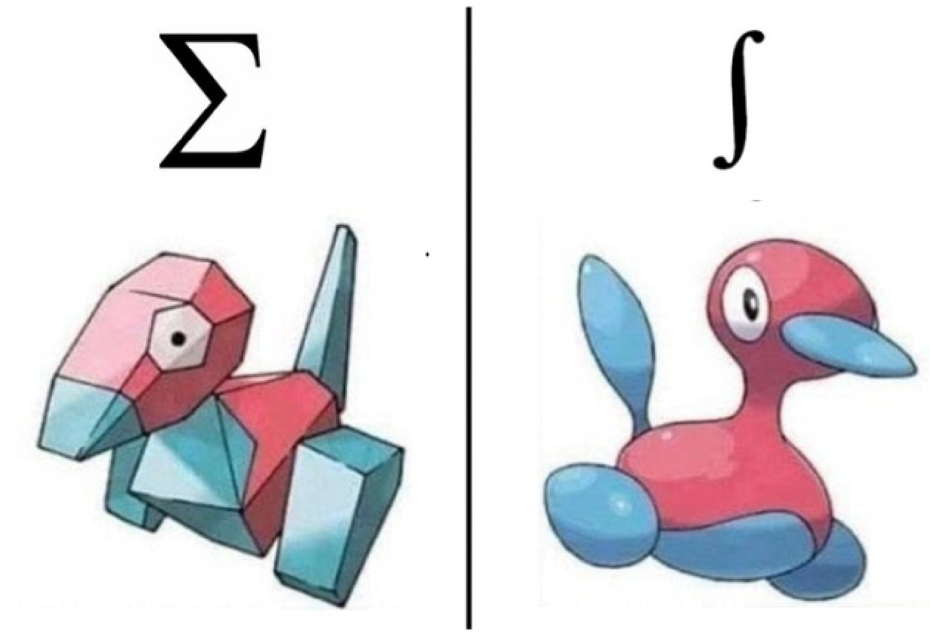
Integration

In this chapter you will learn:

- to reverse the process of differentiation (this process is called integration)
- to find the equation of a curve given its derivative and a point on the curve
- to integrate $\sin x$, $\cos x$ and $\tan x$
- to integrate e^x and $\frac{1}{x}$
- to find the area between a curve and the x- or y-axis
- to find the area enclosed between two curves.



Howie Hua

$\int \quad dx$

$\frac{d}{dx}(x^2) = 2x$	$\int \frac{2x}{2}\, dx = \frac{x^2}{2} + c$	$\int x\, dx = \frac{x^2}{2} + C$
$\frac{d}{dx}(x) = 1$	$\int 1\, dx = x + C$	
$\frac{d}{dx}(x^3) = 3x^2$	$\int \frac{3x^2}{3}\, dx = \frac{x^3}{3} + c$	$\int x^2\, dx = \frac{x^3}{3} + C$
$\frac{d}{dx}(x^4) = 4x^3$	$\int 4x^3\, dx = x^4 + C$	$\int x^3\, dx = \frac{x^4}{4} + C$
$\frac{d}{dx} x^5 = 5x^4$	$\int 5x^4\, dx = x^5 + c$	$\int x^4\, dx = \frac{x^5}{5} + C$
$\frac{d}{dx} x^6 = 6x^5$	$\int 6x^5\, dx = x^6 + c$	$\int x^5\, dx = \frac{x^6}{6} + C$

+1

example 268

Find a possible expression for y in terms of x:

$$\frac{dy}{dx} = 3x^2$$

$y = x^3$

$$\frac{dy}{dx} = 5x^4$$

$y = x^5$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

$\frac{dy}{dx} = -x^{-2}$ $+1$

$y = \frac{-x^{-1}}{-1}$

$y = x^{-1} = \frac{1}{x}$

$$\frac{dy}{dx} = -\frac{4}{x^5} = -4x^{-5}$$

$\frac{-4x^{\overbrace{-5+1}^{-4}}}{-4}$

$y = x^{-4}$

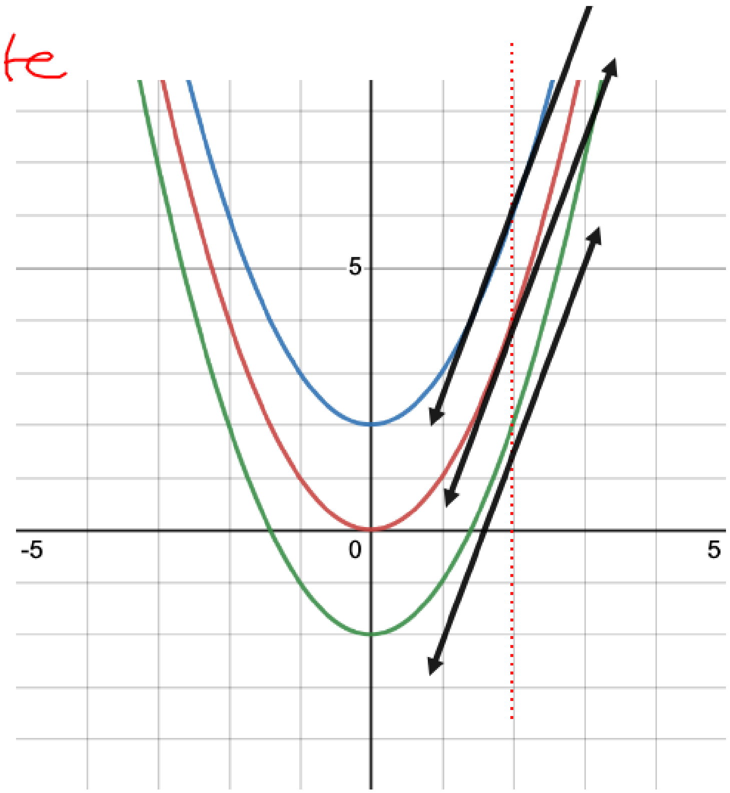
Constant of integration / C for chocolate

$$\int 2x\,dx = x^2$$ $\frac{d}{dx} x^2 = 2x$

$$\int 2x\,dx = x^2 + 1$$ $\frac{d}{dx}(x^2+1) = 2x$

$$\int 2x\,dx = x^2 - \frac{3}{5}.$$ $\frac{d}{dx}(x^2 - \pi) = 2x$

$$\int 2x\,dx = x^2 + c$$



Rules of integration

We know that for $y = x^n$, $\dfrac{dy}{dx} = nx^{n-1}$ or in words:

To differentiate x^n multiply by the old power then decrease the power by 1.

We can express the reverse of this process as follows.

To integrate x^n increase the power by 1 then divide by the new power.

The general rule for integrating x^n for any rational power $n \neq -1$ is:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$$

To integrate multiples of functions:

$$\int kf(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x$$

For the sum of integrals:

$$\int f(x) \pm g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$$

$F(x) + \cancel{c_1} + G(x) + \cancel{c_2} + C$

example 269

$\frac{-4}{3} + 1 = \frac{-1}{3}$

Find (a) $\int 6x^{-3}\,\mathrm{d}x$ (b) $\int (3x^4 - 8x^{-\frac{4}{3}} + 2)\,\mathrm{d}x$

$2x^0$

$= 6\int x^{-3}\,dx$

$= 6\,\frac{x^{-2}}{-2}$

$= -3x^{-2} + C$

$\frac{A}{\frac{2}{3}} = A \times \frac{3}{2}$

$= \frac{3A}{2}$

$= \frac{3}{5}x^5 - \frac{8x^{-\frac{1}{3}}}{-\frac{1}{3}} + 2x$

$= \frac{3}{5}x^5 + 24x^{-\frac{1}{3}} + 2x + C$

$$\int dx = \int 1\, dx = x + C$$

$$\int dn = n + C$$

$$\int x\, dn = x \cdot n + C$$

$$\int aspiri\, \mathrm{dn} =$$

example 270

Find (a) $\int 5x^2 \sqrt[3]{x}\, dx$ (b) $\int \frac{(x-3)^2}{\sqrt{x}}\, dx$

$$= 5\int x^2 \cdot x^{\frac{1}{3}}\, dx$$

$$= 5\int x^{\frac{7}{3}}\, dx$$

$$= \frac{3.\cancel{5}\; x^{\frac{10}{3}}}{\cancel{10}2}$$

$$= \frac{3}{2} x^{\frac{10}{3}} + c$$

$$= \int \frac{x^2 - 6x + 9}{x^{\frac{1}{2}}}\, dx$$

$$= \int \frac{x^2}{x^{\frac{1}{2}}} - \frac{6x}{x^{\frac{1}{2}}} + \frac{9}{x^{1/2}}\, dx$$

$$= \int x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}\, dx$$

$$= \frac{2}{5} x^{\frac{5}{2}} - \frac{\overset{2}{\cancel{6}} \cdot 2x^{\frac{3}{2}}}{\cancel{3}} + 9x^{\frac{1}{2}} \cdot 2$$

$$= \frac{2}{5} x^{\frac{5}{2}} - 4x^{3/2} + 18x^{\frac{1}{2}} + C$$

Integrating x^{-1} and e^x

$$\int x^{-1}\,dx$$ $= \frac{x^0}{0}$!!!!

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\int x^{-1}\,dx = \ln x + c$$

$$\int x^{-1}\,dx = \ln|x| + c$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\int e^x\,dx = e^x + c$$

Integrating trigonometric functions

Differentiating trigonometric functions gives:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

