

TOEPLITZ BLOCK MATRICES IN COMPRESSED SENSING

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ABSTRACT. Recent work in compressed sensing theory shows that $n \times N$ independent and identically distributed (IID) sensing matrices whose entries are drawn independently from certain probability distributions guarantee exact recovery of a sparse signal with high probability even if $n \ll N$. Motivated by signal processing applications, random filtering with Toeplitz sensing matrices whose elements are drawn from the same distributions were considered and shown to also be sufficient to recover a sparse signal from reduced samples exactly with high probability. This paper considers Toeplitz block matrices as sensing matrices. They naturally arise in multichannel and multidimensional filtering applications and include Toeplitz matrices as special cases. It is shown that the probability of exact reconstruction is also high. Their performance is validated using simulations.

1. INTRODUCTION

The central problem in compressed sensing (CS) is the recovery of a vector $x \in \mathbb{R}^N$ from its linear measurements y of the form

$$(1.1) \quad y_i = \langle x, \varphi_i \rangle, \quad 1 \leq i \leq n,$$

where n is assumed to be much smaller than N . Of course, for $n \ll N$, (1.1) posts an under-determined system of equations which has non-unique solutions. Exact recovery of the original vector x needs further prior information. The work by Candés, Donoho, Romberg, Tao, and others (see e.g. [1],[2], and the references therein) showed that under the assumption that x is sparse, one can actually recover x from a sample y which is much smaller in size than x by solving a convex

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program with a suitably chosen sampling basis $\varphi_i, 1 \leq i \leq n$. If we write the linear system (1.1) in the form

$$(1.2) \quad y = \Phi x, \quad \text{where } \Phi \text{ is an } n \times N \text{ matrix,}$$

then the question about what sampling methods guarantee the exact recovery of x becomes the question about what matrices are “good” compressed sensing matrices, meaning that they ensure exact recovery of a sparse x from y with high probability under the condition that $n \ll N$.

In [3] Candès and Tao introduce the *restricted isometry property* as a condition on matrices Φ which provides a guarantee on the performance of Φ in compressed sensing.

Following their definition, we say that a matrix $\Phi \in \mathbb{R}^{n \times N}$ satisfies RIP of order $m \in \mathbb{N}$ and constant $\delta_m \in (0, 1)$ if

$$(1.3) \quad (1 - \delta_m) \|z\|_2^2 \leq \|\Phi_T z\|_2^2 \leq (1 + \delta_m) \|z\|_2^2 \quad \forall z \in \mathbb{R}^{|T|},$$

where $T \subset \{1, 2, \dots, N\}$, $|T| \leq m$, and Φ_T denotes the matrix obtained by retaining only the columns of Φ corresponding to the entries of T .

It was shown in [3] that if Φ satisfies RIP of order $3m$ the decoder given by

$$(1.4) \quad \Delta(y) := \operatorname{argmin} \|x\|_{l_1^N} \quad \text{subject to } \Phi x = y$$

ensures exact recovery of x from y .

Recently Baraniuk et al [4] showed that matrices whose entries are drawn independently from certain probability distribution P satisfy RIP of order m with probability $\geq 1 - e^{-c_2 n}$ for every $\delta_m \in (0, 1)$ provided that $n \geq c_1 m \ln(N/m)$, where $c_1, c_2 > 0$ are some positive constants depending only on δ_m . Motivated by applications in signal processing, Bajwa et al [5] considered (truncated) Toeplitz-structured matrices whose entries are drawn from the same probability distributions P and showed that they satisfy RIP of order $3m$ with probability $\geq 1 - e^{-c_2 n/m^2}$ for every $\delta_{3m} \in (0, 1)$ provided that $n \geq c_1 m^3 \ln(N/m)$.

Some examples of probability distributions that can be used in this context have been studied in [6]. They include

$$(1.5) \quad r_{i,j} \sim N\left(0, \frac{1}{n}\right),$$

$$(1.5) \quad r_{i,j} = \begin{cases} \frac{1}{\sqrt{n}} & \text{with probability } 1/2 \\ -\frac{1}{\sqrt{n}} & \text{with probability } 1/2 \end{cases},$$

$$r_{i,j} = \begin{cases} \sqrt{\frac{3}{n}} & \text{with probability } 1/6 \\ 0 & \text{with probability } 2/3 \\ -\sqrt{\frac{3}{n}} & \text{with probability } 1/6 \end{cases}.$$

Motivated by applications in multichannel sampling, in this paper we will consider Toeplitz block matrices with elements in each block drawn independently from one of the probability distributions in (1.5) and some other block matrices with similar structures. We show that such matrices also satisfy RIP of order $3m$ for every $\delta_{3m} \in (0, 1)$ with high probability, provided that $n \geq c_1 l m \ln(N/m)$, where $l \leq 3m(3m - 1)$ (see (2.1)) and $c_1 > 0$ is some positive constant depending only on δ_{3m} . These Toeplitz block matrices naturally represent the system equation matrices in multichannel sampling applications where a single input signal is recovered from output samples of multiple channels with IID random filters. The result justifies the use of multichannel systems in compressed sensing. The advantages of Toeplitz matrices pointed out in [5], like e.g. efficient implementations, also apply to the matrices considered in this paper.

