Enumerating the Elements of the Eisenstein Array

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Abstract

In [2], we discuss several algorithms that enumerate the elements of the Eisenstein Array [4]. In this document we show and discuss several Haskell implementations of these algorithms.

1 The Eisenstein Array

Given two natural numbers m and n, Stern [4] describes a process (which he attributes to Eisenstein) of generating an infinite sequence of rows of numbers. The *zeroth* row in the sequence ("nullte Entwickelungsreihe") is the given pair of numbers:

$$m \qquad n \quad .$$

Subsequent rows are obtained by inserting between every pair of numbers the sum of the numbers. Thus the *first* row is

$$m \qquad m{+}n \qquad n$$

and the *second* row is

$$m \qquad 2{\times}m+n \qquad m{+}n \qquad m+2{\times}n \qquad n \quad .$$

The process of constructing such rows is repeated indefinitely. The sequence of numbers obtained by concatenating the individual rows in order is what is now called the *Eisenstein array* and denoted by $Ei(m,n)$ (see, for example, [3, sequence A064881]) . Stern refers to each occurrence of a number in rows other than the zeroth row as either a *sum element* ("Summenglied") or a *source element* ("Stammglied"). The sum elements are the newly added numbers. For example, in the first row the number $m{+}n$ is a sum element; in the second row the number $m{+}n$ is a source element.

2 Newman's Algorithm

An interesting question is whether Stern also documents the algorithm currently attributed to Moshe Newman for enumerating the elements of $Ei(1,1)$ [2, section 4.2.3]. Newman's algorithm predicts that each triple of numbers in a given row of $Ei(1,1)$ has the form

$$a \qquad b \qquad (2\left\lfloor \frac{a}{b} \right\rfloor + 1){\times}b - a \quad .$$

In [1, appendix A], we have implemented Newman's algorithm as follows.

$$
\begin{aligned}
&\mathit{cwnEnum} :: [\mathit{Rational}] \\
&\mathit{cwnEnum} = \mathit{iterate}\ \mathit{nextCW}\ 1/1 \\
&\quad \textbf{where}\ \mathit{nextCW} :: \mathit{Rational} \to \mathit{Rational} \\
&\qquad\qquad \mathit{nextCW}\ r = \textbf{let}\ (n, m) = (\mathit{numerator}\ r, \mathit{denominator}\ r) \\
&\qquad\qquad\qquad\qquad\qquad j \qquad = \lfloor n/m \rfloor \\
&\qquad\qquad\qquad\qquad \textbf{in}\ \ m/((2\times j + 1)\times m - n)
\end{aligned}
$$

For the purpose of this document, we are interested in the elements of $Ei(1{,}1)$, i.e., in the sequence of numerators given by *cwnEnum*. Function *newman* enumerates the elements of $Ei(1{,}1)$, using the infinite list created by *cwnEnum* (we have to detect a change in level).

$$
\begin{aligned}
&\mathit{newman} :: [\mathit{Integer}] \\
&\mathit{newman} = \mathit{concatMap}\ \mathit{dlevel}\ \mathit{cwnEnum} \\
&\quad \textbf{where}\ \mathit{dlevel}\ r \mid (\mathit{denominator}\ r) == 1 = [\mathit{numerator}\ r, 1] \\
&\qquad\qquad\qquad\quad \mid \mathit{otherwise} \qquad\qquad\quad = [\mathit{numerator}\ r]
\end{aligned}
$$

3 Enumerating the Elements of $Ei(m{,}n)$

One way of enumerating the elements of the array $Ei(m{,}n)$ is:

$$
\begin{aligned}
&\mathit{ei} :: \mathit{Integer} \to \mathit{Integer} \to [\mathit{Integer}] \\
&\mathit{ei}\ m\ n = m : \mathit{eiloop}\ 1\ 1\ m\ n\ m\ n \\
&\quad \textbf{where}\ \mathit{eiloop}\ a\ 1\ m\ n\ \mathit{cm}\ \mathit{cn} = n : \mathit{cm} : \mathit{eiloop}\ 1\ (a+1)\ \mathit{cm}\ (a\times \mathit{cm} + \mathit{cn})\ \mathit{cm}\ \mathit{cn} \\
&\qquad\qquad \mathit{eiloop}\ a\ b\ m\ n\ \mathit{cm}\ \mathit{cn} = \textbf{let}\ k = 2\times\lfloor a/b \rfloor + 1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textbf{in}\ n : \mathit{eiloop}\ b\ (k\times b - a)\ n\ (k\times n - m)\ \mathit{cm}\ \mathit{cn}
\end{aligned}
$$