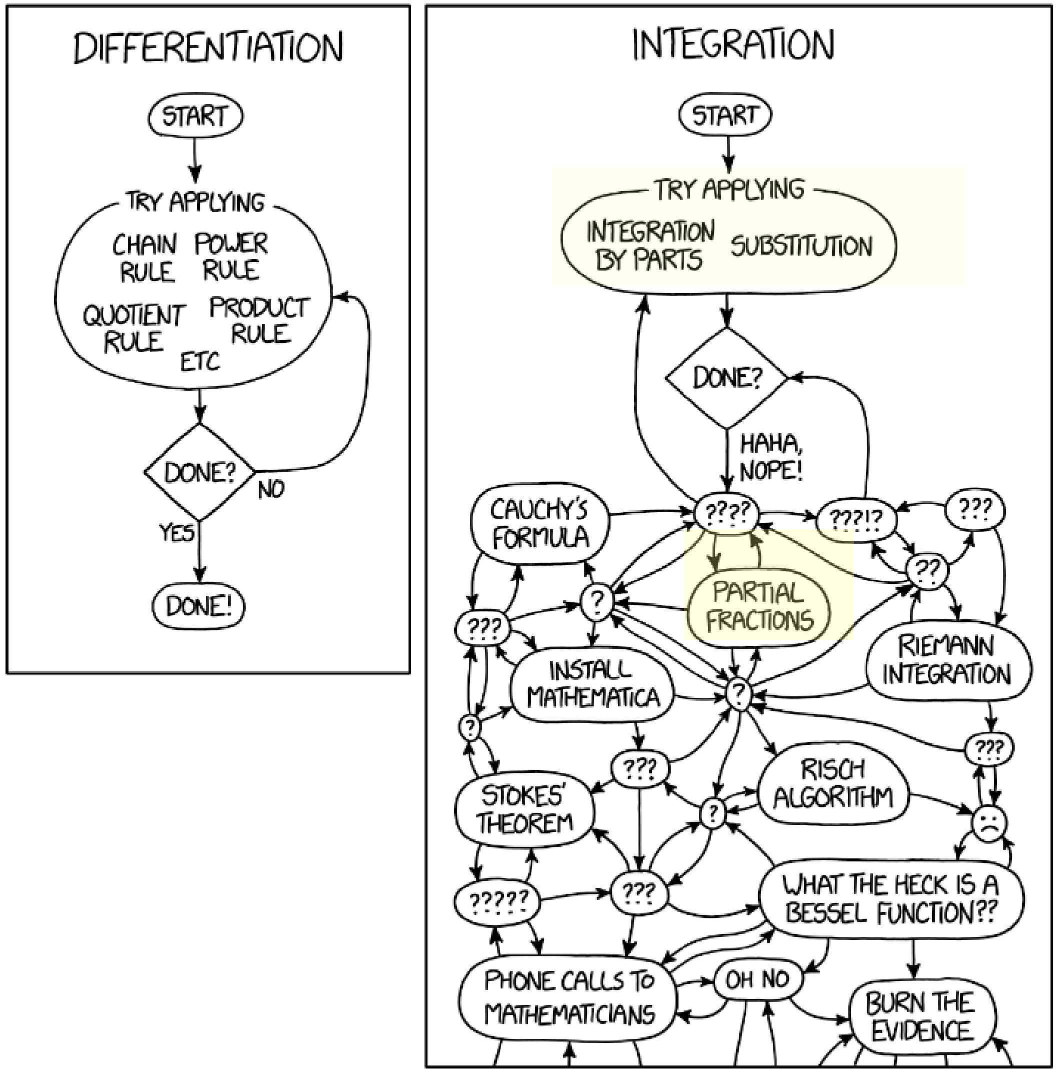
$$\frac{d}{dx}(\tan x) = \sec^2 x$$

The integrals of trigonometric functions:

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \tan x = \ln| \sec x | + c$$

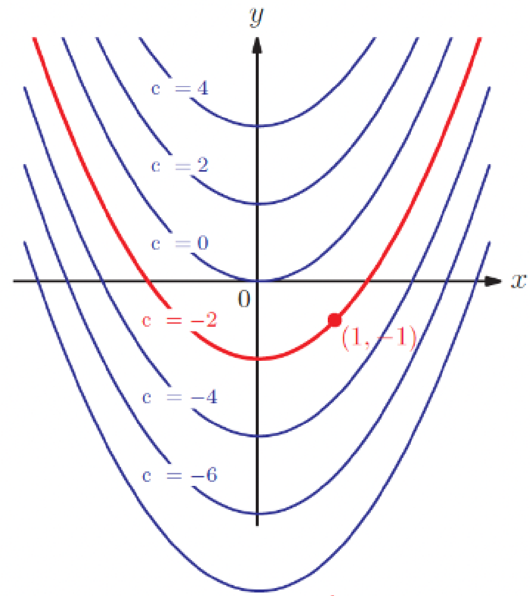

DIFFERENTIATION
START
TRY APPLYING
CHAIN RULE
POWER RULE
QUOTIENT RULE
PRODUCT RULE
ETC
DONE?
NO
YES
DONE!
INTEGRATION
START
TRY APPLYING
INTEGRATION BY PARTS
SUBSTITUTION
DONE?
HAHA, NOPE!
CAUCHY'S FORMULA
????
???!?
???
??
?
???
PARTIAL FRACTIONS
RIEMANN INTEGRATION
INSTALL MATHEMATICA
?
?
???
???
RISCH ALGORITHM
STOKES' THEOREM
?
?????
???
WHAT THE HECK IS A BESSEL FUNCTION??
PHONE CALLS TO MATHEMATICIANS
OH NO
BURN THE EVIDENCE

Finding the equation of a curve

If we again consider $\frac{dy}{dx} = 2x$ which we met at the start of this chapter, we know that the original function must have equation $y = x^2 + c$ for some constant value c.

↳ General Solution

If we are also told that the curve passes through the point $(1, -1)$, we can find c and specify which of the family of curves our function must be. IVP: Initial value problems



$$-1 = 1^2 + c$$

$$c = -2$$

$$\therefore\ y = x^2 - 2$$

Particular Solution.

> To find the equation for y given the gradient $\frac{dy}{dx}$ and one point (p, q) on the curve:
>
> 1. Integrate $\frac{dy}{dx}$, remembering $+c$.
> 2. Find the constant of integration by substituting $x = p$, $y = q$.

example 271

The gradient of a curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = \overbrace{3x^2 - 8x + 5}^{f'(x)}$ and the curve passes through the point (1, –4). Find the equation of the curve.

$$f(x) = \int 3x^2 - 8x + 5 \, dx$$

$$f(x) = x^3 - 4x^2 + 5x + c$$

$(1, -4)$

$$-4 = 1 - 4 + 5 + c$$

$$c = -6$$

$$\therefore f(x) = x^3 - 4x^2 + 5x - 6$$

example 272

Evaluate:

a) $\int \frac{t^3 - 3t^5}{t^5}\, dt$

$$= \int \frac{t^3}{t^5} - \frac{3\cancel{t^5}}{\cancel{t^5}}\, dt$$

$$= \int t^{-2} - 3\, dt$$

$$= \frac{t^{-1}}{-1} - 3t + c$$

$$= -\frac{1}{t} - 3t + c$$

b) $\int \frac{x + 5x^4}{x^2}\, dx$

$$= \int \frac{x}{x^2} + \frac{5x^4}{x^2}\, dx$$

$$= \int \frac{1}{x} + 5x^2\, dx$$

$$= \ln|x| + \frac{5}{3}x^3 + c$$

Definite integration

Until now we have been carrying out a process known as **indefinite integration**: indefinite in the sense that we have an unknown constant each time, for example $\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + c$.

However, there is also a process called **definite integration** which yields a numerical answer without the involvement of the constant of integration, for example

$$\int_a^b x^2\mathrm{d}x = \left[\frac{1}{3}x^3\right]_a^b = \left(\frac{1}{3}b^3\right) - \left(\frac{1}{3}a^3\right).$$

Here a and b are known as the **limits** of integration: a is the lower limit and b the upper limit.

The fundamental theorem of calculus

If f is continuous on $[a, b]$ and if F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$.

example 273

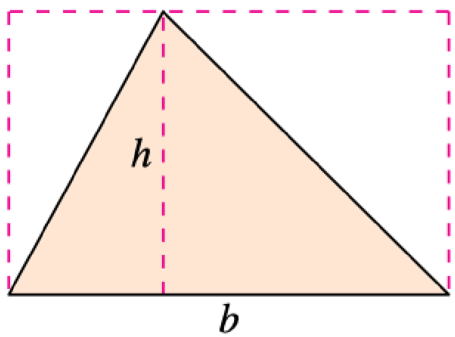
Find the exact value of $\int_1^e \frac{1}{x} + 4\,\mathrm{d}x$.