2. MAIN RESULT

Theorem 2.1. *For Toeplitz block matrices of the form*

$$(2.1) \quad \Phi = \begin{pmatrix} \Phi_k & \Phi_{k-1} & \dots & \Phi_2 & \Phi_1 \\ \Phi_{k+1} & \Phi_k & \dots & \Phi_3 & \Phi_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \Phi_{k+l-1} & \Phi_{k+l-2} & \dots & \dots & \Phi_l \end{pmatrix} \in \mathbb{R}^{n \times N}$$

with blocks $\Phi_i \in \mathbb{R}^{d \times e}$ whose elements are drawn independently from one of the probability distributions in (1.5), there exist constants $c_1, c_2 > 0$ depending only on $\delta_{3m} \in (0, 1)$, such that:

- (i) If $l \leq 3m(3m - 1)$, then for any $n \geq c_1 l m \ln(N/m)$, Φ satisfies RIP of order $3m$ for every $\delta_{3m} \in (0, 1)$ with probability at least

$$1 - e^{-c_2 n/l}.$$

- (ii) If $l > 3m(3m - 1)$, then for any $n \geq c_1 m^3 \ln(N/m)$, Φ satisfies RIP of order $3m$ for every $\delta_{3m} \in (0, 1)$ with probability at least

$$1 - e^{-c_2 n/m^2}.$$

The above theorem gives the requirement for and a lower bound on the probability of exact reconstruction of a m -sparse signal x from a measurement y if Toeplitz block matrices are used. The number of blocks (l) in one column of Φ controls the degrees of freedom of Φ . To compare our result with the known results, note that if $l = 1$, then Φ is an IID matrix, and Theorem 2.1 lower bounds the probability of Φ satisfying RIP of order $3m$ by $1 - e^{-c_2 n}$, which recovers the bound obtained in [4]. And as long as $l \leq 3m(3m - 1)$, a matrix Φ as in (2.1) satisfies RIP of order $3m$ with probability $1 - e^{-c_2 n/l} \geq 1 - e^{-c'_2 n/m^2}$, which is the bound given in [5], since

$$(2.2) \quad -c_2 n/l \leq -c_2 n/(9m^2 - 3m) \leq -c_2 n/9m^2 = -c'_2 n/m^2.$$

As noted in [5, 7], Toeplitz matrices naturally arise in one-dimensional single-channel filtering applications where the matrix elements are filter coefficients. Similarly, the Toeplitz block matrices defined in (2.1) naturally arise in one-dimensional multichannel sampling applications where the length of the filter is at least l points larger than that of the input signal. The conventional multichannel sampling theorem states that the sampling rate reduction over the single channel system cannot exceed the number of channels for exact recovery. While Theorem 2.1 suggests that multichannel systems with IID random filters might be able to reduce the sampling rate by a factor higher than the number of channels.

We remark, that for other block matrices with similar structures, the result in Theorem 2.1 also holds (see IV).

3. PROOF OF MAIN RESULT

Let $T \subset \{1, 2, \dots, N\}$. Denote by $\Phi_{T,i}$ the i -th row of the matrix Φ_T obtained by retaining only those columns of Φ corresponding to the elements in T , and let $\Phi_{T,i} \cap \Phi_j$ denote the set of random variables common to the i -th row of Φ_T and the j -th block of Φ .

We note that, if (1.3) holds for a set $T \subset \{1, 2, \dots, N\}$, then it also holds for any $\tilde{T} \subset T$. To prove that Toeplitz IID block matrices satisfy RIP with high probability, it is therefore enough to consider only those sets T where $|T| = 3m$.

Lemma 3.1. *Define the sets $D_{T,i}$ by*

$$D_{T,i} = \{j \in \{1, 2, \dots, n\} : \Phi_{T,j} \text{ is stochastically dependent on } \Phi_{T,i}, j \neq i\}.$$

- (i) *If T satisfies $|T| < \frac{1+\sqrt{1+4l}}{2}$, then $|D_{T,i}| \leq |T|(|T| - 1) \leq l - 1$.*
- (ii) *If T satisfies $|T| \geq \frac{1+\sqrt{1+4l}}{2}$, then $|D_{T,i}| \leq l - 1$.*

Proof. Fix $\Phi_{T,i}$. T defines a sequence $\{r_{t_s}\}_{s=1}^k$, where r_{t_s} is the number of columns from block Φ_{t_s} in T . Thus $\sum_{s=1}^k r_{t_s} = |T|$. Consider the number of rows that have dependency with the elements in $\Phi_{t_s} \cap \Phi_{T,i}$. Since all elements inside a single block are independent, there can be no dependencies within one block. Moreover, because of the structure of the matrix Φ , there can be at most

$$\begin{cases} 0 & \text{if } \Phi_{t_s} \cap \Phi_{T,i} = \emptyset \\ |T| - r_{t_s} & \text{if } \Phi_{t_s} \cap \Phi_{T,i} \neq \emptyset \end{cases}$$

rows outside the block Φ_{t_s} that depend on any element in $\Phi_{t_s} \cap \Phi_{T,i}$.

