University of Caloocan City

College of Liberal Arts & Sciences

Mathematics Department

Integral Calculus: Practice Problem

Integration By Parts

Evaluate each of the following indefinite integrals using Integration by Parts

1. $\int x^6 cos3x\, dx$
2. $\int e^{2x} cos\frac{1}{4}x\, dx$
3. $\int e^{\frac{1}{3}x}(5x+2)dx$
4. $\int x^7 sin2x^4 dx$
5. $\int 6Arctan\frac{8}{x} dx$

Problem:

$$\int x^6\cos(3x)\,dx$$

Integrate by parts: $\int fg' = fg - \int f'g$

$f = x^6,\quad g' = \cos(3x)$

$\downarrow$ steps $\quad \downarrow$ steps

$f' = 6x^5,\ g = \frac{\sin(3x)}{3}$:

$$= \frac{x^6\sin(3x)}{3} - \int 2x^5\sin(3x)\,dx$$

Now solving:

$$\int 2x^5\sin(3x)\,dx$$

Apply linearity:

$$= 2\int x^5\sin(3x)\,dx$$

Now solving:

$$\int x^5\sin(3x)\,dx$$

Integrate by parts: $\int fg' = fg - \int f'g$

$f = x^5,\quad g' = \sin(3x)$

$\downarrow$ steps $\quad \downarrow$ steps

$f' = 5x^4,\ g = -\frac{\cos(3x)}{3}$:

$$= -\frac{x^5\cos(3x)}{3} - \int -\frac{5x^4\cos(3x)}{3}\,dx$$

Now solving:

$$\int -\frac{5x^4\cos(3x)}{3}\,dx$$

Apply linearity:

$$= -\frac{5}{3}\int x^4\cos(3x)\,dx$$

Now solving:

$$\int x^4\cos(3x)\,dx$$

Integrate by parts: $\int fg' = fg - \int f'g$

$f = x^4,\quad g' = \cos(3x)$

$\downarrow$ steps $\quad \downarrow$ steps

$f' = 4x^3,\ g = \frac{\sin(3x)}{3}$:

$$= \frac{x^4\sin(3x)}{3} - \int \frac{4x^3\sin(3x)}{3}\,dx$$

Now solving:

$$\int \frac{4x^3\sin(3x)}{3}\,dx$$

Apply linearity:

$$= \frac{4}{3}\int x^3\sin(3x)\,dx$$

Now solving:

$$\int x^3\sin(3x)\,dx$$

Integrate by parts: $\int fg' = fg - \int f'g$

$f = x^3,\quad g' = \sin(3x)$

$\downarrow$ steps $\quad \downarrow$ steps

$f' = 3x^2,\ g = -\frac{\cos(3x)}{3}$:

$$= -\frac{x^3\cos(3x)}{3} - \int -x^2\cos(3x)\,dx$$

Now solving:

$$\int x\sin(3x)\,dx$$

Integrate by parts: $\int fg' = fg - \int f'g$

$f = x,\quad g' = \sin(3x)$

$\downarrow$ steps $\quad \downarrow$ steps

$f' = 1,\ g = -\frac{\cos(3x)}{3}$:

$$= -\frac{x\cos(3x)}{3} - \int -\frac{\cos(3x)}{3}\,dx$$

Now solving:

$$\int -\frac{\cos(3x)}{3}\,dx$$

Substitute $u = 3x \longrightarrow du = 3\,dx$ (steps):

$$= -\frac{1}{9}\int \cos(u)\,du$$

Now solving:

$$\int \cos(u)\,du$$

This is a standard integral:

$$= \sin(u)$$

Plug in solved integrals:

$$-\frac{1}{9}\int \cos(u)\,du$$

$$= -\frac{\sin(u)}{9}$$

Undo substitution $u = 3x$:

$$= -\frac{\sin(3x)}{9}$$

Plug in solved integrals:

$$-\frac{x\cos(3x)}{3} - \int -\frac{\cos(3x)}{3}\,dx$$

$$= \frac{\sin(3x)}{9} - \frac{x\cos(3x)}{3}$$

Plug in solved integrals:

$$\frac{2}{3}\int x\sin(3x)\,dx$$

$$= \frac{2\sin(3x)}{27} - \frac{2x\cos(3x)}{9}$$

Plug in solved integrals:

$$\frac{x^2\sin(3x)}{3} - \int \frac{2x\sin(3x)}{3}\,dx$$

$$= \frac{x^2\sin(3x)}{3} - \frac{2\sin(3x)}{27} + \frac{2x\cos(3x)}{9}$$

Plug in solved integrals:

$$-\int x^2\cos(3x)\,dx$$

$$= -\frac{x^2\sin(3x)}{3} + \frac{2\sin(3x)}{27} - \frac{2x\cos(3x)}{9}$$

Plug in solved integrals:

$$-\frac{x^3\cos(3x)}{3} - \int -x^2\cos(3x)\,dx$$

$$= \frac{x^2\sin(3x)}{3} - \frac{2\sin(3x)}{27} - \frac{x^3\cos(3x)}{3} + \frac{2x\cos(3x)}{9}$$

Plug in solved integrals:

$$\frac{4}{3}\int x^3\sin(3x)\,dx$$

$$= \frac{4x^2\sin(3x)}{9} - \frac{8\sin(3x)}{81} - \frac{4x^3\cos(3x)}{9} + \frac{8x\cos(3x)}{27}$$