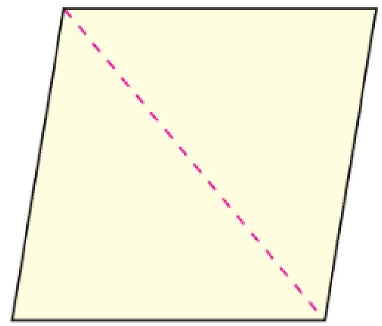
$$= \left[\ln x + 4x\right]_1^e$$

$$= \left[\cancel{\ln e}^{1} + 4e\right] - \left[\cancel{\ln 1}^{0} + 4\right]$$

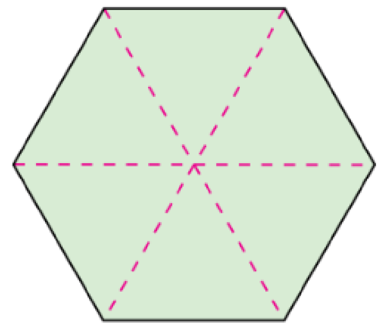
$$= 4e + 1 - 4$$

$$= 4e - 3$$

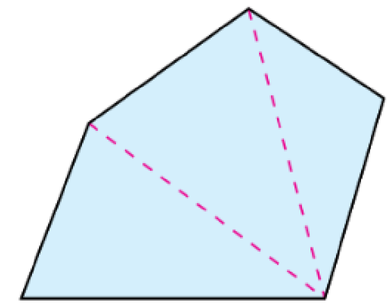




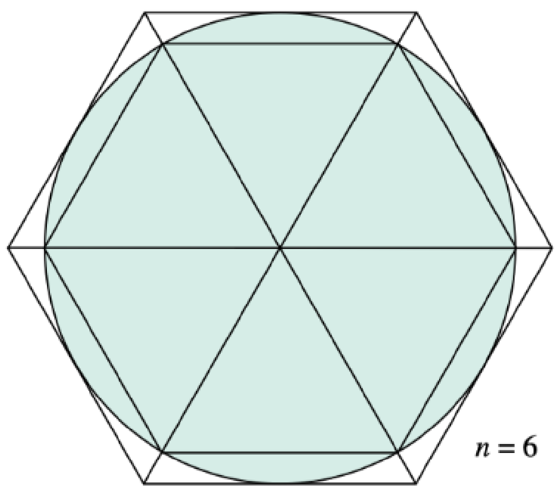
Parallelogram

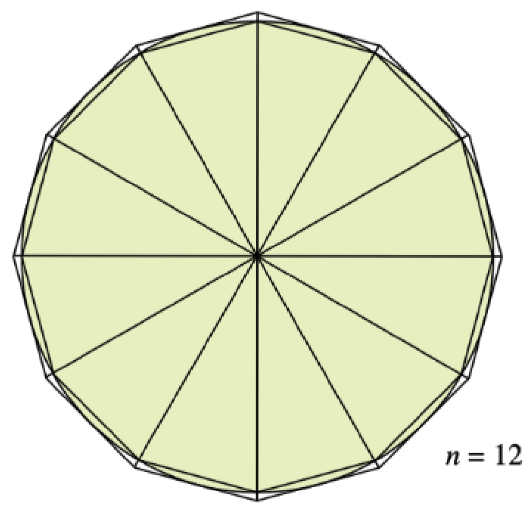


Hexagon



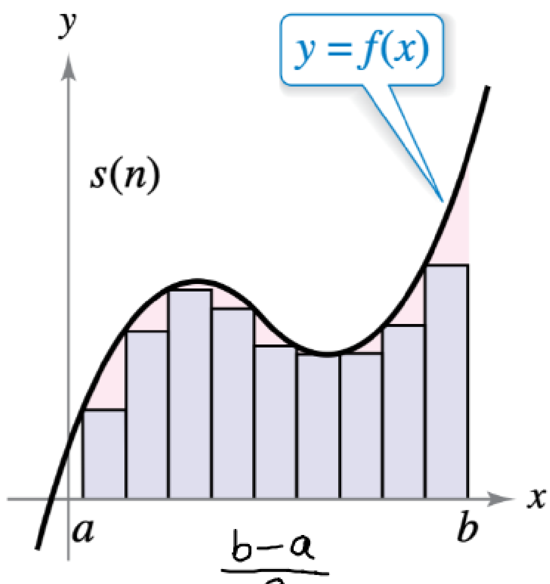
Polygon



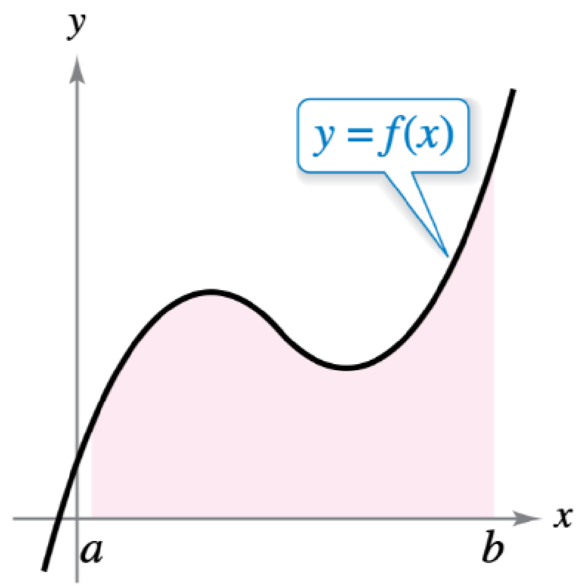


The exhaustion method for finding the area of a circular region

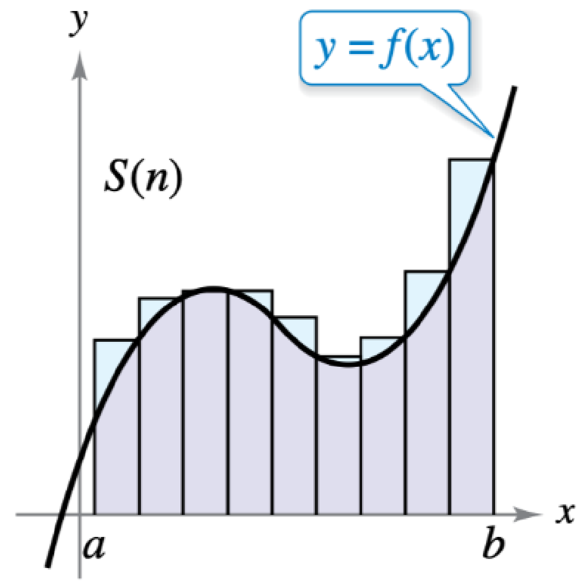
$$s(n) \leq (\text{Area of region}) \leq S(n)$$



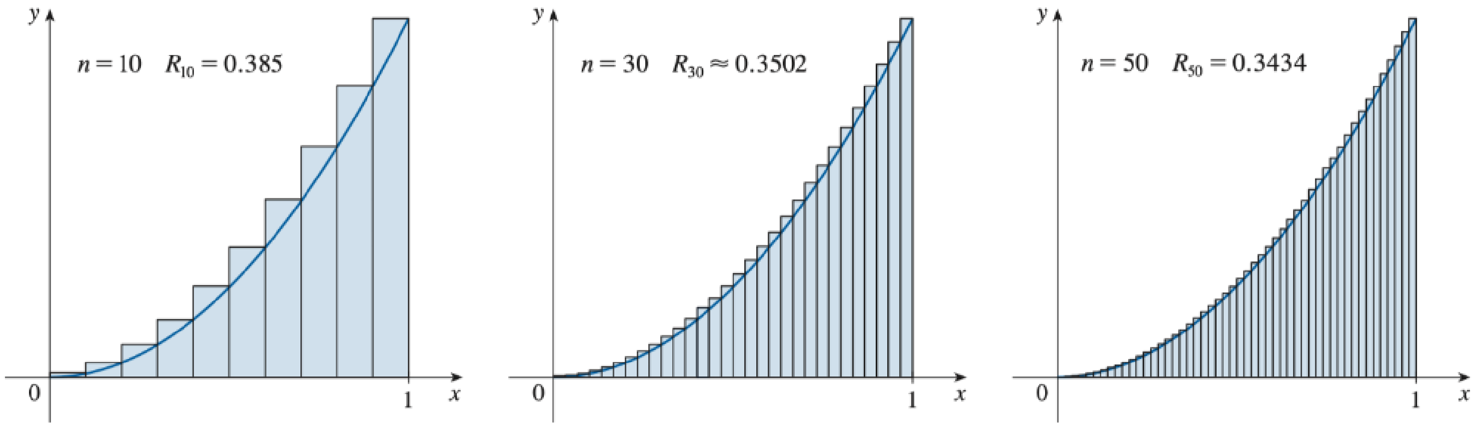
Area of inscribed rectangles is less than area of region.



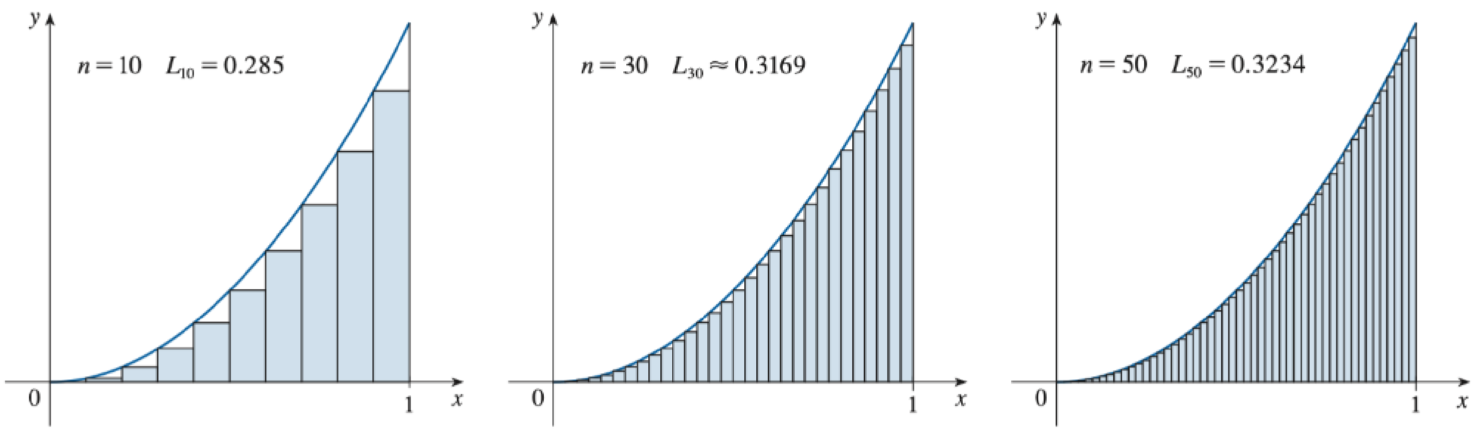
Area of region



Area of circumscribed rectangles is greater than area of region.



Right endpoints produce upper estimates because $f(x) = x^2$ is increasing.



Left endpoints produce lower estimates because $f(x) = x^2$ is increasing.

Definition of the Area of a Region in the Plane

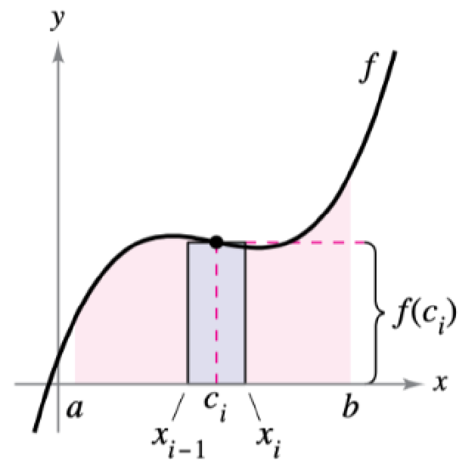
Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

where $x_{i-1} \le c_i \le x_i$ and

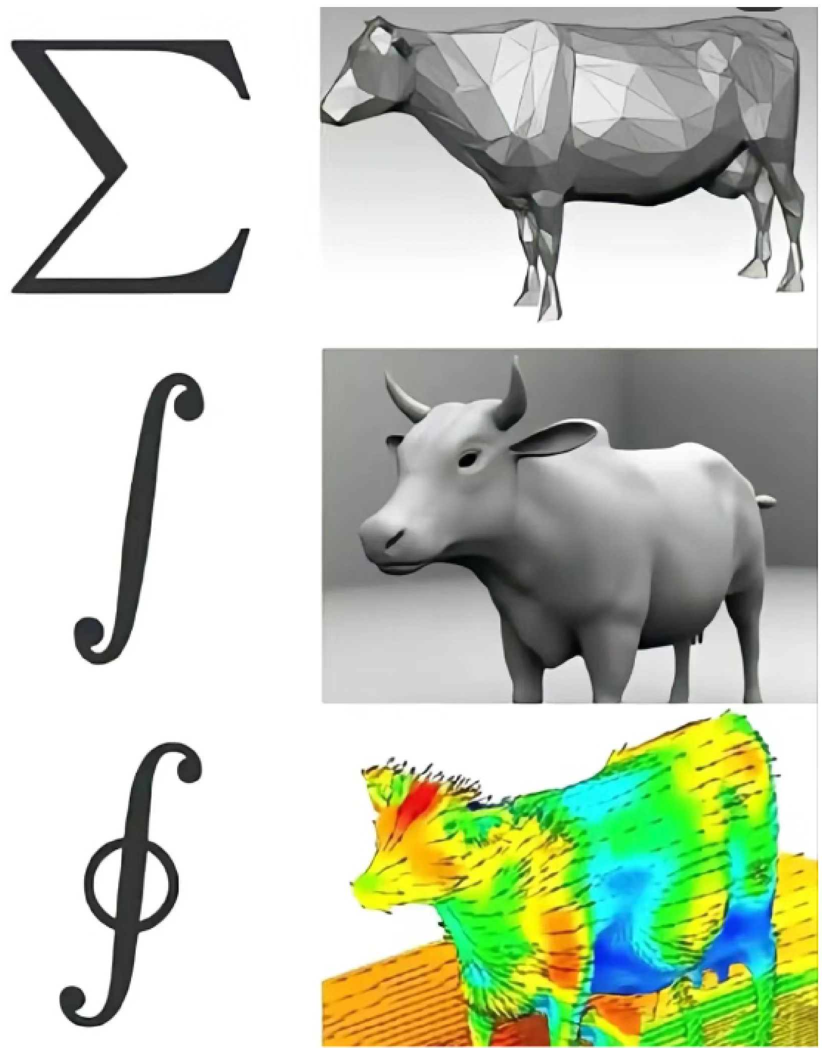
$$\Delta x = \frac{b - a}{n}.$$

Riemann Sum

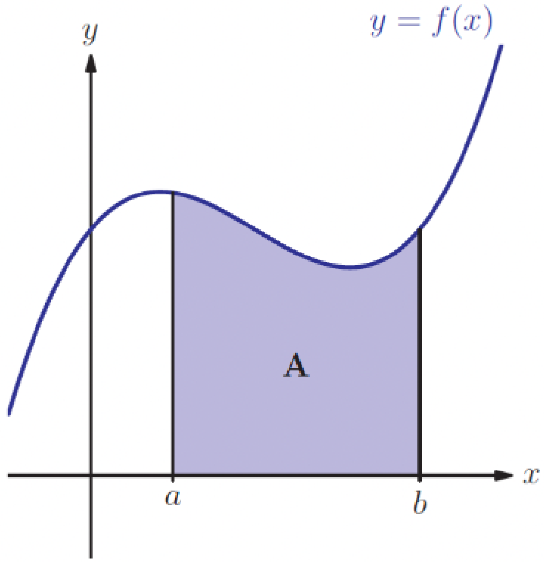


The width of the ith subinterval is $\Delta x = x_i - x_{i-1}$.

