POLS/CS&SS 503:
Advanced Quantitative Political Methodology

RESIDUALS IN OLS: NON-NORMALITY, HETEROSKEDASTICITY

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Overview

Non-Normal Errors

Heteroskedasticity

Non-Normal Errors

Heteroskedasticity

Non-normal errors

Suppose that errors ϵ have $E(\epsilon) = 0$, but not normal distribution

- $\boldsymbol{b}$ still unbiased
- V$(\boldsymbol{b})$ incorrect in small samples
- Still BLUE, but not MVUE

How to diagnose non-normal errors?

Graphical methods

- Plot studentized residuals against theoretical t-quantiles.
  - Studentized residuals (approximately standardized to have std dev of 1)

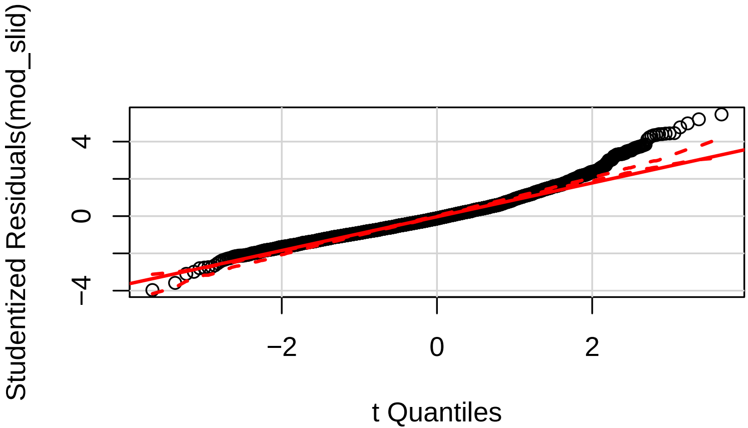
$$E_i^* \approx \frac{E_i}{S_E}$$

- Density plot
- Box-plot

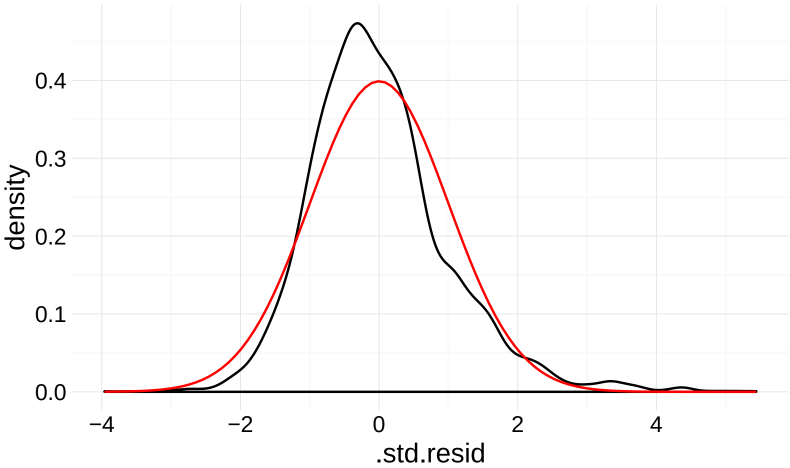
SLID

- Example in Fox
- Survey of Labour Income and Dynamics (Canada)
- 3,997 employed individuals between 16–65 residing in Ontario

```
##
## Call:
## lm(formula = wages ~ sex + age + education, data = SLID)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -26.111  -4.328  -0.792   3.243  35.892
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -7.905243   0.607771  -13.01   <2e-16 ***
## sexMale      3.465251   0.208494   16.62   <2e-16 ***
## age          0.255101   0.008634   29.55   <2e-16 ***
## education    0.918735   0.034514   26.62   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6.602 on 4010 degrees of freedom
##   (3411 observations deleted due to missingness)
## Multiple R-squared:  0.2972,^^IAdjusted R-squared:  0.2967
## F-statistic: 565.3 on 3 and 4010 DF,  p-value: < 2.2e-16
```



