Selected solutions to Kenneth H. Rosen's
Elementary Number Theory and Its Applications (5th edition).

Larkin Wisdom

1.1.5. Use the well-ordering property to show that $\sqrt{3}$ is irrational.

Solution. Assume to the contrary that $\sqrt{3}$ is rational. Then $\sqrt{3} = \frac{a}{b}$ for some $a, b \in \mathbb{N}$. Consider the set $S = \{k\sqrt{3} \mid k, k\sqrt{3} \in \mathbb{N}\}$. This set is nonempty, as $a = b\sqrt{3} \in S$. As S is a nonempty set of positive integers, by the well-ordering property S has a least element, $s\sqrt{3}$. Now, consider $t = (s\sqrt{3} - s)\sqrt{3}$. As $s\sqrt{3} - s$ is a positive integer, $t \in S$. But $s\sqrt{3} - s < s$, and so t is smaller than the supposed smallest element of S ($s\sqrt{3}$), a contradiction.