Functions

Concepts

- Representation
- Relationships

Microconcepts

- Function, domain, range
- Key features of graphs
- Composite functions
- The quadratic function
- Transformations
- Odd and even functions
- Partial fractions

A relation R is a set of ordered pairs (x, y) such that $x \in A, y \in B$ and sets A and B are non-empty.
A function f is a relation in which every x-value has a unique y-value.

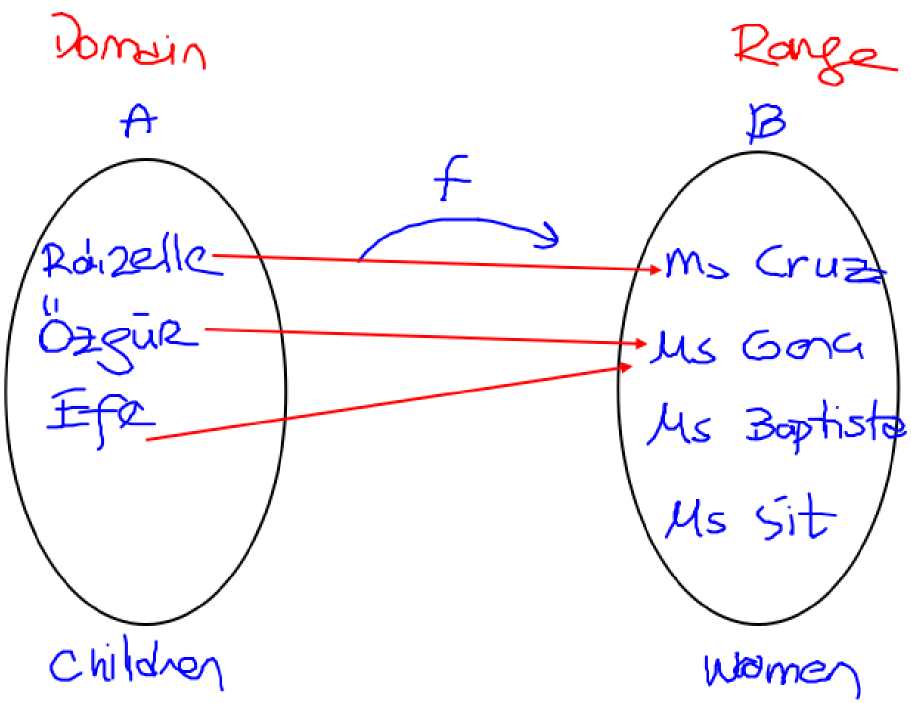
The set of x-values is called the **domain** of a function. The set of y-values that the domain is mapped to is called the **range** of the function. Since the y-values (output) depend on the x-values (input), y is called the dependent variable, and x is the independent variable. The independent variable x is also called the **argument** of the function.

A relation R is a function f if:

- f acts on **all** elements of the domain.
- f is well-defined, ie, it pairs each element of the domain with one and only one element of the range. Therefore, if f contains two ordered pairs (a, b_1) and (a, b_2), then $b_1 = b_2$.

In general, if y is a function of x, you can write $y = f(x)$, or $x \mapsto f(x)$.

$f : A \to B$



[cannot have more than 1]

Each child has a biological mother.
[exactly one]

No child without a mother.

2 or more children might have the same mother

There can be women without any child.

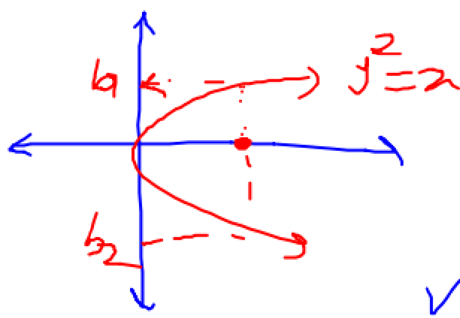
example 1

Determine, giving reasons, which of these relations are functions.
For those that are functions, write down the domain and range.

a $\{(-2, 0), (-3, 3), (-4, 8), (0, 0)\}$

b $\{(5, -1), (2, 2), (5, 3), (10, 4)\}$ $f(5) = -1$ $f(5) = 3$

c $y = 4 - 3x$ **d** $y^2 = x$ **e** $f : \mathbb{R} \to \mathbb{R}; f \mapsto \dfrac{1}{x+1}$



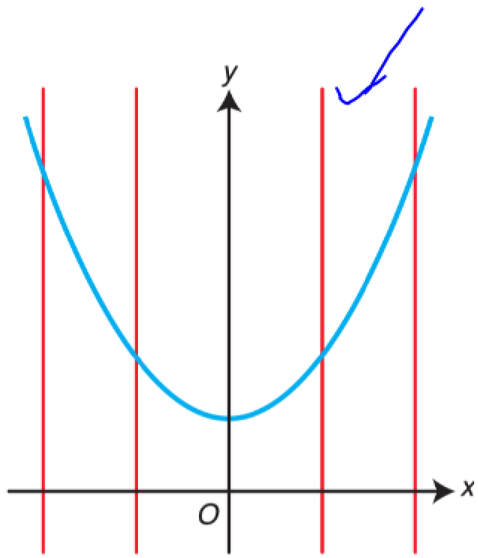
$f : x \longmapsto \dfrac{1}{x+1}$ since $f(-1) = \dfrac{1}{0}$!!!

$f : \mathbb{R} \setminus \{-1\} \longrightarrow \mathbb{R} \setminus \{0\}$

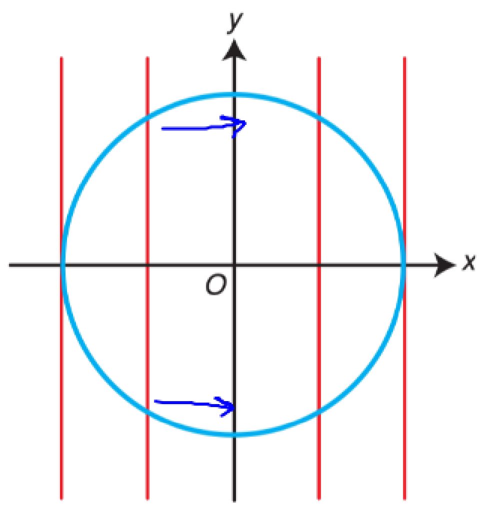
Vertical line test

If a relation is a function, any vertical line will cross its graph no more than once.

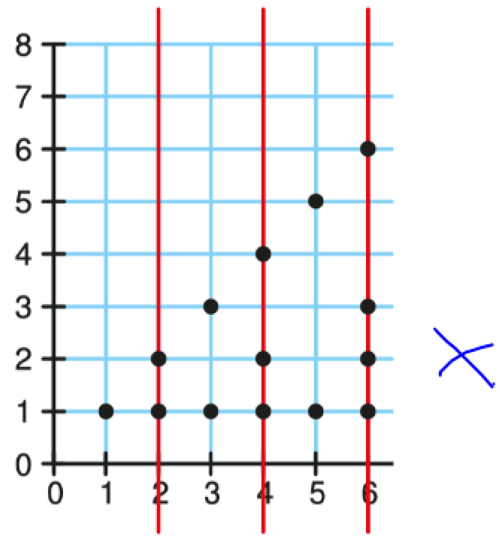
When a mapping is given by its graph, the easiest way to decide whether or not it is a function is to carry out a **vertical line test**:



function



not a function



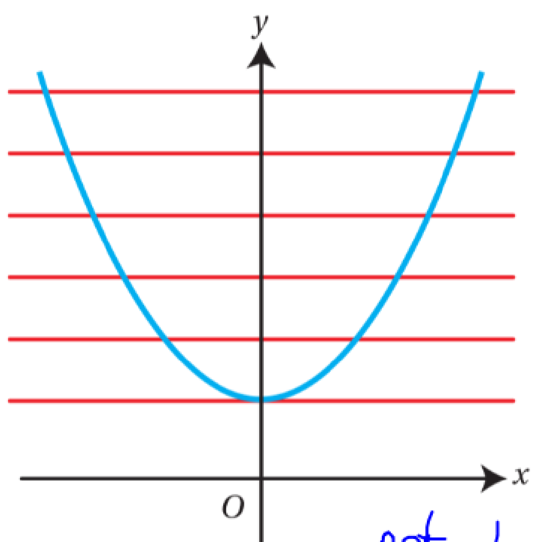
not a function

- A mapping is a function if every input value maps to a single output value.
- Vertical line test: if a mapping is a function, any vertical line will meet its graph at most once.

Horizontal line test

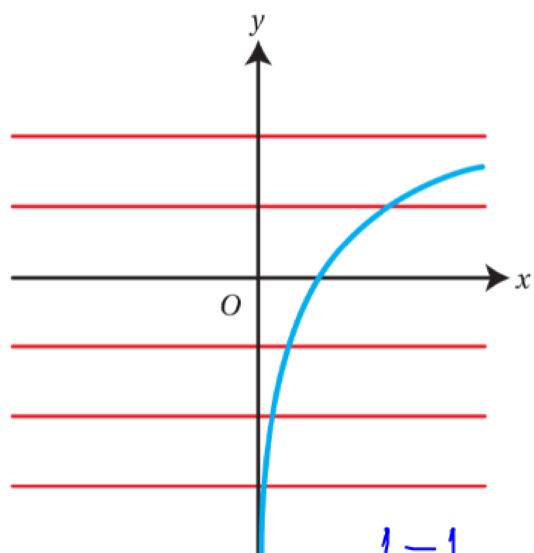
If a function is one-to-one then any horizontal line will cross the graph no more than once.

Having decided that a mapping is a function, you can ask whether each output comes from just one input. To check this, you can apply the **horizontal line test**:



not 1-1

many–one



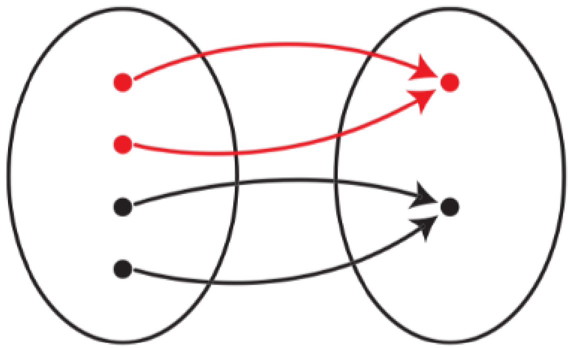
1-1

one–one

A function is:

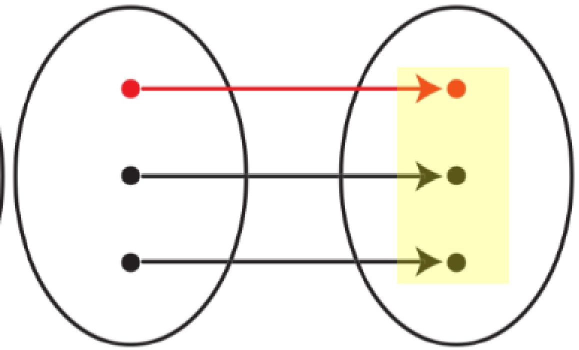
- **one-one** if every y value corresponds to only one x value.
- **many-one** if there is at least one y value that comes from more than one x value.

Horizontal line test: if a function is one-one, any horizontal line will meet the graph at most once.



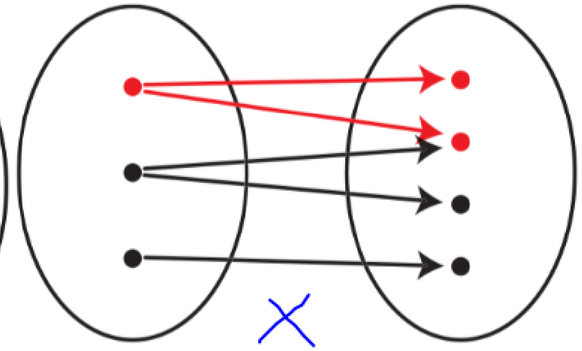
Many–one function

[will not have an inverse]



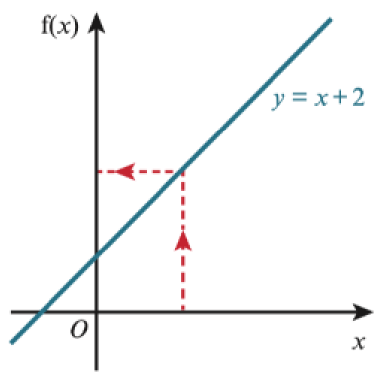
One–one function

[Inverse Exists]

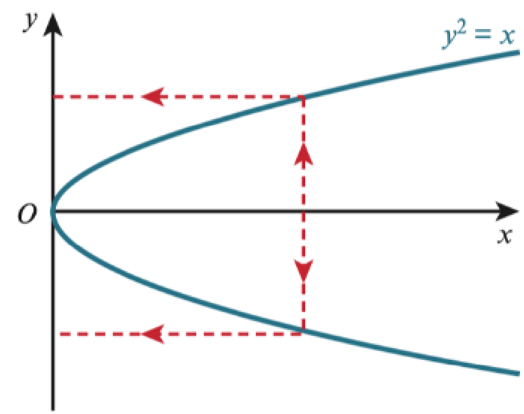


One–many mapping
(not a function)

The function $x \mapsto x + 2$, where $x \in \mathbb{R}$ is an example of a one-one function.