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$



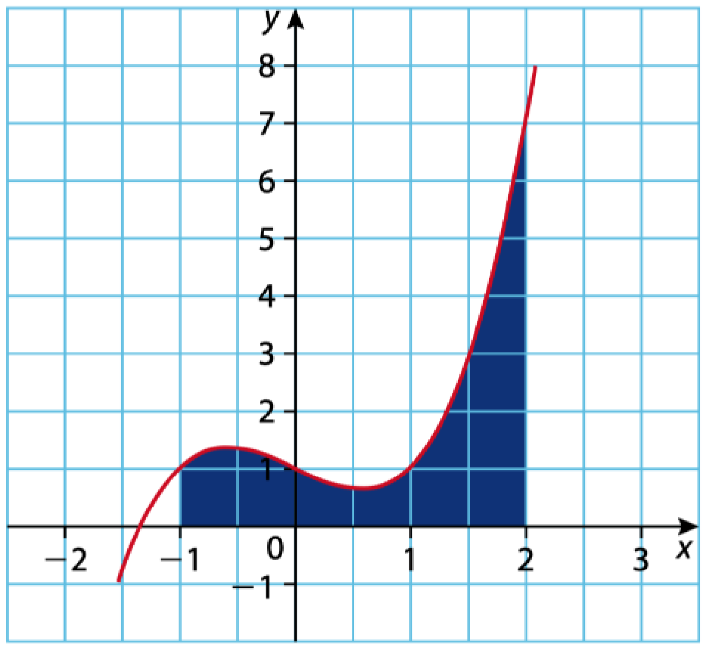
Geometrical significance of definite integration



$$\text{Area} = \int_a^b f(x)\,\mathrm{d}x$$

example 274

Find the area under the curve $f(x) = x^3 - x + 1$ and the x-axis over the interval $[-1, 2]$.



$$\int_{-1}^{2} x^3 - x + 1 \, dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2} + x \right]_{-1}^{2}$$

$$= \left[\frac{16}{4} - \cancel{\frac{4}{2}} + \cancel{2} \right] - \left[\underbrace{\frac{1}{4} - \frac{1}{2}}_{-\frac{1}{4}} - 1 \right]$$

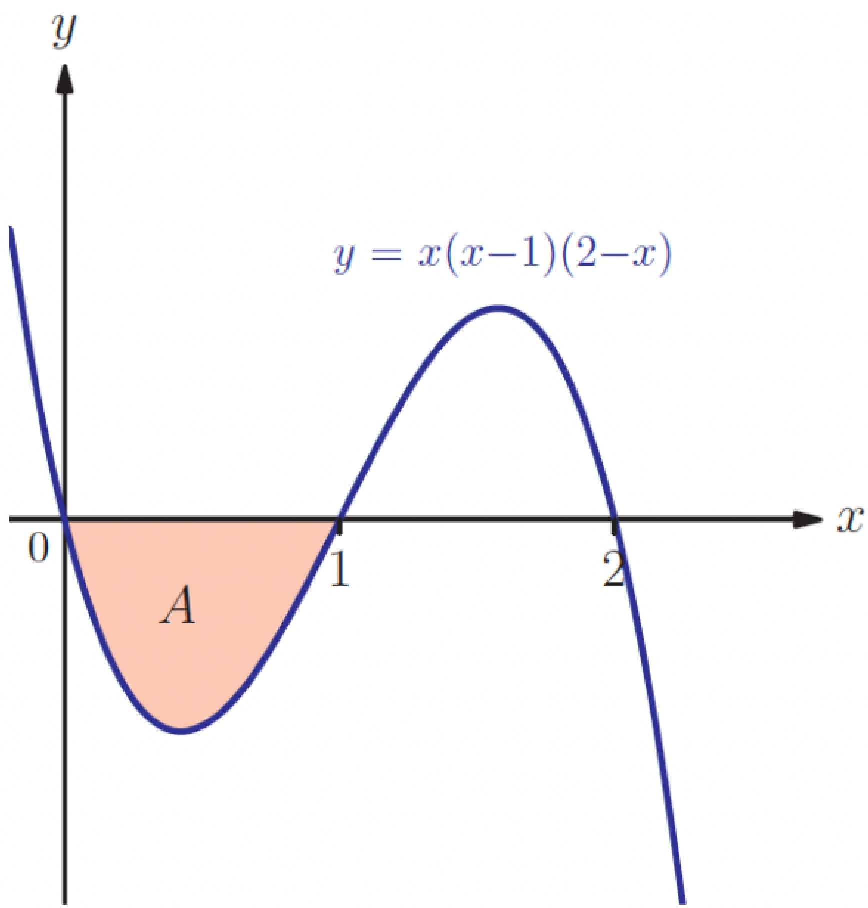
$$= 4 - \left[-\frac{5}{4} \right]$$

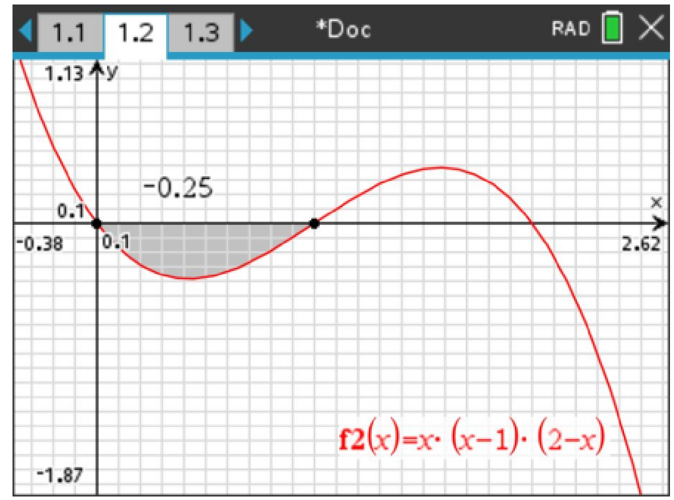
$$= 4 + \frac{5}{4}$$

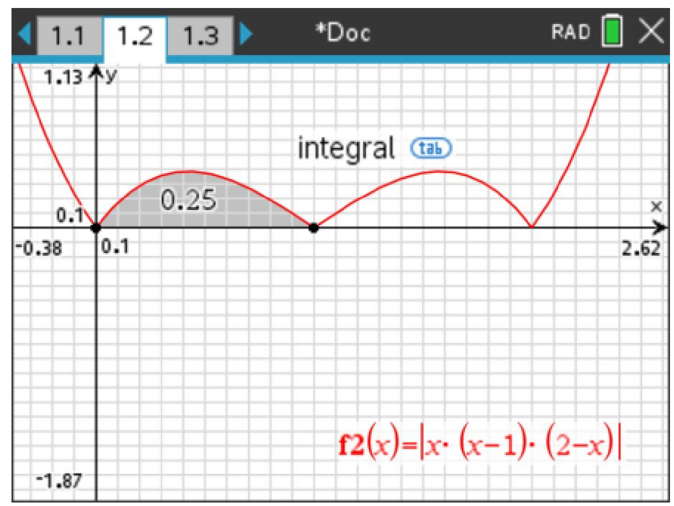
$$= \frac{21}{4}$$

example 275

Find the area A in this graph.



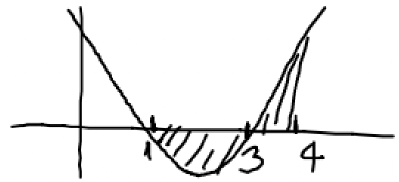




example 276

(a) Find $\int_1^4 x^2 - 4x + 3 \, dx$

$(x-1)(x-3)$



(b) Find the area enclosed between the x-axis, the curve $y = x^2 - 4x + 3$ and the lines $x = 1$ and $x = 4$.

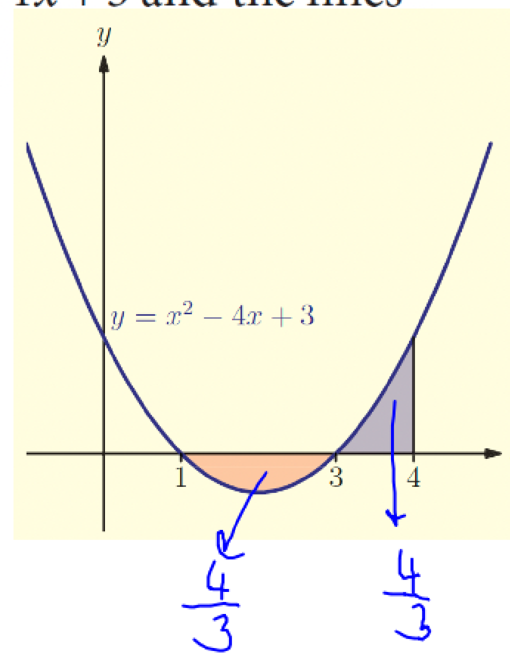
a) $\left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^4$

$= \left[\frac{4^3}{3} - 2\cdot 4^2 + 3\cdot 4\right] - \left[\frac{1}{3} - 2 + 3\right]$

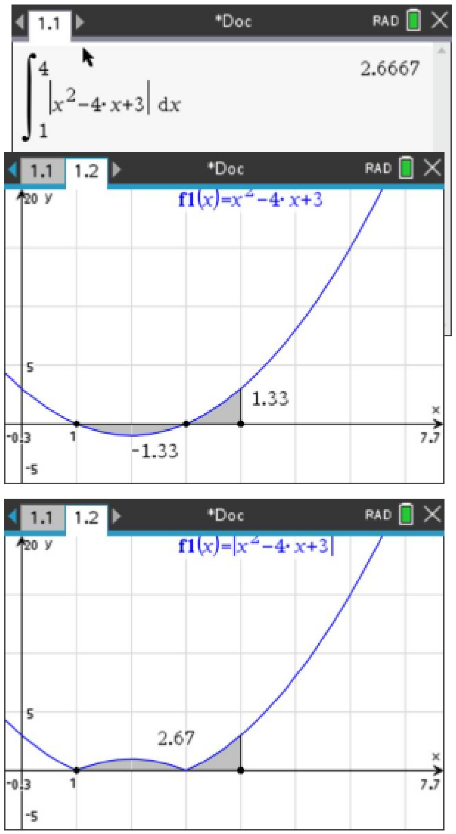
$= 0$

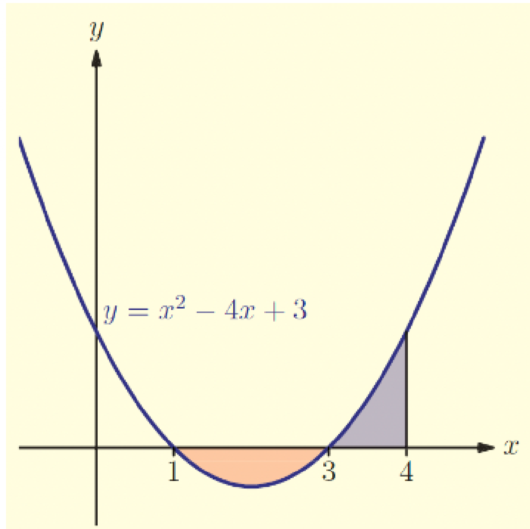
b) $\int_1^3 f(x)\, dx = -\frac{4}{3}$

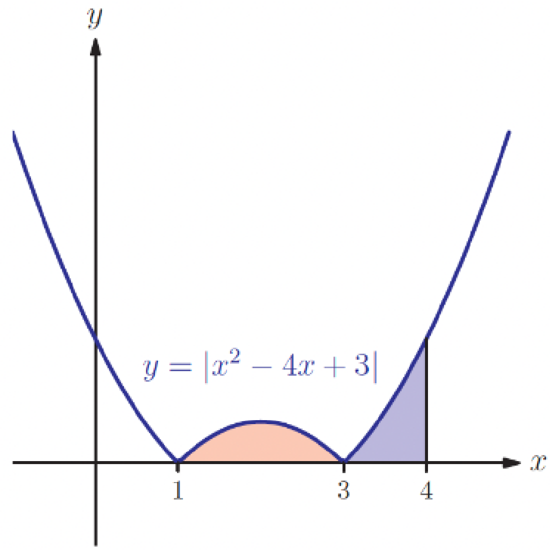
$\text{Area} = 2 \times \frac{4}{3} = \frac{8}{3}$