We can test if the function *newman* is in fact enumerating the elements of $Ei(1{,}1)$. Let's compare the first 1000 elements of both enumerations:

$\gg$ (take 1000 newman) == (take 1000 (ei 1 1))
True
$\gg$ (take 1000 (map (2*) newman)) == (take 1000 (ei 2 2))
True

The part after the prompt, $\gg$, is the Haskell code that `ghci` is executing. The result is shown in the subsequent line. The second command shows an instance of the property:

$$\mathit{map}\ (k\times)\ (\mathit{ei}\ 1\ 1) == \mathit{ei}\ k\ k \qquad .$$

A property that we have not yet proved is that we can replace a and b by m and n in the calculation of k (when m and n are both positive).

$$
\begin{aligned}
&\mathit{ei}' :: \mathit{Integer} \to \mathit{Integer} \to [\mathit{Integer}] \\
&\mathit{ei}'\ m\ n = m : \mathit{eiloop}\ 1\ 1\ m\ n\ m\ n \\
&\quad \textbf{where}\ \mathit{eiloop}\ a\ 1\ m\ n\ \mathit{cm}\ \mathit{cn} = n : \mathit{cm} : \mathit{eiloop}\ 1\ (a+1)\ \mathit{cm}\ (a\times \mathit{cm} + \mathit{cn})\ \mathit{cm}\ \mathit{cn} \\
&\qquad\qquad \mathit{eiloop}\ a\ b\ m\ n\ \mathit{cm}\ \mathit{cn} = \textbf{let}\ k = 2\times\lfloor m/n \rfloor + 1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textbf{in}\ n : \mathit{eiloop}\ b\ (k\times b - a)\ n\ (k\times n - m)\ \mathit{cm}\ \mathit{cn}
\end{aligned}
$$

We now define the function *test*, which compares the first 1000 elements of two enumerations of $Ei(m{,}n)$, with $0 \leqslant m \leqslant x$ and $1 \leqslant n \leqslant x$:

$$\mathit{test}\ f\ g\ x = \mathit{and}\ [\mathit{take}\ 1000\ (f\ m\ n) == \mathit{take}\ 1000\ (g\ m\ n) \mid m \leftarrow [0\,..\,x], n \leftarrow [1\,..\,x]]$$

We can use $test$ to see if the first 1000 elements of $ei\ m\ n$ and $ei'\ m\ n$, for $0 \leqslant m \leqslant 100$ and $1 \leqslant n \leqslant 100$, are the same (we are testing 10100 pairs).

$\gg$ test ei ei' 100
True

The function $extnewman$, defined below, is the same as ei, but it replaces variables a and b by variable r:

$$
\begin{array}{l}
extnewman :: Integer \to Integer \to [Integer] \\
extnewman\ cm\ cn = cm : loop\ 0\ cm\ cn\ cm\ cn \\
\quad \mathbf{where}\ loop\ r\ m\ n\ cm\ cn \mid ((m == (cm + r \times cn)) \wedge (n == cn)) = \\
\qquad\qquad\qquad\qquad\qquad\qquad n : cm : loop\ (r+1)\ cm\ ((r+1) \times cm + cn)\ cm\ cn \\
\qquad\qquad\qquad\qquad \mid otherwise = \mathbf{let}\ k = 2 \times \lfloor m/n \rfloor + 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad \mathbf{in}\ n : loop\ r\ n\ (k \times n - m)\ cm\ cn
\end{array}
$$

We can do a similar test for $extnewman$ as we did for ei':

$\gg$ test ei extnewman 100
True

References

[1] Roland Backhouse and João F. Ferreira. Recounting the rationals: Twice! volume 5133, pages 79–91, 2008.

[2] Roland Backhouse and João F. Ferreira. On Euclid's algorithm and elementary number theory. 2009.

[3] Neil J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. `http://www.research.att.com/~njas/sequences/`.

[4] Moritz A. Stern. Üeber eine zahlentheoretische Funktion. *Journal für die reine und angewandte Mathematik*, 55:193–220, 1858.