Evaluate this expression

$$7\times(3^2-5)-7+3$$

Many people don't believe the answer is 4!

Factorial notation

The product of the first n positive integers is denoted by $n!$ and is called ***n* factorial:**

$$n! = 1 \times 2 \times 3 \times 4 \ldots (n-2) \times (n-1) \times n$$

We also define $0! = 1$.

$$n! = n \times (n-1)(n-2) \ldots \ldots 1$$

$$10! = 10 \times 9 \times 8 \times \ldots \ldots 1$$

$$= 10 \times 9! = 10 \times 9 \times 8!$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

example 61

Simplify the following expressions

$2! + 3! = 2 + 3 \times 2 = 8$

$$\frac{9!}{7!} = \frac{9 \times 8 \times \cancel{7!}}{\cancel{7!}} = 72$$

$$\frac{10! + 9!}{9!} = \frac{10!}{9!} + \frac{\cancel{9!}}{\cancel{9!}}1$$

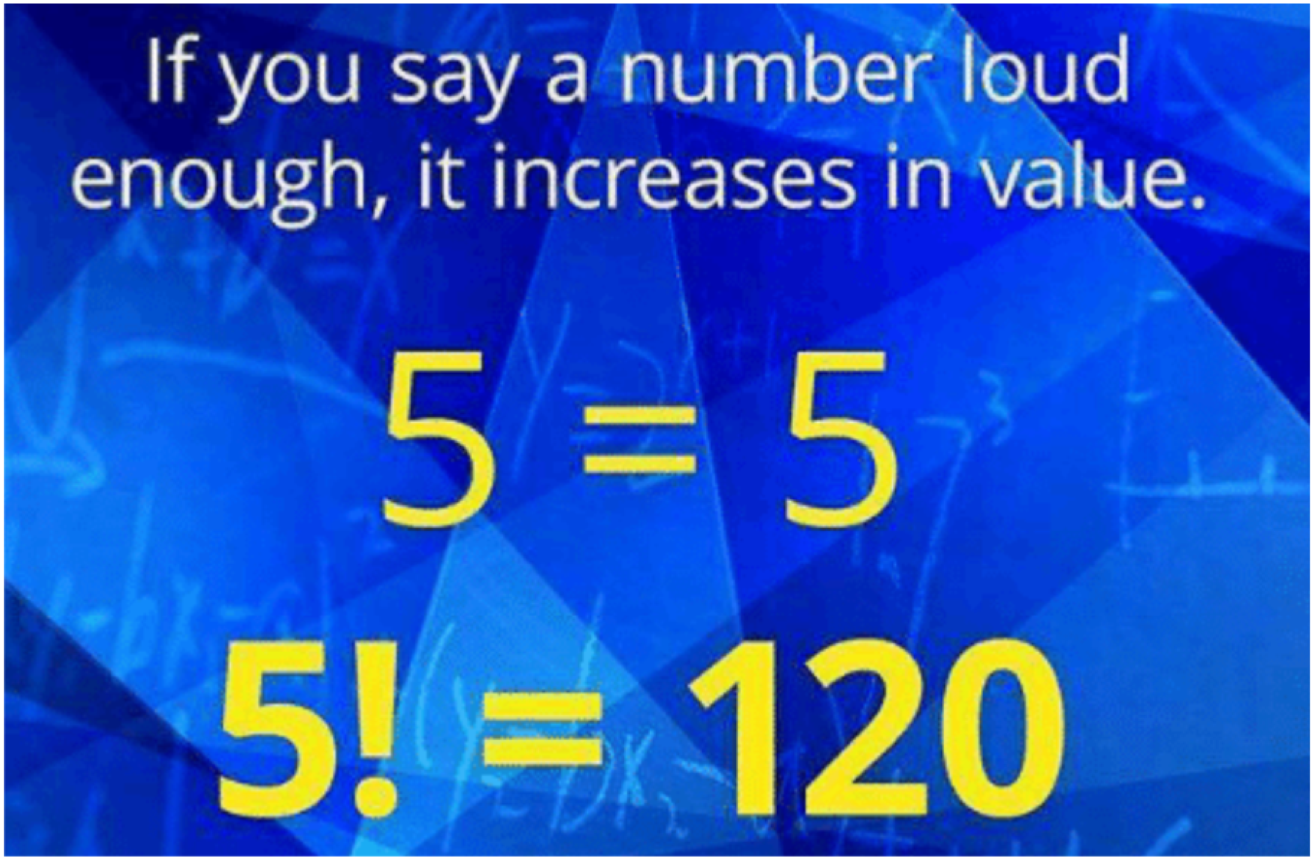
$$= \frac{10 \times \cancel{9!}}{\cancel{9!}} + 1 = 11$$

