

# **GASDYNAMICS**

## **Becker**

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#### Gas Dynamics

This is the only authorized English edition of *Gasdynamik* (a volume in the series "Leitfäden der angewandten Mathematik und Mechanik," edited by Professor Dr. H. Görtler), originally published in the German language by B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1966.

## PREFACE TO THE GERMAN EDITION

This book is intended to be an introduction to the dynamics and thermodynamics of gases. The limitation of space requires a strict choice of the material. The book therefore cannot give a complete survey of the entire field of gas dynamics. It should rather make the reader familiar with the fundamental facts of gas dynamics, thus enabling him to go on to more advanced works and original literature without much difficulty.

In selecting the material, I have strived to take a modern point of view, despite the introductory nature of the book. Such a viewpoint is important if one wishes to study high speeds and extreme altitudes of flight. The inclusion in the theory of the physical effects which then play a role leads beyond the classical dynamics of ideal gases. Some of the important effects are the following: 1. The gas in a flow does not behave like an ideal gas. 2. The local thermodynamic state of the gas can no longer be described as being an unconstrained equilibrium, and *gas dynamic relaxation* appears. 3. The gas does not behave like a continuum, and the effect of the mean free path plays a role. 4. At high temperatures, the gas ionizes and becomes electrically conducting, so that it is affected by electromagnetic forces. Effects 3 and 4 are studied in the dynamics of rarefied gases and in magnetogasdynamics; they will not be considered in this book. However, departures from ideal gas behavior and from unconstrained thermodynamic equilibrium are considered. Effects 1 and 2 have recently made the viewpoint of thermodynamics and physics in gas dynamics much more important than was customary one or two decades ago.

Chapter 1 presents the fundamentals of thermodynamics in as complete a manner as I deemed necessary for a first understanding of many new developments in gas dynamics. In the subsequent chapters, the deductions are specialized as late as possible to an ideal gas as an example—but invariably an impor-

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**NOMENCLATURE\***

<i>A</i>	Area; thermodynamic state variable defined by Eq. (3.1)
<i>a</i>	(Equilibrium) sound velocity
<i>B</i>	Thermodynamic state variable defined by Eq. (3.71)
<i>b</i>	Frozen sound velocity
<i>C<sub>p</sub>, C<sub>v</sub></i>	Heat capacity at constant pressure or constant volume
<i>C</i>	Characteristic curves
<i>C<sub>1</sub>, C<sub>2</sub></i>	Mach lines
<i>C<sub>3</sub></i>	Streamlines in the <i>x, y</i> plane or particle paths in the <i>x, t</i> plane
<i>c</i>	Mass concentration; phase velocity
<i>c<sub>D</sub></i>	Drag coefficient
<i>c<sub>f</sub></i>	Friction coefficient
<i>c<sub>L</sub></i>	Lift coefficient
<i>c<sub>p</sub></i>	Specific heat at constant pressure; pressure coefficient
<i>c<sub>v</sub></i>	Specific heat at constant volume
<i>D</i>	Drag
<i>D<sub>f</sub></i>	Friction drag
<i>D</i>	Deformation tensor
<i>E</i>	Internal energy
<i>E</i>	Unit tensor
<i>e</i>	Specific internal energy
<i>F</i>	Free energy
<i>f</i>	Specific free energy
<i>f(ζ)</i>	Blasius function
<i>G</i>	Gibbs enthalpy
<i>g</i>	Specific Gibbs enthalpy
<i>g(ζ)</i>	Enthalpy function for flat-plate boundary layer
<i>H</i>	Enthalpy
<i>H*</i>	Constant in the equation of state (1.56) of calorically ideal gas

\* In this book, the terms "strong shock" and "weak shock" are used in two different senses: In Section 3.5, these terms have the meaning defined on p. 141, while elsewhere they are used in the usual sense of a "sufficiently strong" or "sufficiently weak" shock.

**Nomenclature**

$h$	Specific enthalpy
$h^*$	Constant in the equation of state (1.58) of calorically ideal gas
$h_r$	Recovery enthalpy
$\kappa$	Isothermal compressibility coefficient; circulation; transonic similarity parameter
$K(T)$	Equilibrium constant in the law of mass action
$k$	Thermal conductivity; wave number
$\mathbf{k}$	Volume force per unit mass (components $k_x, k_y, k_z$ )
$L$	Lift; relaxation function
$\ell$	Shock thickness
$\ell_f$	Width of fully dispersed wave
$l_f$	Mean free path
$M$	Mass of a thermodynamic system; Mach number
$M_a$	Mach number based on equilibrium sound velocity
$M_b$	Mach number based on frozen sound velocity
$\mathfrak{M}$	Molar mass
$m$	Molecular mass
$n$	Molar number
$\mathbf{n}$	Unit vector in normal direction
$\Pr$	Prandtl number
$\hat{\Pr}$	Prandtl number characteristic of flow in a normal shock ( $= c_p \hat{\eta} / k$ )
$p$	Pressure
$p_d$	Characteristic pressure for dissociation
$Q$	Quantity of heat
$q_w$	Heat flux on the wall in boundary layer flow
$\mathbf{q}$	Energy flux vector, in general, heat flux vector (components $q_x, q_y, q_z$ )
$R$	Specific gas constant ( $= \mathfrak{R}/\mathfrak{M} = R^*/M$ )
$R^*$	Gas constant
$R(\zeta)$	Function for flat-plate boundary layer defined by Eq. (4.140)
$Re$	Reynolds number
$\mathfrak{R}$	Universal gas constant
$r$	Recovery factor
$\mathbf{r}$	Position vector
$S$	Entropy
$S(\zeta)$	Function for flat-plate boundary layer defined by Eq. (4.141)
$\Delta S_o$	Entropy introduced from outside
$\Delta S_i$	Entropy generated in system
$\mathbf{S}$	Stress tensor ( $= -p\mathbf{E} + \mathbf{T}$ )
$St$	Stanton number
$s$	Specific entropy
$T$	Absolute temperature
$T^*$	Constant in the equation of state (1.57) of calorically ideal gas
$T_d$	Characteristic temperature for dissociation
$T_r$	Recovery temperature

**Nomenclature**

$\mathbf{T}$	Viscous stress tensor
$t$	Time
$t_d$	Decay time [Eq. (3.103)]
$\mathbf{t}$	Stress vector; unit vector in tangential direction
$U$	Magnitude of velocity vector ( $=  \mathbf{v} $ )
$U_{\max}$	Isentropic maximum speed
$u, v, w$	Components of velocity vector $\mathbf{v}$ in $x, y, z$ directions
$u, v$	Components of velocity perpendicular or parallel to a wavefront or shock front
$V$	Volume
$\bar{v}$	Mean thermal speed of molecules
$\mathbf{v}$	Velocity vector
$W$	Work
$w$	Vorticity vector ( $= \text{curl } \mathbf{v}$ )
$X$	Mole fraction
$x, y, z$	Cartesian coordinates
$Z$	Compressibility factor
$\alpha$	Coefficient of thermal expansion; degree of dissociation; angle of attack
$\beta$	$= (1 - M_\infty^2)^{1/2}$
$\beta_a$	$= (M_{a\infty}^2 - 1)^{1/2}$
$\beta_b$	$= (M_{b\infty}^2 - 1)^{1/2}$
$\Gamma$	Thermodynamic state variable defined by Eq. (1.73) or (1.138)
$\gamma$	Adiabatic exponent ( $= c_p/c_v$ )
$\delta, \epsilon$	Constants in the linearized solution of the flow past a wavy wall
$\delta$	Boundary layer thickness
$\delta^*$	Displacement thickness
$\zeta$	Shock angle; similarity variable for flat-plate boundary layer
$\eta$	Viscosity
$\eta_b$	Bulk viscosity
$\hat{\eta}$	Viscosity characteristic of flow in shock ( $= 4\eta/3 + \eta_b$ )
$\Theta$	Mass flow; momentum thickness
$\theta$	Characteristic temperature for molecular vibrations; angle between velocity and $x$ direction; deflection angle; enthalpy thickness
$\lambda$	Relaxation length; characteristic length for shock wave
$\mu$	Chemical potential; Mach angle ( $= \text{arc sin } M^{-1}$ )
$\mu_0(T)$	Temperature function in equation of state (1.53) for thermally ideal gas
$\nu$	Stoichiometric coefficient; kinematic viscosity ( $= \eta/\rho$ )
$\nu(M)$	Prandtl-Meyer function
$\xi$	Thermodynamic variable to describe constrained equilibrium; affinely distorted $x$ coordinate for subsonic flow past slender body
$\rho$	Density
$\rho_d$	Characteristic density for dissociation
$\sigma$	Entropy generated per unit volume per unit time
$\sigma_t$	Entropy generated by viscous friction

- $\sigma_h$  Entropy generated by heat conduction
- $\sigma_x, \sigma_y, \sigma_z$  Diagonal terms of viscous stress tensor  $T$
- $\bar{\sigma}$  Mean normal stress [ $= p + (\sigma_x + \sigma_y + \sigma_z)/3$ ]
- $\tau, \tau_r$  Relaxation times [see Eq. (3.95)]
- $\tau_w$  Wall shear stress
- $\tau_{xy}, \tau_{xz}, \tau_{yz}$  Shear stresses
- $\phi$  Dissipation function; error integral
- $\psi$  Angle in polar coordinates; potential of perturbation velocity
- $\psi(T)$  Thermodynamic function for thermally ideal gas [Eq. (1.103)]
- $\Psi$  Stream function
- $\Omega$  Potential of volume forces; molecular collision cross section
- $\omega$  Circular frequency; exponent in viscosity law (4.3)
- $\omega(\varrho)$  Density function for unsteady simple waves [Eq. (3.39)]
- $\boldsymbol{\omega}$  Angular velocity vector

#### Subscripts and Superscripts

- $m$  Molar quantities
- $t$  Isentropic stagnation quantities (reservoir quantities)
- $u, l$  Upper or lower side of a profile
- \* Critical quantities (Values of flow variables at Mach number 1 in steady inviscid flow)
- 0 Quantities in undisturbed gas at rest for unsteady wave propagation
- 1, 2 Values ahead or behind a shock wave
- $\infty$  Free stream

Tilde ( $\sim$ ) denotes quantities in unconstrained thermodynamic equilibrium  
 Prime ( $'$ ) denotes perturbations to a gas at rest or in uniform motion

## GAS DYNAMICS

# 1 FUNDAMENTALS OF THERMODYNAMICS

## 1.1 Basic Concepts

In classical hydrodynamics, the compressibility of the flowing media is neglected. For the flow of liquids, this is almost always permissible, while for the flow of gases, this is permissible for relatively low velocities only. In a flow field, the pressure gradient serves as the driving force, and the density and the entire thermodynamic state of the flowing gases change with the pressure. Thermodynamic laws are therefore of basic importance in gas dynamics. This introductory chapter gives a brief survey of those laws of thermodynamics which are necessary for the understanding of gas dynamics.

The essential characteristic of thermodynamics involves the discussion of the relationship between heat and the other forms of energy (mechanical, electrical, etc.). Every system of material bodies whose thermodynamic behavior is being studied will be called a *thermodynamic system*. For our purposes, it is generally sufficient to consider systems of the following type: We take a closed surface enclosing a finite volume filled with a gas. The enclosed gas forms the *system* of interest, and the bounding surface and the space outside form the *environment*. We call a system *closed* when no transport of matter across the boundary of the system is possible, i.e., when the surface of the volume is impervious to gas. The mass of such a system is a constant which remains invariant throughout all changes of the system. When the transport of matter across the boundary is possible, then the system is called *open*. In this case, the boundary of the system can be, for example, an imaginary closed surface in the gas, which we may want to specify for some purposes.

Closed systems will be divided into the following: An *isolated system* is one which cannot have any interaction (e.g., work done or heat transfer) with its environment. An *adiabatic system* is one whose only possible interaction with its environment is through positive or negative work done on the system (the boundary of the system is heat-insulating).

In a sufficiently long time-interval, an isolated system tends toward *thermodynamic equilibrium*, after which no further observable macroscopic changes can be found in the system. Conversely, we say a system is in thermodynamic equilibrium if no macroscopic changes occur in the system after it has been isolated. The system is then completely homogeneous, and for the unique characterization of its state it is sufficient to give the pressure  $p$  and the volume  $V$ . Therefore, if, in two different observations, it is found that the system is in equilibrium and that it has the same pressure and volume, then all other macroscopic properties will also be the same, irrespective of changes the system has undergone in the interim.

When two closed nonadiabatic systems are brought into contact, then the states of both systems will generally change until a new equilibrium state is reached. The two systems are then in mutual *thermal equilibrium*. Experience shows that when two systems are in thermal equilibrium with a third, then the two systems are always also in equilibrium with each other.

We now assume that a system  $A$  is in thermal equilibrium with a system  $B$ . The state of  $A$  can be represented by a point  $A'$  in the  $p, V$  plane (Fig. 1). If we separate the systems from each other and change the pressure and volume of  $A$ , then  $A$  will in general no longer be in equilibrium with  $B$ , except for special combinations of pressures and volumes of  $A$ . If we connect all the

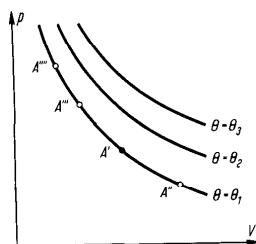


Fig. 1. Isotherms in the  $p, V$  plane.

points  $A', A'', \dots$ , etc., in the  $p, V$  plane of  $A$ , at which states  $A$  is still in equilibrium with  $B$ , then we obtain a curve which we can characterize numerically by a parameter  $\theta$ . Now, we take a system  $C$  and similarly draw in its  $p, V$  plane all points which are in equilibrium with  $B$ , and we characterize the resulting curve by the same value of the parameter  $\theta$ . From the above discussion, we see that systems  $A$  and  $C$  are always in equilibrium when their states correspond to points on these two curves. We call  $\theta$  the temperature of both systems. Two systems are thus in mutual thermal equilibrium when their temperatures coincide. When we now change the state of  $B$  and again connect all the points of  $A$  in the  $p, V$  plane which correspond to thermal equilibrium with the new state of  $B$ , then we obtain in general a new curve, which will be characterized by a numerically different temperature  $\theta$ . Thus, the  $p, V$  plane of a system is covered by a family of constant temperature curves, the isotherms. This family of curves can be represented in implicit form by a relation  $\Phi(p, V, \theta) = 0$ , which we call the *thermal equation of state* of the system. Thus far, the value of the parameter  $\theta$  has been left arbitrary. From now on, we shall adopt as the appropriate temperature scale the absolute temperature  $T$ , since the formulation of thermodynamic laws in terms of  $T$  is particularly simple; the absolute temperature is always positive.

We have thus far mentioned three thermodynamic state variables:  $p, V$ , and  $T$ . In the first law of thermodynamics, another state variable, the internal energy  $E$ , will be introduced (Section 1.2). Since the thermodynamic state of a system of the type being considered here is determined by two state variables, there must exist for each system a relation of the form  $\Psi(E, T, V) = 0$ , the *caloric equation of state*. The thermal and caloric equations of state of a system are not completely independent of each other: Given the thermal equation of state, the possible forms of the caloric equation of state are limited. For a system consisting of a *thermally ideal gas*, we have, for example, the thermal equation of state

$$pV = R^*T \quad (1.1)$$

with constant  $R^*$ ; one may deduce from this that  $E = E(T)$ , or, the internal energy depends only on the temperature and not on the volume or pressure of the system (Section 1.4). In the special case

$$E = C_v T + E_0 \quad (1.2)$$

with constant  $C_v$  and  $E_0$ , the gas is called a *calorically ideal* gas ( $C_v$  is the specific heat at constant volume, see Section 1.4).

In the second law of thermodynamics, the entropy  $S$  will be introduced as another state variable. The five quantities  $p$ ,  $V$ ,  $T$ ,  $E$ , and  $S$  are completely adequate for the formulation of thermodynamic laws for closed systems (for open systems, the number of moles must be given; see Section 1.7). However, it has proved convenient to introduce certain combinations of these five quantities as new state variables, in particular, the enthalpy  $H$ , the free energy  $F$ , and the Gibbs enthalpy  $G$  (Section 1.4).

The state variables defined so far can be divided into *intensive* and *extensive* variables: If one imagines a system in thermodynamic equilibrium divided into two parts by means of a partition, such that each part contains exactly half the mass of the total system, then in each part, the temperature and pressure are the same as in the original undivided system, whereas the volume, internal energy, entropy, enthalpy, free energy, and Gibbs enthalpy all have exactly half the values of the undivided system. In other words,  $p$  and  $T$  do not depend on the mass of a system, and are intensive quantities, while  $V$ ,  $E$ ,  $S$ ,  $H$ ,  $F$ , and  $G$  are proportional to the mass, and are extensive quantities. When these extensive quantities are divided by the total mass  $M$ , we get quantities which, upon subdivision of the system (because of its homogeneity in thermodynamic equilibrium), also will not change, just as  $p$  and  $T$  will not; they are thus also intensive quantities. These intensive quantities are also called *specific* quantities. The specific quantities (specific volume, etc.) corresponding to the variables  $V$ ,  $E$ ,  $S$ ,  $H$ ,  $F$ , and  $G$  will be denoted by  $1/\varrho$ ,  $e$ ,  $s$ ,  $h$ ,  $f$ , and  $g$ , respectively, where  $\varrho$  is the density (mass/volume). In the formulation of the laws of gas dynamics, we will only encounter intensive quantities. Let us remark now that we shall sometimes use *molar* quantities, which are obtained by dividing the extensive quantities by the number of moles (mass of system/molar mass), and which will be denoted by the subscript  $m$ :  $E_m$ ,  $H_m$ , etc. Molar quantities, like specific quantities, are also intensive.

Two further important concepts are *change of state* and *process*. If a system at time  $t_1$  is in a thermodynamic equilibrium state 1, and at time  $t_2 > t_1$  it is in another equilibrium state 2, then a change of state has taken place; the term "change of state" comprises the pair of states, initial state 1 and final state 2. Such a change of state is produced by a thermodynamic process occurring in the time interval between  $t_1$  and  $t_2$ . The same change

of state can be produced by very different kinds of processes; for example, we can increase the temperature of a body of water by a given increment either by heating it in a vessel over a flame or by adding a certain amount of energy through mechanical means. In general, a system does not pass through equilibrium states during a process. (In the heating of water, an uneven distribution of temperature is usually produced, and a convection flow results; a certain time is required for heat transfer to be completed, after which equilibrium is again achieved.) This process is then called *nonstatic*. On the other hand, during an infinitesimally slow process, a system does pass through equilibrium states; such a process is called *quasistatic*. A quasistatic process can be represented in the  $p$ ,  $V$  plane (or the plane of any other two state variables) by a curve connecting the initial and final states and passing through the intermediate states assumed during the process (Fig. 2).

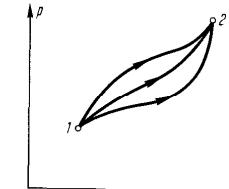
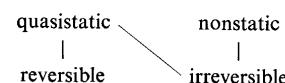


Fig. 2. Diagram of different quasistatic processes in the  $p$ ,  $V$  plane that result in the same change of state (1, 2).

Another classification of processes is into *reversible* and *irreversible* processes. Indeed, we can always make a change of state in the reverse direction, irrespective of the nature of the forward process  $1 \rightarrow 2$ ; but generally, after the reverse process  $2 \rightarrow 1$  has been completed, a change in the environment of the system remains. When it is possible in principle to find a process  $2 \rightarrow 1$  such that after completion both the system and the environment will be in the same state as before the process  $1 \rightarrow 2$  took place, we then call the process  $1 \rightarrow 2$  reversible (and  $2 \rightarrow 1$  is then also reversible). The different kinds of processes are related in the following manner:



While a quasistatic process can be reversible as well as irreversible, a non-static process is always irreversible.

## 1.2. First Law of Thermodynamics

Let us consider a closed adiabatic system. If we take the system from an initial state 1 to a final state 2, then we know from experience that whatever the nature of the process may be, the same work  $\Delta W$  will have been done on the system. For a given initial state 1, the work done on the system is dependent only on the final state 2, and is therefore a state variable of the system. Using an arbitrary choice of the initial value  $E_1$ , we define this state variable as the internal energy  $E$ :

$$E_2 = E_1 + \Delta W. \quad (1.3)$$

The work  $\Delta W$  done on the system through an adiabatic process (process in an adiabatic system) is thus equal to the change  $\Delta E = E_2 - E_1$  of the internal energy of the system.

In the case of a nonadiabatic system, Eq. (1.3) does not apply in general, i.e.,  $\Delta E \neq \Delta W$ . In this case, we have

$$E_2 - E_1 = \Delta W + \Delta Q, \quad (1.4)$$

by which the quantity of heat  $\Delta Q$  transmitted to the system during the process is defined. When a system is changed from an initial state 1 to a final state 2, regardless of the nature of the process used, the sum of the work  $\Delta W$  done on the system and the heat  $\Delta Q$  conducted to the system must remain constant, i.e., this sum is equal to the change  $\Delta E$  of the internal energy of the system. This change depends only on the initial and final states:  $\Delta E = E_2 - E_1$ .

*Example:* A gas enclosed in a box of constant volume  $V$  can be brought from temperature  $T_1$  to temperature  $T_2$  (a) through stirring and (b) through heating over a flame. In the first case,  $E_2 - E_1 = \Delta W$ , while in the second case,  $E_2 - E_1 = \Delta Q$ . If we stir and heat at the same time, then we have  $E_2 - E_1 = \Delta W + \Delta Q$ .

*Note:* In a relation like (1.4), all quantities should naturally be measured

in the same units. We therefore recall the conversion of mechanical and caloric units of energy (mechanical equivalent of heat):

$$1 \text{ kcal} = 4.19 \times 10^3 \text{ kg m}^2 \text{ sec}^{-2}$$

## 1.3 Second Law of Thermodynamics

The starting point for the formulation of the second law of thermodynamics is the differential form

$$dE + p dV = \left( \frac{\partial E(p, V)}{\partial V} + p \right) dV + \frac{\partial E(p, V)}{\partial p} dp. \quad (1.5)$$

Experience shows that (1.5) is not an exact differential: If one integrates in the  $p, V$  plane of a thermodynamic system along a smooth path (which may be assumed to symbolize a quasistatic process 1 → 2, see Fig. 2), the integral  $\int_1^2 (dE + p dV)$  depends on the path connecting state 1 with state 2 in the  $p, V$  plane. But, as is known from calculus, there always exists an infinite number of "integrating factors"  $\phi(p, V)$  such that  $\int_1^2 \phi(p, V) (dE + p dV)$  is independent of the path and hence depends only on the limits of integration, i.e., on initial state 1 and final state 2.

The content of the second law may now be divided into two separate statements:

*First Statement:* There exists for all thermodynamic systems an integrating factor which depends only on the temperature  $\theta$  of the system (as defined on p. 3) and which, moreover, is, for all systems, the same positive function of temperature  $\phi(\theta) > 0$ . We denote this universally valid integrating factor by  $1/T$  and define a new state variable  $S$  by

$$T dS = dE + p dV. \quad (1.6)$$

$T(\theta) > 0$  can be chosen as a convenient measure of temperature. Using that temperature, the thermal equation of state of a thermally ideal gas has the simple form (1.1). Since the integral  $\int_1^2 dS$  is now independent of the path leading from 1 to 2, we can write, with arbitrarily fixed value  $S_1$ :

$$S_2 = S_1 + \int_1^2 \frac{dE + p dV}{T}; \quad (1.7)$$

$S_2$  depends only on state 2, and  $S$  is therefore a state variable, as already mentioned.  $S$  is called the entropy of the system, and it is an extensive variable, since  $E$  and  $V$  are extensive.

In preparation for the second statement of the second law, let us imagine that the system illustrated in Fig. 3 is undergoing a quasistatic process. Here,

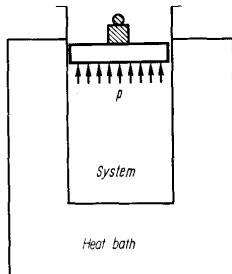


Fig. 3. A system in a heat bath.

heat is conducted into the system and from it through contact with heat baths, whose temperatures are at an infinitesimal value above or below that of the system itself. At the same time, the volume of the system is decreased or increased by making the weight on the piston which closes the cylinder infinitesimally greater or smaller than the force exerted on the piston by the pressure  $p$  of the system. The process is thus quasistatic, and we can easily show that it is also reversible. By a volume change  $dV$  we do work to the system

$$dW = -p dV. \quad (1.8)$$

According to the first law (1.4), therefore,

$$dQ = dE + p dV, \quad (1.9)$$

and, according to the definition (1.6) and using (1.9), we get

$$dS = dQ_{\text{rev}}/T, \quad (1.10)$$

in which the subscript "rev" indicates that the process is a reversible one.

The total change of entropy is therefore

$$S_2 - S_1 = \int_1^2 dQ_{\text{rev}}/T. \quad (1.11)$$

We now assume that the system is brought from an initial state 1 to a final state 2 by means of any process 1 → 2. Then,

$$S_2 - S_1 = \Delta S_o + \Delta S_i. \quad (1.12)$$

Here,  $\Delta S_i$  is the entropy produced in the system during this process and  $\Delta S_o$  is the entropy carried into the system over the boundaries from the outside; we can compute  $\Delta S_o$  in the following manner: When an area element  $dA$  of the boundary is at a temperature  $T$  and a quantity of heat  $dQ$  flows into the system across this element during time  $dt$ , then the contribution to  $\Delta S_o$  from this element will be equal to  $dQ/T$ . These contributions are to be summed over the entire boundary of the system and over the entire duration of the process. (When nonstatic processes are used, the temperature  $T$  will vary with the location and time in the system and on the boundary.)

*Second Statement:* The inequality

$$\Delta S_i \geq 0 \quad (1.13)$$

holds. Only in a reversible process is  $\Delta S_i = 0$ , and  $dQ = dQ_{\text{rev}}$ , so that (1.12) becomes (1.11).

To make this more clear let us look again at the example mentioned at the end of Section 1.2. The final state of the system differs from the initial state by a change of energy  $\Delta E$  at constant volume  $V$ . Independent of the nature of the process, the change of entropy is given by definition (1.7):<sup>1</sup>

$$S_2 - S_1 = \int_1^2 \frac{dE}{T} = \int_1^2 \left( \frac{\partial E}{\partial T} \right)_V \frac{dT}{T}. \quad (1.14)$$

In this manner, we can compute the change of entropy when we know the initial and final temperatures of the system and the caloric equation of state. If we now carry out the process by stirring the gas without heat transfer from

<sup>1</sup> Since the independent variables in thermodynamic relations are changed frequently, it is always desirable to state them explicitly. It is customary in writing the partial derivatives of thermodynamic quantities to specify the variable being held fixed by a suffix.

the outside, then  $S_2 - S_1 = \Delta S_1$ . On the other hand, we can produce the same change of state by means of a quasistatic heat transfer, in which case  $S_2 - S_1 = \Delta S_o$ .

*Example 1:* Entropy generation by viscosity. Consider an annular space between two concentric circular cylinders. The annulus is filled with gas (Fig. 4), the inner cylinder is held fixed, and the outer cylinder rotates with

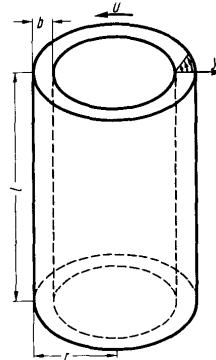


Fig. 4. Flow in the annulus between a fixed and a rotating cylinder.

a peripheral velocity  $U$ . The gas adheres to the cylinders, and its velocity in the space is given by  $u = Uy/b$  if the width of the annulus is small ( $b \ll r$ ) and the peripheral velocity is not too great. To rotate the outer cylinder, a torque  $M = 2\pi rl\tau r$  is required, in which  $2\pi rl$  is the cylinder area,  $\tau$  the shear stress resulting from the viscosity of the gas, and  $r$  the lever arm. The mechanical work to be done per unit time is

$$\dot{W} = MU/r, \quad (1.15)$$

so that the mechanical power per unit volume of the gas is

$$\dot{W} = \frac{\dot{W}}{2\pi rl b} = \frac{\tau U}{b} = \tau \frac{du}{dy}. \quad (1.16)$$

On the other hand, we have  $\tau = \eta du/dy$ , where  $\eta$  is the coefficient of shear viscosity of the gas. If, furthermore, we think of the entire setup as an adiabatic system, then, according to the first law of thermodynamics,

$\Delta E = \Delta W$ , or  $\dot{E} = \dot{W}$  per unit time; since the volume of the gas in the annulus does not change, we have, by (1.6),  $T\dot{S} = \dot{W}$ , or, per unit volume,

$$T\sigma = \dot{W} = \tau du/dy$$

or

$$\sigma = \frac{\eta}{T} \left( \frac{du}{dy} \right)^2 \geq 0, \quad (1.17)$$

where  $\sigma$  is the irreversibly generated entropy per unit time per unit volume of the gas as a result of viscosity. Thus, Eq. (1.17) gives the distributed entropy sources due to viscosity in the flow field produced by the rotation of the outer cylinder.

*Example 2:* Generation of entropy through heat conduction. Let two heat reservoirs at temperatures  $T_1$  and  $T_2 < T_1$  be connected with a heat-conducting material of length  $b$  and cross section  $A$  (Fig. 5). Then a quantity of heat

$$\dot{Q} = kA(T_1 - T_2)/b = -kA dT/dx$$

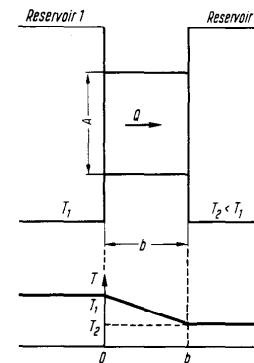


Fig. 5. Heat conduction between two heat reservoirs.

flows from reservoir 1 into reservoir 2 per unit time ( $k$  is the constant thermal conductivity of the material), where we assume that both reservoirs remain in thermodynamic equilibrium during the process. This is the case if the thermal conductivity of the reservoir material is so large that infinitesimal temperature differences can produce finite heat flows. The volumes of all the

parts are assumed to be constant. According to the first law, the internal energy of the reservoirs changes by  $\dot{E}_1 = -\dot{Q}$  and  $\dot{E}_2 = +\dot{Q}$  per unit time. According to (1.6), the following entropy changes occur:  $\dot{S}_1 = -(\dot{Q}/T_1)$  and  $\dot{S}_2 = +(\dot{Q}/T_2)$ . Reservoir 2 gains more entropy than reservoir 1 loses, since  $T_2 < T_1$ . The difference

$$\Delta\dot{S} = \dot{S}_1 + \dot{S}_2 = \dot{Q} \left( \frac{1}{T_2} - \frac{1}{T_1} \right) = \frac{kA}{b} \frac{(T_1 - T_2)^2}{T_1 T_2} \geq 0$$

is the entropy produced in the irreversible heat-conduction process. The entropy generated per unit volume per unit time by the heat conduction is, accordingly,

$$\sigma = \frac{d(\dot{Q}/T)}{A dx} = \frac{k}{T^2} \left( \frac{dT}{dx} \right)^2 \geq 0. \quad (1.18)$$

We readily see that the integral  $A \int_0^b \sigma dx$  directly gives the above value of  $(kA/b)[(T_1 - T_2)^2/T_1 T_2]$  for  $\Delta\dot{S}$  (see Section 4.1.2).

When a thermodynamic system goes through an arbitrary process, heat conduction as well as viscosity will, in general, appear in the system. The total entropy generated in the system during the process is then  $\Delta S_i = \iiint \sigma dV dt$ , where the entropy generated per unit volume per unit time  $\sigma$  is now expressed as a sum of expressions of the forms (1.17) and (1.18), and the integral extends over the entire volume of the system and the entire duration of the process. Other mechanisms of entropy generation are diffusion and relaxation processes, which we shall return to later (see Section 2.4).

#### 1.4 Canonical Equations of State; Heat Capacities

##### 1.4.1 CANONICAL EQUATIONS OF STATE

Although the five variables  $p$ ,  $T$ ,  $V$ ,  $E$ , and  $S$  already introduced are sufficient for the formulation of thermodynamic laws of closed systems, it often becomes desirable to define more state variables, of which the following are frequently used:

Enthalpy:  $H = E + pV$ , (1.19)

Gibbs enthalpy:  $G = H - TS = E + pV - TS$ , (1.20)

Free energy:  $F = E - TS$ . (1.21)

Among the five original state variables, the relation (1.6) holds:

$$dE = T dS - p dV. \quad (1.22)$$

We can establish similar relations for  $H$ ,  $G$ , and  $F$ . By (1.19),

$$dH = dE + p dV + V dp,$$

which becomes, upon the introduction of  $dE$  from (1.22),

$$dH = T dS + V dp. \quad (1.23)$$

In a completely similar manner, we derive

$$dG = -S dT + V dp \quad (1.24)$$

and

$$dF = -S dT - p dV. \quad (1.25)$$

Since the state of thermodynamic equilibrium of a closed system is determined by two state variables,  $E$ ,  $H$ ,  $G$ , and  $F$  must be functions each of two different state variables. The relations (1.22)–(1.25) yield some especially convenient choices of these state variables as arguments:

$$E = E(S, V); \quad H = H(S, p); \quad G = G(T, p); \quad F = F(T, V). \quad (1.26)$$

Each of the relations (1.26) is called a *canonical equation of state*. Each relation defines the complete thermodynamic behavior of the system. If we take any one of these relations, all the thermodynamic state variables can be computed as functions of the two independent variables in the canonical equation of state. In taking a thermal equation of state, this is not so; we have already seen in Section 1.1 that while the choice of a thermal equation of state does somewhat restrict the choice of a caloric equation of state, the caloric equation of state is nevertheless not fixed by the thermal equation of state, but must be added in order to give a complete description of the thermodynamic behavior of a system.

Let us assume that  $G(T, p)$  is given. From (1.24), it follows that

$$(\partial G / \partial p)_T = V, \quad (1.27)$$

$$(\partial G / \partial T)_p = -S. \quad (1.28)$$

Equation (1.27) fixes  $p$ ,  $T$  and  $V$ , and is therefore the thermal equation

of state of our system. On the other hand, (1.28) contains the caloric equation of state, since it follows from the definition (1.20) of  $G$  together with (1.27) and (1.28) that

$$E = G - pV + TS = G - p(\partial G/\partial p)_T - T(\partial G/\partial T)_p, \quad (1.29)$$

whereby  $E(T, p)$  is known if  $G(T, p)$  is given. The relations for  $E$ ,  $H$ , and  $F$  corresponding to (1.27) and (1.28) follow immediately from (1.22), (1.23), and (1.25):

$$(\partial E/\partial S)_V = T; \quad (\partial E/\partial V)_S = -p; \quad (1.30)$$

$$(\partial H/\partial S)_p = T; \quad (\partial H/\partial p)_S = V; \quad (1.31)$$

$$(\partial F/\partial T)_V = -S; \quad (\partial F/\partial V)_T = -p. \quad (1.32)$$

*Supplementary Remarks.* 1. From a given thermal equation of state  $V = V(T, p)$ , we can compute the volume expansivity

$$\alpha = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_p \quad (1.33)$$

as well as the isothermal compressibility

$$K = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T. \quad (1.34)$$

We have

$$(\partial p/\partial T)_V = \alpha/K, \quad (1.35)$$

$$(\partial p/\partial V)_T = -1/KV. \quad (1.36)$$

For a system satisfying the ideal gas equation (1.1), we have

$$\alpha = 1/T, \quad K = 1/p. \quad (1.37)$$

2. From

$$\frac{\partial}{\partial T} \left( \frac{\partial F}{\partial V} \right)_T = \frac{\partial}{\partial V} \left( \frac{\partial F}{\partial T} \right)_V,$$

it follows immediately from observing (1.32) and (1.35) that

$$\left( \frac{\partial S}{\partial V} \right)_T = \left( \frac{\partial p}{\partial T} \right)_V = \frac{\alpha}{K}. \quad (1.38)$$

Likewise, it follows from (1.27), (1.28), and (1.33) that

$$\left( \frac{\partial S}{\partial p} \right)_T = - \left( \frac{\partial V}{\partial T} \right)_p = -\alpha V. \quad (1.39)$$

These *Maxwell's relations* give the dependence of entropy on volume and pressure in terms of the easily measurable quantities  $\alpha$  and  $K$ .

3. From (1.21), we get  $E = F + TS$ , and thus

$$\left( \frac{\partial E}{\partial V} \right)_T = \left( \frac{\partial F}{\partial V} \right)_T + T \left( \frac{\partial S}{\partial V} \right)_T. \quad (1.40)$$

Using (1.32) and (1.38), it follows that

$$\left( \frac{\partial E}{\partial V} \right)_T = \frac{\alpha T - Kp}{K}. \quad (1.41)$$

For a thermally ideal gas, (1.37) is valid, and (1.41) gives  $(\partial E/\partial V)_T = 0$ . From the validity of the thermal equation of state (1.1), it follows that the internal energy  $E$  is dependent only on the temperature  $T$ , a fact already pointed out in Section 1.1.

#### 1.4.2 HEAT CAPACITIES

Imagine the system in Fig. 3 undergoing a quasistatic reversible process. When an addition of heat  $dQ$  changes the temperature of the system by  $dT$ , we call

$$C = dQ/dT \quad (1.42)$$

the heat capacity of the system; heat capacity depends not only on the thermodynamic state of a system but also on the process (heat is not a state variable!). Two kinds of heat capacities are especially important:

a. *Heat capacity at constant pressure*,  $C_p$  (heat capacity for an isobaric process). In such a process,  $p$  is constant, and thus  $dW = -p dV = -d(pV)$ ; consequently, by the first law of thermodynamics,

$$dQ = dE + d(pV) = dH.$$

Moreover, by (1.23),  $dH = T dS$ , so that

$$C_p = \left( \frac{\partial H}{\partial T} \right)_p = T \left( \frac{\partial S}{\partial T} \right)_p. \quad (1.43)$$

b. *Heat capacity at constant volume*,  $C_v$  (heat capacity for an isochoric process). Since in this process  $dV = 0$ , the first law gives  $dQ = dE$ , and, by (1.22),  $dE = T dS$ , so that

$$C_v = \left( \frac{\partial E}{\partial T} \right)_V = T \left( \frac{\partial S}{\partial T} \right)_V. \quad (1.44)$$

The heat capacities  $C_p$  and  $C_v$  are commonly used extensive state variables. Moreover, we can show that  $C_v > 0$  must always hold, for otherwise the thermodynamic system will be unstable (the condition of thermal stability).

We can obtain a relation between the heat capacities  $C_p$  and  $C_v$  in the following manner: We start from  $S = S(T, V)$  and  $V = V(T, p)$  and construct the differential

$$dS = \left[ \left( \frac{\partial S}{\partial T} \right)_V + \left( \frac{\partial S}{\partial V} \right)_T \left( \frac{\partial V}{\partial T} \right)_p \right] dT + \left( \frac{\partial S}{\partial V} \right)_T \left( \frac{\partial V}{\partial p} \right)_T dp.$$

It follows that

$$\left( \frac{\partial S}{\partial T} \right)_p = \left( \frac{\partial S}{\partial T} \right)_V + \left( \frac{\partial S}{\partial V} \right)_T \left( \frac{\partial V}{\partial T} \right)_p.$$

From this, as well as (1.43), (1.44), (1.33), and (1.38), we get the following relation between the heat capacities:

$$C_p = C_v + (\alpha^2 TV)/K = C_v + R^*, \quad (1.45)$$

where the latter equation follows from the use of (1.1) and (1.37) and thus holds only for a thermally ideal gas. A necessary and immediately obvious condition for the stability of a thermodynamic system is  $K > 0$ ; i.e., in an isothermal system, a decrease of volume must accompany an increase of pressure (the condition of mechanical stability). It then follows from (1.45) that we must always have

$$C_p > C_v > 0. \quad (1.46)$$

*Specific heats* are heat capacities divided by the system mass  $M$ :  $c_p = C_p/M$  and  $c_v = C_v/M$  are the specific heats at constant pressure and constant volume, respectively.

Because of the importance of specific quantities in gas dynamics, we summarize the important formulas of this section in terms of specific quantities. We give these formulas the same numbers as before, but distinguish them

by asterisks:

$$h = e + (p/\varrho) \quad (1.19^*)$$

$$g = h - Ts \quad (1.20^*)$$

$$f = e - Ts \quad (1.21^*)$$

$$de = T ds + (p/\varrho^2) d\varrho \quad (1.22^*)$$

$$dh = T ds + (1/\varrho) dp \quad (1.23^*)$$

$$dg = -s dt + (1/\varrho) dp \quad (1.24^*)$$

$$df = -s dt + (p/\varrho^2) d\varrho \quad (1.25^*)$$

$$(\partial e/\partial s)_q = T; \quad (\partial e/\partial \varrho)_s = p/\varrho^2 \quad (1.30^*)$$

$$(\partial h/\partial s)_p = T; \quad (\partial h/\partial p)_s = 1/\varrho \quad (1.31^*)$$

$$(\partial g/\partial t)_p = -s; \quad (\partial g/\partial p)_t = 1/\varrho \quad (1.27^*); (1.28^*)$$

$$(\partial f/\partial t)_q = -s; \quad (\partial f/\partial \varrho)_q = p/\varrho^2 \quad (1.32^*)$$

$$\alpha = - (1/\varrho) (\partial \varrho/\partial t)_p \quad (1.33^*)$$

$$K = (1/\varrho) (\partial \varrho/\partial p)_t \quad (1.34^*)$$

$$(\partial s/\partial \varrho)_t = -\alpha/(K\varrho^2); \quad (\partial s/\partial p)_t = -\alpha/\varrho \quad (1.38^*); (1.39^*)$$

$$(\partial e/\partial \varrho)_t = (Kp - \alpha t)/(K\varrho^2) \quad (1.41^*)$$

$$c_p = (\partial h/\partial t)_p = T (\partial s/\partial t)_p \quad (1.43^*)$$

$$c_v = (\partial e/\partial t)_q = T (\partial s/\partial t)_q \quad (1.44^*)$$

$$c_p = c_v + \alpha^2 T/(K\varrho) \quad (1.45^*)$$

### 1.5 Equations of State for Gases

We know from experience that in a gas at constant temperature, the product  $pV$  is constant to a good approximation (Boyle–Mariotte's law). If the temperature changes at constant pressure  $p$ , the volume changes proportionally to the absolute temperature  $T$  (Gay-Lussac's law). Both laws are incorporated in the thermal equation of state (1.1). If the system contains  $n$

moles of a chemically homogeneous gas (thus, no mixtures), we have, in addition

$$R^* = n\mathfrak{R}, \quad (1.47)$$

where  $\mathfrak{R}$  is the universal gas constant. Relation (1.47) together with (1.1) contain Avogadro's law, which states that at a given temperature and given pressure, equal volumes of different gases contain an equal number of moles (or equivalently, an equal number of molecules  $N = nL$ , where  $L$  is Loschmidt's constant or the number of molecules per mole).

Deviations from the ideal gas law (1.1) become noticeable, for example, under high compression of a gas. We can take these deviations into account by generalizing the thermal equation of state as follows:

$$pV = n\mathfrak{R}T \left(1 + \frac{B}{V} + \frac{C}{V^2} + \dots\right). \quad (1.48)$$

The temperature-dependent quantities  $B$ ,  $C$ , etc., are called the second, third, etc., virial coefficients. In many cases, introducing the second virial coefficient  $B$  alone is sufficient;  $B$  is negative for low temperatures and positive for high temperatures. The temperature  $T_b$  at which  $B(T_b) = 0$  is called the Boyle temperature, since for this temperature the Boyle–Mariotte law holds up to the term of order  $C/V^2$ . For air,  $T_b = 347^\circ\text{K}$ .

Another form of thermal equation of state for gases is van der Waal's equation:

$$\left(p + \frac{n^2a}{V^2}\right)(V - nb) = n\mathfrak{R}T, \quad (1.49)$$

with two positive constants  $a$  and  $b$ . If we expand the product  $pV$  in powers of  $1/V$ , we obtain the virial form (1.48) of the equation of state, where, in particular,

$$B = n(b - (a/\mathfrak{R}T)). \quad (1.50)$$

Qualitatively,  $B$  possesses the temperature dependence mentioned before. It should be mentioned that the quotient  $pV/n\mathfrak{R}T$ , which equals unity for a thermally ideal gas, is called the compressibility factor  $Z$ . The values of  $Z$  for air are presented in Fig. 6b.

We now turn to the caloric properties of a gas system, restricting ourselves to a thermally ideal gas. From the thermal equation of state

$$V = n\mathfrak{R}T/p, \quad (1.51)$$

and from (1.27) it follows that

$$(\partial G/\partial p)_T = n\mathfrak{R}T/p. \quad (1.52)$$

From this, we obtain after integration

$$G(T, p) = n[\mathfrak{R}T \ln(p/p_0) + \mu_0(T)]. \quad (1.53)$$

Here,  $p_0$  is an arbitrary reference pressure, and  $\mu_0(T)$  is a function of temperature, which cannot be more precisely defined from thermodynamics; for each gas,  $\mu_0(T)$  must be either determined experimentally or computed theoretically using statistical mechanics. The subscript 0 on  $\mu_0$  indicates that the value of this function depends on the choice of  $p_0$ . For subsequent use (see Section 1.8), we transform the expression (1.53) for  $G$ , replacing the variable  $p$  by  $V$  by using the thermal equation of state:

$$G(T, V) = n\mathfrak{R}T \left[ \ln \left( \frac{n\mathfrak{R}T}{p_0 V} \right) + \frac{\mu_0}{\mathfrak{R}T} \right]. \quad (1.54)$$

The enthalpy can be calculated from the relation  $H = G + TS$ , i.e., by (1.28):  $H = G - T(\partial G/\partial T)_p$ . We get

$$H = n \left( \mu_0 - T \frac{d\mu_0}{dT} \right) = -nT^2 \frac{d}{dT} \left( \frac{\mu_0}{T} \right). \quad (1.55)$$

Here  $H$  is a function of temperature alone.

We now turn to the special case of a calorically ideal gas, which, by definition, has  $C_p = dH/dT = \text{const}$ , i.e.,

$$H = n(H^* + C_{pm}T) = -nT^2 \frac{d}{dT} \left( \frac{\mu_0}{T} \right), \quad (1.56)$$

where  $C_{pm}$  is the (constant) molar heat capacity and  $H^*$  is a constant which cannot be made further precise. From (1.56), it immediately follows from integration with respect to  $T$  that

$$\mu_0 = H^* - C_{pm}T \ln(T/T^*), \quad (1.57)$$

where  $T^*$  is a constant of integration having the dimension of temperature. If we divide (1.56) by the mass of the system  $M = n\mathfrak{M}$  ( $\mathfrak{M}$  is the molar mass) and introduce the specific heats  $c_p$  and  $c_v$ , the specific gas constant  $R = \mathfrak{R}/\mathfrak{M}$ , and the notation  $h^* = H^*/\mathfrak{M}$ , then we get for the specific enthalpy

$$h = h^* + c_p T. \quad (1.58)$$

The equation of state (1.1) written for specific quantities is

$$p = (\mathfrak{R}/\mathfrak{M}) T \varrho = RT \varrho, \quad (1.59)$$

and, from (1.45), we have

$$c_p = c_v + R. \quad (1.60)$$

Finally, we can calculate  $e$  and  $s$  from the relations  $e = h - (p/\varrho)$  (1.19\*) and  $s = -(\partial g/\partial T)_p$  (1.27\*):

$$e = h^* + c_v T, \quad (1.61)$$

$$s = -R \ln\left(\frac{p}{p_0}\right) + c_p \left[ \ln\left(\frac{T}{T^*}\right) + 1 \right]. \quad (1.62)$$

From (1.62), we can derive another important relation: (1.62) can be written as

$$\frac{p}{p_0} = \left(\frac{T}{T^*}\right)^{\gamma/(\gamma-1)} \exp\left(\frac{c_p - s}{R}\right), \quad (1.63)$$

where

$$\gamma = \frac{c_p}{c_v} = \frac{c_p}{c_p - R} = \frac{C_{pm}}{C_{pm} - \mathfrak{R}} > 1 \quad (1.64)$$

is the adiabatic exponent, which is a constant for a calorically ideal gas (as considered here) and a function of temperature for a thermally ideal gas. We now study isentropic changes of state, i.e., changes of state in which the entropy remains constant. We can introduce a new reference temperature  $T_0$  defined as  $T_0 = T^* \exp[(s - c_v)/c_p]$ , and find, from (1.63), that for isentropic changes of state,

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\gamma/(\gamma-1)} = \left(\frac{\varrho}{\varrho_0}\right)^\gamma; \quad (1.65)$$

This equation follows from the thermal equation of state if we set  $\varrho_0 = p_0/RT_0$ .

Equation (1.62) also permits us to calculate  $T$  as a function of  $s$  and  $p$ . If we substitute the result into (1.58), then we obtain  $h = h(p, s)$ , one of the possible forms of the canonical equation of state. The canonical equation of state  $h = h(p, s)$  may be represented graphically, with the lines  $p = \text{const}$  drawn in the  $s, h$  plane. For a calorically ideal gas, these are simply exponential curves. As a more general example, Fig. 6 gives these curves for air over a wide range of states. Such a figure is called a Mollier diagram.

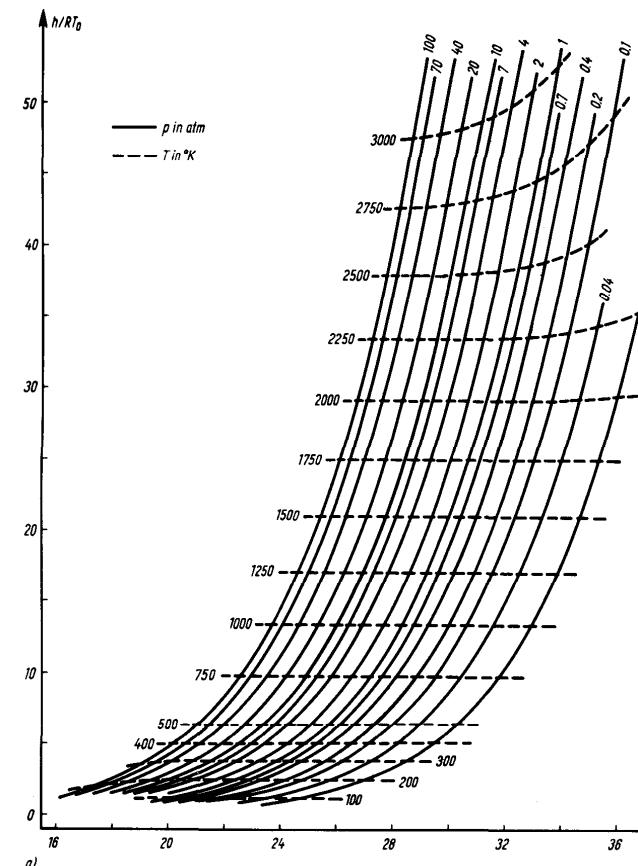


Fig. 6a. Mollier diagram for air:  $R = 6.886 \times 10^{-2} \text{ cal/g } ^\circ\text{K} = 2.882 \times 10^6 \text{ cm}^2/\text{sec}^2 \text{ }^\circ\text{K}$ ;  $T_0 = 273.16 \text{ }^\circ\text{K}$ ;  $p_0 = 1 \text{ atm}$ . (From J. Hilsenrath, C.W. Beckett et al., Tables of thermodynamic and Transport Properties of Air etc., Pergamon Press Oxford 1960).