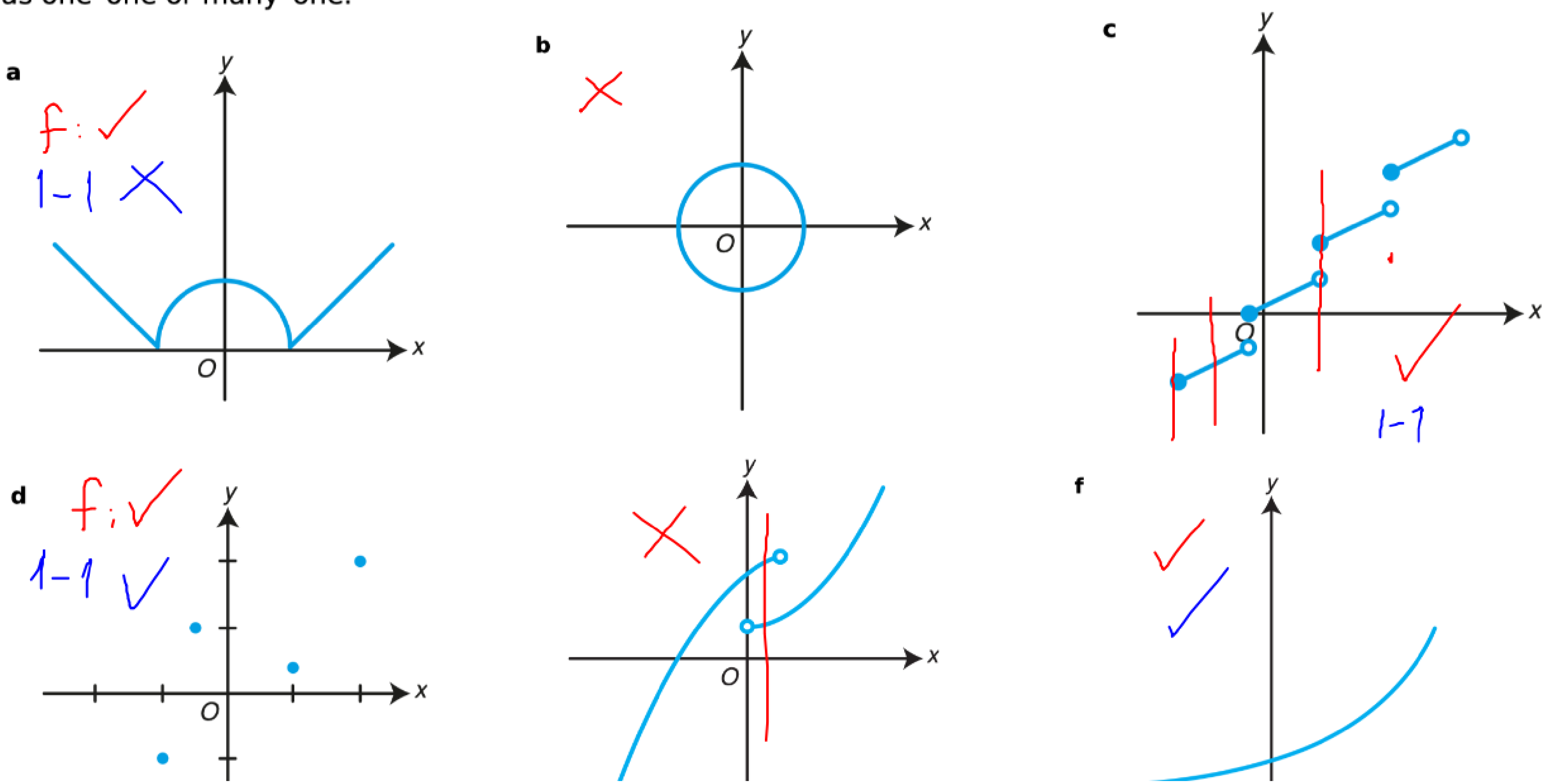


If we now consider the graph of $y^2 = x$:



example 2

Which of the graphs below could represent functions? For those that could be functions, classify them as one–one or many–one.



The set of allowed input values is called the **domain** of the function. Conventionally we write it after the rule using set notation, interval notation or inequalities.

If no domain is explicitly mentioned you can assume that the domain is all real numbers.

EXAM HINT

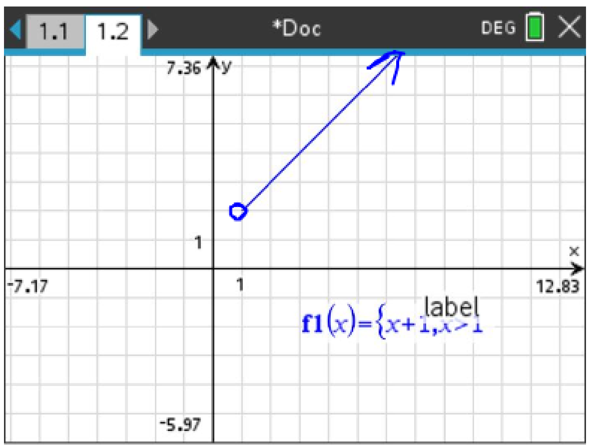
When you are instructed to sketch the graph of $f(x)$, this just means the graph of $y = f(x)$.

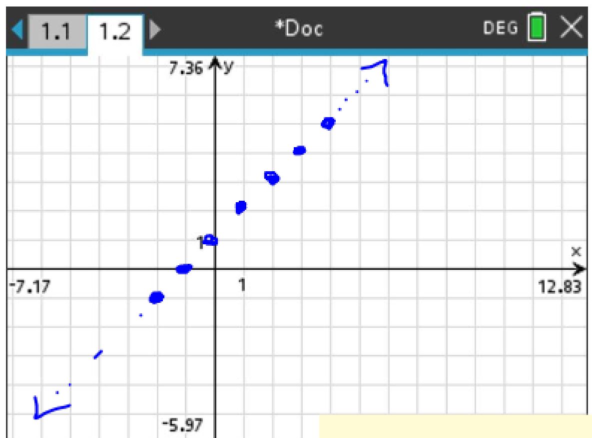
example 3

Sketch the graph of $f(x) = x + 1$ over the domain

(a) $x \in \mathbb{R}, x > 1$

(b) $x \in \mathbb{Z}$ {Integers}

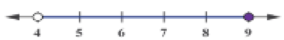




The set of all possible outputs of a function is called the **range**. The easiest way of finding this is to sketch the graph (possibly using your GDC). Be aware that the range will depend upon the domain.

Domain and range notation

The domain of a function and the range of function are often expressed using the notations shown in the table below.

Number line	Inequality notation	Set notation	Interval notation
	$4 < x \leq 9$	$\{x \in \mathbb{R} \mid 4 < x \leq 9\}$	$(4,\ 9]$ or $]4,\ 9]$
	$4 \leq x < 9$	$\{x \in \mathbb{R} \mid 4 \leq x < 9\}$	$[4,\ 9)$ or $[4,\ 9[$
	$x < 9$	$\{x \in \mathbb{R} \mid x < 9\}$	$(-\infty, 9)$ or $]-\infty, 9[$
	$x > 9$	$\{x \in \mathbb{R} \mid x > 9\}$	$(9, +\infty)$ or $]9, +\infty[$

Note that when a number is not included in the domain or range, you use the parenthesis (,) or] , [.

[ ]

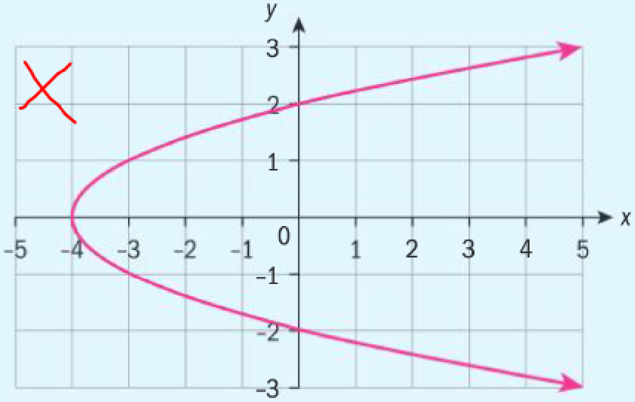
( )

] [

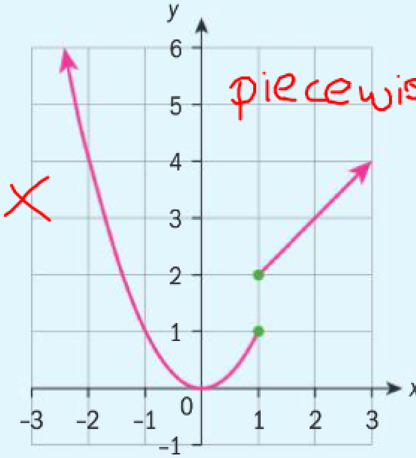
example 4

Determine which of these graphs show relations that are functions. For those that are functions, state the domain and range of the function.

a

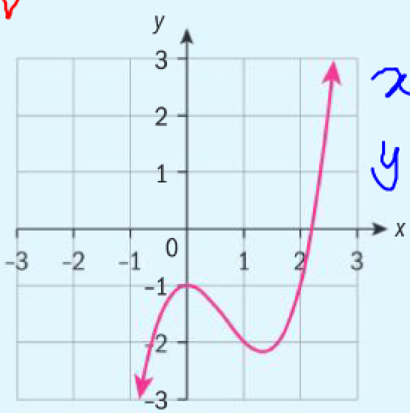


b



piecewise f.

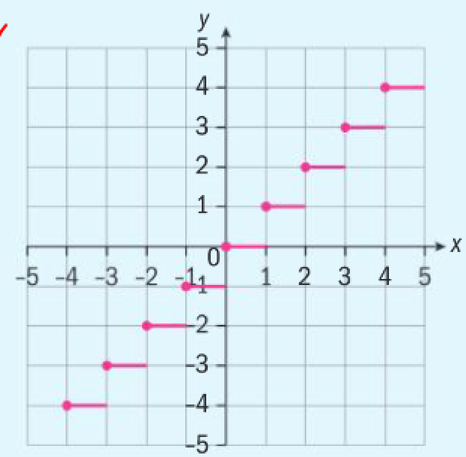
c



$x \in \mathbb{R}$

$y \in \mathbb{R}$

d



$x \in \mathbb{R}, x \in [-4, 5)$

$y \in \mathbb{Z}, y \in [-4, 4]$

Exam tip

There are many ways of writing domain and range notation. Do not worry about notation too much! You can even write in your examination a statement such as 'All x's are allowed except π', which could also be written as:

$$x \in (-\infty, \pi) \cup (\pi, +\infty)$$

$$x \in]-\infty, \pi[\cup]\pi, +\infty[$$

$$x < \pi \text{ or } \pi < x.$$

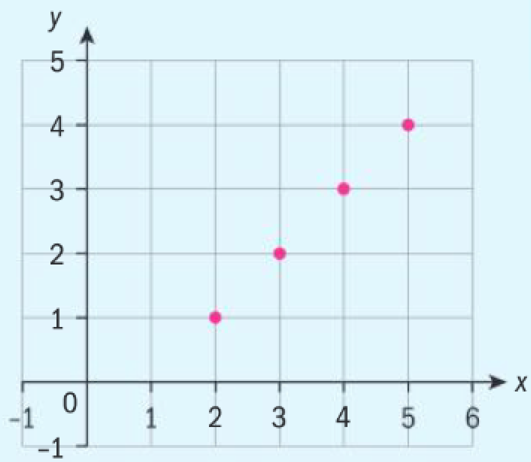
$\cup$: OR (union)

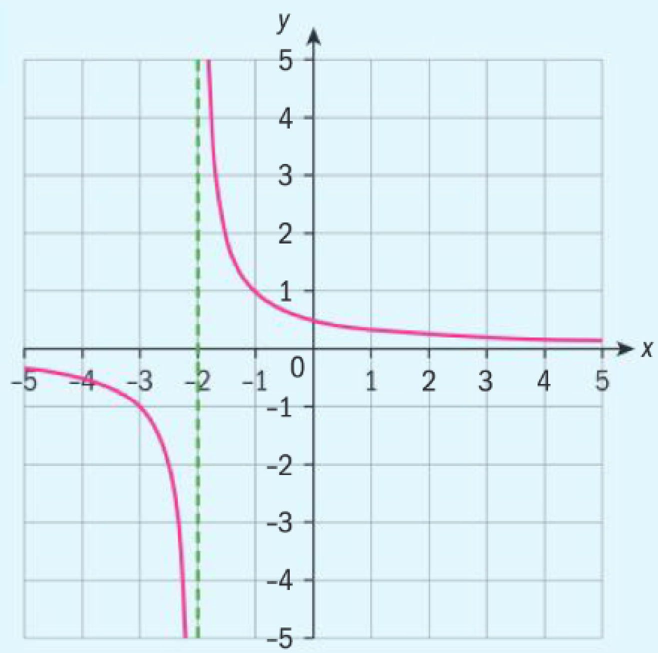
$\cap$: And (intersection)

example

5

Use these graphs to identify the domain and range of the functions they represent.





$x \in \mathbb{Z}$, $\{2, 3, 4, 5\}$

$x \in [2, 5]$

$x \in (1, 6)$

$2 \leq x \leq 5$

$y \in \mathbb{Z}$ $y \in [1, 4]$

$\mathbb{R} \setminus \{-2\} \longrightarrow \mathbb{R} \setminus \{0\}$

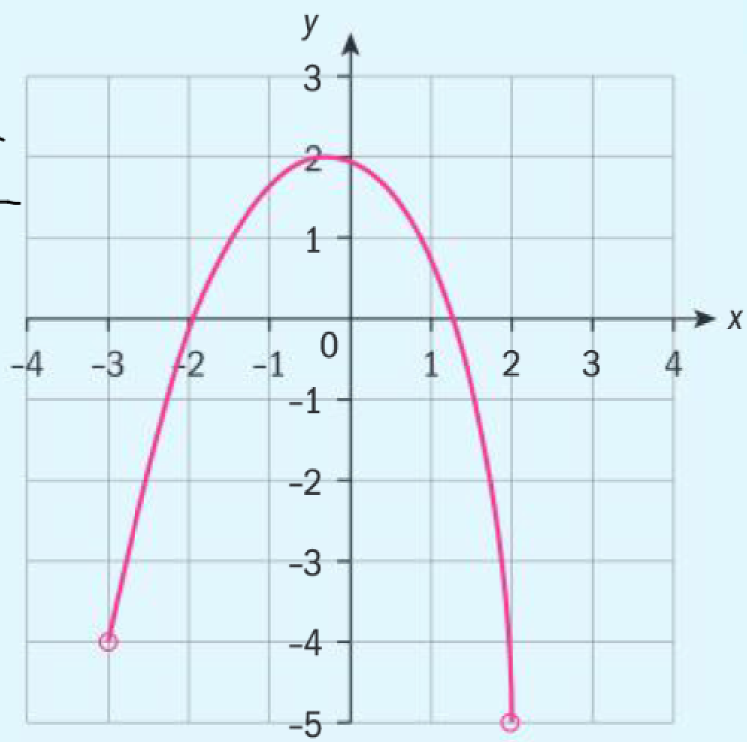
$\mathbb{R} - \{-2\}$

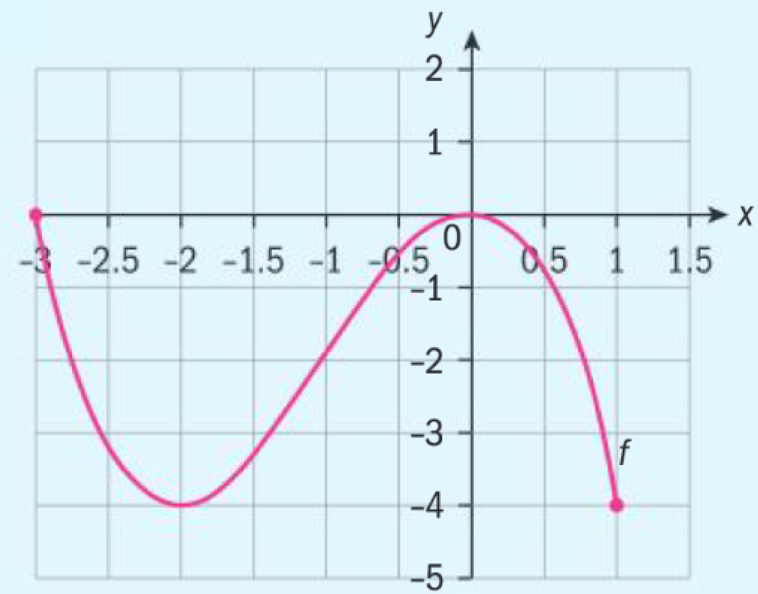
$x \in]-3, 2[$

$-3 < x < 2$

$-5 < y \leq 2$

$y \in]-5, 2]$





Function notation

$f: x \to x+3$

$f(x) = x+3$.

example 6

We define a function $g(x) = x^2 + x$.

