POLS/CS&SS 503:
Advanced Quantitative Political Methodology

MATRIX ALGEBRA, LINEAR REGRESSION

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Agenda

- Linear regression as finding a “best” line
- Linear regression as the conditional expectation function
- How linear regression relates to the normal distribution

What is regression?

Regression

distribution of a **response** (outcome) variable Y — or summary of that distribution — as a function of **explanatory** variables $X_1, \ldots, X_k$.

Ordinary Least Squares

Finds a $\hat{Y} = XB$ that minimizes $\sum(Y_i - \hat{Y}_i)^2)$. This estimates a linear conditional expectation function $E(Y|X_1, \ldots, X_k)$.

OLS Objective Function

One X

Find the line

$$\hat{Y} = A + BX$$

such that

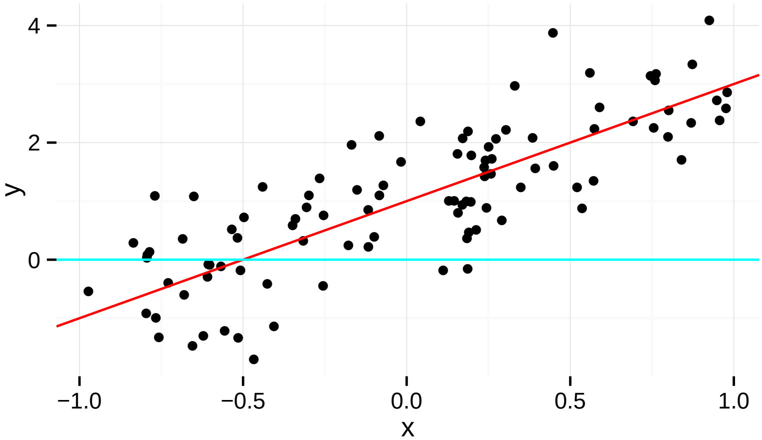
$$A, B = \arg\min_{A,B} S(A, B)$$

where

$$S(A, B) = \sum_i E_i^2 = \sum_i (Y_i - \hat{Y}_i)^2 = \sum_i (Y_i - A - BX_i)^2$$

How do we minimize this?

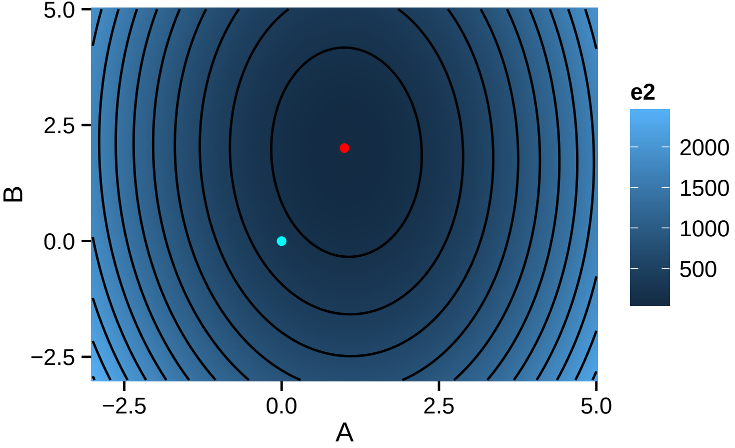
What does the OLS objective function look like?



Data generated by $Y_i = 1 + 2X_i + E_i$. Lines are $A = 1, B = 2$, and $A = 0, B = 0$.