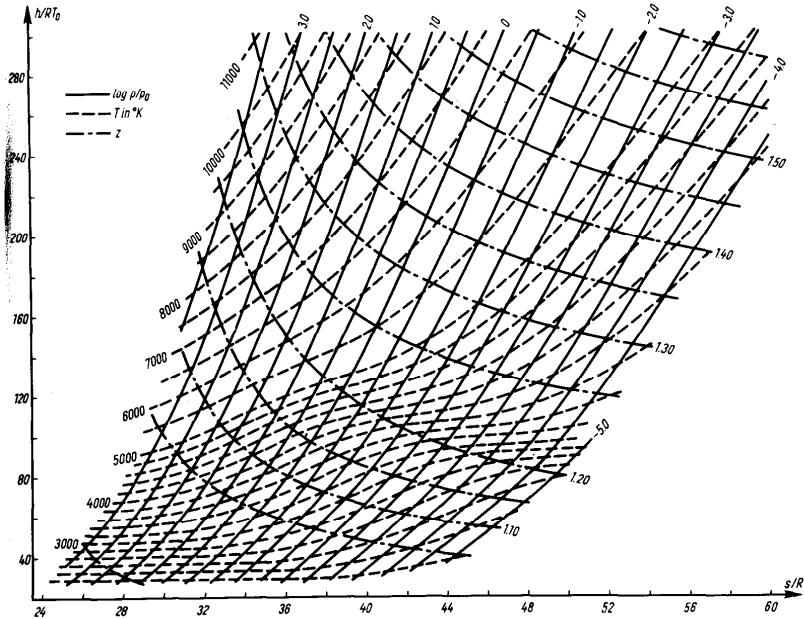


Fig. 6b. (See legend to Fig. 6a). (From S. Feldman, Hypersonic gasdynamic charts for Equilibrium Air. AVCO Res. Lab. Rept. RR40, 1957.)

The Mollier diagram contains all the thermodynamic information on a gas, since all the information is contained in the canonical equation of state.

While monatomic gases (noble gases such as argon) behave like a calorically ideal gas over a wide range of temperatures and pressures, diatomic (and polyatomic) gases do not. Oxygen and nitrogen, for example, at room temperatures and pressures below about 10 atm behave approximately like a calorically ideal gas, the specific heats  $c_p$  and  $c_v$  not changing with temper-

ature; at temperatures of several hundred degrees centigrade, however, the specific heats increase noticeably with temperature. This is caused by the fact that at high temperatures, the vibration of the atoms in the molecules absorbs a portion of the heat added, thus making this heat unavailable for increasing the temperature; this phenomenon does not occur at lower temperatures. Yet another large increase in the specific heats occurs with dissociation, and the gas is then no longer thermally ideal (see Section 1.10). We can take into account the influence of molecular vibration on the caloric behavior of diatomic gases (for sufficiently low temperatures,  $C_{pm} = 7\mathcal{R}/2$ ) to a good approximation by adding another term to the  $\mu_0$  given by (1.57):

$$\mu_0 = H^* - \frac{7}{2}\mathcal{R}T \ln(T/T^*) + \mathcal{R}T \ln[1 - \exp(-\theta/T)], \quad (1.66)$$

where  $\theta$  is a constant characteristic of the particular molecules and having the dimension of temperature. Proceeding from this expression, we see:

$$h = H^* + \frac{7}{2}\mathcal{R}T + \frac{\mathcal{R}\theta}{\exp(\theta/T) - 1}, \quad (1.67)$$

$$s = -R \ln\left(\frac{p}{p_0}\right) + \frac{7}{2}R \left[ \ln\left(\frac{T}{T^*}\right) + 1 \right] - R \ln[1 - \exp(-\theta/T)] + \frac{R\theta/T}{\exp(\theta/T) - 1}. \quad (1.68)$$

In particular, with this expression for  $s$ , we have, in place of (1.65), the relation for isentropic change of states ( $T_0$  being a suitably defined reference temperature):

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{7/2} \frac{1 - \exp(-\theta/T_0)}{1 - \exp(-\theta/T)} \exp\left[\frac{\theta/T}{\exp(\theta/T) - 1} - \frac{\theta/T_0}{\exp(\theta/T_0) - 1}\right]. \quad (1.69)$$

The thermal equation of state (1.59) remains unchanged. We notice from this formula that at  $T \ll \theta$ , the gas behaves like a calorically ideal gas with specific heat  $c_p = \frac{7}{2}R$ , while at  $T \gg \theta$ , it behaves like a calorically ideal gas with  $c_p = \frac{9}{2}R$ . The departure from a calorically ideal gas is most marked at  $T \approx \theta$ . We get for  $\gamma$

$$\gamma = \frac{7[\exp(\theta/T) - 1]^2 + 2(\theta/T)^2 \exp(\theta/T)}{5[\exp(\theta/T) - 1]^2 + 2(\theta/T)^2 \exp(\theta/T)}. \quad (1.70)$$

Thus,  $\gamma = 7/5$  for  $T \ll \theta$ , and  $\gamma = 9/7$  for  $T \gg \theta$ .

### 1.6 Conditions of Equilibrium

A system left to itself will tend toward a state of thermodynamic equilibrium. We now consider the case where, because of an internal constraint, the system is prevented from assuming unconstrained equilibrium, but, instead, has to remain in a constrained equilibrium. As an illustration, we imagine an isolated system in which one half remains at temperature  $T_1$  and the other half at  $T_2 \neq T_1$  (Fig. 7). This state can only be maintained by

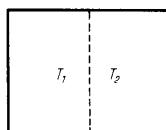


Fig. 7. A system with a heat-insulating dividing wall.

preventing heat flow from 1 to 2 (such as by artificially introducing a heat-insulating dividing wall). We further assume that the quantities  $E$ ,  $S$ , and  $G$  can be defined for such constrained states of equilibrium. This is easily done in our example: these quantities for the total system, being extensive quantities, are the sums of the quantities of the two parts. Another example we can imagine is a box containing a mixture of hydrogen and oxygen in the volume ratio of 2:1 at room temperature. This mixture will remain in a constrained equilibrium practically indefinitely. However, the internal constraint can be removed instantly if we introduce a platinum catalyst; the mixture immediately reacts and forms water, and the system has then reached unconstrained equilibrium.

If we remove the internal constraints (by removing the dividing wall or introducing the catalyst), a process will start in the system that will lead to an unconstrained equilibrium. For an isolated system, according to (1.13), each process will give

$$\Delta S = \Delta S_i \geq 0.$$

In other words, after the completion of the process leading to unconstrained equilibrium, the entropy cannot be less than the entropy in the constrained state of equilibrium. Since we assume the system to be isolated,  $E$  and  $V$

remain constant in the process. Thus, we can state the result: For given values of  $E$  and  $V$ , the entropy of a system in an unconstrained state of equilibrium is a maximum.

More precisely, we can say the following: In an unconstrained thermodynamic equilibrium (as we have seen in the previous sections),  $S$  is a function of its natural variables  $E$  and  $V$ ,  $S = S(E, V)$ . But to describe fully the state in a constrained equilibrium, we must introduce at least one additional parameter  $\xi$  which quantitatively characterizes the departure from unconstrained equilibrium (in our first example,  $\xi = T_2 - T_1$ , or  $\xi = T_2/T_1$ , etc.). Thus,  $S = S(E, V, \xi)$ , and a necessary condition for unconstrained equilibrium is

$$(\partial S / \partial \xi)_{E, V} = 0. \quad (1.71)$$

For a given function  $S(E, V, \xi)$ , this condition permits us to determine the value of  $\xi$  in unconstrained equilibrium when  $E$  and  $V$  are given.

In our second example ( $H_2$ - $O_2$  mixture) the system in constrained equilibrium is also homogeneous, and has a uniquely defined temperature  $T$  and pressure  $p$ . With a properly chosen variable  $\xi$  (for example, relative mass of  $H_2$  to  $H_2O$ ), we can generalize Eq. (1.6) as follows:

$$T dS = dE + p dV + \Gamma d\xi, \quad (1.72)$$

in which the new state variable  $\Gamma$  is obviously

$$\Gamma = T (\partial S / \partial \xi)_{E, V}. \quad (1.73)$$

If  $n$  variables  $\xi_1, \xi_2, \dots, \xi_n$  are needed to describe constrained states of equilibrium, we can generalize as follows:  $S = S(E, V, \xi_1, \dots, \xi_n)$  and

$$T dS = dE + p dV + \sum_{i=1}^n \Gamma_i d\xi_i, \quad (1.74)$$

with

$$\Gamma_i = T \left( \frac{\partial S}{\partial \xi_i} \right)_{E, V, \xi_{k \neq i}}. \quad (1.75)$$

Before proceeding with the study of thermodynamic conditions of equilibrium, let us say a few more words about the significance of the relations

(1.72)–(1.75) in gas dynamics. The gas particles<sup>2</sup> in a flow field undergo changes in their thermodynamic states during motion. In classical gas dynamics, it is always assumed that the processes causing these changes are quasistatic, i.e., that the individual gas particles proceed through unconstrained states of equilibrium. The entropy changes are given by (1.6), since in an unconstrained equilibrium  $\Gamma = 0$ . In the case of very high flow velocities, however, the states of the gas particles change so fast under certain conditions that this assumption no longer holds. The gas then goes through constrained states of equilibrium, and in order to describe these states fully, we need at least one other variable  $\xi$ . The entropy changes are then given in (1.72) or (1.74). Then we are talking about *gasdynamic relaxation*.

The condition derived earlier for unconstrained equilibrium, i.e.  $S = \max$  for  $E$ ,  $V = \text{const}$ , can be expressed in a series of completely equivalent formulations. Among these, we shall use the following: Given a system with pressure  $p$  and temperature  $T$ , the unconstrained state of equilibrium can be distinguished from the constrained equilibria by a minimum in the Gibbs enthalpy  $G$ , i.e.,  $G = \min$  for  $p$ ,  $T = \text{const}$ . The necessary condition for this is

$$\partial G(T, p, \xi)/\partial \xi = 0, \quad (1.76)$$

in which  $\xi$  has the meaning given earlier.

We briefly sketch how this condition is derived from the previous condition on  $S$ . Let us imagine a system  $A$  immersed in a system  $B$  (Fig. 8).

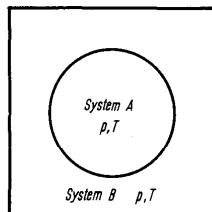


Fig. 8. Diagram for the derivation of formula (1.79).

<sup>2</sup> By a "gas particle," we shall always mean an arbitrary volume in the flow field, the surface of which is moving with the local mass velocity, and which, on the one hand, is sufficiently small so that the thermodynamic variables of pressure, temperature, etc., inside it have essentially no spatial variation, while, on the other hand, it is still large compared to the mean free path of the gas molecules.

Let  $B$  be completely isolated from its environment, and let it be so large compared to  $A$  that its pressure and temperature remain constant regardless of the state changes that may occur in  $A$ . If the boundary between them is movable and heat conducting, the pressures and temperatures in  $A$  and  $B$  will be the same. Let  $A$  first be in a constrained state of equilibrium, then go into an unconstrained state of equilibrium through some appropriate process, while  $B$  is in unconstrained equilibrium at the beginning as well as at the end of the process. As a result of this process, we have  $\Delta V_A = -\Delta V_B$  and  $\Delta E_A = -\Delta E_B$  (the volume and energy of the total system remain constant, since the system is entirely isolated from its environment).

Furthermore,

$$\Delta S = \Delta S_A + \Delta S_B \geq 0. \quad (1.77)$$

On the other hand, since  $p$  and  $T$  are constant, we can integrate (1.6) for system  $B$ :

$$\Delta S_B = (1/T)(\Delta E_B + p \Delta V_B) = -(1/T)(\Delta E_A + p \Delta V_A). \quad (1.78)$$

Substituting this into (1.77), we get

$$\Delta(TS_A - E_A - pV_A) \geq 0,$$

and using the definition (1.20) of  $G$ ,

$$\Delta G_A \leq 0. \quad (1.79)$$

Therefore, the Gibbs enthalpy of system  $A$  decreases during the process toward an unconstrained equilibrium, or at best remains constant.

### 1.7\* Chemical Potentials

So far we have only studied closed systems, i.e., systems that contain a fixed amount of gas. A convenient measure of this quantity is the mole number  $n$ , which is a constant parameter in a closed system. In an open system, material is transported across system boundaries, so that the mole number  $n$  is a variable, and must be used together with the thermodynamic variables in defining the state. Imagine an open system which contains a mixture of  $k$  gases. Then we have to specify all the  $k$  mole numbers  $n_1, n_2, \dots, n_k$  in addition to temperature and pressure (or two other state variables) in order to fully describe the thermodynamic state in an unconstrained

equilibrium. In particular, the entropy is dependent on all the  $n_i$  in addition to  $E$  and  $V$ :  $S = S(E, V, n_1, \dots, n_k)$ , and (1.6) can be replaced by

$$T dS = dE + p dV - \sum_{i=1}^k \mu_i dn_i, \quad (1.80)$$

where

$$\mu_i = -T \left( \frac{\partial S}{\partial n_i} \right)_{E, V, n_j \neq i}. \quad (1.81)$$

We call the intensive state variable  $\mu_i$  the chemical potential of the  $i$ th component of the gas mixture. As intensive variables, the  $\mu_i$  do not change if all the  $n_1, \dots, n_k$  are changed proportionally.<sup>3</sup> Thus, the  $\mu_i$  can depend only on the ratios of the mole numbers. Accordingly, we introduce the mole fractions  $X_i$  as variables

$$X_i = n_i / \sum_{i=1}^k n_i \quad (1.82)$$

where, naturally,  $\sum_{i=1}^k X_i = 1$ . Then,

$$\mu_i = \mu_i(T, p, X_1, \dots, X_{k-1}).$$

Furthermore, by the definition (1.20) for  $G$  and by (1.80), it follows that

$$dG = -SdT + Vdp + \sum_{i=1}^k \mu_i dn_i, \quad (1.83)$$

or, if we regard  $G$  as a function of  $T, p$ , and  $n_1, \dots, n_k$ ,

$$\mu_i = \left( \frac{\partial G}{\partial n_i} \right)_{T, p, n_j \neq i}. \quad (1.84)$$

On the other hand, as an extensive state variable,  $G$  has the following property: If we change all the mole numbers in the system from  $n_i$  to  $\alpha n_i$  with  $T$  and  $p$  fixed, then  $G$  changes to  $\alpha G$ ,<sup>4</sup> i.e.

$$G(T, p, \alpha n_1, \dots, \alpha n_k) = \alpha G(T, p, n_1, \dots, n_k).$$

Differentiating this identity with respect to  $\alpha$ , setting  $\alpha$  equal to 1, and using

<sup>3</sup> Mathematically speaking, the  $\mu_i$  are homogeneous functions of  $n_1, \dots, n_k$  of zeroth degree.

<sup>4</sup>  $G$  as a function of  $n_1, \dots, n_k$  is thus homogeneous of first degree.

(1.84), we obtain<sup>5</sup>

$$\sum_{i=1}^k n_i \frac{\partial G}{\partial n_i} = G, \quad G = \sum_{i=1}^k n_i \mu_i. \quad (1.85)$$

If the system contains only one gas, then  $G = \nu \mu$ , i.e.,  $\mu$  here is the molar Gibbs enthalpy. For a thermally ideal gas, because of (1.53), we have

$$\mu = RT \ln(p/p_0) + \mu_0(T). \quad (1.86)$$

*Supplementary Remarks.* An analog of (1.83) follows from (1.80) and (1.21):

$$dF = -SdT - pdV + \sum_{i=1}^k \mu_i dn_i, \quad (1.87)$$

and, similarly,

$$dE = TdS - pdV + \sum_{i=1}^k \mu_i dn_i, \quad (1.88)$$

$$dH = TdS + Vdp + \sum_{i=1}^k \mu_i dn_i. \quad (1.89)$$

Since in gas dynamics we use specific variables rather than extensive state variables, Eq. (1.80) will be transformed so that it contains only intensive state variables. Introducing the molar mass  $\mathfrak{M}_i$  of the  $i$ th component, the total mass  $M = \sum n_i \mathfrak{M}_i$ , and the mass concentration  $c_i = n_i \mathfrak{M}_i / M$  of the  $i$ th component, we rewrite (1.80) as

$$T d(sM) = d(eM) + p d\left(\frac{M}{\varrho}\right) - \sum_{i=1}^k \frac{\mu_i}{\mathfrak{M}_i} d(c_i M) \quad (1.90)$$

or

$$MT ds + Ts dM = M \left[ de + p d\left(\frac{1}{\varrho}\right) - \sum_{i=1}^k \frac{\mu_i}{\mathfrak{M}_i} dc_i \right] \\ + dM \left[ e + \frac{p}{\varrho} - \sum_{i=1}^k \frac{\mu_i}{\mathfrak{M}_i} c_i \right]. \quad (1.91)$$

But if (1.85) is divided by  $M$ , it can be written as

$$g = \sum_{i=1}^k \frac{\mu_i}{\mathfrak{M}_i} c_i. \quad (1.92)$$

<sup>5</sup> This is a special case of Euler's theorem for homogeneous functions.

On the other hand, according to (1.19\*) and (1.20\*),  $g = e + (p/\varrho) - Ts$ , so that the terms multiplied by  $dM$  on both sides of (1.91) cancel. The result is

$$T ds = de - \frac{p}{\varrho^2} d\varrho - \sum_{i=1}^k \frac{\mu_i}{\mathfrak{M}_i} dc_i. \quad (1.93)$$

Equations (1.83), (1.88), and (1.89) can be written in specific form if the extensive variables are replaced by the corresponding intensive ones, and  $\mu_i dn_i$  by  $(\mu_i/\mathfrak{M}_i) dc_i$ .

### 1.8\* Mixtures of Ideal Gases

We shall now study systems comprised of a mixture of  $k$  thermally ideal gases. Separately, each of the  $k$  components of the mixture satisfies its own thermal equation of state (1.1) of an ideal gas. All the important physical phenomena, for our purposes here, can be summarized in the statement: Let  $\Phi$  be any extensive state variable (except  $V$ ) of the system, which we regard as a function of  $T, V, n_1, \dots, n_k$ ; then,

$$\Phi(T, V, n_1, \dots, n_k) = \sum_{i=1}^k \Phi_i(T, V, n_i), \quad (1.94)$$

where  $\Phi_i$  is the value  $\Phi$  assumes when the  $n_i$  moles of the  $i$ th component alone occupy the volume  $V$  at temperature  $T$ . In particular, (1.94) is valid for the free energy:

$$F(T, V, n_1, \dots, n_k) = \sum_{i=1}^k F_i(T, V, n_i). \quad (1.95)$$

On the other hand, the generalization of (1.32) immediately follows from (1.87), or

$$p = - \left( \frac{\partial F}{\partial V} \right)_{T, n_1, \dots, n_k}. \quad (1.96)$$

If we then substitute for  $F$  the expression on the right side of (1.95), we obtain

$$p = - \sum_{i=1}^k \frac{\partial F_i(T, V, n_i)}{\partial V} = \sum_{i=1}^k p_i, \quad (1.97)$$

where the partial pressures  $p_i$  are obviously the pressures which the  $i$  components would exert individually if alone in volume  $V$  at temperature  $T$ . Equation (1.97) is Dalton's law. Since each component satisfies by itself the thermal equation of state of an ideal gas,  $p_i = n_i \mathfrak{R} T / V$ , we can write (1.97) as

$$p = (n_1 + n_2 + \dots + n_k) \mathfrak{R} T / V. \quad (1.98)$$

Dividing (1.98) by the mass  $M = \sum n_i \mathfrak{M}_i$  (where  $\mathfrak{M}_i$  is the molar mass of the  $i$ th component), we transform it into the equation of state (1.59), now written for specific variables, with the specific gas constant  $R$  for the mixture defined as

$$R = \mathfrak{R} \frac{\sum_{i=1}^k n_i}{\sum_{i=1}^k n_i \mathfrak{M}_i}. \quad (1.99)$$

Applying the relation (1.94) to the Gibbs enthalpy, and using the expression (1.54) for  $G$ , we obtain

$$G(T, V, n_1, \dots, n_k) = \sum_{i=1}^k n_i \mathfrak{R} T \left[ \ln \left( \frac{n_i \mathfrak{R} T}{p_0 V} \right) + \frac{\mu_{0i}(T)}{\mathfrak{R} T} \right]. \quad (1.100)$$

If we then replace  $p$  by  $V$  from (1.98), we obtain

$$G(T, p, n_1, \dots, n_k) = \mathfrak{R} T \sum_{i=1}^k n_i \left[ \ln \left( \frac{p}{p_0} \right) + \ln X_i + \frac{\mu_{0i}(T)}{\mathfrak{R} T} \right], \quad (1.101)$$

where the mole fractions  $X_i$  are defined in (1.82). The subscript  $i$  on  $\mu_{0i}$  indicates that this function can be different for different gases. Comparing (1.101) with (1.85), we finally arrive at an important relation for the chemical potential  $\mu_i$  of the  $i$ th component in a mixture of ideal gases:

$$\mu_i = \mathfrak{R} T [\ln(p/p_0) + \ln X_i + \phi_i(T)], \quad (1.102)$$

with

$$\phi_i(T) = \mu_{0i}(T)/\mathfrak{R} T. \quad (1.103)$$

Thus, in an ideal gas mixture, each  $\mu_i$  depends only on  $T, p$ , and the single mole fraction  $X_i$ . In conclusion, we note that from (1.103),

$$\frac{d\phi_i}{dT} = \frac{1}{\mathfrak{R}} \frac{d}{dT} \left( \frac{\mu_{0i}}{T} \right) = - \frac{H_{mi}(T)}{\mathfrak{R} T^2}, \quad (1.104)$$

where we have used (1.55) and where  $H_{mi}(T)$  is the molar enthalpy (enthalpy per mole) of the  $i$ th component.

### 1.9\* Law of Mass Action

Consider a closed system consisting of a mixture of 4 thermally ideal gases  $A_1, A_2, A_3$ , and  $A_4$  which react with one another according to the reaction equation



(for example:  $\text{CO}_2 + \text{H}_2 \rightleftharpoons \text{CO} + \text{H}_2\text{O}$ , the water-gas reaction). In a closed system, changes of mole numbers  $n_1, n_2, n_3$ , and  $n_4$  can occur only as a result of chemical reaction, where, by (1.105), the changes  $\Delta n_i$  must obviously satisfy the relation

$$\Delta n_1 : \Delta n_2 : \Delta n_3 : \Delta n_4 = v_1 : v_2 : (-v_3) : (-v_4). \quad (1.106)$$

On the basis of this relation, we can express the possible mole numbers in the system in terms of an auxiliary variable  $\xi$ :

$$\begin{aligned} n_1 &= Nv_1(1 - \xi) + n_{10}, \\ n_2 &= Nv_2(1 - \xi) + n_{20}, \\ n_3 &= Nv_3\xi + n_{30}, \\ n_4 &= Nv_4\xi + n_{40}, \end{aligned} \quad (1.107)$$

with the five constants  $N$  and  $n_{10}, \dots, n_{40}$ . A change of  $\Delta\xi$  in  $\xi$  produces changes  $\Delta n_i$  in the  $n_i$  which satisfy (1.106). Now, assume that the reaction given by (1.105) has progressed so far from left to right that all the  $A_1$  molecules have been used up (this of course presumes that there are sufficient  $A_2$  molecules for the  $A_1$  molecules to react with; otherwise, we interchange the subscripts 1 and 2). For this state, we arbitrarily set  $\xi = 1$ , and, by (1.107),  $n_{10}$  must be zero. Conversely, if the reaction from right to left has progressed so far that all the  $A_3$  molecules have been used up, then we define  $\xi = 0$  (and thus,  $n_{30} = 0$ ). With this definition,  $\xi$  is called the degree of reaction. The constant  $N$  is the number of moles of  $A_1$  divided by  $v_1$  when all the moles of  $A_3$  have been used up, or, conversely, it is the mole number of  $A_3$  divided by  $v_3$  when all the  $A_1$  molecules have been used up;  $N$ , as well as the remaining constants  $n_{20}$  and  $n_{40}$ , depends on the amounts of the gases which originally were filled into the system.

For each pressure  $p$  and temperature  $T$ , there will be an unconstrained thermodynamic equilibrium in which the degree of reaction  $\xi$  will have a definite value depending on  $T$  and  $p$ . If  $T$  and  $p$  are changed, but  $\xi$  remains

fixed because of internal constraints, then the new state will, in general, not be one of unconstrained equilibrium. Upon removal of the internal constraint,  $\xi$  will change until the new unconstrained state of equilibrium is reached. It is easy to formulate the conditions for this unconstrained equilibrium. In (1.101),  $G$  is given as a function of  $T, p$ , and  $n_1, \dots, n_4$ . If we put in the  $n_i$  given by (1.107), then  $G = G(T, p, \xi)$ ; for an unconstrained equilibrium with given  $T$  and  $p$ , we must have, by (1.76),

$$\left(\frac{\partial G}{\partial \xi}\right)_{T, p} = \frac{\partial G}{\partial n_1} \frac{dn_1}{d\xi} + \frac{\partial G}{\partial n_2} \frac{dn_2}{d\xi} + \frac{\partial G}{\partial n_3} \frac{dn_3}{d\xi} + \frac{\partial G}{\partial n_4} \frac{dn_4}{d\xi} = 0. \quad (1.108)$$

Comparing (1.84) with (1.107), we obtain

$$v_1\mu_1 + v_2\mu_2 = v_3\mu_3 + v_4\mu_4. \quad (1.109)$$

Using Eq. (1.102) for the chemical potentials in an ideal gas mixture, we can express this condition (1.109) for unconstrained equilibrium as

$$\frac{X_3^{v_3} X_4^{v_4}}{X_1^{v_1} X_2^{v_2}} \left(\frac{p}{p_0}\right)^{(v_3+v_4-v_1-v_2)} = K(T). \quad (1.110)$$

The "equilibrium constant," which depends only on  $T$ , is

$$K(T) = \exp(v_1\phi_1 + v_2\phi_2 - v_3\phi_3 - v_4\phi_4). \quad (1.111)$$

Equation (1.110) is the *law of mass action*.

From (1.111) it immediately follows that

$$\frac{d \ln K}{dT} = v_1 \frac{d\phi_1}{dT} + v_2 \frac{d\phi_2}{dT} - v_3 \frac{d\phi_3}{dT} - v_4 \frac{d\phi_4}{dT}. \quad (1.112)$$

Because of (1.104), this means

$$\frac{d \ln K}{dT} = \frac{1}{\mathfrak{R} T^2} [v_3 H_{m3} + v_4 H_{m4} - v_1 H_{m1} - v_2 H_{m2}] = \frac{\delta H_m(T)}{\mathfrak{R} T^2}. \quad (1.113)$$

This is Van 't Hoff's relation for the equilibrium constant  $K$ .  $\delta H_m$  is a short-hand notation for the sum of the products of the molar enthalpies with the stoichiometric coefficients  $v_i$  (with "+" or "-" as indicated).

These formulas, valid for thermally ideal gases in general, can be further simplified when the gases are in addition calorically ideal. Then, according to

(1.56),  $H_m = H^* + C_{pm}T$ , so that

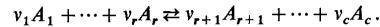
$$\begin{aligned}\delta H_m &= (v_3 H_3^* + v_4 H_4^* - v_1 H_1^* - v_2 H_2^*) T \\ &\quad + (v_3 C_{pm3} + v_4 C_{pm4} - v_1 C_{pm1} - v_2 C_{pm2}) T \\ &= \delta H^* + \delta C_{pm}T.\end{aligned}\quad (1.114)$$

Substituting this into (1.113) and integrating with respect to  $T$ , we obtain

$$K = K_0 \left( \frac{T}{T_d} \right)^{\delta C_{pm}/\mathfrak{R}} \exp \left( - \frac{T_d}{T} \right), \quad (1.115)$$

where  $K_0$  is an integration constant and  $T_d$  denotes  $\delta H^*/\mathfrak{R}$  and has the dimension of temperature.

It is not difficult to generalize these formulas to reactions of the type



The law of mass action (1.110) becomes

$$\frac{X_{r+1}^{v_{r+1}} X_{r+2}^{v_{r+2}} \cdots X_c^{v_c} (p)}{X_1^{v_1} X_2^{v_2} \cdots X_r^{v_r}} = K(T), \quad (1.116)$$

and the generalization of (1.111)–(1.115) is obvious. For example, instead of (1.111), we have

$$K(T) = \exp [(v_1 \phi_1 + \cdots + v_r \phi_r) - (v_{r+1} \phi_{r+1} + \cdots + v_c \phi_c)]. \quad (1.117)$$

*Supplementary Remarks.* We have ascribed the symbol  $\xi$  to the degree of reaction; the same symbol was used in Section 1.6 for the variables which describe the deviations from thermodynamic equilibrium. We use the same symbol for the following reason: In some thermodynamic processes involving chemical reactions (e.g., dissociation; see Section 1.10), unconstrained thermodynamic equilibrium is not attained at all under certain conditions,—for example, if the process occurs with so high a velocity that the reaction “lags” behind the other changes (relaxation; see Section 1.6). The states assumed during such processes can often be described as constrained equilibria, and their degrees of reaction do not agree with the equilibrium values. We then have to use the more general equation (1.72) instead of (1.6). One equation of this form follows immediately from (1.80) if we note that by (1.107),

$$\begin{aligned}dn_1 &= -Nv_1 d\xi; & dn_2 &= -Nv_2 d\xi; \\ dn_3 &= +Nv_3 d\xi; & dn_4 &= +Nv_4 d\xi.\end{aligned}$$

With this, (1.80) becomes

$$T dS = dE + p dV + N(v_1 \mu_1 + v_2 \mu_2 - v_3 \mu_3 - v_4 \mu_4) d\xi.$$

But this is of the form (1.72). At the same time, we have found an expression for the quantity  $\Gamma$  which is valid for chemical reactions of the type characterized by Eq. (1.105):

$$\Gamma = N(v_1 \mu_1 + v_2 \mu_2 - v_3 \mu_3 - v_4 \mu_4).$$

If  $\xi$  coincides with its equilibrium value given by the law of mass action, then, according to (1.109),  $\Gamma = 0$ .

### 1.10\* Dissociation

Let us consider a diatomic gas, which can dissociate into its atoms; for example, oxygen:



This dissociation reaction belongs to the type of reactions described in Section 1.9, if we identify  $O_2$  molecules with  $A_1$  and O atoms with  $A_3$ , so that  $v_1 = 1$ ,  $v_3 = 2$ . The law of mass action (1.110) gives

$$\frac{X_3^2 p}{X_1 p_0} = K_0 \left( \frac{T}{T_d} \right)^{\delta C_{pm}/\mathfrak{R}} \exp \left( - \frac{T_d}{T} \right). \quad (1.119)$$

We now introduce the degree of reaction given by (1.107). In dissociation, it is common to denote this by  $\alpha$ , and it is called the degree of dissociation:

$$n_1 = N(1 - \alpha); \quad n_3 = 2N\alpha. \quad (1.120)$$

From this, we obtain

$$\alpha = \frac{n_3}{n_3 + 2n_1} = \frac{n_3 \mathfrak{M}_3}{n_3 \mathfrak{M}_3 + n_1 \cdot 2\mathfrak{M}_3} = \frac{M_3}{M_3 + M_1}, \quad (1.121)$$

where  $\mathfrak{M}_3$  is the molar mass of the atoms and  $2\mathfrak{M}_3 = \mathfrak{M}_1$  is that of the molecules. Thus, according to (1.121),  $\alpha$  can also be interpreted as the ratio of the mass of atoms in the system  $M_3$  to the total mass  $M_1 + M_3$ . The mole

fractions  $X_1$  and  $X_3$  can be expressed in terms of  $\alpha$  as:

$$X_1 = \frac{n_1}{n_1 + n_3} = \frac{1 - \alpha}{1 + \alpha},$$

$$X_3 = \frac{n_3}{n_1 + n_3} = \frac{2\alpha}{1 + \alpha}.$$

Substituting into (1.119), we get

$$\frac{\alpha^2}{1 - \alpha^2} = \frac{p_d}{p} \left( \frac{T}{T_d} \right)^{\delta C_{pm}/R} \exp\left(-\frac{T_d}{T}\right). \quad (1.122)$$

$p_d = K_0 p_0 / 4$  and  $T_d$  are constants characteristic of the particular dissociation reaction, and have the dimensions of pressure and temperature respectively.

The thermal equation of state of a partially dissociating gas is, according to (1.98),

$$pV = (n_1 + n_3) RT.$$

Dividing this by the system mass  $M = (n_1 + \frac{1}{2}n_3)\mathfrak{M}_1 = N\mathfrak{M}_1$ , we obtain

$$p = (1 + \alpha) R_1 T \varrho, \quad (1.123)$$

with  $R_1 = R/\mathfrak{M}_1$ , the specific gas constant of the molecular gas.

Further discussion of Eq. (1.122) requires an assumption concerning  $\delta C_{pm}$ . We discuss two different assumptions:

(a)  $\delta C_{pm} = 0$ , or  $C_{pm1} = 2C_{pm3}$ . We assume that the molar heat capacity of the molecules at constant pressure is exactly double that of the atoms. This implies that the specific heat  $c_p$  is the same for atomic and molecular gases. Equation (1.122) then becomes

$$\frac{\alpha^2}{1 - \alpha^2} = \frac{p_d}{p} \exp\left(-\frac{T_d}{T}\right). \quad (1.124)$$

(b)  $\delta C_{pm} = R$ , or  $2C_{pm3} - C_{pm1} = R$ . From (1.45),  $C_{pm} = C_{vm} + R$ , so that  $C_{vm1} = 2C_{vm3}$ . Now, we assume that the molar heat capacity of the molecules at constant volume is exactly double that of the atoms, and, accordingly, the specific heats are the same for both. This assumption is to be found in a fundamental paper of Lighthill on the dynamics of dissociating gas,<sup>6</sup> and we therefore call a gas satisfying such an assumption a *Lighthill gas*. We obtain

<sup>6</sup> M.J. Lighthill, Dynamics of a dissociating gas. Part I: Equilibrium flow, *J. Fluid Mech.* **2**, 1-32 (1957).

from (1.122)

$$\frac{\alpha^2}{1 - \alpha^2} = \frac{p_d}{p} \frac{T}{T_d} \exp\left(-\frac{T_d}{T}\right). \quad (1.125)$$

If we then substitute for the pressure  $p$  in terms of the density  $\varrho$  from (1.123), we obtain a relation equivalent to (1.125):

$$\frac{\alpha^2}{1 - \alpha^2} = \frac{\varrho_d}{\varrho} \exp\left(-\frac{T_d}{T}\right), \quad (1.126)$$

where  $\varrho_d = p_d/(R_1 T_d)$  is a characteristic density for the dissociation process.

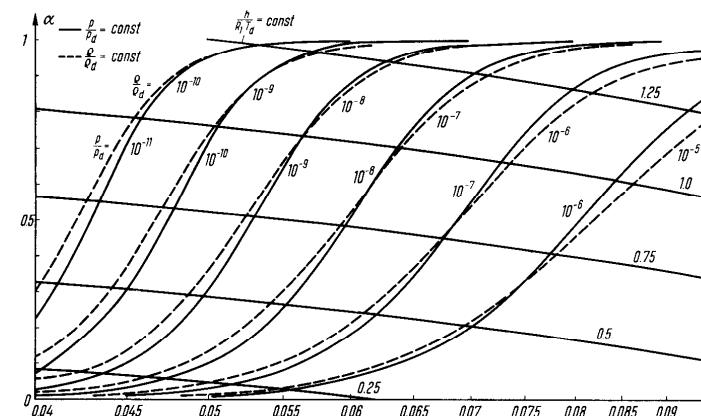


Fig. 9. Degree of dissociation of a Lighthill gas as a function of temperature; pressure, density, and enthalpy are parameters.

In Fig. 9, the degree of dissociation is presented as a function of the pressure  $p$  or density  $\varrho$  and the temperature  $T$ , according to (1.125) or (1.126). The constants  $p_d$ ,  $\varrho_d$  and  $T_d$  are different for different gases. According to Lighthill: for oxygen,  $p_d = 2.3 \times 10^7$  atm,  $\varrho_d = 150 \text{ g cm}^{-3}$ , and  $T_d = 59,000^\circ\text{K}$ ; for nitrogen,  $p_d = 4.5 \times 10^7$  atm,  $\varrho_d = 130 \text{ g cm}^{-3}$ , and  $T_d = 113,000^\circ\text{K}$ . These values provide acceptable agreement with experimental data in the pressure and temperature regimes of practical interest. On the other hand, fairly good agreement with experimental values can also be obtained by the

use of assumption (a),  $\delta C_{pm} = 0$ , with suitably chosen values for  $T_d$  and  $p_d$ .

We conclude this section by giving some caloric properties of a Lighthill gas. For the molar enthalpies of the molecules or atoms, we have

$$H_{m1} = C_{pm1} T, \quad (1.127)$$

$$H_{m3} = C_{pm3} T + \frac{1}{2}\delta H^*. \quad (1.128)$$

Here we have set to zero the unimportant additive constant on the right side of (1.127). But then, we must put a constant in (1.128), which was called  $\delta H^*/2$  in (1.114). Since  $C_{pm3} = \frac{1}{2}C_{pm1} + \frac{1}{2}\mathfrak{R}$ , the total enthalpy of the system is

$$H = n_1 H_{m1} + n_3 H_{m3} = C_{pm1} T (n_1 + \frac{1}{2}n_3) + \frac{1}{2}n_3 (\mathfrak{R}T + \delta H^*).$$

Dividing by the mass  $M = (n_1 + \frac{1}{2}n_3)\mathfrak{M}_1$ , making the assumption (valid for diatomic gases) that

$$C_{pm1} = 4\mathfrak{R}, \quad (1.129)$$

and using the previously defined temperature  $T_d = \delta H^*/\mathfrak{R}$  and specific gas constant  $R_1 = \mathfrak{R}/\mathfrak{M}_1$  of the molecules, we finally obtain the following expression for the specific enthalpy:

$$h = (4 + \alpha) R_1 T + \alpha R_1 T_d. \quad (1.130)$$

From this, we immediately obtain for the specific internal energy  $e = h - (p/\varrho)$

$$e = 3R_1 T + \alpha R_1 T_d. \quad (1.131)$$

In an unconstrained thermodynamic equilibrium,  $\alpha$  is defined as a function of  $\varrho$  and  $T$  by (1.126). Thus,  $h$  and  $e$  are known functions of  $\varrho$  and  $T$ , by (1.130) and (1.131). In Fig. 9, the lines  $h = \text{const}$  are drawn. The constant  $R_1 T_d$  has the meaning of dissociation energy. By (1.131), this is the amount of energy by which the internal energy of a fully dissociated gas exceeds that of the molecular gas at the same temperature. The specific entropy of a Lighthill gas is given by

$$S = - \left( \frac{\partial G}{\partial T} \right)_{p, n},$$

which, upon division by  $M$ , gives

$$\frac{s}{R_1} = 3 \ln \left( \frac{T}{T_d} \right) + \alpha (1 - 2 \ln \alpha) - (1 - \alpha) \ln (1 - \alpha) - (1 + \alpha) \ln \frac{\varrho}{\varrho_d}. \quad (1.132)$$

A Mollier diagram can be found in Lighthill's original paper.



### 1.11 Speed of Sound

In general, the gas particles in a flow field undergo unconstrained thermodynamic equilibrium, i.e., the thermodynamic processes occurring in a flow field are usually quasistatic. For such flows, the state variable  $a$  defined by

$$a^2 = (\partial p / \partial \varrho)_s \quad (1.133)$$

has a special significance;  $a$  has the dimension of velocity. In Section 3.2.1, we shall show that this is the speed with which a small disturbance or sound wave propagates in a gas. Like any other state variable, the sound speed  $a$  is determined by two independent thermodynamic variables. We can show that as a consequence of the thermal and mechanical stability requirements (see Section 1.4), we always have  $a^2 > 0$ ; thus,  $a$  is always a real velocity.

The expression (1.133) for  $a$  can be transformed as follows: From the caloric equation of state,  $h = h(p, \varrho)$ , and we can write (1.23) in the form

$$(\partial h / \partial p) dp + (\partial h / \partial \varrho) d\varrho = T ds + (1/\varrho) dp.$$

Setting  $ds = 0$ , we get

$$\left( \frac{\partial p}{\partial \varrho} \right)_s = a^2 = - \frac{\partial h / \partial \varrho}{(\partial h / \partial p) - 1/\varrho}. \quad (1.134)$$

For a thermally ideal gas,  $h = h(T)$ , so that

$$a^2 = - \frac{(dh/dT)(\partial T / \partial \varrho)_p}{(dh/dT)(\partial T / \partial p)_e - (1/\varrho)}.$$

Using  $dh/dT = c_p(T)$  (the specific heat), and the thermal equation of state (1.59), we get

$$a^2 = \gamma(T) RT = \gamma p / \varrho, \quad (1.135)$$

where  $\gamma(T) = c_p/(c_p - R) = c_p/c_v$ . The sound speed of a thermally ideal gas thus depends on the temperature only. For a calorically ideal gas, moreover,  $\gamma = \text{const}$ , so that  $a \sim \sqrt{T}$ . Figure 10 gives the speed of sound in air as a dimensionless variable as calculated by Hansen and Heims.

We now extend the discussion to systems in constrained equilibrium, which must be described in terms of at least one additional variable  $\xi$ . The caloric equation of state will be  $h = \hat{h}(p, \varrho, \xi)$ . In unconstrained equilibrium,  $\xi$  is a known function of  $p$  and  $\varrho$ ,  $\xi = \tilde{\xi}(p, \varrho)$ , so that  $h = \hat{h}(p, \varrho, \tilde{\xi}(p, \varrho)) = h(p, \varrho)$ .

## 1 Fundamentals of Thermodynamics

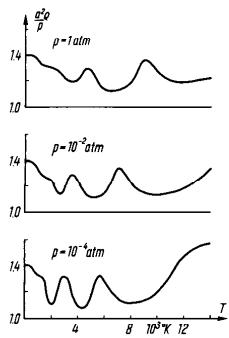


Fig. 10. Speed of sound in air as a function of temperature at different pressures. (From C.F. Hansen and S.P. Heims, A Review of the Thermodynamic, Transport and Chemical Reaction Rate Properties of High Temperature Air. NACA TN 4359, 1958.)

We now define two quantities  $a$  and  $b$ , both having the dimension of velocity:

$$a^2 = \left( \frac{\partial p}{\partial \varphi} \right)_{s=\text{const}; \xi=\tilde{\xi}}, \quad (1.136)$$

$$b^2 = \left( \frac{\partial p}{\partial \varphi} \right)_{s=\text{const}; \xi=\text{const}}. \quad (1.137)$$

These expressions can be transformed. To this end, we start from Eq. (1.72), which is, in specific quantities,

$$T ds = dh - dp/\varphi + \Gamma d\xi. \quad (1.138)$$

[We keep the symbol  $\Gamma$  for simplicity. Here,  $\Gamma$  is  $(\partial s/\partial \xi)_{e,g}$ , whereas in (1.72) it had the meaning given by (1.73).] With  $ds = 0$  and the fact that  $\Gamma = 0$  for  $\xi = \tilde{\xi}$ , we now obtain

$$a^2 = - \frac{\frac{\partial \hat{h}}{\partial \varphi} + \frac{\partial \hat{h}}{\partial \xi} \frac{\partial \tilde{\xi}}{\partial \varphi}}{\frac{\partial \hat{h}}{\partial \xi} \frac{\partial \tilde{\xi}}{\partial p} - \frac{1}{\varphi}} = - \frac{\frac{\partial \hat{h}}{\partial \varphi}}{\frac{\partial \hat{h}}{\partial p} - \frac{1}{\varphi}} \quad (1.139)$$

and

$$b^2 = - \frac{\frac{\partial \hat{h}}{\partial \varphi}}{\frac{\partial \hat{h}}{\partial p} - \frac{1}{\varphi}}. \quad (1.140)$$

The definition of  $a$  is identical to definition (1.133);  $b$  depends naturally on  $\xi$  in addition to  $p$  and  $\varphi$ . We call  $a$  the equilibrium speed of sound, and  $b$  the frozen speed of sound. In general, we shall study flows in which the gas particles are in unconstrained thermodynamic equilibrium; in this case,  $a$  will just be called the speed of sound.

We briefly remark further on the meaning of the speeds  $a$  and  $b$ . In certain gas flows, relaxation phenomena must be considered (Section 1.6); if the processes are sufficiently slow, the gas particles undergo unconstrained equilibrium ( $\xi = \tilde{\xi}$ ), whereas, if the processes are very fast, the equilibrium becomes constrained or frozen ( $\xi = \text{const}$ ). A slow process is one in which the state variables do not change significantly over time intervals comparable to characteristic relaxation times  $\tau$ . Under such circumstances, we can show that  $a$  is the phase velocity of a harmonic sound wave whose frequency is very small compared to  $\tau^{-1}$ , while  $b$  is the phase velocity of a sound wave whose frequency is very large compared to  $\tau^{-1}$ . For unconstrained equilibrium, we always have  $a^2 < b^2$  (see Section 3.3.2). For a Lighthill gas, we obtain<sup>7</sup>

$$a^2 = \frac{p}{\varphi} \left[ 1 + \frac{\frac{2T + \frac{2+3\alpha^2-\alpha^3}{(1-\alpha^2)\alpha} T^2}{1+\frac{3(2-\alpha)}{\alpha(1-\alpha)} T^2}}{1+\frac{3(2-\alpha)}{\alpha(1-\alpha)} T^2} \right]. \quad (1.141)$$

$$b^2 = \frac{p}{\varphi} \frac{4+\alpha}{3}. \quad (1.142)$$

Here,  $b$  is the sound speed for  $\alpha = \text{const}$ , i.e., for frozen dissociation equilibrium.

<sup>7</sup> J.P. Appleton, The Structure of a Centered Rarefaction in an Ideal Dissociating Gas. Rep. No. 136 Univ. of Southampton, Southampton, England, (1960).

## 1.11 Speed of Sound

### 1.12 Application to Systems in Motion

The thermodynamic laws given in the previous sections are relations connecting the states of thermodynamic equilibrium to one another, each state being specified by two state variables. There may be constrained equilibrium states which require the introduction of some finite number of additional variables  $\xi_i$ . However, a gas in motion is not a system in equilibrium describable by a finite number of variables, and the questions are raised of whether and how we can apply thermodynamic laws to these systems.

Such a generalization is made possible by the use of local state variables. Local state variables are functions of position in the flow field and of time, and are defined thus: We imagine a volume element cut out in the neighborhood of a point  $P$  at time  $t$  in the flow field. Now we isolate this volume element completely from its environment and let it move with the mass velocity it possessed just before it was cut out. After a while, all the initial inhomogeneities would have evened out, and the smaller the element, the faster this happens. The element is then in a state of thermodynamic equilibrium, with definite and measurable values of its intensive and specific state variables. The limiting values of these quantities for infinitesimal volume elements (and therefore infinitesimal equilibrating times) are defined as the local state variables at the point  $P$  and time  $t$ .

The starting point of classical gas dynamics is the assumption that these volume elements are in unconstrained equilibrium. It is therefore assumed that the thermodynamic processes undergone by each gas particle<sup>8</sup> moving in the flow field are quasistatic processes, and that the local state variables for such a gas particle are connected to one another by relations valid for unconstrained thermodynamic equilibrium. These assumptions will also be the basis for subsequent discussions in general, unless broader assumptions are stated. It then suffices to use two state variables, e.g., pressure  $p$  and density  $\rho$ , to characterize the local states.

As already mentioned in Section 1.6, nowadays such flow problems assume ever more significance in which the gas particles do not undergo unconstrained thermodynamic equilibrium. To describe the constrained equilibria established in an isolated small volume element, at least another variable  $\xi$  is needed, which also varies with position and time in the flow

<sup>8</sup> See footnote 2 for the meaning of a "gas particle."

field. In this case, the thermodynamic relations must be modified accordingly (see Section 1.6).

Our discussion so far assumes that a gas is a continuum. But every gas is composed of molecules, so that the previous arguments become meaningless if the dimensions of the volume element cut out from the flow field become comparable to the typical lengths of the molecular structure. The important length in limiting the validity of gas dynamics is the *mean free path* of the molecules, and the previous ideas remain meaningful only if the thermodynamic state variables (obtained by isolating and shrinking the volume elements) are already practically independent of the dimensions of the volume element when these dimensions are still large compared to the mean free path. In other words, the local state variables must not change significantly over distances of the order of the mean free path.<sup>9</sup> If this assumption is violated, e.g., in flows of highly rarefied gases (e.g., reentry of space vehicles in the upper atmosphere) or with extremely high gradients of the state variables (e.g., strong shock waves), continuum gas dynamics becomes no longer applicable. We must then resort to kinetic theory of gases, which is outside the scope of the present book.

<sup>9</sup> In addition, we must require that the characteristic time scale for the substantial time derivative (see Section 2.2) of the thermodynamic variables be large compared with the time between the collisions of two molecules.

## 2 FUNDAMENTALS OF CONTINUUM MECHANICS

**Remarks on Notation.** In the following sections, we shall denote vectors by lower-case boldface Roman letters:  $\mathbf{a}$ ,  $\mathbf{b}$ , etc., and square matrices (which always represent second-order tensors) by capital boldface Roman letters:  $\mathbf{A}$ ,  $\mathbf{B}$ , etc. The vector  $(\mathbf{A} \cdot \mathbf{b})$  is the product of the matrix  $\mathbf{A}$  with the vector  $\mathbf{b}$  (written as a column matrix), while the vector  $(\mathbf{a} \cdot \mathbf{B})$  is the product of the vector  $\mathbf{a}$  (written as a row matrix) with the matrix  $\mathbf{B}$ . The scalar  $(\mathbf{a} \cdot \mathbf{b})$ , or the product of the row matrix  $\mathbf{a}$  with the column matrix  $\mathbf{b}$ , is the scalar product of the vectors  $\mathbf{a}$  and  $\mathbf{b}$ .  $(\mathbf{a} \cdot (\mathbf{A} \cdot \mathbf{b}))$  then denotes the scalar product of the vectors  $\mathbf{a}$  and  $\mathbf{b}$ . The vector product of a vector  $\mathbf{a}$  with a vector  $\mathbf{b}$  will be written as  $\mathbf{a} \times \mathbf{b}$ .

Although we are primarily concerned with gases in this book, the discussions of Sections 2.1–2.5 remain valid, with slight exceptions, for deformable media in general; in particular, they are also valid for liquids. As is customary, we shall call each function defined at all the points of a connected region in space and at all the points in a time interval a field function, or, for short, a field.

### 2.1 Kinematics of a Flowing Medium

The state of motion of a flowing gas is characterised by the velocity field  $\mathbf{v}(x, y, z, t)$ . The vector  $\mathbf{v}$  is the velocity with which a gas particle at time  $t$  and location  $x, y, z$  moves.<sup>10</sup> In general, we use rectangular Cartesian co-

<sup>10</sup> In kinetic theory,  $\mathbf{v}$  is defined as the *mean mass velocity*, i.e., it is the weighted (with molecular mass) average of the molecular velocities at the point  $x, y, z$  and time  $t$ . If all the molecules have equal mass, this mean is just an arithmetic mean.

ordinates  $x, y, z$ . The components of the velocity in the  $x, y, z$  directions will be denoted by  $u, v, w$ :

$$\mathbf{v}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)). \quad (2.1)$$

We shall assume for the velocities  $u, v, w$ , as well as for the density  $\varrho$ , the pressure  $p$ , and other field functions to be introduced, that these functions and their derivatives appearing in our equations are all continuous functions of their arguments. When this is not the case, special considerations must be used (see Section 3.4).

When the velocity and all the other flow variables do not depend on time, we say the flow is steady; otherwise, it is unsteady. The velocity field gives a direction at each point in space at any instant. The integral curves of this direction field at any given instant are the *streamlines*. For these curves, we have  $dx:dy:dz = u:v:w$ . By *particle paths*, we mean the space curves traced out by the individual gas particles in the course of time. In steady flows, the streamlines and particle paths are identical; in unsteady flows, they generally are not. If one takes a point  $P$  on a streamline in an unsteady flow, then the path of the particle which coincides with  $P$  at that instant  $t$  will be tangent to the streamline at  $P$  (Fig. 11a). By a *streamtube* we mean the surface formed by all the streamlines passing through a closed curve (Fig. 11b).

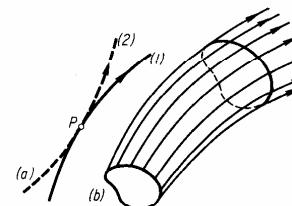


Fig. 11. a. (1) Streamline and (2) particle path. b. Streamtube.