Simplify $\frac{n!}{(n-3)!} = \frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}}$

example 62

Write $10 \times 11 \times 12$ as a ratio of two factorials.

$$\frac{12!}{9!} = \frac{12 \times 11 \times 10 \times \cancel{9!}}{\cancel{9!}}$$



la

Solve the following equations

example 63 $\dfrac{(n+2)!}{(n+1)!} = 15$ $\dfrac{(n+2)\cancel{(n+1)!}}{\cancel{(n+1)!}} = 15$ $n = 13$ $n \in \mathbb{Z}^+$

example 64 $\dfrac{(n+1)!}{(n-1)!} = 20$ $\longrightarrow$ $\dfrac{(n+1) \cdot n \, \cancel{(n-1)!}}{\cancel{(n-1)!}} = 20$ $5 \cdot 4$

$n^2 + n - 20 = 0$

$(n+5)(n-4) = 0$

$n = 4$ $\cancel{n = -5}$

example 65 $\dfrac{(n+2)!}{n! + (n+1)!} = 8$

$= \dfrac{(n+2)!}{n! + (n+1) \cdot n!}$

$= \dfrac{\cancel{(n+2)}(n+1)\cancel{n!}}{\cancel{n!}\cancel{(1+n+1)}} = 8$ $n = 7$

example
66

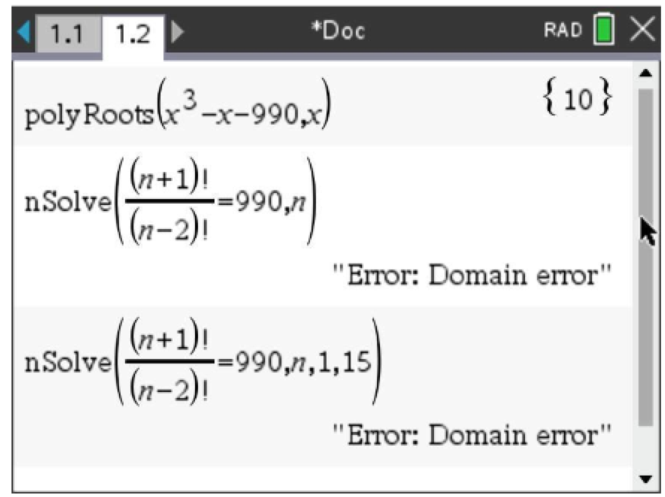
Solve $\dfrac{(n+1)!}{(n-2)!} = 990.$ *[6 marks]*

$$\frac{(n+1)\,n\,(n-1)\,\cancel{(n-2)!}}{\cancel{(n-2)!}} = 990$$

99×10

$9 \times 10 \times 11$

$n = 10$



example

Solve $\dfrac{n!}{(n-2)!} = 20$ where n is a positive integer. *[5 marks]*

HW

example

Solve $n! - (n-1)! = 16(n-2)!$ for $n \in \mathbb{N}$. *[6 marks]*

Instructions: Insert an exclamation point where needed.

I want to kiss you 6 times

☒ I want to kiss you 6 times!

☑ I want to kiss you 6! times

MATH NERDS DO IT BETTER

example Determine the number of zeros at the end of the number 50! **HL Only**

la

M
Fi
i.e

N

HL Only

Counting arrangements

The word 'ARTS' and the word 'STAR' both contain the same letters, but arranged in a different order. They are both arrangements (also known as **permutations**) of the letters R, A, T and S. We can count the number of different permutations.

There are four possibilities for the first letter, then for each choice of the first letter there are three options for the second letter (because one of the letters has already been used). This leaves two options for the third letter and then the final letter is fixed. Using the 'AND rule' the number of possible permutations is $4 \times 3 \times 2 \times 1 = 24$.

The number of permutations of n different objects is equal to the product of all positive integers less than or equal to n. This **expression** is abbreviated to $n!$ (pronounced 'n factorial').

example 67

A test has 12 questions.

How many different arrangements of the questions are possible?

12!

Counting selections

HL Only

Suppose that three pupils are to be selected from a class of 11 to attend a meeting with the Head Teacher. How many different groups of three can be chosen?