Studentized Residuals(mod_slid)
−4
0
4
−2
0
2
t Quantiles


density
0.0
0.1
0.2
0.3
0.4
−4
−2
0
2
4
.std.resid

Non-Normal Errors

Heteroskedasticity

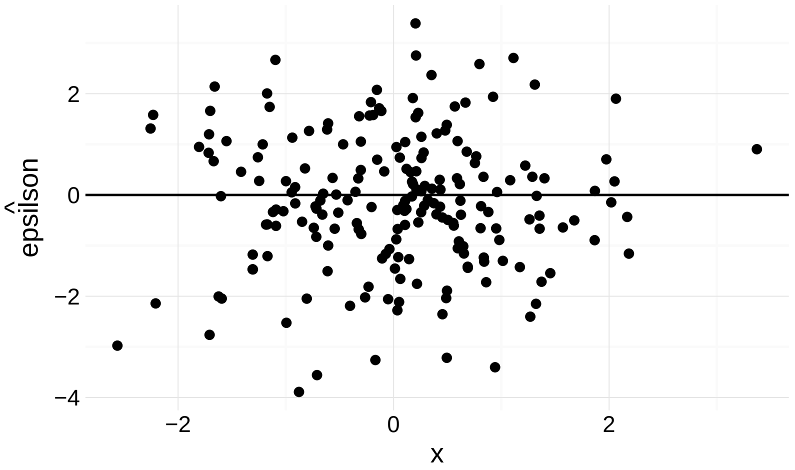
Homoskedasticity

- equal variance
- $\mathsf{V}(\epsilon_i) = \sigma^2$ for all obs

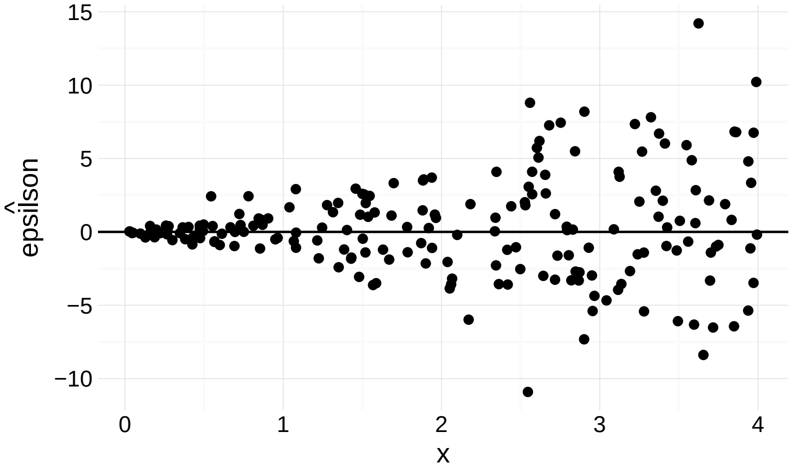
Heteroskedasticity

- unequal variance
- Some $\mathsf{V}(\epsilon_i) \neq \mathsf{V}(\epsilon_j)$
- In both cases, errors are uncorrelated $\mathsf{C}(\epsilon_i, \epsilon_j) = 0$ if $i \neq j$.

