University of Caloocan City
College of Liberal Arts & Sciences

Mathematics Department

Integral Calculus: Learning Module No. 7

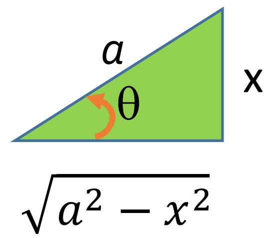
Topic	**METHODS OF INTEGRATION**
Sub-Topic	**Trigonometric Substitution**
Duration	3 hours
Introduction	There are certain types of integrals involving algebraic expressions which can be transformed into a problem of evaluating trigonometric integrals. The transformation will involve appropriate trigonometric substitutions for the original variable of integration.
Theories/Concepts/Formulas Note: This module may contain copyrighted material. The use of which has not been specifically authorized by the copyright owner. This module is for educational purpose only for online instruction and is not used to generate profit. Thus this constitutes a "Fair Use" of the copyrighted material as provided by virtue of Republic Act No. 8293 otherwise known as Intellectual Property Code of the Philippines.	**3. Trigonometric Substitution** The aforesaid trigonometric substitutions which will lead to integrable forms are listed as follows: 1. If the integrand contains $\boldsymbol{a^2 - u^2}$ use the substitution $\boldsymbol{u = a\ sin\theta}$. 2. If the integrand contains $\boldsymbol{a^2 + u^2}$ use the substitution $\boldsymbol{u = a\ tan\theta}$. 3. If the integrand contains $\boldsymbol{u^2 - a^2}$ use the substitution $\boldsymbol{u = a\ sec\theta}$
YouTube Link/s	https://www.youtube.com/watch?v=cmZWy3GmQw4 https://www.youtube.com/watch?v=DWWa7dSZmR8 https://www.youtube.com/watch?v=Fx10QvjpZrc https://www.youtube.com/watch?v=GVL4lHX6DgM https://www.youtube.com/watch?v=3lC5AuCFK4c
Sample Problems	Ex.1. Evaluate $\int \frac{dx}{\sqrt{a^2-x^2}}$ $a^2 = a^2$ $\quad$ $u^2 = x^2$

$a = a$ $\quad$ u = x

use: u = a sinθ

x = a sinθ $\qquad$ dx= a cosθ dθ

$$sin\theta = \frac{x}{a}$$



$$cos\ \theta = \frac{\sqrt{a^2 - x^2}}{a}$$

$$\sqrt{a^2 - x^2} = a\ cos\theta$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{\cancel{a}\ \cancel{cos\ \theta}\, d\theta}{\cancel{a}\ \cancel{cos\ \theta}}$$

$$= \int d\theta$$

$$= \theta + C$$

but

$$sin\theta = \frac{x}{a}$$

and

$$\theta = Arcsin\frac{x}{a}$$

Thus,

$$\boxed{\int \frac{\boldsymbol{dx}}{\sqrt{\boldsymbol{a^2 - x^2}}} = \boldsymbol{Arcsin\ \frac{x}{a} + C}}$$

Ex.2. Evaluate $\int \frac{(x+3)dx}{\sqrt{16 - x^2}}$

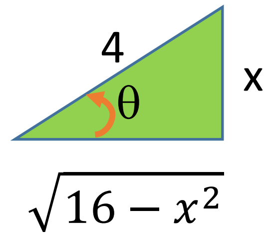
$a^2 = 16$ $\qquad$ $u^2 = x^2$

$a = 4$ $\qquad$ u = x

use: u = a sinθ

$x = 4\sin\theta \qquad dx = 4\cos\theta\, d\theta$

$$\sin\theta = \frac{x}{4}$$



$$\cancel{\cos\theta = \frac{\sqrt{16-x^2}}{4}}$$

$$\sqrt{16-x^2} = 4\cos\theta$$

$$\int \frac{(x+3)dx}{\sqrt{16-x^2}} = \int \frac{(4\sin\theta+3)(\cancel{4\cos\theta}\, d\theta)}{\cancel{4\cos\theta}}$$

$$= \int (4\sin\theta + 3)d\theta$$

$$= 4\int \sin\theta\, d\theta + 3\int d\theta$$

$$= 4(-\cos\theta) + 3\theta + C$$

but

$$-\cos\theta = -\frac{\sqrt{16-x^2}}{4}$$

and

$$\theta = Arcsin\frac{x}{4}$$

$$= \cancel{4}\left(-\frac{\sqrt{16-x^2}}{\cancel{4}}\right) + 3(Arcsin\frac{x}{4}) + C$$