(i) If T satisfies $|T| < \frac{1+\sqrt{1+4l}}{2}$, i.e. if $l > |T|(|T| - 1)$, these rows may be distinct, and we have

$$\begin{aligned} |D_{T,i}| &\leq \sum_{\{t_s, s \in \{1,2,\dots,k\} : \Phi_{t_s} \cap \Phi_{T,i} \neq \emptyset\}} (|T| - r_{t_s}) \\ &\leq \sum_{t \in T} (|T| - 1) = |T|(|T| - 1) \leq l - 1 \end{aligned}$$

dependent rows.

(ii) If T satisfies $|T| \geq \frac{1+\sqrt{1+4l}}{2}$, i.e. if $|T|(|T| - 1) \geq l$, then $|D_{T,i}|$ is upper bounded by the number of blocks, so $|D_{T,i}| \leq l - 1$. \square

In [6] it has been shown that for given n , N , and $T \subset \{1, 2, \dots, N\}$ with $|T| \leq m$, an IID matrix of size $n \times N$ with entries drawn independently from one of the distributions P in (1.5)¹ satisfies (1.3) with probability

$$(3.1) \quad \geq 1 - e^{-f(n,m,\delta_m)},$$

where

$$(3.2) \quad f(n, m, \delta_m) = c_0 n - m \ln(12/\delta_m) - \ln(2).$$

Now consider a (truncated) Toeplitz block matrix $\Phi \in \mathbb{R}^{n \times N}$ as in (2.1), where the blocks $\{\Phi_i\}_{i=1}^{k+l-1}$ are such IID matrices $\in \mathbb{R}^{d \times e}$ with entries drawn independently from the same set of distributions as above.

The following lemma gives an upper bound for the probability that a matrix as in (2.1) with $1 \leq l \leq n$ satisfies (1.3) for any fixed subset T with $|T| = 3m$. Lemma 3.3 gives a tighter bound for the case $l > |T|(|T| - 1)$.

¹These matrices consist of columns whose squared norm is equal to 1 in expectation.

The proof of Lemma 3.2 uses an argument similar to the one in the proof of Lemma 1 in [5].

Lemma 3.2. *For given $T \subset \{1, 2, \dots, N\}$ with $|T| = m$, and $\delta_m \in (0, 1)$, the Toeplitz block submatrix Φ_T satisfies (1.3) with probability at least*

$$1 - e^{-f(d, m, \delta_m) + \ln(l)}.$$

Proof. We can write the matrix Φ_T as

$$(3.3) \quad \Phi_T = \begin{pmatrix} \Phi_T^1 \\ \vdots \\ \Phi_T^l \end{pmatrix},$$

where the blocks Φ_T^i of size $d \times |T|$ are given by the columns determined by T in the i -th row of blocks $(\Phi_{k+i-1}, \Phi_{k+i-2}, \dots, \Phi_i)$ in Φ .

Note that $\forall i \in \{1, 2, \dots, l\}$, Φ_T^i is an IID matrix with entries from one of the distributions in (1.5). If we let $\tilde{\Phi}_T^i = \sqrt{l}\Phi_T^i$, then the matrices $\tilde{\Phi}_T^i$ have columns whose squared norm is equal to 1 in expectation and by (3.1) satisfy (1.3), i.e.

$$(1 - \delta_m)\|z\|_2^2 \leq \|\tilde{\Phi}_T^i z\|_2^2 \leq (1 + \delta_m)\|z\|_2^2, \\ \forall z \in \mathbb{R}^{|T|}, \forall i \in \{1, 2, \dots, l\},$$

with probability at least

$$(3.4) \quad 1 - e^{-f(d, m, \delta_m)}.$$

Now since

$$(3.5) \quad \|\Phi_T z\|_2^2 = \sum_{i=1}^l \|\Phi_T^i z\|_2^2 = \sum_{i=1}^l \frac{1}{l} \|\tilde{\Phi}_T^i z\|_2^2$$

and $\sum_{i=1}^l \frac{1}{l} = 1$, we have

$$(1 - \delta_m)\|z\|_2^2 \leq \|\Phi_T z\|_2^2 \leq (1 + \delta_m)\|z\|_2^2, \quad \forall z \in \mathbb{R}^{|T|}.$$

