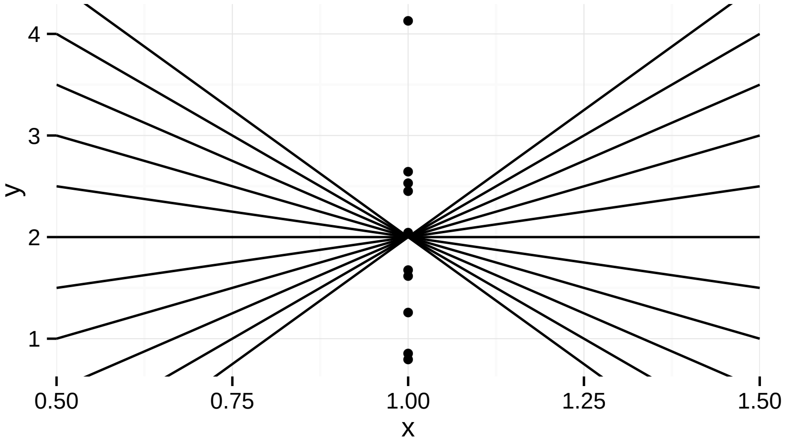
The least squares coefficients are

$$A = \bar{Y} + B\bar{X}$$

$$B = \frac{\sum_i (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2}$$

- State B in terms of covariance of X and Y and variance?
- State B in terms of correlation of X and Y and standard deviations?
- What values can B take if $\mathrm{SD}(X) = \mathrm{SD}(Y) = 1$?
- What is $\hat{Y}$ for $X = \bar{X}$?
- What happens to B as $\mathrm{V}(X)$ decreases? $\mathrm{V}(Y)$ decreases? If $\mathrm{V}(X) = 0$

Least squares when $\mathbb{V}X = 0$



Least squares coefficients are unidentified if $\mathrm{V}\, x = 0$

- If $\mathrm{V}\, x = 0$ then least squares solution is unidentified
- There is no unique value of A, B that $\arg\min_{A,B} \sum_i E_i^2$

```r
y <- c(1, 2, 3, 4, 5)
x <- 1
ybar <- mean(y)
ybar
```

```
## [1] 3
```

```r
# A = 2, B = 1
sum((y - 2 - 1 * x) ^ 2)
```

```
## [1] 10
```

```r
# A = -7, B = 10
sum((y + 7 - 10 * x) ^ 2)
```

```
## [1] 10
```

Coefficients of a multiple regression

$$\vec{Y} = Xb + e$$

- $b = (X'X)^{-1}X'y$. *Not that intuitive!*
- Coefficient b_j is

$$b_k = \frac{\mathrm{C}(y, \tilde{x}_j)}{\mathrm{V}(\tilde{x}_k)}$$

- Where $\tilde{x}_j$ are the residulals of x_j on all X_h where $h \neq j$

$$\tilde{X}_{j,i} = X_{j,i} - \tilde{A} - \sum_{h \neq j} \tilde{B}_h X_h$$

Regression example

See `multiple_regression_anatomy.R`

Least Squares coefficients are unidentified if $(X'X)^{-1}$ does not exist

- Common cases in which $(X'X)^{-1}$ does not exist:
  - Number of observations less than $k+1$
  - X_k is constant
  - X_k is a linear function of other variables: $X_k = \sum_{j \neq k} c_j X_j$.
    - dummy variables for all categories of a categorical variable
    - variable multiplied by the constant of another variable

Which of these would be cases of collinearity and why?

- There is a variable that takes values “white”, “black”, “hispanic”, “asian”, “other”. You include a dummy variable for each category.
- GDP, GDP per capita, and population
- Log GDP, log GDP per capita, and log population
- GDP in millions of dollars; GDP in trillions of dollars
- GDP measured in nominal value; GDP measured in real terms
- Regression with 3 variables and 4 observations

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Population model in a simple regression

$$Y_i = \alpha + \beta X_i + \epsilon_i$$

Assumptions for statistical inference

1. *X is not invariant:* $\mathrm{V}(X) > 0$
2. *Linearity*. Average value of error given x is 0. $E(\epsilon_i) = E(\epsilon_i | x_i) = 0$

$$\mu_i = E(Y_i) = \mathrm{E}(Y|X_i) = \mathrm{E}(\alpha + \beta X_i + \epsilon_i) = \alpha + \beta x_i$$

3. *Constant variance* Variance of the errors is the same regardless of the value of X

$$\mathrm{V}(Y|x_i) = E(\epsilon_i^2) = \sigma_\epsilon^2$$

4. *Independence:* Observations are sampled independently. $\mathrm{Cor}(\epsilon_i, \epsilon_j) = 0$ for all $i \neq j$.
5. Fixed X or X measured without error and independent of the error.
6. Errors are normally distributed $\epsilon_i \sim N\left(0, \sigma_\epsilon^2\right)$

Population model in a multiple regression

$$Y_i = \alpha + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \ldots + \beta_k x_{i,k} + \epsilon_i$$

Assumptions for statistical inference

1. *X is not invariant* and no X is a perfect linear function of the others.
2. *Linearity*. $\mathrm{E}(\epsilon_i) = 0$
3. *Constant variance* $V(\epsilon_i) = \sigma_\epsilon^2$
4. *Independence Observations are sampled independently*. $\mathrm{Cor}(\epsilon_i, \epsilon_j) = 0$ for all $i \neq j$.
5. *Fixed X or X measured without error and independent of the error*
6. *Normality* Errors are normally distributed $\epsilon_i \sim N\left(0, \sigma_\epsilon^2\right)$

Definitions

population The observations of interest. May be theoretical

sample The data you have.

parameter A function of the population distribution

statistic A function of the sample

sampling distribution The distribution of a statistic calculated from the distribution of samples of a given size drawn from a population.

