

POLS/CS&SS 503:
Advanced Quantitative Political Methodology

STATISTICAL INFERENCE FOR REGRESSION

April 21, 2015

Jeffrey B. Arnold



Overview

Overview of Statistical Inference

Difference of Means Example

Significance Tests

Confidence Intervals

Comments on Statistical Inference

Statistical Inference for OLS

Miscellaneous problems with significance testing

Overview of Statistical Inference

Difference of Means Example

Significance Tests

Confidence Intervals

Comments on Statistical Inference

Statistical Inference for OLS

Miscellaneous problems with significance testing

Statistical Inference

- Population: Y
- Parameters of interest: β from $Y = \beta X + \epsilon$.
- Sample: y
- Sample statistics (estimates): \mathbf{b}
- Since samples are random, different samples produce \mathbf{b} ?
 - How do we use the samples to the population parameters?
 - How do we quantify our uncertainty about that estimate?

Science is about Uncertainty

- Knowledge is never certain
- Goal: Estimating unknowns and **quantifying the uncertainty** of those estimates
- Estimates without uncertainty are incomplete at best, useless or biased at worst

The Fundamental Problem of Statistical Inference

- We have methods to calculate the probability of a sample and sample statistics **given** we know the population parameters.
- But we don't know the population parameters, so what do we do?
- Two (three) main methods
 - Frequentist: do not calculate the probability of the parameter
 - Hypothesis testing: Assume a hypothesis and check if data is consistent with it.
 - Confidence intervals: find a plausible range of parameters
 - Bayesian: calculate the probability of the parameter

Overview of Statistical Inference

Difference of Means Example

Significance Tests

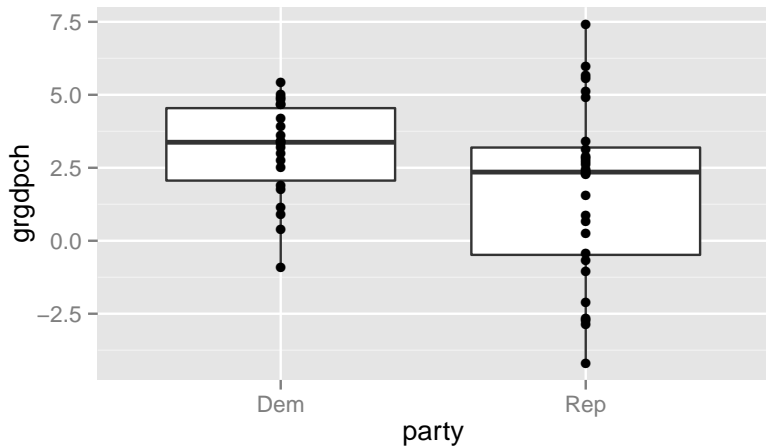
Confidence Intervals

Comments on Statistical Inference

Statistical Inference for OLS

Miscellaneous problems with significance testing

Is US Economic Growth Higher Under Democratic Presidents than Republican Presidents?



Is US Economic Growth Higher Under Democratic Presidents than Republicans?

```
## Source: local data frame [2 x 4]
```

```
##
```

```
##   party      mean      sd length
```

```
## 1   Dem 3.094356 1.672123     22
```

```
## 2   Rep 1.725821 3.014028     28
```

Sampling Distribution of the Difference in Means

- Want to know $\mu_D - \mu_R$? (Difference in population means)
- What is the sample? What is the population?
- We will be making other dubious assumptions in this example: populations are independent, normal (not important).
- Estimate is $\bar{x}_D - \bar{x}_R$ (Difference in sample means)
- But the observed sample is random, so how do we characterize the uncertainty in our estimates?

Sampling Distribution of the Difference in Means

If we knew $\mu_D, \mu_R, \sigma_R, \sigma_D$, we could calculate the distribution of $\bar{x}_D - \bar{x}_R$.

$$(\bar{x}_D - \bar{x}_R) \sim N\left(\mu_D - \mu_R, \frac{\sigma_R^2}{n_R} + \frac{\sigma_D^2}{n_D}\right)$$

But we don't know the population ...