In other words, the event $E_1 = \{\tilde{\Phi}_T^i \text{ satisfies (1.3)} \forall i\}$ implies the event $E_2 = \{\Phi_T \text{ satisfies (1.3)}\}$. Consequently,

$$\begin{aligned} P(E_2) &= 1 - P(E_2^c) \geq 1 - P(E_1^c) \\ &\geq 1 - \sum_{i=1}^l P(\{\tilde{\Phi}_T^i \text{ does not satisfy (1.3)}\}) \\ &\geq 1 - \sum_{i=1}^l e^{-f(d, m, \delta_m)} \quad (\text{by (3.4)}) \\ &= 1 - e^{-f(d, m, \delta_m) + \ln(l)}. \end{aligned}$$

□

Lemma 3.3. *For given $T \subset \{1, 2, \dots, N\}$ with $|T| = m$, and $\delta_m \in (0, 1)$, if $l > |T|(|T| - 1)$, the Toeplitz block submatrix Φ_T satisfies (1.3) with probability at least*

$$1 - e^{-f(\lfloor n/q \rfloor, m, \delta_m) + \ln(q)},$$

where $q = |T|(|T| - 1) + 1$.

Proof. Let $\Phi_{T,i}$ denote the i -th row of Φ_T and construct an undirected dependency graph $G = (V, E)$ such that $V = \{1, 2, \dots, n\}$ and

$$E = \{(i, i') \in V \times V : i \neq i', \Phi_{T,i} \text{ and } \Phi_{T,i'} \text{ are dependent}\}.$$

By Lemma 3.1, $\Phi_{T,i}$ can at most be dependent with $|T|(|T| - 1)$ other rows. Therefore, the maximum degree Δ of G is given by $\Delta \leq |T|(|T| - 1)$, and using the Hajnal-Szemerédi theorem on equitable coloring of graphs (see [8]), we can partition G using $q = |T|(|T| - 1) + 1$ colors. Let $\{C_j\}_{j=1}^q$ be the different color classes, then

$$|C_j| = \lfloor n/q \rfloor \text{ or } |C_j| = \lceil n/q \rceil.$$

Now, let Φ_T^j be the $|C_j| \times |T|$ submatrix obtained from Φ_T retaining the rows corresponding to the indices in C_j and define $\tilde{\Phi}_T^j = \sqrt{n/|C_j|} \Phi_T^j$. Then

$$(3.6) \quad \forall z \in \mathbb{R}^{|T|}, \quad \|\Phi_T z\|_2^2 = \sum_{j=1}^q \|\Phi_T^j z\|_2^2 = \sum_{j=1}^q \frac{|C_j|}{n} \|\tilde{\Phi}_T^j z\|_2^2.$$

Every $\tilde{\Phi}_T^j$ is a $|C_j| \times |T|$ IID matrix whose columns have squared norm equal to 1 in expectation. By (3.1), they satisfy (1.3) with probability at least

$$(3.7) \quad 1 - e^{-f(|C_j|, m, \delta_m)} \geq 1 - e^{-f(\lfloor n/q \rfloor, m, \delta_m)}.$$

Since $\sum_{j=1}^q \frac{|C_j|}{n} = 1$, by (3.6), we have that if

$$(1 - \delta_m) \|z\|_2^2 \leq \|\tilde{\Phi}_T^j z\|_2^2 \leq (1 + \delta_m) \|z\|_2^2, \quad \forall z \in \mathbb{R}^{|T|}, \forall j \in \{1, 2, \dots, q\}$$

then

$$(1 - \delta_m) \|z\|_2^2 \leq \|\Phi_T z\|_2^2 \leq (1 + \delta_m) \|z\|_2^2, \quad \forall z \in \mathbb{R}^{|T|}.$$

In other words, the event $E_1 = \{\tilde{\Phi}_T^j \text{ satisfies (1.3) for all } j\}$ implies the event $E_2 = \{\Phi_T \text{ satisfies (1.3)}\}$. Consequently,

$$\begin{aligned}
P(E_2) &= 1 - P(E_2^c) \geq 1 - P(E_1^c) \\
&\geq 1 - \sum_{j=1}^q P(\{\tilde{\Phi}_T^j \text{ does not satisfy (1.3)}\}) \\
&\geq 1 - \sum_{j=1}^q e^{-f(\lfloor n/q \rfloor, m, \delta_m)} \quad (\text{by (3.7)}) \\
&= 1 - e^{-f(\lfloor n/q \rfloor, m, \delta_m) + \ln(q)}.
\end{aligned}$$

□

Main result in Theorem 2.1.

Proof. (i) From (3.2) and Lemma 3.2 we have that Φ satisfies (1.3) for any $T \subset \{1, 2, \dots, N\}$ such that $|T| = 3m$ with probability at least

$$(3.8) \quad 1 - e^{-c_0 d + 3m \ln(12/\delta_{3m}) + \ln(2) + \ln(l)}.$$

Since there are $\binom{N}{3m} \leq (eN/3m)^{3m}$ such subsets, using Bonferroni's inequality (see e.g. [9]) yields that Φ satisfies RIP of order $3m$ with probability at least

$$(3.9) \quad 1 - e^{-c_0 n/l + 3m[\ln(12/\delta_{3m}) + \ln(N/3m) + 1] + \ln(2) + \ln(l)}.$$