$$\int_1^4 |x^2 - 4x + 3|\,\mathrm{d}x = \frac{8}{3}$$





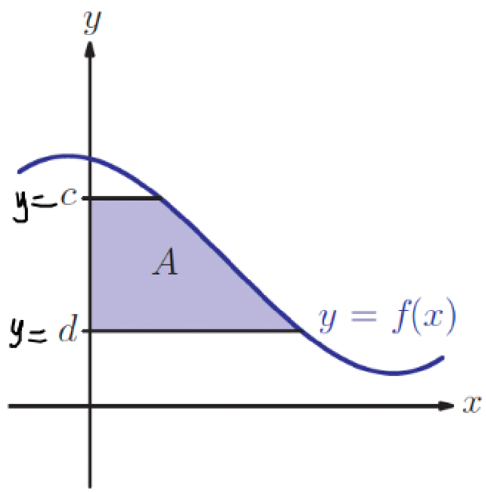


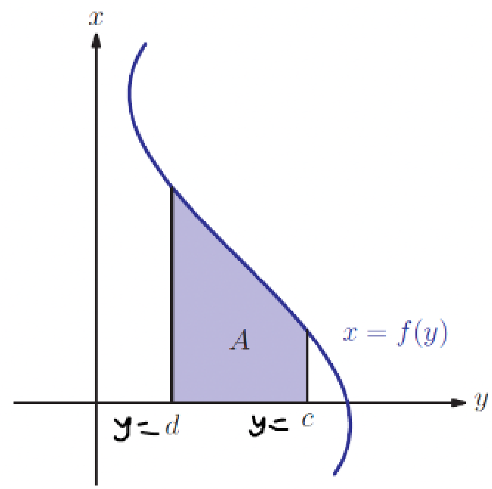
The area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b |f(x)|\,\mathrm{d}x$.

When working without a calculator, if the curve crosses the x-axis between a and b we need to split the area into several parts and find each one separately.

HL ONLY

The area between a curve and the y-axis



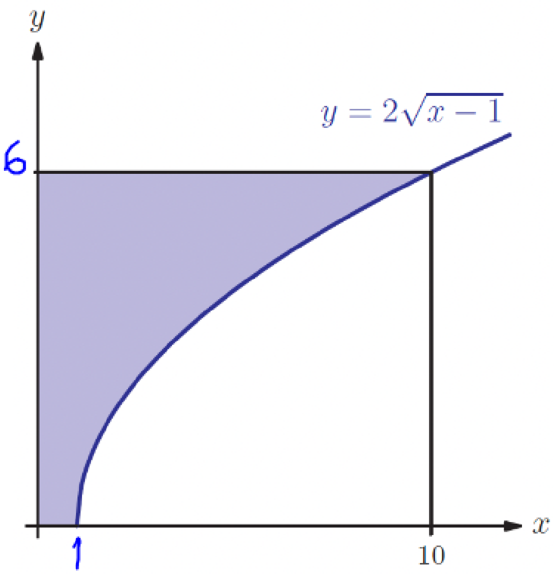


The area bounded by the curve $y = f(x)$, the y-axis and the lines $y = c$ and $y = d$ is given by $\int_c^d g(y)\,\mathrm{d}y$, where $g(y)$ is the expression for x in terms of y.

HL ONLY

example

The curve shown has equation $y = 2\sqrt{x-1}$. Find the shaded area.



$y = 2\sqrt{x-1}$

$y^2 = 4(x-1)$

$\frac{y^2}{4} + 1 = x$

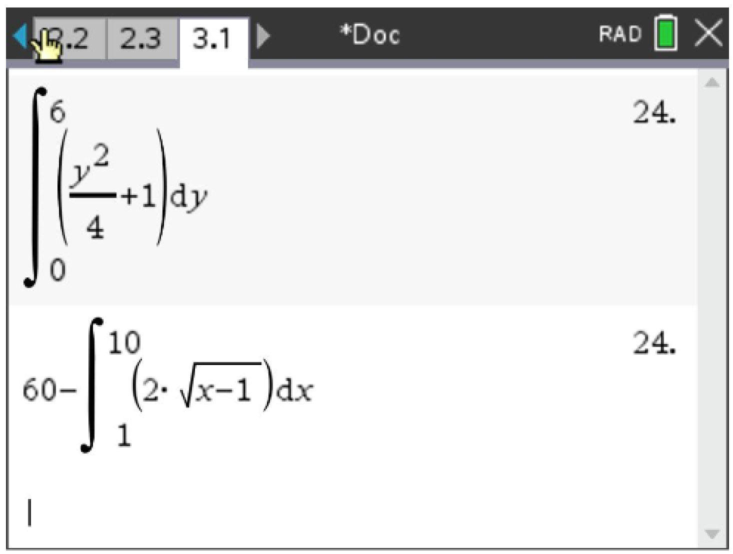
method 1

10×6 – white Area

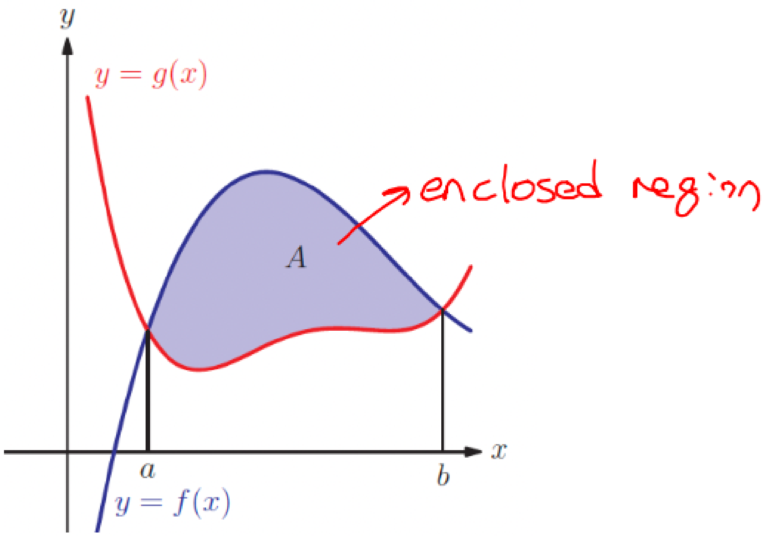
$= 60 - \int_1^{10} 2\sqrt{x-1}\, dx$

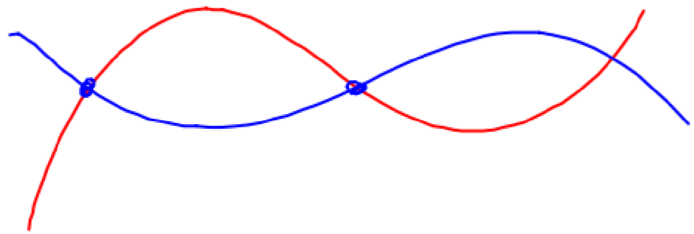
method 2

$\int_0^6 \frac{y^2}{4} + 1 \quad dy$



The area between two curves





$$A = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x$$

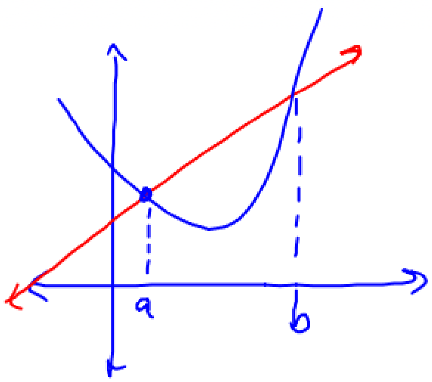
The area A between two curves, $f(x)$ and $g(x)$, is:

$$A = \int_a^b \left| f(x) - g(x) \right| \mathrm{d}x$$

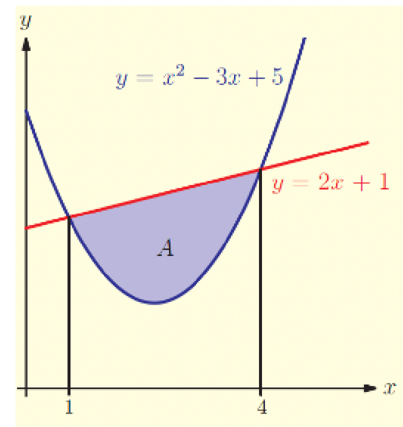
where a and b are the x-coordinates of the intersection points of the two curves.

example 288

Find the area A enclosed between $y = 2x + 1$ and $y = x^2 - 3x + 5$.