In this example we need to *choose* three pupils out of 11, but they are not to be arranged in any specified order. The order is not important; the selection of Ali, Bill and then Cathy is the same as the selection of Bill, Cathy and then Ali. This sort of selection is called a **combination**. In general, the formula for

the number of ways of choosing r objects out of n is given the symbol $\binom{n}{r}$, or ${}^nC_r$ (pronounced 'n C r' or 'n choose r').

EW
EB WB
ED WD BD
EZ WZ BZ DZ

$$\binom{5}{2} = \frac{5!}{2!\ 3!}$$

$$= \frac{5 \cdot 4 \cdot \cancel{3!}}{2 \cdot 1\ \cancel{3!}} = 10$$

> The number of ways of choosing r objects out of n is
> $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Both Groups

example 68

Find the value of a) $\binom{7}{3}$ b) $\binom{7}{4}$ c) $\binom{7}{0}$ d) $\binom{7}{7}$

$$\binom{n}{r} = \binom{n}{n-r}$$

$$= \frac{7!}{3!\cdot 4!} = 35 \qquad = \frac{7!}{4!\cdot 3!} = 35 \qquad = 1 \qquad = \frac{7!}{7!\ 0!} = 1$$

$$\binom{7}{1} = \binom{7}{6}$$

$$\binom{7}{2} = \binom{7}{5}$$

example 69

Calculate the following:

$\binom{6}{0}$, $\binom{6}{1}$, $\binom{6}{2}$, $\binom{6}{3}$, $\binom{6}{4}$, $\binom{6}{5}$, $\binom{6}{6}$

$= 1$ $= 6$ $= 15$ $= 20$ $= 15$ $= 6$ $= 1$

$$\binom{6}{2} = \frac{6\cdot 5\cdot \cancel{4!}}{2!\cdot \cancel{4!}} = \frac{\overset{3}{\cancel{6}}\cdot 5}{\cancel{2}\cdot 1} = 15$$

The number of ways of choosing r objects out of n is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

example

thw

Evaluate:

(a) (i) $\binom{7}{2}$ (ii) $\binom{12}{5}$

(b) (i) $3\times\binom{6}{3}$ (ii) $10\times\binom{6}{5}$

(c) (i) $\binom{5}{0}\times\binom{9}{5}$ (ii) $\binom{10}{8}\times\binom{3}{1}$

(d) (i) $\binom{5}{2}+\binom{9}{5}$ (ii) $\binom{6}{0}+\binom{7}{3}$

The number of ways of choosing r objects out of n is

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}$$

example

HW

Solve:

(a) (i) $\binom{n}{2} = 91$ (ii) $\binom{n}{2} = 351$

(b) (i) $\binom{n}{3} = 1330$ (ii) $\binom{n}{3} = 680$

Binomial expansion

In this chapter you will learn:

- how to expand an expression of the form $(a + b)^n$ for positive integer n
- how to find individual terms in the expansion of $(a + b)^n$ for positive integer n
- how to use partial expansions of $(a + bx)^n$ to find approximate values.

Introductory problem

Without using a calculator, find the value of $(1.002)^{10}$ correct to 8 decimal places.

A **binomial** expression is one which contains two terms, for example, $a + b$.

Expanding a power of such a binomial expression could be performed by expanding brackets; for example $(a+b)^7$ could be found by calculating, at length,

$$(a+b)(a+b)(a+b)(a+b)(a+b)(a+b)(a+b).$$

This is time-consuming, but happily there is a much faster approach.

Let us look at some special cases of the expansion of $(x + y)^n$:

$(x + y)^0 = 1$

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

- There are $n + 1$ terms in the expansion of $(x + y)^n$.
- The degree of each term is n.
- The powers on x begin with n and decrease to 0.
- The powers on y begin with 0 and increase to n.
- The coefficients are symmetric.

example

$$(x+y)^7$$
$$= \boxed{1}x^7 + \boxed{7}x^6y + \boxed{21}x^5y^2 + \boxed{35}x^4y^3 + \boxed{35}x^3y^4 + \boxed{21}x^2y^5 + \boxed{7}xy^6 + \boxed{1}y^7$$