Fix $c_2 > 0$ and pick $c_1 = (3 \ln((12/\delta_{3m})) + 15)/(c_0 - c_2)$. Then for any $n \geq c_1 l m \ln(N/m)$, the exponent of e in (3.9) is upper bounded by

$-c_2 n/l$:

$$\begin{aligned}
& -\frac{c_0 n}{l} + 3m \left[\ln \left(\frac{12}{\delta_{3m}} \frac{N}{3m} \right) + 1 \right] + \ln(2l) \leq -\frac{c_2 n}{l} \\
\Leftrightarrow & 3m \left[\ln \left(\frac{12}{\delta_{3m}} \frac{N}{3m} \right) + 1 \right] + \ln(2l) \leq \frac{n}{l} (c_0 - c_2) \\
\Leftrightarrow & \frac{3lm}{c_0 - c_2} \left[\ln \left(\frac{12}{\delta_{3m}} \frac{N}{3m} \right) + 1 + \frac{\ln(2)}{3m} + \frac{\ln(l)}{3m} \right] \leq n \\
\Leftrightarrow & \frac{3lm \ln \left(\frac{N}{m} \right)}{c_0 - c_2} \left[\frac{\ln \left(\frac{12}{\delta_{3m}} \frac{N}{3m} \right) + 1 + \ln(2) + \ln(l)}{3m \ln \left(\frac{N}{m} \right)} \right] \leq n \\
\Leftarrow & \frac{3lm \ln \left(\frac{N}{m} \right)}{c_0 - c_2} \left[\ln \left(\frac{12}{\delta_{3m}} \right) + 5 \right] \leq n \\
\Leftrightarrow & c_1 l m \ln \left(\frac{N}{m} \right) \leq n
\end{aligned}$$

(ii) From (3.2) and Lemma 3.3 we have that Φ satisfies (1.3) for any $T \subset \{1, 2, \dots, N\}$ such that $|T| = 3m$ with probability at least

$$\begin{aligned}
& 1 - e^{-c_0 \lfloor n/q \rfloor + 3m \ln(12/\delta_{3m}) + \ln(2) + \ln(q)} \\
(3.10) \quad & \geq 1 - e^{-c_0 n/9m^2 + 3m \ln(12/\delta_{3m}) + \ln(2) + \ln(9m^2) + c_0}.
\end{aligned}$$

Since there are $\binom{N}{3m} \leq (eN/3m)^{3m}$ such subsets, using Bonferroni's inequality again yields that Φ satisfies RIP of order $3m$ with probability at least

$$(3.11) \quad 1 - e^{-c_0 k/9m^2 + 3m [\ln(12/\delta_{3m}) + \ln(N/3m) + 1] + \ln(2) + \ln(9m^2) + c_0}.$$

Now fix $c_2 > 0$ and pick $c_1 > 27c_3/(c_0 - 9c_2)$, where $c_3 = \ln(12/\delta_{3m}) + \ln(2) + c_0 + 4$. Then, for any $n \geq c_1 m^3 \ln(N/m)$, the exponent of e in (3.11) is upper bounded by $-c_2 n/m^2$. This completes the proof of the theorem. \square

4. OTHER BLOCK MATRICES

4.1. Circulant matrices. The above consideration can be applied to (truncated) circulant block matrices of the form

$$(4.1) \quad \Phi = \begin{pmatrix} \Phi_k & \Phi_{k-1} & \dots & \Phi_2 & \Phi_1 \\ \Phi_1 & \Phi_k & \dots & \Phi_3 & \Phi_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \Phi_{l-1} & \Phi_{l-2} & \dots & \dots & \Phi_l \end{pmatrix} \in \mathbb{R}^{n \times N},$$

where the blocks Φ_i are all IID matrices.

Similar to (2.1), the circulant matrices in (4.1) also represent the system equation matrices in multichannel sampling, but the convolution is a circular one. They usually arise in applications where convolutions are implemented by multiplications in Fourier domain.

Before we present the theorem for this type of matrices, we first comment on the maximum number of stochastically dependent rows in a (truncated) circulant matrix of the form

$$(4.2) \quad A = \begin{pmatrix} a_q & a_{q-1} & \dots & a_2 & a_1 \\ a_1 & a_q & \dots & a_3 & a_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{p-1} & a_{p-2} & \dots & \dots & a_p \end{pmatrix} \in \mathbb{R}^{p \times q}.$$

Again, we denote by $A_{T,i}$ the i -th row of the matrix A_T , which is obtained by retaining only those columns of A corresponding to $T \subset \{1, 2, \dots, N\}$.