To study the deformation of a moving medium, we consider two infinitesimally close points  $P_1$  and  $P_2$  which at time  $t$  are separated from each other by a displacement vector  $d\mathbf{r}$  with components  $dx, dy, dz$ . The velocity components  $u_1$  and  $u_2$  at the two points differ by

$$u_2 - u_1 = du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz, \quad (2.2)$$

and similarly for  $v$  and  $w$ . Introducing the square matrix

$$\text{grad } \mathbf{v} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix} \quad (2.3)$$

and regarding the vector  $\mathbf{v}$  as a column matrix, we can combine the three component equations (2.2) into a single equation:

$$\mathbf{v}_2 - \mathbf{v}_1 = d\mathbf{v} = (\mathbf{dr} \cdot \text{grad } \mathbf{v}), \quad (2.4)$$

where the product of the matrix  $\text{grad } \mathbf{v}$  with the vector  $d\mathbf{r}$  is defined in the preliminary remarks on notation.

Every matrix, which gives a linear relation between two vectors (as in Eq. (2.4), the relation between  $d\mathbf{r}$  and  $d\mathbf{v}$  is given by  $\text{grad } \mathbf{v}$ ), is the representation of a second-order tensor in the coordinate system chosen. Thus, we call  $\text{grad } \mathbf{v}$  the *gradient tensor* of the velocity field. Just as vectors have significance independent of the choice of the coordinate system—namely, that they can be regarded as directed line segments—tensors also have a significance independent of the choice of coordinates. If we introduce a new Cartesian coordinate system by a rotation of the original coordinate system, then all the components of  $d\mathbf{r}$  and  $\text{grad } \mathbf{v}$  will indeed be changed, but the relation (2.4) remains unchanged in the new coordinate system. From the known transformation for vector components under rotation of coordinates, we can derive the transformation formulas for tensor components.

Let the points  $P_1$  and  $P_2$  move with the flowing medium (Fig. 12); then

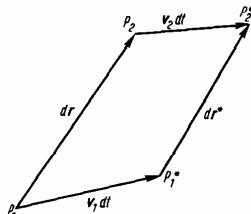


Fig. 12. Translation of the material points  $P_1$  and  $P_2$  in time  $dt$ .

the displacement vector  $d\mathbf{r}^*$  at time  $t + dt$  will be given by

$$d\mathbf{r}^* = d\mathbf{r} + (\mathbf{v}_2 - \mathbf{v}_1) dt = d\mathbf{r} + (d\mathbf{r} \cdot \text{grad } \mathbf{v}) dt. \quad (2.5)$$

The scalar distance between the points at time  $t + dt$ , or the absolute value  $|d\mathbf{r}^*|$  of the vector  $d\mathbf{r}^*$ , is in general different from the distance  $|d\mathbf{r}|$  at time  $t$ . As a measure of this change, we introduce the difference  $\Delta$  in the squares of these distances:

$$\Delta = (d\mathbf{r}^* \cdot d\mathbf{r}^*) - (d\mathbf{r} \cdot d\mathbf{r}). \quad (2.6)$$

Substituting  $d\mathbf{r}^*$  from Eq. (2.5) into this, we get the first-order term in  $dt$ :

$$\Delta = 2 dt (d\mathbf{r} \cdot (d\mathbf{r} \cdot \text{grad } \mathbf{v})). \quad (2.7)$$

For further discussion, we split the tensor  $\text{grad } \mathbf{v}$  into the sum of a symmetric part  $\mathbf{D}$  and an antisymmetric part  $\mathbf{R}$ :

$$\text{grad } \mathbf{v} = \mathbf{D} + \mathbf{R}, \quad (2.8)$$

where, using the notations  $\partial u / \partial x = u_x$ , etc., we have

$$\mathbf{D} = \begin{bmatrix} u_x & \frac{1}{2}(v_x + u_y) & \frac{1}{2}(w_x + u_z) \\ \frac{1}{2}(u_y + v_x) & v_y & \frac{1}{2}(w_y + v_z) \\ \frac{1}{2}(u_z + w_x) & \frac{1}{2}(v_z + w_y) & w_z \end{bmatrix}, \quad (2.9)$$

$$\mathbf{R} = \begin{bmatrix} 0 & \frac{1}{2}(v_x - u_y) & \frac{1}{2}(w_x - u_z) \\ \frac{1}{2}(u_y - v_x) & 0 & \frac{1}{2}(w_y - v_z) \\ \frac{1}{2}(u_z - w_x) & \frac{1}{2}(v_z - w_y) & 0 \end{bmatrix}. \quad (2.10)$$

Since  $\mathbf{R}$  is antisymmetric, for any vector  $\mathbf{a}$ , and in particular for  $\mathbf{a} = d\mathbf{r}$ , we have:

$$(\mathbf{a} \cdot (\mathbf{a} \cdot \mathbf{R})) = 0,$$

so that (2.7) becomes just

$$\Delta = 2 dt (d\mathbf{r} \cdot (d\mathbf{r} \cdot \mathbf{D})). \quad (2.11)$$

From this, it follows that when, in a flow, at any point at any instant  $\mathbf{D} = 0$ , (i.e., when all the components of  $\mathbf{D}$  vanish), then the distance between any two infinitesimally close points moving with the medium there and then will be instantaneously unchanged. The local motion of the medium must then be a rigid translation, rotation, or a combination of the two. More precisely, when  $\mathbf{D} = 0$ , then (2.5) becomes

$$d\mathbf{r}^* = d\mathbf{r} + dt (d\mathbf{r} \cdot \mathbf{R}). \quad (2.12)$$

Using the definition of  $\text{curl } \mathbf{v}$ , this can also be written as

$$d\mathbf{r}^* = d\mathbf{r} + \frac{1}{2} dt (\text{curl } \mathbf{v} \times d\mathbf{r}). \quad (2.13)$$

This is easily seen if Eqs. (2.12) and (2.13) are both written in terms of their components. On the other hand, we know that if the points  $P_1$  and  $P_2$  are regarded as rigidly connected, and this "dumbbell" is rotating with vector angular velocity  $\boldsymbol{\omega}$  in space, then, independent of the translatory motion, the displacement  $d\mathbf{r}$  between these two particles will change in time  $dt$  to  $d\mathbf{r}^*$ , where

$$d\mathbf{r}^* = d\mathbf{r} + dt (\boldsymbol{\omega} \times d\mathbf{r}). \quad (2.14)$$

Comparing (2.13) and (2.14), we obtain an important relationship between the angular velocity  $\boldsymbol{\omega}$  of a particle in a moving medium and the vector  $\text{curl } \mathbf{v}$ :

$$\boldsymbol{\omega} = \frac{1}{2} \text{curl } \mathbf{v}. \quad (2.15)$$

On the other hand, if  $\mathbf{R} = 0$ , then we have the following: Because of the symmetry of  $\mathbf{D}$ , a well-known theorem of analytic geometry permits us to find a special Cartesian coordinate system in which the matrix  $\mathbf{D}$  is diagonal:

$$\mathbf{D} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \quad (2.16)$$

In this coordinate system, Eq. (2.5) becomes

$$\begin{aligned} dx^* &= dx(1 + \alpha dt), \\ dy^* &= dy(1 + \beta dt), \\ dz^* &= dz(1 + \gamma dt). \end{aligned} \quad (2.17)$$

This means that a small element of the flowing medium will be stretched or compressed in these three directions, depending on whether  $\alpha$ ,  $\beta$ , and  $\gamma$  are positive or negative. Thus, the element undergoes a deformation, and  $\mathbf{D}$  is called the deformation tensor.

In general, neither  $\mathbf{D}$  nor  $\mathbf{R}$  vanishes, and an element experiences, in addition to pure translation, a rotation given by  $\mathbf{R}$  and a deformation given by  $\mathbf{D}$  (stretching in three mutually perpendicular directions).

We illustrate this by an example of plane shear flow (Fig. 13).<sup>11</sup> The velocity

<sup>11</sup> A flow is said to be plane if an  $x, y, z$  coordinate system can be chosen such that all the flow variables are independent of the  $z$  direction and the  $w$  component of the velocity field is identically zero.

field has an  $x$  component  $u = cy$  with constant  $c$ , and the other components  $v = w = 0$ .  $\text{curl } \mathbf{v}$  has only one component, which is perpendicular to the  $x, y$  plane and has the value  $c$ , so that the angular velocity of any element in this flow is  $c/2$ . The upper diagrams of Fig. 13 show the motion and deformation of a square element during time  $dt$ ; we easily see that the angle  $d\phi = c dt$ . The lower diagrams show the same element, apart from pure translation, being stretched in direction 1, compressed in direction 2, and rotated through  $d\phi/2$ , i.e., since  $d\phi/2dt = c/2$ , the angular velocity is  $c/2$ .

## 2.2 Derivatives with Respect to Time

### 2.2.1 SUBSTANTIAL DERIVATIVE WITH RESPECT TO TIME

Let a scalar field function  $\Phi(x, y, z, t)$  be given. We imagine an observer moving in space and measuring the quantity  $\Phi$  while moving. The time rate of change of  $\Phi$  as registered by this observer is given by

$$\frac{d\Phi}{dt} = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\Phi(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) - \Phi(x, y, z, t)],$$

i.e.,

$$\frac{d\Phi}{dt} = \dot{x} \frac{\partial \Phi}{\partial x} + \dot{y} \frac{\partial \Phi}{\partial y} + \dot{z} \frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial t},$$

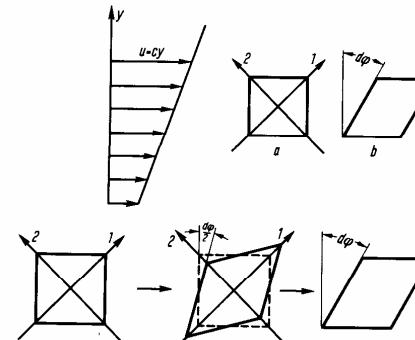


Fig. 13. Deformation of an originally square fluid element in plane shear flow.

where  $\Delta x$  is the distance traveled by the observer during time  $\Delta t$  in the  $x$  direction. Hence  $\dot{x}$  is the  $x$  component of the observer's velocity. The same applies to  $\dot{y}$  and  $\dot{z}$ . Let the space now be filled with a moving medium of velocity field  $\mathbf{v}$ , and let the observer move along with an element of the medium. The speed of the observer will be the same as  $\mathbf{v}$  at every instant and at every location. If we denote the time rate of change in this case by the symbol  $D/Dt$ , then we have

$$\frac{D\Phi}{Dt} = u \frac{\partial \Phi}{\partial x} + v \frac{\partial \Phi}{\partial y} + w \frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial t} + (\mathbf{v} \cdot \operatorname{grad} \Phi), \quad (2.18)$$

with the vector  $\operatorname{grad} \Phi = (\partial \Phi / \partial x, \partial \Phi / \partial y, \partial \Phi / \partial z)$ .

The time derivative defined by Eq. (2.18) is called a substantial derivative. The total rate of change of  $\Phi$  is seen to be the sum of the local rate of change  $\partial \Phi / \partial t$  and the convective rate of change  $(\mathbf{v} \cdot \operatorname{grad} \Phi)$ .

The assumption that  $\Phi$  is a scalar quantity can be relaxed. If  $\Phi$  is one of the three components of a vector field  $\mathbf{a}(x, y, z, t)$ , e.g.,  $\Phi = a_x$ , then Eq. (2.18) holds unchanged:

$$\frac{Da_x}{Dt} = \frac{\partial a_x}{\partial t} + u \frac{\partial a_x}{\partial x} + v \frac{\partial a_x}{\partial y} + w \frac{\partial a_x}{\partial z}. \quad (2.19)$$

Likewise, there are two similar equations in which  $a_x$  is replaced by  $a_y$  and  $a_z$ , respectively. These three equations may be combined into a single vector equation:

$$\frac{D\mathbf{a}}{Dt} = \frac{\partial \mathbf{a}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad} \mathbf{a}), \quad (2.20)$$

with the meaning of the gradient  $\operatorname{grad} \mathbf{a}$  of a vector field  $\mathbf{a}$  explained in Section 2.1. In particular, for  $\mathbf{a} = \mathbf{v}$ , we have:

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad} \mathbf{v}) \quad (2.21)$$

or

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad} \frac{\mathbf{v}^2}{2} - (\mathbf{v} \times \operatorname{curl} \mathbf{v}). \quad (2.22)$$

The identity of Eqs. (2.21) and (2.22) is readily shown if the right-hand sides

of both relations are written out in terms of their components [ $\mathbf{v}^2$  denotes the scalar product  $(\mathbf{v} \cdot \mathbf{v})$ ].  $D\mathbf{v}/Dt$  is the acceleration of an element of the moving medium. In a steady flow,  $\partial \mathbf{v} / \partial t = 0$ , but the acceleration  $D\mathbf{v}/Dt$  is in general nonzero.

### 2.2.2 INTEGRAL TIME RATE OF CHANGE

In the following discussion, we shall derive a three-dimensional generalization of the differentiation formula for integrals depending on a parameter  $\alpha$ , well known from elementary analysis:

$$\frac{d}{d\alpha} \int_{\xi_1(\alpha)}^{\xi_2(\alpha)} f(\xi, \alpha) d\xi = \int_{\xi_1(\alpha)}^{\xi_2(\alpha)} \frac{\partial f}{\partial \alpha} d\xi + \frac{d\xi_2}{d\alpha} f(\xi_2, \alpha) - \frac{d\xi_1}{d\alpha} f(\xi_1, \alpha). \quad (2.23)$$

For further use, we next note the trivial generalization of (2.23) to the case when the integral depends not only on  $\alpha$  but also on additional parameters  $\beta, \gamma, \dots$ . We then have

$$\begin{aligned} \frac{\partial}{\partial \alpha} \int_{\xi_1(\alpha, \beta, \dots)}^{\xi_2(\alpha, \beta, \dots)} f(\xi, \alpha, \beta, \dots) d\xi &= \int_{\xi_1(\alpha, \beta, \dots)}^{\xi_2(\alpha, \beta, \dots)} \frac{\partial f}{\partial \alpha} d\xi \\ &\quad + \frac{\partial \xi_2}{\partial \alpha} f(\xi_2, \alpha, \beta, \dots) - \frac{\partial \xi_1}{\partial \alpha} f(\xi_1, \alpha, \beta, \dots). \end{aligned} \quad (2.23^*)$$

We now consider a volume  $V$  enclosed by a smooth surface  $A$  in a moving medium. Let each point on the surface move with the flow velocity  $\mathbf{v}$ , so that the surface will always consist of the same material points. In addition, let a scalar field  $\Phi(x, y, z, t)$  be given. Integration of  $\Phi$  over this volume  $V$  gives

$$\Psi(t) = \iiint_V \Phi dV = \int_{x_1(t)}^{x_2(t)} \left\{ \int_{y_1(x, t)}^{y_2(x, t)} \left[ \int_{z_1(x, y, t)}^{z_2(x, y, t)} \Phi(x, y, z, t) dz \right] dy \right\} dx. \quad (2.24)$$

The meaning of the limits of integration  $x_1, x_2, y_1, y_2, z_1, z_2$  is evident from Fig. 14. Now let us construct the time derivative of  $\Psi$ , which we shall denote by  $D\Psi/Dt$ ; the symbol  $D/Dt$  implies that the surface moves with the fluid velocity and thereby always encloses the same fluid mass. Application

of formula (2.23) to the integral in  $x$  gives

$$\begin{aligned} \frac{D\Psi}{Dt} = & \int_{x_1(t)}^{x_2(t)} \left\{ \frac{\partial}{\partial t} \left[ \int_{y_1(x, t)}^{y_2(x, t)} \left( \int_{z_1(x, y, t)}^{z_2(x, y, t)} \Phi(x, y, z, t) dz \right) dy \right] \right\} dx \\ & + \frac{dx_2}{dt} \int_{y_1(x_2, t)}^{y_2(x_2, t)} \left( \int_{z_1(x_2, y, t)}^{z_2(x_2, y, t)} \Phi dz \right) dy - \frac{dx_1}{dt} \int_{y_1(x_1, t)}^{y_2(x_1, t)} \left( \int_{z_1(x_1, y, t)}^{z_2(x_1, y, t)} \Phi dz \right) dy. \end{aligned}$$

Here, the latter two terms drop out, since the lower and upper limits of the  $y$  integral (and also the  $z$  integral) coincide (see Fig. 14):  $y_1(x_2, t) = y_2(x_2, t)$

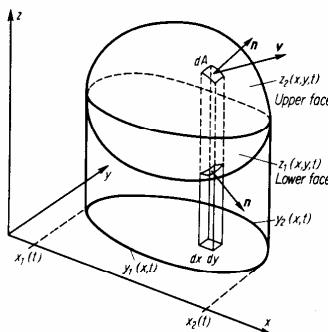


Fig. 14. Diagram for the derivation of Eq. (2.25).

and  $y_1(x_1, t) = y_2(x_1, t)$ . To carry out the partial differentiation with respect to  $t$ , we can now use formula (2.23\*). Applying this formula to the integral in  $y$ , we get next (here,  $x$  plays the role of a parameter),

$$\frac{D\Psi}{Dt} = \int_{x_1(t)}^{x_2(t)} \left\{ \int_{y_1(x, t)}^{y_2(x, t)} \left[ \frac{\partial}{\partial t} \left( \int_{z_1(x, y, t)}^{z_2(x, y, t)} \Phi(x, y, z, t) dz \right) \right] dy \right\} dx.$$

The two terms involving multiplication by  $\partial y_1/\partial t$  and  $\partial y_2/\partial t$  that result from applying (2.23\*) can be omitted. They vanish, since the lower and upper

limits of the integrals over  $z$  appearing in these terms coincide:  $z_1(x, y_1, t) = z_2(x, y_1, t)$  and  $z_1(x, y_2, t) = z_2(x, y_2, t)$  (see Fig. 14). A repeated application of (2.23\*) to the integral over  $z$  finally gives

$$\begin{aligned} \frac{D\Psi}{Dt} = & \int_{x_1(t)}^{x_2(t)} \left\{ \int_{y_1(x, t)}^{y_2(x, t)} \left[ \int_{z_1(x, y, t)}^{z_2(x, y, t)} \frac{\partial \Phi}{\partial t} dz \right] dy \right\} dx \\ & + \int_{x_1(t)}^{x_2(t)} \left\{ \int_{y_1(x, t)}^{y_2(x, t)} \Phi(x, y, z_2, t) \frac{\partial z_2}{\partial t} dy \right\} dx \\ & - \int_{x_1(t)}^{x_2(t)} \left\{ \int_{y_1(x, t)}^{y_2(x, t)} \Phi(x, y, z_1, t) \frac{\partial z_1}{\partial t} dy \right\} dx. \end{aligned}$$

The double integral can be written in a somewhat simpler form: We take an element  $dA$  of the surface, and denote by  $\mathbf{n}$  its outward normal (unit vector in normal direction); the projection of this element in the  $x, y$  plane shall be  $dx dy$ . The velocity of this element in the normal direction is  $\mathbf{v} \cdot \mathbf{n}$ , and the  $z$  component of the velocity is  $\partial z_2/\partial t$  or  $\partial z_1/\partial t$  [depending on whether the normal  $\mathbf{n}$  has a positive or negative  $z$  component, i.e., whether it is on the upper or lower surface of the volume  $V$  (see Fig. 14)]. On the other hand, the velocity of the element in the  $z$  direction can be written as  $(\mathbf{v} \cdot \mathbf{n})/\cos(\mathbf{n}, z)$ , where  $\cos(\mathbf{n}, z)$  is the cosine of the angle between the normal and the positive  $z$  direction. Considering the fact that  $dA \cos(\mathbf{n}, z) = + dx dy$  for an element of the upper surface, and  $dA \cos(\mathbf{n}, z) = - dx dy$  for an element of the lower surface, we see that  $(\partial z_2/\partial t) dx dy = (\mathbf{n} \cdot \mathbf{v}) dA$  for the element of the upper surface and  $-(\partial z_1/\partial t) dx dy = (\mathbf{n} \cdot \mathbf{v}) dA$  for the element of the lower surface. Thus, the final result can be written as:

$$\frac{D\Psi}{Dt} = \iiint_V \frac{\partial \Phi}{\partial t} dV + \iint_A (\mathbf{n} \cdot \mathbf{v}) dA. \quad (2.25)$$

The first integral is the contribution of the local change of the field  $\Phi$  to the time rate of change of  $\Psi$ , while the surface integral is the contribution due to the motion of the surface at the fluid velocity, resulting in the enclosing of new space by the volume  $V$  as time goes on.

Upon transforming the surface integral into a volume integral by means

of Gauss's theorem,<sup>12</sup> Eq. (2.25) becomes

$$\frac{D\Psi}{Dt} = \iiint_V \left[ \frac{\partial \Phi}{\partial t} + \operatorname{div}(\Phi \mathbf{v}) \right] dV \quad (2.26)$$

or

$$\frac{D\Psi}{Dt} = \iiint_V \left[ \frac{D\Phi}{Dt} + \Phi \operatorname{div} \mathbf{v} \right] dV. \quad (2.27)$$

Equation (2.27) is obtained from Eq. (2.26) by the substitution of  $\operatorname{div}(\Phi \mathbf{v}) = (\mathbf{v} \cdot \operatorname{grad} \Phi) + \Phi \operatorname{div} \mathbf{v}$ . It is not necessary to restrict  $\Phi$  to a scalar field. If we replace  $\Phi$  by the three components of a vector field  $\mathbf{a}$  in Eqs. (2.25) and (2.27) and combine the three corresponding equations into a vector equation, then we readily get the following formulas: If

$$\mathbf{b}(t) = \iiint_V \mathbf{a}(\mathbf{r}, t) dV, \quad (2.28)$$

then

$$\frac{D\mathbf{b}}{Dt} = \iiint_V \frac{\partial \mathbf{a}}{\partial t} dV + \iint_A \mathbf{a}(\mathbf{n} \cdot \mathbf{v}) dA \quad (2.29)$$

or

$$\frac{D\mathbf{b}}{Dt} = \iiint_V \left[ \frac{D\mathbf{a}}{Dt} + \mathbf{a} \operatorname{div} \mathbf{v} \right] dV. \quad (2.30)$$

### 2.2.3 CONTINUITY EQUATION

If we identify the quantity  $\Phi$  in the previously derived formulas with the density  $\varrho$  of the flowing medium, then  $\Psi$  becomes the total mass  $M$  of the volume considered. Since by assumption the surface moves with the fluid velocity, then no mass can flow in or out across the surface; at the same

<sup>12</sup> Gauss's theorem states that for a differentiable vector field  $\mathbf{a}$  and a volume  $V$  with smooth surface  $A$ :

$$\iint_A (\mathbf{n} \cdot \mathbf{a}) dF = \iiint_V \operatorname{div} \mathbf{a} dV$$

where

$$\operatorname{div} \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}.$$

time, spontaneous generation or annihilation of mass in the volume shall be excluded. Thus,  $M$  will not change with time. Equation (2.27) then gives

$$\iiint_V \left[ \frac{D\varrho}{Dt} + \varrho \operatorname{div} \mathbf{v} \right] dV = 0. \quad (2.31)$$

Since the volume  $V$  can be chosen completely arbitrary, the integrand must vanish wherever it is continuous:

$$\frac{D\varrho}{Dt} + \varrho \operatorname{div} \mathbf{v} = 0 \quad (2.32)$$

or

$$\partial\varrho/\partial t + \operatorname{div}(\varrho \mathbf{v}) = 0. \quad (2.33)$$

Equation (2.32) [or Eq. (2.33)] is called the continuity equation. Equation (2.31) may be written in the form [corresponding to Eq. (2.25)]

$$\iiint_V \frac{\partial \varrho}{\partial t} dV = - \iint_A \varrho (\mathbf{n} \cdot \mathbf{v}) dA, \quad (2.34)$$

which permits the following interpretation: If we regard  $V$  as a volume fixed in space and unchanging in time, then the left side is the time rate of change of the mass enclosed in this volume, and the right side is the net of the mass inflow per unit time over the mass outflow.

We apply the continuity equation to the following useful transformation: Let  $\phi$  be a field function and let

$$\Psi = \iiint_V \varrho \phi dV.$$

Then, from Eq. (2.27),<sup>13</sup> we have

$$\begin{aligned} \frac{D\Psi}{Dt} &= \iiint_V \left[ \frac{D(\varrho\phi)}{Dt} + \varrho\phi \operatorname{div} \mathbf{v} \right] dV \\ &= \iiint_V \left[ \varrho \frac{D\phi}{Dt} + \phi \left( \frac{D\varrho}{Dt} + \varrho \operatorname{div} \mathbf{v} \right) \right] dV, \end{aligned}$$

<sup>13</sup> One can readily convince oneself that for the substantial derivative with respect to time the usual rules of differentiation regarding the sum and product of two functions all hold, as does the chain rule.

i.e.

$$\frac{D\Psi}{Dt} = \iiint_V \varrho \frac{D\phi}{Dt} dV. \quad (2.35)$$

$\phi$  can be either a scalar field or a vector field.

#### 2.2.4 CHANGE OF CIRCULATION WITH TIME

We consider a closed smooth curve  $\mathcal{C}$  in a flow field, and let  $ds$  be a vector line element of it (Fig. 15). We fix a positive sense of rotation, and define

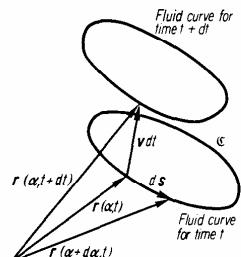


Fig. 15. Translation of a fluid curve in time  $dt$ . Diagram for the derivation of Eq. (2.39).

the circulation around this curve by the line integral

$$K = \oint_{\mathcal{C}} (\mathbf{v} \cdot d\mathbf{s}). \quad (2.36)$$

We now assume that every point of the curve moves with the velocity of the fluid; in other words, the curve always carries the same fluid particles. We shall call such a curve a *fluid curve*. We characterize the points on the fluid curve by a parameter  $\alpha$  such that a fixed  $\alpha$  value corresponds to a fixed material particle at all time  $t$ . A convenient choice of  $\alpha$ , for example, would be the arc length from some arbitrary reference point to the material point on the fluid curve at some initial time  $t_0$  in the positive direction of rotation. In this case,  $\alpha$  varies between the values of 0 and  $l$ , where  $l$  is the total length of  $\mathcal{C}$  at the initial time  $t_0$ . This fluid line is then described for all time by a

vector equation of the form  $\mathbf{r} = \mathbf{r}(\alpha, t)$ . We have  $(\partial \mathbf{r} / \partial \alpha) d\alpha = ds$  (see Fig. 15); moreover  $\partial \mathbf{r} / \partial t = \mathbf{v}$  and  $\partial^2 \mathbf{r} / \partial t^2 = D\mathbf{v}/Dt$ . Differentiation with respect to time for a fixed  $\alpha$  gives the time rate of change of a quantity in question for a material point.

The circulation  $K$  is in general a function of time. Using the above relations, we can write  $K$  as

$$K = \int_0^l \left( \frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial \mathbf{r}}{\partial \alpha} \right) d\alpha. \quad (2.37)$$

We now construct the derivative of  $K$  with respect to time, and denote this derivative by  $DK/Dt$  for obvious reasons; thus:

$$\frac{DK}{Dt} = \int_0^l \left( \frac{\partial^2 \mathbf{r}}{\partial t^2} \frac{\partial \mathbf{r}}{\partial \alpha} \right) d\alpha + \int_0^l \left( \frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial^2 \mathbf{r}}{\partial t \partial \alpha} \right) d\alpha. \quad (2.38)$$

The second integral can be transformed as follows:

$$\int_0^l \left( \frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial^2 \mathbf{r}}{\partial t \partial \alpha} \right) d\alpha = \frac{1}{2} \int_0^l \frac{\partial}{\partial \alpha} \left( \frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial \mathbf{r}}{\partial t} \right) d\alpha = \frac{1}{2} \left( \frac{\partial \mathbf{r}}{\partial t} \right)^2 |_0^l.$$

Since the initial point  $\alpha = 0$  and the endpoint  $\alpha = l$  coincide, this integral vanishes and (2.38) becomes

$$\frac{DK}{Dt} = \oint_{\mathcal{C}} \left( \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{s} \right). \quad (2.39)$$

#### 2.3 Momentum Equation

If we identify the vector  $\mathbf{a}$  in Eq. (2.28) with  $\varrho \mathbf{v}$ , then  $\mathbf{b}$  is the total momentum of the fluid enclosed by the volume in question. By Newton's law in mechanics, the time rate of change of this momentum is equal to the total force acting on the volume. This force can be divided into those forces acting on the surface of this volume and those acting on the individual volume elements. When such *volume forces* exist, e.g., inertial forces, gravitational forces, electromagnetic forces, etc., it is convenient to refer them to unit

mass by introducing a specific force vector  $\mathbf{k}(x, y, z, t)$ . The total volume force is then  $\iiint_V \rho \mathbf{k} dV$ .

The surface forces must be studied in somewhat greater detail. We single out a surface element of area  $\Delta A$  (Fig. 16). On this surface element, the

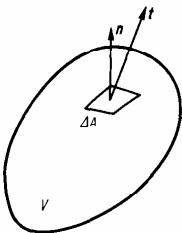


Fig. 16. Stress vector  $\mathbf{t}$  and normal vector  $\mathbf{n}$ .

surrounding fluid outside the volume  $V$  will exert a force  $\Delta \mathbf{K}$ . We define the **stress vector** at each point of the surface to be

$$\mathbf{t} = \lim_{\Delta A \rightarrow 0} \frac{\Delta \mathbf{K}}{\Delta A},$$

where  $\Delta A$  contracts toward the point in question. Using this definition, the total surface force is  $\iint_A \mathbf{t} dA$ . The stress vector  $\mathbf{t}$  does not in general lie along the normal direction given by the unit vector  $\mathbf{n}$ , but has both a normal and a tangential component. In gases, such tangential components (and also a certain contribution to the normal components) enter only when the gas is moving and not when it is at rest, and are caused by viscosity. A relationship between these viscous stresses and the velocity field will be given in Section 4.1.1.

In many cases, one may neglect viscosity, at least in those regions of the flow fields where no large velocity gradients appear. Then the entire surface force will consist of only the thermodynamic pressure, and we have

$$\mathbf{t} = -p\mathbf{n}. \quad (2.40)$$

(The negative sign must be introduced, since  $\mathbf{n}$  has been defined as the outward-pointing normal, whereas the pressure causes a force in the direction opposite to this normal.) It is already clear in this special case that the stress

vector  $\mathbf{t}$  does not form a vector field, i.e., does not depend only on space and time;  $\mathbf{t}$  also depends, in addition to space and time ( $p$  can be space- and time-dependent), on  $\mathbf{n}$ , i.e., on a direction in space. This leads to the conclusion that in general, when viscous stresses appear in addition to the pressure, a complete description of the state of stress requires the introduction of a tensor field  $\mathbf{S}(x, y, z, t)$ , the **stress tensor**. The stress vector  $\mathbf{t}$  at the point  $x, y, z$  and time  $t$  for a normal direction  $\mathbf{n}$  is then given by

$$\mathbf{t} = (\mathbf{n} \cdot \mathbf{S}). \quad (2.41)$$

The proof of the fact that the relation between  $\mathbf{t}$  and  $\mathbf{n}$  is given by a tensor  $\mathbf{S}$  according to Eq. (2.41) is found in basic texts on continuum mechanics. It is also shown in these texts that  $\mathbf{S}$  is a symmetric tensor.

When the viscous stresses drop out completely, then the stress tensor  $\mathbf{S}$  reduces to the spherically symmetric tensor

$$\mathbf{S} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} = -p\mathbf{E} \quad (2.42)$$

( $\mathbf{E}$  = unit tensor), and Eq. (2.41) becomes Eq. (2.40), since  $(\mathbf{n} \cdot \mathbf{E}) = \mathbf{n}$ . In the general case, we split  $\mathbf{S}$ :

$$\mathbf{S} = -p\mathbf{E} + \mathbf{T}. \quad (2.43)$$

Here,  $\mathbf{T}$  is the viscous stress tensor, which, like  $\mathbf{S}$ , is also symmetric, and  $p$  is the thermodynamic state variable, pressure. The relation (2.41) becomes

$$\mathbf{t} = -p\mathbf{n} + (\mathbf{n} \cdot \mathbf{T}), \quad (2.44)$$

$(\mathbf{n} \cdot \mathbf{T})$  being the viscous stress vector. We introduce the following representation in components:<sup>14</sup>

$$\mathbf{T} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}. \quad (2.45)$$

Symmetry of  $\mathbf{T}$  means

$$\tau_{xy} = \tau_{yx}; \quad \tau_{yz} = \tau_{zy}; \quad \tau_{zx} = \tau_{xz}. \quad (2.46)$$

The components of the stress tensor can be readily interpreted: If we

<sup>14</sup> Subscripts on the stress components do not denote partial derivatives.

consider a surface element with a normal in the positive  $x$  direction, then the stress vector is given by Eq. (2.44) as

$$\mathbf{t}^{(x)} = (-p + \sigma_x, \tau_{xy}, \tau_{xz}).$$

Analogous results can be obtained for surface elements with normals in the  $y$  or  $z$  direction. On an infinitesimal cube, the components of the stress tensor hence appear as components of the stress vector on the three mutually perpendicular surfaces directed along the coordinate axes, as shown in Fig. 17. Through the symmetry of the stress tensor, we can always introduce a coordinate system in which the stress tensor has diagonal elements only (see Section 2.1). Then, only normal stresses appear on the surfaces of the

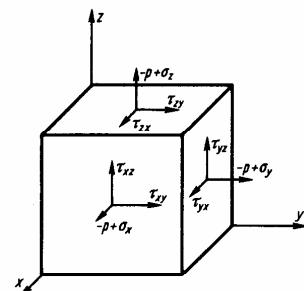


Fig. 17. The elements of the stress tensor as stress components in three planes perpendicular to the coordinate axes.

cube in Fig. 17. The off-diagonal stresses of  $T$  are called the *shear stresses*.

Having made these preparations, we can now write down Newton's second law of motion for an arbitrary volume of a flowing gas:

$$\frac{D}{Dt} \int \int \int \rho \mathbf{v} dV = - \int \int p \mathbf{n} dA + \int \int (\mathbf{n} \cdot \mathbf{T}) dA + \int \int \int \rho \mathbf{k} dV. \quad (2.47)$$

This equation can be written in many other forms which are convenient for various applications. Transformation of the left side on the basis of Eq. (2.29)

leads to

$$\begin{aligned} \int \int \int \frac{\partial(\rho \mathbf{v})}{\partial t} dV &= - \int \int \mathbf{v} (\mathbf{n} \cdot \rho \mathbf{v}) dA \\ &\quad - \int \int p \mathbf{n} dA + \int \int (\mathbf{n} \cdot \mathbf{T}) dA + \int \int \int \rho \mathbf{k} dV. \end{aligned} \quad (2.48)$$

Equation (2.48) can be interpreted [similar to the interpretation of Eq. (2.34)] as follows: Let us think of the volume  $V$  as fixed in space and unchanging in time, a "control volume." The left side of the equation gives the time rate of change of the momentum of the fluid contained in this volume. The first term on the right gives the net of the momentum inflow per unit time over the momentum outflow, since  $\rho \mathbf{v}$  is the mass flow vector,  $-(\mathbf{n} \cdot \rho \mathbf{v})$  is the mass inflow per unit time across a unit surface with outward pointing normal  $\mathbf{n}$ , and  $-\mathbf{v}(\mathbf{n} \cdot \rho \mathbf{v})$  is the momentum of this mass. The other three terms are the forces acting on the volume—namely, the pressure and viscous stress on the surface and the volume force. In the important special case of steady flow without volume forces, (2.48) becomes

$$\int \int [\mathbf{v}(\mathbf{n} \cdot \rho \mathbf{v}) + p \mathbf{n} - (\mathbf{n} \cdot \mathbf{T})] dA = 0. \quad (2.49)$$

If we use Eq. (2.35) to transform the left side of (2.47), and convert the surface integral on the right side by means of Gauss's theorem<sup>15</sup> to a volume integral,

<sup>15</sup> For a differentiable tensor field, Gauss's theorem holds in the form

$$\int \int (\mathbf{n} \cdot \mathbf{T}) dA = \int \int \int \operatorname{div} \mathbf{T} dV.$$

Here,  $\operatorname{div} \mathbf{T}$  is a vector with the components

$$\begin{aligned} (\operatorname{div} \mathbf{T})_x &= \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}, \\ (\operatorname{div} \mathbf{T})_y &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}, \\ (\operatorname{div} \mathbf{T})_z &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}. \end{aligned}$$

This follows immediately from Gauss's theorem in the form given in footnote 12, which we apply to each component of the above vector equation.

we get

$$\iiint_V \left[ \rho \frac{D\mathbf{v}}{Dt} + \text{grad } p - \text{div } \mathbf{T} - \rho \mathbf{k} \right] dV = 0. \quad (2.50)$$

Since  $V$  is arbitrarily chosen, the integrand must vanish everywhere where it is continuous:

$$\rho \frac{D\mathbf{v}}{Dt} = -\text{grad } p + \text{div } \mathbf{T} + \rho \mathbf{k}. \quad (2.51)$$

This differential form of the momentum theorem is of fundamental significance in the theory of all flow processes. For this reason, we shall give it in detail in component form:

$$\left. \begin{aligned} \rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] &= -\frac{\partial p}{\partial x} + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho k_x, \\ \rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] &= -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho k_y, \\ \rho \left[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right] &= -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho k_z. \end{aligned} \right\} \quad (2.52)$$

If we neglect the viscous stresses in Eqs. (2.51) or (2.52), the resulting equations are called the Euler equations. With the expression for the relationship between the viscous stresses and the velocity field (to be explained further in Section 4.1.1), Eqs. (2.51) or (2.52) become the Navier-Stokes equations.

*Supplementary Remarks.* One can also write the first term on the right side of Eq. (2.48) in the form  $-\iint_A (\mathbf{n} \cdot \mathbf{J}) dA$ , where  $\mathbf{J}$  denotes the momentum flow tensor with components of the form  $\rho u^2, \rho uv$ , etc. While the flow of mass (a scalar) is defined by a mass flow vector, a tensor is necessary to define the flow of momentum (a vector). It is of interest to note that the viscous stress tensor  $\mathbf{T}$  for a thermally ideal gas can also be written in the form of a momentum flow tensor. Instead of the products of the macroscopic velocity components  $u, v$ , and  $w$  as in  $\mathbf{J}$ , we have the mean values of the products of the components of the thermal or peculiar velocities of molecules relative to the flow velocity  $\mathbf{v}$ . Finally,  $\mathbf{J} - \mathbf{T}$  can also be interpreted as a single momentum flow tensor, the components of which are now mean values of the products of the components of the absolute molecular velocities (flow velocity + peculiar velocity).

## 2.4 Energy Equation

We now set up an energy balance for an arbitrary volume in a flowing gas. The total energy of such a volume of gas is the sum of its internal energy and its kinetic energy. The time rate of change of this total energy must equal the work done per unit time by all the forces acting on the volume, i.e., the volume forces and surface forces (pressure and viscous stress), plus the influx of energy into the volume per unit time. We characterize this energy flux by an energy flow vector  $\mathbf{q}$ , such that the flow of energy per unit time across a surface element  $dA$  with outward-pointing normal  $\mathbf{n}$  is given by  $-(\mathbf{n} \cdot \mathbf{q}) dA$ . Generally,  $\mathbf{q}$  will simply be the heat flow due to temperature gradients in the gas. However, in the flow of gas mixtures, energy can be transported independently of normal heat conduction, e.g., by diffusion processes, and these also contribute to  $\mathbf{q}$ . The transport of energy by radiation will not be considered here.

The energy balance can now be formulated as follows:

$$\begin{aligned} \frac{D}{Dt} \iiint_V \rho \left( e + \frac{\mathbf{v}^2}{2} \right) dV &= - \iint_A p(\mathbf{v} \cdot \mathbf{n}) dA \\ E &\quad L_1 \\ &+ \iint_A (\mathbf{v} \cdot (\mathbf{n} \cdot \mathbf{T})) dA + \iiint_V \rho (\mathbf{v} \cdot \mathbf{k}) dV - \iint_A (\mathbf{n} \cdot \mathbf{q}) dA. \\ L_2 &\quad L_3 & L_4 \end{aligned} \quad (2.53)$$

where  $E$  is the sum of internal and kinetic energies,  $L_1$  the work done by pressure,  $L_2$  the work done by viscous stresses,  $L_3$  the work done by volume forces, and  $L_4$  the energy influx per unit time. To explain the terms  $L_1, L_2$ , and  $L_3$ , we recall that, in general, a force  $\mathbf{K}$  acting on a point moving with velocity  $\mathbf{v}$  produces the work  $(\mathbf{v} \cdot \mathbf{K})$  per unit time.

We now specify several equivalent formulations of the energy balance, in a similar way as was done with the momentum equation in Section 2.3 which can be derived from Eq. (2.53):

- Transforming the left side of Eq. (2.53) with the use of (2.25) and combining the resulting surface integral with the term  $L_1$  of (2.53), we obtain

$$\iiint_V \frac{\partial}{\partial t} \left[ \rho \left( e + \frac{\mathbf{v}^2}{2} \right) \right] dV = - \iint_A \rho (\mathbf{v} \cdot \mathbf{n}) \left[ e + \frac{p}{\rho} + \frac{\mathbf{v}^2}{2} \right] dA + L_2 + L_3 + L_4. \quad (2.54)$$

The quantity in square brackets [ ] in the first integral on the right side will be denoted by  $h_t$ :

$$e + (p/\rho) + \frac{1}{2}v^2 = h + \frac{1}{2}v^2 = h_t; \quad (2.55)$$

$h_t$  is the sum of the specific enthalpy  $h$  and the specific kinetic energy  $v^2/2$  and is called the specific total enthalpy, (hence the subscript  $t$ ) or the stagnation enthalpy (see Section 3.1). We can now interpret Eq. (2.54) in the same way as Eqs. (2.34) and (2.48), by regarding the volume  $V$  as a "control volume" fixed in space and unchanging in time. The left side is the time rate of change of the total energy in the control volume, the first integral on the right is the net inflow of total enthalpy across the surface per unit time, while the remaining terms are the same as already given.

When we can neglect both the energy flux and the viscous stresses, we call the flow an ideal fluid flow or an inviscid flow. If the flow is inviscid, steady, and without volume forces, then Eq. (2.54) assumes the following simple form:

$$\iint_A \rho h_t (\mathbf{n} \cdot \mathbf{v}) dA = 0. \quad (2.56)$$

We now apply this equation to a streamtube with infinitesimal cross section (Fig. 18); since no flow crosses the sides of the streamtube, we have

$$h_{t1}\rho_1(\mathbf{n}_1 \cdot \mathbf{v}_1) dA_1 = h_{t2}\rho_2(\mathbf{n}_2 \cdot \mathbf{v}_2) dA_2.$$

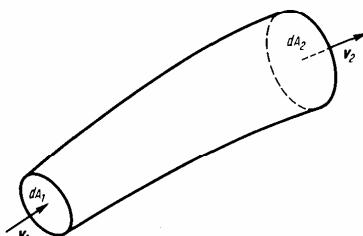


Fig. 18. Streamtube of infinitesimal cross section.

On the other hand, conservation of mass [applying Eq. (2.34) to the streamtube] requires

$$\rho_1(\mathbf{n}_1 \cdot \mathbf{v}_1) dA_1 = \rho_2(\mathbf{n}_2 \cdot \mathbf{v}_2) dA_2.$$

Thus, we have  $h_{t1} = h_{t2}$ . In inviscid steady flows without body forces, therefore,

$$h_t = \text{const} \quad (2.57)$$

on each streamline; naturally, this constant can vary from streamline to streamline. In Eq. (2.63) we shall give a somewhat generalized form of Eq. (2.57).

2. Having become familiar with the notion of specific total enthalpy  $h_t$  in the special case of inviscid steady flow without body forces, we shall now derive from Eq. (2.53) a generally valid relation for  $h_t$  including all the terms on the right side of (2.53). To this end, we add the term  $D/Dt \iiint_V p dV$  to both sides of Eq. (2.53), and, at the same time, we transform the right side of this equation using Eq. (2.25). The result is:

$$\begin{aligned} \frac{D}{Dt} \iiint_V \rho h_t dV &= \iiint_V \rho \frac{Dh_t}{Dt} dV \\ &= \iiint_V \frac{\partial p}{\partial t} dV + \iint_A (\mathbf{n} \cdot \mathbf{v}) dA + L_1 + L_2 + L_3 + L_4. \end{aligned} \quad (2.58)$$

The second integral on the right will cancel  $L_1$ . The terms  $L_2$  and  $L_4$  will be transformed into volume integrals by Gauss's theorem. In the transformation, we note that since  $\mathbf{T}$  is symmetric, we have:  $(\mathbf{v} \cdot (\mathbf{n} \cdot \mathbf{T})) = (\mathbf{n} \cdot (\mathbf{v} \cdot \mathbf{T}))$ . Then Gauss's theorem gives

$$\iint_A (\mathbf{v} \cdot (\mathbf{n} \cdot \mathbf{T})) dA = \iiint_V \operatorname{div}(\mathbf{v} \cdot \mathbf{T}) dV.$$

Finally, we get

$$\iiint_V \left[ \rho \frac{Dh_t}{Dt} - \frac{\partial p}{\partial t} - \operatorname{div}(\mathbf{v} \cdot \mathbf{T}) - \rho(\mathbf{v} \cdot \mathbf{k}) + \operatorname{div} \mathbf{q} \right] dV = 0. \quad (2.59)$$

Since  $V$  can be chosen arbitrarily, the integrand must vanish:

$$\rho \frac{Dh_t}{Dt} = \frac{\partial p}{\partial t} + \operatorname{div}(\mathbf{v} \cdot \mathbf{T}) + \rho(\mathbf{v} \cdot \mathbf{k}) - \operatorname{div} \mathbf{q}. \quad (2.60)$$

In a steady inviscid flow without body forces, this means that

$$Dh_t/Dt = 0. \quad (2.61)$$

Thus, the specific total enthalpy of a particle does not change; then, along each streamline, which coincides with a particle path in this case, the specific total enthalpy must be constant, as we had already established in Eq. (2.57) for this case. Equation (2.61) may be slightly generalized: We still assume inviscid flow, but now admit a body force which is derived from a potential  $\Omega$ :  $\mathbf{k} = -\nabla \Omega$  (a conservative force field). Since, in steady flow ( $\mathbf{v} \cdot \nabla \Omega = 0$ ) is identical to  $D\Omega/Dt$  by definition (2.18), we get from Eq. (2.60) for this case

$$D(h_t + \Omega)/Dt = 0. \quad (2.62)$$

Thus, for each streamline,

$$h_t + \Omega = \text{const.} \quad (2.63)$$

If the constant has the same value on all the streamlines, the flow is called isoenergetic flow.<sup>16</sup>

3. We now undertake a third important transformation of the energy balance (2.53). On the basis of Eq. (2.35), the left side becomes

$$\frac{D}{Dt} \iiint_V \varrho \left( e + \frac{\mathbf{v}^2}{2} \right) dV = \iiint_V \varrho \left[ \frac{De}{Dt} + \left( \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} \right) \right] dV.$$

We can now substitute  $D\mathbf{v}/Dt$  from the momentum equation. Transforming the surface integrals into volume integrals at the same time, we change (2.53) to

$$\begin{aligned} & \iiint_V \left[ \varrho \frac{De}{Dt} - (\mathbf{v} \cdot \nabla p) + (\mathbf{v} \cdot \nabla T) + \varrho(\mathbf{v} \cdot \mathbf{k}) \right] dV \\ &= \iiint_V [-\nabla \cdot (\varrho \mathbf{v}) + \nabla \cdot (\mathbf{v} T) + \varrho(\mathbf{v} \cdot \mathbf{k}) - \nabla \cdot \mathbf{q}] dV. \end{aligned} \quad (2.64)$$

But  $-(\mathbf{v} \cdot \nabla p) = p \nabla \cdot \mathbf{v} - \nabla \cdot (p \mathbf{v})$ , and by the continuity equation (2.32),  $\nabla \cdot \mathbf{v} = -\varrho^{-1} D\varrho/Dt$ . Substituting these into Eq. (2.64) and simplifying, we

<sup>16</sup> We should note that it would be consistent, in analogy to the difference between isentropic flow and homentropic flow as explained later, to use the term isoenergetic flow for those flows for which Eq. (2.63) holds along the streamlines and to call the flows for which the constant in (2.63) is the same on all streamlines homenergetic flows. Since isoenergetic flows (in this sense) which are not at the same time homenergetic will play no role in the following discussion, we keep the term isoenergetic flow in the special sense defined above.

have

$$\iiint_V \varrho \left( \frac{De}{Dt} - \frac{p}{\varrho^2} \frac{D\varrho}{Dt} \right) dV = \iiint_V (\Phi - \nabla \cdot \mathbf{q}) dV, \quad (2.65)$$

where  $\Phi$  is defined as

$$\begin{aligned} \Phi &= \nabla \cdot (\mathbf{v} T) - (\mathbf{v} \cdot \nabla T) \\ &= \sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \sigma_z \frac{\partial w}{\partial z} \\ &\quad + \tau_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \tau_{yz} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) + \tau_{zx} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right). \end{aligned} \quad (2.66)$$

The scalar field quantity  $\Phi$  is called the dissipation function. It can be interpreted as the irreversible dissipation of mechanical energy into heat caused by the viscosity per unit time per unit volume.

Again, the arbitrariness in the choice of  $V$  requires that both integrands in (2.65) be equal. We consider this equation for the integrands, multiply it by a scalar field  $1/T$ , and use the transformation

$$\frac{\nabla \cdot \mathbf{q}}{T} = \nabla \cdot \left( \frac{\mathbf{q}}{T} \right) + \frac{(\mathbf{q} \cdot \nabla T)}{T^2}.$$

Integrating over the volume  $V$  again, we finally obtain

$$\begin{aligned} & \iiint_V \frac{\varrho}{T} \left( \frac{De}{Dt} - \frac{p}{\varrho^2} \frac{D\varrho}{Dt} \right) dV \\ &= - \iint_A \frac{(\mathbf{n} \cdot \mathbf{q})}{T} dA + \iiint_V \left[ \frac{\Phi}{T} - \frac{(\mathbf{q} \cdot \nabla T)}{T^2} \right] dV. \end{aligned} \quad (2.67)$$

The volume integral for  $\nabla \cdot (\mathbf{q}/T)$  has been transformed to a surface integral here.

Up to this point, we have only sparingly made use of the thermodynamic laws; thermodynamic concepts have entered our formulation only through the specific internal energy  $e$  (or the enthalpy  $h$ ). We shall now stipulate in addition that the field function  $T$  used in Eq. (2.67) is to be the absolute temperature, and shall introduce the entropy into the energy equation (2.67) under various simplifying assumptions, thereby gaining some new insight.

First, we make the following basic assumptions, which are satisfied in

many practically important cases and which form the basis for classical gas dynamics:

1. The moving gas particles are at all time in thermodynamic equilibrium. The thermodynamic state variables for the gas are therefore everywhere and at all time connected by relations valid for equilibrium.

2. Each volume being considered, the surfaces of which are moving with the fluid velocity, may be regarded as a closed thermodynamic system; in other words, no exchange of different gases through diffusion occurs at the boundary surface.

Under these two assumptions, the entropy equation (1.6) holds, as does the specific form resulting from (1.22):  $T ds = de - (p/\varrho^2)d\varrho$ . Substituting into Eq. (2.67), we get

$$\iiint_V \varrho \frac{Ds}{Dt} dV = \frac{D}{Dt} \iiint_V \varrho s dV = - \iint_A \left( \frac{\mathbf{n} \cdot \mathbf{q}}{T} \right) dA + \iiint_V \sigma dV. \quad (2.68)$$

The time rate of change of the total entropy  $\iiint_V \varrho s dV$  in the volume appears in two parts: The surface integral on the right side is the inflow of entropy per unit time into the volume, while the volume integral is the entropy generated by the system; according to the second law of thermodynamics,  $\sigma \geq 0$ . Equation (2.68) is none other than a special form of Eq. (1.12), with the above two assumptions incorporated. The entropy source  $\sigma$  is given by

$$\sigma = \frac{\Phi}{T} - \frac{(\mathbf{q} \cdot \text{grad } T)}{T^2}. \quad (2.69)$$

The term  $\Phi/T$  gives the entropy generated by viscosity, while the second term is the entropy generated by heat conduction. Under the plausible assumption that viscosity and heat conduction are independent of each other, each term must be  $\geq 0$ .