Plug in solved integrals:

$$\frac{x^4\sin(3x)}{3} - \int \frac{4x^3\sin(3x)}{3}\,dx$$

$$= \frac{x^4\sin(3x)}{3} - \frac{4x^2\sin(3x)}{9} + \frac{8\sin(3x)}{81} + \frac{4x^3\cos(3x)}{9} - \frac{8x\cos(3x)}{27}$$

Plug in solved integrals:

$$-\frac{5}{3}\int x^4\cos(3x)\,dx$$

$$= -\frac{5x^4\sin(3x)}{9} + \frac{20x^2\sin(3x)}{27} - \frac{40\sin(3x)}{243} - \frac{20x^3\cos(3x)}{27} + \frac{40x\cos(3x)}{81}$$

Plug in solved integrals:

$$-\frac{x^5\cos(3x)}{3} - \int -\frac{5x^4\cos(3x)}{3}\,dx$$

$$= \frac{5x^4\sin(3x)}{9} - \frac{20x^2\sin(3x)}{27} + \frac{40\sin(3x)}{243} - \frac{x^5\cos(3x)}{3} + \frac{20x^3\cos(3x)}{27} - \frac{40x\cos(3x)}{81}$$

Plug in solved integrals:

$$2\int x^5\sin(3x)\,dx$$

$$= \frac{10x^4\sin(3x)}{9} - \frac{40x^2\sin(3x)}{27} + \frac{80\sin(3x)}{243} - \frac{2x^5\cos(3x)}{3} + \frac{40x^3\cos(3x)}{27} - \frac{80x\cos(3x)}{81}$$

Plug in solved integrals:

$$\frac{x^6\sin(3x)}{3} - \int 2x^5\sin(3x)\,dx$$

$$= \frac{x^6\sin(3x)}{3} - \frac{10x^4\sin(3x)}{9} + \frac{40x^2\sin(3x)}{27} - \frac{80\sin(3x)}{243} + \frac{2x^5\cos(3x)}{3} - \frac{40x^3\cos(3x)}{27} + \frac{80x\cos(3x)}{81}$$

The problem is solved:

$$\int x^6\cos(3x)\,dx$$

$$= \frac{x^6\sin(3x)}{3} - \frac{10x^4\sin(3x)}{9} + \frac{40x^2\sin(3x)}{27} - \frac{80\sin(3x)}{243} + \frac{2x^5\cos(3x)}{3} - \frac{40x^3\cos(3x)}{27} + \frac{80x\cos(3x)}{81} + C$$

Rewrite/simplify:

$$= \frac{(81x^6 - 270x^4 + 360x^2 - 80)\sin(3x) + (162x^5 - 360x^3 + 240x)\cos(3x)}{243} + C$$

Problem:

$$\int \cos\left(\frac{x}{4}\right) e^{2x}\, dx$$

We will integrate by parts twice in a row: $\int fg' = fg - \int f'g$.

First time:

$$f = e^{2x},\quad g' = \cos\left(\frac{x}{4}\right)$$

↓ steps ↓ steps

$$f' = \boxed{2e^{2x}},\ g = \boxed{4\sin\left(\frac{x}{4}\right)}:$$

$$= 4\sin\left(\frac{x}{4}\right) e^{2x} - \int 8\sin\left(\frac{x}{4}\right) e^{2x}\, dx$$

Second time:

$$f = \boxed{2e^{2x}},\ g' = \boxed{4\sin\left(\frac{x}{4}\right)}$$

↓ steps ↓ steps

$$f' = 4e^{2x},\quad g = -16\cos\left(\frac{x}{4}\right):$$

$$= 4\sin\left(\frac{x}{4}\right) e^{2x} - \left(-32\cos\left(\frac{x}{4}\right) e^{2x} - \int -64\cos\left(\frac{x}{4}\right) e^{2x}\, dx\right)$$

Apply linearity:

$$= 4\sin\left(\frac{x}{4}\right) e^{2x} - \left(-32\cos\left(\frac{x}{4}\right) e^{2x} + 64\int \cos\left(\frac{x}{4}\right) e^{2x}\, dx\right)$$

$\int \cos\left(\frac{x}{4}\right) e^{2x}\, dx$ appears again on the right side of the equation, we

$$= -\frac{16\left(-\frac{\sin\left(\frac{x}{4}\right) e^{2x}}{4} - 2\cos\left(\frac{x}{4}\right) e^{2x}\right)}{65}$$

The problem is solved:

$$\int \cos\left(\frac{x}{4}\right) e^{2x}\, dx$$

$$= -\frac{16\left(-\frac{\sin\left(\frac{x}{4}\right) e^{2x}}{4} - 2\cos\left(\frac{x}{4}\right) e^{2x}\right)}{65} + C$$

Rewrite/simplify:

$$= \frac{4\left(\sin\left(\frac{x}{4}\right) + 8\cos\left(\frac{x}{4}\right)\right) e^{2x}}{65} + C$$

Problem:

$$\int (5x+2)\, e^{\frac{x}{3}}\, dx$$

Integrate by parts: $\int fg' = fg - \int f'g$

$$f = 5x+2,\ g' = e^{\frac{x}{3}}$$

↓ steps ↓ steps

$$f' = 5,\quad g = 3e^{\frac{x}{3}}:$$

$$= 3(5x+2)\, e^{\frac{x}{3}} - \int 15e^{\frac{x}{3}}\, dx$$

Now solving:

$$\int 15e^{\frac{x}{3}}\, dx$$

Substitute $u = \frac{x}{3} \longrightarrow du = \frac{1}{3}\, dx$ (steps):

$$= 45\int e^u\, du$$

Now solving:

$$\int e^u\, du$$

Apply exponential rule:

$$\int a^u\, du = \frac{a^u}{\ln(a)} \text{ with } a = e:$$

$$= e^u$$

Plug in solved integrals:

$$45\int e^u\, du$$

$$= 45e^u$$

Undo substitution $u = \frac{x}{3}$:

$$= 45e^{\frac{x}{3}}$$

Plug in solved integrals:

$$3(5x+2)\, e^{\frac{x}{3}} - \int 15e^{\frac{x}{3}}\, dx$$

$$= 3(5x+2)\, e^{\frac{x}{3}} - 45e^{\frac{x}{3}}$$

The problem is solved:

$$\int (5x+2)\, e^{\frac{x}{3}}\, dx$$

$$= 3(5x+2)\, e^{\frac{x}{3}} - 45e^{\frac{x}{3}} + C$$

Rewrite/simplify:

$$= (15x-39)\, e^{\frac{x}{3}} + C$$

Problem:

$$\int x^7 \sin(2x^4)\, dx$$

Substitute $u = x^4 \longrightarrow du = 4x^3\, dx$ (steps):

$$= \frac{1}{4}\int u \sin(2u)\, du$$

Now solving:

$$\int u \sin(2u)\, du$$

Integrate by parts: $\int \mathtt{f}\mathtt{g}' = \mathtt{f}\mathtt{g} - \int \mathtt{f}'\mathtt{g}$

$$\mathtt{f} = u, \quad \mathtt{g}' = \sin(2u)$$

↓ steps ↓ steps

$$\mathtt{f}' = 1, \quad \mathtt{g} = -\frac{\cos(2u)}{2}:$$

$$= -\frac{u\cos(2u)}{2} - \int -\frac{\cos(2u)}{2}\, du$$

Now solving:

$$\int -\frac{\cos(2u)}{2}\, du$$

Substitute $v = 2u \longrightarrow dv = 2\, du$ (steps):

$$= -\frac{1}{4}\int \cos(v)\, dv$$

Now solving:

$$\int \cos(v)\, dv$$

This is a standard integral:

$$= \sin(v)$$

Plug in solved integrals:

$$-\frac{1}{4}\int \cos(v)\, dv$$

$$= -\frac{\sin(v)}{4}$$

Undo substitution $v = 2u$:

$$= -\frac{\sin(2u)}{4}$$

Plug in solved integrals:

$$-\frac{u\cos(2u)}{2} - \int -\frac{\cos(2u)}{2}\, du$$

$$= \frac{\sin(2u)}{4} - \frac{u\cos(2u)}{2}$$

Plug in solved integrals:

$$\frac{1}{4}\int u \sin(2u)\, du$$

$$= \frac{\sin(2u)}{16} - \frac{u\cos(2u)}{8}$$

Undo substitution $u = x^4$:

$$= \frac{\sin(2x^4)}{16} - \frac{x^4\cos(2x^4)}{8}$$

The problem is solved:

$$\int x^7 \sin(2x^4)\, dx$$

$$= \frac{\sin(2x^4)}{16} - \frac{x^4\cos(2x^4)}{8} + C$$

Rewrite/simplify:

$$= \frac{\sin(2x^4) - 2x^4\cos(2x^4)}{16} + C$$

Problem:

$$\int 6\arctan\left(\frac{8}{x}\right) dx$$

Apply linearity:

$$= 6\int \arctan\left(\frac{8}{x}\right) dx$$

Now solving:

$$\int \arctan\left(\frac{8}{x}\right) dx$$

Integrate by parts: $\int \mathtt{f}\mathtt{g}' = \mathtt{f}\mathtt{g} - \int \mathtt{f}'\mathtt{g}$

$$\mathtt{f} = \arctan\left(\frac{8}{x}\right), \quad \mathtt{g}' = 1$$

↓ steps ↓ steps

$$\mathtt{f}' = -\frac{8}{x^2+64}, \quad \mathtt{g} = x:$$

$$= \arctan\left(\frac{8}{x}\right)x - \int -\frac{8x}{x^2+64}\, dx$$

Now solving:

$$\int -\frac{8x}{x^2+64}\, dx$$

Substitute $u = x^2 + 64 \longrightarrow du = 2x\, dx$ (steps):

$$= -4\int \frac{1}{u}\, du$$

Now solving:

$$\int \frac{1}{u}\, du$$

This is a standard integral:

$$= \ln(u)$$

Plug in solved integrals:

$$-4\int \frac{1}{u}\, du$$

$$= -4\ln(u)$$

Undo substitution $u = x^2 + 64$:

$$= -4\ln(x^2+64)$$

Plug in solved integrals:

$$\arctan\left(\frac{8}{x}\right)x - \int -\frac{8x}{x^2+64}\, dx$$

$$= 4\ln(x^2+64) + \arctan\left(\frac{8}{x}\right)x$$

Plug in solved integrals:

$$6\int \arctan\left(\frac{8}{x}\right) dx$$

$$= 24\ln(x^2+64) + 6\arctan\left(\frac{8}{x}\right)x$$

The problem is solved:

$$\int 6\arctan\left(\frac{8}{x}\right) dx$$

$$= 24\ln(x^2+64) + 6\arctan\left(\frac{8}{x}\right)x + C$$

Algebraic Substitution

Evaluate each of the following indefinite integrals using Algebraic Substitution

1. $\int \frac{(8x+1)}{\sqrt{4x-3}}\,dx$
2. $\int x^3\sqrt{2x^2+1}\,dx$
3. $\int \frac{x^3}{(x^2+1)^3}\,dx$
4. $\int \frac{x}{\sqrt[4]{2x+1}}\,dx$
5. $\int \frac{(6x-1)}{(2x+1)^{3/2}}\,dx$