Lemma 4.1. *Define the sets $D_{T,i}$ by $D_{T,i} = \{j \in \{1, 2, \dots, p\} : A_{T,j} \text{ is stochastically dependent on } A_{T,i}, j \neq i\}$. Then $D_{T,i}$ has cardinality at most $|T|(|T| - 1)$.*

Proof. Note first, that an upper bound for the case $p = q$ clearly upper bounds the case where $p < q$. We may therefore assume that $p = q$ and A is a square circulant matrix. Then the number of rows stochastically dependent on $A_{T,i}$ is independent of i and we can, w.l.o.g., assume that $i = 1$. Let $\mathbf{t} \in \{0, 1\}^q$ be a q -tuple defined by

$$\mathbf{t}_j = \begin{cases} 0 & \text{if } j \notin T \\ 1 & \text{if } j \in T \end{cases}, \quad j=1, \dots, q,$$

and consider the matrix

$$(4.3) \quad \tilde{A} = \begin{pmatrix} \mathbf{t} \\ \sigma(\mathbf{t}) \\ \vdots \\ \sigma^{q-1}(\mathbf{t}) \end{pmatrix} \in \mathbb{R}^{q \times q},$$

where $\sigma : \{0, 1\}^q \rightarrow \{0, 1\}^q$ defines the right-shift $(\mathbf{t}_1, \dots, \mathbf{t}_{q-1}, \mathbf{t}_q) \rightarrow (\mathbf{t}_q, \mathbf{t}_1, \dots, \mathbf{t}_{q-1})$. Denote by \tilde{A}_T the matrix obtained by retaining only those columns of \tilde{A} corresponding to $T \subset \{1, 2, \dots, q\}$. It is now easy to see that

$$\begin{aligned} |D_{T,i}| &= |\{\tilde{A}_{T,i}, i \in \{2, \dots, q\} : h(\tilde{A}_{T,1}, \tilde{A}_{T,i}) < |T|\}| \\ &\leq \{\# \text{ of ones in } \mathbf{t}\} \cdot (\{\# \text{ of ones in } \mathbf{t}\} - 1) \\ &= |T|(|T| - 1), \end{aligned}$$

where $h : \{0, 1\}^q \times \{0, 1\}^q \rightarrow \mathbb{N}$ is the Hamming distance defined by

$$h(x, y) = |\{j \in \{1, 2, \dots, q\} : x_j \neq y_j\}|.$$

□

The following theorem gives lower bounds for the probability that a circulant block matrix as in 4.2 satisfies the RIP of order $3m$. Note that the bounds obtained are the same as in 2.1 although the number of independent entries in Φ is greater than before. This is due to the nature of the proof using the number of stochastically dependent rows of Φ which is the same for both Toeplitz and circulant matrices.

Theorem 4.1. *Let Φ be as in (4.1). Then there exist constants $c_1, c_2 > 0$ depending only on $\delta_{3m} \in (0, 1)$, such that:*

- (i) *If $l \leq 3m(3m - 1)$, then for any $n \geq c_1 l m \ln(N/m)$, Φ satisfies RIP of order $3m$ for every $\delta_{3m} \in (0, 1)$ with probability at least*

$$1 - e^{-c_2 n/l}.$$

- (ii) *If $l > 3m(3m - 1)$, then for any $n \geq c_1 m^3 \ln(N/m)$, Φ satisfies RIP of order $3m$ for every $\delta_{3m} \in (0, 1)$ with probability at least*

$$1 - e^{-c_2 n/m^2}.$$

Proof. A similar argument as the one in the proof of Lemma 3.1 shows that the upper bound for the maximum number of rows stochastically dependent on any row of a (truncated) circulant block matrix is the same as for the (truncated) Toeplitz block matrices (use Lemma 4.1). Then the proof of Theorem 2.1 directly applies to the setting at hand. □

4.2. Circulant-circulant Matrices. We also consider matrices that are (truncated) circulant block matrices whose blocks are themselves circulant:

$$(4.4) \quad \Phi = \begin{pmatrix} \Phi_k & \Phi_{k-1} & \dots & \Phi_2 & \Phi_1 \\ \Phi_1 & \Phi_k & \dots & \Phi_3 & \Phi_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \Phi_{l-1} & \Phi_{l-2} & \dots & \dots & \Phi_l \end{pmatrix} \in \mathbb{R}^{n \times N},$$

$$(4.5) \quad \Phi_i = \begin{pmatrix} \varphi_p^i & \varphi_{p-1}^i & \dots & \varphi_2^i & \varphi_1^i \\ \varphi_1^i & \varphi_p^i & \dots & \varphi_3^i & \varphi_2^i \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \varphi_{q-1}^i & \varphi_{q-2}^i & \dots & \dots & \varphi_q^i \end{pmatrix} \in \mathbb{R}^{q \times p}.$$