Overview of Statistical Inference

Difference of Means Example

Significance Tests

Confidence Intervals

Comments on Statistical Inference

Statistical Inference for OLS

Miscellaneous problems with significance testing

Logic of Significance Tests

- Assume null H_0 and alternative H_a hypotheses
- Calculate the sampling distribution of the test statistic assuming H_0 is true
- p -value is the probability of data (test statistics) equal or more extreme than the sample
- (optional) At a pre-defined significance level (α), reject H_0 if p -value less than α , fail to reject if p -value greater than α .

Significance Test for Difference in Means

- Null hypothesis: $H_0 : \mu_D - \mu_R = 0$
- Alternative hypothesis: $H_a : \mu_D - \mu_R \neq 0$
- The test statistic is

$$t = \frac{\bar{x}_D - \bar{x}_R}{SE(\bar{x}_D - \bar{x}_R)} \quad (1)$$

where

$$SE(\bar{x}_D - \bar{x}_R) = \sqrt{\frac{\sigma_D^2}{n_D} + \frac{\sigma_R^2}{n_R}}$$

- Since we don't know σ_R^2 and σ_D^2 , use sample variances: s_R^2, s_D^2 as estimators.
- Use t distributed Student's t to account for uncertainty from estimating standard deviations. It would be distributed standard Normal if the population standard deviations were known.

t-distribution

See | <https://jrnold.shinyapps.io/tdist> |

t-tests for difference of means in R

```
t.test(grgdpch ~ party, # y ~ x
       data = gdp, # dataset
       mu = 0, # H_0: mu_1 - mu_2
       conf.level = 0.95 # confidence level to use for CI
)

##
## Welch Two Sample t-test
##
## data: grgdpch by party
## t = 2.0366, df = 43.679, p-value = 0.04778
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  0.01400377 2.72306582
## sample estimates:
## mean in group Dem mean in group Rep
##          3.094356          1.725821
```


Significance Tests

- Two approaches:
 - Fisher: p -value represents the level of evidence against H_0
 - Neyman-Pearson: choose a significance level α and reject null hypothesis if p -value is less than α .
- When making a decision of reject / not reject:
 - Type I error: H_0 true, reject H_0
 - Type II error: H_0 false, fail to reject H_0
- Power: $1 - \Pr(\text{Type II error})$
- Tests generally focus on Type I error
 - “conservative” – is it really? It prioritizes a hypothesis, and usually the null hypothesis has less evidence than the alternative.
 - much harder to calculate Type II errors

Overview of Statistical Inference

Difference of Means Example

Significance Tests

Confidence Intervals

Comments on Statistical Inference

Statistical Inference for OLS

Miscellaneous problems with significance testing

Logic of Confidence Intervals

- Find a plausible range of values of the parameter: $[\bar{x}_{lower}, \bar{x}_{upper}]$
- Only know probability of data given parameter value, so cannot calculate a probability distribution for a parameter value (Bayesian approach)
- Frequentist approach: method to generate intervals which contain the true parameter μ in $C\%$ of the samples.

What a $100(1 - \alpha)\%$ confidence interval means

Coverage A $100(1 - \alpha)\%$ confidence interval for a parameter θ , is an interval generated by a method that generates intervals that include the true parameter θ in $100(1 - \alpha)\%$ of samples.

Rejection Region A $100(1 - \alpha)\%$ confidence interval such that $H_0 : \theta = \theta'$ cannot be rejected at the α significance level for all values of θ' in the interval, and $H_0 : \theta = \theta'$ is rejected for all values of θ' outside the interval. (not all confidence intervals have this property).

Confidence levels for difference in means

To get a $100(1 - \alpha)\%$ confidence interval for a difference of means

$$\bar{x}_D - \bar{x}_R \pm t_{\alpha/2, \nu} \sqrt{\frac{s_D^2}{n_D} + \frac{s_R^2}{n_R}}$$

where $t_{\alpha/2, \nu}$ is a critical value of the t distribution such that the tails area of the distribution is α . The value of ν is complicated.