$$\int_a^b (2x+1) - (x^2 - 3x + 5)\, dx$$

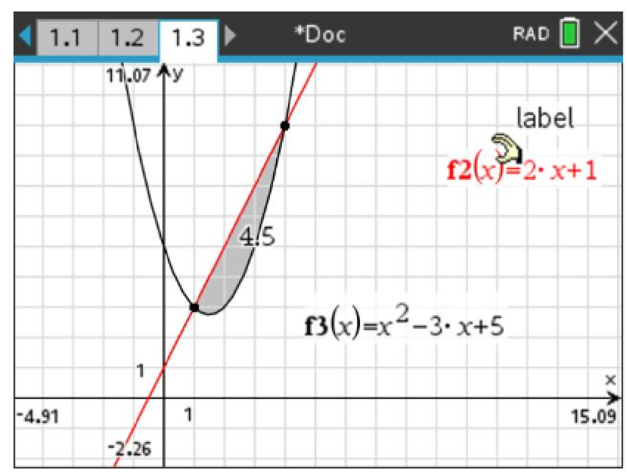


$2x + 1 = x^2 - 3x + 5$

$0 = x^2 - 5x + 4$

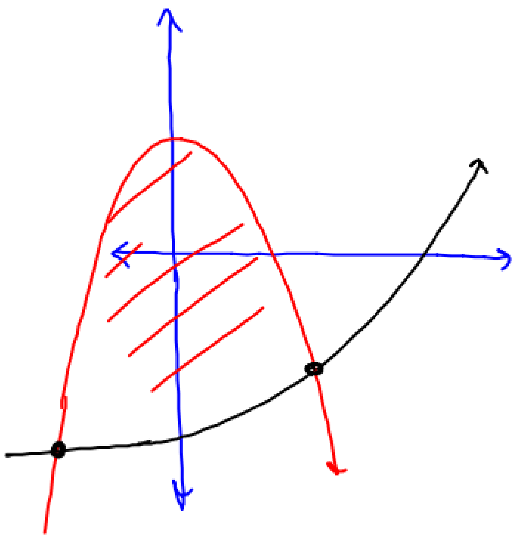
$= (x-4)(x-1)$

$x = 4 \quad x = 1$

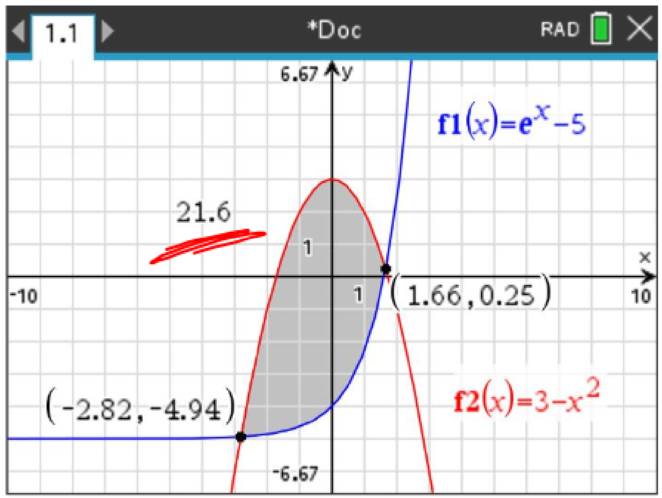


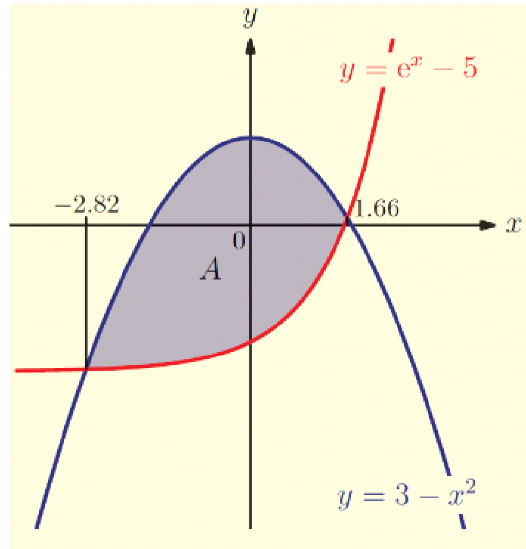
example 289

Find the area bounded by the curves $y = e^x - 5$ and $y = 3 - x^2$.



$$\text{Area} = \int_{-2.82}^{1.66} (3 - x^2 - e^x + 5)\, dx$$





Properties of the definite integral

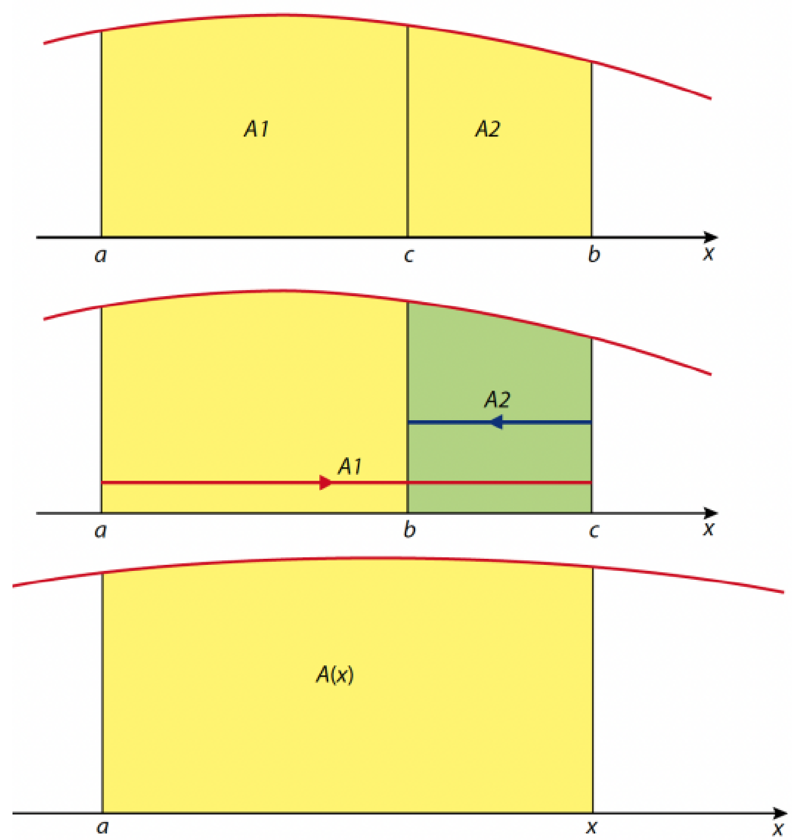
1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b c\,dx = c(b-a)$

4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5. $\int_a^b cf(x)\,dx = c\int_a^b f(x)dx$, where c is any constant



6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$$\frac{d}{dx}\sin x = \cos x$$

$$\int \cos x \, dx = \sin x + c$$

$$\frac{d}{dx}\sin \underbrace{2x} = 2\cos 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x) = 2\cos 2x$$

$$\int 2\cos\underbrace{(2x)}_{\text{Linear}}\,\mathrm{d}x = \sin 2x + c$$

$$\int 2\cos(2x)\,\mathrm{d}x = \sin 2x + c$$

$$\frac{d}{dx}\left(\frac{1}{5}(2x-3)^5\right) = 2(2x-3)^4$$

The reverse chain rule

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+c$$

where $F(x)$ is the integral of $f(x)$.

$u = ax+b$

$\frac{1}{a}\,du = \cancel{a}\cdot dx$

$$\int (\underline{2}x-3)^4\,dx \quad \begin{cases} u = 2x-3 \\ du = 2\cdot dx \\ \frac{1}{2}du = dx \end{cases}$$

$$\int (2x-3)^4\,dx = \frac{1}{2}\times\frac{1}{5}(2x-3)^5 + c = \frac{1}{10}(2x-3)^5 + c.$$

example 290

Find the following:

(a) $\int \frac{1}{2} e^{4x} dx$ (b) $\int \frac{2}{5-x} dx$

$\int \frac{1}{x} dx = \ln|x| + c$

EXAM HINT

This rule only applies when the 'inside' function is of the form $(ax+b)$!

$= \frac{1}{2} \int e^{4x} dx$

$= \frac{1}{2} \cdot e^{4x} \cdot \frac{1}{4} + c$

$= \frac{1}{8} e^{4x} + c$

check $\frac{d}{dx} \frac{1}{8} e^{4x} = \frac{1}{\cancel{8}_2} \cdot e^{4x} \cdot \cancel{4}$

$= 2 \cdot \int \frac{1}{5-x} dx$

$= -2 \cdot \ln|5-x| + c$

check

$\frac{d}{dx} -2 \ln|5-x|$

$= -2 \cdot \frac{1}{5-x} \cdot (-1)$

$= \frac{2}{5-x}$

example 291

Evaluate each integral.

a) $\int\sqrt{6x+11}\,dx$

$= \int (6x+11)^{\frac{1}{2}}\,dx$

$= \frac{1}{\cancel{6}_3}\cdot\frac{\cancel{2}}{3}(6x+11)^{\frac{3}{2}} = \frac{1}{9}(6x+11)^{\frac{3}{2}} + c$

$\frac{1}{\frac{3}{2}} = \frac{2}{3}$

b) $\int(\underbrace{5x^3+2})^8 x^2\,dx$

$$\boxed{\begin{aligned} u &= 5x^3+2 \\ du &= 15x^2\,dx \\ \frac{1}{15}du &= x^2\,dx \end{aligned}}$$

$\frac{1}{15}\int u^8\cdot du = \frac{u^9}{9}\cdot\frac{1}{15} + c$

$\therefore \frac{(5x^3+2)^9}{135} + c$

example 292

Evaluate each integral.

c) $\displaystyle\int \frac{x^3 - 2}{\sqrt[5]{x^4 - 8x + 13}}\,dx$

d) $\displaystyle\int \sin^4(3x^2)\cos(3x^2)x\,dx$

$u = x^4 - 8x + 13$

$du = (4x^3 - 8)\,dx$

$\frac{1}{4}du = \cancel{4}(x^3 - 2)\,dx$

$$\frac{1}{4}\int \frac{1}{\sqrt[5]{u}}\,du = \frac{1}{4}\int u^{-\frac{1}{5}}\,du$$

$$= \frac{1}{4} \cdot \frac{u^{\frac{4}{5}}}{4} \cdot 5$$

$$= \frac{5}{16}u^{\frac{4}{5}} = \frac{5}{16}(x^4 - 8x + 13)^{\frac{4}{5}} + C$$

293

d) $\int \sin^4(3x^2)\cos(3x^2)x\,dx$ $= \frac{1}{6}\int u^4 \cdot du$

$\frac{\pi}{3}$

Choose most difficult looking expression

$u = \sin(3x^2)$

$\frac{1}{6}du = \cos(3x^2)\cdot \cancel{6}x\,dx$

$= \frac{u^5}{30} + c$

$= \frac{\sin^5(3x^2)}{30} + C$

Integration by substitution

$$\int \cos 2x \, dx = \int \frac{1}{2} \times 2\cos 2x \, dx = \frac{1}{2}\int 2\cos 2x \, dx = \frac{1}{2}\sin 2x + c$$

This cannot be done with a variable: $\int x \sin x \, dx$ is not the same as $x\int \sin x \, dx$. So we need a different rule for integrating a product of two functions. In some cases this can be done by extending the principle of reversing the chain rule, leading to the method of **integration by substitution**.

When using the chain rule to differentiate a composite function, we differentiate the outer function and multiply this by the derivative of the inner function; for example

$$\frac{d}{dx}\left(\sin\left(x^2+2\right)\right) = \cos\left(x^2+2\right) \times 2x$$

$$\int x\cos(x^2+2)\,\mathrm{d}x. \qquad = \frac{1}{2}\int \cos u \; du = \frac{1}{2}\cdot \sin u + C$$

$$= \frac{1}{2}\sin(x^2+2) + C$$

$$u = x^2 + 2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x.$$

$$du = 2x\,dx$$

$$\frac{1}{2}\,du = x\,dx$$

A word of warning here: $\frac{\mathrm{d}u}{\mathrm{d}x}$ is not really a fraction, so it is not clear that the above 'rearrangement' is valid. However, it can be shown that it follows from the chain rule that it is valid to replace $\mathrm{d}x$ by $\frac{1}{f'(x)}\mathrm{d}u$.

Integration by substitution

1. Select a substitution (if not already given).
2. Differentiate the substitution and write dx in terms of du.
3. Replace dx by the above expression, and replace any obvious occurrences of u.
4. Simplify as far as possible.
5. If any terms with x remain, write them in terms of u.
6. Work out the new integral in terms of u.
7. Write the answer in terms of x.

example 293

Find the following:

(a) $\int \sin^5 x \cos x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$\int u^5 \cdot du = \frac{u^6}{6} + c$

$= \frac{\sin^6 x}{6} + c$

(b) $\int x^2 e^{x^3+4} \, dx$ $= \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + c$

$u = x^3 + 4$

$\frac{1}{3} du = \cancel{3} x^2 dx$

$= \frac{1}{3} e^{x^3+4} + c$

example

295

Find $\int x\sqrt{4x-1}\,dx$ using the substitution $u=\sqrt{4x-1}$.