Denote by $\tau : \{0, 1\}^{kp} \rightarrow \{0, 1\}^{kp}$ the right-shift of *blocks* Φ_i and by $\sigma : \{0, 1\}^{kp} \rightarrow \{0, 1\}^{kp}$ the right-shift of *elements inside a block* Φ_i ,

both by one position. These matrices arise in two-dimensional imaging applications where the independent elements are the coefficients of the point spread function of the imaging system (see [10]). Replacing (4.3) in the proof of Lemma 4.1 by

$$\bar{A} = \begin{pmatrix} \mathbf{t} \\ \sigma^1 \tau^0(\mathbf{t}) \\ \vdots \\ \sigma^{(i-1)(\bmod p)} \tau^{\lfloor \frac{i-1}{p} \rfloor}(\mathbf{t}) \\ \vdots \\ \sigma^{p-1} \tau^{l-1}(\mathbf{t}) \end{pmatrix} \in \mathbb{R}^{lq \times kp},$$

readily yields the upper bound $|T|(|T| - 1)$ for the number of rows stochastically dependent on any one row of Φ . Applying Lemma 3.3 and Theorem 4.1 shows that the probability for perfect reconstruction is no less than $1 - e^{-c_2 n/m^2}$. This says that imaging systems with IID random point spread functions can significantly reduce the number of acquired samples, while still being able to reconstruct the original sparse image if the above conditions hold.

4.3. Circulant-circulant Block Matrices. As a generalization of the matrices defined by (4.4) and (4.5), the following matrices are also considered:

$$(4.6) \quad \Phi = \begin{pmatrix} \Phi_{k_1} & \Phi_{k_1-1} & \cdots & \Phi_2 & \Phi_1 \\ \Phi_1 & \Phi_{k_1} & \cdots & \Phi_3 & \Phi_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \Phi_{l_1-1} & \Phi_{l_1-2} & \cdots & \cdots & \Phi_{l_1} \end{pmatrix} \in \mathbb{R}^{n \times N},$$

$$\Phi_i = \begin{pmatrix} \Upsilon_{k_2} & \Upsilon_{k_2-1} & \cdots & \Upsilon_2 & \Upsilon_1 \\ \Upsilon_1 & \Upsilon_{k_2} & \cdots & \Upsilon_3 & \Upsilon_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \Upsilon_{l_2-1} & \Upsilon_{l_2-2} & \cdots & \cdots & \Upsilon_{l_2} \end{pmatrix},$$

where the blocks Υ_j are all IID matrices. These matrices arise in multichannel two-dimensional imaging applications where the number of rows in Υ_j corresponds to the $n/(l_1 l_2)$ independent channels. We show next that these matrices are also good compressed sensing matrices.

Corollary 4.1. *Let Φ be as in (4.6). Then there exist constants $c_1, c_2 > 0$ depending only on $\delta_{3m} \in (0, 1)$, such that:*

- (i) *If $l_1 l_2 \leq 3m(3m - 1)$, then for any $n \geq c_1 l_1 l_2 m \ln(N/m)$, Φ satisfies RIP of order $3m$ for every $\delta_{3m} \in (0, 1)$ with probability*

at least

$$1 - e^{-c_2 n / l_1 l_2}.$$

- (ii) If $l_1 l_2 > 3m(3m - 1)$, then for any $n \geq c_1 m^3 \ln(N/m)$, Φ satisfies RIP of order $3m$ for every $\delta_{3m} \in (0, 1)$ with probability at least

$$1 - e^{-c_2 n / m^2}.$$

This follows directly from Lemma 4.1 and Theorem 4.1.

4.4. Deterministic Construction. The CS matrices we have considered so far are based on randomized constructions. However, in certain applications, deterministic constructions are preferred. In [11] DeVore provided a deterministic construction of CS matrices using polynomials over finite fields. We will consider deterministic block matrices based on DeVore's construction. Let us first recall the construction in [11].

Consider the set $\mathbb{Z}_p \times \mathbb{Z}_p$, where \mathbb{Z}_p denotes the field of integers modulo p , p a prime. This set has $n := p^2$ elements. Define $P_r := \{f \in \mathbb{Z}_p[x] : \deg(f) \leq r\}$, $0 < r < p$. This set has $N := p^{r+1}$ elements. For every $f \in P_r$, define the graph of f by

$$\mathcal{G}(f) = \{(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p : y = f(x), x \in \mathbb{Z}_p\} \subset \mathbb{Z}_p \times \mathbb{Z}_p$$

and consider the column vector $v(f) \in \{0, 1\}^n$, indexed by the elements of $\mathbb{Z}_p \times \mathbb{Z}_p$ ordered lexicographically, given by

$$v(f) := (\mathbf{1}_{(0,0) \in \mathcal{G}(f)}, \dots, \mathbf{1}_{(0,p-1) \in \mathcal{G}(f)}, \mathbf{1}_{(1,0) \in \mathcal{G}(f)}, \dots, \mathbf{1}_{(p-1,p-1) \in \mathcal{G}(f)})^t,$$

where

$$\mathbf{1}_{(a,b) \in \mathcal{G}(f)} = \begin{cases} 1 & \text{if } (a, b) \in \mathcal{G}(f) \\ 0 & \text{if } (a, b) \notin \mathcal{G}(f) \end{cases}$$

Construct the matrix $\Phi_0 = (v(f_1), v(f_2), \dots, v(f_N))$, where the polynomials f_i are ordered lexicographically with respect to their coefficients. It was shown in [11], that the matrix $\Phi = \frac{1}{\sqrt{p}} \Phi_0$ satisfies RIP for any $m < p/r + 1$ with $\delta = (m - 1)r/p (< 1)$.