Problem:

$$\int \frac{8x+1}{\sqrt{4x-3}}\,dx$$

Substitute $u = 4x-3 \longrightarrow du = 4\,dx$ (steps):

$$= \frac{1}{4}\int \frac{2u+7}{\sqrt{u}}\,du$$

Now solving:

$$\int \frac{2u+7}{\sqrt{u}}\,du$$

Expand:

$$= \int \left(2\sqrt{u} + \frac{7}{\sqrt{u}}\right) du$$

Apply linearity:

$$= 2\int \sqrt{u}\,du + 7\int \frac{1}{\sqrt{u}}\,du$$

Now solving:

$$\int \sqrt{u}\,du$$

Apply power rule:

$$\int u^n\,du = \frac{u^{n+1}}{n+1} \text{ with } n = \frac{1}{2}:$$

$$= \frac{2u^{\frac{3}{2}}}{3}$$

Now solving:

$$\int \frac{1}{\sqrt{u}}\,du$$

Apply power rule with $n = -\frac{1}{2}$:

$$= 2\sqrt{u}$$

Plug in solved integrals:

$$2\int \sqrt{u}\,du + 7\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{4u^{\frac{3}{2}}}{3} + 14\sqrt{u}$$

Plug in solved integrals:

$$\frac{1}{4}\int \frac{2u+7}{\sqrt{u}}\,du$$

$$= \frac{u^{\frac{3}{2}}}{3} + \frac{7\sqrt{u}}{2}$$

Undo substitution $u = 4x-3$:

$$= \frac{(4x-3)^{\frac{3}{2}}}{3} + \frac{7\sqrt{4x-3}}{2}$$

The problem is solved:

$$\int \frac{8x+1}{\sqrt{4x-3}}\,dx$$

$$= \frac{(4x-3)^{\frac{3}{2}}}{3} + \frac{7\sqrt{4x-3}}{2} + C$$

Rewrite/simplify:

$$= \frac{\sqrt{4x-3}\,(8x+15)}{6} + C$$

Problem:

$$\int x^3\sqrt{2x^2+1}\,\mathrm{d}x$$

Prepare for substitution:

$$=\int 2x\cdot\frac{x^2\sqrt{2x^2+1}}{2}\,\mathrm{d}x$$

Substitute $u=x^2 \longrightarrow \mathrm{d}u=2x\,\mathrm{d}x$ (steps):

$$=\frac{1}{2}\int u\sqrt{2u+1}\,\mathrm{d}u$$

Now solving:

$$\int u\sqrt{2u+1}\,\mathrm{d}u$$

Substitute $v=2u+1 \longrightarrow \mathrm{d}v=2\,\mathrm{d}u$ (steps):

$$=\frac{1}{4}\int\left(v^{\frac{3}{2}}-\sqrt{v}\right)\mathrm{d}v$$

Now solving:

$$\int\left(v^{\frac{3}{2}}-\sqrt{v}\right)\mathrm{d}v$$

Apply linearity:

$$=\int v^{\frac{3}{2}}\,\mathrm{d}v-\int\sqrt{v}\,\mathrm{d}v$$

Now solving:

$$\int v^{\frac{3}{2}}\,\mathrm{d}v$$

Apply power rule:

$$\int v^{\mathrm{n}}\,\mathrm{d}v=\frac{v^{\mathrm{n}+1}}{\mathrm{n}+1}\text{ with }\mathrm{n}=\frac{3}{2}:$$

$$=\frac{2v^{\frac{5}{2}}}{5}$$

Now solving:

$$\int\sqrt{v}\,\mathrm{d}v$$

Apply power rule with $\mathrm{n}=\frac{1}{2}$:

$$=\frac{2v^{\frac{3}{2}}}{3}$$

Plug in solved integrals:

$$\int v^{\frac{3}{2}}\,\mathrm{d}v-\int\sqrt{v}\,\mathrm{d}v$$

$$=\frac{2v^{\frac{5}{2}}}{5}-\frac{2v^{\frac{3}{2}}}{3}$$

Plug in solved integrals:

$$\frac{1}{4}\int\left(v^{\frac{3}{2}}-\sqrt{v}\right)\mathrm{d}v$$

$$=\frac{v^{\frac{5}{2}}}{10}-\frac{v^{\frac{3}{2}}}{6}$$

Undo substitution $v=2u+1$:

$$=\frac{(2u+1)^{\frac{5}{2}}}{10}-\frac{(2u+1)^{\frac{3}{2}}}{6}$$

Plug in solved integrals:

$$\frac{1}{2}\int u\sqrt{2u+1}\,\mathrm{d}u$$

$$=\frac{(2u+1)^{\frac{5}{2}}}{20}-\frac{(2u+1)^{\frac{3}{2}}}{12}$$

Undo substitution $u=x^2$:

$$=\frac{(2x^2+1)^{\frac{5}{2}}}{20}-\frac{(2x^2+1)^{\frac{3}{2}}}{12}$$

The problem is solved:

$$\int x^3\sqrt{2x^2+1}\,\mathrm{d}x$$

$$=\frac{(2x^2+1)^{\frac{5}{2}}}{20}-\frac{(2x^2+1)^{\frac{3}{2}}}{12}+C$$

Rewrite/simplify:

$$=\frac{(2x^2+1)^{\frac{3}{2}}(3x^2-1)}{30}+C$$

Problem:

$$\int\frac{x^3}{\left(x^2+1\right)^3}\,\mathrm{d}x$$

Substitute $u=x^2+1 \longrightarrow \mathrm{d}u=2x\,\mathrm{d}x$ (steps):

$$=\frac{1}{2}\int\frac{u-1}{u^3}\,\mathrm{d}u$$

... or choose an alternative:

Substitute x^2

Don't substitute

Now solving:

$$\int\frac{u-1}{u^3}\,\mathrm{d}u$$

Expand:

$$=\int\left(\frac{1}{u^2}-\frac{1}{u^3}\right)\mathrm{d}u$$

Apply linearity:

$$=\int\frac{1}{u^2}\,\mathrm{d}u-\int\frac{1}{u^3}\,\mathrm{d}u$$

Now solving:

$$\int\frac{1}{u^2}\,\mathrm{d}u$$

Apply power rule:

$$\int u^{\mathrm{n}}\,\mathrm{d}u=\frac{u^{\mathrm{n}+1}}{\mathrm{n}+1}\text{ with }\mathrm{n}=-2:$$

$$=-\frac{1}{u}$$

Now solving:

$$\int\frac{1}{u^3}\,\mathrm{d}u$$

Apply power rule with $\mathrm{n}=-3$:

$$=-\frac{1}{2u^2}$$

Plug in solved integrals:

$$\int\frac{1}{u^2}\,\mathrm{d}u-\int\frac{1}{u^3}\,\mathrm{d}u$$

$$=\frac{1}{2u^2}-\frac{1}{u}$$

Plug in solved integrals:

$$\frac{1}{2}\int\frac{u-1}{u^3}\,\mathrm{d}u$$

$$=\frac{1}{4u^2}-\frac{1}{2u}$$

Undo substitution $u=x^2+1$:

$$=\frac{1}{4\left(x^2+1\right)^2}-\frac{1}{2\left(x^2+1\right)}$$

The problem is solved:

$$\int\frac{x^3}{\left(x^2+1\right)^3}\,\mathrm{d}x$$

$$=\frac{1}{4\left(x^2+1\right)^2}-\frac{1}{2\left(x^2+1\right)}+C$$

Rewrite/simplify:

$$=-\frac{2x^2+1}{4\left(x^2+1\right)^2}+C$$

Problem:

$$\int \frac{x}{\sqrt[4]{2x+1}}\,\mathrm{d}x$$

Substitute $u = 2x + 1 \longrightarrow \mathrm{d}u = 2\,\mathrm{d}x$ (steps):

$$= \frac{1}{4}\int \frac{u-1}{\sqrt[4]{u}}\,\mathrm{d}u$$

Now solving:

$$\int \frac{u-1}{\sqrt[4]{u}}\,\mathrm{d}u$$

Expand:

$$= \int \left(u^{\frac{3}{4}} - \frac{1}{\sqrt[4]{u}}\right)\mathrm{d}u$$

Apply linearity:

$$= \int u^{\frac{3}{4}}\,\mathrm{d}u - \int \frac{1}{\sqrt[4]{u}}\,\mathrm{d}u$$

Now solving:

$$\int u^{\frac{3}{4}}\,\mathrm{d}u$$

Apply power rule:

$$\int u^{\mathrm{n}}\,\mathrm{d}u = \frac{u^{\mathrm{n}+1}}{\mathrm{n}+1} \text{ with } \mathrm{n} = \frac{3}{4}:$$

$$= \frac{4u^{\frac{7}{4}}}{7}$$

Now solving:

$$\int \frac{1}{\sqrt[4]{u}}\,\mathrm{d}u$$

Apply power rule with $\mathrm{n} = -\frac{1}{4}$:

$$= \frac{4u^{\frac{3}{4}}}{3}$$

Plug in solved integrals:

$$\int u^{\frac{3}{4}}\,\mathrm{d}u - \int \frac{1}{\sqrt[4]{u}}\,\mathrm{d}u$$

$$= \frac{4u^{\frac{7}{4}}}{7} - \frac{4u^{\frac{3}{4}}}{3}$$

Plug in solved integrals:

$$\frac{1}{4}\int \frac{u-1}{\sqrt[4]{u}}\,\mathrm{d}u$$

$$= \frac{u^{\frac{7}{4}}}{7} - \frac{u^{\frac{3}{4}}}{3}$$

Undo substitution $u = 2x + 1$:

$$= \frac{(2x+1)^{\frac{7}{4}}}{7} - \frac{(2x+1)^{\frac{3}{4}}}{3}$$

The problem is solved:

$$\int \frac{x}{\sqrt[4]{2x+1}}\,\mathrm{d}x$$

$$= \frac{(2x+1)^{\frac{7}{4}}}{7} - \frac{(2x+1)^{\frac{3}{4}}}{3} + C$$

Rewrite/simplify:

$$= \frac{2(2x+1)^{\frac{3}{4}}(3x-2)}{21} + C$$

Problem:

$$\int \frac{6x-1}{(2x+1)^{\frac{3}{2}}}\,\mathrm{d}x$$

Substitute $u = 2x + 1 \longrightarrow \mathrm{d}u = 2\,\mathrm{d}x$ (steps):

$$= \frac{1}{2}\int \frac{3u-4}{u^{\frac{3}{2}}}\,\mathrm{d}u$$

Now solving:

$$\int \frac{3u-4}{u^{\frac{3}{2}}}\,\mathrm{d}u$$

Expand:

$$= \int \left(\frac{3}{\sqrt{u}} - \frac{4}{u^{\frac{3}{2}}}\right)\mathrm{d}u$$

Apply linearity:

$$= 3\int \frac{1}{\sqrt{u}}\,\mathrm{d}u - 4\int \frac{1}{u^{\frac{3}{2}}}\,\mathrm{d}u$$