Thus,

$$\boxed{\int \frac{(x+3)dx}{\sqrt{16-x^2}} = 3Arcsin\,\frac{x}{4} - \sqrt{16-x^2} + C}$$

Ex.3. Evaluate $\int \frac{dx}{(x^2+16)^{\frac{3}{2}}}$

$$= \int \frac{dx}{\left(\sqrt{x^2+16}\right)^3}$$

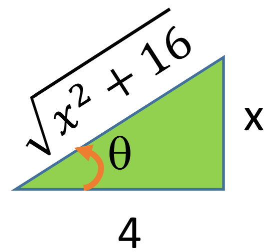
$a^2 = 16$ $\quad$ $u^2 = x^2$

$a = 4$ $\quad$ $u = x$

use: $u = a \tan\theta$

$x = 4\tan\theta$ $\quad$ $dx = 4\sec^2\theta\, d\theta$

$$\tan\theta = \frac{x}{4}$$



$$\sec\theta = \frac{\sqrt{x^2+16}}{4}$$

$$\sqrt{x^2+16} = 4\sec\theta$$

$$\int \frac{dx}{\left(\sqrt{x^2+16}\right)^3} = \int \frac{4\sec^2\theta\, d\theta}{(4\sec\theta)^3}$$

$$= \int \frac{4\sec^2\theta\, d\theta}{64\sec^3\theta}$$

$$= \cancel{\frac{4}{64}} \int \left(\frac{\cancel{\sec^2\theta}}{\sec^{\cancel{3}}\theta}\right) d\theta$$

$$= \frac{1}{16} \int \frac{d\theta}{\sec\theta}$$

from reciprocal identity

$\frac{1}{sec\theta} = cos\theta$

$$= \frac{1}{16}\int cos\theta d\theta$$

$$= \frac{1}{16}(sin\theta) + C$$

but

$$sin\theta = \frac{x}{\sqrt{x^2+16}}$$

$$= \frac{1}{16}\left(\frac{x}{\sqrt{x^2+16}}\right) + C$$

Thus,

$$\boxed{\int \frac{dx}{(x^2+16)^{\frac{3}{2}}} = \frac{x}{16(x^2+16)^{\frac{1}{2}}} + C}$$