1

1 1

1 2 1

1 3 3 1

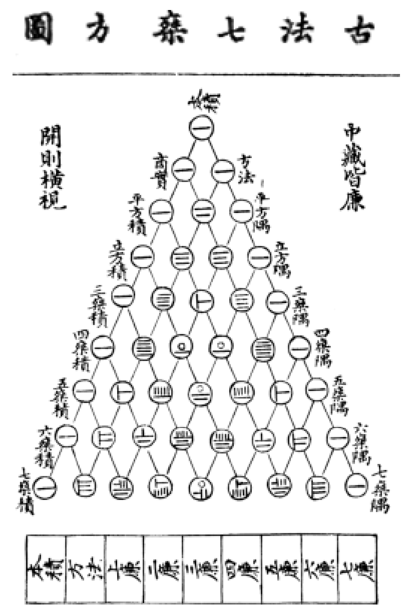
1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Pascal's triangle was known in China in the early 11th century through the work of the Chinese mathematician Jia Xian (1010–1070).In the 13th century, Yang Hui (1238–1298) presented the triangle and hence it is still called **Yang Hui's triangle** (杨辉三角; 楊輝三角) in China.[12]



$(x+y)^0$	1							row 0	
$(x+y)^1$	1	1						row 1	
$(x+y)^2$	1	2	1					row 2	n
$(x+y)^3$	1	3	3	1				row 3	
$(x+y)^4$	1	4	6	4	1			row 4	
$(x+y)^5$	1	5	10	10	5	1		row 5	
$(x+y)^6$	1	6	15	20	15	6	1	row 6	
	column 0	column 1	column 2	column 3	column 4	column 5	column 6		

r

Pascal's triangle

Every entry in a row is the sum of the term directly above it and the entry diagonally above and to the left of it. When there is no entry, the value is considered zero.

Pascal's triangle was known in China in the early 11th century through the work of the Chinese mathematician Jia Xian (1010–1070).In the 13th century, Yang Hui (1238–1298) presented the triangle and hence it is still called **Yang Hui's triangle** (杨辉三角; 楊輝三角) in China.[12]

example Use Pascal's triangle to expand $(2k-3)^5$.

70

1 5 10 10 5 1

$$= 1(2k)^5(-3)^0 + 5(2k)^4(-3)^1 + 10(2k)^3(-3)^2 + 10(2k)^2(-3)^3 + 5(2k)^1(-3)^4 + 1(2k)^0(-3)^5$$

$$= 32k^5 - 240k^4 + - - - -$$

$$\begin{aligned}(2k-3)^5 &= \binom{5}{0}(2k)^5 + \binom{5}{1}(2k)^4(-3) + \binom{5}{2}(2k)^3(-3)^2 + \binom{5}{3}(2k)^2(-3)^3 \\ &\quad + \binom{5}{4}(2k)(-3)^4 + \binom{5}{5}(-3)^5 \\ &= 32k^5 - 240k^4 + 720k^3 - 1080k^2 + 810k - 243\end{aligned}$$

$(x+y)^0$	1							row 0
$(x+y)^1$	1	1						row 1
$(x+y)^2$	1	2	1					row 2
$(x+y)^3$	1	3	3	1				row 3
$(x+y)^4$	1	4	6	4	1			row 4
$(x+y)^5$	1	5	10	10	5	1		row 5
$(x+y)^6$	1	6	15	20	15	6	1	row 6
	column 0	column 1	column 2	column 3	column 4	column 5	column 6	

In $\binom{n}{r}$, n is the row number and r is the column number.

The binomial coefficient

With n and r as non-negative integers such that $n \geqslant r$, the binomial coefficient $\binom{n}{r}$ is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Using the binomial theorem

We are now prepared to state the binomial theorem. The proof of the theorem is optional and will require mathematical induction.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

In a compact form, we can use sigma notation to express the theorem as follows:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Binomial theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

example 71 Find the coefficient of $x^5 \boxed{y^3}$ in the expansion of $(x+y)^8$.

$n=8$

$\boxed{r=3}$

$8-3$

$$\binom{8}{\boxed{3}} \cdot x^5 \, y^{\boxed{3}} = \frac{8 \cdot 7 \cdot \cancel{6}}{\cancel{3 \cdot 2} \cdot 1} x^5 y^3$$

$$= 56 x^5 y^3$$

example 72 Find the term containing a^3 in the expansion $(2a - 3b)^9$.

$a^3 b^6 \qquad r=6$

$$\binom{9}{6} \cdot (2a)^3 \cdot (-3b)^6$$

$$= \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} \cdot 2^3 \cdot 3^6 \, a^3 b^6 = 489888 a^3 b^6$$

$\text{nCr}(9,6) \cdot 2^3 \cdot 3^6 \qquad 489888$

example 73

Find the term in x^6y^4 in the expansion of $\left(x+3y^2\right)^8$.