$\sum E_i^2$ as a function of A and B

Least squares is the minimum of this function



Finding the best A, B in Least Squares

One X

To minimize, set partial derivatives equal to 0 and solve:

$$\frac{\partial S(A,B)}{\partial A} = \sum(-1)(2)(Y_i - A - BX_i) = 0$$
$$\frac{\partial S(A,B)}{\partial B} = \sum(-X_i)(2)(Y_i - A - BX_i) = 0$$

Rearrange to get

$$A = \bar{Y} - B\bar{X}$$
$$B = \frac{\sum(X - \bar{X})(Y - \bar{Y})}{\sum(X_i - \bar{X})^2} = \frac{\mathrm{C}(X,Y)}{\mathrm{V}(X)}$$

Implications of the OLS Solution

Least squares A and B

$$A = \bar{Y} - B\bar{X}$$
$$B = \frac{\sum(X - \bar{X})(Y - \bar{Y})}{\sum(X_i - \bar{X})^2} = \frac{\mathrm{C}(X, Y)}{\mathrm{V}(X)}$$

- $\bar{X}, \bar{Y}$ is in the regression line
- $\sum X_i E_i = 0$

$$\begin{aligned}\sum X_i E_i &= \sum X_i (Y_i - A - BX_i) \\ &= \sum X_i Y_i - A \sum X_i - B \sum X_i = 0\end{aligned}$$

- $\sum \hat{Y}_i E_i = 0$
- Errors E uncorrelated with $\hat{Y}$ and X

OLS Objective Function

Multiple X

Find plane

$$Y = A + B_1X_1 + B_2X_2 + \ldots + B_kX_k$$

such that

$$A, B_1, \ldots, B_k = \underset{A, B_1, \ldots, B_k}{\arg\min}\, S(A, B_1, \ldots, B_k)$$

where

$$S(A, B_1, \ldots, B_k) = \sum_i E_i^2 = \sum_i (Y_i - \hat{Y}_i)^2$$

$$= \sum_i (Y_i - A - \sum_{j=1}^{k} B_j X_{i,j})$$

How do we minimize this?

Finding the best A, B in Least Squares Regression

Multiple X

Set partial derivatives equal to 0 and solve system of equations for

$$\begin{aligned}
\frac{\partial S(A, B_1, B_2, \dots, B_k)}{\partial A} &= \sum(-1)(2)(Y_i - A - BX_i) = 0 \\
\frac{\partial S(A, B_1, B_2, \dots, B_k)}{\partial B_1} &= \sum(-X_{i,1})(2)(Y_i - A - B_1X_{i,1} - \ldots - B_2X_{i,k}) = 0 \\
\vdots &= \vdots \\
\frac{\partial S(A, B_1, B_2, \dots, B_k)}{\partial B_k} &= \sum(-X_{i,k})(2)(Y_i - A - B_1X_{i,1} - \ldots - B_2X_{i,k}) = 0
\end{aligned}$$

Not as easy ...

Linear Regression in Matrix Form

Scalar representation

$$Y_i = B_0 + B_1 X_{i,1} + B_2 X_{i,2} + \dots B_k X_{i,k} + E_i$$

Equivalent matrix representation

$$\underset{n\times 1}{y} = \underset{n\times(k+1)}{X} \quad \underset{(k+1)\times 1}{b} + \underset{n\times 1}{e}$$

or

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_{1,1} & X_{2,1} & \dots & X_{k,1} \\ 1 & X_{1,2} & X_{2,2} & \dots & X_{k,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{1,n} & X_{2,n} & \dots & X_{k,n} \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_k \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix}$$

Linear Regression in Matrix Form

Objective Function

The linear regression is

$$y = Xb + e$$

Want to find the b that minimizes the squared errors:

$$\arg\min_b S(b)$$

where

$$\begin{aligned} S(b) &= \sum E_i^2 = e'e \\ &= (y - Xb)'(y - Xb) \end{aligned}$$

Why does e need to be transposed?

Linear Regression in Matrix Form

Transpose of Sums

$$(A + B)' = A' + B'$$

$$\left(\begin{bmatrix} 10 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}\right)' = ?$$

$$? = ?$$

Linear Regression in Matrix Form

Transpose of a product

$$(XB)' = B'X'$$

$$\begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = ?$$

$$? = ?$$

Simplify $e'c$

$$
\begin{aligned}
e'e &= (y - Xb)'(y - Xb) && \\
&= (y' - (Xb)')(y - Xb) && \text{distribute the transpose} \\
&= (y - b'X)(y - Xb) && \text{substitute } b'X' \text{ for } (Xb)' \\
&= y'y - b'X'y - y'Xb + b'X'Xb && \text{multiply out} \\
&= y'y - 2b'X'y + b'X'Xb && \text{simplify}
\end{aligned}
$$