Find and simplify: (a) $g(2)$ (b) $g(y)$

$= 2^2 + 2$

$= 6$

$= y^2 + y$

(c) $g(x+1)$ (d) $g(3x)$ (e) $4[g(x-1)] - 3$

$= (x+1)^2 + (x+1)$

$= (x+1)(x+2)$

$= (3x)^2 + (3x)$

$= 9x^2 + 3x$

$= 4[(x-1)^2 + (x-1)] - 3$

Try

example 7

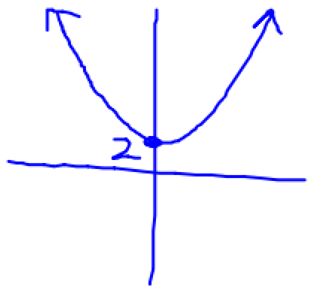
Find the largest possible domain for each function and state the corresponding range.

a $f(x) = 3x - 1$

$x \in \mathbb{R}$

$y \in \mathbb{R}$

b $f(x) = \underbrace{x^2} + 2$



$x \in \mathbb{R}$

$y \geq 2$

c $f(x) = 2^x$

$2^{-1} = \frac{1}{2}$

$2^{-2} = \frac{1}{4}$

$\vdots$

$x \in \mathbb{R}$

$y > 0$

d $f(x) = \frac{1}{x}$

$x \neq 0$

$y \neq 0$

e $f(x) = \frac{1}{x - 2}$

$x \neq 2$

$y \neq 0$

f $f(x) = \sqrt{x - 3} - 2$

$x \geq 3$

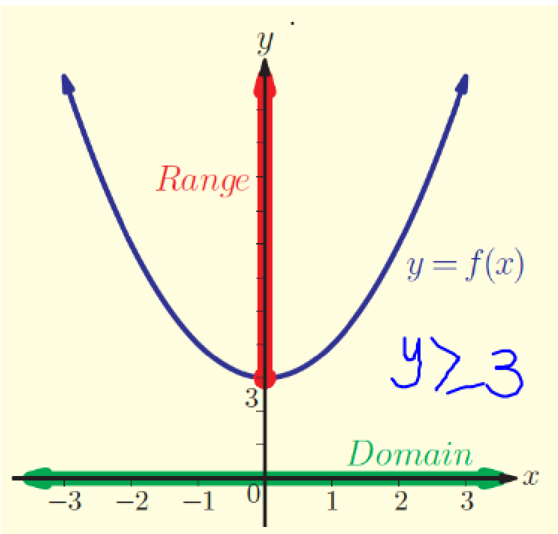
$y \geq -2$

example 8

Find the range of $f(x) = x^2 + 3$ if the domain is:

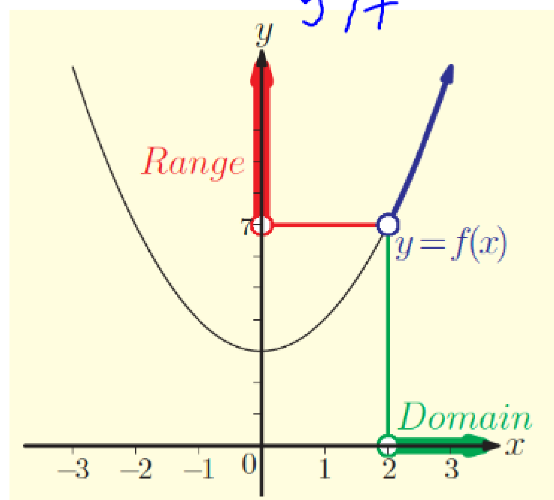
(a) $x \in \mathbb{R}$

(b) $x > 2$



$y \geq 3$

$y > 7$



When working with real numbers, the three most important reasons to restrict the domain are

- you cannot divide by zero
- you cannot square root a negative number
- you cannot take the logarithm of a negative number or zero.

log

example 9

What is the largest possible domain of $h : x \to \frac{1}{x-2} + \sqrt{x+3}$?

$\{x \in \mathbb{R} \mid x \geqslant -3 \text{ and } x \neq 2\}$ $x \neq 2$ And $-3 \geq x$

example 10

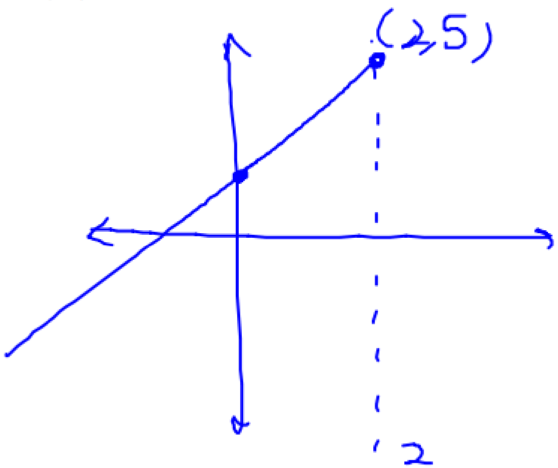
A function is defined by

$$f(x)=\begin{cases}2x+1, & x\le 2\\ k-x, & x>2\end{cases}$$

piecewise functions

(a) If $k=8$, find the range of $f(x)$.

(b) Find the value of k for which $f(x)$ is continuous.

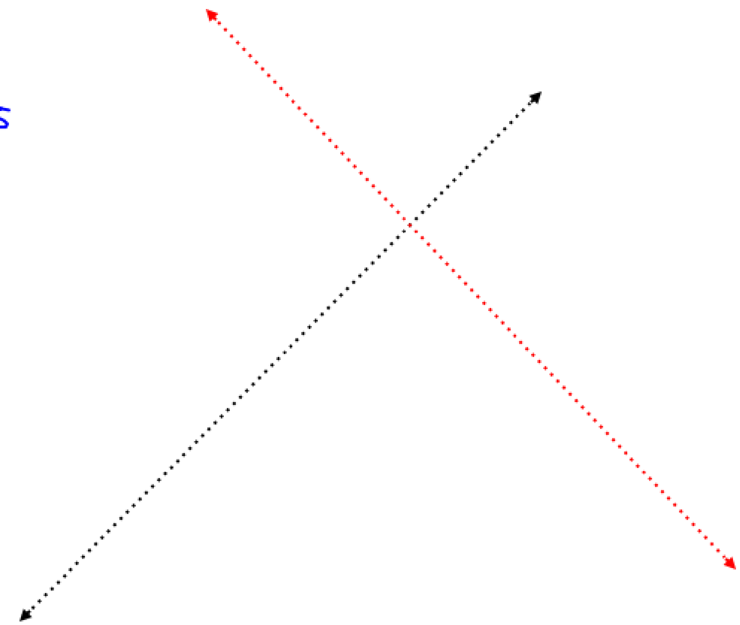


$f(2)=5$

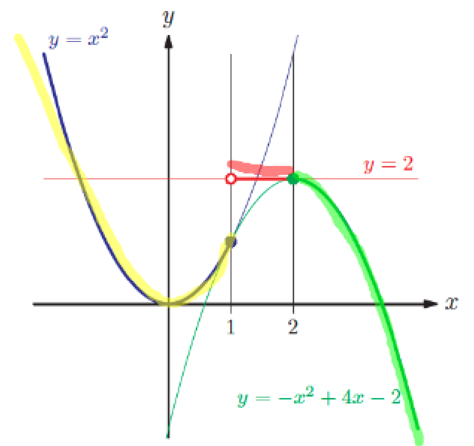
$k-\overset{2}{x}=5$

$k-2=5$

$k=7$

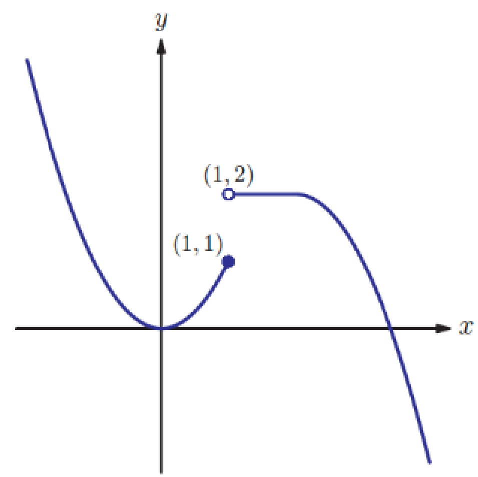


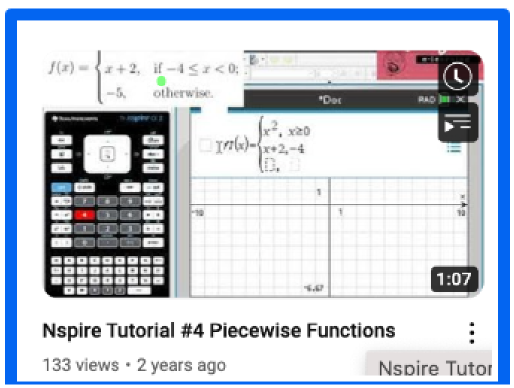
A function does not have the same rule over all of its domain. For example, we could define a function $f(x)$ over all of $\mathbb{R}$ as follows:



$$f(x) = \begin{cases} x^2, & x \le 1 \\ 2, & 1 < x < 2 \\ -x^2 + 4x - 2, & x \ge 2 \end{cases}$$

A function defined in this way is called a **piecewise** function. To sketch the graph of $f(x)$ we need to sketch all three graphs and then select the relevant graph for each part of the domain. Remember to use a closed circle when the end point is included and an open circle when it is not.





Transformations of graphs

- how some changes to functions affect their graphs
- how to use the modulus functions to transform graphs
- how to sketch complicated functions by considering them as transformations of simpler functions
- about graphs of reciprocal functions
- how the symmetries of a graph can be seen from its equation.

Translation

The graph $y = f(x) + c$ is the graph of $y = f(x)$ translated up by c units.

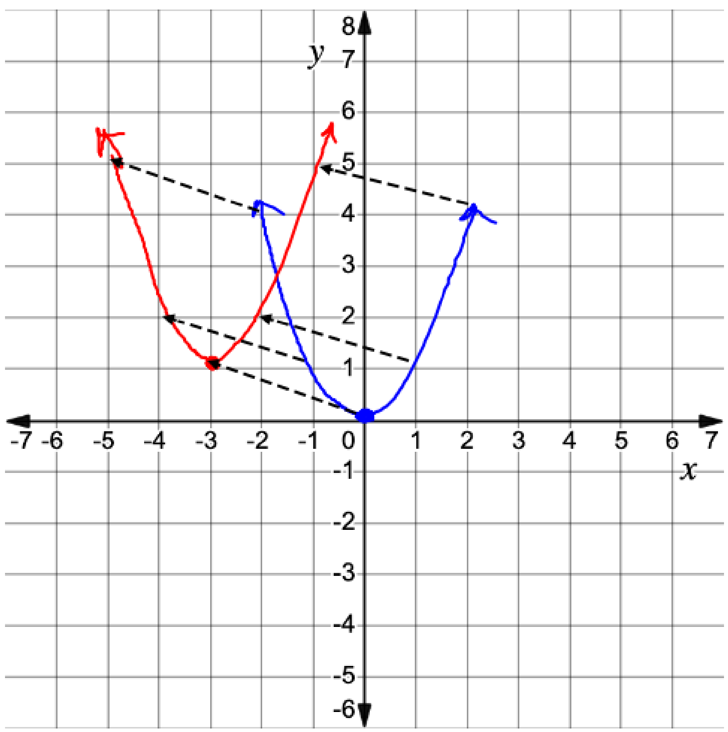
If c is negative the graph is translated down.

The graph $y = f(x + d)$ is the graph of $y = f(x)$ translated left by d units. If d is negative the graph is translated right.

example 11

Describe the transformations necessary to obtain the graph of $y = (x + 3)^2 + 1$ from the graph of $y = x^2$. State the coordinates of the image of the vertex under this translation. Sketch both graphs on the same set of axes.

$(-3, 1)$



translate the graph 3 unit left

& 1 unit up

translate by $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

Stretch / Dilation / Enlargement

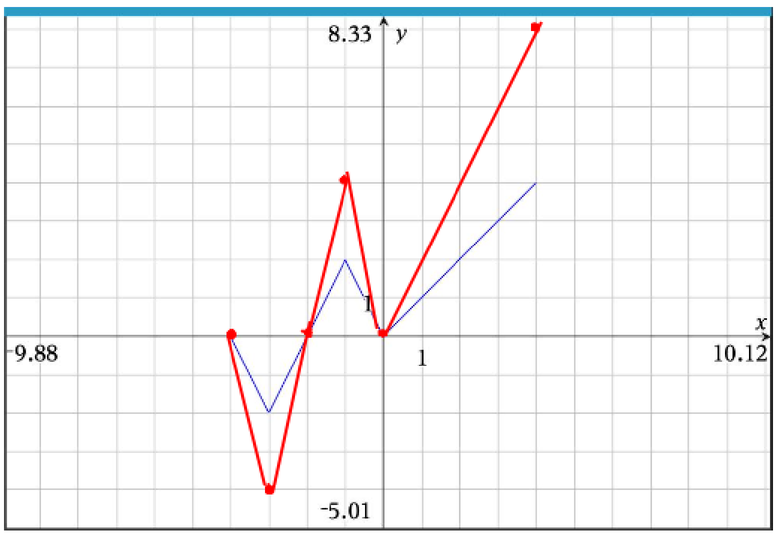
The graph $y = pf(x)$, where $p > 0$, is the graph of $y = f(x)$ *stretched* vertically relative to the x-axis (away from) with scale factor p. If $0 < p < 1$, then $y = f(x)$ is *compressed* vertically relative to the x-axis (towards). If $p < 0$ the scale factor is negative $(-p)$ and it might be easier to think of the transformation as a stretch/compression by scale factor p followed by reflection in the x-axis.