~~$r=4$~~

$\cancel{\binom{8}{4}\cdot x^4\cdot(3y^2)^4}$

$\cancel{=\binom{8}{4}\cdot 3^4\, x^4\, y^8}$

$r=2$

$\binom{8}{2}(x)^6(3y^2)^2$

$= \frac{{}^{4}\cancel{8}\cdot 7}{\cancel{2}\cdot 1}\; 3^2\cdot x^6 y^4$

$= 252\, x^6 y^4$

example 74

The expansion of $(1+x)^n$ up to the third term is given by $1+6x+ax^2$. Find the value of n and of a.

$$\binom{n}{0}1^n x^0 + \binom{n}{1}1^{n-1}x + \binom{n}{2}1^{n-2}x^2 + \dots$$

$$= 1 + n \cdot x + \binom{6}{2} \cdot x^2$$

$$a = \frac{6 \cdot 5}{2 \; 1} = 15$$

example
75 Find the coefficient of x^2y^5 in the expansion of $(x-y)(x+y)^6$.

$$(x-y)\left(x^6+\binom{6}{1}x^5y+\binom{6}{2}x^4y^2+\binom{6}{3}x^3y^3+\binom{6}{4}x^2y^4+\binom{6}{5}xy^5+y^6\right)$$

x^2y^5

$$x\cdot\left[xy^5\right]=\binom{6}{5}\cdot xy^5\cdot x=6x^2y^5$$

$$-y\cdot\left[x^2y^4\right]=\binom{6}{4}x^2y^4\cdot(-y)=-15x^2y^5$$

$$\binom{6}{2}=\frac{6\cdot 5}{2}=15$$

$+$

$$-9x^2y^5$$

example 76

Find the coefficient of a^4b^3 in the expansion of $(a-5b)(a+b)^6$.

$$a \cdot \left[a^3b^3\right] = a \cdot \binom{6}{3} a^3b^3$$

$$-5b\left[a^4b^2\right] = -5b \cdot \binom{6}{2} a^4b^2$$

$+$

$$\frac{\cancel{6}\cdot 5\cdot 4}{\cancel{3}\cdot\cancel{2}\cdot 1} a^4b^3 - 5 \times \frac{\overset{3}{\cancel{6}}\cdot 5}{\cancel{2}} a^4b^3$$

$$= -55\, a^4b^3$$

example 77

Use the Binomial theorem to expand and simplify $(5-3x)(2-x)^4$.

Let's find x^3 term only

$$5 \cdot [x^3 \text{ term}] = 5 \cdot \binom{4}{3} \cdot 2\,(-x)^3 = -40x^3$$

$$-3x\,[x^2 \text{ term}] = -3x \cdot \binom{4}{2} \cdot 2^2\,(-x)^2 = -72x^3$$

$3 \times 6 \times 4$

$$+ \quad \overline{-112x^3}$$

$80 - 208x + 216x^2 - 112x^3 + 29x^4 - 3x^5$

Exit Ticket

example 78

Find the coefficient of x^5 in the expansion of $\left(2x^2 - \frac{1}{x}\right)^7$.

$(2x^2)^7 - 7\cdot(2x^2)^6 \frac{1}{x} + \cdots$

focus on numbers only!

focus on variables only. Find r

$\left(2x^2 - \frac{1}{x}\right)^7$

$\left(2x^2 - \frac{1}{x}\right)^7$

$(x^2)^{7-r}(x^{-1})^r = x^5$

$x^{14-2r} \cdot x^{-r} = x^5$

$x^{14-3r} = x^5$

$14-3r=5$

$3r=9$

$r=3$

$\binom{7}{3}\cdot 2^4 (-1)^3$

$= -\frac{7\cdot\cancel{6}\cdot 5}{\cancel{3}\cdot\cancel{2}\cdot 1}\cdot 2\times 8$

$= -560$

example 79

constant term : x^0

Find the term independent of x in $\left(4x^3 - \frac{2}{x^2}\right)^5$.