When we assume a flow is inviscid, which by definition means  $\mathbf{q} = 0$  and  $T = 0$ , then  $\sigma = 0$ , and Eq. (2.68) gives

$$Ds/Dt = 0. \quad (2.70)$$

Thus, in this case, the entropy of a gas particle is constant, and the flow is *isentropic*. If  $s$  is, in addition, constant throughout the entire flow field, then the flow is called *homentropic*.

We now keep assumption 2, but modify assumption 1 to the extent that

relaxation phenomena are included; with relaxation phenomena, the gas will undergo an equilibrium lag, as explained in Section 1.6. In the case where this lag can be described by one additional variable  $\xi$ , Eq. (1.72) becomes, in terms of specific quantities,

$$T ds = de - (p/\varrho^2)d\varrho + \Gamma d\xi \quad (2.71)$$

[Here we have kept the notation  $\Gamma$  and  $\xi$  for simplicity, even though we are now using specific quantities as against the total quantities in (1.72);  $\Gamma$  now has the meaning  $\Gamma = T(\partial s/\partial \xi)_{e,q}$ .] Substitution of Eq. (2.71) into Eq. (2.67) gives yet another equation of the form (2.68), except that  $\sigma$  now has a different meaning from (2.69), namely

$$\sigma = \frac{\Phi}{T} - \frac{(\mathbf{q} \cdot \text{grad } T)}{T^2} + \frac{\varrho}{T} \Gamma \frac{D\xi}{Dt}. \quad (2.72)$$

As a result of the relaxation process, an additional entropy generation appears, given by the last term of Eq. (2.72). This entropy generation will vanish if the gas attains unconstrained equilibrium at all time, since then  $\Gamma = 0$ . However, it will also vanish if the variable  $\xi$  for the individual gas particles does not change, so that the flow is frozen with respect to the variable  $\xi$ . In case  $n$  variables are necessary to describe the equilibrium lag, the last term in (2.72) will have to be replaced by  $\sum_{i=1}^n (\varrho \Gamma_i / T) D\xi_i / Dt$ .

In the flow of a mixture of different gases which are chemically reacting, we must discern two cases:

1. When diffusion processes may be neglected, we can still consider a volume whose surface moves with fluid velocity as a closed system. As explained in Section 1.9, we can then introduce a degree of reaction  $\xi$  (for many mutually independent reactions, many  $\xi_i$  must be introduced). Then, Eq. (2.67) still holds (see supplementary remarks at the end of Section 1.9), and the entropy generation due to chemical reaction is again given by the last term of formula (2.72). (Of course, we should remark that in those regimes where diffusion processes can be neglected, the viscosity and heat conductivity can in general also be neglected, so that the other two terms in (2.72) will be absent.) In case the degree of reaction is everywhere the same as the equilibrium value (i.e., given by the law of mass action), or  $\Gamma = 0$  everywhere, then the chemical reaction does not contribute to entropy generation. Likewise, it will not contribute if  $\xi$  remains constant for each gas particle, so that the reaction is not occurring at all. This is the case of a

reacting mixture in which, for example, the reaction speed is negligibly small in comparison with the flow velocity, which governs the rate of change of thermodynamic state of a gas particle.

2. If we cannot neglect diffusion processes in a flow field, then we must take formula (1.93) into account, which is valid for open systems. In addition, the entropy flux over the system boundary must be supplemented by a term accounting for the material transport over the boundary. Then, another term, the entropy source due to diffusion, must be added to the entropy source  $\Gamma$ .

The details cannot be given in this introduction (see special references<sup>17)</sup>. We repeat, however, that the formulas of this section through Eq. (2.67) hold irrespective of the thermodynamic behavior of a flowing gas.

**Supplementary Remarks.** One can readily show, by applying Eqs. (2.53) and (2.65), the validity of the following formula:

$$\frac{D}{Dt} \iiint_V \frac{\rho v^2}{2} dV = - \iint_A p(\mathbf{n} \cdot \mathbf{v}) dA + \iint_A (\mathbf{v} \cdot (\mathbf{n} \cdot \mathbf{T})) dA + \iiint_V \left[ \rho(\mathbf{v} \cdot \mathbf{k}) + \frac{p D\rho}{\rho Dt} + \Phi \right] dV.$$

Applying this formula to a streamtube with infinitesimal cross section (Fig. 18), we see that for an inviscid steady flow without body forces in a medium of constant density (i.e.,  $D\rho/Dt = 0$ ), Bernoulli's equation holds on each streamline:

$$p + \frac{1}{2}\rho v^2 = \text{const.}$$

From this it follows that for an inviscid flow of a constant density fluid,

$$De/Dt = 0.$$

## 2.5 Vorticity Theorems

In the following we shall, for the sake of simplicity, use  $\mathbf{w}$  to denote the vorticity vector  $\text{curl } \mathbf{v}$ . The vector  $\mathbf{w}$  defines a direction at each point in

<sup>17</sup> For example: S.R. de Groot, "Thermodynamik irreversibler Prozesse." Mannheim, Germany, 1960.

space (provided that  $\mathbf{w} \neq 0$ ). The integral curves of this direction field are called *vortex lines*; they correspond to streamlines for a velocity field. A *vortex tube* is formed by all the vortex lines passing through a closed curve  $\mathcal{C}$ . The circulation

$$K = \oint_{\mathcal{C}} (\mathbf{ds} \cdot \mathbf{v}) \quad (2.73)$$

is the same along all the curves  $\mathcal{C}$  generating a vortex tube, i.e., it is a constant of the vortex tube. This is easily shown by integration of the expression  $\text{div } \mathbf{w} = 0$  over an arbitrary piece of the vortex tube and transforming surface integrals into line integrals by Stokes' theorem.

### 2.5.1 KELVIN'S THEOREM

We substitute the expression (2.51) for  $Dv/Dt$  into the formula (2.39), which governs the time rate of change of the circulation around a fluid line. We shall neglect the viscous stresses ( $T = 0$ ), and assume that the body force possesses a potential:  $\mathbf{k} = -\text{grad } \Omega$ . Then,

$$\frac{DK}{Dt} = - \oint_{\mathcal{C}} \frac{(ds \cdot \text{grad } p)}{\rho} - \oint_{\mathcal{C}} (ds \cdot \text{grad } \Omega) = - \oint_{\mathcal{C}} \frac{dp}{\rho} - \oint_{\mathcal{C}} d\Omega,$$

where  $dp$  and  $d\Omega$  are the changes in  $p$  and  $\Omega$  over  $ds$  along the fluid line  $\mathcal{C}$ . Since the second integral vanishes if we assume a single-valued potential,<sup>18</sup> we obtain Kelvin's theorem

$$\frac{DK}{Dt} = - \oint_{\mathcal{C}} \frac{dp}{\rho}. \quad (2.74)$$

If along the contour  $\mathcal{C}$  the density is a single-valued function of the pressure alone,  $\rho = \rho(p)$ , then the integral in (2.74) vanishes, and we get  $DK/Dt = 0$ . A flow field in which the density is a single-valued function of pressure is called a *barotropic* flow. Thus, in a barotropic inviscid flow field, the circulation around a fluid line remains constant for all time. Moreover, it follows from this that a gas particle which is rotating with a certain speed

<sup>18</sup> In this way, we have excluded cyclic potentials, which may appear in multiply-connected regions.

(vorticity  $w \neq 0$ ) at any instant will continue to rotate at all time. Conversely, a gas particle which is rotation-free at any instant will always remain rotation-free under our assumption of a barotropic and inviscid flow field.

Equation (2.74) can be transformed under various assumptions. If we assume that the gas attains immediate thermodynamic equilibrium everywhere, then Eq. (1.23\*) [ $dp/\rho = dh - T ds$ ] holds, and

$$\frac{DK}{Dt} = \oint_C T \, ds. \quad (2.75)$$

In a homentropic flow, the specific entropy is uniform by definition, so that  $DK/Dt = 0$ .

### 2.5.2 CROCCO'S THEOREM

Under the assumption that the body force possesses a potential  $\Omega$ , and taking Eq. (2.22) into account, we can write the momentum equation (2.51) as

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \mathbf{w} = -(1/\rho) \operatorname{grad} p - \operatorname{grad}(\frac{1}{2} \mathbf{v}^2 + \Omega) + (1/\rho) \operatorname{div} \mathbf{T}. \quad (2.76)$$

We utilize the relation (1.23\*), valid for equilibrium flow, in the form  $dp/\rho = dh - T ds$ , and obtain from (2.76), for steady inviscid flow, Crocco's theorem:

$$-\mathbf{v} \times \mathbf{w} = -\operatorname{grad}(h_t + \Omega) + T \operatorname{grad} s. \quad (2.77)$$

Since in a steady, inviscid flow  $h_t + \Omega$  is constant along streamlines and since  $\mathbf{v} \times \mathbf{w}$  has no component in the direction  $\mathbf{v}$  and hence along the streamline, it follows that  $\operatorname{grad} s$  cannot have a component along the streamline. Thus,  $s$  is constant along the streamline, which has been established for a special case in Eq. (2.70).

In Section 2.4, we called a flow isoenergetic when the quantity  $h_t + \Omega$  has the same value throughout the entire flow field. For such flows, Eq. (2.77) reduces to

$$-\mathbf{v} \times \mathbf{w} = T \operatorname{grad} s. \quad (2.78)$$

It follows from this that every irrotational, isoenergetic flow is a homentropic flow ( $\operatorname{grad} s = 0$ ). Conversely, every nonhomentropic, isoenergetic flow must be rotational. In a plane flow,  $\mathbf{w}$  is perpendicular to  $\mathbf{v}$ , so the product  $\mathbf{v} \times \mathbf{w}$  for nonzero  $\mathbf{v}$  can only vanish when  $\mathbf{w} = 0$ . Thus, for plane flow, we have

the theorem: Every isoenergetic, homentropic flow is irrotational. We can summarize all these consequences of Crocco's theorem for steady, inviscid, isoenergetic flows as follows:

$$\begin{aligned} \text{Irrotational} &\rightarrow \text{homentropic} \\ \text{Nonhomentropic} &\rightarrow \text{rotational} \\ \text{Homentropic, plane} &\rightarrow \text{irrotational.} \end{aligned}$$

If we now include a relaxation process, so that  $dp/\rho = dh - T ds + \Gamma d\xi$ , then instead of (2.77), we have the following for steady inviscid flows:

$$-\mathbf{v} \times \mathbf{w} = -\operatorname{grad}(h_t + \Omega) + T \operatorname{grad} s - \Gamma \operatorname{grad} \xi. \quad (2.79)$$

If  $\xi$  changes along a streamline, so that  $\operatorname{grad} \xi$  has a component along it, then  $\operatorname{grad} s$  also has a component along the streamline. The change of  $s$  along the streamline is then given by  $ds = (\Gamma/T)d\xi$ ; this is in agreement with the term given in Eq. (2.72), in which entropy generation by relaxation was considered.

# 3 INVISCID FLOWS

## 3.1 Relation between Velocity and State Variables in Inviscid Steady Flow

From the results of Section 2.4 we can draw certain conclusions which have great significance in the theory of steady gas flows. We assume that the flow field is steady and inviscid, and that the gas is in unconstrained thermodynamic equilibrium everywhere, so that its thermodynamic state is defined by two variables. We shall also neglect volume forces. For simplicity, the magnitude of the velocity vector  $|\mathbf{v}|$  will be denoted by  $U$ .

First of all, we make an assumption on the thermodynamic behavior of gases, namely<sup>10</sup>

$$A^2 = \left( \frac{\partial^2 p}{\partial \varrho^2} \right)_s = \left( \frac{\partial a^2}{\partial \varrho} \right)_s > 0. \quad (3.1)$$

While we can conclude from general thermodynamic considerations that  $(\partial p / \partial \varrho)_s - a^2 > 0$  must always hold (see Section 1.11), the condition (3.1) does not follow from thermodynamic laws. However, for all gases of practical interest, (3.1) is satisfied.

From Eqs. (2.57) or (2.61) and from Eq. (2.70), we see that along each streamline

$$\frac{1}{2} U^2 + h = h_t = \text{const}, \quad (3.2)$$

$$s = s_t = \text{const}, \quad (3.3)$$

where  $h_t$  and  $s_t$  can vary from streamline to streamline. If  $h_t$  is constant in the entire flow field, the flow is called isoenergetic; if  $s_t$  has the same value

<sup>10</sup> We can replace the relation (3.1) by the more general condition  $(\partial^2 p / \partial (1/\varrho)^2)_s > 0$  without changing the essence of the conclusions. Since we want to present basic concepts in a single way rather than in complete generality, we use the somewhat simpler condition (3.1).

everywhere, the flow is called homentropic. When  $h = h_t$  and  $s = s_t$ , we define this special thermodynamic state of the gas as the isentropic stagnation state or reservoir state; the values of the thermodynamic variables in this state are also called "total values" (hence the subscript  $t$  to denote this state). The gas attains this state at the points of a streamline where  $U = 0$ .

From Eqs. (3.2) and (3.3) we obtain the relations for the change of the thermodynamic state variables on a streamline: As the speed  $U$  increases, the enthalpy  $h$  decreases. Since  $dp = \varrho dh$  for  $s = \text{const}$ , the pressure  $p$  also decreases; this also follows naturally from the momentum theorem that an acceleration must be accompanied by a pressure drop. Because  $d\varrho = a^{-2} dp$ , the density  $\varrho$  also drops. Then, by Eq. (3.1), the speed of sound  $a$  must also decrease with the density. We now define a dimensionless velocity parameter, the Mach number  $M$ , to be the ratio of the flow velocity to the sound speed:

$$M = U/a. \quad (3.4)$$

Since  $a$  decreases as  $U$  increases,  $M$  increases with  $U$ . For  $U < a$ ,  $M$  is  $< 1$  (subsonic flow), while for  $U > a$ ,  $M$  is  $> 1$  (supersonic flow). Finally, we can use the additional assumption that the thermal expansivity [see Eq. (1.33)] be positive to show that the temperature  $T$  must decrease with an increase in  $U$ .

The magnitude  $|\mathbf{v}|$  of the mass-flow vector will be denoted by  $\Theta$ , which we just call mass-flow. For a streamtube of infinitesimal cross section  $dA$ ,  $\Theta dA$  is constant along the entire length, by continuity; the reciprocal  $\Theta^{-1}$  is thus everywhere proportional to the cross section of the tube, the greatest mass flow being at the narrowest cross section. From  $\Theta = \varrho U$ , we have

$$d\Theta = \varrho dU + U d\varrho. \quad (3.5)$$

We now consider the change  $d\Theta$  along a streamline. From (3.2), and  $dh = \varrho^{-1} dp = a^2 \varrho^{-1} d\varrho$  (since  $s = \text{const}$ ), we obtain:  $U dU + dh = U dU + a^2 \varrho^{-1} d\varrho = 0$ , or  $d\varrho = -\varrho U a^{-2} dU$ . Substituting this into (3.5), we get

$$d\Theta/dU = \varrho(1 - M^2). \quad (3.6)$$

This relation holds along every streamline, and in the special case of isoenergetic, homentropic flow, it holds in the entire flow field. By Eq. (3.6),  $\Theta$  increases monotonically with increasing  $U$  when  $M < 1$ , but decreases monotonically with increasing  $U$  when  $M > 1$ . The mass flow  $\Theta$  as a function of the velocity has the shape shown in Fig. 19: For  $U = 0$ , it is obvious that

$\Theta = 0$ . The velocity is bounded above; the gas particles accelerate to the maximum velocity  $U_{\max}$  when the pressure drops to  $p = 0$ . Since in an isentropic expansion of a gas,  $\varrho$  tends to zero with  $p$ , then for  $U = U_{\max}$ ,  $\Theta$  is again 0. In practice,  $p = 0$  is unattainable, since this must be accompanied

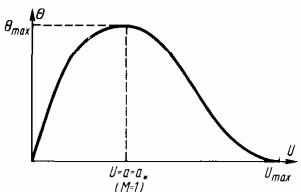


Fig. 19. Dependence of the mass flow  $\Theta$  on the speed  $U$ .

by the temperature tending to zero, and, in a real gas, condensation would have occurred.

The behavior of the mass flow as shown in Fig. 19 explains the flow out of a slender convergent nozzle (Fig. 20); as the pressure difference  $p_t - p$

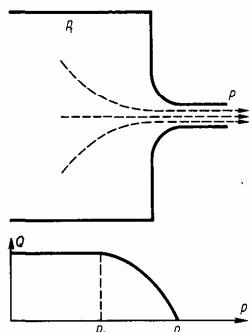


Fig. 20. Flow out of a reservoir through a convergent nozzle. Dependence of the mass flow per unit time  $Q$  on the external pressure  $p$  (schematic);  $p_t$  is the reservoir pressure.

increases, the mass flow per unit time  $Q$  first increases, but when the pressure difference exceeds a certain critical value  $p_t - p_*$ , the mass flow remains

constant. At the critical pressure difference, the exit velocity is exactly the sonic velocity, and the mass flow attains its maximum. With further decrease in the exit pressure the nozzle exit velocity remains at the sonic value, since the exit cross section is the narrowest cross section along the streamtube. Outside the nozzle, there is at first an expansion of the jet, with a corresponding widening in the cross section. This initial widening will stop after a certain distance. The jet then contracts over a certain distance, then expands again, etc., (see Fig. 86).

To summarize, as the velocity  $U$  increases along a streamline: the Mach number  $M$  increases, the pressure  $p$ , density  $\varrho$ , enthalpy  $h$ , temperature  $T$ , and sound speed  $a$  decrease, and the mass flow  $\Theta$  increases when  $M < 1$  and decreases when  $M > 1$ .

Before we study these facts more precisely for a calorically ideal gas, we shall draw another important conclusion from Eq. (3.2) on the dependence of the pressure on the speed along a streamline. Since from Eq. (3.3),  $dh = \varrho^{-1} dp$ , Eq. (3.2) can be written as

$$\frac{U^2}{2} + \int_{p_t}^p \frac{dp}{\varrho} = 0 \quad (3.7)$$

(This equation can also be obtained, without recourse to Eq. (3.3), directly from the momentum equation (2.51) for steady inviscid flows without volume forces.) For sufficiently small velocities  $U$ , the thermodynamic state of the gas is not much different from the reservoir state, and one may use the following approximation:

$$\varrho = \varrho_t + \left( \frac{\partial \varrho}{\partial p} \right)_s (p - p_t) = \varrho_t \left( 1 + \frac{p - p_t}{\varrho_t a_t^2} \right),$$

where  $a_t$  is the velocity of sound for the reservoir state. Then, restricting ourselves to the linear term in the integrand of (3.7), we get

$$\frac{U^2}{2} + \int_{p_t}^p \frac{dp}{\varrho_t} \left( 1 - \frac{p - p_t}{\varrho_t a_t^2} \right) = \frac{U^2}{2} + \frac{p - p_t}{\varrho_t} - \frac{(p - p_t)^2}{2\varrho_t a_t^2} = 0.$$

From this equation, we can express  $(p - p_t)/\varrho_t$  as  $-U^2/2$  in linear approximation, and then substitute this for the square term in  $p - p_t$ , thereby obtaining

$$p = p_t - \frac{1}{2} \varrho_t U^2 [1 - (U^2/4a_t^2) + \dots], \quad (3.8)$$

where the ... indicates higher-order terms in  $U/a_t$ . For very small flow velocity, Eq. (3.8) becomes the Bernoulli equation for incompressible fluids:  $p = p_t - \frac{1}{2}\rho_t U^2$ . The error in this equation for compressible fluids will be small, by Eq. (3.8), if  $U^2/(4a_t^2)$  is small compared to 1. If  $U < 0.2 a_t$ ,  $U^2/(4a_t^2) < 0.01$ . In air at room temperature,  $a_t = 340$  m/sec. Thus, for flow speeds below about 70 m/sec, we may regard the flow of air at room temperature as incompressible with negligible error.

We now consider the special case of a calorically ideal gas. From Eq. (1.58),  $h = c_p T + \text{const}$  with constant specific heat  $c_p$ . Equation (3.2) then becomes

$$\frac{1}{2}U^2 + c_p T = c_p T_t. \quad (3.9)$$

One again sees that with increase in velocity  $U$  the temperature  $T$  decreases. By Eq. (1.65), the pressure  $p = 0$  corresponds to the temperature  $T = 0$  in an isentropic change of state. Thus, for  $T = 0$ , the maximum velocity is attained:

$$U_{\max} = (2c_p T_t)^{\frac{1}{2}}. \quad (3.10)$$

Since from Eq. (1.135) the speed of sound is given by

$$a^2 = (\gamma - 1)c_p T \quad (3.11)$$

with constant adiabatic coefficient  $\gamma$ , we can write Eq. (3.9) in the alternate form

$$\frac{U^2}{2} + \frac{a^2}{\gamma - 1} = \frac{a_t^2}{\gamma - 1}. \quad (3.12)$$

Dividing through by  $a^2$  and introducing the Mach number, we obtain

$$\frac{a^2}{a_t^2} = \frac{T}{T_t} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1}. \quad (3.13)$$

However, for a calorically ideal gas at constant entropy, Eq. (1.65) holds, i.e.,  $p/p_t = (T/T_t)^{\gamma/\gamma-1}$ , which, when combined with Eq. (3.13), gives

$$\frac{p}{p_t} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\gamma/(\gamma-1)}. \quad (3.14)$$

Since by Eq. (1.65)  $p \sim \rho^\gamma$ , we finally get from (3.14)

$$\frac{\rho}{\rho_t} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1/(\gamma-1)}. \quad (3.15)$$

Considering Eq. (3.10), we can also write Eq. (3.9) as:

$$U^2/U_{\max}^2 = 1 - (T/T_t).$$

This, together with Eq. (3.13), yields

$$\frac{U}{U_{\max}} = M \left( \frac{2}{\gamma - 1} + M^2 \right)^{-1/2}. \quad (3.16)$$

As  $T \rightarrow 0$ , by Eq. (3.13),  $a \rightarrow 0$ ; and, by Eq. (3.12), we have the expression:

$$U_{\max} = a_t [2/(\gamma - 1)]^{\frac{1}{2}}. \quad (3.17)$$

We now define the critical sound velocity, or simply, critical velocity,  $a_*$  as the sound velocity at the location where it just equals the flow velocity, or where  $M = 1$ . From Eq. (3.13), it follows that

$$a_* = a_t [2/(\gamma + 1)]^{\frac{1}{2}}. \quad (3.18)$$

We shall also call the values of the other state variables for  $M = 1$  the "critical" values, and denote them by a subscript (\*). The critical value  $\Theta_*$  of the mass flow is in addition the maximum value of the mass flow, as explained above. By Eqs. (3.15) and (3.18), we have

$$\Theta_* = \varrho_* a_* = \varrho_t a_t \left( \frac{2}{\gamma + 1} \right)^{(y+1)/2(y-1)}. \quad (3.19)$$

This equation, together with Eqs. (3.15) and (3.16), finally gives us

$$\frac{\Theta}{\Theta_*} = \frac{\varrho U}{\varrho_* a_*} = M \left( \frac{2 + (\gamma - 1) M^2}{\gamma + 1} \right)^{-(y+1)/2(y-1)}. \quad (3.20)$$

The formulas (3.13)–(3.16) and (3.20) give the dependence of the temperature, speed of sound, pressure, density, velocity, and mass flow on the Mach number  $M$  and the stagnation values  $T_t$ ,  $a_t$ ,  $p_t$  and  $\varrho_t$ . These relations are shown in Fig. 21 for  $\gamma = 1.2$  and  $\gamma = 1.4$ . It is a fortunate circumstance that the ratios of the variables to the stagnation values (or to the critical values, as with  $\Theta$ , or to the maximum values, as with  $U$ ) depend only on the Mach number and not on the stagnation values; this is a result of the assumption of a calorically ideal gas. If we assume that the gas is thermally ideal but not calorically ideal, then  $p/p_t$ ,  $\varrho/\varrho_t$ , etc., will depend not only on the Mach number  $M$  but also on the stagnation temperature  $T_t$  (see supplementary remark 3, p. 85). In the general case, in addition to the stagnation temper-

ature  $T_t$ , the stagnation density  $\rho_t$  also appears in these relations (or two other appropriate state variables).

The curves corresponding to those in Fig. 21 for a general gas can readily be constructed if we know the thermodynamic properties of the gas. Particularly useful is a Mollier diagram of the type shown in Fig. 6. For a given stagnation state and any velocity  $U$ , we can immediately read off the temperature and pressure. Since the difference in ordinates corresponds to  $\frac{1}{2}U^2$  [by Eq. (3.2)], it is advantageous to use a straight paper edge with a  $U$ -scale marked on it to read off values from such a diagram. Furthermore, the density  $\rho$  is given by formula (1.31\*) as  $\rho^{-1} = (\partial h / \partial p)_s$ .

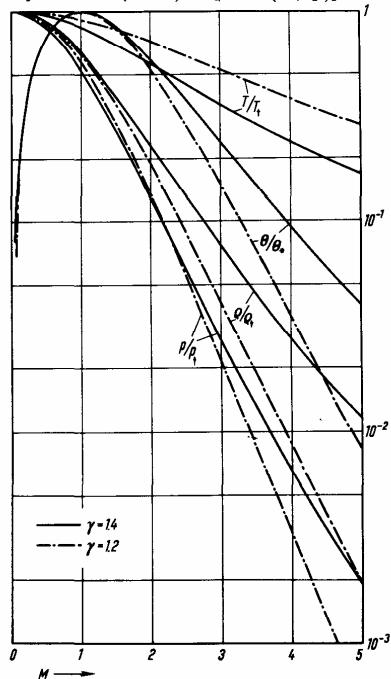


Fig. 21. Dependence of pressure, density, and temperature of a calorically ideal gas on the Mach number in steady flow.

To clarify the principles described here, we shall now discuss the flow of a gas from a reservoir through a Laval nozzle, i.e., a convergent-divergent nozzle (Fig. 22). If the nozzle is relatively slender, then the flow variables

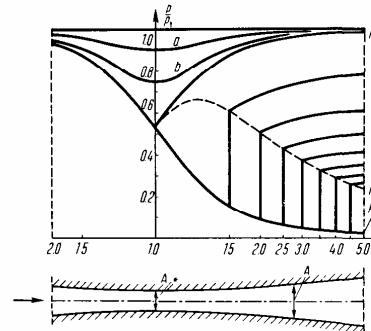


Fig. 22. Pressure distribution in a Laval nozzle for different pressures at the exit section (calorically ideal gas with  $\gamma = 1.4$ ).

over a cross section perpendicular to the nozzle axis are approximately constant, and we can treat the nozzle as if it were a single streamtube over whose cross section the flow quantities do not vary significantly. The nature of the flow established in the nozzle depends on the external pressure  $p_e$  at the exit section of the nozzle. For a specific exit pressure  $p_e = p_3$  (Fig. 22), the gas will accelerate smoothly from the stagnation state in the reservoir toward sonic velocity at the throat (i.e., the narrowest section); beyond the throat, the velocity continues to increase and the pressure continues to decrease smoothly. The velocity and pressure distributions are easily calculated for a calorically ideal gas from the formulas given above. At every point along the nozzle, the mass-flow ratio is given by the area ratio  $A/A_*$ , thus:  $\Theta/\Theta_* = A_*/A$  (continuity equation). Equation (3.20) then determines the Mach number  $M$ , and Eq. (3.14) gives  $p/p_t$  (and accordingly all the other state variables). But if  $p_e \neq p_3$ , then one of the following flows would take place:

1.  $p_e = p_t$ : The gas remains everywhere at rest.
2.  $p_t > p_e > p_3$ : The exit pressure is only slightly below the reservoir

pressure  $p_1$ , and the gas accelerates to a speed below the sound speed at the throat. The entire flow is subsonic (Fig. 22, curves *a*, *b*).

3.  $p_e = p_1$ : The speed of sound is only attained at the throat. Beyond the throat, the pressure rises again, and we again have subsonic flow. The flow is symmetrical with respect to the throat: At cross sections of the same area before or after the throat, we have the same flow velocities and the same thermodynamic states.

4.  $p_1 > p_e > p_2$ : Beyond the throat, the flow is supersonic, but at a definite location between the throat and the exit section, a normal shock wave appears (see Section 3.4), across which the flow velocity suddenly becomes subsonic again. In the shock wave, the change of state is not isentropic; therefore, the dependence of the velocity and the state variables on the Mach number behind the shock is different from that in front, since the stagnation states of the gas are different on the two sides of the shock.

5.  $p_e = p_2$ : The shock wave has moved to the exit section.

6.  $p_2 > p_e > p_3$ : Oblique shock waves appear beyond the exit section (see Fig. 87).

7.  $p_e = p_3$ : The flow is everywhere continuous (see the introductory paragraph).

8.  $p_3 > p_e$ : The flow inside the nozzle is the same as in cases 6 and 7; outside the nozzle, there is an over-expansion, as already mentioned in connection with Fig. 20 (see Fig. 86).

A high-pressure reservoir with a Laval nozzle obviously offers a simple means of producing a supersonic gas jet. Such a jet can be used in aerodynamic research, such as with flight models. The entire assembly of a reservoir-Laval nozzle is thus a simple wind tunnel (intermittent-operation blowdown tunnel). If we want to attain high Mach numbers with such a wind tunnel, we shall invariably encounter one practical difficulty: Because of the drop in temperature with the increase in Mach number, the gas in the tunnel will condense when a certain limiting Mach number is exceeded. For operation with air at room temperature and a reservoir pressure of the order of a few atmospheres, the practically attainable Mach numbers are below  $M = 4-5$ . Higher Mach numbers require the heating of the gas in the reservoir, or the use of gases with very low condensation temperatures (e.g., helium).

*Supplementary Remarks.* 1. Let us consider a circularly symmetric, plane,

steady flow. It is then convenient to introduce polar coordinates  $r, \phi$  (Fig. 23a). Because of circular symmetry, the velocity components  $u_r, u_\phi$  depend only on  $r$ , as do all the thermodynamic variables. We thus have the same stagnation state and the same entropy on all the streamlines, so that

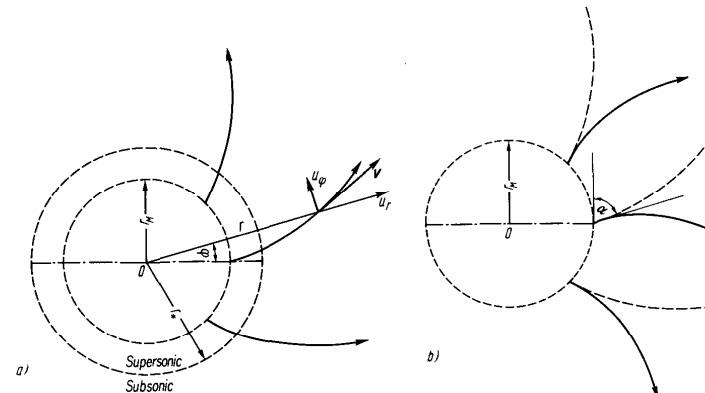


Fig. 23. Streamlines in a circularly symmetric plane flow.

the flow is isoenergetic and homentropic. (Exceptions are those flows with circular streamlines  $r = \text{const}$ . For these flows, we must further assume that they be isoenergetic and homentropic.) By Eq. (2.78), an isoenergetic, homentropic, plane flow must be irrotational. This implies that in this case

$$u_\phi = c_1/r, \quad (3.21)$$

with constant  $c_1$  (since then the circulation for each circle of radius  $r$  and center at the origin is constant:  $K = 2\pi r u_\phi = 2\pi c_1$ , and by Stokes theorem the flow is irrotational in every annular region  $0 < r_1 \leq r \leq r_2$ ). On the other hand, the continuity equation requires that

$$u_r = c_2/\varrho r \quad (3.22)$$

(since then the same mass flows across each circle of radius  $r$  and center at the origin:  $Q = 2\pi r \varrho u_r = 2\pi c_2$ ).

Using the previously derived relation between the density  $\varrho$  and the speed

$U$ , we can easily determine all the flow variables as functions of  $r$ : First of all, we have

$$U^2 = u_r^2 + u_\phi^2 = (c_2^2/\rho^2 r^2) + (c_1^2/r^2) \quad (3.23)$$

or

$$r^2 = (c_2^2/\Theta^2) + (c_1^2/U^2),$$

with  $\Theta = \rho U$ . Using the relation  $\Theta(U)$  given in Fig. 19, we sketch the function  $r^2(U^2)$  in Fig. 24. This naturally permits us to know the inverse

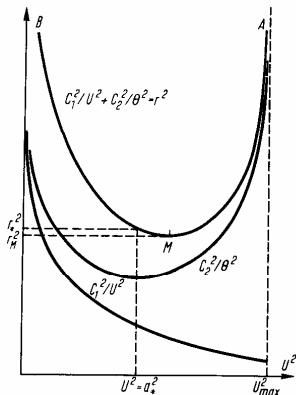


Fig. 24. Circularly symmetric plane flow; explanation of Eq. (3.23).

function  $U$  as a function of  $r$ . Since the velocity  $U$  must be a single-valued function of  $r$ , it is obvious that two completely different flows are possible, corresponding to the two branches  $MA$  and  $MB$  of the curve  $r^2(U^2)$  in Fig. 24. In the case  $MA$ , we have a purely supersonic flow (Fig. 23b), whereas in the case  $MB$ , we have a supersonic flow for  $r_M \leq r < r_*$  and a subsonic flow for  $r > r_*$  (Fig. 23a). In both cases, the flow is possible only for  $r \geq r_M$ . The limiting circle  $r = r_M$  is determined from the condition  $d(r^2)/dU^2 = 0$ . Applying this to Eq. (3.23) and taking Eq. (3.6) into account, we get the relation satisfied on the limiting circle:

$$\sin \alpha = M^{-1}. \quad (3.24)$$

Thus, on the limiting circle,  $\alpha$  is identical to the Mach angle  $\mu$  (see Section 3.5); accordingly, the limiting circle coincides with a Mach line.

2. Using Eq. (1.65), we see that for a calorically ideal gas

$$A^2 = \gamma(\gamma - 1) p \rho^{-2}.$$

3. For a thermally ideal gas described by Eqs. (1.66)–(1.70), the ratios  $p/p_t$ ,  $\rho/\rho_t$ , etc., depend on the parameter  $\theta/T_t$  in addition to the Mach number  $M$ . For  $\theta/T_t \rightarrow \infty$ , we get the previously given formula with  $\gamma = 7/5$ , while for  $\theta/T_t \rightarrow 0$ , we have the same with  $\gamma = 9/7$ . From Eq. (3.2) and using Eq. (1.67), we first have

$$U^2 = 2RT_t \left[ \frac{7}{2} \left( 1 - \frac{T}{T_t} \right) + \frac{\theta}{T_t} \left( \frac{1}{\exp(\theta/T_t) - 1} - \frac{1}{\exp(\theta/T) - 1} \right) \right].$$

Dividing by  $a^2 = \gamma RT$ , with  $\gamma$  given by Eq. (1.70), we then get  $M^2$  as a function of  $T$ . The pressure is then determined as a function of  $T$  by Eq. (1.69) (with  $p_0 = p_t$ ,  $T_0 = T_t$ ).<sup>20</sup>

### 3.2 One-Dimensional Unsteady Flow

In what follows we shall study flows in which only a single velocity component  $u$  is nonzero; furthermore, all the flow variables shall depend only on one space coordinate  $x$  and on time  $t$ . One such example is the flow in a cylindrical tube, filled with gas, produced by the motion of a piston in the axial direction. The continuity equation (2.32) reduces to

$$\rho_t + u \rho_x + \rho u_x = 0 \quad (3.25)$$

(here and hereafter, we shall denote the partial derivatives with respect to  $x$  and  $t$  by subscripts). The momentum equation (2.52) is, in the absence of volume forces and friction,

$$\rho u_t + qu u_x + p_x = 0, \quad (3.26)$$

while Eq. (2.70), which expresses the isentropy of the change of state of each gas particle, becomes

$$s_t + us_x = 0. \quad (3.27)$$

We shall first study homentropic flow (Sections 3.2.1–3.2.3), in which the specific entropy  $s$  has a constant value  $s_0$  independent of  $x$  and  $t$ . If the gas

<sup>20</sup> A.J. Eggers, Jr., One-dimensional Flows of an Imperfect Diatomic Gas. NACA Report No. 959 (1950); Ames Research Staff, Equations, Tables and Charts for Compressible Flow. NACA Report No. 1135 (1935).

is at rest and in thermodynamic equilibrium (and hence homentropic) at any time, then it is homentropic at all times (except for those flows in which shock waves later appear; see Section 3.2.4). Thus, if we know  $\varrho(x, t)$ , all the thermodynamic state variables of the gas are known as functions of  $x$  and  $t$ , since density  $\varrho$  and entropy  $s$  determine the thermodynamic state. (We assume that the gas is in thermodynamic equilibrium everywhere; relaxation processes will be treated later in Section 3.3.) We can then eliminate the pressure from (3.26) with  $dp = a^2 d\varrho$ , and obtain

$$\varrho(u_t + uu_x) + a^2 \varrho_x = 0. \quad (3.28)$$

### 3.2.1 LINEAR WAVE EQUATION

Let the gas be initially at rest; in this state, the variables (independent of  $x$  and  $t$ ) will be denoted by the subscript 0:  $p_0$ ,  $\varrho_0$ ,  $a_0$ , etc. (We discern between the stagnation state in steady flow, denoted by the subscript "t," and the state of rest in unsteady flow, denoted by "0".) We shall now perturb this state of rest, so that  $p = p_0 + p'$ ,  $\varrho = \varrho_0 + \varrho'$ ,  $a = a_0 + a'$ , etc. We assume that these perturbations remain small, i.e.,  $|\varrho'| \ll \varrho_0$ ,  $|p'| \ll p_0$ ,  $|a'| \ll a_0$ , etc. This also implies that we must have  $|u| \ll a_0$ , as we shall see later [see Eq. (3.35)]. We can then linearize Eqs. (3.25) and (3.28) by neglecting the products of the perturbation quantities  $\varrho'$ ,  $a'$ , and  $u$ :

$$\varrho'_t + \varrho_0 u_x = 0, \quad (3.29)$$

$$\varrho_0 u_t + a_0^2 \varrho'_x = 0. \quad (3.30)$$

Eliminating  $\varrho'$  from these equations, we obtain the linear wave equation for  $u$ :

$$u_{tt} - a_0^2 u_{xx} = 0 \quad (3.31)$$

or, eliminating  $u$  from (3.29) and (3.30), we obtain the analogous equation for  $\varrho'$ :

$$\varrho'_{tt} - a_0^2 \varrho'_{xx} = 0. \quad (3.32)$$

The general solution of the system of Eqs. (3.29) and (3.30) is

$$u = f\left(t - \frac{x}{a_0}\right) + g\left(t + \frac{x}{a_0}\right), \quad (3.33)$$

$$\varrho' = \frac{\varrho_0}{a_0} \left[ f\left(t - \frac{x}{a_0}\right) - g\left(t + \frac{x}{a_0}\right) \right]. \quad (3.34)$$

Inserting expressions (3.33) and (3.34) into Eqs. (3.29) and (3.30), we easily see that the equations are satisfied if  $f$  and  $g$  are differentiable functions of their arguments. Likewise, Eqs. (3.31) and (3.32) are satisfied if  $f$  and  $g$  are twice differentiable. Equations (3.29) and (3.30), or (3.31) and (3.32), are the basic equations of acoustics for the one-dimensional propagation of sound waves in a gas at rest.

If  $g \equiv 0$ , the disturbance propagates into the gas at rest by a simple translation of points of equal states with a velocity  $a_0$  in the positive  $x$  direction. We call this motion a *forward-running* wave (or just a forward wave for short). This shows that  $a_0$  is indeed the propagation velocity of small disturbances, and thus the velocity of sound. When  $f \equiv 0$ , we have a *backward-running* wave. In a forward wave, an increment  $\delta u$  between the velocity at location  $x_1$  and time  $t_1$  and the velocity at  $x_2, t_2$ , corresponds, according to (3.33) and (3.34), to a density difference of

$$\delta\varrho = (\varrho_0/a_0) \delta u. \quad (3.35)$$

For a backward-running wave, a minus sign appears on the right side of (3.35). The functions  $f$  and  $g$  in the general solution (3.33), (3.34) are determined in each specific problem by the initial and boundary conditions. Let us consider the following case: The gas is at rest in the region  $x > 0$  at  $t = 0$ . For  $t > 0$ , we prescribe the gas velocity at  $x = 0$  to be  $u(0, t) = u_0(t)$  (either by blowing or suction of the gas at this point). If we define  $f(t)$  in the following way:

$$f(t) = 0 \quad \text{for } t \leq 0, \quad f(t) = u_0(t) \quad \text{for } t > 0,$$

then the solution of the problem is given by a forward wave

$$u = f(t - (x/a_0)).$$

Let us now assume more specifically that the velocity  $u_0(t)$  is 0 at  $t = 0$  and

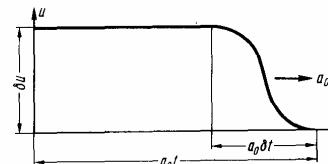


Fig. 25. Forward-running wave generated by a continuous piston motion.

increases in a short time  $\delta t$  to  $\delta u$  and then remains constant; then the forward wave will have the form shown in Fig. 25.

To prepare ourselves to solve the nonlinear equations (3.25) and (3.28) by the method of characteristics, let us consider the following process: Let the gas be at rest in a cylindrical tube at time  $t = 0$ . The tube is closed at  $x = 0$  by a piston, which at the instant  $t_1 = 0$  suddenly starts to move with velocity  $\delta u_1 < 0$  (toward the left). For a sufficiently small  $|\delta u_1|$ , such that linear theory is valid, an unsteady Mach wave propagates to the right into the gas at rest and has a form that is the limiting case (as  $\delta t \rightarrow 0$ ) of the wave shown in Fig. 25. As the gas particles cross the wavefront, their velocity jumps from 0 to  $\delta u_1$ . This value remains constant behind the wavefront, so that at the piston surface the boundary condition: gas velocity = piston velocity is satisfied (Fig. 26). In the meantime, by (3.35), passage across the wavefront

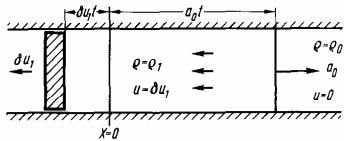


Fig. 26. Wave generated by a suddenly started piston.

changes the gas density from  $\rho_0$  to  $\rho_1 = \rho_0 + \rho_0(\delta u_1/a_0)$ . The other state variables change accordingly. To an observer moving with the velocity  $\delta u_1$  in the  $x$  direction, the gas between the piston and the wavefront is at rest. Thus, at time  $t_2 > 0$ , if the piston again suddenly accelerates toward the left by  $\delta u_2 < 0$  to a total velocity of  $\delta u_1 + \delta u_2$ , the observer will see a second wave of the type described propagating into the gas between the piston and the first wavefront. The velocity of the second wave relative to the observer is  $a_1 = a_0 + \delta a_1$ , and that relative to the fluid at rest is  $a_1 + \delta u_1$ . Since  $\delta u_1 < 0$ ,  $\delta a_1 < 0$ ; with assumption (3.1), we also have  $\delta a_1 < 0$ , so that  $a_1 + \delta u_1 < a_0$ , i.e., the second wave remains behind the first. (Had we not assumed  $\delta u_1 < 0$ , we would have to contend with the possibility of the second wave overtaking the first. Then we must consider shock waves, which we shall do in Section 3.2.4.) Across the second wave, the gas density decreases to  $\rho_2 = \rho_1 + \rho_1(\delta u_2/a_1)$ .

We can imagine these repeated backward accelerations to continue, and

obtain the following picture in the  $x, t$  plane (Fig. 27): The path of the piston is a polygonal curve with corners at the times  $t_k$ , at which the piston receives the sudden accelerations. The straight lines issuing from these corners have slopes given by  $dx/dt = a_k + \sum_{i=1}^k \delta u_i$ , and represent the individual wavefronts. They divide the  $x, t$  plane to the right of the piston path into strip-like areas, in each of which the gas variables and flow velocity are constant. Indeed,

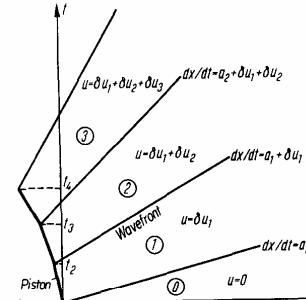


Fig. 27. Sequence of waves in the  $x, t$  plane generated by sudden movements of a piston.

in the  $k$ th strip, we have

$$u_k = \sum_{i=1}^k \delta u_i; \quad \rho_k = \rho_0 + \sum_{i=1}^k \rho_{i-1} \frac{\delta u_i}{a_{i-1}}. \quad (3.36)$$

The remaining state variables  $p_k, a_k$ , etc., are determined from  $\rho_k$ . Using the density jumps  $\delta \rho_i = \rho_i - \rho_{i-1}$ , we can also write the following instead of (3.36):

$$\rho_k = \rho_0 + \sum_{i=1}^k \delta \rho_i; \quad u_k = \sum_{i=1}^k \frac{a_{i-1}}{\rho_{i-1}} \delta \rho_i. \quad (3.37)$$

If we now imagine the velocity changes  $\delta u_i$  to become increasingly smaller and to follow each other increasingly more often, then we approach in the limit a continuous piston motion. Then (3.37) for  $u_k$  becomes

$$u = \omega(\rho), \quad (3.38)$$

where

$$\omega(\varrho) = \int_{\varrho_0}^{\varrho} \frac{a(\varrho)}{\varrho} d\varrho. \quad (3.39)$$

Equation (3.38) gives the relation between  $u$  and  $\varrho$  (and hence with all the other state variables) in the flow resulting from the piston motion. Thus, at each point of the  $x, t$  plane, we can find the flow velocity  $u(x, t)$  and density  $\varrho(x, t)$ , when we realize that the strips in Fig. 27 will collapse to straight lines in the limit of reductions in velocity jumps  $\delta u_i$  and time steps  $t_{i+1} - t_i$ . The velocity  $u$  is then constant on each straight line, being equal to the piston velocity at the initial point of each such straight line on the piston path. According to (3.38), the density is also constant and can be computed from the known function  $\omega(\varrho)$ . Thus, the sound velocity  $a(\varrho)$  is also known, and the slope of each straight line is  $dx/dt = u + a$ . We call these straight lines *Mach lines*. A flow of this type, in which the velocity and the thermodynamic state are constant along the Mach lines, is called a *simple wave*. We shall return to the details later, after we put these ideas on a firmer basis by discussing the nonlinear equations (3.25) and (3.28).

*Supplementary Remark.* When the velocity changes by  $\delta u$  in a forward wave in a calorically ideal gas, then the following relations govern the changes of the other variables in the linear approximation:

$$\frac{\delta u}{a_0} = \frac{\delta \varrho}{\varrho_0} = \frac{1}{\gamma} \frac{\delta p}{p_0} = \frac{1}{\gamma - 1} \frac{\delta T}{T_0} = \frac{2}{\gamma - 1} \frac{\delta a}{a_0}. \quad (3.40)$$

### 3.2.2 METHOD OF CHARACTERISTICS FOR HOMENTROPIC FLOW

Let us return to Eqs. (3.25) and (3.28), which we shall transform as follows: We multiply (3.25) by  $a$  and then add or subtract the product from (3.28); this gives us the two equations

$$a[\varrho_t + (u + a)\varrho_x] + \varrho[u_t + (u + a)u_x] = 0, \quad (3.41)$$

$$a[\varrho_t + (u - a)\varrho_x] - \varrho[u_t + (u - a)u_x] = 0. \quad (3.42)$$

We now define two families of curves,  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ , in the  $x, t$  plane thus: Each curve of the family  $\mathfrak{C}_1$  is defined by

$$dx/dt = u + a \quad (3.43)$$

and each curve of the family  $\mathfrak{C}_2$  by

$$dx/dt = u - a. \quad (3.44)$$

These two families of curves are called the characteristic curves, or, for short, the characteristics of the systems (3.41) and (3.42).  $\mathfrak{C}_1$  are the *forward-running* characteristics and  $\mathfrak{C}_2$  the *backward-running* characteristics. In gas dynamics,  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  are also called *Mach lines*. In the previous example, the forward-running Mach lines  $\mathfrak{C}_1$  are of special form: they are straight lines. In any specific problem, the Mach lines are obviously known only if the solutions  $u(x, t)$  and  $\varrho(x, t)$ , and hence also  $a(x, t)$ , are known.

Equations (3.41) and (3.42) imply the following relations between  $u$  and  $\varrho$  along the curves  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ :

$$d\varrho + \varrho du = 0 \quad \text{along } \mathfrak{C}_1, \quad (3.45)$$

$$d\varrho - \varrho du = 0 \quad \text{along } \mathfrak{C}_2 \quad (3.46)$$

[In Eq. (3.45), we recognize (3.35)!].

To deduce the relations (3.45) and (3.46) from Eqs. (3.41) and (3.42) we can proceed as follows: We designate the curves of the family  $\mathfrak{C}_1$  by a parameter  $\lambda$  such that each curve of this family corresponds to a definite value of this parameter. In the same way, we designate the curves of the family  $\mathfrak{C}_2$  by a parameter  $\mu$ . Then the  $x, t$  plane will be covered by a curvilinear system of coordinates  $\lambda, \mu$ . Although this coordinate net is not defined beforehand, but has to be constructed gradually as each specific problem is being solved (in a way to be described later), the following considerations are nevertheless valid: We interpret  $u$  and  $\varrho$  as functions of  $\lambda$  and  $\mu$ . Then,

$$\frac{\partial u}{\partial \mu} = u_x \frac{\partial x}{\partial \mu} + u_t \frac{\partial t}{\partial \mu}.$$

The partial derivatives with respect to  $\mu$ , i.e., derivatives at fixed  $\lambda$ , are directional derivatives in the direction of  $\mathfrak{C}_1$ . According to (3.43), we have

$$\frac{\partial x}{\partial \mu} = (u + a) \frac{\partial t}{\partial \mu},$$

so that

$$\frac{\partial u}{\partial \mu} = [u_t + (u + a)u_x] \frac{\partial t}{\partial \mu}.$$

In the same way, we also get:

$$\frac{\partial \varrho}{\partial \mu} = [\varrho_t + (u + a)\varrho_x] \frac{\partial t}{\partial \mu}.$$

Multiplying Eq. (3.41) by  $\partial t/\partial\mu$  and taking into account the two relations just derived, we obtain

$$a \frac{\partial \varrho}{\partial \mu} + \varrho \frac{\partial u}{\partial \mu} = 0. \quad (3.45^*)$$

But this is identical to relation (3.45). In a completely analogous manner, we also obtain

$$a \frac{\partial \varrho}{\partial \lambda} - \varrho \frac{\partial u}{\partial \lambda} = 0, \quad (3.46^*)$$

which is identical to Eq. (3.46).

Introducing the function  $\omega(\varrho)$  defined by Eq. (3.39), we can simplify Eqs. (3.45) and (3.46) to

$$d\omega + du = 0 \quad \text{along } \mathfrak{C}_1,$$

$$d\omega - du = 0 \quad \text{along } \mathfrak{C}_2,$$

or

$$\omega + u = \text{const} \quad \text{on } \mathfrak{C}_1, \quad (3.47)$$

$$\omega - u = \text{const} \quad \text{on } \mathfrak{C}_2. \quad (3.48)$$

The quantities  $\omega + u$  and  $\omega - u$  are called Riemann invariants.