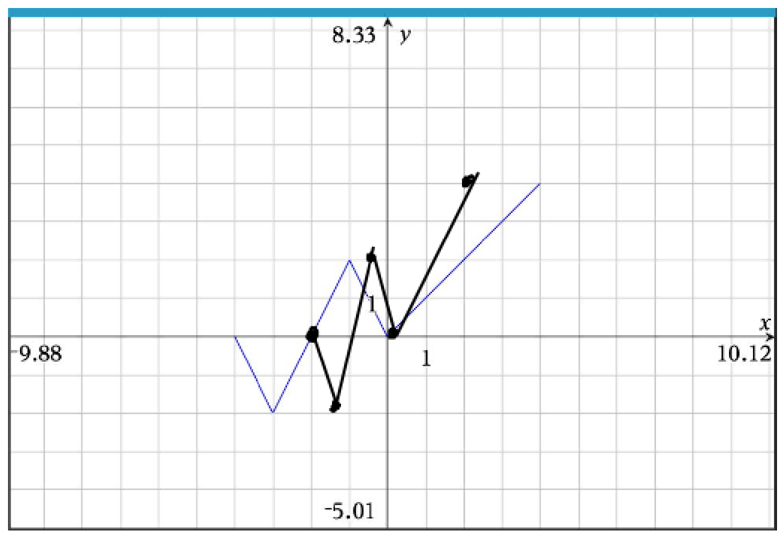
The graph $y = f(qx)$ is the graph of $y = f(x)$ stretched horizontally relative to the y-axis by scale factor $\frac{1}{q}$. This can be considered a compression relative to the y-axis (towards) when $q > 0$. When $0 < q < 1$, it is considered a stretch relative to the y-axis (away from) and when $q < 0$, the scale factor is negative $\left(-\frac{1}{q}\right)$ and it is easier to think of the transformation as a stretch/compression by scale factor $\frac{1}{q}$ followed by a reflection in the y-axis.

example 12

Graph the piecewise function $y=\begin{cases}-(2x+8), & -4\le x<-3\\ 2x+4, & -3<x\le -1\\ -2x, & -1\le x<0\\ x, & 0\le x\le 4\end{cases}$. Clearly label the zeros, intercepts and any maximum or minimum points with their coordinates. Graph the transformations and indicate the coordinates of all the labelled points of the original graph.

a $y=2f(x)$ **b** $y=f(2x)$ c) $y=2\cdot f(2x)$??





HL Only

example

Describe a transformation which transforms the graph of $y = \ln x - 1$ to the graph of $y = \ln x^4 - 4$.

example 13

Consider the graph of the function $y = f(x)$, where $-2 \leq x \leq 6$.

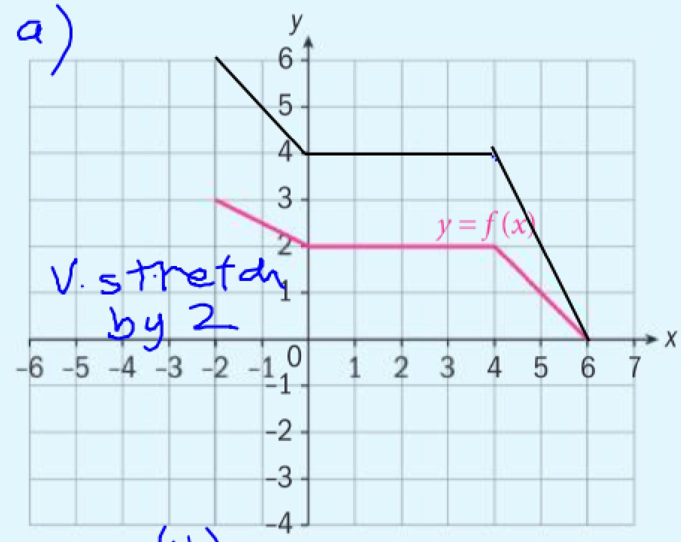
Sketch the graph of:

a $y = 2f(x)$ **b** $y = f(-x)$

c $y = f(2x)$ **d** $y = f(x - 1) - 3$

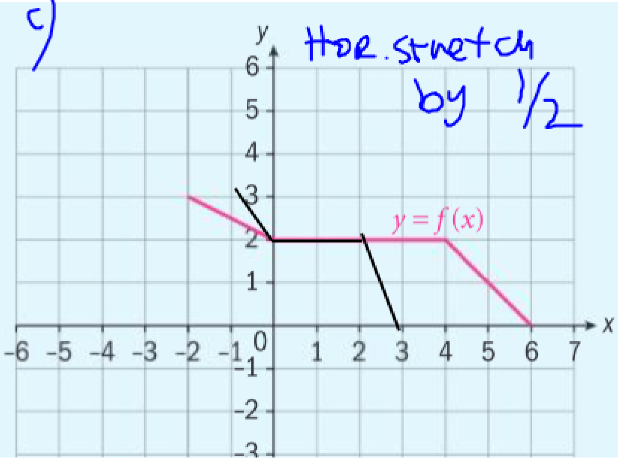
a)

V. stretch by 2



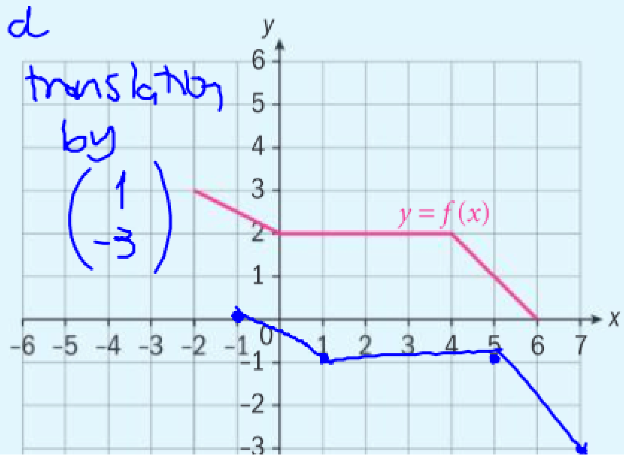
c)

Hor. stretch by 1/2



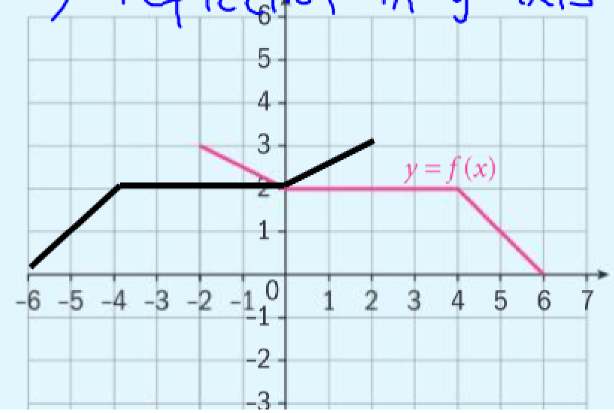
d

translation by $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$



(H)

b) Reflection in y-axis

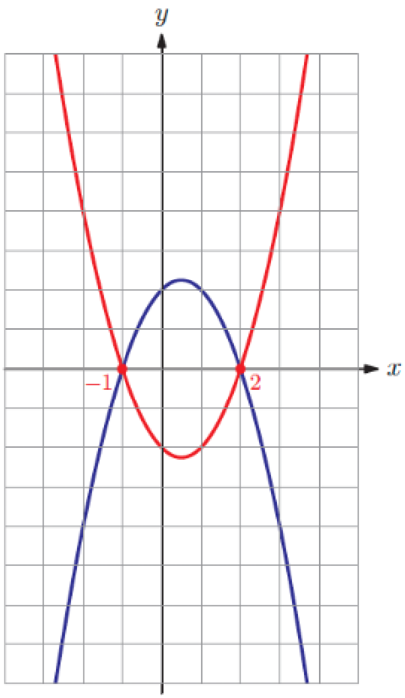


Reflection

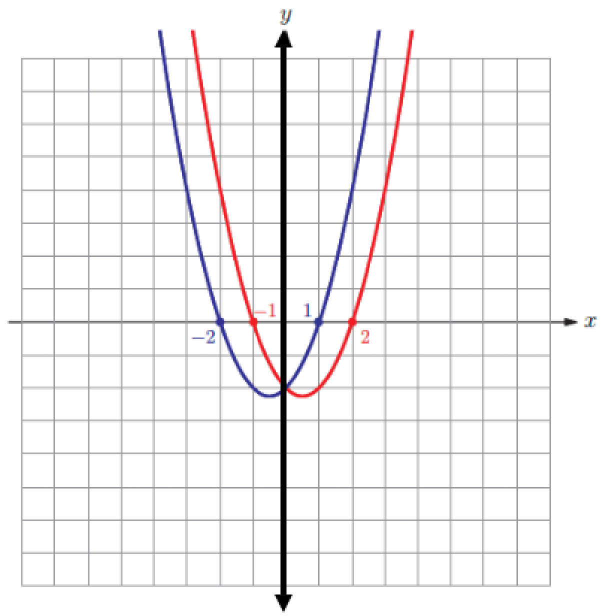
The graph $y = -f(x)$ is the graph of $y = f(x)$ reflected in the x-axis.

$y = x^2 - x - 2$

$y = -(x^2 - x - 2)$



The graph $y = f(-x)$ is the graph of $y = f(x)$ reflected in the y-axis.



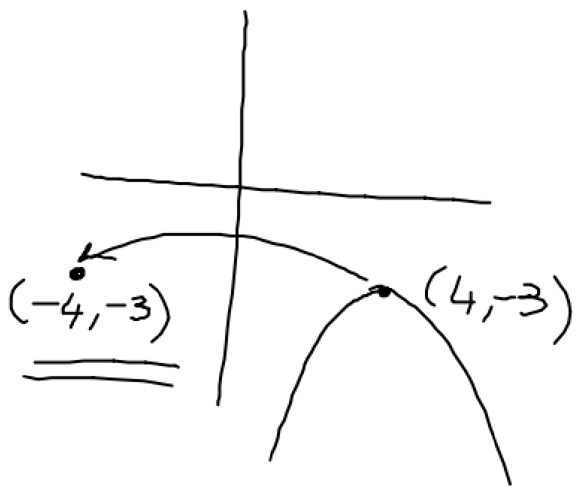
$y = x^2 - x - 2$

$y = (-x)^2 - (-x) - 2$

example 14

The graph of $y = f(x)$ has a single maximum point with coordinates $(4, -3)$. Find the coordinates of the maximum point on the graph of $y = f(-x)$.

Reflection in y-axis

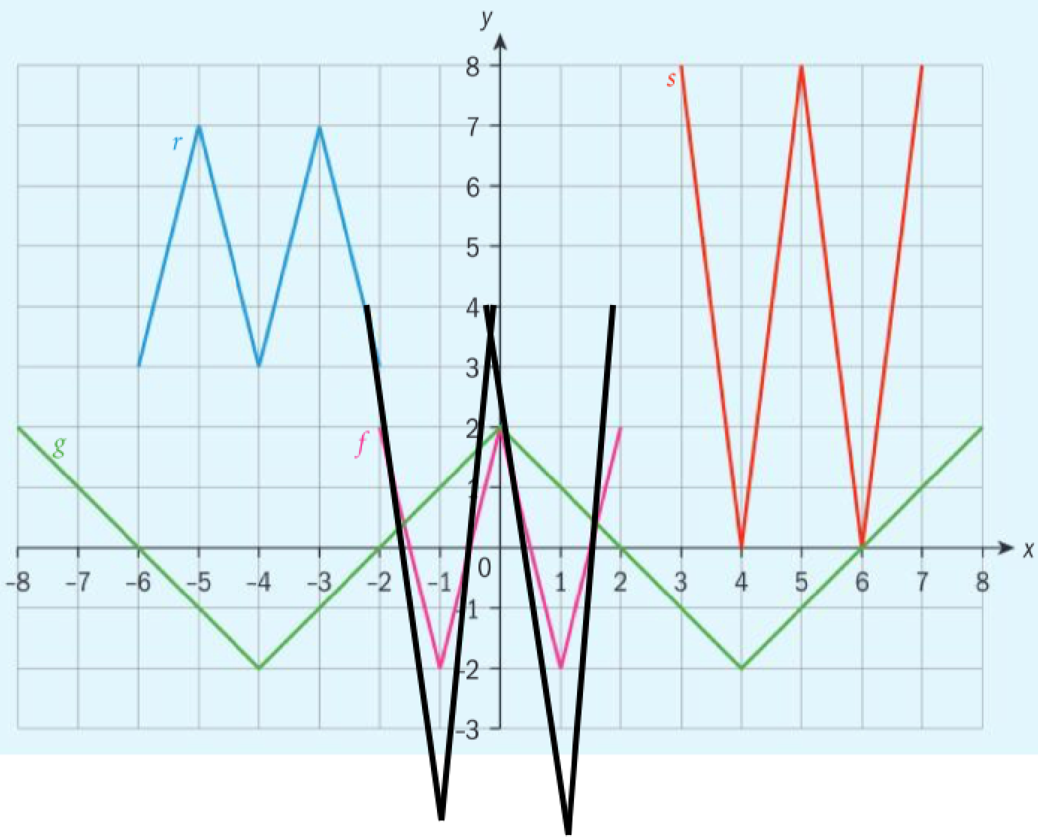


example 15

Exercise 2Q

Exercise 2S

Functions g, r and s are transformations of the graph of f. Write the functions g, r and s in terms of f.



8

$$g(x) = f\left(\frac{1}{4}x\right)$$

$$r(x) = -f(x+4)+5$$

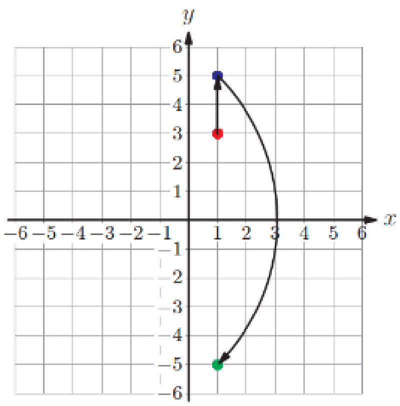
$x-(-4)$

$$s(x) = 2f(x-5)+4$$

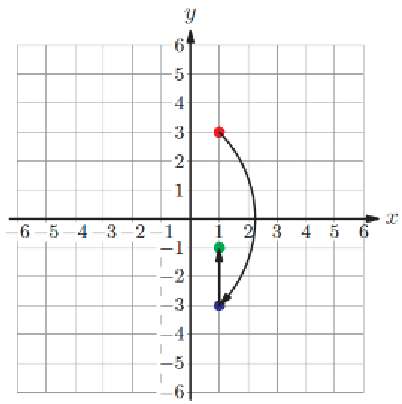
Exercise 3J

Exercise 3K

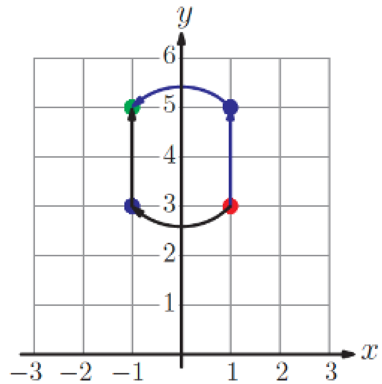
Consecutive transformations



If the point $(1,3)$ is translated two units up followed by a reflection in the x-axis (two vertical transformations) the new point is $(1,-5)$.