See `Sampling_Distributions.Rmd`

Sampling Distribution of Simple Regression Coefficients

The sampling distributions of A, B given $Y_i = \alpha + \beta X_i + \epsilon_i$

- expected values (linearity)

$$E(A) = \alpha$$
$$E(B) = \beta$$

- variances (linearity, constant variance, independence)

$$V(A) = \frac{\sigma_\epsilon^2}{n} \cdot \frac{\sum x_i^2}{\sum (x_i - \bar{x})^2}$$
$$V(B) = \frac{\sigma_\epsilon^2}{\sum (x_i - \bar{x})^2} = \frac{\sigma_\epsilon^2}{n\, V(x)}$$

- normal distribution (normal errors)

$$A \sim N(E(A), V(A))$$
$$B \sim N(E(B), V(B))$$

Coefficient sampling distributions in multiple regression

The sampling distributions of B_k given $Y_i = \alpha + \sum \beta_j X_{j,i} + \epsilon_i$

- Expected value: $\mathrm{E}(B_K) = \beta_k$
- Variance:

$$\mathrm{V}(B_j) = \frac{1}{(1 - R_j^2)} \frac{\sigma_\epsilon^2}{\sum (x_{i,j} - \bar{x}_j)^2}$$
$$= \frac{\sigma_\epsilon^2}{\sum_i \left(x_{i,j} - \hat{x}_{i,j}\right)^2}$$

  Where R_j^2 is R^2 from regression of X_j on other X, and $\hat{x}_{ij}$ are fitted values from that regression.

- Normally distributed if errors are normally distributed or as $n \to \infty$.
- b is multivariate normally distributed

$$b \sim N\left(\beta, \sigma_\epsilon^2 (X'X)^{-1}\right)$$

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Let's define some things

statistic: Function of a sample, e.g. Sample mean $\bar{x} = \frac{1}{n} \sum x_i$

parameter: Function of the population distribution, e.g. Expected value μ of the normal distribution.

estimator: Method to use a sample statistic (estimate) to infer a population parameter (estimand)

How to determine if an estimator is good?

- Is $\hat{\beta} = \left(X'X\right)^{-1}X'y$ a good estimator for β?
- Would another estimator be better?
- First, need criteria to by which to judge estimators

What makes an estimator good?

- Bias
- Variance
- Efficiency (mean squared error)
- Consistency

Bias and Variance

Bias

On average how far off is the estimator?

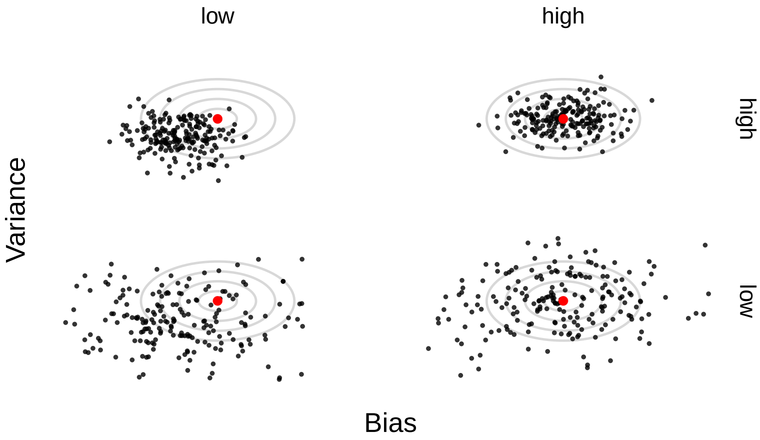
$$\text{bias}(\hat{\beta}) = \mathrm{E}(\hat{\beta}) - \beta$$

Variance

Does the estimator give similar results in different samples?

$$\mathrm{V}(\hat{\beta}) = \mathrm{E}\left((\beta - \mathrm{E}(\hat{\beta}))^2\right)$$

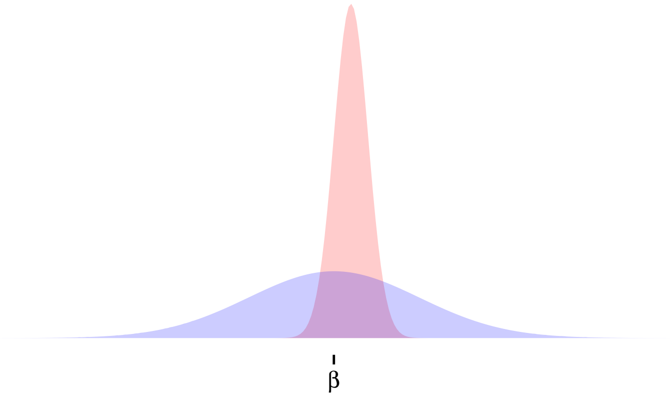
Bias and Variance Visualized



What makes an estimator good?