Let us now point out the connection between formulas (3.47) and (3.48) and the results of linear theory in Eqs. (3.33) and (3.34). If the state of the gas does not deviate much anywhere from a state of rest, then we can make a linear approximation:

$$\omega = \int_{\varrho_0}^{\varrho} a d\varrho = \frac{a_0}{\varrho_0} \int_{\varrho_0}^{\varrho+u'} d\varrho = \frac{a_0}{\varrho_0} u'.$$

Now, multiplying Eq. (3.34) by  $a_0/\varrho_0$ , adding the result to Eq. (3.33), and using the expression for  $\omega$  just derived, we obtain  $\omega + u = 2f[t - (x/a_0)]$ . Thus,  $\omega + u$  is always constant on the straight lines  $t - (x/a_0) = \text{const}$ . These straight lines are the characteristics  $\mathfrak{C}_1$  in linear theory. In a completely similar way, we also obtain  $\omega - u = 2g[t + (x/a_0)]$ , i.e.,  $\omega - u$  is constant on the characteristics  $\mathfrak{C}_2$  of linear theory.

The two relations (3.47) and (3.48) form the starting point for computing  $\varrho(x, t)$  and  $u(x, t)$  in a specific problem, i.e., with given initial values. We shall study three different possibilities for initial value problems:

*The First Initial-Value Problem (Cauchy Initial-Value Problem).* The values of  $u$  and  $\varrho$ , and thus also  $\omega$  (where an arbitrary but convenient lower limit in the integral (3.39) is chosen, that does not have to be the rest-state density

$\varrho_0$ ), are prescribed on a segment of a noncharacteristic curve  $A_0B_0$  in the  $x, t$  plane (Fig. 28). Noncharacteristic in this context means that the curve is nowhere tangent to a Mach line. If the  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  characteristics issuing from all the points of the segment  $A_0B_0$  were known, then we could get the solution at each interior point  $P$  of the triangle  $A_0P_0B_0$  (and also of the triangle formed by  $A_0B_0$  and the two characteristics forming a vertex on the

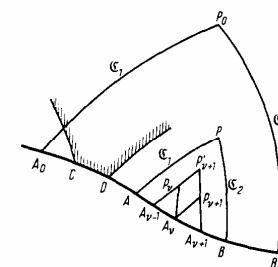


Fig. 28. Diagram for the explanation of the method of characteristics for the first initial-value problem.

other side of  $A_0B_0$  from the point  $P$ ; in general, we are interested only in the solution in one of these two triangles). According to Eqs. (3.47) and (3.48),

$$\omega(P) + u(P) = \omega(A) + u(A),$$

$$\omega(P) - u(P) = \omega(B) - u(B),$$

and thus  $\omega(P)$  and  $u(P)$  are uniquely determined. Of course, as already stated, the characteristics and thus the point  $P$  are not known beforehand, but must be constructed together with the solution. This can be done in the following manner of approximation: We divide the curve  $A_0B_0$  into small segments by a large number of points  $A_v$ , where the number of these points  $A_v$  will determine the accuracy of the approximate solution to be constructed. We approximate the characteristics  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  through the points  $A_v$  by their tangents at  $A_v$ . The directions of these tangents are known because  $u$  and  $\varrho$ , and thus  $a(\varrho)$ , are given on  $A_0B_0$ . The intersections of these tangents determine a row of points  $P_v$  (Fig. 28). We obtain the values of  $u$  and  $\varrho$  at  $P_v$  approximately by applying relations (3.47) and (3.48), valid for  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ , to the straight tangential segments; thus  $\omega(P_v) + u(P_v) = \omega(A_{v-1}) +$

$u(A_{v-1})$  and  $\omega(P_v) - u(P_v) = \omega(A_v) - u(A_v)$ . The solution is thus determined at the points  $P_v$ ; by repeating the same process, it can be extended to a further row of points  $P'_v$ . It can be shown that in the limit, as the distance between the points  $A_v$  tends to zero, the approximate solution thus obtained converges to the true solution (in the regions where a continuous solution exists).<sup>21</sup>

This construction of the solution immediately shows that the solution at point  $P$  (Fig. 28) depends only on the initial values on the portion of the curve between  $A$  and  $B$ ; a change of the initial values on the curve outside of  $AB$  has no influence on the solution at  $P$ . The initial values on the segment  $CD$ , for example, influence the solution only in the area bounded by the two characteristics, i.e., the shaded area in Fig. 28. We can interpret this physically: Small perturbations in the flow propagate with sound velocity relative to the gas, i.e., with velocity  $u + a$  in the positive  $x$  direction and  $u - a$  in the negative  $x$  direction. Therefore, the sound wavefronts correspond to the characteristics  $C_1$  and  $C_2$  in the  $x, t$  plane. A change in the initial values on  $CD$  does not influence  $P$  if  $P$  is outside the shaded region in Fig. 28.

*The Second Initial-Value Problem (Characteristic Initial-Value Problem).* The values of  $u$  and  $\varrho$  are given on a segment  $AB$  of a  $C_1$  characteristic and on a segment  $AC$  of a  $C_2$  characteristic (Fig. 29). Of course,  $u$  and  $\varrho$  cannot be prescribed completely arbitrarily: they must satisfy the compatibility relations (3.47) on  $AB$  and (3.48) on  $AC$ . We can then start from the corner  $A$

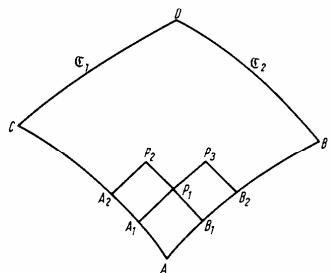


Fig. 29. Diagram for the second initial-value problem.

<sup>21</sup> We return to this in Section 3.8.2; see also R. von Mises, "Mathematical Theory of Compressible Fluid Flow". Academic Press, New York, 1958.

and determine the solution in the characteristic rectangle  $ABCD$ . For example, the solution at the point  $P_1$  is found from  $\omega(P_1) + u(P_1) = \omega(A_1) + u(A_1)$  and  $\omega(P_1) - u(P_1) = \omega(B_1) - u(B_1)$ . Subsequently, the solution at  $P_2$  is found from the initial values at  $A_2$  and the solution at  $P_1$ , the solution at  $P_3$  from the initial values at  $B_2$  and the solution at  $P_1$ , etc. The characteristics are again replaced by their local tangents piece by piece.

*The Third Initial-Value Problem.*  $u$  and  $\varrho$  [satisfying Eq. (3.47)] are prescribed on a segment of the characteristic curve  $AB$ ; on a segment of a noncharacteristic curve  $AC$ , only  $u$  or only  $\varrho$  or  $\omega$ , or more generally, a functional relation  $f(u, \varrho) = 0$ , is given. We can then determine the solution in the triangle  $ABC$  (Fig. 30). First,

$$\omega(A_1) - u(A_1) = \omega(B_1) - u(B_1) \quad \text{and} \quad (u(A_1), \omega(A_1)) = 0.$$

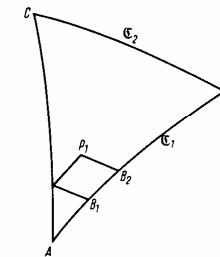


Fig. 30. Diagram for the third initial-value problem.

From this we obtain  $\omega(A_1)$  and  $u(A_1)$ . Then we immediately obtain the solution at  $P_1$  from  $\omega(P_1) + u(P_1) = \omega(A_1) + u(A_1)$  and  $\omega(P_1) - u(P_1) = \omega(B_2) - u(B_2)$ , etc.

All three initial-value problems play a role in solving the following problem (Fig. 31): A cylindrical tube is closed at right and at left by pistons  $K_1$  and  $K_2$ , respectively. The gas is at rest between the pistons at time  $t = 0$ ; i.e.,  $u = 0$  and  $\varrho = 0$  on the segment  $AB$  in the  $x, t$  plane. (We choose as lower limit  $\varrho_0$  in Eq. (3.39) the rest-state density of the gas in the cylinder). The two pistons are set in motion at time  $t = 0$  with continuous velocities;  $u$  is prescribed on the piston paths  $R_1$  and  $R_2$  in the  $x, t$  plane, since the gas velocity and piston velocity must be the same there. In the triangle  $AEB$ , we must solve the first initial-value problem. It is immediately clear that  $u$

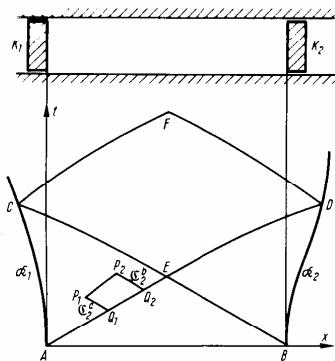


Fig. 31. Gas motion generated by two pistons; location of the characteristics.

and  $\omega$  vanish everywhere in  $AEB$ , including the boundaries: The gas remains undisturbed and at rest, which is physically clear, since the effect of the piston motion is transmitted inward from both sides at sound speed, i.e., the initial signals propagate into the gas at rest along the characteristics  $AE$  and  $BE$ . Thus, these characteristics are straight lines with slopes  $dx/dt = +a_0$  and  $-a_0$ . In the triangle  $AEC$  we must solve the third initial-value problem. Since the disturbance in the originally stationary gas resulting from the motion of the piston  $K_2$  propagates leftward along the characteristic  $BEC$  and cannot be observed at all in  $AEC$ , the flow in  $AEC$  is the same one we would see if the tube were open on the right without the piston  $K_2$ . Thus, we have in  $AEC$  the forward-running simple wave described before. In the same way, we have in  $BED$  a backward-running simple wave.

We can also demonstrate the fact that these are simple waves in a purely formal way: Let us take an arbitrary forward-running characteristic in  $AEC$  and choose two arbitrary points on it,  $P_1$  and  $P_2$  (Fig. 31). Joining these two points by two backward-running characteristics to the points  $Q_1$  and  $Q_2$  on  $AE$ , we then have

$$\begin{aligned}\omega(P_1) - u(P_1) &= \omega(Q_1) - u(Q_1) = 0, \\ \omega(P_2) - u(P_2) &= \omega(Q_2) - u(Q_2) = 0, \\ \omega(P_1) + u(P_1) &= \omega(P_2) + u(P_2).\end{aligned}$$

From this, it follows that  $\omega(P_1) = \omega(P_2)$  and  $u(P_1) = u(P_2)$ ; i.e.,  $\omega$  and  $u$  are constant on  $CE$ , so  $CE$  is a straight line. This conclusion also holds when  $\omega$  and  $u$  do not vanish on  $AE$ , but are constant there.

The two simple waves emanating from the two pistons intersect above  $CED$ ; from there on it is not so easy to describe the flow without constructing the explicit solution. In  $CEDF$  we must solve the second initial-value problem.

### 3.2.3 SIMPLE EXPANSION WAVES IN CALORICALLY IDEAL GASES

We shall now study simple waves in greater detail; without loss of generality, we shall confine ourselves to forward-running waves. Let the wave propagate into a gas at rest, whose state we denote by the subscript 0. We can imagine such a wave to have been produced by the motion of a piston  $K$  in a tube open at the right. According to Eq. (3.38), the relation  $u = \omega(\varrho)$  obtains between the density and the velocity in the wave. For a calorically ideal gas,  $\omega(\varrho)$  is easily given explicitly: With  $a^2 = \gamma RT$ , it follows that

$$a^2 = \gamma RT_0 \frac{T}{T_0} = a_0^2 \left( \frac{\varrho}{\varrho_0} \right)^{\gamma-1}.$$

Substituting this into Eq. (3.39) and integrating, we obtain

$$\omega(\varrho) = \frac{2a_0}{\gamma-1} \left[ \left( \frac{\varrho}{\varrho_0} \right)^{(\gamma-1)/2} - 1 \right], \quad (3.49)$$

and the following for a forward-running wave:

$$\frac{u}{a_0} = \frac{2}{\gamma-1} \left[ \left( \frac{\varrho}{\varrho_0} \right)^{(\gamma-1)/2} - 1 \right] = \frac{2}{\gamma-1} \left( \frac{a}{a_0} - 1 \right) \quad (3.50)$$

or,

$$\frac{a}{a_0} = 1 + \frac{\gamma-1}{2} \frac{u}{a_0}. \quad (3.51)$$

Let us now restrict ourselves to the case where the piston moves leftward with a speed that monotonically increases with time (Fig. 32a). As the wave passes through a given point  $x_1$ , the density decreases monotonically with time (since  $u$  decreases monotonically, and, with it, according to (3.50), so does  $\varrho$ ), and we have an *expansion wave*. We can imagine a special expansion wave, called a *centered expansion wave*, which is produced by impulsively

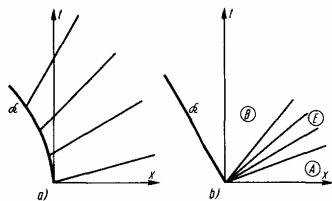


Fig. 32. Simple expansion waves generated by piston motion. (a) Piston motion with continuous velocity change. (b) Piston suddenly set in motion, centered simple wave.

accelerating the piston backwards at time  $t = 0$  so that for  $t > 0$  it moves with constant velocity  $u_0$  (Fig. 32b). The point  $x = t = 0$  is a singular point, from which the forward characteristics of the expansion wave emanate in a fan formation, and at which  $u$  and  $\rho$  are not defined. There is a state of rest in front of the expansion wave ( $E$ ) in the region ( $A$ ), but behind the wave ( $B$ ) there is a constant state with constant flow velocity equal to the piston velocity.

We now introduce the Mach number  $M$  as a parameter to describe the states in an expansion wave; with  $|u| = -u = U$  (since  $u < 0$ ), the Mach number is  $U/a$ . But

$$\frac{u}{a_0} = \frac{U}{a} = -M \frac{a}{a_0} = -M \left(1 + \frac{\gamma - 1}{2} \frac{U}{a_0}\right)$$

[by Eq. (3.51)]. We have from this

$$\frac{U}{a_0} = -\frac{u}{a_0} = M \left(1 + \frac{\gamma - 1}{2} M\right)^{-1}; \quad (3.52)$$

now, since  $a/a_0 = M^{-1}U/a_0$ ,

$$\frac{a}{a_0} = \left(1 + \frac{\gamma - 1}{2} M\right)^{-1}, \quad (3.53)$$

and, since  $a/a_0 = (\rho/\rho_0)^{(\gamma-1)/2}$ ,

$$\frac{\rho}{\rho_0} = \left(1 + \frac{\gamma - 1}{2} M\right)^{-2/(\gamma-1)}, \quad (3.54)$$

and, since  $p/p_0 = (\rho/\rho_0)^\gamma$ ,

$$\frac{p}{p_0} = \left(1 + \frac{\gamma - 1}{2} M\right)^{-2\gamma/(\gamma-1)}, \quad (3.55)$$



and finally, since  $T/T_0 = (a/a_0)^2$ ,

$$\frac{T}{T_0} = \left(1 + \frac{\gamma - 1}{2} M\right)^{-2}. \quad (3.56)$$

These formulas express the velocity  $U$  and the most important thermodynamic state variables in an expansion wave as functions of the Mach number  $M$ . The relations (3.52)–(3.56) can be applied to simple compression waves if everywhere we substitute  $-M$  for  $M$ , where, as before,  $M = U/a$ . In Fig. 33, these relations are given graphically for  $\gamma = 1.2$  and  $1.4$ ; in

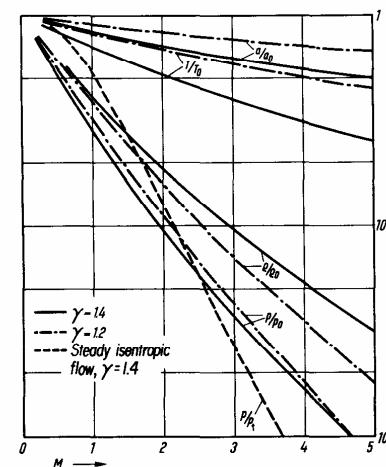


Fig. 33. State variables in a simple expansion wave.

addition, relation (3.14) for steady flow is also shown for comparison. We see that in order to achieve a given Mach number  $M$  by a simple wave, a higher pressure drop is needed than in a steady expansion (e.g., Laval nozzle) for small  $M$ ; for large  $M$ , the converse is true (see supplementary remark 1 below). Obviously, a maximum velocity exists for expansion through a simple wave; when  $M = \infty$ , we get from Eq. (3.52)

$$U_{\max} = 2a_0/(\gamma - 1). \quad (3.57)$$

Corresponding to this maximum speed,  $p = \rho = T = a = 0$ . The slope of the characteristic is  $dx/dt = u = -U_{\max}$ , and the characteristic is tangent to the piston path in the  $x, t$  plane (Fig. 34). If the piston is accelerated to a speed greater than  $U_{\max}$ , then a vacuum is created between this last characteristic and the piston.

*Supplementary Remarks.* 1. For each point of a simple expansion wave we can define an adiabatic reservoir state (denoted by a subscript “(1)”), which is connected to the local state variables  $p$ ,  $\rho$ ,  $T$ , and  $a$  through the formulas (3.13)–(3.15). We can show that  $p_1 \leq p_0$ ,  $\rho_1 \leq \rho_0$ ,  $T_1 \leq T_0$ , and  $a_1 \leq a_0$  if  $M \leq 4/(3 - \gamma)$ .

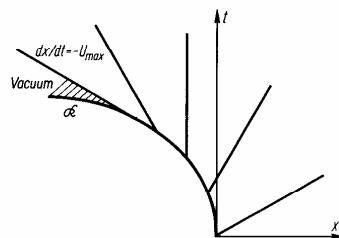


Fig. 34. Formation of vacuum for sufficiently large piston speed.

2. A special form of intermittent blow down wind tunnel mentioned at the end of Section 3.1 is the Ludwieg tube<sup>22</sup> (Fig. 35). It consists of a thin cylindrical tube closed at the right and connected at the left to a Laval nozzle which is initially sealed off from the outside by a thin diaphragm or membrane. The tube is initially filled with a gas at high pressure. If the nozzle length is negligible compared with the tube length, then when the diaphragm is punctured, a centered expansion wave propagates into the gas and generates behind it a uniform subsonic flow in the direction toward the nozzle. The Mach number  $M_1 < 1$  of this flow is just large enough for the steady flow in the nozzle to attain sound velocity at the throat, and the flow in the divergent part of the nozzle has supersonic velocity (assuming that the exit pressure is sufficiently low; see the discussion on Laval nozzle flow in Section 3.1). In a calorically ideal gas,  $M_1$  is determined by the ratio of the nozzle throat area to the tube area [formula (3.20)]. Figure 35 gives the

<sup>22</sup> H. Ludwieg, Der Rohrwindkanal, *Z. Flugwiss.* 3, 205–216 (1955).

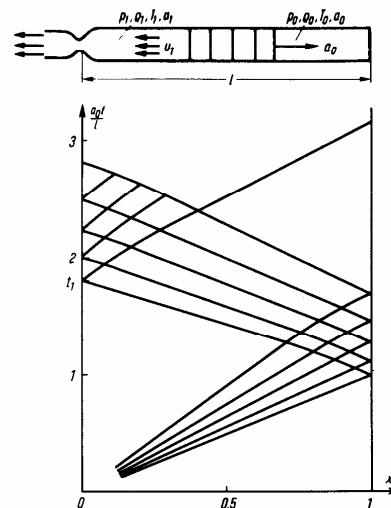


Fig. 35. Characteristics for the flow of a calorically ideal gas in a Ludwieg tube;  $\gamma=1.4$ ,  $M_1=0.4$ .

characteristics diagram of a Ludwieg tube for  $M_1 = 0.4$  and  $\gamma = 7/5$ . The centered expansion wave running to the right will be reflected at the closed end of the tube; here, the boundary condition  $u = 0$  must be satisfied at all times (in the region of interaction of the incident and reflected waves, we must solve initial-value problem 3). After leaving the interaction region, the reflected wave (again a simple wave, but no longer centered) runs toward the nozzle. Behind the reflected wave, the gas is again at rest, but its state has been changed from the original state of rest by a reduction in pressure, temperature, and density. At time  $t_1$ , the head of the wave reaches the nozzle, and the wave will then reflect from the Laval nozzle. If the dimensions of the nozzle are negligibly small compared to the length of the tube, then the flow in the nozzle during the reflection process may be treated as quasisteady. To calculate the reflection, we must again solve the third initial-value problem, where now the boundary condition  $u = -M_1 a$  must be satisfied at the nozzle ( $x = 0$ ), since in quasisteady flow the Mach number  $M_1$  is still

determined by the area ratio of the nozzle. In general, only the flow for  $0 < t < t_1$  is of interest, since, when using this arrangement as a wind tunnel, a strictly steady flow is desired.

Using the relations derived above for simple waves, we can show after some calculation that

$$t_1 = \frac{2l}{a_0} \left( 1 + \frac{\gamma - 1}{2} M_1 \right)^{(y+1)/2(y-1)} (1 + M_1)^{-1}.$$

Furthermore, the pressure  $p_0'$  in the gas at rest behind the reflected wave (reflected from the closed end) is

$$p_0' = p_0 \left[ \frac{2 + (\gamma - 1) M_1}{2 - (\gamma - 1) M_1} \right]^{-2y/(y-1)}$$

### 3.2.4 SIMPLE COMPRESSION WAVES. FORMATION OF SHOCK WAVES

We again consider a simple wave propagating to the right into a gas at rest, caused by a piston moving to the right (Fig. 36). This time, instead of an expansion wave, we obtain a compression wave. We write the piston path as  $x_p = x_p(t)$ , the piston velocity as  $u_p = \dot{x}_p(t)$ , and the piston acceleration as  $b_p = \ddot{x}_p(t)$ . The slope of a forward characteristic starting from the piston path at time  $t = \tau$  is given by

$$dx/dt = a + u = a_0 + \frac{1}{2}(\gamma + 1) u(x, t) = a_0 + \frac{1}{2}(\gamma + 1) u_p(\tau),$$

where Eq. (3.51) has been used. Thus, the equation of this characteristic is

$$x = x_p(\tau) + (t - \tau) [a_0 + \frac{1}{2}(\gamma + 1) u_p(\tau)]. \quad (3.58)$$

This one-parameter family of straight characteristics forms an envelope from some definite point  $S$  onward; the equation of the envelope is readily found in the usual manner by differentiating Eq. (3.58) with respect to the parameter  $\tau$  and then eliminating  $\tau$ . This elimination cannot always be carried out in closed form for arbitrary piston paths. Only the initial point  $S$  of the envelope can always be found, provided we assume that it lies on the characteristic emanating from the origin ( $x = 0, t = 0$ ) (Fig. 36)<sup>23</sup>; this point is determined,

<sup>23</sup> This is always the case when  $b_p(0) \geq b_p(t)$ .

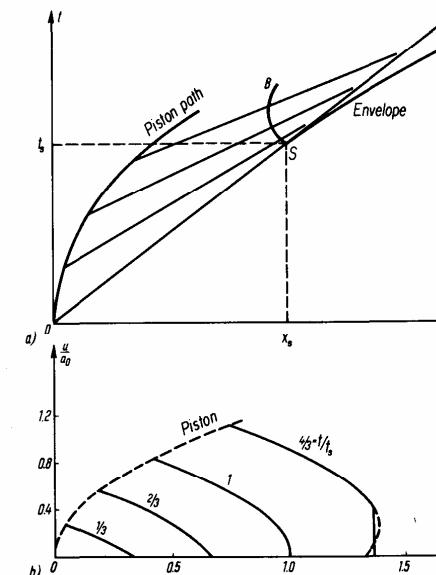


Fig. 36. (a) Formation of envelope of forward-running characteristics. (b) Velocity distribution at different time  $t$  for the piston motion  $x_p = Bt^2/2$ .

by setting the parameter  $\tau = 0$ , to be

$$x_s = \frac{2a_0^2}{(\gamma + 1) b_p(0)} \quad \text{and} \quad t_s = \frac{2a_0}{(\gamma + 1) b_p(0)}. \quad (3.59)$$

We see from these formulas that, in particular, the coordinates of the initial point of the envelope ( $x_s, t_s$ ) depend only on the initial acceleration  $b_p(0)$  of the piston, and with increasing  $b_p(0)$  this point can approach the origin arbitrarily closely.

For the special piston motion  $x_p = \frac{1}{2}Bt^2$ , the envelope can be given explicitly; it is a parabolic arc. For the initial point of the envelope, the value  $b_p(0) = B$  is substituted into Eq. (3.59). As is evident from Fig. 36a,



a wedge-shaped region is formed in the  $x, t$  plane for  $x > x_s, t > t_s$ , in which the characteristics overlap and in which, therefore, no unique solution can be found. On each characteristic,  $u = B\tau$ ; replacing  $\tau$  in Eq. (3.58) by  $u/B$  and eliminating  $B$  with Eq. (3.59), we obtain for  $u(x, t)$  in the present case the relation

$$\frac{\gamma(\gamma+1)}{4} \left( \frac{u}{a_0} \right)^2 + \frac{\gamma+1}{2} \left( 1 - \frac{t}{t_s} \right) \frac{u}{a_0} + \frac{x}{x_s} - \frac{t}{t_s} = 0, \quad (3.60)$$

from which  $u(x, t)$  can be calculated. Figure 36b shows the velocity distribution at different times  $t$  as found from this equation. When  $t > t_s$  and  $x > x_s$ , three values appear for  $u$  [i.e.,  $u=0$  and two more from (3.60)]; obviously, this cannot occur physically.

Experience shows that in this case a compression shock forms, i.e., an extremely thin region of width  $l$ , in which the velocity, pressure, density, etc., change very rapidly. Since in this region very large gradients of these quantities appear, viscosity and thermal conductivity play a role, and the entropy of a gas particle going through the shock increases, as we shall establish in detail later (see Section 3.4). On the other hand, experimental observations and theory (see Section 4.2) both show that the width  $l$  of a shock is in general so small that it can be neglected. Then we treat a shock as a discontinuity in an otherwise inviscid flow field; in Fig. 36, the discontinuity lies between the envelope and the extension of the straight line  $OS$ .

Now, the following considerations should be noted: While the solution in the triangle  $OSB$  of the  $x, t$  plane is not changed by the appearance of the shock wave, we expect it to be changed to the right of the backward Mach line  $SB$  from the point  $S$ . Thus, the solution is no longer a simple wave. Since the backward characteristics emanating from the region of rest intersect the shock wave, the arguments of Section 3.2.2, on which the existence of simple waves adjacent to a region of rest was based, become invalid; in fact, the previous assumption that  $s$  is constant must be abandoned, as the entropy undergoes a jump in the shock wave. To be sure, it will be shown in Section 3.4 that the entropy of a gas particle changes significantly only if it goes through a relatively strong shock. For a weak shock, such as the shock in our example for  $t$  not much greater than  $t_s$ , we can still consider the flow field as approximately homentropic. Therefore, the flow to the right of the characteristic  $SB$  is also only slightly different from a simple wave. In sketching the velocity distribution for  $t/t_s = 4/3$  in Fig. 36, this has been tacitly assumed, and,

behind the shock, the velocity distribution corresponding to a simple wave given by (3.60) has been retained.

### 3.2.5 METHOD OF CHARACTERISTICS FOR NONHOMENTROPIC FLOW

To find the solution in the region to the right of  $SB$  (Fig. 36) there are two problems to be solved: First, we must determine the location of the shock wave in the  $x, t$  plane and know the discontinuous changes in the flow variables across the shock. The discussion of the relevant shock relations will be postponed to Section 3.4. Second, we must extend the method of characteristics to the case of variable entropy. While in a homentropic flow it suffices to find the density  $\varrho(x, t)$  along with the velocity  $u(x, t)$ , since the thermodynamic state is completely determined by the density and the known constant entropy  $s_0$ , it is now necessary to determine an additional thermodynamic variable besides  $\varrho$ . For this variable, we shall select the pressure  $p$ , and we start our discussion from Eqs. (3.25)–(3.27). Since, by Eq. (3.27), the entropy of each individual gas particle remains constant, we have from  $a^2 = (\partial p / \partial \varrho)_s$ ,

$$Dp/Dt = a^2 D\varrho/Dt;$$

i.e., (3.27) may be replaced by

$$p_t + up_x - a^2(\varrho_t + u\varrho_x) = 0. \quad (3.61)$$

Using Eq. (3.25), we can rewrite this as

$$p_t + up_x + a^2\varrho u_x = 0. \quad (3.62)$$

Multiplying (3.26) by  $a$  and adding the result to Eq. (3.62) yields

$$p_t + (u+a)p_x + \varrho a[u_t + (u+a)u_x] = 0, \quad (3.63)$$

while multiplying (3.26) by  $a$  and subtracting the result from (3.62) yields

$$p_t + (u-a)p_x - \varrho a[u_t + (u-a)u_x] = 0. \quad (3.64)$$

In addition to the characteristics  $C_1$  and  $C_2$  defined before (the Mach lines), we now define a third family of characteristic curves  $C_3$ , for which at any point in the  $x, t$  plane

$$dx/dt = u. \quad (3.65)$$

The curves  $C_3$  are none other than the particle paths in the  $x, t$  plane.

Equations (3.63), (3.64), and (3.61) can now be written in the following form [see the analogous derivation for Eqs. (3.45) and (3.46)]:

$$dp + \varrho a du = 0 \quad \text{along } \mathfrak{C}_1, \quad (3.66)$$

$$dp - \varrho a du = 0 \quad \text{along } \mathfrak{C}_2, \quad (3.67)$$

$$dp - a^2 d\varrho = 0 \quad \text{along } \mathfrak{C}_3. \quad (3.68)$$

In homentropic flow, as studied in Section 3.2.2, Eq. (3.68) holds not only along  $\mathfrak{C}_3$ , but along every arbitrary curve as well, and in particular along  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ ; we can thus replace  $dp$  by  $a^2 d\varrho$  in Eqs. (3.66) and (3.67) and recover relations (3.45) and (3.46).

To illustrate the method of characteristics based on Eqs. (3.66)–(3.68), we now discuss the first initial-value problem only; the other two initial-value problems can be solved in an analogous way. The  $p$ ,  $\varrho$ , and  $u$  are to be continuously prescribed on a noncharacteristic curve  $\mathfrak{R}$  (which is nowhere tangent to  $\mathfrak{C}_1$ ,  $\mathfrak{C}_2$ , or  $\mathfrak{C}_3$ ) in the  $x$ ,  $t$  plane (Fig. 37). The values of  $p$ ,  $\varrho$ , and  $u$

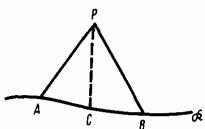


Fig. 37. Diagram for the explanation of the method of characteristics for a nonhomentropic flow; first initial-value problem.

at a point in the neighborhood of  $\mathfrak{R}$  is found approximately as follows: We take two neighboring points  $A$ ,  $B$  on  $\mathfrak{R}$  and approximate the characteristic  $\mathfrak{C}_1$  through  $A$  by its tangent at  $A$ ;  $a(p, \varrho)$  is known from  $p$  and  $\varrho$ , so that the direction of the tangent to  $\mathfrak{C}_1$  at  $A$  is known. Similarly, we approximate the characteristic  $\mathfrak{C}_2$  through  $B$  by its tangent. Let the two tangents intersect at the point  $P$ . Now we replace the differential equations (3.66) and (3.67) by difference equations, using for the quantities  $\varrho$  and  $a$  their values on  $\mathfrak{R}$ , and obtain the approximate relations

$$p(P) - p(A) + \varrho(A) a(A) [u(P) - u(A)] = 0$$

and

$$p(P) - p(B) - \varrho(B) a(B) [u(P) - u(B)] = 0.$$

We can calculate  $p(P)$  and  $u(P)$  from these equations. Next, we approximate the characteristic  $\mathfrak{C}_3$  through  $P$  by its tangent at  $P$ , this being defined by the value  $u(P)$  just calculated. Let  $C$  be the intersection of this tangent with  $\mathfrak{R}$ . The differential equation (3.68) is replaced by the difference equation  $p(P) - p(C) - a^2(C) [\varrho(P) - \varrho(C)] = 0$ , from which  $\varrho(P)$  is found.

This procedure can be improved by iteration, by which we construct a new point and calculate the state at this point by repeating the process using the arithmetic means of the variables at  $A$  and  $P$  or  $B$  and  $P$ .

### 3.3\* One-Dimensional Wave Propagation in a Gas with Relaxation

#### 3.3.1\* METHOD OF CHARACTERISTICS<sup>24</sup>

From the study of nonhomentropic wave propagation in Section 3.2.5 it is only a short step to the study of wave propagation in a gas with relaxation processes, i.e., thermodynamic processes leading through constrained equilibrium states (see Section 1.6). Then, at least one more variable  $\xi$  is needed to describe the thermodynamic states, and here we shall restrict ourselves to the case where one single additional variable  $\xi$  suffices. The variable  $\xi$  is to be determined as a function of  $x$  and  $t$ , just as the velocity  $u$  and the variables  $p$  and  $\varrho$  are.

To this end, we must observe that the hitherto useful relation  $Ds/Dt = 0$  no longer holds. In Section 2.4 it was indeed shown that relaxation processes and entropy changes are connected [see formula (2.72)]. While the continuity equation (3.25) and the momentum equation (3.26) still hold, we must abandon Eq. (3.27) as well as the relations (3.61) and (3.62) derived from it.

<sup>24</sup> L.J.F. Broer, On the influence of acoustic relaxation on compressible flow, *Appl. Sci. Res.* **A2**, 447–468 (1950). L.J.F. Broer, Characteristics of the equations of motion of a reacting gas, *J. Fluid Mech.* **4**, 276–282 (1958). E.L. Resler, Jr., Characteristics and sound speed in nonisentropic gas flows with nonequilibrium thermodynamic states, *J. Aero-Space Sci.* **24**, 785–791 (1957). W.W. Wood and J.G. Kirkwood, Characteristic equations for reactive flow, *J. Chem. Phys.* **27**, 596 (1957). B.T. Chu, Wave Propagation and the Method of Characteristics in Reacting Gas Mixtures with Applications to Hypersonic Flow. Brown Univ., Providence, Rhode Island, WADC TN 57-213 (1957). B.T. Chu, Wave Propagation in a Reacting Mixture. Heat Transfer and Fluid Mechanics Institute, Stanford, California, 1958. T.Y. Li, Recent advances in nonequilibrium dissociating gas-dynamics, *Am. Rocket Soc. J.*, 170–178 (1961).

Instead, we return to Eq. (2.65), which is valid independently of any assumption on the thermodynamic behavior. Neglecting viscosity and heat conduction, we see that the right side vanishes, and since the volume  $V$  in (2.65) is arbitrary, and since  $\varrho > 0$ , we must have

$$\frac{De}{Dt} - \frac{p}{\varrho^2} \frac{D\varrho}{Dt} = \frac{Dh}{Dt} - \frac{1}{\varrho} \frac{Dp}{Dt} = 0, \quad (3.69)$$

where the enthalpy  $h = e + (p/\varrho)$  has been introduced to replace the internal energy  $e$ ;  $h$  is now a function of  $p$ ,  $\varrho$ , and  $\xi$ , i.e.,  $h = \hat{h}(p, \varrho, \xi)$ . (We use the notation  $\hat{h}$  for this function, just as we did in Section 1.11). Equation (3.69) can now be written as

$$\left( \frac{\partial \hat{h}}{\partial p} - \frac{1}{\varrho} \right) \left( \frac{Dp}{Dt} - b^2 \frac{D\varrho}{Dt} + B \frac{D\xi}{Dt} \right) = 0, \quad (3.70)$$

where the frozen sound speed  $b$  was introduced in Eq. (1.140) and the short hand notation

$$B(p, \varrho, \xi) = \frac{\partial \hat{h}/\partial \xi}{(\partial \hat{h}/\partial p) - (1/\varrho)} = -b^2 \frac{\partial \hat{h}/\partial \xi}{\partial \hat{h}/\partial p} \quad (3.71)$$

has been used. Since the first factor in (3.70) does not vanish identically, Eq. (3.70) becomes [with  $D/Dt = \partial/\partial t + u(\partial/\partial x)$ ]

$$p_t + up_x - b^2(\varrho_t + u\varrho_x) = -B(\xi_t + u\xi_x), \quad (3.72)$$

or, taking (3.25) into account, we have

$$p_t + up_x + \varrho b^2 u_x = -B(\xi_t + u\xi_x). \quad (3.73)$$

These two equations now take the place of Eqs. (3.61) and (3.62).

Furthermore, we must realize that we are determining a fourth variable  $\xi$  as a function of  $x$  and  $t$  in addition to the three variables  $u$ ,  $p$ , and  $\varrho$ , and therefore we need a fourth equation in addition to the three equations (3.25), (3.26), and (3.73). To establish this equation, we observe the following: Imagine a gas at rest in a fixed state  $p$ ,  $\varrho$ ,  $\xi$ , which does not coincide with any unconstrained state of equilibrium. In general, the variable  $\xi$  will change with time as long as unconstrained equilibrium has not yet been reached. We now assume that the time rate of change  $\dot{\xi}$  of the variable  $\xi$  depends only on the instantaneous thermodynamic state  $p$ ,  $\varrho$ ,  $\xi$ , i.e.,  $\dot{\xi} = L(p, \varrho, \xi)$ . The function  $L$  must either be determined experimentally or calculated theoreti-

cally with the help of theories which go beyond simple thermodynamics (such as statistical mechanics and reaction kinetics). If  $\xi$  coincides with the equilibrium value  $\tilde{\xi}(p, \varrho)$  corresponding to  $p$  and  $\varrho$ , then, by definition,

$$L(p, \varrho, \tilde{\xi}(p, \varrho)) = 0. \quad (3.74)$$

We now transfer these statements on the rate of change  $\dot{\xi}$  of  $\xi$  to a moving gas, in which, for each gas particle, we set

$$D\xi/Dt = \xi_t + u\xi_x = L(p, \varrho, \xi). \quad (3.75)$$

Then, Eq. (3.73) can be written as

$$p_t + up_x + \varrho b^2 u_x = -BL. \quad (3.76)$$

Everything else now continues exactly as in Section 3.2: We multiply Eq. (3.26) by  $b$  and add the result to Eq. (3.76):

$$p_t + (u + b)p_x + b\varrho[u_t + (u + b)u_x] = -BL. \quad (3.77)$$

Multiplying by  $b$  and subtracting yields

$$p_t + (u - b)p_x - b\varrho[u_t + (u - b)u_x] = -BL. \quad (3.78)$$

The families of curves  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  are defined by

$$dx/dt = u + b \quad \text{for } \mathfrak{C}_1, \quad (3.79)$$

$$dx/dt = u - b \quad \text{for } \mathfrak{C}_2. \quad (3.80)$$

As before, the family  $\mathfrak{C}_3$  remains identical to the particle paths [formula (3.65)]. Equations (3.72), (3.75), (3.77), and (3.78) are in characteristic form, i.e., in each case, they contain differentiation in one direction only: Eqs. (3.72) and (3.75) in the direction of  $\mathfrak{C}_3$ , (3.77) in the direction of  $\mathfrak{C}_1$ , and (3.78) in the direction of  $\mathfrak{C}_2$ . Of course, the form of the relations (3.66)–(3.68) will be changed because of the inhomogeneous term  $-BL$ . Nevertheless, the method of characteristics is still applicable with a minor modification: If we go forward, say, along a  $\mathfrak{C}_1$  characteristic, by a line element whose projection is  $dt$  on the  $t$  axis, we would have

$$dp/dt = p_t + p_x(dx/dt) = p_t + (u + b)p_x.$$

The other characteristics are treated similarly, and we can finally write:

$$dp + \varrho b du = -BLdt \quad \text{along } \mathfrak{C}_1, \quad (3.81)$$

$$dp - \varrho b du = -BLdt \quad \text{along } \mathfrak{C}_2, \quad (3.82)$$

$$dp - b^2 d\varrho = -BLdt \quad \text{along } \mathfrak{C}_3, \quad (3.83)$$

$$d\xi = Ldt \quad \text{along } \mathfrak{C}_3. \quad (3.84)$$

[One should compare the derivation of Eqs. (3.45) and (3.46)]. For given boundary values, we can use these equations and construct a solution piece by piece in exactly the same way as in Section 3.2.5. The method should be evident from this point on, and no further detailed explanation is required.

An important result of this discussion is the fact that in the definitions (3.79) and (3.80) of the characteristics  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  the frozen sound speed  $b$  appears instead of  $a$ . Therefore,  $b$  is the speed with which small disturbances propagate relative to the gas. If we set  $L \equiv 0$ , then  $\xi$  for each particle cannot change. We call this a frozen flow. Equations (3.81)–(3.83) then become Eqs. (3.66)–(3.68), but with  $b$  replacing  $a$ . The limiting transition to a flow which is everywhere in thermodynamic equilibrium, i.e., starting from Eqs. (3.81)–(3.84) and obtaining Eqs. (3.66)–(3.68), is likewise possible, but not so simple to carry out. Therefore, rather than carrying out this transition in full generality, we shall discuss it in the following section after first linearizing the equations.

### 3.3.2\* LINEAR WAVE EQUATION<sup>25</sup>

We now apply the discussion of Section 3.2.1 to a gas with relaxation. Let this gas be initially at rest and in unconstrained thermodynamic equilibrium:  $u = 0$ ,  $p = p_0$ ,  $\varrho = \varrho_0$ , and  $\xi = \xi_0 = \tilde{\xi}(p_0, \varrho_0)$ . We now perturb this state:  $u \neq 0$ ,  $p = p_0 + p'$ ,  $\varrho = \varrho_0 + \varrho'$ , and  $\xi = \xi_0 + \xi'$ . Under the assumption that the perturbation quantities  $u$ ,  $p'$ ,  $\varrho'$ , and  $\xi'$  remain sufficiently small, Eqs. (3.26), (3.72), (3.75), and (3.76), which describe the propagation of disturbances in full generality, will now be linearized in these perturbation quantities. In linearizing  $L(p, \varrho, \xi)$  we make several observations: First of all, in

<sup>25</sup> E.V. Stupochenko and I.P. Stakhanov, The equations of relaxation hydrodynamics, *Soviet Physics Doklady* **5** (1961), 957–960 [Engl. transl. of *Doklady Akademii Nauk SSSR* **134** (1960), 782–785]. F.K. Moore and W.E. Gibson, Propagation of weak disturbances in a gas subject to relaxation effects, *J. Aero-Space Sci.* **27**, 117–127 (1960).

the linear approximation

$$\begin{aligned} L(p_0 + p', \varrho_0 + \varrho', \xi_0 + \xi') \\ = L(p_0, \varrho_0, \xi_0) + \left(\frac{\partial L}{\partial p}\right)_0 p' + \left(\frac{\partial L}{\partial \varrho}\right)_0 \varrho' + \left(\frac{\partial L}{\partial \xi}\right)_0 \xi'. \end{aligned} \quad (3.85)$$

The partial derivatives of  $L$  with respect to its arguments are taken at  $p_0$ ,  $\varrho_0$ , and  $\xi_0$ , and are denoted by the subscript 0. We should note that because of Eq. (3.74), the first term on the right side of Eq. (3.85) vanishes:  $L(p_0, \varrho_0, \xi_0) = 0$ , since, by assumption,  $\xi_0 = \tilde{\xi}(p_0, \varrho_0)$ .

In formula (3.85),  $L$  is already linearized, but it is more convenient to express  $L$  in terms of the departure from thermodynamic equilibrium. We first write, by (3.74),

$$L(p_0 + p', \varrho_0 + \varrho', \tilde{\xi}(p_0 + p', \varrho_0 + \varrho')) = 0. \quad (3.86)$$

We expand to terms of the first order to get, since  $L(p_0, \varrho_0, \xi_0) = 0$ ,

$$\left(\frac{\partial L}{\partial p}\right)_0 p' + \left(\frac{\partial L}{\partial \varrho}\right)_0 \varrho' = - \left(\frac{\partial L}{\partial \xi}\right)_0 \left[ \left(\frac{\partial \tilde{\xi}}{\partial p}\right)_0 p' + \left(\frac{\partial \tilde{\xi}}{\partial \varrho}\right)_0 \varrho' \right]. \quad (3.87)$$

Combining Eqs. (3.87) and (3.85), we obtain the following linearized expression for  $L$ :

$$L = \left(\frac{\partial L}{\partial \xi}\right)_0 \left[ \xi' - \left(\frac{\partial \tilde{\xi}}{\partial p}\right)_0 p' - \left(\frac{\partial \tilde{\xi}}{\partial \varrho}\right)_0 \varrho' \right]. \quad (3.88)$$

This formula can be interpreted in an obvious manner, if we add to the square brackets the term  $\xi_0 - \xi_0 = 0$ :

$$L = \left(\frac{\partial L}{\partial \xi}\right)_0 \left[ \xi_0 + \xi' - \left\{ \xi_0 + \left(\frac{\partial \tilde{\xi}}{\partial p}\right)_0 p' + \left(\frac{\partial \tilde{\xi}}{\partial \varrho}\right)_0 \varrho' \right\} \right].$$

Now,  $\xi_0 + \xi' = \xi$ , while the expression in the curly brackets is  $\tilde{\xi}$ , the equilibrium value of  $\xi$  corresponding to  $p = p_0 + p'$  and  $\varrho = \varrho_0 + \varrho'$ . Using the shorthand notation  $(\partial L / \partial \xi)_0^{-1} = -\tau_r$ , we thus have in the linear approximation

$$D\xi/Dt = L = -(\xi - \tilde{\xi})/\tau_r. \quad (3.89)$$

The time rate of change of  $\xi$  in the linear approximation is thus directly proportional to the deviation of  $\xi$  from the equilibrium value  $\tilde{\xi}$  corresponding to  $p$  and  $\varrho$ . The quantity  $\tau_r$  has the meaning of a relaxation time.

It is a measure of the speed with which the gas approaches unconstrained thermodynamic equilibrium.

Linearization of the remaining terms in Eqs. (3.26), (3.72), (3.75), and (3.76) offers no difficulty. We obtain finally

$$\text{from Eq. (3.26)} \quad \varrho_0 u_t + p'_x = 0, \quad (3.90)$$

$$\text{from Eq. (3.72)} \quad p'_t - b_0^2 \varrho'_t = -B_0 L, \quad (3.91)$$

$$\text{from Eq. (3.75)} \quad \xi'_t = L, \quad (3.92)$$

$$\text{from Eq. (3.76)} \quad p'_t + \varrho_0 b_0^2 u_{xx} = -B_0 L, \quad (3.93)$$

where the linearized expression (3.89) is used for  $L$  and  $B_0$  denotes  $B(p_0, \varrho_0, \xi_0)$  [see Eq. (3.71)]. Eliminating  $p'$ ,  $\varrho'$ , and  $\xi'$ , we can derive from this first-order system of four equations a single third-order equation for  $u$ . Considering definitions (1.139) and (1.140) for  $a$  and  $b$ , we obtain after some lengthy but trivial calculations

$$\tau \frac{\partial}{\partial t} (u_{tt} - b_0^2 u_{xx}) + (u_{tt} - a_0^2 u_{xx}) = 0. \quad (3.94)$$

Here

$$\tau = \tau_r \left[ 1 + B_0 \left( \frac{\partial \xi}{\partial p} \right)_0 \right]^{-1} = \tau_r \frac{\hat{h}_p - (1/\varrho)}{h_p - (1/\varrho)}. \quad (3.95)$$

The factor by which  $\tau$  and  $\tau_r$  differ is in general of the order of unity. In subsequent discussions, we shall always use the term relaxation time to mean the quantity  $\tau$ .

Various interesting conclusions may be drawn from (3.94): If  $\tau_r = 0$  (and thus  $\tau = 0$ ), then, by Eq. (3.89),  $\xi = \tilde{\xi}$ . Obviously, this implies that for a vanishingly small relaxation time, the gas is always in unconstrained thermodynamic equilibrium. Equation (3.94) then becomes the wave equation (3.31). On the other hand, dividing Eq. (3.94) by  $\tau$  and passing to the limit  $\tau \rightarrow \infty$ , we cancel the term  $u_{tt} - a_0^2 u_{xx}$  (which is divided by  $\tau$ ) and obtain

$$\frac{\partial}{\partial t} (u_{tt} - b_0^2 u_{xx}) = 0.$$

Under the additional assumption that the motion of the gas started from a state of rest, so that for time  $t < t_0$  ( $t_0$  = start of motion)  $u(x, t) = 0$ , it

follows that

$$u_{tt} - b_0^2 u_{xx} = 0. \quad (3.96)$$

Thus, we see that for a completely frozen flow, for which Eq. (3.89) gives  $D\xi/Dt = 0$  and thus the gas particles retain their initial values of  $\xi$  at all time, the linear wave equation again holds, except that  $a_0$  is replaced by  $b_0$ .

Let us now consider a particular solution of Eq. (3.94) in the form of a forward-moving wave periodic in time (harmonic sound wave):

$$u = A \exp[i(\omega t - kx)]. \quad (3.97)$$

As is customary, only the real part of (3.97) has physical meaning.  $A$  is the amplitude,  $\omega$  is a real circular frequency ( $T = 2\pi\omega^{-1}$  is the period of oscillation), and  $k$  is a complex number determined by  $\omega$ ,  $k = k_r + ik_i$ ;  $k_r = 2\pi \lambda^{-1}$  is the real wave number ( $\lambda$  is the wave length), and  $k_i$  is a measure of the damping of the wave. Upon substituting (3.97) into Eq. (3.94) and canceling the common factor  $A \exp[i(\omega t - kx)]$  from each term, we get the following relation between  $k$  and  $\omega$ :

$$k^2 = \frac{\omega^2}{a_0^2} \frac{1 + i\omega\tau}{1 + i\omega\tau(b_0^2/a_0^2)}. \quad (3.98)$$

For  $\tau \rightarrow 0$ , we have  $k_r \rightarrow \pm \omega/a_0$ ,  $k_i \rightarrow 0$ , corresponding to undamped waves with a phase velocity  $a_0$ . In  $\pm \omega/a_0$ , we must choose the + sign for forward waves, and the - sign for backward waves. In the limit of  $\tau \rightarrow \infty$ , on the other hand, we have  $k_r \rightarrow \pm \omega/b_0$ ,  $k_i \rightarrow 0$ . Thus, we again get an undamped wave, only this time with the phase velocity  $b_0$ . For intermediate values of  $\tau$ ,  $k_i \neq 0$ , so that the waves are damped in the direction of propagation. (In calculating the square root of the expression on the right side of Eq. (3.98), two things should be noted:  $\omega k_r > 0$  and  $\omega k_r < 0$  correspond to forward and backward waves respectively; moreover,  $b_0^2$  is always greater than  $a_0^2$ , as already mentioned in Section 1.12. If  $b_0^2 < a_0^2$ , we would have a wave with increasing amplitude, or the gas would be unstable; this is contrary to experience.) Thus, we have

$$u = A \exp(\mp |k_i| x) \exp[i\omega(t \mp |k_r/\omega| x)].$$

The amplitude of the wave decreases exponentially in the direction of propagation. Moreover, the phase velocity of the wave  $c = |\omega/k_r|$  depends on the frequency  $\omega$ , a phenomenon which we call dispersion (see Fig. 38).

The fact that harmonic sound waves experience damping and dispersion

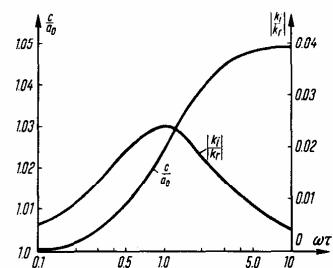


Fig. 38. Phase velocity  $c$  and the ratio of the imaginary part  $k_i$  to the real part  $k_r$  of the wave number for sound waves with relaxation as a function of the frequency  $\omega$ ;  $\tau$  is the relaxation time,  $a_0$  the equilibrium sound speed;  $b_0^2/a_0^2 = 1.1$ .

is well known. Studies in which only shear viscosity and heat conduction (both generate damping and dispersion) are included lead, in general, to too small values of damping, particularly in ultrasonic waves. Only by considering the relaxation processes can we explain theoretically the observed damping.<sup>26</sup> Relaxation damping is caused, above all, by the molecular vibrational and rotational degrees of freedom, which require a considerably longer relaxation time  $\tau$  for adjustment to a change in thermodynamic state than do the translational degrees of freedom. At ultrasonic frequencies, the product  $\omega\tau$  can reach the order of magnitude of unity. Then, pronounced damping and dispersion phenomena appear. Just as a knowledge of the relaxation time permits us to calculate the dispersion and damping of sound waves, so naturally the measurements of dispersion and damping permit us to determine the relaxation time.