If $(1,3)$ is reflected in the x-axis followed by a translation two units up (two vertical transformations) the new point is $(1,-1)$.



However, if the transformations were a translation two units up and a reflection in the y-axis (one vertical transformation and one horizontal transformation), irrespective of the order the result would be $(-1,5)$.

When two horizontal or two vertical transformations are combined, the outcome depends on the order.

When one vertical and one horizontal transformation are combined, the outcome does not depend on the order.

There is a very important rule to remember when resolving horizontal or vertical transformations:

- vertical transformations follow the 'normal' order of operations as applied to arithmetic
- horizontal transformations are resolved in the opposite order to the 'normal' order of operations.

V. translation

Bedmas

V. stretch

Let us combine two vertical transformations to transform the graph of $y = f(x)$ into the graph of:

$$y = pf(x) + c$$

We can achieve this by first multiplying $f(x)$ by p and then adding on c, so this form represents a stretch / reflection followed by a translation.

$$\begin{tikzcd}
\boxed{f(x)} \arrow[r, "\text{Stretch Factor } p", "\textit{multiply function by } p"'] & \boxed{pf(x)} \arrow[r, "\text{Translate } \begin{pmatrix} 0 \\ c \end{pmatrix}", "\textit{add } c \textit{ to function}"'] & \boxed{pf(x) + c}
\end{tikzcd}$$

If we combine two horizontal transformations, we can transform the graph of $y = f(x)$ into the graph of:

$$y = f(qx + d)$$

We can achieve this by first replacing x with $x + d$ and then replacing all occurrences of x by qx, so this represents a translation followed by a stretch / reflection.

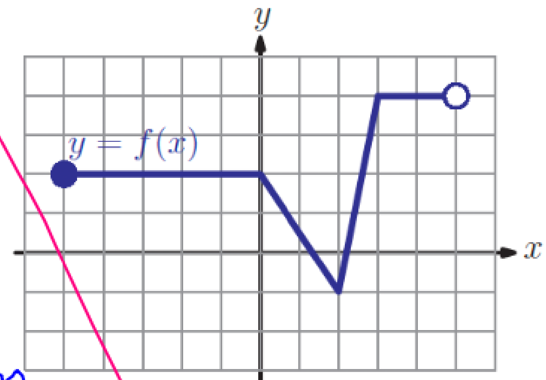
$$\begin{tikzcd}[column sep=huge]
\boxed{f(x)} \arrow[r, "\text{Translate } \begin{pmatrix} -d \\ 0 \end{pmatrix}", "\textit{Replace } x \textit{ with } x+d"'] & \boxed{f(x+d)} \arrow[r, "\text{Stretch Factor } q", "\textit{Replace } x \textit{ with } qx"'] & \boxed{f(qx+d)}
\end{tikzcd}$$

Following the normal order of operations, you would expect to resolve 'qx' before '$+d$' but you resolve the transformation in the opposite order.

example 15

Below is the graph of $y = f(x)$. Sketch the corresponding graph for $y = 3 - 2f(2x+1)$.

$y = 3 - 2f(2x+1)$



V. translation upward by 3 4

V. strecth by factor 2 / V. Reflection 3

H. stretch by factor $\frac{1}{2}$ 2

H. translation to left by 1 1

$2\left(x + \frac{1}{2}\right)$

example

16 Let $f(x) = 2x^2 - 8x + 6$, for $x \in \mathbb{R}$.

(a) Write down the value of $f(0)$. $=6$

(b) Solve the equation $f(x) = 0$.

$2x^2-8x+6=0$

$2(x^2-4x+3)=0$

$(x-1)(x-3)=0$

$x=1$

$x=3$

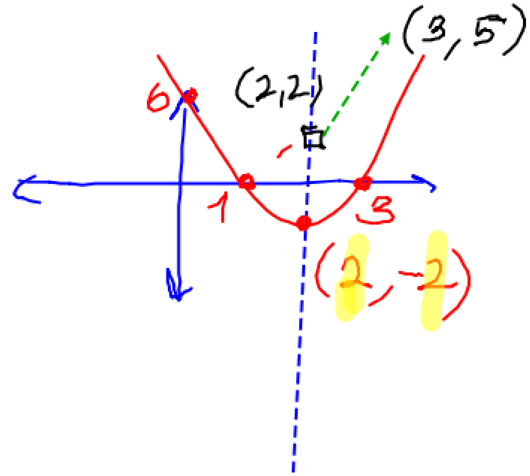
The function f can be written in the form $f(x) = a(x-h)^2 + k$.

(c) Find the values of a, h and k. $=2(x-2)^2-2$

(d) For the graph of f, write down:

(i) the coordinates of the vertex;

(ii) the equation of the axis of symmetry. $x=2$



$g(x)=-f(x)$

The graph of a function g is obtained from the graph of f by a reflection in the x-axis, followed by a translation by the vector $\begin{array}{cl}\begin{pmatrix}1\\3\end{pmatrix}. & \begin{array}{l}\leftarrow x\\ \leftarrow y\end{array}\end{array}$

$\Rightarrow -f(x-1)+3$

(e) Find $g(x)$, giving your answer in the form $g(x) = px^2 + qx + r$.

$=-\left[2(x-1-2)^2+3\right]$

$g(x)=-2(x-3)^2+5$

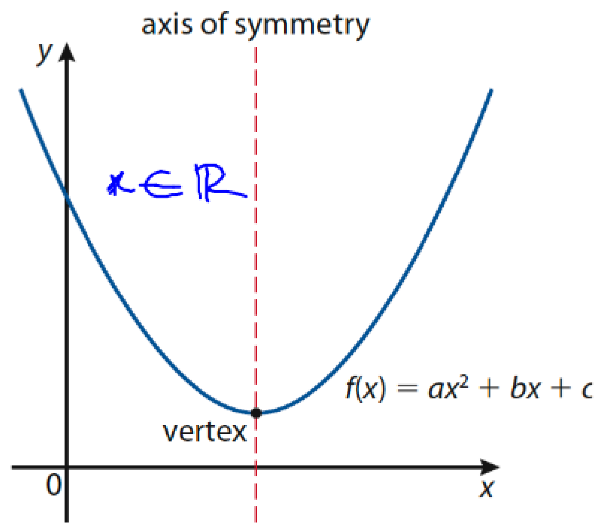
Polynomials

Microconcepts

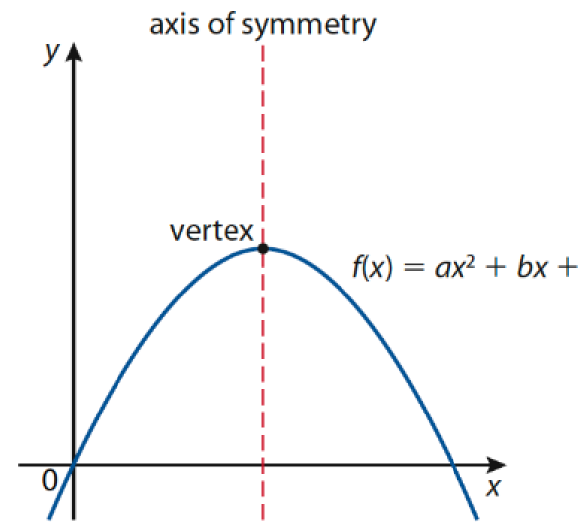
- Quadratic function and graph
- Quadratic equations and inequalities
- Discriminant
- Complex numbers
- Modulus of a complex number
- Operations with complex numbers
- Powers and roots of complex numbers
- Polynomial functions and their graphs
- Operations on polynomials
- Linear combination of two polynomials
- Factor and remainder theorem
- The fundamental theorem of algebra
- Polynomial equations
- Sum and product of the roots of polynomial equations
- Polynomial inequalities

Quadratics Equations and Inequalities

General Form (Standard Form)



If $a > 0$ then the parabola opens upward.



If $a < 0$ then the parabola opens downward.

a: leading coefficient

example

17

Exercise 3A

Solve the following quadratic equations by factorization.

a $x^2 + 3x - 40 = 0$ **b** $2x^2 - 7x - 4 = 0$

$x \quad +8$

$x \quad -5$

$(x-5)(x+8)=0$

$x=5$

$x=-8$

$2x \quad -4$

$x \quad +1$

$-8x$

$+ \quad x$

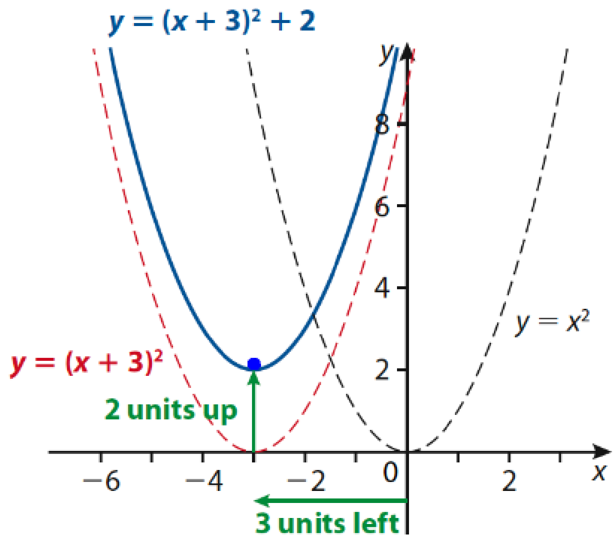
$-7x$

$(2x+1)(x-4)=0$

$x=-\frac{1}{2} \qquad x=4$

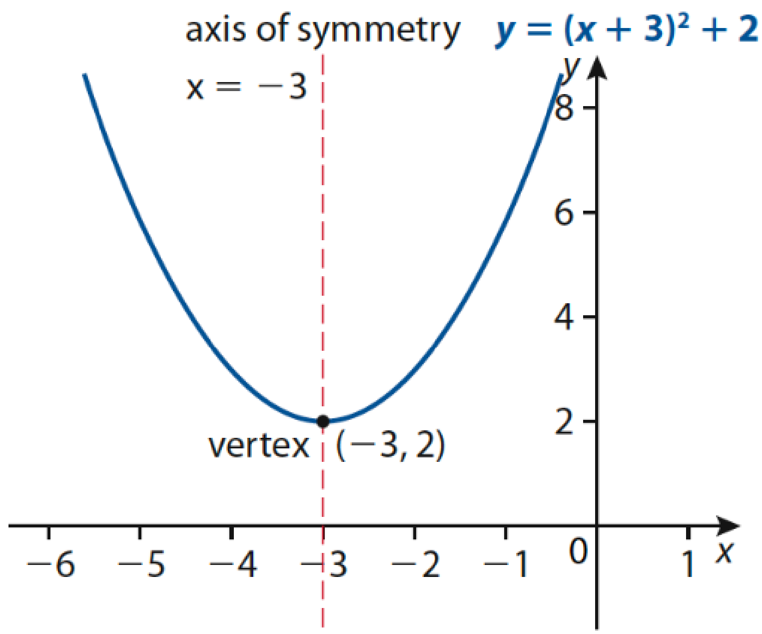
Exercise 3P

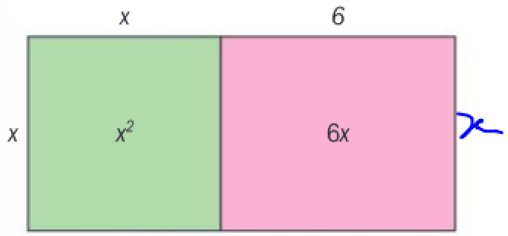
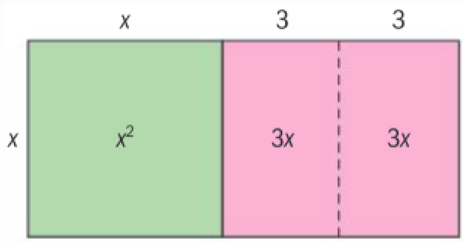
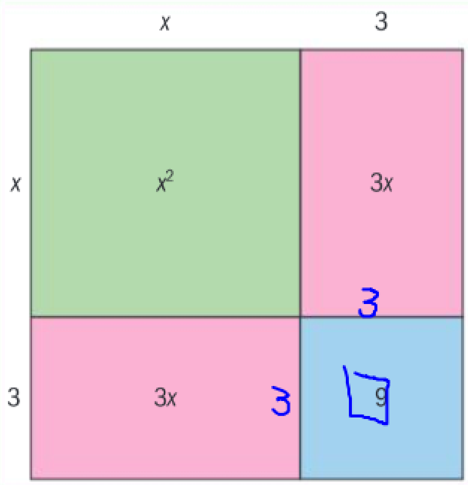
Vertex Form



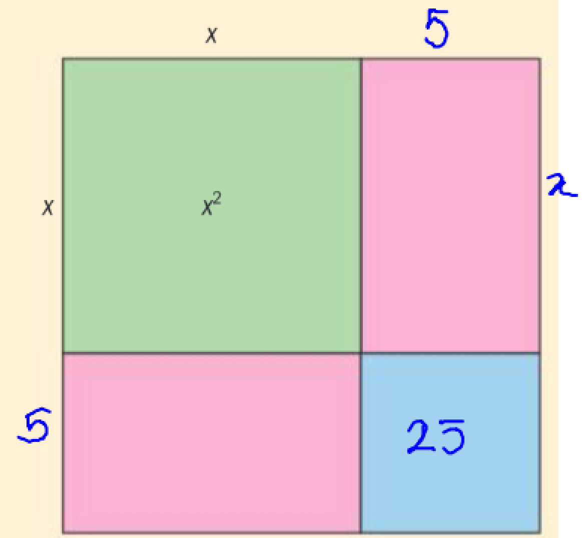
$x \in \mathbb{R}$

$y \geq 2$



	The area of the region shaded in green is x^2 and the area of the region shaded in pink is $6x$. Thus, the area of the whole rectangle is $x^2 + 6x$.
	Divide the pink region into two equal parts.
	Rearrange the parts. The large square formed has area $(x + 3)^2$ So the expression $x^2 + 6x$ becomes a perfect square when you add 9.