Now solving:

$$\int \frac{1}{\sqrt{u}}\,\mathrm{d}u$$

Apply power rule:

$$\int u^{\mathrm{n}}\,\mathrm{d}u = \frac{u^{\mathrm{n}+1}}{\mathrm{n}+1} \text{ with } \mathrm{n} = -\frac{1}{2}:$$

$$= 2\sqrt{u}$$

Now solving:

$$\int \frac{1}{u^{\frac{3}{2}}}\,\mathrm{d}u$$

Apply power rule with $\mathrm{n} = -\frac{3}{2}$:

$$= -\frac{2}{\sqrt{u}}$$

Plug in solved integrals:

$$3\int \frac{1}{\sqrt{u}}\,\mathrm{d}u - 4\int \frac{1}{u^{\frac{3}{2}}}\,\mathrm{d}u$$

$$= 6\sqrt{u} + \frac{8}{\sqrt{u}}$$

Plug in solved integrals:

$$\frac{1}{2}\int \frac{3u-4}{u^{\frac{3}{2}}}\,\mathrm{d}u$$

$$= 3\sqrt{u} + \frac{4}{\sqrt{u}}$$

Undo substitution $u = 2x + 1$:

$$= 3\sqrt{2x+1} + \frac{4}{\sqrt{2x+1}}$$

The problem is solved:

$$\int \frac{6x-1}{(2x+1)^{\frac{3}{2}}}\,\mathrm{d}x$$

$$= 3\sqrt{2x+1} + \frac{4}{\sqrt{2x+1}} + C$$

Rewrite/simplify:

$$= \frac{6x+7}{\sqrt{2x+1}} + C$$

Trigonometric Substitution

Evaluate each of the following indefinite integrals using Algebraic Substitution

1. $\int \frac{\sqrt{x^{2}+16}}{x^{4}}\,dx$
2. $\int x^{3}(3x^{2}-4)^{5/2}\,dx$
3. $\int \frac{2}{x^{4}\sqrt{x^{2}-25}}\,dx$
4. $\int 2x^{5}\sqrt{9x^{2}+2}\,dx$
5. $\int \frac{(x+3)^{5}}{(40-6x-x^{2})^{3/2}}\,dx$

Simplification:

1. **Simplify the integrand:**

$$\frac{\sqrt{x^{18}}}{x^{4}}=\frac{x^{9}}{x^{4}}=x^{9-4}=x^{5}$$

2. **Integrate the simplified function:**

$$\int x^{5}\,dx$$

3. **Apply the power rule for integration:**

The power rule for integration is $\int x^{n}\,dx=\frac{x^{n+1}}{n+1}+C$

.

So, for $n=5$:

$$\int x^{5}\,dx=\frac{x^{6}}{6}+C$$

Conclusion:

Thus, the integral $\int \frac{\sqrt{x^{18}}}{x^{4}}\,dx$ simplifies to:

$$\int x^{5}\,dx=\frac{x^{6}}{6}+C$$

Simplification using substitution:

1. **Substitute $u=3x^{2}-4$:**

   Then $du=6x\,dx$ or $\frac{du}{6}=x\,dx$.

2. **Rewrite the integral:**

   Notice that $x^{3}=x\cdot x^{2}$, and from the substitution $u=3x^{2}-4$, we have $x^{2}=\frac{u+4}{3}$. So, we can rewrite the integral in terms of u.

$$\int x^{3}(3x^{2}-4)^{5/2}\,dx=\int x\cdot x^{2}(3x^{2}-4)^{5/2}\,dx=\int x\cdot\frac{u+4}{3}\cdot u^{5/2}\,dx$$

3. **Substitute $x\,dx$ with $\frac{du}{6}$:**

$$x\,dx=\frac{du}{6}$$

   So the integral becomes:

$$\int\frac{u+4}{3}\cdot u^{5/2}\cdot\frac{du}{6}$$

   Simplify the constants:

$$\int\frac{1}{18}(u+4)\cdot u^{5/2}\,du$$

   Distribute $u^{5/2}$:

$$\int\frac{1}{18}(u^{7/2}+4u^{5/2})\,du$$

4. **Apply linearity of the integral:**

$$\frac{1}{18}\left(\int u^{7/2}\,du+4\int u^{5/2}\,du\right)$$

5. **Apply the power rule for integration:**

   The power rule for integration is $\int u^{n}\,du=\frac{u^{n+1}}{n+1}+C$.

   For $n=7/2$:

$$\int u^{7/2}\,du=\frac{u^{9/2}}{9/2}=\frac{2}{9}u^{9/2}$$

   For $n=5/2$:

$$\int u^{5/2}\,du=\frac{u^{7/2}}{7/2}=\frac{2}{7}u^{7/2}$$

6. **Combine the results:**

$$\frac{1}{18}\left(\frac{2}{9}u^{9/2}+4\cdot\frac{2}{7}u^{7/2}\right)$$
$$=\frac{1}{18}\left(\frac{2}{9}u^{9/2}+\frac{8}{7}u^{7/2}\right)$$
$$=\frac{2}{18\cdot 9}u^{9/2}+\frac{8}{18\cdot 7}u^{7/2}$$
$$=\frac{1}{81}u^{9/2}+\frac{4}{63}u^{7/2}$$

7. **Substitute back $u=3x^{2}-4$:**

$$=\frac{1}{81}(3x^{2}-4)^{9/2}+\frac{4}{63}(3x^{2}-4)^{7/2}+C$$

Conclusion:

So, the integral $\int x^{3}\cdot(3x^{2}-4)^{5/2}\,dx$ simplifies to:

$$\frac{1}{81}(3x^{2}-4)^{9/2}+\frac{4}{63}(3x^{2}-4)^{7/2}+C$$

Problem:

$$\int \frac{2}{x^4\sqrt{x^2-25}}\,dx$$

Apply linearity:

$$= 2\int \frac{1}{x^4\sqrt{x^2-25}}\,dx$$

Now solving:

$$\int \frac{1}{x^4\sqrt{x^2-25}}\,dx$$

Perform trigonometric substitution:

Substitute $x = 5\sec(u) \longrightarrow u = \operatorname{arcsec}\left(\frac{x}{5}\right)$, $dx = 5\sec(u)\tan(u)\,du$ (steps):

$$= \int \frac{\tan(u)}{125\sec^3(u)\sqrt{25\sec^2(u)-25}}\,du$$

Simplify using $25\sec^2(u) - 25 = 25\tan^2(u)$:

$$= \frac{1}{625}\int \frac{1}{\sec^3(u)}\,du$$

... or choose an alternative:

Perform hyperbolic substitution

Now solving:

$$\int \frac{1}{\sec^3(u)}\,du$$

Rewrite/simplify using trigonometric/hyperbolic identities:

$$= \int \cos^3(u)\,du$$

Prepare for substitution:

$$= \int \cos(u)\left(1-\sin^2(u)\right)\,du$$

Substitute $v = \sin(u) \longrightarrow dv = \cos(u)\,du$ (steps):

$$= \int \left(1-v^2\right)dv$$

Apply linearity:

$$= \int 1\,dv - \int v^2\,dv$$

Now solving:

$$\int 1\,dv$$

Apply constant rule:

$$= v$$

Now solving:

$$\int v^2\,dv$$

Apply power rule:

$$\int v^n\,dv = \frac{v^{n+1}}{n+1} \text{ with } n = 2:$$

$$= \frac{v^3}{3}$$

Plug in solved integrals:

$$\int 1\,dv - \int v^2\,dv$$

$$= v - \frac{v^3}{3}$$

Undo substitution $v = \sin(u)$:

$$= \sin(u) - \frac{\sin^3(u)}{3}$$

Plug in solved integrals:

$$\frac{1}{625}\int \frac{1}{\sec^3(u)}\,du$$

$$= \frac{\sin(u)}{625} - \frac{\sin^3(u)}{1875}$$

Undo substitution $u = \operatorname{arcsec}\left(\frac{x}{5}\right)$, use:

$$\sin\left(\operatorname{arcsec}\left(\frac{x}{5}\right)\right) = \frac{5\sqrt{\frac{x^2}{25}-1}}{x}$$

$$= \frac{\sqrt{\frac{x^2}{25}-1}}{125x} - \frac{\left(\frac{x^2}{25}-1\right)^{\frac{3}{2}}}{15x^3}$$

Plug in solved integrals:

$$2\int \frac{1}{x^4\sqrt{x^2-25}}\,dx$$

$$= \frac{2\sqrt{\frac{x^2}{25}-1}}{125x} - \frac{2\left(\frac{x^2}{25}-1\right)^{\frac{3}{2}}}{15x^3}$$

The problem is solved:

$$\int \frac{2}{x^4\sqrt{x^2-25}}\,dx$$

$$= \frac{2\sqrt{\frac{x^2}{25}-1}}{125x} - \frac{2\left(\frac{x^2}{25}-1\right)^{\frac{3}{2}}}{15x^3} + C$$

Rewrite/simplify:

$$= \frac{\sqrt{x^2-25}\left(4x^2+50\right)}{1875x^3} + C$$

Problem:

$$\int 2x^5\sqrt{9x^2+2}\,dx$$

Apply linearity:

$$= 2\int x^5\sqrt{9x^2+2}\,dx$$

Now solving:

$$\int x^5\sqrt{9x^2+2}\,dx$$

Substitute $u = 9x^2 + 2 \longrightarrow du = 18x\,dx$ (steps), use:

$$x^4 = \frac{(u-2)^2}{81}$$

$$= \frac{1}{1458}\int (u-2)^2\sqrt{u}\,du$$

... or choose an alternative:

Substitute x^2

Don't substitute

Now solving:

$$\int (u-2)^2\sqrt{u}\,du$$

Expand:

$$= \int \left(u^{\frac{5}{2}} - 4u^{\frac{3}{2}} + 4\sqrt{u}\right)du$$

Apply linearity:

$$= \int u^{\frac{5}{2}}\,du - 4\int u^{\frac{3}{2}}\,du + 4\int \sqrt{u}\,du$$

Now solving:

$$\int u^{\frac{5}{2}}\,du$$

Apply power rule:

$$\int u^n\,du = \frac{u^{n+1}}{n+1} \text{ with } n = \frac{5}{2}:$$

$$= \frac{2u^{\frac{7}{2}}}{7}$$

Now solving:

$$\int u^{\frac{3}{2}}\,du$$

Apply power rule with $n = \frac{3}{2}$:

$$= \frac{2u^{\frac{5}{2}}}{5}$$

Now solving:

$$\int \sqrt{u}\,du$$

Apply power rule with $n = \frac{1}{2}$:

$$= \frac{2u^{\frac{3}{2}}}{3}$$

Plug in solved integrals:

$$\int u^{\frac{5}{2}}\,du - 4\int u^{\frac{3}{2}}\,du + 4\int \sqrt{u}\,du$$

$$= \frac{2u^{\frac{7}{2}}}{7} - \frac{8u^{\frac{5}{2}}}{5} + \frac{8u^{\frac{3}{2}}}{3}$$