$\cancel{x^6} \quad \frac{1}{\cancel{x^6}}$

$\left(4x^3 - \frac{2}{x^2}\right)^5$

$(x^3)^{5-r}(x^{-2})^r = x^0$

$x^{15-3r} \cdot x^{-2r} = x^0$

$15-5r = 0$

$r = 3$ //

$\left(4x^3 - \frac{2}{x^2}\right)^5$

$\binom{5}{3} 4^2 \cdot (-2)^3$

$= -\frac{5 \cdot 4}{2 \cdot 1} \cdot 4^2 \cdot 2^3$

$= -1280$

example 80

Find the term independent of x in the binomial expansion of $\left(2x^2 + \frac{1}{2x^3}\right)^{10}$.

$$(x^2)^{10-r} \cdot (x^{-3})^r = x^0$$

$$x^{20-2r} \cdot x^{-3r} = x^0$$

$$20 - 5r = 0$$

$$r = 4$$

$$\binom{10}{4} \cdot 2^6 \cdot \left(\frac{1}{2}\right)^4$$

$$= \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8}^{4} \cdot 7}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} \cdot \frac{\cancel{2^{6}}^{2}}{\cancel{2^4}}$$

$$= 840$$

example

81 Find the coefficient of b^6 in the expansion of $\left(2b^2 - \frac{1}{b}\right)^{12}$.

$(b^2)^{12-r}(b^{-1})^r = b^6$

$b^{24-2r-r} = b^6$

$24-3r = 6$

$3r = 18$

$r = 6$

$\binom{12}{6} 2^6 \cdot (-1)^6$

$= 59136$

menu

↓

5

↓

3

nCr(12,6)

example

HL Only

Find the coefficient of x^4 in the expansion of $(1+3x-x^2)(2+x)^5$.

$1 \cdot x^4$

$3x \cdot x^3$

$-x^2 \cdot x^2$

$$1\cdot\binom{5}{4}\cdot 2^1\cdot x^4 + 3x\cdot\binom{5}{3}\cdot 2^2\cdot x^3 - x^2\cdot\binom{5}{2}\cdot 2^3\cdot x^2$$

$$= 10x^4 + 120x^4 - 80x^4$$

$$= 50x^4$$

example

HL Only

Calculate $\binom{n}{r-1} + \binom{n}{r}$.

This is called Pascal's rule.

r

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

n

$n+1$

$\binom{n+1}{r}$

$$\frac{r}{r}\cdot\frac{n!}{(r-1)!\,(n-r+1)!} + \frac{n!}{r!\,(n-r)!}\cdot\frac{n-r+1}{n-r+1}$$

$$= n!\left(\frac{\cancel{r} + n-\cancel{r}+1}{r!\cdot(n-r+1)!}\right)$$

$$= \frac{(n+1)\cdot n!}{r!\,(n+1-r)!} = \frac{(n+1)!}{r!\,(n+1-r)!} = \binom{n+1}{r}$$

HL Only

If the value of x is much less than one, large powers of x will be extremely small.

31.207960999

example

Find the first 3 terms in ascending powers of x of the expansion of $(2-x)^5$.
By setting $x = 0.01$, use your answer to find an approximate value of 1.99^5.

$$x = 0.01 = \frac{1}{100}$$

$$(2-x)^5 = 2^5 - \binom{5}{1} \cdot 2^4 \cdot x + \binom{5}{2} 2^3 x^2 + \ldots$$

$$= 32 - 80x + 80x^2$$

$$= 32 - 8\cancel{0} \cdot \frac{1}{10\cancel{0}} + 8\cancel{0} \cdot \frac{1}{1000\cancel{0}}$$

$$= 32$$

$$0.008$$

$$-0.8$$

$$= 31.208$$

example What is the sum of the coefficients in the expansion of $(3x - 2y)^{20}$? **HL Only**

Let's think simple:

$x=1$
$y=1$

$$(3x-2y)^2 = 9x^2 - 12xy + 4y^2$$

$$(3-2)^2 = 9 - 12 + 4 = 1$$

$$(3-2)^{20} = 1$$

HL Only

example What is the sum of the coefficients in the expansion of $(2x^2 - x + 1)^4(3x - 2)^2$?

$x = 1$

$2^4 \cdot 1^2 = 16$ //

example

HL Only

What's the coefficient of $x^3y^2z^3$ term in the expansion of $(x+y+z)^8$?

$$\left((x+y)+z\right)^8$$

$r=3$

$$\binom{8}{3}\cdot(x+y)^5 z^3$$

$\downarrow$

Find x^3y^2 term

example

HL Only

Find the constant term in the expansion of $\left(\frac{x^2}{y^2} - \frac{y}{x}\right)^9$

Let $\frac{x}{y} = t$

$\left(t^2 - \frac{1}{t}\right)^9$