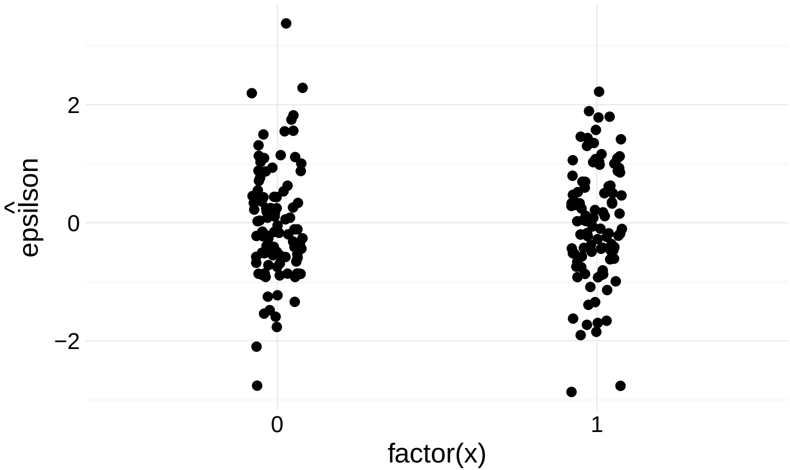
Homoskedasticity for a continuous X



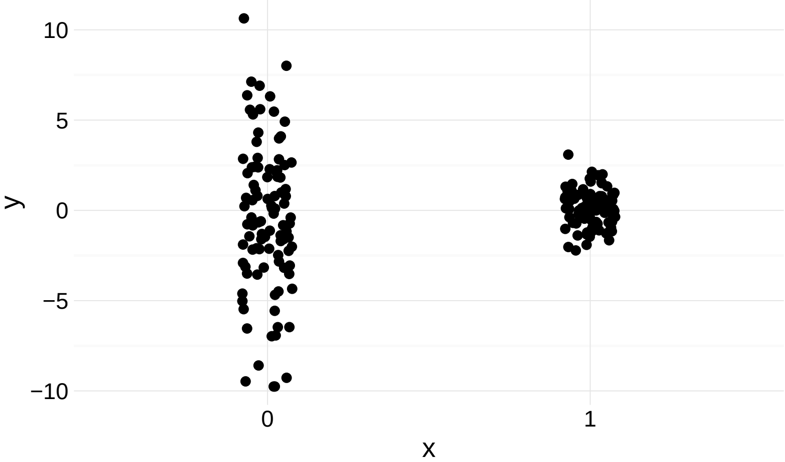
Heteroskedasticity for a continuous X



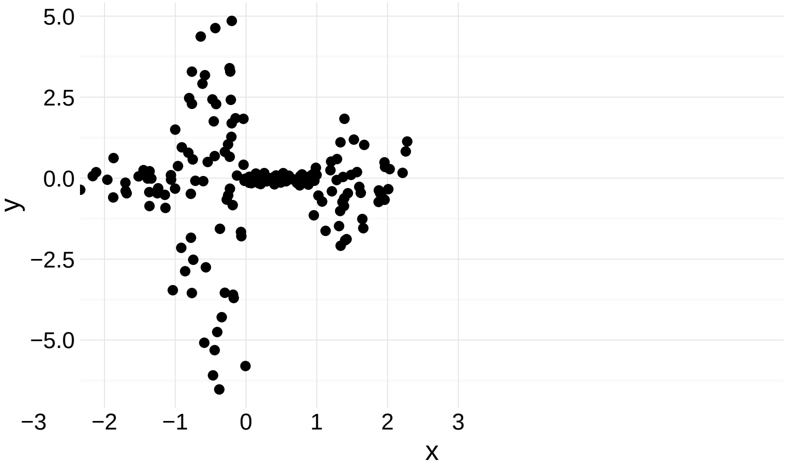
Homoskedasticity with Binary Explantory Variables



Heteroskedasticity with Binary Explantory Variables



Unusual Heteroskedasticity



Diagnosing heteroskedasticity?

Diagnosing

- Plot E or $|E|^2$ against against $\hat{Y}$ X
- R function `residualPlots`

Tests

- All tests of the form regress residuals on functions of X (Breusch-Pagan, White) `car::ncvTest`
- Are robust standard errors different from classic standard errors?

What does heteroskedasticity do?

Violates some of the Gauss-Markov Assumptions.

- Point estimate is still unbiased: $E(\boldsymbol{b}) = \beta$
- But, variance wrong: $V(\boldsymbol{b}) \neq \sigma_{\epsilon}^{2}(\boldsymbol{X}'\boldsymbol{X})^{-1}$
- And OLS is not BLUE or MVUE

What to do about Heteroskedasticity

Weighted Least Squares

If you know the form of the heteroskedasticity

Heteroskedasticity consistent standard errors

If you don't.

Weighted Least Squares

Like OLS, but weight each observation

$$\hat{\beta}_{WLS} = (\boldsymbol{X}'\boldsymbol{W}\boldsymbol{X})^{-1}\boldsymbol{X}\boldsymbol{W}\boldsymbol{y}$$

where $\boldsymbol{W}$ is a diagonal matrix with diag$(\boldsymbol{W}) = (w_1^2, w_2^2, \ldots, w_n^2)$
This minimizes the weighted sum of squares

$$\hat{\beta}_{WLS} = \arg\min_{\beta} \sum w_i^2 (y_i - \boldsymbol{x}_i\beta)^2$$

Note $\hat{\beta}_{WLS} \neq \hat{\beta}_{OLS}$, but both are unbiased if form of heteroskedasticity known.

Where do the weights in WLS come from?

Weights are such that

$$y_i \sim N(\boldsymbol{X}\beta, \sigma_\epsilon^2 / w_i^2)$$

Example

You have a survey and are using average values from counties. What weights should you use? What is the justification?

What if you had no idea what σ_i^2 was?

What would you use as an estimates?

, robust

- Use $\boldsymbol{b}$ from OLS, only correct the $V(\boldsymbol{b})$
- Since E$(\epsilon_i) = 0$, V$(\epsilon_i) = \sigma_i^2 = E(\epsilon_i^2)$
- So use E_i^2 as estimate of σ_i^2
- Then
$$\tilde{V}(\boldsymbol{b}) = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\hat{\Sigma}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}$$
- Where $\hat{\Sigma} =$ diag$(E_1^2, \ldots, E_n^2)$
- R functions `hccm` in **car**.

Thoughts on Heteroskedasticity

- Affects standard errors, not bias
- Non-constant error variance only an issue when ratio of largest to smallest variance is ≥ 4 (Fox)
- If using robust standard errors, always compare them to classical standard errors
- Angrist and Pischke suggest using max of robust and classical standard errors
- Tests tell you if it is a problem, visualization needed to get ideas how to fix it
- **MOST IMPORTANT**: problems with residuals point to misspecification issues

Residuals related to specifications of the model

Diagnostics for Simple Linear Regression