How to report a confidence interval

- Either of
 - Democratic presidents enjoyed growth rates 1.37 points higher (95% CI: 0.01 to 2.72) than their Republican counterparts.
 - Democrats enjoyed 1.37 points higher growth than Republicans, with a 95 percent confidence interval of 0.01 to 2.72.
- We could calculate any CI we wish: 90 percent, 80 percent, 50 percent, etc.
- The most commonly used are: 90, 95, and 99.

Overview of Statistical Inference

Difference of Means Example

Significance Tests

Confidence Intervals

Comments on Statistical Inference

Statistical Inference for OLS

Miscellaneous problems with significance testing

Fun Stuff

- <https://xkcd.com/882/>
- <https://xkcd.com/1478/>

Confidence Intervals vs. Significance Tests

- Problems with **both**
 - Simply commitment to a certain error rate, given **assumptions**. Does not account for **model uncertainty**.
 - “File drawer problem”, “fishing”: even if it makes sense on an individual test, multiple testing within a research project + selecting on significant results can result in biases.
- Problems with significance tests that CI overcome
 - tests are “weak” - only show one result
 - confidence intervals focus more on substantive significance (parameter values); p -values ignore all substantive significance.

Statistical and Substantive Significance

- p -values are a function of estimated effect size (B) but also the sample size
- p -values only show statistical significance, not substantive significance.
- Confidence intervals can be more useful for displaying substantive significance

Confidence Intervals vs. Significance Tests

- Confidence intervals often misinterpreted
- Definition of confidence interval is awkward and not exactly what we want, so often interpreted as probability interval
- But which is clearer?
 - Compared to Republicans, the effect of Democratic presidents on the economy is significantly positive at the 0.05 level.
 - Democratic presidents enjoyed 1.37 points higher growth than Republicans, with a 95 percent confidence interval of 0.01 to 2.72.

Bayesian vs. Frequentist Statistics

- Confidence intervals and significance tests do not calculate the probability of hypotheses (parameters)
- Bayesian statistics attempts to do so, but
 - requires prior probability of the hypotheses
 - computationally, mathematically more difficult

Conditional Probability

$$p(A|B) = \frac{p(A \& B)}{P(B)}$$

- What if A and B are independent? $P(A|B) = P(A)$
- What is the sampling distribution?

Bayes Rule

$$\begin{aligned} p(A|B) &= \frac{p(B|A)p(A)}{\sum_{A'} p(B|A')p(A')} \\ &= \frac{p(B|A)p(A)}{\sum_{A'} p(B|A')p(A')} \\ &\propto p(B|A)p(A) \end{aligned}$$

Inference and Bayes Rule

Want to find the probability of a hypothesis H given the data D :

$$\begin{aligned} p(H|D) &= \frac{p(H|D)p(D)}{\sum_{H'} p(D|H')p(H')} \\ &= \frac{p(D|H)p(H)}{p(D)} \\ &\propto p(D|H)p(H) \end{aligned}$$

- $p(D|H)$ is the likelihood (related to the sampling distribution).
- Where does $p(H)$ come from?

Bayesian and Frequentist Statistics

In many research questions we are interested in the probability of the hypothesis H , given the data D : $p(H|D)$.

Frequentist Assume a hypothesis, e.g. null hypothesis H_0 , and calculate the probability of the data: $p(D|H_0)$

Bayesian Assume a prior distribution $p(H)$ and calculate the probability of the hypothesis given the data:

$$p(H|D) \propto p(D|H)p(H)$$

My Claim: Even if using frequentist tests $p(D|H)$, a paper assigns prior probabilities to hypotheses, e.g. lit review, logical arguments, etc. to make a Bayesian argument, $p(D|H)p(H)$.