Now consider

$$(4.7) \quad \Psi_0 = \begin{pmatrix} \Psi_t & \Psi_{t-1} & \dots & \Psi_2 & \Psi_1 \\ \Psi_{t+1} & \Psi_t & \dots & \Psi_3 & \Psi_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \Psi_{t+s-1} & \Psi_{t+s-2} & \dots & \dots & \Psi_s \end{pmatrix} \in \mathbb{R}^{sp^2 \times tl},$$

where $tl \leq p^{r+1}$, and each block $\Psi \in \mathbb{R}^{p^2 \times l}$ is constructed from the first tl vectors $v(f)$, $f \in P_r$, as above.

Theorem 4.2. *The matrix $\Psi = \frac{1}{\sqrt{sp}} \Psi_0$ satisfies RIP with $\delta = (m - 1)r/p$ for any $m < p/r + 1$.*

Proof. As before, we only have to consider the case where $|T| = m$. Let $T \subset \{1, 2, \dots, tl\}$ such that $|T| = m$, and let Ψ_T be the matrix obtained by retaining only those columns of Ψ corresponding to the elements in T . Consider the matrix $G_T = \Psi_T^t \Psi_T$. Since every column of Ψ_0 has exactly sp ones, the diagonal elements of G_T are all one. An off diagonal element of G_T has the form $g_{ij}^T = \sum_{x=1}^s \langle v_{x,i}, v_{x,j} \rangle$, where $i, j \in \{1, 2, \dots, m\}$, and $v_{x,i}$ denotes the vector $(\Psi_{T,(x-1)n+1,i}, \dots, \Psi_{T,(x-1)n+n,i})^t \in \{0, 1\}^n$ that represents some polynomial $f \in P_r$. Since the graphs of two different polynomials in P_r have at most r elements in common, $g_{ij}^T \leq sr/sp = r/p$ for any $i \neq j$. Therefore, the sum of all off diagonal elements in any row or column of G_T is $\leq (m-1)r/p = \delta < 1$ whenever $m < p/r + 1$. We can, therefore, write

$$(4.8) \quad G_T = I + B_T,$$

where $\|B_T\|_1 \leq \delta$ and $\|B_T\|_\infty \leq \delta$. Since $\|B_T\|_2^2 \leq \|B_T\|_1 \|B_T\|_\infty$, we have that $\|B_T\|_2 \leq \delta$ and so the spectral norms of B_T and B_T^{-1} are $\leq 1 + \delta$ and $\leq (1 - \delta)^{-1}$, respectively. This shows that Ψ satisfies (1.3). \square

5. NUMERICAL RESULTS

To validate that the probability of exact recovery for Toeplitz block CS matrices is high, the performance of Toeplitz block, IID, and Toeplitz CS matrices is compared empirically. In our simulation, a length $n = 2048$ signal with randomly placed $m = 20$ non-zero entries drawn independently from the Gaussian distribution was generated. Each such generated signal is sampled using $n \times N$ IID, Toeplitz and Toeplitz block matrices with entries drawn independently from the Bernoulli distribution and reconstructed using the log barrier solver from [12]. The experiment is declared a success if the signal is exactly recovered, i.e., the error is within the range of machine precision. The empirical probability of success is determined by repeating the reconstruction experiment 1000 times and calculating the fraction of success. This empirical probability of success is plotted as a function of the number of measurement samples n in Fig. 1. The simulation results suggest that Toeplitz block matrices perform similar to IID matrices in applications. However, we do not know how to prove that at this point.

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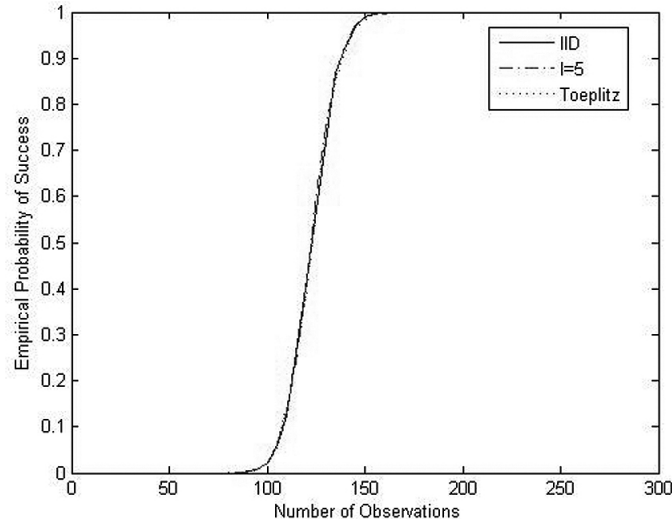


FIGURE 1. Empirical probability of success plotted against the number of observations for IID, Toeplitz block, and Toeplitz matrices.

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