To conclude this section, we yet have to discuss how a wave of the type sketched in Fig. 25 in the limit of  $\delta t \rightarrow 0$  behaves in a gas with relaxation. Such a wave propagating to the right in the region  $x > 0$  can be generated in the following manner: For  $t < 0$ , let the gas be at rest in the region  $x > 0$ ; for  $t > 0$ , let it be arranged so that at the point  $x=0$  a constant velocity  $u(0, t) = \delta u$  obtains. This can be achieved, for example, by blowing ( $\delta u > 0$ ) or by suction ( $\delta u < 0$ ). In the linear approximation, we can also imagine the motion to be generated by a piston which is started suddenly; for, although

<sup>26</sup> See H.O. Kneser, Schallabsorption und -dispersion in Gasen. In "Handbuch der Physik" (S. Flügge, ed), vol. XI/1, p. 129–201, Springer, Berlin, 1961.

the boundary condition: gas velocity = piston velocity should be satisfied at the piston location, for sufficiently small  $\delta u$  the piston position is arbitrarily close to  $x = 0$ . In the flow without relaxation as sketched in Fig. 26, this really makes no difference, since the gas velocity is everywhere constant in the region between the piston and the wave front.

Considering now the fact that the function

$$f(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp(i\omega t) \frac{d\omega}{\omega}, \quad (3.99)$$

(where the integration is taken over the entire real axis of the complex  $\omega$  plane with an arc below and around the singular point  $\omega = 0$ ) possesses the following properties:

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ \frac{1}{2} & \text{for } t = 0, \\ 1 & \text{for } t > 0, \end{cases} \quad (3.100)$$

we can write the solution of our problem in the following form:

$$u(x, t) = \frac{\delta u}{2\pi i} \int_{-\infty}^{\infty} \exp[i(\omega t - k(\omega)x)] \frac{d\omega}{\omega}, \quad (3.101)$$

with  $k(\omega)$  given by (3.98) also extended to complex values of  $\omega$ .

The expression (3.101) satisfies the wave equation (3.94), since the exponential function in the integrand satisfies this equation. Moreover, it satisfies the boundary condition at  $x = 0$ , since the expression (3.101) reduces to  $\delta u f(t)$  at  $x = 0$ . Further consideration of the integral in (3.101) gives, moreover,  $u \equiv 0$  for  $x > b_0 t$  (compare with the detailed discussion in Section 3.10.3). For  $t < 0$ , the condition  $u = 0$  in  $x > 0$  is thus satisfied. On the straight line  $x = b_0 t$  in the  $x, t$  plane (Fig. 39),  $u$  jumps from the value 0 for  $x > b_0 t$  to the value  $\varepsilon \delta u$  (see Section 3.10.3), where

$$\varepsilon = \exp \left[ - \left( 1 - \frac{a_0^2}{b_0^2} \right) \frac{t}{2\tau} \right]. \quad (3.102)$$

This discontinuity on the straight line  $x = b_0 t$  thus decays exponentially with time, where

$$t_d = 2\tau [1 - (a_0^2/b_0^2)]^{-1} \quad (3.103)$$

measures the decay time. In the region  $x < b_0 t$ ,  $u$  increases continuously to

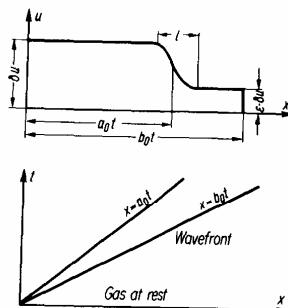


Fig. 39. Waves generated by an impulsively accelerated piston in a gas with relaxation.

its final value  $\delta u$  at  $x=0$ . For times  $t \gg t_d$ , we find that this increase is practically complete in a region of width  $l$  located around the straight line  $x=a_0 t$  (Fig. 39), where

$$l = [2t\tau(b_0^2 - a_0^2)]^{1/2}. \quad (3.104)$$

The solution (3.101) for  $t \gg t_d$  becomes asymptotically

$$u(x, t) \approx \frac{\delta u}{2} \left[ 1 - \Phi\left(\frac{x - a_0 t}{l}\right) \right], \quad (3.105)$$

where  $\Phi$  is the error integral

$$\Phi(\xi) = \frac{2}{\sqrt{\pi}} \int_0^\xi \exp(-\eta^2) d\eta.$$

Figure 39 shows the important features of this solution. With the help of formulas (3.101)–(3.105), we can easily discuss how the solution tends to the one without relaxation as  $\tau \rightarrow 0$  (Fig. 26).

In conclusion, we point out the following: If  $\delta u > 0$ , i.e., when we are dealing with compression waves, then the widening of the transition region as predicted by Eq. (3.104) does not go on indefinitely. The nonlinearity of the process works against this widening. As we have seen before in Section 3.2.4, this nonlinearity in a gas without relaxation always causes a steepening of an originally continuous compression wave in time (see Fig. 36). This

steepening tendency also exists in a gas with relaxation. While the widening tendency is dominating for small time  $t$ , for large time  $t$  a final state is reached where both influences balance each other exactly. Then, a continuous wave propagates with constant velocity into the gas at rest without further changing its form ("fully dispersed wave").<sup>27</sup> A measure of the width of this wave is the length

$$l = \frac{\tau(b_0^2 - a_0^2)}{\delta u}. \quad (3.106)$$

The velocity profile in the wave is given by a hyperbolic tangent function; in Eq. (4.47) a corresponding formula will be given for the velocity profile of a weak shock with viscosity and heat conductivity considered.<sup>27</sup> (See below for the connection between relaxation processes and bulk viscosity.)

*Supplementary Remarks.* If the period of oscillation  $2\pi\omega^{-1}$  is large compared to the relaxation time  $\tau$ , i.e.,  $\omega\tau \ll 1$ , the state of the gas will deviate only slightly from thermodynamic equilibrium at any instant. On the other hand, we obtain for  $\omega\tau \ll 1$ , by expanding (3.98) in  $\omega\tau$ , the approximate expression

$$k^2 = \frac{\omega^2}{a_0^2} \left[ 1 - i\omega\tau \frac{b_0^2 - a_0^2}{a_0^2} \right].$$

An analogous formula results if we calculate the propagation of sound waves in a gas with finite bulk viscosity  $\eta_b$  (see Section 4.1.1) but negligible shear viscosity and thermal conductivity. In this case, we get

$$k^2 = \frac{\omega^2}{a_0^2} \left[ 1 - i\eta_b \frac{\omega}{\rho_0 a_0^2} \right].$$

This formula is identical to the one above if we set  $\eta_b = \rho_0 \tau (b_0^2 - a_0^2)$ . The agreement is not incidental. Indeed, one can show in general that, when the time scale of the state changes is large compared to the relaxation time, the effect of the relaxation process can be accounted for by assuming that the thermodynamic state variables are connected by relations valid for unconstrained equilibrium, but with a finite bulk viscosity included. Since the

<sup>27</sup> M.J. Lighthill, Viscosity Effects in Sound Waves of Finite Amplitudes, in *Surveys in Mechanics* (G.K. Batchelor and R.M. Davies, eds.), 250–351, Cambridge Univ. Press, London, 1956.

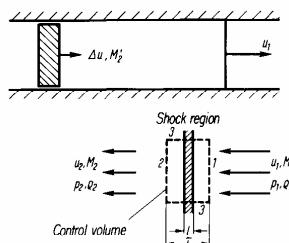
relaxation time of the rotational degrees of freedom of a molecule is very small, we can often describe the effect of this relaxation by a corresponding bulk viscosity, while with vibrational degrees of freedom, this is usually not permissible because of the much longer relaxation times.

### 3.4 Normal Shock Waves

#### 3.4.1 SHOCK RELATIONS

In Section 3.2.4, the formation of a shock wave was explained for a special example. The shock wave that appeared in the flow treated in that example is unsteady, i.e., its strength, measured, say, by the jump of the pressure across the shock, changes with time. A simpler behavior is found if the piston is not accelerated continuously from zero velocity, as was done in Section 3.2.4, but is suddenly accelerated to a constant final velocity  $\Delta u$ . Then, a shock of constant strength will propagate into the stationary gas with constant velocity  $u_1$ . Introducing a coordinate system which moves with the shock, we transform the flow to a steady flow in this coordinate system: In front of the shock the gas has velocity  $u_1$ , and behind it a velocity  $u_2 = u_1 - \Delta u$  (Fig. 40).

In a shock wave, viscosity and heat conduction play a role. As we shall establish more exactly later on (Section 4.2), these effects in general remain confined to a very small region of width  $l$  (shock thickness). We now imagine



**Fig. 40.** Shock wave generated by a piston suddenly set in motion. In the upper figure, the gas in front of the shock is at rest; in the lower figure, the coordinate system is chosen so that the shock is at rest.

a control volume as sketched in Fig. 40. The surfaces 1 and 2 of the control volume completely enclose the shock region of width  $l$ . On these surfaces, therefore, neither shear stress nor heat flow occurs.<sup>28</sup> The side surface 3 of the control volume is a streamtube, i.e., the normal to the surface is everywhere perpendicular to the streamlines. Now we must note that in a one-dimensional flow, the gradients of all the flow variables only have components in the flow direction. In an isotropic medium, the energy flux vector  $\mathbf{q}$  introduced in Section 2.4 is then also in the streamline direction; moreover, on the side surface 3 the viscous stress is perpendicular to the flow direction, and does not enter the momentum equation. In applying the three conservation laws of mass, momentum, and energy to our control volume, viscous stress and heat conduction drop out of these equations.

In each equation, we assume the steadiness of the flow, and we divide by the area  $A_1 = A_2$ ; the continuity equation (2.34) becomes

$$\varrho_1 u_1 = \varrho_2 u_2. \quad (3.107)$$

(The subscript “1” denotes quantities in front of the shock, the subscript “2” those behind it.) With body forces neglected, the momentum equation (2.49) becomes

$$\varrho_1 u_1^2 + p_1 = \varrho_2 u_2^2 + p_2. \quad (3.108)$$

The energy equation (2.54) or (2.56) [since in (2.54) the terms  $L_2$ ,  $L_3$ , and  $L_4$  all drop out for reasons given above] becomes

$$\frac{1}{2} u_1^2 + h_1 = \frac{1}{2} u_2^2 + h_2. \quad (3.109)$$

For given velocity  $u_1$  and given thermodynamic state 1 before the shock, we can use the caloric equation of state  $h = h(p, \varrho)$  and calculate from Eqs. (3.107)–(3.109) the velocity  $u_2$  and the thermodynamic state 2 behind the shock without knowing the detail processes in the shock region. Thus, without express reference to viscosity or heat conduction, we can consider the shock (because of its small thickness  $l$ ) as a discontinuous surface in an

<sup>28</sup> In a shock wave, relaxation processes of the type considered in Section 3.3. can also play a role and considerably influence the shock thickness (consider, for example, the thickness  $\hat{l}$  given by (3.106) for a fully dispersed wave, which is nothing more than a weak shock; see Section 4.2.3). The shock relations about to be derived, however, remain unchanged if the control surface fully encloses the relaxation region and thermodynamic equilibrium obtains outside the control volume.

inviscid flow field on which the three shock relations (3.107)–(3.109) are to be satisfied. The assumptions of steadiness and negligible body forces are, moreover, not very essential; in nearly all practical cases, these relations can also be applied to unsteady shocks, as well as to shocks with volume forces present.

The effect of unsteadiness can, for example, be estimated in the continuity equation (2.34) as follows: The contribution to the surface integral in Eq. (2.34) from surface 1 is (apart from the sign),  $\varrho_1 u_1 A$  ( $A = A_1 = A_2$ ) and the contribution from surface 2 is  $\varrho_2 u_2 F$ . The volume integral yields a contribution  $A\bar{l}\langle\partial\varrho/\partial t\rangle$ , where  $\langle\partial\varrho/\partial t\rangle$  is an appropriate mean value of  $\partial\varrho/\partial t$ . We now set  $\langle\partial\varrho/\partial t\rangle = (\varrho_1 + \varrho_2)/2\bar{l}$ , where the time  $\bar{l}$  is the time scale measuring the local time-variation of the density. The volume integral can then be neglected when compared with the contributions from the two surfaces if  $A\bar{l}(\varrho_1 + \varrho_2)/2\bar{l} \ll A(\varrho_1 u_1 + \varrho_2 u_2)$ , i.e., if

$$\bar{l} \gg \bar{l}/\bar{u}, \quad (3.110)$$

where  $\bar{u} = (2\varrho_1 u_1 + 2\varrho_2 u_2)/(\varrho_1 + \varrho_2)$ . Since  $\bar{l}$  can be chosen to be of the order of the shock thickness  $l$ , this means that the characteristic time  $\bar{l}$  of the unsteady changes must be large compared to the time  $l/\bar{u}$ , which is the order of magnitude of the duration of stay of a gas particle inside the shock region. Because of the smallness of the shock thickness, this condition is almost always satisfied. Similar consideration of the momentum and energy equations including the presence of body forces leads to the same conclusion.

#### 3.4.2 CHANGE IN THE THERMODYNAMIC STATE ACROSS A SHOCK WAVE

We now study the change of the thermodynamic state across a shock by eliminating the velocities  $u_1$  and  $u_2$  from Eqs. (3.107)–(3.109). We first get, from (3.107) and (3.108),

$$u_1^2 = \frac{\varrho_2 p_2 - p_1}{\varrho_1 \varrho_2 - \varrho_1} \quad \text{and} \quad u_2^2 = \frac{\varrho_1 p_2 - p_1}{\varrho_2 \varrho_2 - \varrho_1}. \quad (3.111)$$

Substituting this into Eq. (3.109) and making a short calculation, we obtain

$$h_2 - h_1 = \frac{1}{2} \left( \frac{1}{\varrho_1} + \frac{1}{\varrho_2} \right) (p_2 - p_1). \quad (3.112)$$

This equation, which is fundamental in shock theory, is called the Hugoniot

relation. For very weak shocks in which state 1 differs only very slightly from state 2, we can set  $h_2 - h_1 = dh$ ,  $p_2 - p_1 = dp$ , and  $\frac{1}{2}(1/\varrho_1 + 1/\varrho_2) = 1/\varrho$ , and Eq. (3.112) becomes  $dh = dp/\varrho$  or  $ds = 0$ . Thus, in very weak shocks, the entropy change of the gas particles is negligible. In this limiting case, we get the discontinuous waves discussed in Section 3.3.2.

For further discussion, we imagine the fixed initial state to be the state  $p_1, \varrho_1$  in a  $p, \varrho$  plane (point  $A$ , Fig. 41). For a given caloric equation of state

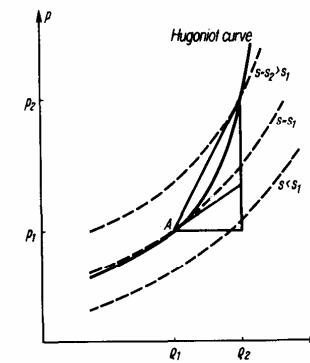


Fig. 41. Isentropes (---) and Hugoniot curve in the  $p, \varrho$  plane.

$h = h(p, \varrho)$ , all the attainable final states can be determined by calculating  $p_2$  from (3.112) with given values of  $\varrho_2$ . These states  $p_2, \varrho_2$  lie on a curve passing through  $A$ , the "Hugoniot adiabatic." In Fig. 41, the isentropic curves  $s(p, \varrho) = \text{const}$  are drawn. It is assumed here that the isentropes always rise with density and are convex with respect to the  $\varrho$  axis, which is always true when relation (3.1) is fulfilled. Moreover, it is assumed that the entropy increases with pressure, i.e.

$$(\partial s / \partial p)_\varrho > 0. \quad (3.113)$$

Applying the relations given in Section 1.4, we can show that Eq. (3.113) is satisfied if the volume expansivity  $\alpha > 0$ , which will always be assumed in the following.

Since the change of state is isentropic for weak shocks, the Hugoniot adiabatic must be tangent to the isentrope passing through  $A$ . This tangency

is even of second order, i.e., at the point of tangency, not only are the directions of both curves the same, but so are the curvatures. This can be shown as follows: The relation  $p_2 = f(\varrho_2)$  for the Hugoniot adiabatic through  $A$  is given by Eq. (3.112). We drop the subscript "2" for simplicity and differentiate Eq. (3.112) with respect to  $\varrho$ :

$$\frac{dh}{d\varrho} = -\frac{p - p_1}{2\varrho^2} + \frac{1}{2} \left( \frac{1}{\varrho_1} + \frac{1}{\varrho} \right) \frac{dp}{d\varrho}.$$

Substituting from (1.23)  $dh = T ds + (1/\varrho) dp$ , we get

$$T \frac{ds}{d\varrho} = -\frac{p - p_1}{2\varrho^2} + \frac{1}{2} \left( \frac{1}{\varrho_1} - \frac{1}{\varrho} \right) \frac{dp}{d\varrho}, \quad (3.114)$$

from which follows the already known fact that  $ds/d\varrho = 0$  at  $p = p_1$ ,  $\varrho = \varrho_1$ . Differentiation again of (3.114) with respect to  $\varrho$  yields

$$T \frac{d^2s}{d\varrho^2} + \frac{dT}{d\varrho} \frac{ds}{d\varrho} = \frac{p - p_1}{\varrho^3} + \frac{1}{2} \left( \frac{1}{\varrho_1} - \frac{1}{\varrho} \right) \frac{d^2p}{d\varrho^2}. \quad (3.115)$$

This shows that at the point  $A$ ,  $d^2s/d\varrho^2$  is also zero, so that at this point the curvatures of the isentrope and Hugoniot adiabatic coincide, as asserted before. Another differentiation with respect to  $\varrho$  finally leads to the result which is valid specially for the point  $A$ :

$$T_1 \frac{d^3s}{d\varrho^3} \Big|_{\varrho=\varrho_1} = \frac{1}{\varrho_1^3} \frac{dp}{d\varrho} \Big|_{\varrho=\varrho_1} + \frac{1}{2\varrho_1^2} \frac{d^2p}{d\varrho^2} \Big|_{\varrho=\varrho_1}. \quad (3.116)$$

However, because of the second-order tangency of the Hugoniot curve with the isentrope at point  $A$ , the derivatives on the right side are identical to the derivatives  $(\partial p/\partial\varrho)_s$  and  $(\partial^2 p/\partial\varrho^2)_s$ . Using definitions (1.133) and (3.1), we thus get from (3.116)

$$\frac{d^3s}{d\varrho^3} \Big|_{\varrho=\varrho_1} = \frac{1}{T_1 \varrho_1^2} \left[ \frac{a_1^2}{\varrho_1} + \frac{A_1^2}{2} \right]. \quad (3.117)$$

Expanding the entropy  $s$  on the Hugoniot adiabatic as a Taylor series in the density  $\varrho$  about the point  $A$ , we get from the above results

$$s_2 - s_1 = \frac{1}{6T_1 \varrho_1^2} \left[ \frac{a_1^2}{\varrho_1} + \frac{A_1^2}{2} \right] (\varrho_2 - \varrho_1)^3 + 0[(\varrho_2 - \varrho_1)^4]. \quad (3.118)$$

This shows that in a weak shock,  $s_2 > s_1$  always when  $\varrho_2 > \varrho_1$ , i.e., the

entropy increases across a shock when the density increases. By the second law of thermodynamics, positive entropy is generated in the shock region, so that it must increase. Then we must have  $\varrho_2 > \varrho_1$ , so that only the branch of the Hugoniot adiabatic to the right of  $A$  has physical meaning: A shock must actually be a compression, under the assumed thermodynamic behavior of gases [e.g., satisfying Eq. (3.1)], and expansion shocks are to be excluded. Of course, this conclusion drawn from Eq. (3.118) is thus far good only for weak shocks, for which  $\varrho_2 - \varrho_1$  is so small that the term  $0[(\varrho_2 - \varrho_1)^4]$  in Eq. (3.118) plays a negligible role. Nevertheless, we can show by a more general procedure that in every shock, the entropy increases (or decreases) as the density increases (or decreases), so that only compression shocks can exist.<sup>29</sup>

Using this fact, we can now easily show that the flow in front of a compression shock is always supersonic, i.e.,  $u_1 > a_1$ . To this end, we observe that

$$a_1^2 = \left( \frac{\partial p}{\partial \varrho} \right)_{s=s_1, \varrho=\varrho_1}$$

is the slope of the isentrope at the initial point  $A$ , and hence also of the Hugoniot adiabatic there. But  $\varrho_2/\varrho_1 > 1$ , and Fig. 41 leads us to the conclusion that  $(p_2 - p_1)/(\varrho_2 - \varrho_1) > (dp/d\varrho)_1 = a_1^2$ . Thus, by Eq. (3.111),  $u_1^2 > a_1^2$ . In a gas at rest, a shock wave thus always propagates with a velocity greater than the sound velocity. In an entirely similar manner, we can show that  $u_2^2$  is always  $< a_2^2$ , i.e., the flow behind the shock is always subsonic. To this end we regard the final state 2 as fixed, and seek all possible initial states 1 that will lead to this final state across a shock. When we interchange the meaning of the subscripts "1" and "2" in Fig. 41 and identify point  $A$  with the final state 2, then all possible initial states 1 must lie on the Hugoniot curve to the left of  $A$ . The same arguments as before then lead to  $u_2^2 < a_2^2$ .

Let us now briefly sketch how one can determine the shock wave from a given Hugoniot adiabatic, for example, in the problem of the suddenly started piston (with constant final velocity  $\Delta u$ ) in a gas-filled cylindrical tube. From Eq. (3.111), we can calculate for each point on the Hugoniot adiabatic

<sup>29</sup> Relations (3.1) and (3.113) suffice as assumptions for the validity of this statement; Eq. (3.1) can even be replaced by the weaker requirement  $(\partial^2 p/\partial(1/\varrho)^2) > 0$ . See J. Serrin, Mathematical Principles of Classical Fluid Mechanics, in "Handbuch der Physik" (S. Flügge, ed.), vol. VIII/1, p. 125–263. Springer, Berlin, 1959.

through  $p_1, \varrho_1$  (the pressure and density of the stationary gas in front of the shock) the velocities  $u_1$  and  $u_2$ , and the difference  $u_1 - u_2$ . The state of the gas behind the shock produced by the piston motion corresponds to that point on the Hugoniot adiabatic at which the difference  $u_1 - u_2$  equals the piston velocity. Using assumptions (3.1) and (3.113) on the thermodynamic behavior of the gas, we can, moreover, show that  $u_1$  increases monotonically when we proceed along the Hugoniot adiabatic away from the initial point. This means, therefore, that the normal shock is uniquely determined by  $p_1, \varrho_1$ , and  $u_1$ .

#### 3.4.3 SPECIAL FORMULAS FOR A CALORICALLY IDEAL GAS

For a calorically ideal gas the caloric equation of state can be written in the form  $h = h^* + [\gamma/(\gamma-1)] p/\varrho$ . Substituting this into Eq. (3.112) and solving for the pressure ratio  $p_2/p_1$ , we obtain the equation of the Hugoniot adiabatic:

$$\frac{p_2}{p_1} = \frac{(\varrho_2/\varrho_1)(\gamma+1) - (\gamma-1)}{\gamma+1 - (\gamma-1)(\varrho_2/\varrho_1)}. \quad (3.119)$$

The density ratio of the gas that can be attained across a compression shock is accordingly bounded by the value

$$(\varrho_2/\varrho_1)_{\max} = (\gamma+1)/(\gamma-1), \quad (3.120)$$

since  $\varrho_2/\varrho_1 \rightarrow (\varrho_2/\varrho_1)_{\max}$  corresponds to  $p_2/p_1 \rightarrow \infty$ . Together with the pressure  $p_2$ , the temperature  $T_2$  also increases without bounds, since the thermodynamic equation of state gives  $T_2/T_1 = (p_2/p_1)(\varrho_1/\varrho_2)$ . The change of specific entropy of a gas particle across the shock is given from (1.62):

$$\begin{aligned} s_2 - s_1 &= c_p \ln(T_2/T_1) - R \ln(p_2/p_1) \\ &= c_v [\ln(p_2/p_1) - \gamma \ln(\varrho_2/\varrho_1)]. \end{aligned} \quad (3.121)$$

If we realize that the stagnation temperature  $T_t$  (see Section 3.1) of a calorically ideal gas does not change across a shock (we shall return to this later), i.e.,  $T_{t1} = T_{t2}$ , we can write in place of Eq. (3.121)

$$s_2 - s_1 = -R \ln(p_{t2}/p_{t1}). \quad (3.122)$$

As a convenient parameter to express the connection of the velocity and state variables in front of the shock to those behind the shock, we now

introduce the Mach number  $M_1 = u_1/a_1$  of the flow in front of the shock. From Eq. (3.111), we get

$$\frac{u_1^2}{a_1^2} = M_1^2 = M_1^2 \gamma \frac{p_1}{\varrho_1} = \frac{\varrho_2}{\varrho_1} \frac{p_2 - p_1}{\varrho_2 - \varrho_1}$$

or

$$\gamma M_1^2 = \frac{(p_2/p_1) - 1}{(\varrho_2/\varrho_1) - 1} \frac{\varrho_2}{\varrho_1}.$$

Substituting for  $p_2/p_1$  according to (3.119) and then solving for  $\varrho_2/\varrho_1$ , we get

$$\frac{\varrho_2}{\varrho_1} = \frac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2}. \quad (3.123)$$

The maximum value of  $\varrho_2/\varrho_1$ , (3.120), is attained for  $M_1 \rightarrow \infty$ . Substituting (3.123) into Eq. (3.119), we get

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}, \quad (3.124)$$

and we obtain from (3.123) and (3.124) together

$$\frac{p_2 \varrho_1}{p_1 \varrho_2} = \frac{T_2}{T_1} = \frac{[2\gamma M_1^2 - (\gamma-1)][2 + (\gamma-1) M_1^2]}{(\gamma+1)^2 M_1^2}. \quad (3.125)$$

The Mach number  $M_2$  behind the shock can be calculated as follows: By Eq. (3.107),  $u_2^2 = \varrho_1^2 u_1^2 / \varrho_2^2$ , so that

$$M_2^2 = \frac{u_2^2}{a_2^2} = \frac{\varrho_1^2 u_1^2}{\varrho_2^2} \frac{\varrho_2}{\gamma p_2} = \frac{M_1^2}{(\varrho_2/\varrho_1)(p_2/p_1)}.$$

Hence, with Eqs. (3.123) and (3.124),

$$M_2^2 = \frac{\gamma+1+(\gamma-1)(M_1^2-1)}{\gamma+1+2\gamma(M_1^2-1)}. \quad (3.126)$$

As  $M_1 \rightarrow \infty$ ,  $M_2$  tends to the finite limit

$$M_{2\min} = [(\gamma-1)/2\gamma]^{1/2}. \quad (3.127)$$

We now derive a relation between the velocities  $u_1$  and  $u_2$ , commonly called the Prandtl relation. Using the definition of the stagnation, or reservoir, quantities given in Section 3.1, Eq. (3.109) can also be written as  $h_{t1} = h_{t2}$ . For a calorically ideal gas,  $h$  is a pure function of temperature so that we also have  $T_{t1} = T_{t2}$  and  $a_{t1} = a_{t2}$ . By Eq. (3.18), the critical sound

velocity  $a_*$  is proportional to the stagnation sound velocity  $a_t$ , so that we also have  $a_{*1} - a_{*2} = a_*$ . From Eq. (3.111), it follows that

$$u_1 u_2 = \frac{p_1 (p_2/p_1) - 1}{\varrho_1 (\varrho_2/\varrho_1) - 1} = \frac{a_1^2 (p_2/p_1) - 1}{\gamma (\varrho_2/\varrho_1) - 1}.$$

Again substituting  $p_2/p_1$  from Eq. (3.124) and  $\varrho_2/\varrho_1$  from Eq. (3.123), we obtain

$$u_1 u_2 = \frac{2a_1^2}{\gamma + 1} \left[ 1 + \frac{\gamma - 1}{2} M_1^2 \right]. \quad (3.128)$$

From relations (3.13) and (3.18), it immediately follows that Eq. (3.128) can be written in the form given by Prandtl:

$$u_1 u_2 = a_*^2. \quad (3.129)$$

From Eq. (3.128) we can derive an expression for the velocity difference  $\Delta u = u_1 - u_2$ , and thus for the gas velocity behind the shock as seen by an observer at rest with the gas in front of the shock. This difference is

$$\Delta u = u_1 - u_2 = M_1 a_1 [(1 - (u_1 u_2/u_1^2))].$$

Substituting for  $u_1 u_2$  from Eq. (3.128) and making a short transformation, we get

$$\Delta u = \frac{2a_1}{\gamma + 1} \left( M_1 - \frac{1}{M_1} \right). \quad (3.130)$$

We now define  $M'_2 = \Delta u/a_2$  as the Mach number of the flow behind the shock as seen by an observer fixed to the gas at rest in front of the shock. Since  $a_1/a_2 = (T_2/T_1)^{-1/2}$ , we get, from Eq. (3.130) with (3.125),

$$M'_2 = \frac{2(M_1^2 - 1)}{[2\gamma M_1^2 - (\gamma - 1)]^{1/2} [2 + (\gamma - 1) M_1^2]^{1/2}}. \quad (3.131)$$

$M'_2$  is bounded from above by

$$M'_{2\max} = \lim_{M_1 \rightarrow \infty} M'_2 = [2/\gamma(\gamma - 1)]^{1/2}. \quad (3.132)$$

For  $\gamma = 7/5$ , for example,  $M'_{2\max} = 1.89$ . Thus, although by (3.130) the velocity  $\Delta u$  increases without bounds with  $M_1$ , the Mach number  $M'_2$  remains bounded, since the temperature  $T_2$  and thus the sound velocity  $a_2$  also increase without bounds.

Finally, we must point out the following: Although the changes of state in a shock wave (for a calorically ideal gas) indeed give  $T_{t1} = T_{t2}$  and therefore  $p_{t2}/\varrho_{t2} = p_{t1}/\varrho_{t1}$ , nevertheless,  $p_{t2} \neq p_{t1}$  and  $\varrho_{t2} \neq \varrho_{t1}$ . In fact, the stagnation pressure and stagnation density both decrease across a shock. Combining formulas (3.124), (3.126), and (3.14) and performing a short calculation for the ratio of stagnation pressures, we get

$$\frac{p_{t2}}{p_{t1}} = \left[ \frac{(\gamma + 1) M_1^2}{2 + (\gamma - 1) M_1^2} \right]^{\gamma/(\gamma-1)} \left[ \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1} \right]^{-1/(\gamma-1)} = \left[ \frac{(\varrho_2/\varrho_1)^\gamma}{p_2/p_1} \right]^{1/(\gamma-1)}. \quad (3.133)$$

The most important of the relations derived here are shown graphically in Fig. 42.

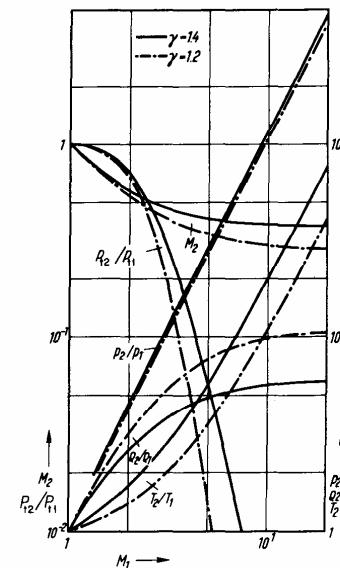


Fig. 42. Pressure, density, temperature, stagnation pressure, and Mach number  $M_2$  behind a normal shock wave as a function of the Mach number  $M_1$  in front of the shock; calorically ideal gas.

*Supplementary Remarks.* For weak shocks in a calorically ideal gas, the following expansions from (3.122) and (3.133) hold:

$$\frac{s_2 - s_1}{R} = \frac{2\gamma}{3(\gamma + 1)^2} (M_1^2 - 1)^3 - \frac{2\gamma^2}{(\gamma + 1)^3} (M_1^2 - 1)^4 + \dots, \quad (3.122^*)$$

$$\frac{p_{t2}}{p_{t1}} = 1 - \frac{2\gamma}{3(\gamma + 1)^2} (M_1^2 - 1)^3 + \frac{2\gamma^2}{(\gamma + 1)^3} (M_1^2 - 1)^4 + \dots. \quad (3.133^*)$$

Introducing in place of the flow Mach number  $M_1$  the pressure ratio  $p_2/p_1$  as independent variable, we obtain, after applying (3.124),

$$\frac{s_2 - s_1}{R} = \frac{\gamma + 1}{12\gamma^2} \left( \frac{p_2}{p_1} - 1 \right)^3 - \frac{\gamma + 1}{8\gamma^2} \left( \frac{p_2}{p_1} - 1 \right)^4 + \dots, \quad (3.122^{**})$$

$$\frac{p_{t2}}{p_{t1}} = 1 - \frac{\gamma + 1}{12\gamma^2} \left( \frac{p_2}{p_1} - 1 \right)^3 + \frac{\gamma + 1}{8\gamma^2} \left( \frac{p_2}{p_1} - 1 \right)^4 + \dots. \quad (3.133^{**})$$

#### 3.4.4 SIMPLE APPLICATIONS

Using the relations deduced above, we can calculate the pressure distribution along a slender Laval nozzle when a shock wave occurs in the diverging part of the nozzle (see the discussion of nozzle flow in Section 3.1). If we assume that the shock wave occurs at a certain location of the nozzle where the cross-sectional area is  $A_{sh}$ , then the ratio  $A_{sh}/A_*$  ( $A_*$  = throat area) will determine the Mach number  $M_1$  of the flow just in front of the shock wave. We can then calculate  $M_2$ ,  $p_2$ , and  $p_{t2}$  from the above relations, and from  $M_2$  we can determine  $\Theta_2/\Theta_{*2}$  from Eq. (3.20). By continuity, at a section with cross-sectional area  $A$  downstream of the shock,  $\Theta A = \Theta_2 A_{sh}$ , or

$$\frac{\Theta}{\Theta_{*2}} = \frac{\Theta_2}{\Theta_{*2}} \frac{A_{sh}}{A}.$$

In this way, we therefore obtain the ratio  $\Theta/\Theta_{*2}$  for each cross section  $A$ , and get from Eq. (3.20) the corresponding Mach number  $M < 1$ . With this Mach number, we can calculate the pressure  $p$  at the cross section  $A$  from Eq. (3.14), in which we set  $p_1 = p_{t2}$ . The curves shown in Fig. 22, for example, were calculated in this manner; they give the pressure distribution as a function of  $A/A_*$  for  $\gamma = 7/5$ . The different curves correspond to different positions of the shock wave, i.e., to different pressures at the nozzle exit.

As a further application, let us now consider the following problem: A gas is flowing past a rotationally-symmetric, or, in plane flow, an axisymmetric, solid, blunt-nosed body with constant velocity  $u_\infty$  (and hence constant Mach number  $M_\infty$ ) in the direction of the axis of symmetry (Fig. 43). We shall designate the constant state far upstream, which is undisturbed by the body, by the subscript “ $\infty$ ,” just as we designate the freestream velocity. Upstream of the body, the axis of symmetry is a streamline, while at point  $S$  on the body, the stagnation point, the velocity is zero. When the flow is subsonic or sonic,  $M_\infty \leq 1$ , we can calculate the state variables at the stagnation point

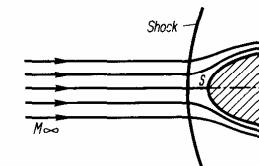


Fig. 43. Supersonic flow past a blunt-nose body with a detached shock.

(denoted by the subscript “ $s$ ”) from the isentropic relations given in Section 3.1, since the quantities at the stagnation point will be identical to the corresponding total or reservoir quantities defined in Section 3.1. Thus, for example, we get from Eq. (3.14),

$$p_s = p_\infty [1 + \frac{1}{2}(\gamma - 1) M_\infty^2]^{1/(\gamma-1)}. \quad (3.134)$$

This changes when the flow around the body has supersonic velocity ( $M_\infty > 1$ ), since then a shock wave will be created in front of the body (Fig. 43), as will be explained later (Section 3.5.1). The flow in front of the shock is completely uninfluenced by the body. On the axis of symmetry, we have a normal shock (since there the streamline is perpendicular to the shock surface) with upstream Mach number  $M_\infty$ . The pressure at the stagnation point is calculated as follows: First we obtain the total pressure  $p_{t\infty}$  of the flow in front of the shock from  $M_\infty$  and  $\rho_\infty$  using Eq. (3.14), and then, using Eq. (3.133), we obtain the total pressure behind the shock, which will be identical to the pressure  $p_s$  at the stagnation point. The result is

$$p_s = p_\infty \left( \frac{\gamma + 1}{2} M_\infty^2 \right)^{\gamma/(\gamma-1)} \left[ 1 + \frac{2\gamma}{\gamma + 1} (M_\infty^2 - 1) \right]^{-1/(\gamma-1)} \quad (3.135)$$

Formulas (3.134) and (3.135) are useful in experimental determinations of the Mach number of a flow. By measuring the pressure  $p_s$  at the stagnation point and simultaneously the pressure  $p_\infty$  far ahead of the body, we can calculate the flow Mach number  $M_\infty$  from these formulas.

In contrast to the total pressure (and total density), the total enthalpy does not change when the gas passes through a shock wave; in a thermally ideal gas, the total temperature also does not change. This is why we can use the particular relations (3.13) given in Section 3.1 for calculating the stagnation-point temperature of a calorically ideal gas:

$$T_s = T_\infty [1 + \frac{1}{2}(\gamma - 1) M_\infty^2]. \quad (3.136)$$

Instead of thinking of a gas flowing with velocity  $u_\infty$  past a body at rest, we can imagine the body to move with velocity  $u_\infty$  in a gas at rest. The relation between the state variables of a gas at the stagnation point and those in the stationary gas far ahead of the moving body (denoted by the subscript “ $\infty$ ”) can again be determined from the above formulas. In particular, we find from Eq. (3.136) that the temperature  $T_s$  of the gas at the stagnation point increases with the flight Mach number  $M_\infty$  (the ratio of flight velocity  $u_\infty$  to the sonic velocity  $a_\infty$  of the undisturbed gas far away from the body). With  $\gamma = 1.4$  and  $M_\infty = 2.24$ , we have  $T_s = 2T_\infty$ . Since at an altitude of 10 km the sound velocity in the atmosphere will be approximately 1100 km/h (corresponding to a temperature of 230 °K there), this means, for example, that at the stagnation point of a flight vehicle with a speed of  $1100 \times 2.24 = 2464$  km/h, a temperature of  $2 \times 230$  °K = 460 °K = 187 °C will occur! Temperatures such as these do not only appear at stagnation points. A more careful examination of the hitherto neglected friction and heat conduction in the boundary layer at the body surface shows that the gas near the entire body surface experiences temperatures of the order of magnitude of the stagnation-point temperature provided that no heat transfer occurs between the body and the gas (thermal steady state, see also Section 4.3.2). Much higher temperatures appear at velocities of importance to space flight (orbit velocity near the earth is 7.9 km/sec = 28,440 km/h, for example). The heating of the body as a result of these high gas temperatures plays a crucial role in the reentry of space vehicles into the upper atmosphere. On the other hand, we can no longer regard air at high temperatures as a calorically ideal gas, so that formula (3.136) is not applicable. As already explained in Section 1.5, the specific heat of the gas rises appreciably at higher temperatures, which results

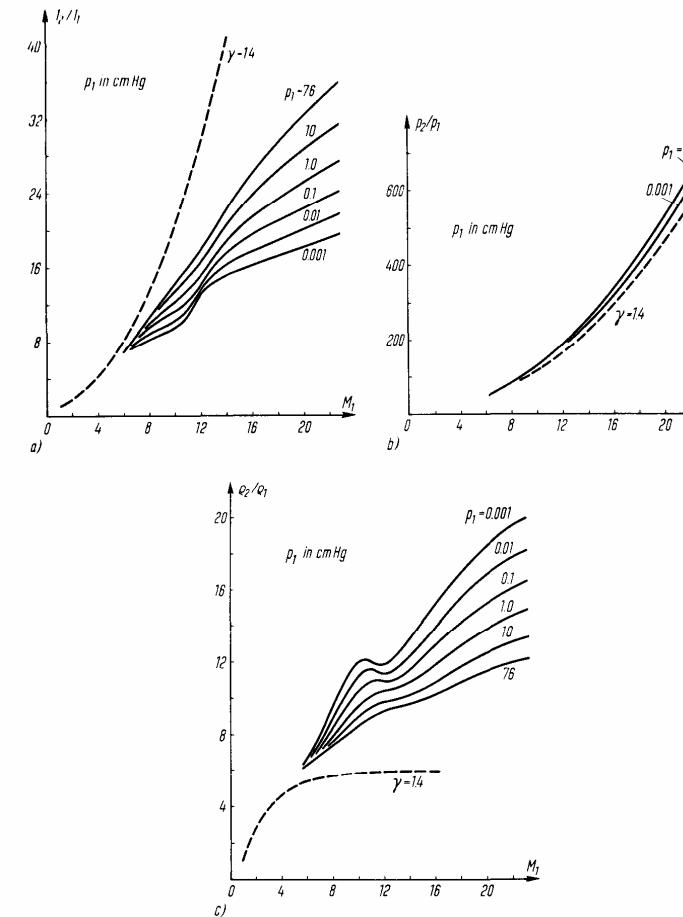


Fig. 44. Temperature, pressure, and density behind a normal shock wave in air as a function of the Mach number  $M_1$  and the pressure  $p_1$  in front of the shock.  $T_1 = 300$  °K. Comparison with a calorically ideal gas with  $\gamma = 1.4$ . (a) Temperature, (b) pressure, (c) density.

in lower temperatures behind the shock than those calculated for a calorically ideal gas. Therefore, it is particularly important in calculating this heating process that we study the normal shock relations for real gases, of the type shown in Fig. 42 for a calorically ideal gas.

Figure 44 contains the important relations for air based on the calculations of S. Feldman. In particular, we can see from the density ratio curve how the departure from the behavior of a calorically ideal gas with  $\gamma = 1.4$  is caused by three different processes: At relatively small Mach numbers  $M_1$  and thus relatively low temperatures  $T_2$ , molecular vibrations first become noticeable behind the shock, these vibrations being unexcited at the initial temperature of 300°K. This constitutes the first departure from the curve for  $\gamma = 1.4$ , and it is independent of pressure  $p_1$ . At higher temperatures, dissociation sets in, which is strongly pressure-dependent (see Section 1.10). Finally, at Mach numbers of the order of 20 or higher, the gas ionizes because of the high temperature  $T_2$ , resulting in yet another departure from the ideal gas curve. Moreover, the ionized gas becomes electrically conducting, which may permit the flow processes to be influenced by electromagnetic forces. These phenomena form the subject of magnetogasdynamics, which is beyond the scope of the present work.

#### 3.4.5 SHOCK TUBE

For experimental gas dynamics and for many physical-chemical studies of high-temperature gas properties, such as those involving relaxation processes, etc., a device known as a shock tube is of great value.<sup>30</sup> In its simplest form, the shock tube is a cylindrical tube which is partitioned by a membrane (uppermost part of Fig. 45). To the right of the membrane is a stationary gas at pressure  $p_1$ , and to the left, a stationary gas at pressure  $p_4 > p_1$ . At time  $t = 0$ , the membrane is instantaneously removed—for example, by puncturing it with a needle, so that the pressure difference causes the membrane to burst. For  $t > 0$ , the initial pressure difference will then tend to equalize through a flow shown in Fig. 45: In the low-pressure stationary gas, a shock wave  $S$

<sup>30</sup> I.I. Glass, W. Martin, and G.N. Patterson, A Theoretical and Experimental Study of the Shock Tube. University of Toronto, Institute of Aerophysics (UTIA) Rep. No. 2, Toronto, Canada, 1953. A. Ferri (ed.), Fundamental data obtained from shock tube experiments. AGARDograph No. 41, Oxford-London-New York-Paris, 1961. J.K. Wright, Shock tubes, London-New York, 1961.

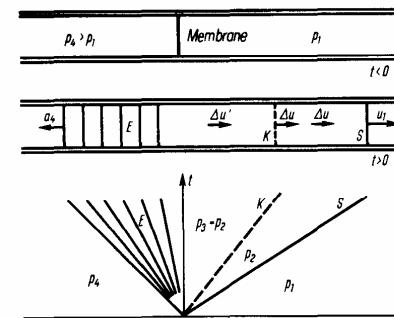


Fig. 45. Flow in a shock tube.

propagates to the right with velocity  $u_1$ , raising the pressure from  $p_1$  to  $p_2$  and generating a velocity  $\Delta u$ . In the high-pressure stationary gas, a centered expansion wave  $E$  propagates to the left, reducing the pressure from  $p_4$  to  $p_3$  and generating a velocity  $\Delta u'$ . At the contact surface  $K$ , the original high-pressure gas and the original low-pressure gas adjoin each other. At the location of the contact surface, we can imagine an infinitely thin piston moving to the right and producing exactly the same flow with the same shock and expansion wave. On purely kinematic grounds, the gas velocity on the right and on the left of the contact surface must be the same ( $\Delta u' = \Delta u$ ). The same must be true for the pressure ( $p_3 = p_2$ ), since a pressure discontinuity (i.e., shock) cannot be convected with the flow (since it must propagate with supersonic velocity relative to the stream!). However, all the other thermodynamic quantities can be discontinuous across the contact surface  $K$ , and it is therefore also called a “contact discontinuity.”

For further discussion, let us confine ourselves to calorically ideal gases. Let the adiabatic coefficient and sound velocity in the stationary low-pressure gas be  $\gamma_1$  and  $a_1$ , respectively, and those in the high pressure gas  $\gamma_4$  and  $a_4$ , respectively. (Since we can select two different gases on the two sides of the membrane, we generally have  $\gamma_1 \neq \gamma_4$ ; also, even in the same gas,  $a_1 \neq a_4$  is possible when the initial temperatures  $T_1$  and  $T_4$  are different.) We introduce the shock Mach number  $M_1 = u_1/a_1$  as a parameter. The pressure increase across the shock is then given by Eq. (3.124), while the density increase and temperature increase are given by Eq. (3.123) and (3.125),

respectively. The gas velocity  $\Delta u$  behind the shock is determined from Eq. (3.130). On the other hand, for the expansion wave  $E$ ,

$$\frac{p_3}{p_4} = \left( 1 - \frac{\gamma_4 - 1}{2} \frac{\Delta u'}{a_4} \right)^{2\gamma_4/(\gamma_4 - 1)}. \quad (3.137)$$

This formula is obtained by eliminating  $M$  from Eqs. (3.52) and (3.55) and substituting in this case  $\Delta u'$  for  $\Delta u$ ,  $p_4$  for  $p_0$ , and  $a_4$  for  $a_0$ . Now we set  $\Delta u = \Delta u'$  and  $p_3 = p_2$ ; dividing Eq. (3.124) by Eq. (3.137), we obtain

$$\frac{p_4}{p_1} = \frac{2\gamma_1 M_1^2 - (\gamma_1 - 1)}{\gamma_1 + 1} \left[ 1 - \frac{\gamma_4 - 1}{\gamma_1 + 1} \frac{a_1}{a_4} \left( M_1 - \frac{1}{M_1} \right) \right]^{-2\gamma_4/(\gamma_4 - 1)}. \quad (3.138)$$

From this equation, we can calculate for each shock Mach number  $M_1$  the required pressure ratio  $p_4/p_1$ ; conversely, for a given pressure ratio, we can find the resulting Mach number  $M_1$ , from which all the other quantities in the shock tube can be calculated. Formula (3.138) shows that for a given pressure ratio  $p_4/p_1$ , the larger  $a_4/a_1$ , the larger the attainable shock Mach number  $M_1$ . To generate strong shock waves, it is therefore advantageous to use for the high-pressure gas (the driver gas) a gas with the smallest possible molecular weight (e.g., helium), since, in such a case, the specific gas constant  $R = \mathfrak{R}/M$  will be large, and, by formula (1.135), so will the sound velocity at a given temperature. Naturally, we can also increase the sound velocity  $a_4$  by heating the high-pressure gas. In Fig. 46, the pressure

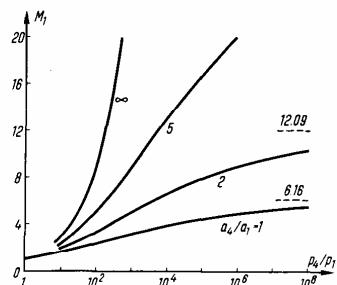


Fig. 46. Shock Mach number  $M_1$  in a shock tube as a function of the pressure ratio  $p_4/p_1$  and the ratio of the sound velocities  $a_4/a_1$ ; calorically ideal gas with  $\gamma = 1.4$ .

ratio  $p_4/p_1$  is shown as a function of the Mach number  $M_1$ , with  $a_4/a_1$  as parameter and  $\gamma_1 = \gamma_4 = 7/5$ .

The main purpose of a shock tube is generally to produce a slug of very hot gas (temperature  $T_2$ ) between the shock  $S$  and the contact surface  $K$ . This temperature can be increased even further in a very simple manner, just by making the right end of the tube rigidly closed. The incident shock will then reflect at the closed end, and the reflected shock runs toward the left. The strength of the reflected shock is just great enough to stop the gas motion which has been caused by the incident shock, since the gas cannot flow into or out of the closed end. We denote by  $M_r$  the Mach number of the reflected shock relative to the gas motion resulting from the incident shock. Applying Eq. (3.130) to the reflected shock, we get for the resulting velocity  $\Delta u_r$  relative to the moving gas behind the incident shock the value

$$\Delta u_r = \frac{2a_2}{\gamma_1 + 1} \left( M_r - \frac{1}{M_r} \right).$$

Since  $\Delta u_r = \Delta u$  must hold, we get, after using  $\Delta u$  from (3.130),

$$\left( M_1 - \frac{1}{M_1} \right) \frac{a_1}{a_2} = M_r - \frac{1}{M_r}. \quad (3.139)$$

This determines  $M_r$  for every  $M_1$ , and  $M_r$  in turn determines all the properties of the reflected shock.

The shock tube can be modified in various ways. If, for example, we adjoin the tube at the right end to a Laval nozzle, the throat cross section of which is very small in comparison to the tube cross section, then the incident shock will reflect from the nozzle in practically the same way as from a closed end. Following the reflection, we have a practically stationary gas of very high temperature in front of the Laval nozzle and flowing through it. The combination of a shock tube and a Laval nozzle is then a simple blowdown tunnel of the type already mentioned at the end of Section 3.1. The time usable for aerodynamic measurements is of course limited to the time during which the flow through the nozzle is steady, i.e., the time between the arrival at the nozzle of the first incident shock  $S$  and the arrival of the disturbance  $S'$  resulting from the interaction of the reflected shock with the contact surface (Fig. 47). This type of apparatus plays a great role in studying the processes for high-speed flight, for which very high stagnation temperatures result. If

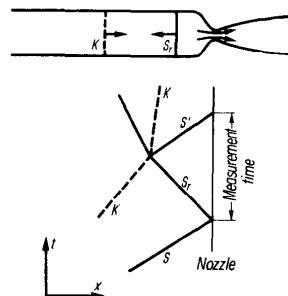


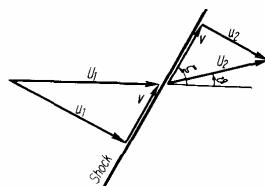
Fig. 47. Reflection of a shock wave in a shock tube.

we are to study these processes in a wind tunnel, then the reservoir temperature of the gas must be very high. In the device discussed here, this heating is achieved by the incident and reflected shock waves.

### 3.5 Oblique Shocks and Mach Waves in Steady Supersonic Flow

#### 3.5.1 OBLIQUE SHOCKS

Let there be a normal shock in a steady flow, as shown in Fig. 40. We now take the point of view of an observer moving with constant velocity  $-v$  perpendicular to the flow, i.e., along the shock. As seen by this observer, the gas enters the shock with velocity  $U_1$  and leaves it with velocity  $U_2$  (Fig. 48; for the sake of convenience, we have here changed the direction of the flow

Fig. 48. Derivation of an oblique shock wave from a normal shock by translating the coordinate system at the velocity  $v$ .

given in Fig. 40). The shock wave forms an angle  $\zeta$  with the upstream flow, with

$$\tan \zeta = u_1/v. \quad (3.140)$$

The downstream flow is deflected at an angle  $\theta$  from the upstream flow:

$$\tan(\zeta - \theta) = u_2/v. \quad (3.141)$$

The normal shock has become an oblique shock in the coordinate system moving with velocity  $-v$ . In this way, we can associate with each normal shock a family of oblique shocks by superimposing on the normal shock different velocities  $v$ .