$u = (4x-1)^{\frac{1}{2}}$

$du = \frac{1}{2}(4x-1)^{-\frac{1}{2}} \cdot 4 \; dx$

$du = 2(4x-1)^{-\frac{1}{2}} \; dx$

$\int \; u$

Not efficient

$u^2 = 4x-1$

$\cancel{2}u\,du = \cancel{4}\,dx$

$\frac{u}{2}\,du = dx$

$\frac{u^2+1}{4} = \frac{\cancel{4}x}{\cancel{4}}$

$x = \frac{u^2+1}{4}$

$\int \frac{u^2+1}{4} \cdot u \cdot \frac{u}{2}\,du$

$= \frac{1}{8}\int u^4+u^2 \; du$

$= \frac{1}{8}\left(\frac{u^5}{5} + \frac{u^3}{3}\right) + c$

$$=\frac{1}{8}\left(\frac{1}{5}\left(\sqrt{4x-1}\right)^5+\frac{1}{3}\left(\sqrt{4x-1}\right)^3\right)+c$$

example

296

Evaluate $\int_{x=0}^{x=1} \frac{x-3}{x^2-6x+7}\,dx$ giving your answer in the form $a \ln p$.

$u = x^2 - 6x + 7$

$du = (2x-6)\,dx$

$\frac{1}{2} du = (x-3)\,dx$

$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|$

$= \frac{1}{2}\left[\ln|x^2-6x+7|\right]_0^1$

$= \frac{1}{2}\left[\ln 2 - \ln 7\right]$

$= \frac{1}{2}\ln\left(\frac{2}{7}\right)$

OR alternatively

$x=1 \quad u(1) = 1^2 - 6 + 7$

$= 2$

$x=0 \quad u(0) = 7$

$\frac{1}{2}\int_7^2 \frac{1}{u}\,du$

$= \frac{1}{2}\left[\ln u\right]_7^2$

$= \frac{1}{2}(\ln 2 - \ln 7)$

$= \frac{1}{2}\ln\left(\frac{2}{7}\right)$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

example

297 Integrate $\int \frac{x+4}{12-5x-2x^2}\,dx. = \int \frac{1\,\cancel{(x+4)}}{\cancel{(x+4)}(-2x+3)} = -\frac{1}{2}\ln|-2x+3| + C$

$u = 12-5x-2x^2$

$du = (-5-4x)\,dx$

$\cancel{du = -4\left(x+\frac{5}{4}\right)dx}$

$-2x^2-5x+12 = (x+4)(-2x+3)$

$-2x \longrightarrow 4$

$x \longrightarrow 3$

Oxford

13 F Trig

13 G Subst.

13 H Subs + Def. Int.

Integration with Partial Fractions

HL ONLY

example Find the partial fraction decomposition of $\dfrac{x+1}{x^2+5x+6}$.

$$\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} = \frac{2}{x+3} - \frac{1}{x+2}$$

$x = -2 \quad A = \frac{-1}{1} = -1$

$x = -3 \quad B = \frac{-2}{-1} = 2$

example Find

$$\int \frac{x+1}{x^2+5x+6}\,dx,$$

$$\frac{x+1}{x^2+5x+6} \equiv \frac{a}{x+2} + \frac{b}{x+3}$$

$$= \int \frac{2}{x+3} - \frac{1}{x+2}\,dx$$

$$= \cancel{2}\ln|x+3|^{2} - \ln|x+2| + c$$

$$= \ln\left|\frac{(x+3)^2}{x+2}\right| + c$$

example

(a) Find constants A, B and C such that:

$$\frac{x^2+5}{x+2} = Ax + B + \frac{c}{x+2} = x - 2 + \frac{9}{x+2}$$

(b) Hence find $\int \frac{x^2+5}{x+2}\,\mathrm{d}x$. $= \int x - 2 + \frac{9}{x+2}\,dx$

$$\begin{array}{r|l} & x - 2 \\ \hline x+2 & \cancel{x^2} + 5 \\ & \cancel{-x^2} \mp 2x \\ \hline & -2\cancel{x} + 5 \\ & \mp 2\cancel{x} \mp 4 \\ \hline & 9 \end{array}$$

$$= \frac{x^2}{2} - 2x + 9 \ln|x+2| + c$$

example

Find the indefinite integral $\int \frac{3x-1}{x^2+4x+4}dx$.

$$\frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$(x+2)^{-2}$

$$\int \frac{3}{x+2} - \frac{7}{(x+2)^2}\, dx$$

$$= 3 \cdot \ln|x+2| - 7 \cdot \frac{(x+2)^{-1}}{-1} + c$$

$$= 3 \ln|x+2| + \frac{7}{x+2} + c$$

Possible cases for partial fractions

1 **Denominator is a quadratic** that factorises into two distinct linear factors, and numerator $p(x)$ is a constant or linear.

$$\frac{p(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

2 **Denominator is a quadratic** that factorises into two repeated linear factors, and numerator $p(x)$ is a constant or linear.

$$\frac{p(x)}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

3 **Denominator is a cubic** that factorises into three repeated linear factors, and numerator $p(x)$ is a constant, linear or quadratic.

$$\frac{p(x)}{(ax+b)^3} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}$$

4 **Denominator is a cubic** that factorises into one linear factor and one quadratic factor (that cannot be factorised), and numerator $p(x)$ is a constant, linear or quadratic.

$$\frac{p(x)}{(ax+b)(cx^2+dx+e)} = \frac{A}{ax+b} + \frac{Bx+C}{cx^2+dx+e}$$

5 **Denominator is a cubic** that factorises into three distinct linear factors, and numerator $p(x)$ is a constant, linear or quadratic.

$$\frac{p(x)}{(ax+b)(cx+d)(ex+f)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f}$$

example

Find the indefinite integral $\int \frac{2}{x^3 + 2x^2 + 2x} dx$.

HL ONLY

$$\frac{2}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}$$

$$= \frac{A(x^2+2x+2) + Bx^2 + Cx}{\text{- - - - - -}}$$

$A+B=0$ $\quad B=-1$

$2A+C=0$ $\quad C=-2$

$2A=2$

$A=1$

$$= \int \frac{1}{x} - \frac{x+2}{x^2+2x+2}$$

$u = x^2+2x+2$

$du = 2x+2$

$$= \int \frac{1}{x} - \frac{x+1}{x^2+2x+2} - \frac{1}{x^2+2x+2} \, dx$$

$$= \ln x - \frac{1}{2} \ln|x^2+2x+2| - \int \frac{1}{x^2+2x+2} dx$$

LATER

HOMEWORK EXERCISE

HL ONLY

example

Find the indefinite integral $\int \frac{5x^2 + 16x + 17}{2x^3 + 9x^2 + 7x - 6} dx$.

example

Evaluate $\int \frac{3x - 1}{x^3 + 8} dx$.

Integration by parts

$$\int x\sin x\,\mathrm{d}x \text{ or } \int x^2 e^x\,\mathrm{d}x$$

- $x\ln x$
- $x\arcsin x$
- $x\cos x$
- xe^x
- $e^x\sin x$

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Integrating with respect to x we get:

$$uv = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

$$\Rightarrow \int u\frac{\mathrm{d}v}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x} = uv - \int v\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x}$$

$$\int u\,dv = uv - \int v\,du$$

HL ONLY

The **integration by parts** formula.

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

example

Find $\int x \sin x \, dx$.

$x = u \quad dv = \sin x \, dx$

$dx = du \quad v = -\cos x$

$\int u \, dv = u \cdot v - \int v \cdot du$

$= -x\cos x - \int -\cos x \, dx$

$= -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x + c$

L A P T E

logs, arc trig., polyn., trig., exp.

Decide your u in this way,

$(*)$ Find $\int x \sin x \, dx$.

$u = \sin x \quad dv = x \, dx$

$du = \cos x \, dx \quad v = \frac{x^2}{2}$

$(*) = \frac{x^2}{2} \sin x - \int \frac{x^2}{2} \cdot \cos x \, dx$

Harder than the original question!

HL ONLY

example

Evaluate $\int xe^{-x}dx$. ⊛

$u = x$ $\quad dv = e^{-x}dx$

$du = dx$ $\quad V = -e^{-x}$

⊛ $= -x \cdot e^{-x} + \int e^{-x}dx$

$= -x \cdot e^{-x} - e^{-x} + c$

example ⊛

Evaluate $\int x \ln x \, dx$.

$u = \ln x$ $\quad dv = x\,dx$

$du = \frac{1}{x}dx$ $\quad v = \frac{x^2}{2}$

⊛ $= \frac{x^2}{2} \cdot \ln x - \int \frac{1}{\cancel{x}} \frac{x^{\cancel{2}}}{2} dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \frac{x^2}{2} + c$

$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + c$

L A P T E

logs, arc. trig., polyn., trig., exp.

Decide your u in this way,

HL ONLY

$$\int x \ln x \, dx = \frac{1}{2}\left(x^2 \ln x - \frac{x^2}{2} + c\right) = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

example

Repeated use of integration by parts

Find the exact value of $\int_0^{\ln 2} x^2 e^x \, dx$. $= x^2 \cdot e^x - 2\int x e^x \, dx$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$u = x^2 \quad dv = e^x dx$

$du = 2x dx \quad v = e^x$

$= x^2 \cdot e^x - 2\left[x \cdot e^x - \int e^x \, dx\right]$

$= \left[x^2 e^x - 2x e^x + 2e^x\right]_0^{\ln 2}$

$= (\ln 2)^2 \cdot 2 - 2 \cdot \ln 2 \cdot 2 + 2 \cdot 2 - 2$

$= 2(\ln 2)^2 - 4\ln 2 + 2$

L A P T E

logs, arc. trig., polyn., trig., exp.

Decide your u in this way,

Remember

$\ln e^2 = 2$

$e^{\ln 2} = 2$

$\ln a^2 = 2 \cdot \ln a$

$(\ln a)^2 = \ln a \cdot \ln a$

HL ONLY

example

Using integration by parts to find unknown integrals

Use integration by parts to find $\int e^x \cos x \, dx$. $= e^x \cos x + \int e^x \sin x \, dx$

$u = \sin x \quad dv = e^x dx$

$du = \cos x \, dx \quad v = e^x$

$u = \cos x \quad dv = e^x dx$

$du = -\sin x \, dx \quad v = e^x$

$= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$

Same as the question.

$2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x$

Combine them on one side

$$\int e^x \cos x \, dx = \frac{e^x \cos x + e^x \sin x}{2}$$

L A P T E

logs, arc. trig., polyn., trig., exp.

HL ONLY

example Find $\int \ln x \, dx$.

$u = \ln x \quad dv = 1\,dx$

$du = \frac{1}{x}dx \quad v = x$

$$= \int \ln x \cdot 1 \, dx$$

$$= x \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= x \ln x - x + c$$

L A P T E

logs, arc. trig., polyn., trig., exp.

Decide your u in this way.

HL ONLY

example

Find $\int 3^x \, dx$.