$x^2 + 10x + c$



$x^2 + 10x + C$

$\downarrow$

$\frac{10}{2} = 5$

$\downarrow$

$5^2 = 25$

example 18

For each quadratic expression, find the value of k which makes the expression a perfect square. Hence, write each in a factorized form.

a $x^2 + \overset{2 \cdot 4 \cdot x}{8x} + k$ **b** $x^2 - 7x + k$ **c** $9x^2 + 3x + k$

$\left(\frac{8}{2}\right)^2 = 16$

$\left(-\frac{7}{2}\right)^2 = \frac{49}{4}$

$(3x)^2 + 2 \cdot (3x)\left(\frac{1}{2}\right) + k$

$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\left(3x + \frac{1}{2}\right)^2$

$(a+b)^2 = a^2 + 2ab + b^2$

example 19

Exercise 3B

Solve the following equations by completing the square. Give your answers in exact form.

a $x^2 + 5x - 24 = 0$ **b** $2x^2 - 7x + 1 = 0$ HW

$$x^2+5x+\left(\frac{5}{2}\right)^2-24 = \frac{25}{4}$$

$$\left(x+\frac{5}{2}\right)^2 = 24+\frac{25}{4}$$

$$\sqrt{\left(x+\frac{5}{2}\right)^2} = \sqrt{\frac{121}{4}}$$

$$x+\frac{5}{2} = \pm\frac{11}{2}$$

$$x = -\frac{5}{2}+\frac{11}{2} \qquad x = -\frac{5}{2}-\frac{11}{2}$$

$$= 3 \qquad x = -8$$

Exercise 3S

HL Only

Solve

$$ax^2 + bx + c = 0$$

$$\cancel{a}\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$\boxed{x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

4a

4a

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\mp\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \mp \frac{\sqrt{b^2 - 4ac}}{2a}$$

Exercise 3C

The zeros of $ax^2 + bx + c$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

example 20

Use the quadratic formula to solve the following equations. Leave your answers in exact form.

a $x^2 + 11x + 24 = 0$ **b** $3x^2 + 8x + 4 = 0$ **c** $5x^2 - 2x - 1 = 0$ **d** $2x^2 + x + 1 = 0$

$a=1$
$b=11$
$c=24$

$$x_1 = \frac{-11 + \sqrt{11^2 - 4 \cdot 1 \cdot 24}}{2}$$

$$= \frac{-11 + \sqrt{25}}{2}$$

$$= -3$$

$$x_2 = \frac{-11 - \sqrt{25}}{2}$$

$$= -8$$

$a=2 \quad b=1 \quad c=1$

$$x = \frac{-1 \mp \sqrt{1 - 4 \cdot 2}}{4}$$

$$x_1 = \frac{-1 + \sqrt{-7}}{4} \qquad x_2 = \frac{-1 - \sqrt{-7}}{4}$$

No REAL solution

Exercise 3T
Exercise 3U

example 21

HL Only

Solve the quadratic equation $px^2 + 3 = 3px + x$, $p \neq 0$, by using the quadratic formula. Express x in terms of the real parameter p.

$$px^2 - x(3p+1) + 3 = 0$$

$$x_{1,2} = \frac{(3p+1) \mp \sqrt{(3p+1)^2 - 12p}}{2p}$$

$$9p^2 + 6p + 1 - 12p$$
$$= 9p^2 - 6p + 1$$
$$= (3p-1)^2$$

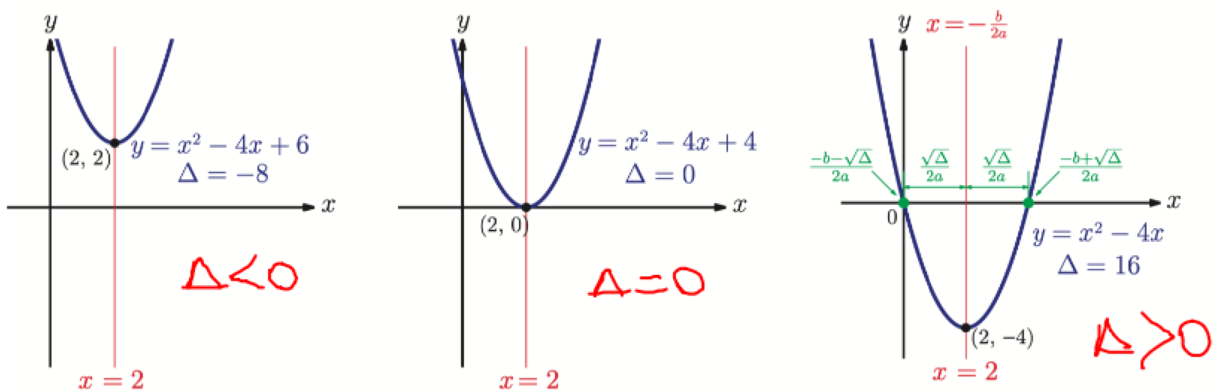
$$x_{1,2} = \frac{3p+1 \mp (3p-1)}{2p}$$

$$x_1 = \frac{6p}{2p} = 3 \qquad x_2 = \frac{2}{2p} = \frac{1}{p}$$

The line of symmetry of $y = ax^2 + bx + c$ is

$$x = -\frac{b}{2a}$$

The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic (often symbolised by the Greek letter Δ) and Δ $\frac{\sqrt{b^2 - 4ac}}{2a}$ is the distance of the zeros from the line of symmetry $x = -\frac{b}{2a}$.



For a quadratic expression $ax^2 + bx + c$, the discriminant is:
$\Delta = b^2 - 4ac$.

- If $\Delta < 0$ the expression has no real zeros
- If $\Delta = 0$ the expression has one (repeated) zero double roots.
- If $\Delta > 0$ the expression has two distinct real zeros

example 22

Without solving, determine the nature of the roots of each equation. [ie Δ]

a $5x^2 + 7x - 1 = 0$ **b** $25x^2 + 49 = 70x$ **c** $2x^2 + \frac{7}{4}x + \frac{1}{2} = 0$

$a = 5$
$b = 7$
$c = -1$

$\Delta = 49 - 4 \cdot 5 \cdot (-1)$
$= 49 + 20$
$= 69 > 0$

It has 2 slns.
roots
zeros
x-intercepts

$25x^2 - 70x + 49 = 0$

$\Delta = 0$
Double Root

$\Delta = \frac{49}{16} - 4 \cdot \cancel{2} \cdot \frac{1}{\cancel{2}} < 0$

No Real solution

example 23

Find the value(s) of the real parameter p so that

a $2x^2 - 3x + p = 0$ has two real roots $\Delta > 0$

b $px^2 + p = 13x$ has one real repeated root $\Delta = 0$

c $(p + 2)x^2 + 2px = 1 - p$ has no real roots. $\Delta < 0$

a) $\Delta = (-3)^2 - 4 \cdot 2 \cdot p > 0$

$9 - 8p > 0$

$9 > 8p$

$p < \frac{9}{8}$

c) $\Delta = (2p)^2 - 4 \cdot (p+2) \cdot (p-1) > 0$

$4p^2 - 4(p^2 + p - 2) > 0$

$\cancel{4p^2} - \cancel{4p^2} - 4p + 8 > 0$

$4p < 8$

$p < 2$

b) $px^2 - 13x + p = 0$

$a = p$

$b = -13$

$c = p$

$\Delta = (-13)^2 - 4 \cdot p^2 = 0$

$169 - 4p^2 = 0$

$4p^2 = 169$

$p^2 = \frac{169}{4}$

$p = \mp \frac{13}{2}$

example 24

Exercise 3D

Find the values of r for which the equation $x^2 + 3rx + 1 = 0$ has $\Delta = 9r^2 - 4$

a two distinct real roots   **b** one real repeated root   **c** no real roots.

$\Delta > 0$ $\Delta = 0$ $\Delta < 0$

$a = 1$
$b = 3r$
$c = 1$

$\Delta = 9r^2 - 4$

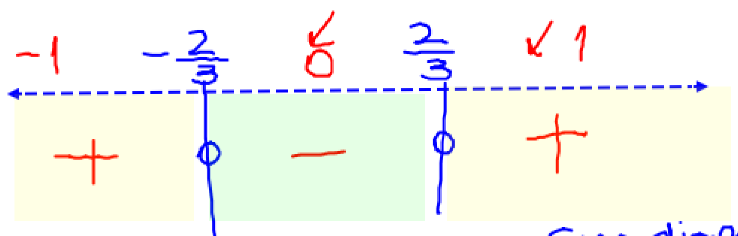
$9r^2 - 4 = 0$

$9r^2 = 4$

$r^2 = \frac{4}{9}$

$r = \mp \frac{2}{3}$

$\boxed{-\frac{2}{3} < r < \frac{2}{3}}$



Sign diagram

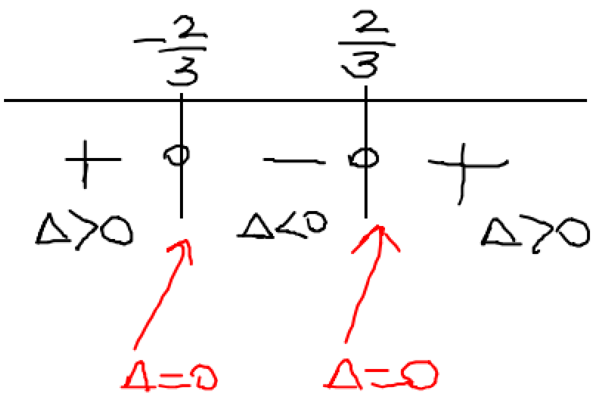
a) $r < -2/3$

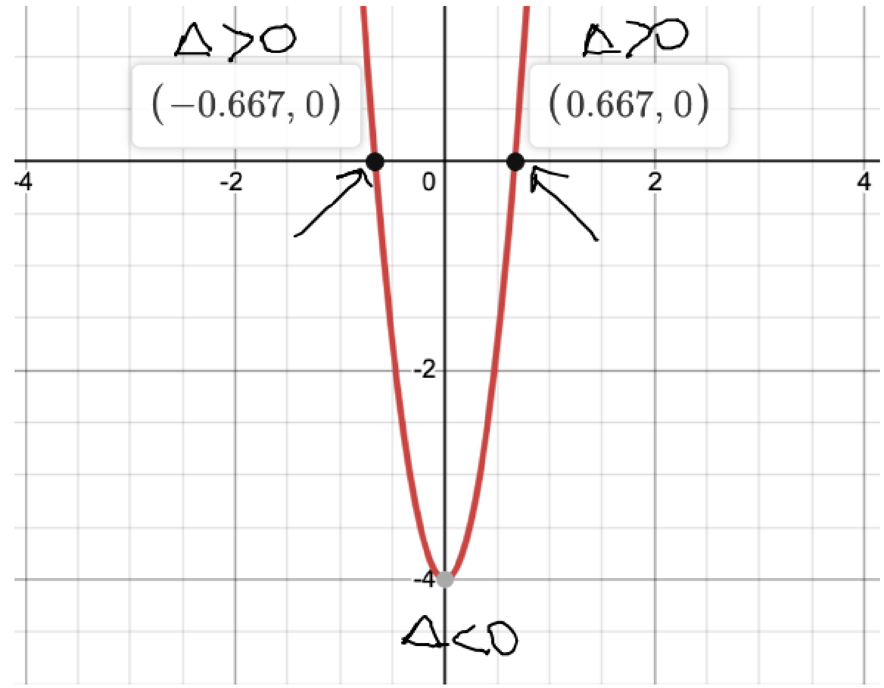
OR

$r > 2/3$

Exercise 3V

$\Delta = 9r^2 - 4$ $\qquad r = \mp \frac{2}{3}$





example 25

Find the exact values of k for which the quadratic equation $kx^2 - (k+2)x + 3 = 0$ has a repeated root. $\Delta = 0$

$a = k$
$b = -(k+2)$
$c = 3$

$$\Delta = (k+2)^2 - 4 \cdot 3 \cdot k = 0$$

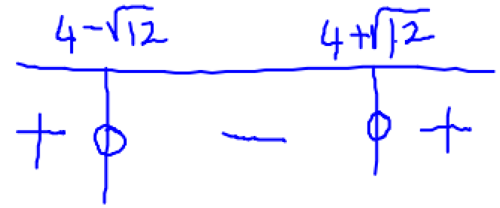
$$k^2 + 4k + 4 - 12k = 0$$

$$k^2 - 8k + 4 = 0$$

$$k^2 - 8k + 16 + 4 = 16$$

$$(k-4)^2 = 12$$

$$k = 4 \mp \sqrt{12}$$



example 26

Solve the quadratic inequality $2x^2 - 5x - 3 \geq 0$.

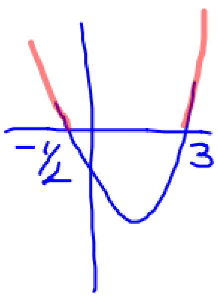
Exercise 3E

$2x \longrightarrow -3$

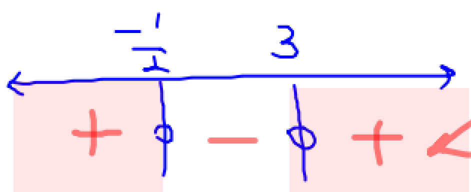
$x \longrightarrow +1$

$(2x+1)(x-3) \geq 0$

$x = -\frac{1}{2} \quad x = 3$



Sign Diagram



$x \leq -\frac{1}{2}$ OR $x \geq 3$

Exercise 3W

For a quadratic function with $\Delta < 0$:

if $a > 0$ then $y > 0$ for all x

if $a < 0$ then $y < 0$ for all x

example 27

Let $y = -3x^2 + kx - 12$.

Find the values of k for which $y < 0$ for all x.