Ex.4. Evaluate $\int \frac{dx}{x^2+8x+17}$

$$=\int \frac{dx}{x^2+8x+16+1}$$

$$=\int \frac{dx}{(x^2+8x+16)+1}$$

$$=\int \frac{dx}{(x+4)^2+1}$$

$a^2 = 1$ $\quad$ $u^2 = (x+4)^2$

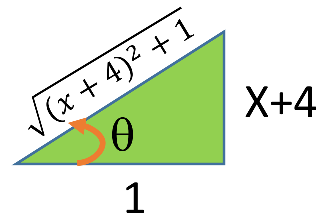
$a = 1$ $\quad$ $u = x+4$

use: $u = a\tan\theta$

$x+4 = \tan\theta$ $\quad$ $dx = \sec^2\theta\, d\theta$

$(x+4)^2 = \tan^2\theta$

$$tan\theta = x + 4$$



$$\int \frac{dx}{x^2+8x+17} = \int \frac{sec^2\theta d\theta}{tan^2\theta+1}$$

but from trigonometric identities

$tan^2\theta + 1 = sec^2\theta$

$$= \int \frac{\cancel{sec^2\theta}\, d\theta}{\cancel{sec^2\theta}}$$

$$= \int d\theta$$

$$= \theta + C$$

from

$$x + 4 = tan\theta$$

and

$$\theta = Arctan\,(x + 4)$$

Thus,

$$\boxed{\int \frac{dx}{x^2 + 8x + 17} = Arctan\,(x + 4) + C}$$

Ex.5. Evaluate $\int \frac{x^2 dx}{\sqrt{4x^2-9}}$

$a^2 = 9$ $\qquad$ $u^2 = 4x^2 = (2x)^2$

$a = 3$ $\qquad$ $u = 2x$

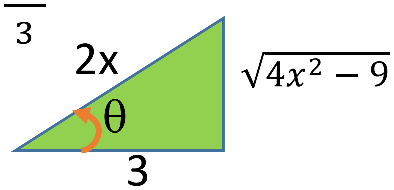
use: $u = a\,sec\theta$

2x = 3secθ,   2dx= 3secθ tanθ dθ

$$x = \frac{3}{2}sec\theta, \; dx = \frac{3}{2}sec\theta tan\theta d\theta$$

$$x^2 = \frac{9}{4}sec^2\theta$$

$$sec\theta = \frac{2x}{3}$$



$$tan\,\theta = \frac{\sqrt{4x^2-9}}{3}, \quad \sqrt{4x^2-9} = 3tan\theta$$

$$\int \frac{x^2dx}{\sqrt{4x^2-9}} = \int \frac{\left(\frac{9}{4}sec^2\theta\right)\left(\frac{\cancel{3}}{2}sec\theta \cancel{tan\theta} d\theta\right)}{\cancel{3\,tan\,\theta}}$$

$$= \frac{9}{8}\int sec^3\theta d\theta$$

$$= \frac{9}{8}\int sec\theta . sec^2\theta d\theta$$

u = secθ   dv = sec²θ dθ

du = secθ tanθ   v = tanθ

$$\int sec^3\theta d\theta = sec\theta tan\theta - \int (\tan\theta)(\sec\theta \tan\theta d\theta)$$

$$\int sec^3\theta d\theta = sec\theta tan\theta - \int tan^2\theta . sec\theta d\theta$$

but from trigonometric identities

$\tan^2\theta = \sec^2\theta - 1$

$$\int sec^3\theta d\theta = sec\theta tan\theta - \int (sec^2\theta - 1)\, sec\theta d\theta$$

$$\int sec^3\theta d\theta = sec\theta tan\theta - \int sec^3\theta d\theta + \int sec\theta\, d\theta$$

$$2\int sec^3\theta d\theta = sec\theta tan\theta + \int sec\theta\, d\theta$$

$$2\int sec^3\theta d\theta = sec\theta tan\theta + \ln(sec\theta + \tan\theta) + C$$

$$\int sec^3\theta d\theta = \frac{1}{2}[sec\theta tan\theta + \ln(sec\theta + \tan\theta)] + C$$

$$= \frac{9}{8}\cdot\frac{1}{2}[sec\theta tan\theta + \ln(sec\theta + \tan\theta)] + C$$

$$= \frac{9}{16}\left[\left(\frac{2x}{3}\right)\left(\frac{\sqrt{4x^2-9}}{3}\right) + \ln\left(\frac{2x}{3} + \frac{\sqrt{4x^2-9}}{3}\right)\right] + C$$

$$= \frac{9}{16}\left[\frac{2x\sqrt{4x^2-9}}{9} + \ln\left(\frac{2x+\sqrt{4x^2-9}}{3}\right)\right] + C$$

$$= \frac{\cancel{9}}{\cancel{16}}\cdot\frac{\cancel{2}x\sqrt{4x^2-9}}{\cancel{9}} + \frac{9}{16}ln\left(\frac{2x+\sqrt{4x^2-9}}{3}\right) + C$$

$$= \frac{x\sqrt{4x^2-9}}{8} + \frac{9}{16}ln\left(\frac{2x+\sqrt{4x^2-9}}{3}\right) + C$$

Thus, $\int \frac{x^2dx}{\sqrt{4x^2-9}}$

$$\boxed{= \frac{x\sqrt{4x^2-9}}{8} + ln\left(\frac{2x+\sqrt{4x^2-9}}{3}\right)^{\frac{9}{16}} + C}$$