Exercise 7J

HL ONLY

$$\frac{d}{dx} e^x = e^x$$

$$\int e^x \, dx = e^x + c$$

$$\frac{d}{dx} \cdot a^x = a^x \cdot \ln a$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x + c$$

$$(a^m)^n = a^{mn}$$

$$3^x = (e^{\ln 3})^x = e^{x \cdot \ln 3}$$

$$\int e^{2x} dx = \frac{e^{2x}}{2} + c$$

$$\int 3^x dx = \int e^{x \ln 3} dx$$

$$= \frac{e^{x \cdot \ln 3}}{\ln 3} + c$$

$$= \frac{3^x}{\ln 3} + c$$

"I'll change him."

example (a) Find $\int x(\ln x)^2 dx$. $= \frac{x^2}{2}(\ln x)^2 - \int x \ln x \, dx = \frac{x^2}{2}(\ln x)^2 - \left(\frac{x^2}{2}\ln x - \int \frac{x}{2} dx\right)$

$= \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$

$u = (\ln x)^2$ $\quad dv = x\,dx$

$du = \frac{2}{x}\ln x\,dx$ $\quad v = \frac{x^2}{2}$

one more time

$u = \ln x$ $\quad dv = x\,dx$

$du = \frac{1}{x}dx$ $\quad v = \frac{x^2}{2}$

(b) Hence, show that $\int_1^4 x(\ln x)^2 dx = 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$.

$\int_1^4 ----dx = \left[\frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4}\right]_1^4$

$= 8(\ln 4)^2 - 8\ln 4 + 4 - \frac{1}{4}$

$= 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$

$(\ln 2^2)^2$

$= (2\ln 2)^2$

$= 4\cdot(\ln 2)^2$

HL ONLY

example [Maximum mark: 7]

By using the substitution $u = \sin x$, find $\int \frac{\sin x \cos x}{\sin^2 x - \sin x - 2} dx$.

$u = \sin x$

$du = \cos x \, dx$

$$\frac{A}{x-2} + \frac{B}{x+1}$$

$$A = \frac{2}{3}$$

$$B = \frac{1}{3}$$

$$= \int \frac{u}{u^2 - u - 2} du = \int \frac{u}{(u-2)(u+1)} du$$

$$= \int \frac{2}{3} \frac{1}{(u-2)} + \frac{1}{3} \frac{1}{(u+1)} du$$

$$= \frac{2}{3} \ln |u-2| + \frac{1}{3} \ln |u+1| + c$$

$$= \frac{2}{3} \ln |\sin x - 2| + \frac{1}{3} \ln |\sin x + 1| + c$$

example Find the value of $\int_0^1 t\ln(t+1)\,dt$.

$z = t+1$

$dz = dt$

$$= \int_1^2 (z-1)\ln z\, dz = \left(\frac{z^2}{2} - z\right)\ln z - \int \left(\frac{z}{2} - 1\right) dz$$

$u = \ln z \qquad dv = (z-1)\,dz$

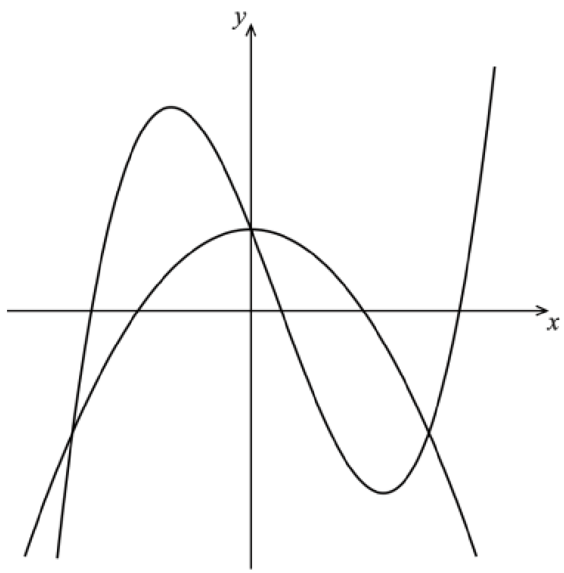
$du = \frac{1}{z}\,dz \qquad v = \frac{z^2}{2} - z$

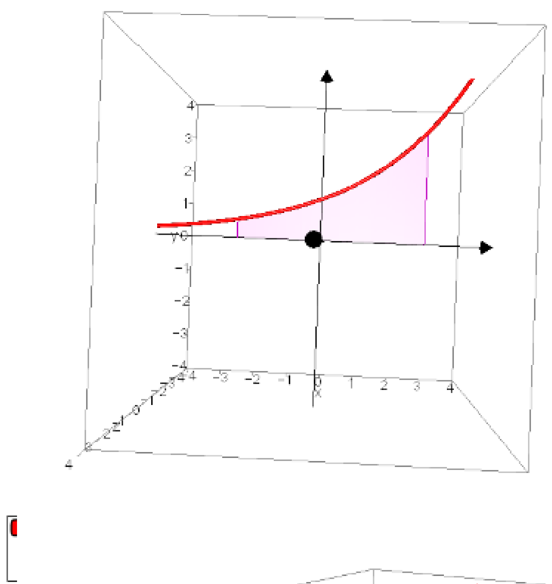
$$= \left[\left(\frac{z^2}{2} - z\right)\ln z - \frac{z^2}{4} + z\right]_1^2$$

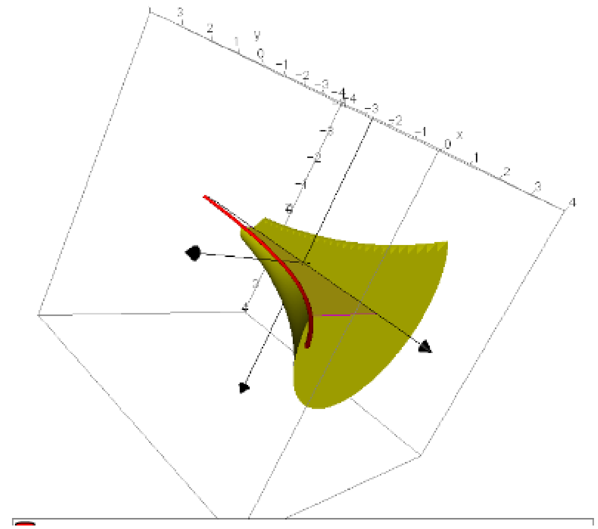
$$= \cancel{-1+2} - \left(-\frac{1}{4} \cancel{+1}\right) = \frac{1}{4}$$

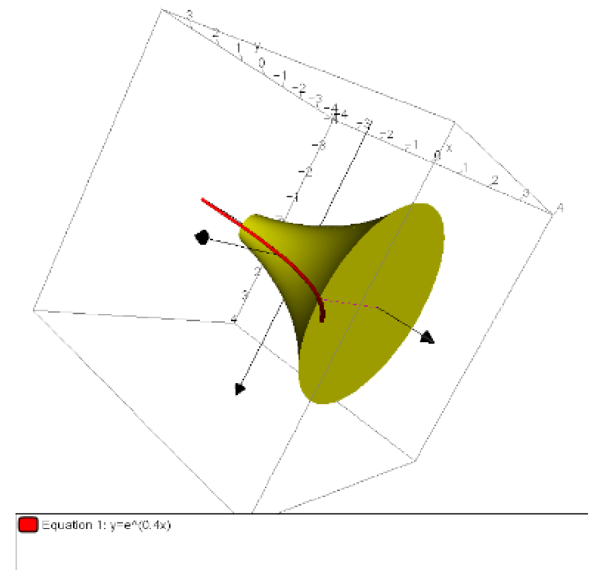
example

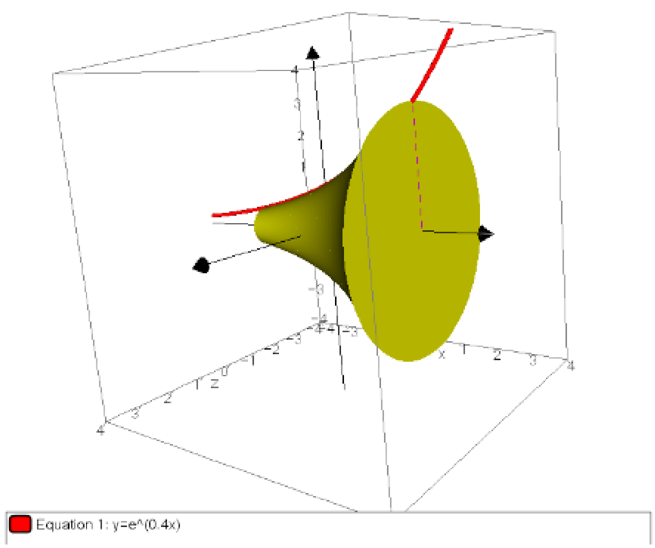
The graphs of $f(x) = -x^2 + 2$ and $g(x) = x^3 - x^2 - bx + 2$, $b > 0$, intersect and create two closed regions. Show that these two regions have equal areas.

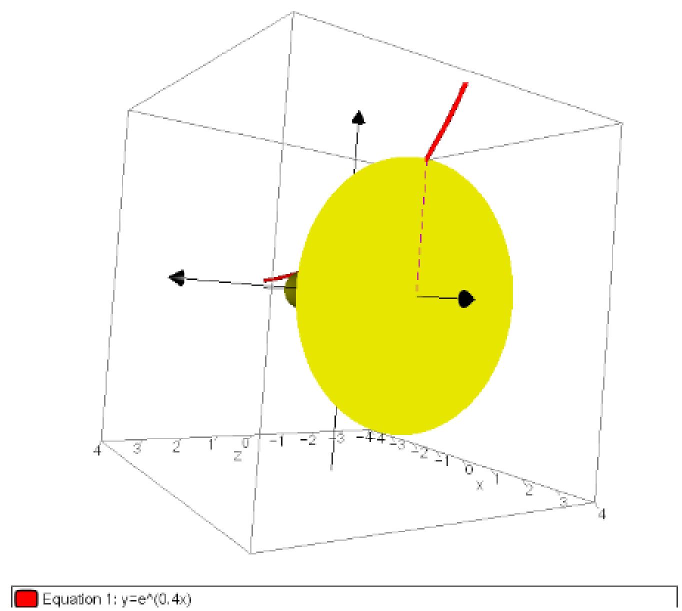




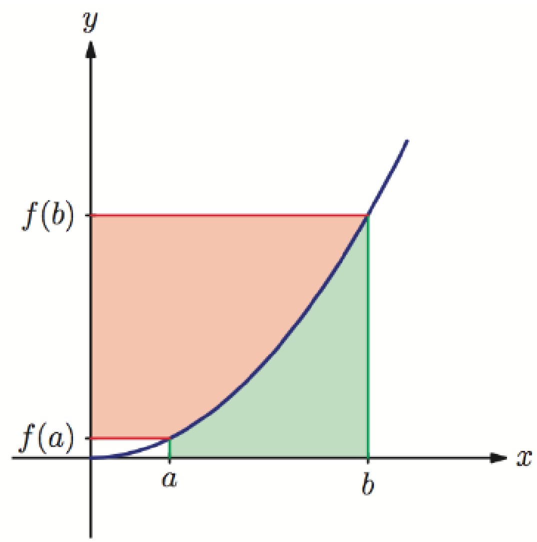



Equation 1: y=e^(0.4x)


Equation 1: y=e^(0.4x)


Equation 1: y=e^(0.4x)

Volumes of Revolution



- The volume of a solid of revolution formed when $y = f(x)$, which is continuous in the interval $[a, b]$, is rotated 2π radians about the x-axis is $V = \pi \int_a^b y^2 \mathrm{d}x$.
- The volume of a solid of revolution formed when $y = f(x)$, which is continuous in the interval $y = c$ to $y = d$, is rotated 2π radians about the y-axis is $V = \pi \int_c^d x^2 \mathrm{d}y$.