$a = -3$

$b = k$

$c = -12$

$\Delta < 0$

$\Delta = b^2 - 4ac$

$k^2 - 4\cdot(-3)(-12) < 0$

$k^2 - 144 < 0$

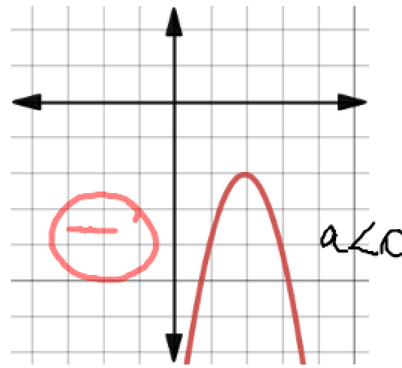
$(k-12)(k+12) < 0$

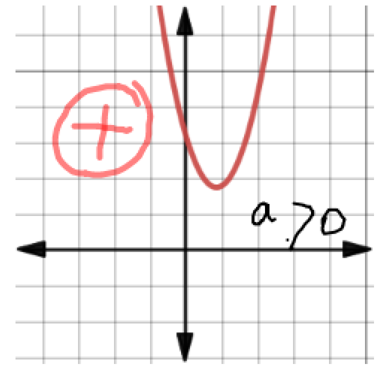
$k = 12$

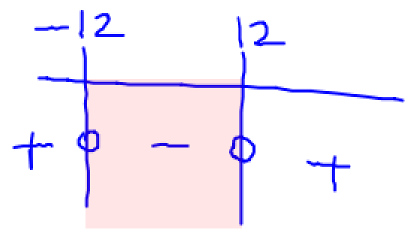
$k = -12$

$a^2 - b^2 = (a-b)(a+b)$

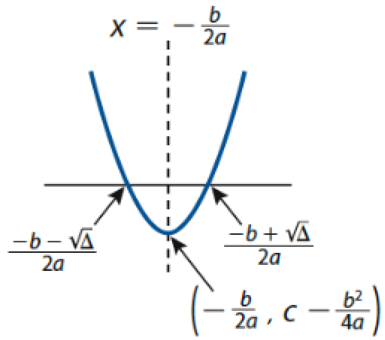
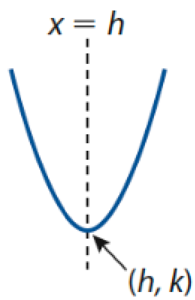
$\Delta < 0$

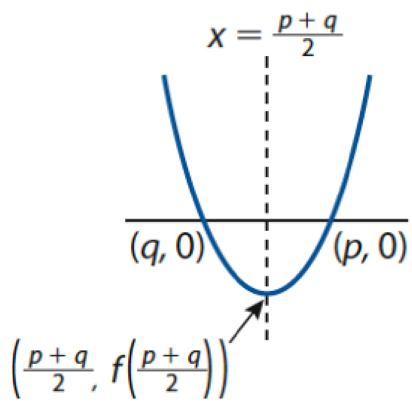
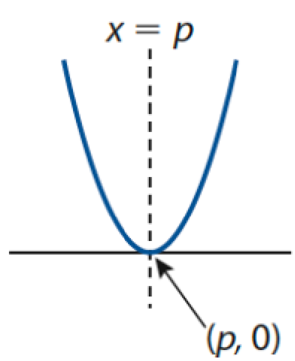






$-12 < k < 12$

Quadratic function, $a \neq 0$	Graph of function	Results
General form $f(x) = ax^2 + bx + c$ $\Delta = b^2 - 4ac$ (discriminant)	Parabola opens up if $a > 0$ Parabola opens down if $a < 0$  If $\Delta \geqslant 0$, f has x-intercept(s) If $\Delta < 0$, f has no x-intercept(s)	Axis of symmetry is $x = -\dfrac{b}{2a}$ If $\Delta \geqslant 0$, f has x-intercept(s): $\left(\dfrac{-b \pm \sqrt{\Delta}}{2a}, 0\right)$ Vertex is: $\left(-\dfrac{b}{2a}, c - \dfrac{b^2}{4a}\right)$
Vertex form $f(x) = a(x - h)^2 + k$		Axis of symmetry is $x = h$ Vertex is (h, k)

Quadratic function, $a \neq 0$	Graph of function	Results
Factorized form (two distinct rational zeros) $f(x) = a(x-p)(x-q)$		Axis of symmetry is $x = \dfrac{p+q}{2}$ x-intercepts are: $(p, 0)$ and $(q, 0)$
Factorized form (one rational zero) $f(x) = a(x-p)^2$		Axis of symmetry is $x = p$ Vertex and x-intercept is $(p, 0)$

Exercise 3N Exercise 3O

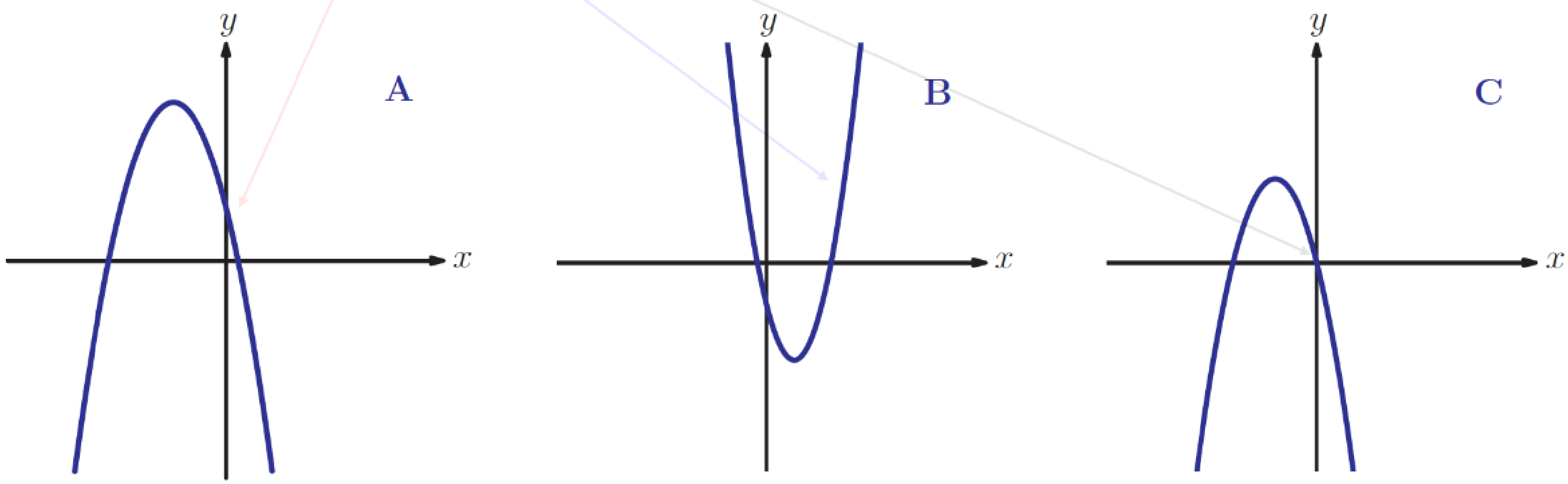
example 28

Match each equation to the corresponding graph, explaining your reasons.

(a) $y = 3x^2 - 4x - 1$

(b) $y = -2x^2 - 4x$

(c) $y = -x^2 - 4x + 2$

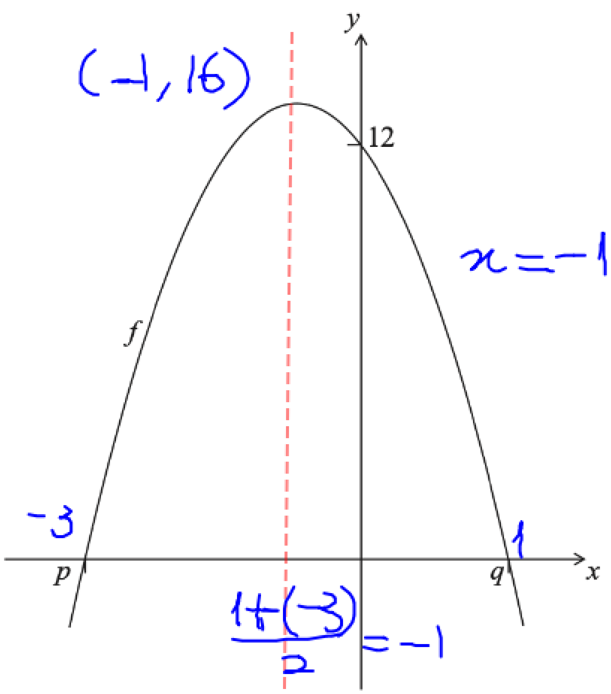


example 29

[Maximum mark: 15]

Let $f(x) = a(x+3)(x-1)$. The following diagram shows part of the graph of f.

-4



The graph has x-intercepts at $(p, 0)$ and $(q, 0)$, and a y-intercept at $(0, 12)$.

(a) (i) Write down the value of p and of q.

(ii) Find the value of a.

(b) Find the equation of the axis of symmetry of the graph of f.

$x = -1$

(c) Find the largest value of f. $f(-1) = -4(2)(-2) = 16$

The function f can also be written as $f(x) = a(x-h)^2 + k$.

(d) Find the value of h and of k.

$= -4(x+1)^2 + 16$

example
30

There are five values of x that satisfy the equation $(x^2 - 5x + 5)^{x^2+4x-60} = 1$.

Determine these five values of x.

$$\boxed{\begin{gathered}(\text{nonzero})^{0} = 1 \\ (1)^{\text{nonzero}} = 1 \\ (-1)^{\text{even}} = 1\end{gathered}}$$

$\longrightarrow (-1)^{\text{odd}} = -1$

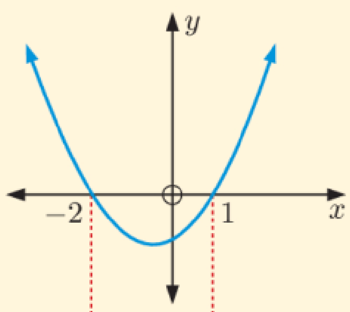
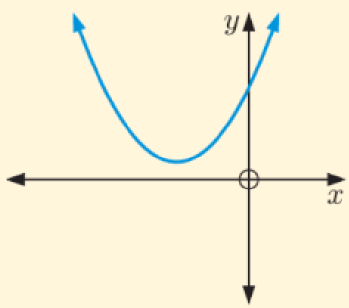
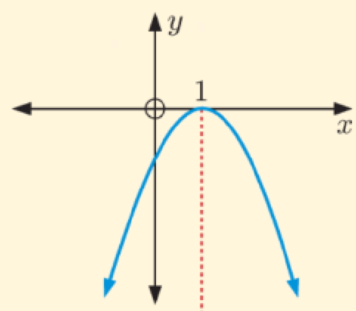
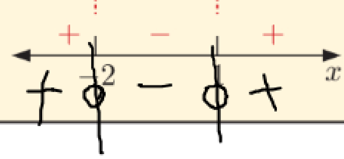
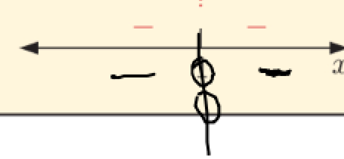
SIGN DIAGRAMS

A **sign diagram** is a number line which indicates the values of x for which a function is negative, zero, positive, or undefined.

A sign diagram consists of:

- a **horizontal line** which represents the x-axis
- **positive** (+) and **negative** (−) signs indicating where the graph is **above** and **below** the x-axis respectively
- the **zeros** of the function, which are the x-intercepts of its graph.

Consider the three functions below:

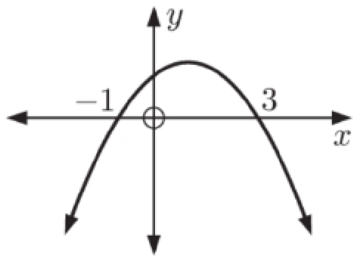
Function	$y = (x+2)(x-1)$	$y = (x+3)^2 + 2$	$y = -2(x-1)^2$
Graph			
Sign diagram			

example

31

Draw a sign diagram for each graph:

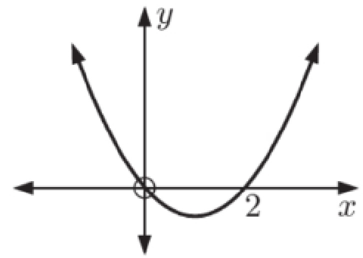
a



−1 3

− 0 + 0 −

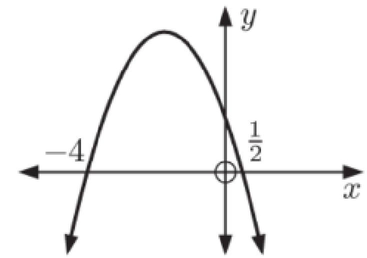
b



0 2

+ 0 − 0 +

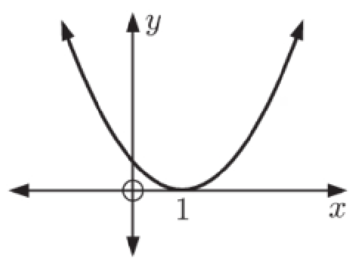
c



−4 1/2

− 0 + 0 −

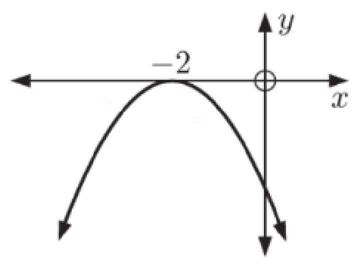
d



1

+ 0 0 +

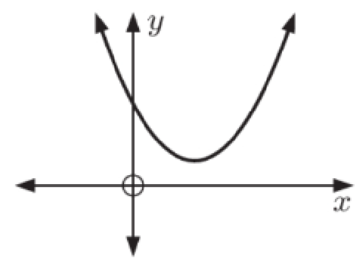
e



−2

− 0 0 −

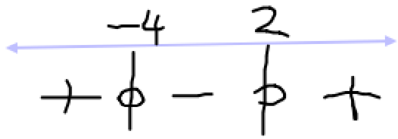
f



+

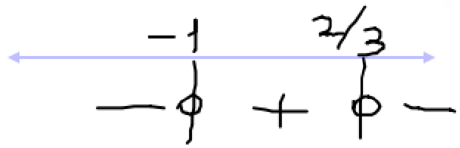
example 32 Draw a sign diagram for:

$(x+4)(x-2)$

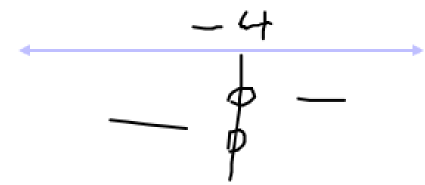


$(2-3x)(x+1)$

$-(3x-2)(x+1)$



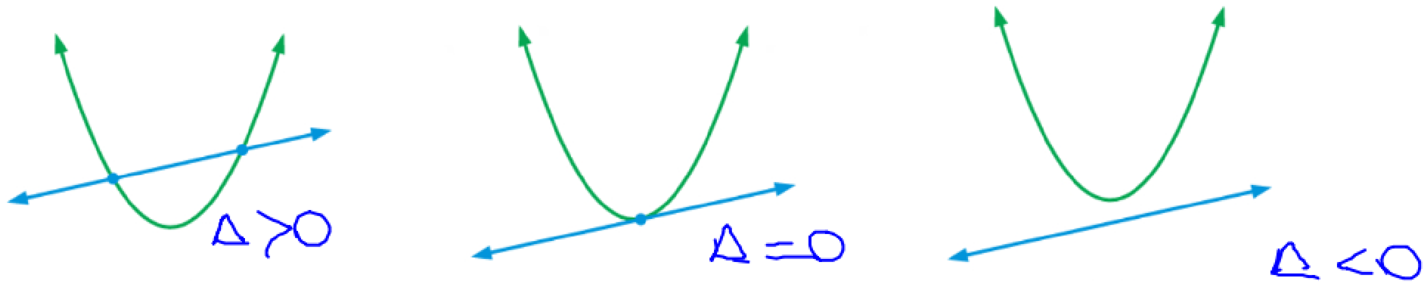
$-3(x+4)^2$



Intersection of graphs

Consider the graphs of a quadratic function and a linear function on the same set of axes.