- Unbiased methods may still miss the truth by a large amount, just direction not systematic
- Unbiased estimates can be horrible: random draw from numbers 0–24 for time of day
- Biased estimates are not necessarily terrible: a clock that's 2 minutes fast

You may prefer a biased, low variance estimator to an unbiased, high variance estimator



Mean Squared Error (MSE)

- MSE is

$$MSE(\hat{\beta}) = \mathrm{E}\left((\hat{\beta} - \beta)^2\right)$$

- MSE trades off bias and variance

$$\begin{aligned} MSE(\hat{\beta}) &= \mathrm{E}((\hat{\beta} - \mathrm{E}(\hat{\beta}))^2) + \mathrm{E}(\mathrm{E}(\hat{\beta}) - \beta))^2 \\ &= \mathrm{V}(\hat{\beta}) + \left(\text{bias}(\hat{\beta}, \beta)\right)^2 \end{aligned}$$

- root mean squared error (RMSE) $\sqrt{\text{MSE}}$: on average how far is an estimate from the truth
- An **efficient** estimator has the smallest MSE
- What is the MSE of an unbiased estimator?

$$MSE(\hat{\beta}) = \mathrm{V}(\hat{\beta}) + \left(\text{bias}(\hat{\beta}, \beta)\right)^2 = \mathrm{V}(\hat{\beta}) + 0 = \mathrm{V}(\hat{\beta})$$

MSE Example

- Suppose population parameter $\beta = 1$
- Consider two estimators $\hat{\beta}_1$ and $\hat{\beta}_2$.
  - $\hat{\beta}_1 \sim N(1, 1^2)$
  - $\hat{\beta}_2 \sim N(0.5, 0.5^2)$
- What are the bias, variance, and MSE of each estimator?

Consistency

- A consistent estimator converges to the parameter value as the number of observations grows

$$E(\hat{\beta} - \beta) \to 0 \qquad \text{as } n \to \infty$$

- A concern of econometricians
- May not be as much a concern in finite, small sample sizes
- We will mainly be concerned with efficiency, secondarily with bias, rarely with consistency

LS assumptions and consequences of violations

	Assumption		Consequence of violation
1	No perfect collinearity	$\text{rank}(X) = k, k < n$	Coefficients unidentified
2	X is exogenous	$\text{E}(X\epsilon) = 0$	Biased, even as $n \to \infty$
3	Disturbances have mean 0	$\text{E}(\epsilon) = 0$	Biased, even as $n \to \infty$
4	No serial correlation	$\text{E}(\epsilon_i \epsilon_j) = 0, i \neq j$	Unbiased but ineff. Wrong se.
5	Homoskedastic errors	$\text{E}(\epsilon' \epsilon') = \sigma^2 I$	Unbiased but ineff. Wrong se.
6	Normal errors	$\epsilon \sim N(0, \sigma^2)$	se wrong unless $n \to \infty$

Assumptions stronger from top to bottom, 4 and 5 could be combined

Unbiasedness of LS

- Only need assumptions 1-3 (no collinearity, X exogenous, $\mathrm{E}(\epsilon) = 0$
- Start with

$$\begin{aligned}\hat{\beta} &= (X'X)^{-1}X'(X\beta + \epsilon) \\ &= \beta + (X'X)^{-1}X'\epsilon\end{aligned}$$

- Take the expectation

$$\begin{aligned}\mathrm{E}(\hat{\beta}) &= \mathrm{E}(\beta) + \mathrm{E}(X'X)^{-1}X'\epsilon) \\ &= \mathrm{E}(\beta) + X'X)^{-1}X'\,\mathrm{E}(\epsilon) \\ &= \mathrm{E}(\beta)\end{aligned}$$

- Since $\mathrm{E}(\hat{\beta}) = \mathrm{E}(\beta)$, LS is unbiased.

Gauss-Markov

- If make assumptions 1–5: LS is the best linear unbiased estimator (BLUE)
- LS estimator is **linear** because $\hat{\beta} = My$, where $M = (X'X)^{-1}X'$
- **best** is best mean squared error (MSE).
- If LS is unbiased, then its mean squared error is the same as its ...?
- Could exist other non-linear unbiased estimators with smaller MSE, e.g. Robust regression when population has fat tailed errors
- If errors are Gaussian, LS is Minimum Variance Unbiased (MVU).
- MVU = for *all* estimators that are unbiased. $\hat{\beta}$ has smallest variance (and MSE).

References

- Some slides derived from Christopher Adolph *Linear Regression in Matrix Form / Propoerties & Assumptions of Linear Regression*. Used with permission. <http://faculty.washington.edu/cadolph/503/topic3.pw.pdf>
- Material included from
    - Fox Ch 6, 9.3
    - Angrist and Pischke, Chapter 3

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