Plug in solved integrals:

$$\frac{1}{1458}\int (u-2)^2\sqrt{u}\,du$$

$$= \frac{u^{\frac{7}{2}}}{5103} - \frac{4u^{\frac{5}{2}}}{3645} + \frac{4u^{\frac{3}{2}}}{2187}$$

Undo substitution $u = 9x^2 + 2$:

$$= \frac{\left(9x^2+2\right)^{\frac{7}{2}}}{5103} - \frac{4\left(9x^2+2\right)^{\frac{5}{2}}}{3645} + \frac{4\left(9x^2+2\right)^{\frac{3}{2}}}{2187}$$

Plug in solved integrals:

$$2\int x^5\sqrt{9x^2+2}\,dx$$

$$= \frac{2\left(9x^2+2\right)^{\frac{7}{2}}}{5103} - \frac{8\left(9x^2+2\right)^{\frac{5}{2}}}{3645} + \frac{8\left(9x^2+2\right)^{\frac{3}{2}}}{2187}$$

The problem is solved:

$$\int 2x^5\sqrt{9x^2+2}\,dx$$

$$= \frac{2\left(9x^2+2\right)^{\frac{7}{2}}}{5103} - \frac{8\left(9x^2+2\right)^{\frac{5}{2}}}{3645} + \frac{8\left(9x^2+2\right)^{\frac{3}{2}}}{2187} + C$$

Rewrite/simplify:

$$= \frac{2\left(9x^2+2\right)^{\frac{3}{2}}\left(1215x^4-216x^2+32\right)}{76545} + C$$

$$\frac{(x+3)^5}{(-x^2-6x+40)^{\frac{3}{2}}}$$

Note: Your input has been rewritten/simplified.

Simplify/rewrite:

$$\frac{(x+3)^5}{(-(x-4)(x+10))^{\frac{3}{2}}}$$

Problem:

$$\int \frac{(x+3)^5}{(-x^2-6x+40)^{\frac{3}{2}}}\,dx$$

Substitute $u=(x+3)^2 \longrightarrow du=2(x+3)\,dx$ (steps):

$$=\frac{1}{2}\int \frac{u^2}{(49-u)^{\frac{3}{2}}}\,du$$

... or choose an alternative:

Don't substitute

Now solving:

$$\int \frac{u^2}{(49-u)^{\frac{3}{2}}}\,du$$

Substitute $v=49-u \longrightarrow dv=-du$ (steps), use:

$u^2=(49-v)^2$

$$=-\int \frac{(v-49)^2}{v^{\frac{3}{2}}}\,dv$$

Now solving:

$$\int \frac{(v-49)^2}{v^{\frac{3}{2}}}\,dv$$

Expand:

$$=\int \left(\sqrt{v}-\frac{98}{\sqrt{v}}+\frac{2401}{v^{\frac{3}{2}}}\right)dv$$

Apply linearity:

$$=\int \sqrt{v}\,dv-98\int \frac{1}{\sqrt{v}}\,dv+2401\int \frac{1}{v^{\frac{3}{2}}}\,dv$$

Now solving:

$$\int \sqrt{v}\,dv$$

Apply power rule:

$$\int v^n\,dv=\frac{v^{n+1}}{n+1} \text{ with } n=\frac{1}{2}:$$

$$=\frac{2v^{\frac{3}{2}}}{3}$$

Now solving:

$$\int \frac{1}{\sqrt{v}}\,dv$$

Apply power rule with $n=-\frac{1}{2}$:

$$=2\sqrt{v}$$

Now solving:

$$\int \frac{1}{v^{\frac{3}{2}}}\,dv$$

Apply power rule with $n=-\frac{3}{2}$:

$$=-\frac{2}{\sqrt{v}}$$

Plug in solved integrals:

$$\int \sqrt{v}\,dv-98\int \frac{1}{\sqrt{v}}\,dv+2401\int \frac{1}{v^{\frac{3}{2}}}\,dv$$

$$=\frac{2v^{\frac{3}{2}}}{3}-196\sqrt{v}-\frac{4802}{\sqrt{v}}$$

Plug in solved integrals:

$$-\int \frac{(v-49)^2}{v^{\frac{3}{2}}}\,dv$$

$$=-\frac{2v^{\frac{3}{2}}}{3}+196\sqrt{v}+\frac{4802}{\sqrt{v}}$$

Undo substitution $v=49-u$:

$$=-\frac{2(49-u)^{\frac{3}{2}}}{3}+196\sqrt{49-u}+\frac{4802}{\sqrt{49-u}}$$

Plug in solved integrals:

$$\frac{1}{2}\int \frac{u^2}{(49-u)^{\frac{3}{2}}}\,du$$

$$=-\frac{(49-u)^{\frac{3}{2}}}{3}+98\sqrt{49-u}+\frac{2401}{\sqrt{49-u}}$$

Undo substitution $u=(x+3)^2$:

$$=-\frac{\left(49-(x+3)^2\right)^{\frac{3}{2}}}{3}+98\sqrt{49-(x+3)^2}+\frac{2401}{\sqrt{49-(x+3)^2}}$$

The problem is solved:

$$\int \frac{(x+3)^5}{(-x^2-6x+40)^{\frac{3}{2}}}\,dx$$

$$=-\frac{\left(49-(x+3)^2\right)^{\frac{3}{2}}}{3}+98\sqrt{49-(x+3)^2}+\frac{2401}{\sqrt{49-(x+3)^2}}+C$$

Rewrite/simplify:

$$=-\frac{x^4+12x^3+250x^2+1284x-17363}{3\sqrt{-(x-4)(x+10)}}+C$$