There are three possible scenarios for intersection:



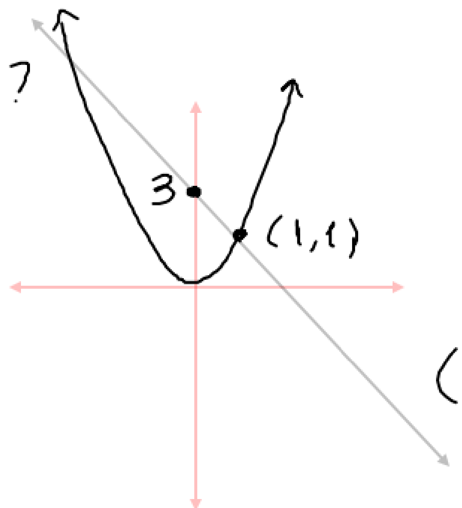
cutting (2 points of intersection) | **touching** (1 point of intersection) | **missing** (no points of intersection)

If the line *touches* the curve, we say that the line is a **tangent** to the curve.

The x-coordinates of any intersection points of the graphs can be found by solving the two equations **simultaneously**.

example 33

The graph of $y_1 = x^2$ intersects the graph of $y_2 = 3 - 2x$ at the point (1 , 1) and one other point. Find the coordinates of the other point.



$y_1 = y_2$

$x^2 = 3 - 2x$

$x^2 + 2x - 3 = 0$

$(x-1)(x+3) = 0$

$x = 1 \quad x = -3$

$y = (-3)^2 = 9$

example 34

Find the coordinates of the point(s) of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$x^2 - x - 18 = x - 3$

$x^2 - 2x - 15 = 0$

$(x-5)(x+3) = 0$

$x = 5$ $\quad$ $x = -3$

$y = 2$ $\quad$ $y = -6$

$(5, 2)$ $\quad$ $(-3, -6)$

example

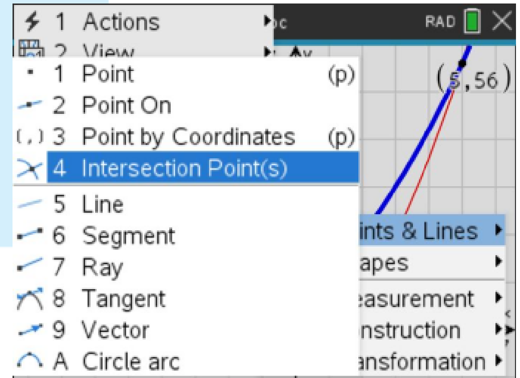
35

Consider the curves $y = x^2 + 5x + 6$ and $y = 2x^2 + 2x - 4$.

a Solve for x: $x^2 + 5x + 6 = 2x^2 + 2x - 4$.

b Graph the two curves on the same set of axes.

c Hence solve for x: $x^2 + 5x + 6 > 2x^2 + 2x - 4$.



$f_1 = f_2$

$\cancel{x^2} + \cancel{5x} + \cancel{6} = 2x^2 + 2x - 4$

$-x^2 \quad -5x \quad -6 \qquad -x^2 - 5x - 6$

$\boxed{0 > f_2 - f_1}$

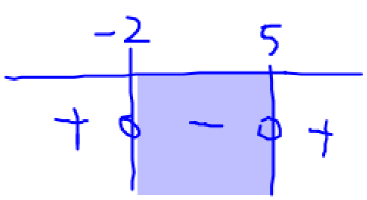
$0 = x^2 - 3x - 10$

$= (x-5)(x+2)$

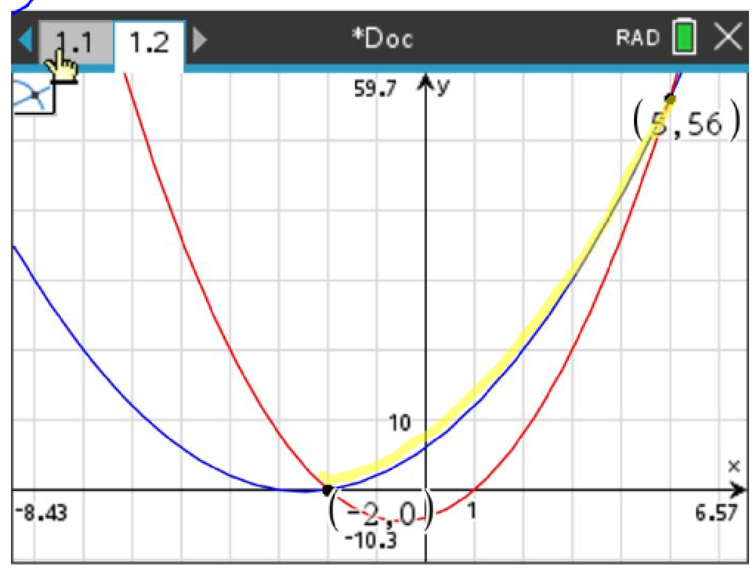
$x = 5 \quad x = -2$

$-2 < x < 5$

c)



c)



example 36

$y = 2x + k$ is a tangent to $y = 2x^2 - 3x + 4$. Find k.

$$2x^2 - 3x + 4 = 2x + k$$

$$2x^2 - 5x + 4 - k = 0$$

$$\Delta = b^2 - 4ac = 0$$

$$(-5)^2 - 4 \cdot 2 \cdot (4-k) = 0$$

$$25 - 8(4-k) = 0$$

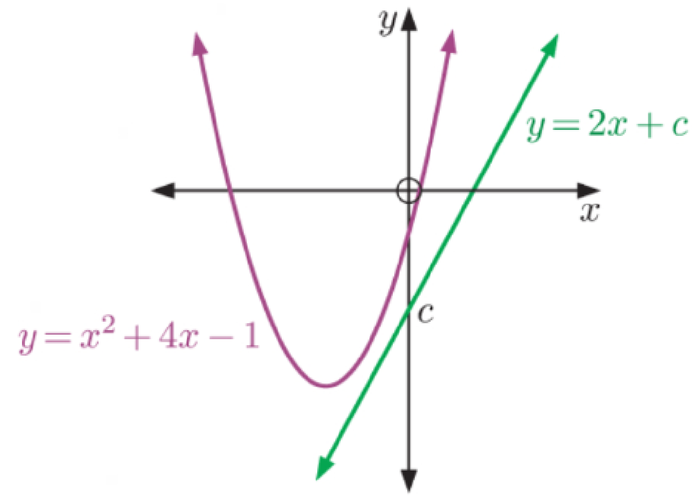
$$25 - 32 + 8k = 0$$

$$8k = 7$$

$$k = 7/8$$

example 37

Consider the curve $y = x^2 + 4x - 1$ and the line $y = 2x + c$. Find the values of c for which the line:



a meets the curve twice $x > -2$

b is a tangent to the curve $x = -2$

c does not meet the curve. $x < -2$

$x^2 + 4x - 1 = 2x + c$

$x^2 + 4x - 2x - 1 - c = 0$

$x^2 + 2x - 1 - c = 0$

$\Delta = 4 - 4(-1 - c) = 0$

$4 + 4 + 4c = 0$

$4c = -8$

$c = -2$

-2

$-\ 0\ +$

example

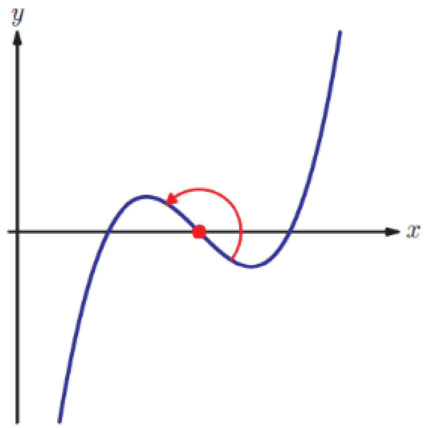
[Maximum mark: 6]

Find the range of possible values of k such that $\mathrm{e}^{2x} + \ln k = 3\mathrm{e}^{x}$ has at least one real solution.

?

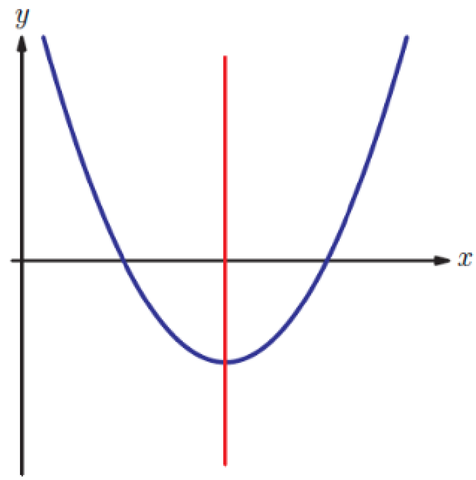
Symmetries of graphs and functions

HL Only



- Two-fold rotational symmetry

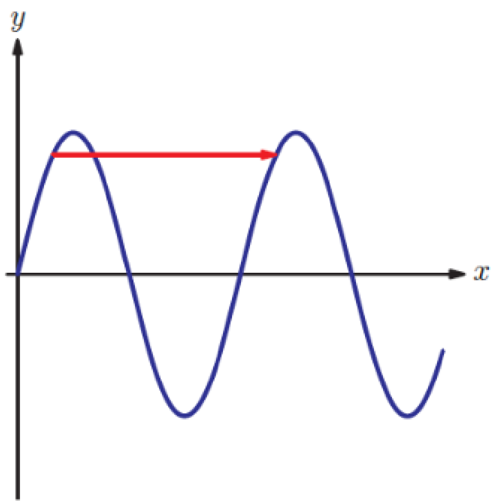
  When the graph is rotated by a 180° about a given point, it becomes the same graph.



- Reflection symmetry

  When the graph is reflected in a given line it becomes the same graph.

HL Only



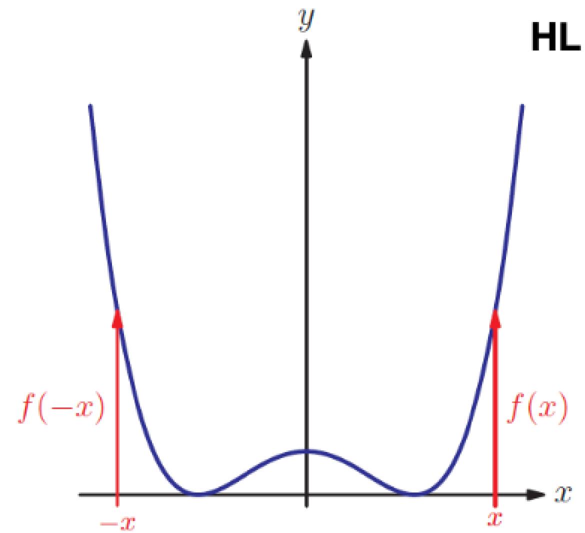
- Translational symmetry

  When the graph is shifted it becomes the same graph.

HL Only

When the graph has reflection symmetry in the y-axis, its function is called an **even function**. The height of the graph at every value of x must be the same as the height of the graph at $-x$:

> For an even function $f(x)$:
>
> $$f(x) = f(-x)$$
>
> This means that $y = f(x)$ has reflection symmetry in the y-axis.



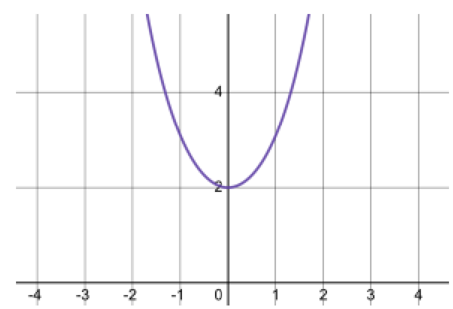
example

Consider the function $h(x) = e^x + e^{-x}$.

(a) Find $h(0)$. $h(0) = 2$

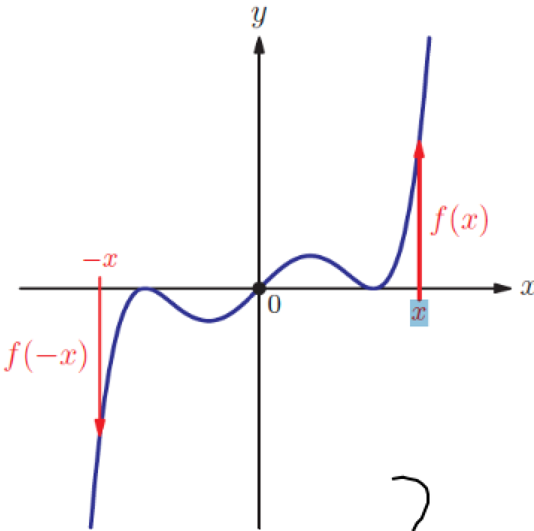
(b) Prove that $h(x)$ is even. $h(-x) = e^{-x} + e^{-(-x)} = h(x)$

(c) Sketch the graph of $y = h(x)$.



HL Only

When the graph has two-fold rotation symmetry about the origin its function is called an **odd function**. The height above the x-axis of the graph at every value of x must be the same as the height below the x-axis of the graph at x, therefore:



For an odd function $f(x)$:

$$f(-x) = -f(x)$$

This means that $y = f(x)$ has two-fold rotation symmetry about the origin.

$-h(x) = h(-x)$?

example

$f(x)$ is an even function and $g(x)$ is an even function. Prove that $h(x) = f(x)g(x)$ is an odd function.

$$\left.\begin{aligned} f(-x) &= -f(x) \\ g(x) &= g(-x) \end{aligned}\right\}$$

$$\begin{aligned} h(-x) &= f(-x)\, g(-x) \\ &= -f(x) \cdot g(x) \\ &= -h(x) \end{aligned}$$