Overview of Statistical Inference

Difference of Means Example

Significance Tests

Confidence Intervals

Comments on Statistical Inference

Statistical Inference for OLS

Miscellaneous problems with significance testing

Statistical Inference for OLS

- Individual Coefficients:
 - Significance tests: β_k
 - Confidence intervals
- Multiple coefficients:
 - Significance test
 - F-test on all slopes: $H_0 : \beta_1 = \dots = \beta_k = 0$
 - F-test on subset of slopes: $H_0 : \beta_1 = \dots = \beta_k = 0$
 - F-test on linear combinations of slopes: e.g $H_0 : \beta_1 - \beta_2 = 0$.
 - Confidence regions

Statistical Inference for Individual Coefficients for Simple Regression

If all assumptions of Gauss-Markov hold, variance of sample coefficient B is

$$V(B) \sim N \left(\beta, \frac{\sigma_{\epsilon}^2}{\sum (x_i - \bar{x})^2} \right)$$

If ϵ not normal, approximate as $n \rightarrow \infty$

- Test statistic for $H_0 : \beta = \beta_0$ distributed Student's t with $n - k - 1$ df.

$$t = \frac{B - \beta_0}{\text{SE}(B)}$$

- Confidence interval is

$$B \pm t_{\alpha/2, n-k-1} \text{SE}(B)$$

- Where $\text{SE}(B)$ is $V(B)$ with $\hat{\sigma}_{\epsilon}^2$ as an estimate of σ_{ϵ} .

Estimate of σ_ϵ^2

Estimate σ_ϵ^2 from the regression squared errors:

$$\hat{\sigma}_\epsilon^2 = \frac{\sum E_i^2}{n - k - 1}$$

- where $n - k - 1 = (\text{observations}) - (\text{variables}) - (\text{intercept})$ is the degrees of freedom.
- Similar to mean squared error, but to estimate population divide by degrees of freedom.

Statistical Inference for Individual Coefficients for Multiple Regression

If multiple variables, and Gauss-Markov assumptions hold, then

$$\mathbf{b} \sim MVN \left(\boldsymbol{\beta}, \sigma_{\epsilon}^2 (\mathbf{X}'\mathbf{X})^{-1} \right)$$

- analogous to the bivariate version
 - calculates all standard errors simultaneously
 - covariances: B_i and B_j can be correlated

Statistical Inference for Individual Coefficients for Multiple Regression

Standard error for a single coefficient

$$SE(B_j) = \sqrt{\hat{\sigma}_\epsilon (X'X)^{-1}_{jj}} = \frac{1}{\sqrt{1 - R_j^2}} \frac{\hat{\sigma}_\epsilon}{\sqrt{\sum (x_{ij} - \bar{x}_j)^2}}$$

- Test statistic for $H_0 : \beta = \beta_0$ distributed Student's t with $n - k - 1$ df.

$$t = \frac{\beta - \beta_0}{se}$$

- Confidence interval is

$$B \pm t_{\alpha/2, n-k-1} se$$

- Where $\hat{\sigma}_\epsilon = \frac{\sum_i E_i^2}{n-k-1}$

Confidence Interval

General Definition

In repeated samples, $C\%$ of samples have a $C\%$ confidence interval that contains the population (true) parameter θ .

- Not a statement about a sample interval, statement about the method
- Each confidence interval either contains θ or not, there is no probability. Parameters are fixed, only samples are random.

Overview of Statistical Inference

Difference of Means Example

Significance Tests

Confidence Intervals

Comments on Statistical Inference

Statistical Inference for OLS

Miscellaneous problems with significance testing

Overlapping confidence intervals does not mean
difference is not statistically significant

See [https:](https://www.cscu.cornell.edu/news/statnews/Stnews73insert.pdf)

[//www.cscu.cornell.edu/news/statnews/Stnews73insert.pdf](https://www.cscu.cornell.edu/news/statnews/Stnews73insert.pdf)

Significant and Not significant are not statistically significant

- Common example:
 - Regression with several dummy variables
 - Coefficient of dummy variable of category A (β_A) is significant at 5% level, dummy variable of category B (β_B) is not significant at the 5% level.
 - Common (wrong) interpretation: A and B are different
 - Correct procedures:
 - Significance test with $H_0 : \beta_A = \beta_B$
 - calculate confidence interval of $\beta_A - \beta_B$.

References

- Some slides derived from Christopher Adolph *Inference and Interpretation of Linear Regression*. Used with permission. <http://faculty.washington.edu/cadolph/503/topic4.pw.pdf>
- Material included from
 - Fox Ch 6, 9.3