- To minimize need to calculate derivative of $e'e$ with respect to b.
- Need two know two things
  - derivative of scalar with respect to vector ($2b'X'y$)
  - derivative of quadratic form ($b'X'Xb$)

What is the derivative of scalar with respect to vector

- Need to take derivative of $e'e$ with respect to b to find b that min the sum of squared.
- A derivative of a scalar with respect to a vector

$$
\begin{aligned}
y = a'x &= a_1x_1 + a_2x_2 + \ldots + a_nx_n \\
\frac{\partial y}{\partial x} &= \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}' \\
\frac{\partial y}{\partial x} &= a
\end{aligned}
$$

Derivative of a quadratic form

- Equivalent to x^2 is inner product $x'x$
- Vector analogue of ax^2 is $x'Xx$, where A is $n \times n$ matrix

$$\frac{\partial ax^2}{\partial x} = 2ax$$
$$\frac{\partial x'Ax}{\partial x} = 2Ax$$

OLS in Matrix Form

Minimizing the objective function

1. Take partial derivative of $S(b)$:

$$\begin{aligned}\frac{\partial S(b)}{b} &= \frac{\partial}{b}(y'y - 2b'X'y + b'X'Xb) \\ &= 0 - (2y'X) + 2(X'X)b\end{aligned}$$

2. Set to 0, and solve for b:

$$\begin{aligned}X'Xb &= X'y \\ b &= (X'X)^{-1}X'y\end{aligned}$$

What $(X'X)^{-1}$ implies

- For b to be defined $(X'X)^{-1}$ needs to exist
- $X'X$ must be full rank
- rank of $X'X$ is the same as the rank of X
- The rank of X is between n and $k+1$, means that $n \geqslant k+1$ (obs > variables)
- $k+1$ columns of X must be linearly indepdendent?
    - Can you have a full set of dummies?
    - Can you include a variable that is always equal to 3?

Takeaways

- Linear regression is the $A, B_1, \ldots, B_k$ that solve $\arg\min_{A,B_1,\ldots,B_k} \sum E_i^2$
- Solving for linear regression coefficients is relatively **easy**; linear equations; there's an explicit solution. No iteration required.

Linear Regression and CEF

Linear Regression and Normal Distribution

Interpretation

CEF justification for linear regression justification

- Conditional Expectation Function is $\mathrm{E}(Y_i|X_i = x)$ for all x
- The CEF is the Min Mean Squared Error (MMSE) predictor of Y_i given X_i
- If the population CEF is linear, then the least squares population regression is the CEF
- If the population CEF is not linear, then the least squares line is the MMSE linear estimate of the CEF.
- See Angrist and Pischke, Ch 3.1

Linear Regression and CEF

Linear Regression and Normal Distribution

Interpretation

But I thought linear regression had to do with the normal distribution?

- Linear regression often presented as

$$y_i = X_i\beta + \epsilon_i \qquad \epsilon_i \sim N(0, \sigma^2)$$

- Why? We haven't had to assume normal distributions before now.
- Helps with statistical inference results.
- However, the CLT handles asymptotic sampling distribution of parameters

Interpreting Regression Coefficients β

How the average outcome variable differs, on average:

predictive
: between **groups of units** that differ by 1 in the relevant explanatory variable while being identical in all other explanatory variables the same

counterfactual
: in the **same individual** when chaning the relevant explanatory variable 1 unit while holding all other explanatory variables the same

See Gelman and Hill, p. 34; Fox, p. 81

References

- Some slides derived from Christopher Adolph *Linear Regression in Matrix Form / Propoerties & Assumptions of Linear Regression*. Used with permission.
- Material included from
    - Fox Ch 2, 5, 9.1–9.2
    - Angrist and Pischke, Chapter 3.1
    - Gelman and Hil, Chapter 2