The shock relations (3.107)–(3.109) continue to be valid, since they are invariant with respect to a uniform translation of the coordinate system (Galilean transformation). Therefore, all the consequences of these relations also hold as before; in particular, for example, formulas (3.111), (3.112), and (3.117)–(3.121) all hold. But, in (3.123)–(3.125) and (3.133), we must remember that the Mach number  $M_1$  is the ratio  $u_1/a_1$ ; with oblique shocks, it is more convenient to define  $M_1$  to be  $U_1/a_1$ , so that  $u_1/a_1 = M_1 \sin \zeta$ . This means that formulas (3.123)–(3.125) and (3.133) also hold for oblique shocks if  $M_1$  is replaced by  $M_1 \sin \zeta$ . In particular, we have:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_1^2 \sin^2 \zeta}{2 + (\gamma - 1) M_1^2 \sin^2 \zeta}, \quad (3.142)$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 \sin^2 \zeta - (\gamma - 1)}{\gamma + 1}, \quad (3.143)$$

$$\frac{T_2}{T_1} = \frac{a_2^2}{a_1^2} = \frac{[2\gamma M_1^2 \sin^2 \zeta - (\gamma - 1)][2 + (\gamma - 1) M_1^2 \sin^2 \zeta]}{(\gamma + 1)^2 M_1^2 \sin^2 \zeta}. \quad (3.144)$$

Since for a shock wave we must always have  $u_1 > a_1$ , then, for a given Mach number  $M_1$ , we can only have shock angles  $\zeta$  for which  $\zeta > \mu$ , where  $\mu = \text{arc sin } M_1^{-1}$  is the Mach angle corresponding to the Mach number  $M_1$ .  $\mu$  is the limiting value of the shock angle for infinitesimally weak shocks:

$$\lim_{p_2/p_1 \rightarrow 1} \zeta = \mu$$

The Mach number  $M_2$  in the relations for normal shocks must be replaced by  $M_2 \sin(\zeta - \theta)$  for oblique shocks; then, they continue to hold for oblique

shocks, and  $M_2$  now means  $U_2/a_2$ . In particular, from (3.126), which is valid for calorically ideal gases, we have

$$M_2^2 \sin^2(\zeta - \theta) = \frac{\gamma + 1 + (\gamma - 1)(M_1^2 \sin^2 \zeta - 1)}{\gamma + 1 + 2\gamma(M_1^2 \sin^2 \zeta - 1)}. \quad (3.145)$$

A relation between the deflection angle  $\theta$  and shock angle  $\zeta$  is obtained as follows: From Eqs. (3.140), (3.141), and (3.107), it follows that

$$\frac{\tan \zeta}{\tan(\zeta - \theta)} = \frac{u_1}{u_2} = \frac{\rho_2}{\rho_1}, \quad (3.146)$$

which becomes, after trigonometric transformations,

$$\tan \theta = \frac{(\rho_2/\rho_1) - 1}{(\rho_2/\rho_1) + \tan^2 \zeta} \tan \zeta. \quad (3.147)$$

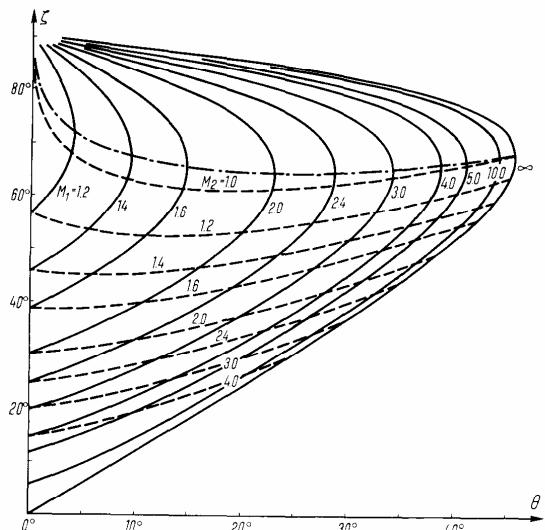


Fig. 49. Relation between shock angle  $\zeta$  and deflection angle  $\theta$  for an oblique shock; calorically ideal gas with  $\gamma = 1.4$ .

For the special case of calorically ideal gases, we can use (3.142), and we get

$$\tan \theta = \frac{M_1^2 \cos^2 \zeta - \cot^2 \zeta}{1 + \frac{1}{2} M_1^2 (\gamma + \cos 2\zeta)} \tan \zeta. \quad (3.148)$$

Shown in Fig. 49 is the relation between  $\theta$  and  $\zeta$  according to (3.148), with the Mach number  $M_1$  as a parameter (and  $\gamma = 1.4$ ). The deflection angle  $\theta$  is zero for  $\zeta = \mu$  (weak shock limit) and for  $\zeta = 90^\circ$  (normal shock) while for values in between, there is one maximum deflection  $\theta_{\max}(M_1)$  corresponding to each upstream Mach number  $M_1$ . The curve connecting the points of maximum deflection is drawn in Fig. 49 as a dot-dash line; it differs only slightly from the curve  $M_2=1$ , which connects all the points at which the flow behind the shock is exactly sonic. To find the thermodynamic quantities behind an oblique shock, we can use Fig. 42 for normal shocks, in which  $M_1 \sin \zeta$  must be substituted for  $M_1$  for the abscissa. This quantity can be first found from Fig. 50 for a given upstream Mach number  $M_1$  and a given deflection (for calorically ideal gases with  $\gamma = 1.4$ ). Using Figs. 50 and 42, we can readily find the ratios  $p_2/p_1$ ,  $\rho_2/\rho_1$ ,  $T_2/T_1$ , and  $p_{t2}/p_{t1}$  as functions of  $M_1$

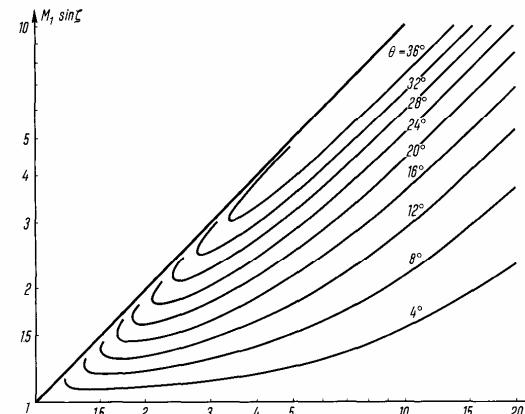


Fig. 50. The quantity  $M_1 \sin \zeta$ , which determines thermodynamic changes of state in an oblique shock, as a function of the Mach number  $M_1$  ahead of the shock and the deflection angle  $\theta$ ; calorically ideal gas with  $\gamma = 1.4$ .

## 3 Inviscid Flows

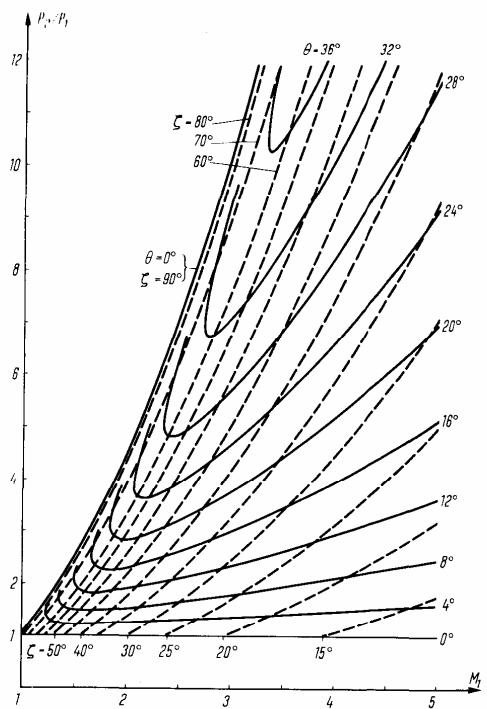


Fig. 51. Pressure ratio for an oblique shock as a function of the Mach number  $M_1$  ahead of the shock and the deflection angle  $\theta$  or the shock angle  $\zeta$ ; calorically ideal gas with  $\gamma=1.4$ . (From L. Rosenhead (ed.), A selection of Graphs for Use in Calculations of Compressible Airflow, Oxford, 1954.)

and  $\theta$ . As an example, Fig. 51 shows  $p_2/p_1$  as a function of  $M_1$  and  $\theta$  (or  $\zeta$ ).

These results permit us to determine in a simple fashion the symmetric two-dimensional flow with uniform supersonic upstream velocity ( $M_1 > 1$ ) past a wedge-shaped body with vertex angle  $2\theta$  (Fig. 52). Two oblique shocks issue from the vertex, each toward one side of the wedge and each deflecting the flow to an angle  $\theta$  with respect to the free stream. The shock

## 3.5 Oblique Shocks and Mach Waves in Steady Supersonic Flow

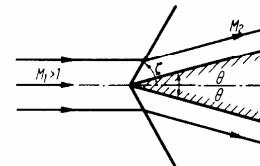


Fig. 52. Supersonic flow past a wedge.

directed to the right of the flow direction (lower part of Fig. 52) will be called "right-running" and the other shock "left-running." In front of the shocks, the flow is completely undisturbed by the wedge. The shock angle  $\zeta$  and the uniform flow states behind the shocks can be found from Figs. 49–51 for calorically ideal gases. Two completely different flow patterns result, depending on whether the wedge angle  $\theta$  is smaller or greater than the maximum possible deflection  $\theta_{\max}$  corresponding to the particular  $M_1$ :

1. When the wedge angle is  $\theta < \theta_{\max}$ , then, according to Fig. 49, there are two shock angles  $\zeta$  for each given  $\theta$ , i.e., the solutions can have either a strong shock or a weak shock. We call an oblique shock a strong shock if  $\zeta$  is greater than the value of  $\zeta$  corresponding to the maximum deflection  $\theta_{\max}$ , and a weak shock for the opposite case. Since in Fig. 49 the curves  $\theta = \theta_{\max}$  and  $M_2 = 1$  almost coincide, a weak shock will generally have supersonic flow behind it, while a strong shock will always have subsonic flow behind it. Which of these two solutions occurs in reality can only be decided experimentally, and, indeed, experiments show that, but for a few exceptions, the weak-shock solution is generally realized. Therefore, we shall always assume a weak shock from now on. Moreover, it is not necessary that the wedge be symmetric to the direction of flow. If we place the axis of the wedge at a certain angle to the flow (Fig. 53), then on the upper surface a shock will

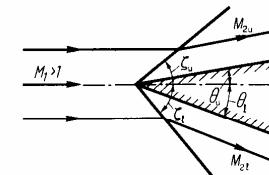


Fig. 53. Supersonic flow past a wedge at an angle of attack.

appear corresponding to the deflection  $\theta_u$ , and on the lower surface a shock corresponding to  $\theta_l$ . We assume that both  $\theta_u$  and  $\theta_l$  are smaller than  $\theta_{\max}$ . The flows on the upper and lower surfaces of the wedge are completely independent of one another.

Of course, a wedge extending to infinity to the right cannot be realized in practice, since a real wedge must somehow be terminated—for example, as is shown in Fig. 54 (double wedge). In the case considered with  $\theta < \theta_{\max}$ , the

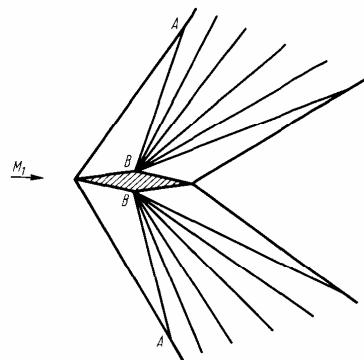


Fig. 54. Supersonic flow past a double-wedge profile; attached shocks.

flow near the vertex is actually observed to be the same as that described before for the infinite wedge. To be sure, the shock waves emanating from the vertex extend outward with constant strength only to the points *A* where they intersect the expansion waves emanating from the corners *B* (Prandtl-Meyer flow, see Section 3.6) and thereby become weakened; this process will be studied in greater detail later (Section 3.6.2). In these considerations, we tacitly assume that the flows behind the weak shocks from the vertex are supersonic ( $M_2 > 1$ ), so that the flows in the triangles formed by the vertex, points *A*, and points *B* are uniform. If we had chosen an upstream Mach number  $M_1$  and wedge angle  $\theta$  in such a manner that the *weak* shock solution in Fig. 49 corresponds to a point between the curves  $M_2 = 1$  and  $\theta = \theta_{\max}$ , then we would have subsonic flow behind the shocks and no uniform flows anywhere in that region.

2. If the vertex angle of the wedge is  $\theta > \theta_{\max}$ , then in each case we no

longer observe two attached shock waves pointing to the two sides of the wedge from the vertex, but instead a *detached shock wave* (Fig. 55). This detached shock is curved, i.e., the shock angle  $\zeta$  varies along the shock, and the flow behind the shock is not uniform. On the line of symmetry of the flow,  $\zeta = 90^\circ$ , so that the shock is normal. Away from this line of symmetry, the shock angle  $\zeta$  decreases monotonically, and at sufficiently large distances away, it tends to the Mach angle  $\mu$ , corresponding to  $M_1$  (the latter also obtains for the attached shock in Fig. 54 after it becomes weak and curved as result of the interaction with the expansion wave from the point *A* onwards).

A curved shock cannot be derived from a normal shock by a translation of the coordinate system parallel to the shock, as was done for a plane oblique shock. The question therefore arises: what are the shock relations for a curved shock? Moreover, we would like to be able to deal with flows which are variable in space in front of the shock, rather than uniform, as in

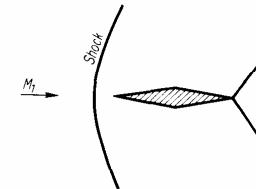


Fig. 55. Supersonic flow past a double-wedge profile; the bow shock is detached.

the example discussed above. Applying the conservation laws for mass, momentum, and energy to an appropriate control volume, we conclude, after considerations similar to those used in Section 3.4.1 for unsteady shocks, that the shock relations obtained before (in which  $M_1$  and  $\zeta$  are now local, position-dependent quantities) also hold for curved shocks provided that the thickness of the shock is small in comparison with the characteristic dimensions over which the flow changes significantly, both in front and behind the shock. These assumptions are almost always valid, and we can treat the shock as a discontinuity surface in an inviscid flow across which the shock relations hold. Thus far, we have only considered plane steady flow or one-dimensional unsteady flow with shocks. But it is clear that the shock relations also hold for three-dimensional unsteady flows when the above requirements and those of Section 3.4.1 are satisfied.

While for wedge-shaped, pointed bodies in supersonic flow either attached or detached shocks can appear, depending on the Mach number and wedge angle, for a blunt-nosed body in supersonic flow only detached shocks are possible. In this sense, a wedge in supersonic flow is a blunt body when  $\theta > \theta_{\max}$ . It is not difficult in principle to calculate the plane flow-field past a two-dimensional body if only attached shocks appear and there are no regions of subsonic flow. On the other hand, such a calculation is difficult if detached shocks and subsonic regions do appear, unless certain simplifying assumptions can be made (e.g., Newtonian theory in hypersonic flow; see the following remarks and Section 3.7.).

We add the following to the discussion of the flow past a wedge: Let  $\theta \ll 1$  (in radians) and  $\tan \theta \approx \theta$ . Since for an attached shock we must always have  $\zeta > \theta$ , then, for fixed  $\theta$  and increasing upstream Mach number  $M_1$ , we will eventually have  $M_1 \sin \zeta \gg 1$ . Flows with upstream Mach number  $M_1 \gg 1$  are in general called *hypersonic flows*. In hypersonic flow past a wedge, the condition  $M_1 \sin \zeta \gg 1$  need not hold, even when  $M_1 \gg 1$ , if  $\zeta$  is sufficiently small and therefore the nose shocks are sufficiently weak (weak-shock assumption). To differentiate it from the general concept of hypersonic flow defined above, we shall call those flows for which  $M_1 \theta \approx M_1 \sin \theta \gg 1$  *strong hypersonic flows*. Since  $\zeta > \theta$ , we have  $M_1 \sin \zeta \gg 1$  for strong hypersonic flows, and the density ratio across the shock becomes, at least in a calorically ideal gas, independent of the Mach number:  $\rho_2/\rho_1 = (\rho_2/\rho_1)_{\max}$ . Considering this fact and the assumption that the weaker of the two possible shocks appears, we get, from Eq. (3.147),

$$\zeta = \frac{(\rho_2/\rho_1)_{\max}}{(\rho_2/\rho_1)_{\max} - 1} \theta = \frac{\gamma + 1}{2} \theta \quad (3.149)$$

(where  $\tan^2 \zeta$  has been neglected in comparison to  $\rho_2/\rho_1$ ). The shock angle  $\zeta$  and the wedge angle  $\theta$  are thus, for high enough Mach numbers, proportional to each other, with the proportionality constant independent of the Mach number and determined only by the thermodynamic properties of the gas (by  $\gamma$ ). This results in the fact that the streamline patterns become completely independent of the Mach number; to be sure, changing the freestream Mach number changes the thermodynamic state at the various points of the flow field, but the purely geometrical features of the flow field no longer change. Since the density ratio  $\rho_2/\rho_1$  has also become independent of Mach number [ $=(\rho_2/\rho_1)_{\max}$ ], this means that the ratios of the velocities behind the

shock to the freestream velocity are also independent of the Mach number. The greater the attainable compression in the shock (i.e., the smaller the  $\gamma$  of the calorically ideal gas), the closer the shock lies to the surface of the wedge, as can be seen in (3.149).<sup>31</sup> For  $\gamma \rightarrow 1$ , i.e.,  $(\rho_2/\rho_1)_{\max} \rightarrow \infty$ , the shock coincides with the wedge surface, and the gas flows in an infinitesimally thin layer with infinite density along the surface of the wedge. This agrees with the model which Newton postulated for the flow past a solid body: He assumed that the flowing gas consists of particles which impinge on the wedge surface with the flow velocity and then lose their velocities normal to the wedge surface and glide along the surface with their tangential

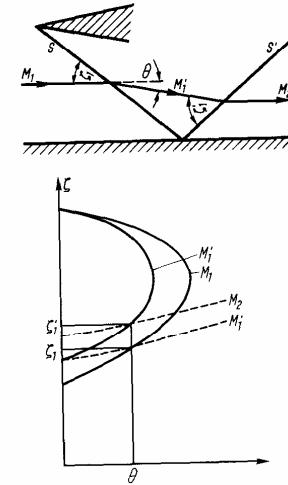


Fig. 56. Top: Reflection of an oblique shock  $S$  on a solid wall. Below: Representation of the reflection in the  $\theta$ ,  $\zeta$  plane.

<sup>31</sup> These ideas are valid not only for the flow past a wedge, but, with obvious extensions, for flow past other bodies as well. In particular, in the flow past a blunt-nose body (Fig. 43), the distance between the detached shock and the body (standoff distance) is independent of the upstream Mach number when this Mach number is sufficiently high. The distance is then dependent only on  $\gamma$  for a calorically ideal gas, and the closer  $\gamma$  is to 1, the smaller this distance will be.

velocities. Of course, this Newtonian theory agrees at best only in hypersonic flows approximately with the reality; for flows with smaller Mach numbers, particularly subsonic flows, the real flow pattern is significantly different from the Newtonian model.

Let us now direct our attention to the reflection of a shock from a solid wall: We imagine a weak shock  $S$  (in the sense defined on p. 141) formed by a wedge in a uniform supersonic flow (of Mach number  $M_1$ ) (see Fig. 56). The flow behind the shock does not satisfy the condition that the flow direction be tangential to the wall. In many cases, however, this boundary condition can be satisfied by the assumption of a reflected shock  $S'$ , which straightens out the flow previously deflected by the shock  $S$ . There are again two possibilities for the reflected shock: The shock can be either a strong or a weak shock. Usually, for a weak incident shock, we also observe a weak reflected shock. If we now fix the Mach number  $M_1$  and enlarge the wedge angle  $\theta$ , the shock  $S$  then becomes stronger, and the Mach number  $M_1'$  behind the shock decreases. For a sufficiently large wedge angle, the maximum deflection corresponding to  $M_1'$  will finally become smaller than that required to satisfy the wall boundary condition, and the wall boundary condition can no longer be satisfied by a normal reflection of the shock, as shown in Fig. 56. Experiments show that a shock configuration of the form

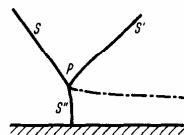


Fig. 57. Mach reflection of an oblique shock  $S$  on a solid wall.

shown in Fig. 57 then appears. Near the wall, we have an approximately normal shock  $S''$  intersecting the incident shock at the point  $P$ . From there, an oblique reflected shock  $S'$  also emanates, which, just like  $S''$ , is also not of constant strength, but is curved near the point  $P$ . Also emanating from there is a contact surface (the dot-dash line in Fig. 57), across which the tangential component of the velocity is discontinuous. We shall go into greater detail concerning these points later. This type of reflection is called Mach reflection.

The reflection of an oblique shock on a plane wall can also be regarded



as the intersection of two oblique shocks of equal strengths (Fig. 58). The flow is symmetric, and the line of symmetry, being a streamline, can be replaced by a plane wall, whereupon we return to the reflection just discussed. The intersection of two shocks of unequal strengths (Fig. 59) can also be treated by the theory of oblique shocks explained before. We must determine

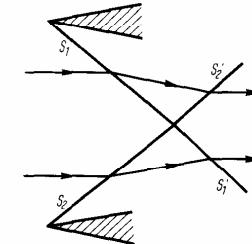


Fig. 58. Intersection of two oblique shocks  $S_1$  and  $S_2$  of equal strengths.

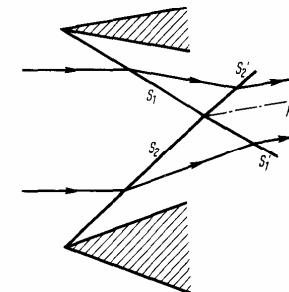


Fig. 59. Intersection of two oblique shocks  $S_1$  and  $S_2$  of different strengths.

the shocks  $S_1'$  and  $S_2'$  such that behind both shocks the same flow direction and the same pressure occur. While in the intersection of two equal shocks the state of the gas in the entire region downstream of  $S_1'$  and  $S_2'$  is uniform and the flow direction is the same as that upstream of  $S_1$  and  $S_2$ , this is no longer so in the case of two unequal shocks. For that case, the streamline through the intersection point (indicated by the dot-dash curve in Fig. 59)

now separates two regions of uniform states in which the same pressure occurs, but all other thermodynamic variables and, in particular, the velocity are different. The streamline separating the two regions is thus a contact discontinuity, similar to that which appeared, for example, in the shock-tube flow problem (see Section 3.4.5). In the present case, the contact discontinuity has in addition the role of a vortex sheet, i.e., the tangential velocity along this discontinuity surface changes discontinuously. Another such discontinuity also appears in Mach reflection, with the added complication that the states downstream of  $S''$  and  $S'$  are not constant on either side of the contact surface. The contact surface is thus itself curved, as are the shocks. The theory of this particular phenomenon is therefore difficult. It is self-evident that in the intersection of two shocks, the phenomenon corresponding to Mach reflection can also appear.

In the flow sketched in Fig. 60, two shocks  $S_1$  and  $S_2$  emanating from the

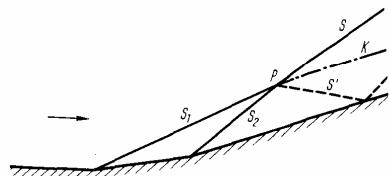


Fig. 60. Merging of two oblique shocks  $S_1$  and  $S_2$  into a single shock  $S$ .

two corners of a wall combine into a single shock  $S$  at the point  $P$ . At the same time, there issues from  $P$  a right-running wave  $S'$ , which need not always be a compression shock of the type hitherto considered (although it will be a shock if  $S_2$  is much weaker than  $S_1$ ), but can also be an expansion wave (Prandtl-Meyer wave, Section 3.6). The strengths of  $S$  and  $S'$  are again determined by the condition that along the streamline  $K$  through  $P$  (which is again a contact discontinuity), the pressures and flow directions must be equal on the two sides. If the shocks  $S_1$  and  $S_2$  are so weak that entropy changes across them are negligible, then we can also neglect the entropy changes across the shock  $S$  and the wave  $S'$ . The flow is then everywhere isoenergetic and homentropic, so that the vortex sheet from the intersection point vanishes (see also Section 2.5), i.e., the flow velocities will be the same on the two sides of the sheet. We assert without proof that, in the same

approximation, the reflected wave  $S'$  may be neglected. Since the entropy rise across a shock increases as the third power of the shock strength [expressed, e.g., as  $\Delta\varrho/\varrho_1 = (\varrho_2 - \varrho_1)/\varrho_1$ ], we can concisely state: Up to terms of the third order in the shock strength, the reflected wave and the vortex sheet may be neglected.

*Supplementary Remarks.* The symmetric supersonic flow past a wedge is related to the axisymmetric supersonic flow past a circular cone.<sup>32</sup> Since for an infinite cone the flow has no characteristic length, the flow state must be constant on each straight line emanating from the vertex of the cone (conical flow). Because of axisymmetry, the flow state must then be constant on each circular cone having the same vertex and axis as the given cone. In particular,

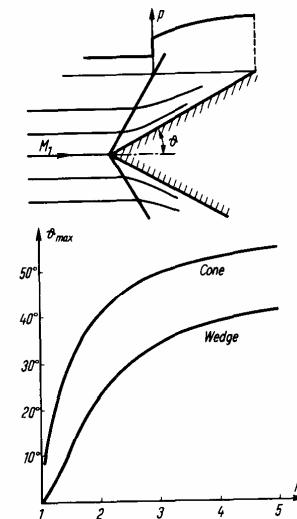


Fig. 61. Axisymmetric supersonic flow past a circular cone. (From L. Rosenhead (ed.), A Selection of Graphs for Use in Calculations of Compressible Airflow. Oxford, 1954)

<sup>32</sup> G. I. Taylor and J. W. MacColl, The air pressure on a cone moving at high speed, Proc. Roy. Soc. A **139**, 278 (1933). Z. Kopal, Tables of Supersonic Flow Around Cones M.I.T. Center of Analysis, Tech. Rep. No. 1, 1947.

the attached shock must be one of these cones, so that this shock must be of constant strength (Fig. 61). Also, the flow state on the surface of the given cone must be constant. In contrast to the analogous flow past a wedge, however, we do not have a constant state in the region between the shock and the cone. Through the shock the pressure will increase to a fraction of its value on the surface of the cone, and the gas is compressed isentropically behind the shock. We shall not go into the detailed results of the theory, except to show one important result in Fig. 61 for a calorically ideal gas with  $\gamma=1.4$ . Just as in wedge flow, there is for each upstream Mach number a maximum vertex angle  $\theta_{\max}$  for the cone beyond which an attached shock wave is no longer possible. Since the flow behind the shock does not immediately attain the asymptotic flow direction given by the vertex angle  $\theta$ , the shock is weaker than the corresponding one for a wedge of the same vertex angle, and the maximum possible vertex angle  $\theta_{\max}$  is accordingly higher for the cone. For angles  $\theta > \theta_{\max}$ , we have a detached shock, just as in the case of the wedge. In actuality, the distance of this shock from the vertex will be determined by the finite length of the cone. This length is a characteristic length of the flow field, so that the flow field is no longer conical in the sense defined above.

### 3.5.2 MACH WAVES

In Section 3.5.1, the relations for oblique shocks were derived from those for normal shocks by a constant velocity translation parallel to the discontinuity surface. Instead of a shock let us now consider a discontinuous wave of the type sketched in Fig. 25 (with  $\delta t = 0$ ), which we can assume to be either a compression or expansion wave, provided it is sufficiently weak such that linearization of the equations of motion (as in Section 3.2.1) is valid. The relations of Section 3.2.1 as well as those to be derived below strictly hold only in the limit of infinitesimal waves. In a coordinate system moving with the wave, i.e., with sound velocity, the flow will be stationary; superposing a velocity  $v$  parallel to the discontinuity surface, we obtain a stationary Mach wave inclined to the flow direction at the Mach angle  $\mu$ . From Fig. 62, we read off the relations corresponding to (3.140) and (3.141) for such waves:

$$\tan \mu = a/v; \quad \tan(\mu - \delta\theta) = (a + \delta u)/v. \quad (3.150)$$

Here,  $a$  is the sound speed of the gas in front of the wave and is equal to

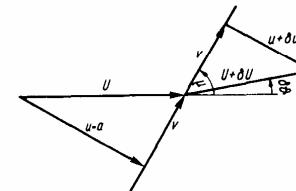


Fig. 62. Diagram for the derivation of formula (3.151) for Mach waves.

the normal component of the flow velocity  $U$ ;  $\delta u$  is the change in the velocity component normal to the wave; and  $\delta\theta$  is the angle of deflection of the flow direction, taken as positive for a flow deflection to the left. Moreover, we use the notation of right- and left-running waves in exactly the same fashion as for oblique shocks (see Section 3.5.1).

From Eq. (3.150), we obtain, by limiting ourselves to terms linear in  $\delta\theta$  and using  $\sin \mu = M^{-1} = a/U$ ,

$$\frac{\delta u}{a} = -\delta\theta \left( \tan \mu + \frac{1}{\tan \mu} \right) = -\delta\theta \frac{M^2}{(M^2 - 1)^{\frac{1}{2}}}. \quad (3.151)$$

Since  $U^2 = u^2 + v^2$ , or  $U \delta U = u \delta u = a \delta u$  (since  $\delta v = 0$  and  $u = a$ ), Eq. (3.151) can also be written as

$$\delta U/U = -\delta\theta (M^2 - 1)^{-\frac{1}{2}}, \quad (3.152)$$

(where  $U$  is the magnitude of the velocity vector). Now, considering the relation (3.35) between  $\delta u$  and the density change  $\delta\varrho$  (altered slightly by taking into account a minus sign, as explained below) we obtain from Eq. (3.151)

$$\frac{\delta\varrho}{\varrho} = \delta\theta \frac{M^2}{(M^2 - 1)^{\frac{1}{2}}} = \frac{1}{a^2} \frac{\delta p}{\varrho}, \quad (3.153)$$

where  $\varrho$  is the density in front of the wave and  $\delta p$  is the pressure change. A minus sign has been introduced because (3.35) is valid for a nonstationary right-running wave. Figure 62, however, is based on a nonstationary left-running wave, for which Eq. (3.35) holds with a minus sign. As explained above, this nonstationary left-running wave is first made stationary by introducing a coordinate system moving with wave velocity; finally, adding the velocity  $v$  in the suitable direction leads to the left-running wave shown in Fig. 62. For this wave, relations (3.151)–(3.153) [and also

(3.154)] obtain. By superposing the velocity  $v$  in the opposite direction, one can, in the same manner, deduce a stationary right-running wave from a nonstationary left-running wave; for such a wave, we must replace  $\delta\theta$  by  $-\delta\theta$  in formulas (3.151)–(3.153), as well as in (3.154). Since  $M^2 = U^2/a^2$ , we obtain from Eq. (3.153)

$$\delta p = \varrho U^2 (M^2 - 1)^{-\frac{1}{2}} \delta\theta = -\varrho U \delta U. \quad (3.154)$$

Using these relations valid for Mach waves, we can construct an approximate theory for the plane steady supersonic flow of a gas approaching a thin airfoil in an isoenergetic, homentropic parallel stream with velocity  $U_\infty$  and Mach number  $M_\infty$ . This flow can also be interpreted as the flow around a profile moving with velocity  $U_\infty$  through a gas of infinite extent at rest. An observer moving with the profile will then see the steady flow considered here. In the linear theory (often called the acoustic approximation or Ackeret theory<sup>33</sup>) we consider the upstream parallel flow distorted by Mach waves which emanate from the profile and deflect the flow in such a way that the boundary condition of a tangential flow direction on the profile is just satisfied. In considering the direction and strength of a single Mach wave, we neglect the disturbances created by all the other waves. We thus assume the same direction for all the waves, given by the Mach angle  $\mu_\infty = \text{arc sin } M_\infty^{-1}$  corresponding to the unperturbed parallel flow, and in calculating the relation between the deflection  $\delta\theta$  and the pressure change  $\delta p$  we replace the local values  $\varrho$ ,  $U$ ,  $M$  in (3.154) by the values  $\varrho_\infty$ ,  $U_\infty$ ,  $M_\infty$ .

We first consider a polygon-shaped profile. A Mach wave emanates from each corner of the polygon; these waves are left-running on the upper surface of the profile and right-running on the lower surface. The wave issuing from the  $i$ th corner turns the flow to a direction determined by  $\delta\theta_i$ . In a region above the profile where the flow direction is determined by  $\delta\theta_u$  (the subscript "u" shall denote upper, the subscript "l" lower), the pressure is changed from that of the unperturbed stream by an amount

$$\delta p_u = \varrho_\infty U_\infty^2 (M_\infty^2 - 1)^{-\frac{1}{2}} \delta\theta_u. \quad (3.155)$$

This expression immediately results if we sum up the contributions to  $\delta p_u$  from all the Mach waves lying upstream of the region in question. A similar

<sup>33</sup> J. Ackeret, Luftkräfte auf Flügel, die mit größerer als Schallgeschwindigkeit bewegt werden, *Z. Flugtechnik und Motorluftschiffahrt* **16**, 72–74, (1925).

formula obtains below if we replace  $\delta\theta_u$  by  $-\delta\theta_l$  and  $\delta p_u$  by  $\delta p_l$ . We thus obtain the flow past the polygon as sketched in Fig. 63. The perturbations to the flow created by the profile propagate with undiminished strength out to infinity on both sides. At the trailing edge of the airfoil, the flow must again be deflected to the (common) exit direction, for otherwise a pressure difference will arise between the gas streams from above and from below. The flow behind the two Mach waves issuing from the trailing edge is exactly the same as the flow upstream.

A solution of this flow problem can also be given in terms of Mach waves running upstream (i.e., right-running above the profile, left-running below).

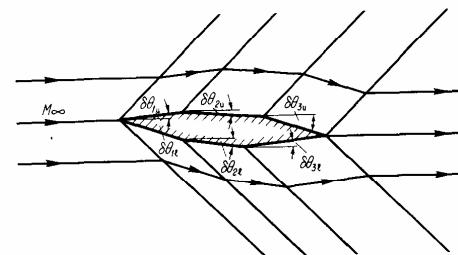


Fig. 63. Supersonic flow past a polygon-shaped profile according to Ackeret theory.

This solution, however, is physically without meaning. In real cases, the flow must somehow be established from rest—for example, by accelerating the profile in a gas at rest from zero velocity to a velocity of  $-U_\infty$  in a finite time interval; an observer moving with the profile then sees the gas flowing toward the profile with velocity  $U_\infty$ , and, after a sufficiently long time, the transient starting process passes over to the steady flow being considered. The disturbances of the gas at rest, created by the profile as it starts to move, propagate in all directions with the velocity of sound. The end result is that these disturbances cannot catch up with the profile, since it is moving with a supersonic velocity, and they will remain behind, as shown in Fig. 63. A supersonic flow behaves completely differently in this respect: the moving body can no longer keep the disturbances from moving upstream, and the gas is thus also disturbed there after the steady state has been attained. To illustrate

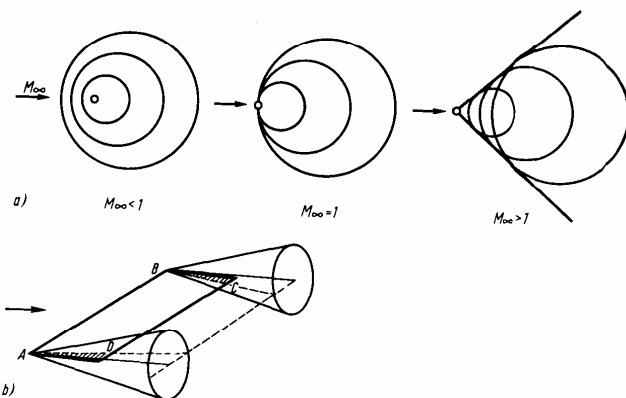


Fig. 64. (a) Propagation of a small disturbance in parallel flow. (b) Rectangular plate in supersonic flow; Mach cones.

these facts, Fig. 64a shows how the disturbances from a point source of sound waves propagate in a parallel stream. Relative to the gas, the disturbances spread out in all directions with the sound speed  $a_\infty$ . In a gas at rest, the disturbance created at  $t = 0$  reaches a circle of radius  $a_\infty t$  about the point source at any later time  $t > 0$ . If the gas now moves with velocity  $U_\infty$  past a fixed point source, then the circular wavefronts will be transported downstream with the velocity  $U_\infty$ . If  $U_\infty > a_\infty$ , then a cone-shaped envelope of these circles will result. We call this envelope the Mach cone issuing from the point source. The half-vertex angle of the Mach cone is the Mach angle  $\mu_\infty$ ; the disturbances created by the point source are confined to the interior of the Mach cone, i.e., the Mach cone is the boundary of the region of influence of the point source. In plane flow, the Mach cone is replaced by the wedge formed by the two Mach lines emanating from the point source.

On the basis of this illustration, we can, moreover, easily see that a plane supersonic flow can be realized even when the body is not infinite in length in the direction perpendicular to the flow. As an example, Fig. 64b shows a rectangular plate placed in a supersonic flow perpendicular to the edge  $AB$ . The influence of the finite plate length is only noticeable inside the

Mach cones emanating from  $A$  and  $B$ . The flow past the nonshaded region  $ABCD$  of the plate is no different from a plane flow. This is still qualitatively true when the disturbances of the parallel flow are so large that linear acoustic theory is no longer valid and the Mach cones are no longer purely circular cones, since the contribution of the perturbation velocities to the convection of wavefronts and the dependence of the sound velocity on the flow velocity can both no longer be ignored.

By increasing the number of corners of the polygon indefinitely, and at the same time decreasing the length of each segment, we pass over from a polygonal profile to a smooth profile; from the condition that the inclination of the profile contour to the stream direction be everywhere small (thin airfoil), the leading edge and trailing edge of the profile must both be sharp. Using the more precisely defined notation of Fig. 65, we have on the upper

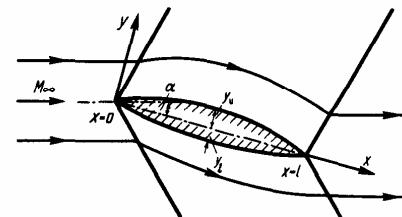


Fig. 65. Supersonic flow past a thin profile according to Ackeret theory.

surface of the profile

$$\delta\theta_u = -\alpha + dy_u/dx, \quad (3.156)$$

and on the lower surface

$$\delta\theta_l = -\alpha + dy_l/dx, \quad (3.157)$$

where  $\alpha$  is called the angle of attack of the profile.

The resultant force on the profile may be found from the integral of the perturbation pressure  $\delta p$  over the entire surface of the profile (the force due to the unperturbed pressure  $p_\infty$  will cancel out in this integration). We decompose this force into a lift  $L$  perpendicular to the flow direction and a drag  $D$  in the flow direction. If we consider the profile to be the cross section of a wing of width  $b$  in the  $z$  direction which is perpendicular to the  $x, y$  plane,

then the lift on this wing is

$$L = -b \int_u \delta p_u \cos \delta \theta_u ds + b \int_l \delta p_\ell \cos \delta \theta_\ell ds, \quad (3.158)$$

where the first integral is taken over the upper surface and the second over the lower surface of the profile;  $ds$  is the element of arc length of the contour. Because of the thin-profile assumption and for small angles of attack we may set  $\cos \delta \theta_u \approx 1 \approx \cos \delta \theta_\ell$  and  $ds \approx dx$ :

$$L = b \int_0^l (\delta p_\ell - \delta p_u) dx = \frac{\rho_\infty U_\infty^2 b}{(M_\infty^2 - 1)^{1/2}} \int_0^l (2\alpha - y_u' - y_\ell') dx.$$

The integral of  $y_u' + y_\ell'$  vanishes, since  $y_u(0) = y_\ell(0) = y_u(l) = y_\ell(l) = 0$ . Using the definition of the lift coefficient

$$c_L = L / (\frac{1}{2} \rho_\infty U_\infty^2 b l), \quad (3.159)$$

we thus obtain

$$c_L = 4\alpha (M_\infty^2 - 1)^{-1/2}. \quad (3.160)$$

Therefore, for a given flow Mach number  $M_\infty$ , the lift coefficient  $c_L$  in the linear approximation depends only on the angle of attack  $\alpha$ ; in fact,  $c_L$  is completely independent of the shape of the profile. Here lies a difference between supersonic flow and subsonic flow past a profile: For example, in subsonic flow, an infinitesimally thin curved plate in the shape of a circular arc experiences a finite lift at  $\alpha = 0$ , and this lift depends on the curvature; in supersonic flow, this lift is zero. To explain this further, Fig. 66 shows

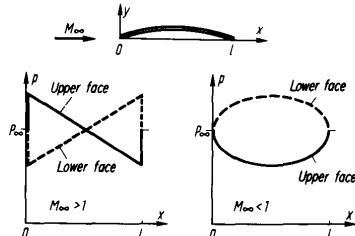


Fig. 66. Pressure distribution on a curved plate of the shape of a circular arc in supersonic flow and subsonic flow.

the pressure distribution in the linear approximation for both cases. The pressure distribution for supersonic flow follows from our formulas, but we shall not enter into the pressure calculations for subsonic flow.<sup>34</sup>

The drag  $D$  and the drag coefficient

$$c_D = D / \frac{1}{2} \rho_\infty U_\infty^2 b l \quad (3.161)$$

are obtained by setting  $\sin \delta \theta \approx \delta \theta$  in

$$D = b \int_0^l (-\delta p_\ell \delta \theta_\ell + \delta p_u \delta \theta_u) dx.$$

This gives

$$\begin{aligned} c_D &= \frac{2}{l(M_\infty^2 - 1)^{1/2}} \int_0^l (\delta \theta_\ell^2 + \delta \theta_u^2) dx \\ &= \frac{2}{(M_\infty^2 - 1)^{1/2}} \left[ 2\alpha^2 + \frac{1}{l} \int_0^l (y_u'^2 + y_\ell'^2) dx \right] \end{aligned} \quad (3.162)$$

or

$$c_D = c_L \alpha + \frac{2}{l(M_\infty^2 - 1)^{1/2}} \int_0^l (y_u'^2 + y_\ell'^2) dx. \quad (3.163)$$

The second term on the right side vanishes for a flat plate:  $y_u = y_\ell = 0$ . It differs from zero for a finite profile curvature, finite thickness, or both. As  $M_\infty \rightarrow 1$ ,  $c_L$  and  $c_D$  both become infinite. This does not correspond to actual physical behavior, but is due to the fact that linear theory is inapplicable for Mach numbers near unity, i.e., for transonic flows (see Section 3.7).

The fact that a flat plate must have  $c_D = c_L \alpha$ , i.e.,  $D = L \alpha$ , is readily understood. The pressure forces and their resultant are everywhere perpendicular to the plate, which is inclined to the flow direction at an angle  $\alpha$ . Since  $\alpha$  is small, the resultant force has a magnitude equal to  $L$ , and its component in the flow direction is  $L \alpha$ . Here is an essential difference between the supersonic and subsonic flows past a flat plate. While in subsonic,

<sup>34</sup> See, e.g., F.W. Riegels, "Aerodynamische Profile," München, 1958. L.M. Milne-Thomson, "Theoretical Aerodynamics", London-New York, 1958.

irrotational steady plane flow the pressure forces are still everywhere perpendicular to the plate, there is absolutely no drag force on a flat plate. This *d'Alembert paradox* is resolved by the fact that in subsonic flow, the gas flows from the pressure side (below) to the suction side (above) around the leading edge, thereby creating in a finite suction force at the leading edge which exactly balances the component of the resultant pressure force in the flow direction. In supersonic flow, there is no flow around the leading edge, and this suction force is absent.

It should be pointed out that the relations derived in this section, and particularly the Ackeret theory of airfoils, are completely independent of the thermal and calorific properties of the gas. These properties will begin to play a role when the perturbations in the variables of the flow field have become so large that linear theory fails. We shall go into the details of the limits of validity of the linear theory in Section 3.7.

The Ackeret theory for airfoils is an important example of linear acoustic theory of plane supersonic flows which deviate only slightly from a parallel flow. In many cases, the application of linear theory requires a knowledge of the simple laws for the reflection and intersection of Mach waves, which will now be stated (see Fig. 67): At a solid wall, a right- (or left-) running

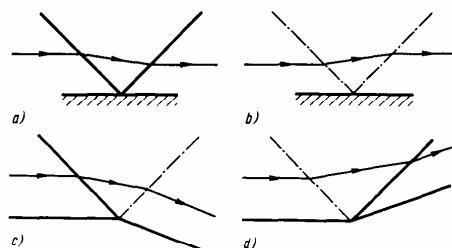


Fig. 67. Reflection of Mach waves at a solid wall (a, b) and at a free-stream surface (c, d).

wave will reflect as a left- (or right-) running wave of the same strength, since this just satisfies the condition that the flow be tangential at the wall. This reflection can also be interpreted as the intersection of two equal-strength, oppositely-running (i.e., one left- and one right-running) waves. Also, waves of different strengths intersect without disturbing each other. In

a uniform supersonic flow along a plane "free-stream" surface (vortex sheet) separating a region of gas at rest, a Mach wave reflects at the free boundary as an oppositely-running, opposite-type Mach wave of equal strength (i.e., an incident compression wave reflects as an expansion wave, and vice versa; an incident right-running wave reflects as a left-running wave, and vice versa.) The flow and also the boundary behind the second wave are thereby deflected by twice the angle of flow deflection behind the incident wave, and the pressure downstream of the reflected wave is equal to that upstream of the incident wave. This happens because the gas at rest on the other side of boundary, and hence on the boundary itself, is at constant pressure.

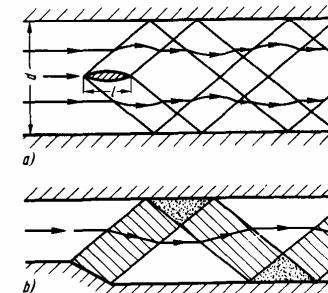


Fig. 68. Example of supersonic flow according to the acoustic theory. (a) Flow past a thin profile between two solid walls. (b) Flow at a widening of the channel.

As an example, we sketch in Fig. 68a the supersonic flow past a profile located in a finite-width channel bounded by fixed walls. The disturbance generated by the profile propagates downstream with undiminished strength through a series of reflections on the walls of the channel. The flow on the profile itself is no different from that on a profile located in a parallel stream of infinite width, as long as the reflected waves from the channel walls do not again intersect the profile. For a given channel width  $d$  and profile length  $i$ , this intersection will take place if the Mach angle  $\mu_\infty$  exceeds a certain value i.e., when the Mach number  $M_\infty$  is below a certain lower limit. For a profile located in the center of the channel, this limit is given by  $\sin \mu_\infty =$

$d(l^2 + d^2)^{-1/2}$ . The same thing holds for a flow bounded not by fixed walls, but by free-stream surfaces; the only difference is that the Mach waves are then reflected to the opposite type each time (i.e., compression to expansion and vice versa). This explains why, in supersonic wind-tunnel tests at sufficiently high Mach numbers, the finite width of the tunnel exerts no influence, and finite-width corrections are totally unneeded, while for Mach numbers only slightly higher than one (transonic flow), they are very important. Subsonic flows behave completely differently: The disturbances on the parallel flow caused by the profile propagate upstream as well as downstream but die down quite rapidly in both directions. At any event, the finite-width correction plays a role here. — In Fig. 68b, the influence of a slight widening of the channel on the parallel incoming flow is shown. The flow downstream of the change of section is distorted by the Mach waves reflecting back and forth from the channel walls, and, at least according to the linear approximation, the flow does not return to an undisturbed parallel flow, as would be the case in subsonic flow. Downstream of the section-change, the gas density in the mean is lower than that upstream; however, the gas density is not constant, and there are periodically alternating regions of density lower than that upstream with regions of density equal to that upstream. In Fig. 68b, the lower pressure regions are shown as shaded and dotted areas; in the dotted regions, the perturbations in pressure and density from their original values are twice as great as in the shaded regions.

In conclusion, we comment on the intersection between a shock and a Mach wave of the same family (i.e., both left-running or both right-running): This corresponds completely to the combination of two shocks of the same family. We can amend Fig. 60 by regarding  $S_1$  as still a shock but  $S_2$  as a Mach wave. The wave  $S_2$  must intersect the shock  $S_1$ , since the component of the velocity normal to the shock downstream is subsonic, so that the Mach wave  $S_2$  is inclined more steeply to the stream than is  $S_1$ . The shock  $S_1$  and the Mach wave  $S_2$  combine into a shock  $S$ , and at the intersection point, a reflected Mach wave and a vortex sheet also emanate. The vortex sheet may be neglected if the shock is so weak that the entropy change across it is negligible; in this approximation, the strength of the reflected wave is also negligible compared to that of the incident wave  $S_2$ . We can also regard  $S_1$  as a Mach wave and  $S_2$  as a shock in Fig. 60; the above remarks analogously apply to this case.

*Supplementary Remarks.* 1. For an oblique shock, we have, instead of the

Prandtl relation (3.129), the equation

$$u_1 u_2 = a_*^2 - \frac{(\gamma - 1)}{(\gamma + 1)} v^2. \quad (3.164)$$

2. The maximum deflection for an oblique shock is given by

$$\lim_{M_\infty \rightarrow \infty} \theta_{\max} = \arcsin \gamma^{-1}. \quad (3.165)$$

3. It is not unimportant to point out that the solutions to the various problems in steady supersonic flow given in this section are not the only ones possible. The flow past a symmetric wedge shown in Fig. 52 can also be regarded as the flow past a wall with a concave corner if we reinterpret the symmetric streamline as a wall; as was already shown, it is theoretically possible to have a strong or a weak shock in this flow. Apart from this, there are still infinitely many other solutions imaginable, as shown in Fig. 69

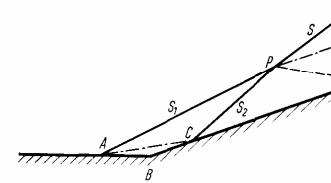


Fig. 69. A possible solution of the flow near a corner.

By assuming a shock  $S_1$  from  $A$  and a vortex sheet  $AC$ , with the gas through the shock  $S_1$  at the same pressure as the gas at rest in  $ABC$ , one can arbitrarily construct many solutions, all satisfying the tangential flow condition at the wall. In many cases, the type of solution shown in Fig. 69 is actually realized in experiments, and the simple solution with a single shock from  $A$  does not occur. In such cases, the boundary layer established along the wall plays a decisive role (see Section 4.3). Within the framework of the steady irrotational theory, however, it is impossible to decide which of the possible solutions will be realized in practice. Such a decision is possible only by resorting to experimental evidence, or to refined analyses in which the viscous and heat-conduction processes, or, possibly the circumstances under which an unsteady initial flow finally attains the steady flow, are all taken into account.

### 3.6 Prandtl-Meyer Flow

#### 3.6.1 FUNDAMENTAL RELATIONS

In the same way as in Section 3.2.1, where we presented the unsteady flow generated by a piston as a succession of infinitesimal waves, we shall now consider the deflection of a steady-plane supersonic flow at a finite corner as a succession of infinitesimal deflections achieved by Mach waves (Fig. 70). The incident parallel flow with constant velocity  $U_1 > a_1$  (i.e., Mach number  $M_1 > 1$ ) shall be isoenergetic, and thus also homentropic (see Section 2.5). Upon crossing the Mach wave emanating from the corner  $A_1$  of the boundary

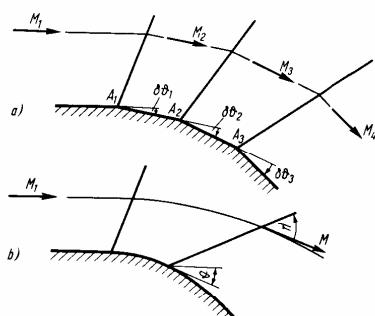


Fig. 70. Diagram for the derivation of Prandtl-Meyer flow. (a) Mach waves on a multicornered wall. (b) Prandtl-Meyer waves on a continuously curved wall.

wall, the flow turns an angle of  $\delta\theta_1$ . The velocity behind this first wave is  $U_2 = U_1 + \delta U_1$ , and in general,  $U_{k+1} = U_1 + \sum_{i=1}^k \delta U_i$ . On the other hand, by Eq. (3.152),

$$\delta\theta_i = -(M_i^2 - 1)^{\frac{1}{2}} \delta U_i / U_i, \quad (3.166)$$

where  $M_i$  (or  $U_i$ ) is the Mach number (or velocity) in the strip between the  $(i-1)$ th and  $i$ th wavefronts. The  $i$ th wavefront is inclined to the flow direction in this strip at the Mach angle  $\mu_i = \arcsin M_i^{-1}$ . By (3.166), we have

$$\theta_k = \sum_{i=1}^k \delta\theta_i = - \sum_{i=1}^k (M_i^2 - 1)^{\frac{1}{2}} \frac{\delta U_i}{U_i}. \quad (3.167)$$

For the case shown in Fig. 70a, all  $\delta\theta_i < 0$ . All the Mach waves are therefore expansion waves, for which  $\delta U_i > 0$ ; according to the results of Section 3.1,  $\delta a_i < 0$ . This implies that the wavefronts issuing from the individual corners will diverge with increasing distance from the wall.

By increasing the number of corners indefinitely and reducing the angles at the corners simultaneously, we can attain a continuous smooth contour (Fig. 70b). From Eq. (3.167), we get, after a corresponding limiting process,

$$\theta = v(M_1) - v(M), \quad (3.168)$$

where the function  $v(M)$  is defined as follows:

$$v(M) = \int_{M=1}^M (M^2 - 1)^{\frac{1}{2}} \frac{dU}{U}. \quad (3.169)$$

Before we discuss this function  $v(M)$ , we first remark: Equation (3.168) gives the connection between  $M$  and  $\theta$ . We can thus use (3.168) to determine the appropriate Mach number  $M$  from  $\theta$  at each point on the wall. This fixes the direction of the Mach line issuing from this point, the Mach line being a straight line inclined at  $\mu = \arcsin M^{-1}$  to the flow and thus at  $\theta + \mu$  to the original upstream flow direction. The strip-like regions of constant states for the wall with corners now shrink into Mach lines, on each of which the flow state is constant and determined by the state at the wall. Thus, the flow field is known everywhere. The flow is plane, steady, and inviscid, and since it is isoenergetic and homentropic in front of the first Mach wave, it remains so everywhere. Thus, by Section 2.5, it is everywhere irrotational.

This flow is called a Prandtl-Meyer flow,<sup>35</sup> after the two scientists who first posed and studied it. The function  $v(M)$  is called the Prandtl-Meyer function. In Section 3.8, we shall use another method to rederive the relations (3.168) and (3.169). We shall then see that there exists a far-reaching analogy between plane, steady, supersonic flow and one-dimensional unsteady flow, and the Prandtl-Meyer flow plays the same role for steady plane flow as does the simple wave treated in Section 3.2 for unsteady flow, where time takes the place of the second space coordinate.

For steady inviscid flow, the dependence of the velocity  $U$  on the Mach

<sup>35</sup> L. Prandtl, Neue Untersuchungen über die strömende Bewegung der Gase und Dämpfe. *Phys. Z.* **8**, 23 (1907). Th. Meyer, Über zweidimensionale Bewegungsvorgänge in einem Gas, das mit Überschallgeschwindigkeit strömt. *Mitt. Forsch. Arb. VDI* **62** (1908).

number  $M$  is known (Section 3.1), and we can calculate  $v(M)$  in Eq. (3.169). In general, the stagnation quantities (e.g.,  $h_t$  and  $s_t$  or  $p_t$  and  $T_t$ ) enter into  $v(M) = v(M; h_t, s_t)$ . For a calorically ideal gas,  $v(M)$  is independent of these stagnation quantities. If, for a calorically ideal gas, we substitute for the  $U$  in integral (3.169) in terms of  $M$  from Eq. (3.16), then the integration can be carried out explicitly with the result

$$v(M) = \mu(M) - \frac{\pi}{2} + \left( \frac{\gamma+1}{\gamma-1} \right)^{\frac{1}{2}} \arctan \left[ \frac{\gamma-1}{\gamma+1} (M^2 - 1) \right]^{\frac{1}{2}}, \quad (3.170)$$

where  $\mu = \arcsin M^{-1}$ . In Fig. 71,  $v(M)$  and  $\mu(M)$  are given for  $\gamma = 1.2$  and 1.4. In the insert, the meaning of the quantities  $v$  and  $\mu$  are again summarized; it is assumed there that the gas approaches with a Mach number  $M = 1$  and

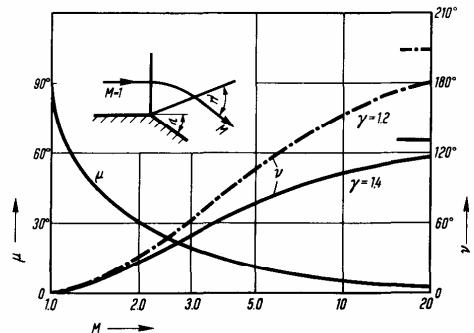


Fig. 71. Prandtl-Meyer function  $v(M)$  for a calorically ideal gas and Mach angle  $\mu(M)$ .

that the curved part of the wall is shrunk into a corner. We call this a centered Prandtl-Meyer wave, in analogy to the centered unsteady expansion wave treated in Section 3.2. From Eq. (3.170), we obtain the upper limit for  $v$  and thus also for the deflection:

$$-\theta_{\max} = v_{\max} = \left[ \left( \frac{\gamma+1}{\gamma-1} \right)^{\frac{1}{2}} - 1 \right] \frac{\pi}{2}. \quad (3.171)$$

When the wall bends beyond this limiting angle (for  $\gamma = 1.4$ , it is  $130.5^\circ$ ), a vacuum occurs between the wall and the gas (Fig. 72; also see the analogous

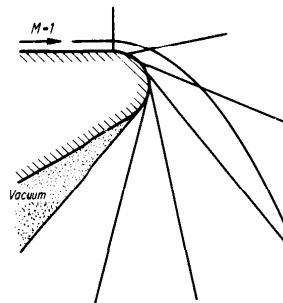


Fig. 72. Occurrence of vacuum for sufficiently large deflection of a supersonic flow.

Fig. 34). The streamline separating the vacuum from the gas is at the same time a Mach line.

Through a Prandtl-Meyer wave, a gas can either be expanded or be compressed. Figures 70 and 72 assume an expansion of the gas. If we bend the wall not into a convex curve, but into a concave one, we then obtain a compression. In this case, of course, the Mach lines can converge and form an envelope at a certain distance from the wall, as in unsteady flow (Fig. 73). This formation of the envelope shows the building up of a shock wave. In a concave wall with a corner, this shock does not form at some distance from the wall, but at the corner itself; we then have the same flow as the flow past a wedge, where the streamline hitting the vertex of the wedge is to be interpreted as a part of the wall.

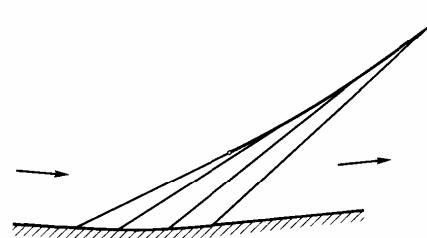


Fig. 73. Formation of a shock on a concave curved wall.

Of practical importance, particularly in flows past wing airfoils, is a comparison of the pressure change across a Prandtl-Meyer wave and the pressure jump across an oblique shock: A parallel flow with Mach number  $M_1 \geq 1$  will be deflected through an angle  $\theta$  by a Prandtl-Meyer wave and leave the wave with a Mach number  $M_2$  which can be calculated immediately from the known function  $v(M)$ :  $v(M_2) = v(M_1) - \theta$ . The pressure changes from  $p_1$  to  $p_2$ ; with  $p_1$ ,  $M_1$ , and  $M_2$  (for a calorically nonideal gas, we need, in addition, the stagnation quantities  $h_i$  and  $s_i$ ),  $p_2$  is determined. For a calorically ideal gas, we use formula (3.14) to calculate  $p_2$ . From this formula,  $p_2/p_1$  comes out as a function of  $M_1$  and  $M_2$  only. Since, on the other hand,  $M_2$  can be expressed in terms of  $M_1$  and  $\theta$ , we obtain a relation of the form  $p_2/p_1 = f(M_1, \theta)$ . Similar results hold for the deflection through an oblique shock, for which this relation is given graphically in Fig. 51.

For applications to the flow past profiles, which we shall discuss further later on, it is convenient to introduce, instead of  $p_2/p_1$ , a pressure coefficient  $c_p$ <sup>36</sup> defined as:

$$c_p = \frac{p_2 - p_1}{\frac{1}{2} \rho_1 U_1^2} = \frac{2p_1}{\rho_1 U_1^2} \left( \frac{p_2}{p_1} - 1 \right) = \frac{2}{\gamma M_1^2} \left( \frac{p_2}{p_1} - 1 \right), \quad (3.172)$$

where the last of these expressions holds only for calorically ideal gases, to which we shall now confine our discussion. In Fig. 74,  $c_p$  for a Prandtl-Meyer wave is presented as function of  $M_1$  with  $\theta$  as parameter;  $c_p > 0$  signifies compression,  $c_p < 0$  expansion. Drawn in dotted lines next to the curves for Prandtl-Meyer flow are the corresponding curves for deflection through a shock (of the two possible solutions, only the weak shock is considered; see Section 3.5.1.). The shock curves end at the Mach numbers for which the given deflections are the maximum possible, and the Prandtl-Meyer curves end at  $M_2 = 1$ .

We see that the compression through a shock for deflections not too large is practically identical to compression through a Prandtl-Meyer wave. This fact will be confirmed through an expansion of  $c_p$  as function of  $\theta$ :

$$c_p = C_1 \theta + C_2 \theta^2 + C_3 \theta^3 + \dots \quad (3.173)$$

It turns out that in this expansion the coefficients  $C_1$  and  $C_2$  for a shock and for a Prandtl-Meyer wave are identical, a fact which we shall not prove

<sup>36</sup> In what follows, there should be no confusion with the specific heat  $c_p$ .

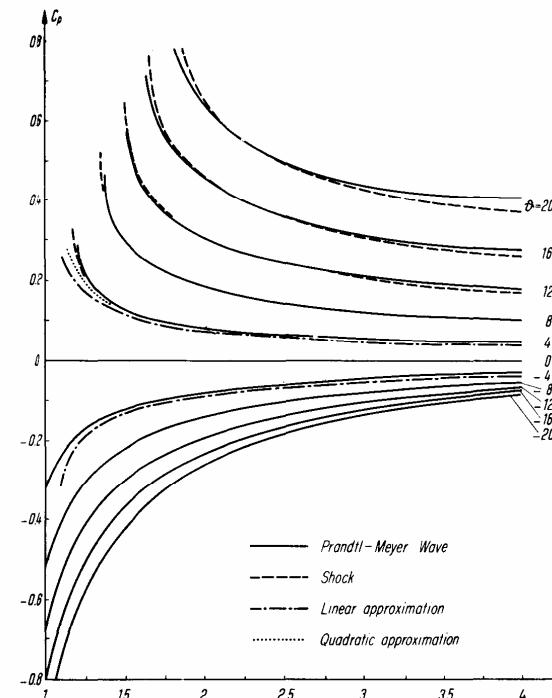


Fig. 74. Pressure coefficient  $c_p$  for the deflections through Prandtl-Meyer waves and through shocks as functions of the upstream Mach number  $M_1$  and the deflection angle; calorically ideal gas with  $\gamma = 1.4$ .

here; thus, in both cases, for left-running waves or shocks, we have<sup>37</sup>

$$C_1 = \frac{2}{(M_1^2 - 1)^{\frac{1}{2}}} \quad \text{and} \quad C_2 = \frac{(M_1^2 - 2)^2 + \gamma M_1^4}{2(M_1^2 - 1)^2}. \quad (3.174)$$

<sup>37</sup> A. Busemann, Aerodynamischer Auftrieb bei Überschallgeschwindigkeit, *Luftfahrtforschung* 12, 210, (1935).

(For right-running waves, we change the signs on  $C_1$ ,  $C_3$ , etc.) A difference can first be seen in the third term, but the difference in the coefficients  $C_3$  for a Prandtl-Meyer wave and for a shock is quite small, which explains the remarkable coincidence of the curves in Fig. 74. For Prandtl-Meyer flow,<sup>38</sup>

$$C_3 = \frac{(\gamma + 1) M_1^8 - (5 + 7\gamma - 2\gamma^2) M_1^6 + 10(\gamma + 1) M_1^4 - 12M_1^2 + 8}{6(M_1^2 - 1)^{7/2}}, \quad (3.175)$$

while for the shock we shall substitute  $C_3 - X$  for  $C_3$ , with

$$X = \frac{(\gamma + 1) M_1^4 [(5 - 3\gamma) M_1^4 - (12 - 4\gamma) M_1^2 + 8]}{48(M_1^2 - 1)^{7/2}}. \quad (3.176)$$

Also drawn in Fig. 74 are the approximate values of  $c_p$  for  $\theta = 4^\circ$ , obtained from Eq. (3.173) by truncating up to the second term. Moreover, we also give in dot-dash lines the results of linear theory for  $\theta = \pm 4^\circ$ , which is equivalent to truncating the series (3.173) to the first term. The coefficient  $C_1$  is furthermore independent of the gas properties, and is therefore also good for gases not calorically ideal. This comes directly out of the linear theory discussed in Section 3.5.2.

### 3.6.2 APPLICATIONS

Through combinations of Prandtl-Meyer waves and shocks, we can construct a whole series of plane supersonic flows of practical importance. An almost trivial example is the isentropic conversion of a parallel flow of Mach

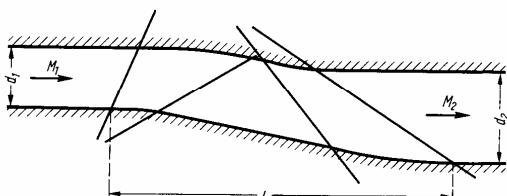


Fig. 75. Expansion of a parallel flow of Mach number  $M_1 > 1$  into another parallel flow of Mach number  $M_2 > M_1$  in the same direction through two Prandtl-Meyer waves.

<sup>38</sup> A. Kahane and L. Lees, The flow at the rear of a two-dimensional supersonic airfoil, *J. Aero. Sci.* **15** 167-170, (1948).

number  $M_1 > 1$  into a parallel flow in the same direction with a different Mach number  $M_2 > 1$ . This can be achieved in every case through two Prandtl-Meyer waves in the manner shown in Fig. 75. We can imagine any two streamlines to be replaced by fixed walls, and thereby obtain a parallel flow in a channel of width  $d_1$ , which then goes through a curved section of length  $l$  and exits in a channel of width  $d_2$ . If we change the channel width from  $d_1$  to  $d_2$  arbitrarily, then, in general, the gas will not leave this variable-area region as an undistorted parallel flow. There will be distortions of the parallel flow downstream, as was shown, for example, in Fig. 68b in the linear approximation.

An arrangement as described above can be used in combination with a converging nozzle, which brings the flow from subsonic velocity through sonic velocity to produce a parallel supersonic flow. The shortcoming of such a nozzle is its lack of symmetry with respect to the flow direction. Besides, we must realize that the flow in the narrowest section is not exactly one-dimensional, and the Mach number of 1 is not attained on a straight line perpendicular to the flow but on a curve whose shape depends on the nozzle shape upstream of the throat, i.e., the subsonic part. Only when the subsonic part of the nozzle is relatively narrow can we assume that the Mach number 1 is attained on each streamline at the narrowest cross section of the nozzle. In Section 3.8.2 we shall give, in connection with the method of characteristics, another procedure for nozzle design in which these shortcomings are avoided (see Fig. 88).

We again return to the plane supersonic flow past a profile, for which a linear theory was already discussed in Section 3.5.2. We shall now treat the general nonlinear problem. We assume that subsonic velocity does not occur anywhere in the flow field. To realize such flows, the freestream Mach number must be sufficiently greater than 1, and the profile shape must be suitable. In particular, no detached shock waves must be permitted to occur. This requires that the profile has sharp leading and trailing edges. As the simplest profile of this type, we first consider a flat plate inclined at an angle of attack  $\alpha$  toward the free stream. The following regions of the flow field are then to be distinguished in the vicinity of the plate (Fig. 76): 1. The unperturbed flow with Mach number  $M_\infty$ . 2. A centered Prandtl-Meyer wave starting from the leading edge  $A$ , which deflects the flow by an angle  $\theta = -\alpha$ , so that the gas above the plate flows tangentially along the plate. 3. A uniform parallel flow downstream of the expansion wave, where the flow direction

## 3 Inviscid Flows

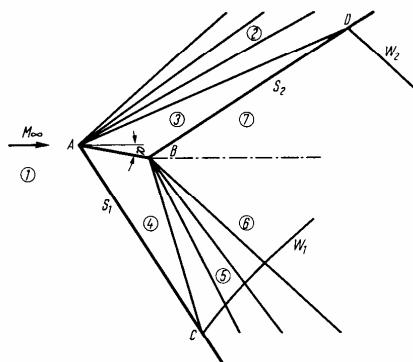


Fig. 76. Supersonic flow on a flat plate at an angle of attack.

is tangential to the plate. 4. A similar flow on the lower side of the plate, which results from the incoming stream 1 being deflected through a shock wave  $S_1$  emanating from the leading edge  $A$ . The flow 4 is then turned back into a uniform parallel flow 6 through a Prandtl-Meyer expansion 5 centered at the trailing edge  $B$ . Similarly, the flow 3 is bent back into a uniform flow 7 by a shock  $S_2$  from the trailing edge. The magnitudes of the deflections in the shock  $S_2$  and in the expansion wave 5 are dictated by the requirement that the pressures and flow directions in regions 6 and 7 must be the same; in particular, this direction is not exactly the same as that of the incoming flow. In addition, from the trailing edge  $B$  there is a contact discontinuity (vortex sheet, shown by dot-dash line in Fig. 76) separating regions 6 and 7.

The flow in the vicinity of the profile remains as described downstream up to the two Mach lines  $W_1$  and  $W_2$ .  $W_1$  starts from point  $C$ , where the shock  $S_1$  intersects the Prandtl-Meyer flow 5;  $W_2$  starts from point  $D$ , the intersection of the expansion wave 2 with the shock  $S_2$ . Downstream of the curves formed by the Mach lines  $W_1$  and  $W_2$ , as well as in the outer portions of the shocks  $S_1$  and  $S_2$ , the flow field is no longer a simple combination of parallel flow and Prandtl-Meyer flow. The flow field downstream of these curves can be numerically calculated by the method of characteristics, as we shall explain in a subsequent section. To calculate the pressure on the plate, however, the knowledge of this flow field is not necessary at all, since the

## 3.6 Prandtl-Meyer Flow

pressure is completely determined by the pressure changes across the wave 2 and shock  $S_1$ .

Entirely analogous is the flow past a double-wedge (or diamond) profile as sketched in Fig. 77. On the other hand, the flow past a continuously curved profile is a different situation. Taking the profile in Fig. 78 as basis for discussion and letting the angle of attack be  $\alpha = 0$ , we see that shocks emanate from the leading edge  $A$  on both sides of the profile. However, these shocks are no longer straight but are curved. The different streamlines cross the bow shock at locations with different shock strengths, so that the entropy increase across the shock differs from streamline to streamline. The flow

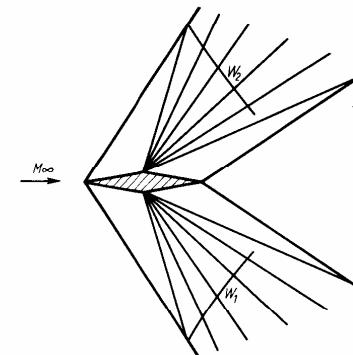


Fig. 77. Supersonic flow past a double-wedge profile.

downstream of the shock is thus no longer homentropic, and, consequently, it is no longer a Prandtl-Meyer flow, since for a Prandtl-Meyer flow the flow quantities, and, in particular, the entropy must be constant along the Mach lines. From the Crocco theorem (Section 2.5), this implies, moreover, that the flow behind the shock is rotational. Crocco's theorem may be applied here, since the flow is assumed to be isoenergetic, and this property remains valid across the shock.

We can also interpret the flow past a curved profile by a wave picture: At the beginning of Section 3.6.1, Prandtl-Meyer flow was derived from a succession of Mach waves through a limiting process. The Mach waves

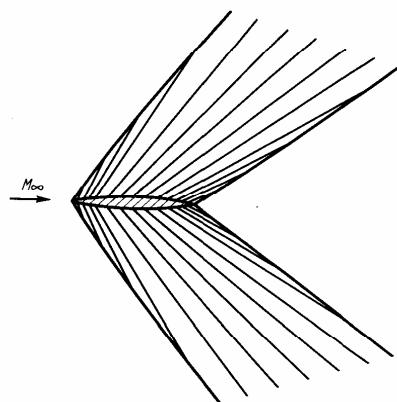


Fig. 78. Supersonic flow past a thin profile.

emanate from the curved part of the fixed wall (Fig. 70) and continuously bend the flow to the wall direction. In the flow past a profile (Fig. 78), left-running and right-running Mach waves leave the upper and lower surfaces of the profile, respectively. These will partially intersect the two shocks from the leading edge; this results in reflected waves, as was explained more precisely in Section 3.5. Consequently, the flow field downstream of the bow shock above the profile contains not only left-running waves, but also reflected right-running waves, while the flow field below the profile contains reflected left-running waves in addition to the right-running waves. (The Mach lines corresponding to the reflected waves have not been drawn in Fig. 78.) From this we again infer that the flow is not a Prandtl-Meyer flow, since only waves of one kind can appear in it.

To calculate the complete flow field numerically, we can apply the method of characteristics. Frequently, however, we are less interested in an exact calculation of the complete flow field than in a practically adequate calculation of the pressure on the profile. To this end, there are several approximate methods at our disposal, which we shall present in decreasing order of accuracy:

a). *Shock-Expansion Theory.* The reflected Mach waves are neglected, and the strength of the shock immediately at the leading edge is determined

uniquely by the deflection. The flow along the upper or lower surface of the profile, up to the trailing edge, is calculated as a Prandtl-Meyer flow, using the flow state immediately behind the shock as the initial state (Fig. 78). The pressure on the two surfaces is then easily calculated.

The following remarks are in order for this approximation: Neglecting the reflected waves and replacing the flow between the leading and trailing shocks by a Prandtl-Meyer flow would be exactly correct if the change of state across a shock were isentropic (as a Prandtl-Meyer wave is homentropic), and if a shock and a Prandtl-Meyer wave gave the same pressure rise for the same flow and same deflection (i.e., if the curves for shock and Prandtl-Meyer flow in Fig. 74 coincide exactly); for only then can a shock and a Prandtl-Meyer wave cancel exactly. Both conditions are satisfied only to second order in the deflection (and hence in shock strength).<sup>39</sup> It is natural to conjecture that the errors in the approximation are of the order of the third power of some characteristic angle of the deflection. In fact, it is even smaller, being of the order of the fourth power in such an angle; this is connected with the fact that only a small portion of the waves reflected from the bow shock hit the profile again.<sup>40</sup>

b). *Simple Wave Theory.* In addition to neglecting the reflected waves, we consider the change of state in the shock as isentropic, so that the entire flow field is homentropic, and we assume that the pressure coefficients  $c_p$  for Prandtl-Meyer waves and for shocks depend in the same way on the Mach number  $M_\infty$  and the deflection angle  $\theta$  (see Section 3.6.1; the curves for shocks and Prandtl-Meyer waves given in Fig. 74 are assumed to coincide). There then exists for any given flow Mach number  $M_\infty$  a unique relation in the entire flow field between  $c_p$  and the flow direction specified by  $\theta: c_p = c_p(\theta; M_\infty)$ . In general, this relation contains the stagnation thermodynamic quantities in addition to the Mach number  $M_\infty$ . For a calorically ideal gas, this dependence is absent, and we can take  $c_p$  from Fig. 74 (where  $M_1$  is the present  $M_\infty$ ). The lift and drag coefficients of the profile result from formulas (3.158) and (3.161), when we divide through by  $\frac{1}{2} \rho_\infty U_\infty^2$  and introduce the

<sup>39</sup> Compare this with the discussion of the interaction of Mach wave and shock wave in Section 3.5.2.

<sup>40</sup> Cf. M. J. Lighthill, Higher Approximations, in "General Theory of High Speed Aerodynamics", Sect. E, High Speed Aerodynamics and Jet Propulsion, vol. VI, Princeton Univ. Press, Princeton, New Jersey, 1954.

pressure coefficient  $c_p$ :

$$c_L = - \int_u c_p(\theta_u) \cos \theta_u \frac{ds}{l} + \int_i c_p(\theta_i) \cos \theta_i \frac{ds}{l}, \quad (3.177)$$

$$c_D = \int_u c_p(\theta_u) \sin \theta_u \frac{ds}{l} - \int_i c_p(\theta_i) \sin \theta_i \frac{ds}{l}. \quad (3.178)$$

(Instead of  $\delta\theta$ , we write  $\theta$  for a finite deflection.)

c). *Busemann Approximation.*<sup>41</sup> Next, we approximate  $c_p$ , which was assumed in (b) to be a function of  $\theta$ , by taking the first two terms in the expansion (3.173). By consistently neglecting all terms which give a third-order contribution (in the deflection) to (3.177) and a fourth-order contribution to (3.178), we finally obtain the following, in the notation of Fig. 65:

$$c_L = 2C_1\alpha + \frac{C_2}{l} \int_0^l (y'_i{}^2 - y'_u{}^2) dx, \quad (3.179)$$

$$c_D = C_1 \left[ 2\alpha^2 + \frac{1}{l} \int_0^l (y'_i{}^2 + y'_u{}^2) dx \right] + \frac{C_2}{l} \left[ 3\alpha \int_0^l (y'_i{}^2 - y'_u{}^2) dx + \int_0^l (y'_u{}^3 - y'_i{}^3) dx \right]. \quad (3.180)$$

Whereas in the linear Ackeret theory (Section 3.5.2), the lift coefficient  $c_L$  is independent of the profile shape, here, the second term on the right side of Eq. (3.179) gives a dependence on the profile shape. Of course, this second term is nonzero only when  $y'_u{}^2 \neq y'_i{}^2$ . Thus, it vanishes for infinitesimally-thin curved profiles ( $y'_u = y'_i$ ) and for finite-thickness symmetric profiles ( $y'_u = -y'_i$ ).

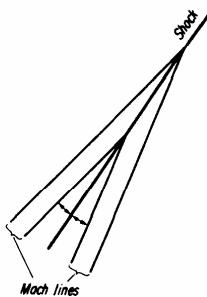
d). *Ackeret Approximation.* We now consider the linear term only in the relation between  $c_p$  and  $\theta$ . Then formulas (3.179) and (3.180) become formulas (3.160) and (3.163), and we have returned to the linearized Ackeret theory.

<sup>41</sup> A. Busemann, footnote 37.

The calculation of the entire flow field in the linear approximation has already been discussed in Section 3.5.2. We should now mention that the entire flow field in the approximation corresponding to assumptions (c) above can be found as follows: Since the Prandtl-Meyer flow is determined by the profile shape and the free stream, and since the flow behind the rear shock is the same as the free stream (this follows from the fact that the same pressure must exist above and below the streamline starting from the trailing edge, which in this approximation is possible only for  $\theta = 0$  and then also  $c_p = 0$ ) we have only to calculate the shape of the shock wave. Using the appropriate shock relations [Eq. (3.148) for calorically ideal gases], we select at each point on the bow shock the shock angle  $\zeta$  such that the flow direction behind the shock is exactly the same as that produced by the Prandtl-Meyer flow at the same location. Then, the pressure jump across the shock leads, apart from terms of order  $\theta^3$ , to the pressure in the Prandtl-Meyer wave, a condition that obviously must hold. In a similar way, we choose the rear shock so that the Prandtl-Meyer flow is everywhere bent back to the original flow direction. In this construction of the shock and in the framework of this approximation, it is meaningful to develop the function  $\zeta = \zeta(\theta)$  [for example, Eq. (3.148)] in powers of  $\theta$  and only keep terms in  $\theta$  and  $\theta^2$ , since the shock relations are only satisfied with errors of order  $\theta^3$ . We have thus obtained the approximate flow field construction due to Friedrichs.<sup>42</sup>

The shape of the shock at sufficiently large distance from the body is easily found from the fact that a weak shock just bisects the angle formed by the Mach lines in front of and behind the shock (Fig. 79). Let us consider the leading-edge shock above the profile: In the uniform flow in front of the shock, the Mach lines are a family of parallel straight lines. At large distance from the body, the Prandtl-Meyer wave behind the shock behaves like a centered wave, with all the Mach lines originating from the same point. The curve which everywhere bisects the angle between two such families of straight lines is well known to be a parabola (the Mach lines of the upstream uniform flow are parallel to the parabola axis, while those of the Prandtl-Meyer wave are the focal rays of the parabola). A similar discussion is also possible for all other shocks, so that, with complete generality, all shocks originating from a profile have this parabolic form at large distances away

<sup>42</sup> K.O. Friedrichs, Formation and decay of shock waves, *Comm. Pure and Appl. Math.* 1, 211 (1948).



**Fig. 79.** A weak shock bisecting the angle between the Mach lines of the flows before and after the shock.

from the body. This implies that the shock strength, and consequently the strength of the disturbances on the parallel stream caused by the profile, decreases as the square root of the distance from the profile, a fact first established by Busemann.

### 3.7 Limits of Validity of Linear Theory in Transonic and Hypersonic Regimes

In the following discussion, the limits of validity of linear theory for plane supersonic flow will be illustrated through the consideration of special examples. The starting point is the deflection of a parallel stream of Mach number  $M_1$  by a Prandtl-Meyer wave or by a shock. Using relation (3.14) for calorically ideal gases, the pressure coefficient  $c_p$  defined by Eq. (3.172) can be written as follows for shock-free flow:

$$c_p = \frac{2}{\gamma M_1^2} \left[ \left( \frac{2 + (\gamma - 1) M_1^2}{2 + (\gamma - 1) M_2^2} \right)^{\gamma/(\gamma-1)} - 1 \right]. \quad (3.181)$$

a. *Transonic Prandtl-Meyer Wave.* We now assume that  $0 \leq M_1 - 1 \ll 1$  and  $0 < M_2 - 1 \ll 1$ , i.e., the Mach numbers  $M_1$  and  $M_2$  are only slightly different from 1. Upon neglecting the terms of higher order in  $M - 1$ , Eq. (3.181) becomes

$$c_p = 4(M_1 - M_2)/(\gamma + 1). \quad (3.182)$$

On the other hand, for  $M - 1 \ll 1$ , we can expand the Prandtl-Meyer function  $v(M)$  in powers of  $M - 1$ . The first term of this expansion gives

$$v(M) = \frac{4\sqrt{2}}{3(\gamma + 1)} (M - 1)^{3/2}. \quad (3.183)$$

By combining Eqs. (3.182) and (3.183), we get for the pressure coefficient  $c_p$  for deflection through a small angle  $\theta = v(M_2) - v(M_1)$  (we limit ourselves to expansions for which, according to the present definition,  $\theta > 0$ ):

$$c_p = 4\theta^{2/3}(\gamma + 1)^{-1/3} \left[ K - \left( \frac{3}{4\sqrt{2}} + K^{3/2} \right)^{2/3} \right]. \quad (3.184)$$

Here we have introduced the transonic similarity parameter  $K$ , defined as:

$$K = (M_1 - 1) [(\gamma + 1) \theta]^{-2/3}. \quad (3.185)$$

If  $K \gg 1$ , we can set in Eq. (3.184)

$$\left( \frac{3}{4\sqrt{2}} + K^{3/2} \right)^{2/3} = K + \frac{1}{2(2K)^{1/2}}.$$

Equation (3.184) then becomes

$$c_p = -\sqrt{2} \theta (M_1 - 1)^{-1/2}.$$

This result agrees with the result of linear theory, since, when  $M_1 - 1 \ll 1$ ,  $(M_1^2 - 1)^{1/2} = \sqrt{2}(M_1 - 1)^{1/2}$  is a good approximation.<sup>43</sup> This also shows that for freestream Mach numbers which are only slightly greater than 1, linear theory can be applied only when  $K \gg 1$ . For a given fixed deflection  $\theta$ , however small, if we decrease the Mach number  $M_1$  we shall eventually exceed the limit of validity of linear theory.

b. *Hypersonic Prandtl-Meyer Wave.* We now assume  $M_1 \gg 1$ ,  $M_2 \gg 1$ . We then obtain from Eq. (3.181)

$$c_p = \frac{2}{\gamma M_1^2} \left[ \left( \frac{M_1}{M_2} \right)^{2\gamma/(\gamma-1)} - 1 \right]. \quad (3.186)$$

(Here we have neglected in the square brackets terms of order  $M^{-2}$ .) On the other hand, expanding Eq. (3.170) in powers of  $1/M$  and neglecting terms

<sup>43</sup> For many purposes, an expansion in  $M^2 - 1$  is advantageous; we have here chosen the direct course of expanding in  $M - 1$ .

of order  $M^{-3}$  gives

$$\nu = \frac{\pi}{2} \left[ \left( \frac{\gamma+1}{\gamma-1} \right)^{\frac{1}{2}} - 1 \right] - \frac{2}{(\gamma-1) M}. \quad (3.187)$$

Combination of (3.186) and (3.187) results in

$$c_p = \frac{20^2}{\gamma \tilde{K}^2} \left[ \left( 1 - \frac{\gamma-1}{2} \tilde{K} \right)^{2\gamma/(\gamma-1)} - 1 \right], \quad (3.188)$$

where  $\tilde{K}$  is the hypersonic similarity parameter, defined by

$$\tilde{K} = M_1 \theta, \quad (3.189)$$

When  $\tilde{K} \ll 1$ , we can expand  $c_p$  in Eq. (3.188) in powers of  $\tilde{K}$ . For the first term of the expansion, we obtain  $c_p = -20^2/\tilde{K} = -20M_1^{-1}$ . This again agrees with the results of linear theory, since when  $M_1 \gg 1$ , we have  $(M_1^2 - 1)^{1/2} = M_1$ . Linear theory is thus applicable to large Mach numbers only when  $M_1 \theta \ll 1$ . For a fixed  $\theta$ , however small, increase in the Mach number  $M_1$  will eventually cause the limit of validity of linear theory to be exceeded.

The reason for the failure of linear theory in the hypersonic regime is readily understood if we recall the derivation of the formulas in Section 3.5.2. The Mach waves considered there result from the nonstationary waves discussed in Section 3.2.1. For the linear theory of Section 3.2.1 to be valid, we must have  $|\delta u/a| \ll 1$ . According to Eq. (3.151), however, this implies that  $|M \delta \theta| \ll 1$  must hold, which is identical to the condition derived above. The analogous discussion for Mach numbers near 1 is somewhat complicated, and will not be given. We shall only mention that for the formulas in Section 3.5.2 to be valid, the change  $\delta u$  in the normal component of the velocity must remain small compared to the difference between the free-stream velocity and the sound velocity. This leads to the condition  $K \gg 1$ .

c. *Hypersonic Shock*. We shall now consider deflection through a hypersonic shock wave, employing arguments similar to those used for deflection through an expansion wave in the hypersonic regime. Again let  $\theta$  denote the deflection angle. Combining the definition (3.172) for  $c_p$  and the result (3.143) for the pressure ratio across an oblique shock, we first get

$$c_p = \frac{4\theta^2}{\gamma+1} \frac{M_1^2 \sin^2 \zeta - 1}{\tilde{K}^2}, \quad (3.190)$$

where  $\tilde{K}$  is again defined as in (3.189). Let us now confine ourselves to small

deflections  $\theta$  and large Mach numbers  $M_1 \gg 1$ . Since for such Mach numbers the shock angle  $\zeta$  (for *weak* shocks) remains small with  $\theta$ , we can in all the formulas replace  $\sin \theta$  or  $\sin \zeta$  by  $\theta$  or  $\zeta$ , respectively, and  $\cos \theta$  or  $\cos \zeta$  by 1. Then, we first get from (3.190)

$$c_p = \frac{4\theta^2}{\gamma+1} \frac{K_\zeta^2 - 1}{\tilde{K}^2}, \quad (3.191)$$

with the notation  $K_\zeta = M_1 \zeta$ . From (3.148) we get, after neglecting the terms as described before and after a short computation,

$$K_\zeta^2 - 1 = \frac{1}{2} (\gamma + 1) \tilde{K} K_\zeta, \quad (3.192)$$

and then, from Eq. (3.191),

$$c_p = 2\theta^2 (K_\zeta/\tilde{K}). \quad (3.193)$$

Solving Eq. (3.192) for  $K_\zeta$ , on the other hand, we get

$$K_\zeta = \frac{\tilde{K}}{2} \left\{ \frac{\gamma+1}{2} + \left[ \left( \frac{\gamma+1}{2} \right)^2 + \frac{4}{\tilde{K}^2} \right]^{\frac{1}{2}} \right\}. \quad (3.194)$$

Substitution of this into (3.193) results in

$$c_p = \frac{\gamma+1}{2} + \left[ \left( \frac{\gamma+1}{2} \right)^2 + \frac{4}{\tilde{K}^2} \right]^{\frac{1}{2}} \theta^2. \quad (3.195)$$

Figure 80 shows this expression for  $c_p$  plotted together with the exact results and the results of linear theory. What has been said about the validity of linear theory in connection with expansion waves is also true here: The higher the Mach number  $M_1$ , the smaller must be the angle  $\theta$  in order for linear theory to hold. From Eq. (3.195), it follows that for  $\tilde{K} \ll 1$ ,  $c_p = 2\theta^2/\tilde{K}$ , which corresponds to linear theory [see Eq. (3.154)]. On the other hand, from Eq. (3.194) we get for  $\tilde{K} \gg 1$  the result derived earlier, (3.149), while Eq. (3.195) in this case becomes

$$c_p = (\gamma + 1) \theta^2. \quad (3.196)$$

In Section 3.5.1 we called a flow satisfying the condition  $\tilde{K} \gg 1$  (i.e.,  $M_1 \theta \gg 1$ ) a strong hypersonic flow, and we established that the geometric properties of this flow are independent of the Mach number. Equation (3.196) now states that the pressure coefficient is similarly independent of the Mach number.

In this connection, we should also mention the main features of an approx-

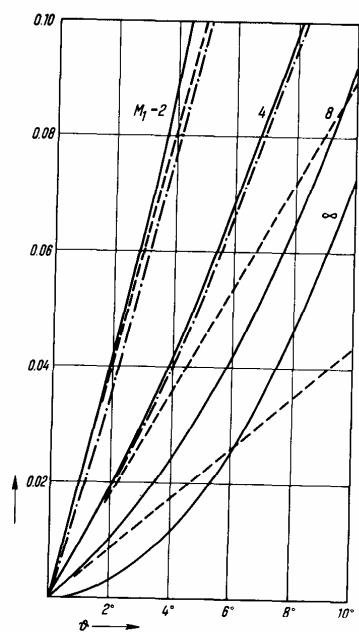


Fig. 80. Pressure coefficient  $c_p$  for the deflection in a shock; exact (—), according to the hypersonic approximation (3.195) (— · —), and according to the linear approximation (---); calorically ideal gas with  $\gamma = 1.4$ .

imate theory for hypersonic flow past solid bodies known as Newtonian theory. Thereby we return to the Newtonian concepts already mentioned in Section 3.5.1. We consider a flow past a body with Mach number  $M_1 \gg 1$  (Fig. 81). If we assume  $\gamma = 1$ , then, under strong hypersonic flow, the shock will coincide with the body surface [see Eq. (3.149)], and the infinitely compressed gas flows in an infinitesimally thin layer along the body surface. The pressure coefficient immediately behind the shock is given for  $\gamma = 1$ ,  $\zeta = \theta$ , and  $M_1 \sin \zeta \gg 1$  by Eq. (3.190) to be

$$c_p = 2 \sin^2 \theta. \quad (3.197)$$

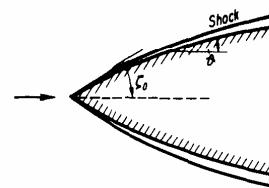


Fig. 81. Hypersonic flow with attached shock.

If the body surface is not curved (wedge), then this is also the pressure coefficient on the upper surface of the body. If, on the other hand, the body is curved, so that the gas between the shock and the body flows along a curved streamline, then a centrifugal force will occur and produce a pressure gradient perpendicular to the body surface. Although in the limit of  $\gamma = 1$  the layer between the shock and the body is infinitesimally thin, the density, and hence also the pressure gradient, in the layer are infinitely high, and therefore a finite difference in pressure between the body and a point immediately behind the shock is generated. We can estimate this pressure difference and take it into account in the theory. It has been shown, however, that a frequently useful approximation to the pressure coefficient on the body surface can be obtained as follows: If we denote by  $\theta_0$  the angle at the nose of the body, then, by (3.197), the pressure coefficient  $c_{p0}$  there is  $c_{p0} = 2 \sin^2 \theta_0$ . Then, (3.197) can also be written as

$$c_p = c_{p0} \frac{\sin^2 \theta}{\sin^2 \theta_0}. \quad (3.198)$$

Formula (3.198) has proven to be a truly useful approximation for the pressure coefficient in hypersonic flow.<sup>44</sup> On the basis of its derivation as indicated here, it is clear that it holds only under the assumption  $M_1 \sin \theta \gg 1$ . In particular, it is applicable to blunt bodies ( $\theta_0 = 90^\circ$ ) on the front surface facing the flow, when only  $M_1 \gg 1$  is assumed. It cannot say anything about the pressure on the back of the body. This is not an important limitation on the usefulness of the formula, since the pressure on the back never exceeds the order of magnitude of the freestream pressure  $p_1$ , while the pressure on the front is very much greater in hypersonic flow.

<sup>44</sup> L. Lees, Hypersonic Flow. Proc. 5th Int. Aero. Conf., Los Angeles, California, pp. 241–276. Inst. Aero. Sci., New York, 1955.

*Supplementary Remarks.* The formulas in this section have been derived for calorically ideal gases. However, they can be generalized in part to arbitrary gases without much difficulty. This point will be illustrated in the example of transonic Prandtl–Meyer waves: From definition (3.172) for the pressure coefficient, it follows that approximately for Mach numbers  $M_1$  and  $M_2$  close to 1:

$$\begin{aligned} c_p &= \frac{(\partial p/\partial M)_* (M_2 - M_1)}{\frac{1}{2}\rho_* a_*^2} = \frac{2}{\rho_*} \left( \frac{\partial \rho}{\partial M} \right)_* (M_2 - M_1) \\ &= 2 \left( \frac{\partial \ln \rho}{\partial M} \right)_* (M_2 - M_1). \end{aligned} \quad (3.182^*)$$

Here we have used the relation

$$\frac{\partial p}{\partial M} = \left( \frac{\partial p}{\partial \rho} \right) \left( \frac{\partial \rho}{\partial M} \right) = a^2 \frac{\partial \rho}{\partial M}.$$

The subscript (\*) means that the corresponding quantities are taken at locations of  $M = 1$ . On the other hand, from definition (3.169) for the Prandtl–Meyer function  $v(M)$ , we get, for Mach numbers near 1, the approximate result:

$$\begin{aligned} v &= \int_{M=1}^M (M^2 - 1)^{1/2} \frac{dU}{U} = \sqrt{2} \left( \frac{\partial \ln U}{\partial M} \right)_* \int_{M=1}^M (M - 1)^{1/2} dM \\ &= \frac{2^{3/2}}{3} \left( \frac{\partial \ln U}{\partial M} \right)_* (M - 1)^{3/2}. \end{aligned}$$

At the locations of  $M = 1$ , however,  $d(\rho U) = 0$  [see Eq. (3.6)]; thus

$$(\partial \ln U / \partial \ln \rho)_* = -1 \quad \text{and} \quad (\partial \ln U / \partial M)_* = -(\partial \ln \rho / \partial M)_*.$$

With this, we obtain

$$v = -\frac{2^{3/2}}{3} \left( \frac{\partial \ln \rho}{\partial M} \right)_* (M - 1)^{3/2}. \quad (3.183^*)$$

Combining (3.182\*) and (3.183\*), we get, after suitable transformation,

$$c_p = 2^{5/3} \theta^{2/3} \left( -\frac{\partial \ln \rho}{\partial M} \right)_*^{1/3} \left[ K - \left( \frac{3}{4\sqrt{2}} + K^{3/2} \right)^{2/3} \right]. \quad (3.184^*)$$

Here the transonic similarity parameter has the meaning

$$K = (M_1 - 1) (20)^{-2/3} (-\partial \ln \rho / \partial M)_*^{2/3}. \quad (3.185^*)$$

For a calorically ideal gas, we have from (3.15) the result

$$(\partial \ln \rho / \partial M)_* = -2/(\gamma + 1).$$

With this, Eqs. (3.182\*)–(3.185\*) again become (3.182)–(3.185), the former being generalizations of the latter. For  $K \gg 1$ , we get the results of linear theory from (3.184\*); as was stated earlier, these results are independent of the thermodynamic properties of the gas. The previous discussion on the validity limit of linear theory in the transonic regime for calorically ideal gases can be carried over without change to arbitrary gases.

### 3.8 Plane Steady Supersonic Flow

#### 3.8.1 BASIC EQUATIONS FOR THE VELOCITY FIELD

Before specializing to plane steady flow, we take the continuity equation (2.33) or

$$\partial \rho / \partial t + (\mathbf{v} \cdot \nabla \rho) + \rho \operatorname{div} \mathbf{v} = 0 \quad (3.199)$$

and transform it as follows: We take the scalar product of  $\mathbf{v}$  with the momentum equation (2.51) for inviscid flow without body forces and get

$$\rho \left( \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} \right) = -(\mathbf{v} \cdot \nabla p). \quad (3.200)$$

If the flow field is homentropic, we can replace  $\nabla p$  by  $a^2 \nabla \rho$ . If the flow field is nonhomentropic but steady, then, by Eq. (2.18),

$$(\mathbf{v} \cdot \nabla p) = Dp/Dt. \quad (3.201)$$

and since for each gas particle the entropy always remains constant,

$$Dp/Dt = a^2 D\rho/Dt = a^2 (\mathbf{v} \cdot \nabla \rho).$$

In both cases, Eq. (3.200) will become

$$\rho \left( \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} \right) = -a^2 (\mathbf{v} \cdot \nabla \rho). \quad (3.202)$$

If we now substitute  $(\mathbf{v} \cdot \nabla \varrho)$  into (3.199), we get the following form of the continuity equation:

$$\frac{1}{\varrho} \frac{\partial \varrho}{\partial t} - \frac{1}{a^2} \left( \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} \right) + \operatorname{div} \mathbf{v} = 0. \quad (3.203)$$

Now, specializing to steady flows ( $\partial \varrho / \partial t = 0$ ), we can write Eq. (3.203) explicitly as

$$\begin{aligned} u_x \left( 1 - \frac{u^2}{a^2} \right) + v_y \left( 1 - \frac{v^2}{a^2} \right) + w_z \left( 1 - \frac{w^2}{a^2} \right) - \frac{uv}{a^2} (u_y + v_x) \\ - \frac{uw}{a^2} (u_z + w_x) - \frac{vw}{a^2} (v_z + w_y) = 0. \end{aligned} \quad (3.204)$$

Let us now confine our attention to plane flows, for which  $w \equiv 0$  and  $u$  and  $v$  depend only on  $x$  and  $y$ . For the following discussion, it is advantageous to introduce the magnitude of the velocity  $U$  and the angle  $\theta$  as dependent variables in place of  $u$  and  $v$ , the angle  $\theta$  being defined as the angle between the velocity vector  $\mathbf{v}$  and the positive  $x$  axis:

$$\begin{aligned} u &= U \cos \theta, & u/a &= M \cos \theta, \\ v &= U \sin \theta, & v/a &= M \sin \theta. \end{aligned} \quad (3.205)$$

The derivatives appearing in Eq. (3.204) are of the type  $u_x = U_x \cos \theta - U \theta_x \sin \theta$ . After a short transformation and combination of terms, Eq. (3.204) becomes

$$U_x (M^2 - 1) \cos \theta + U_y (M^2 - 1) \sin \theta + U \theta_x \sin \theta - U \theta_y \cos \theta = 0. \quad (3.206)$$

By far the most important case, and the one we consider first, is isoenergetic homentropic flow, which, according to Section 2.5 is also irrotational. Then,  $v_x - u_y = 0$ , or, in terms of  $U$  and  $\theta$ ,

$$U_x \sin \theta - U_y \cos \theta + U \theta_x \cos \theta + U \theta_y \sin \theta = 0. \quad (3.207)$$

Now we further confine our study to  $M > 1$ , i.e., to supersonic flows. This restriction is very important for our discussion, since only supersonic flows permit the use of the method of characteristics which will be described. Mathematically, this is due to the fact that only for  $M > 1$  is the system of equations (3.206), (3.207), a hyperbolic system. For  $M < 1$  it is elliptic, and

there exist no real characteristic curves (denoted below by  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ ) in the  $x, y$  plane. The equations for one-dimensional unsteady flow are always hyperbolic, and can therefore always be solved by the method of characteristics (see Section 3.2).

For  $M > 1$ ,  $(M^2 - 1)^{1/2}$  is real. Multiplying Eq. (3.207) by  $U^{-1}(M^2 - 1)^{1/2}$  and adding the result to Eq. (3.206) multiplied by  $U^{-1}$ , we get

$$\begin{aligned} [U_x U^{-1} (M^2 - 1)^{1/2} + \theta_x] [(M^2 - 1)^{1/2} \cos \theta + \sin \theta] \\ + [U_y U^{-1} (M^2 - 1)^{1/2} + \theta_y] [(M^2 - 1)^{1/2} \sin \theta - \cos \theta] = 0. \end{aligned} \quad (3.208)$$

We now define the family of curves  $\mathfrak{C}_1$  in the  $x, y$  plane; for each curve of this family, we require:

$$\frac{dy}{dx} = \frac{(M^2 - 1)^{1/2} \sin \theta - \cos \theta}{(M^2 - 1)^{1/2} \cos \theta + \sin \theta} = \tan(\theta - \mu). \quad (3.209)$$

Here,  $\mu$  is the Mach angle:  $\mu = \arcsin M^{-1} = \arctan(M^2 - 1)^{-1/2}$ . According to (3.209), the curves  $\mathfrak{C}_1$  cut the streamlines everywhere at angle  $\mu$ , and point to the right of each streamline along the flow direction (Fig. 82). We

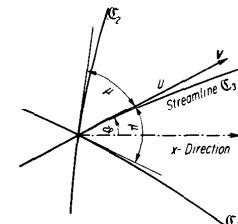


Fig. 82. Characteristics of steady inviscid supersonic flow in the flow plane ( $x, y$  plane).

again call these right-running Mach lines or right-running characteristics. In a manner entirely analogous to Section 3.2.2 for unsteady flow, we conclude from Eqs. (3.208) and (3.209) that along a curve  $\mathfrak{C}_1$  the changes  $d\theta$  and  $dU$  are connected by the following relation (also see the more detailed derivation of Eq. (3.219) in Section 3.8.3):

$$(M^2 - 1)^{1/2} \frac{dU}{U} + d\theta = 0. \quad (3.210)$$

Using the definition (3.169) for the function  $v(M)$ , we can interpret this as

$$dv + d\theta = 0 \quad \text{along} \quad \mathcal{C}_1. \quad (3.211)$$

We now define the family of curves  $\mathcal{C}_2$  (left-running Mach lines, Fig. 82) by

$$dy/dx = \tan(\theta + \mu), \quad (3.212)$$

and get, by subtracting the product of Eq. (3.207) with  $U^{-1}(M^2 - 1)^{1/2}$  from the product of Eq. (3.206) with  $U^{-1}$ :

$$dv - d\theta = 0 \quad \text{along} \quad \mathcal{C}_2. \quad (3.213)$$

The results (3.211) and (3.213) can be expressed in the following form:

$$v + \theta = \text{const} \quad \text{on} \quad \mathcal{C}_1, \quad (3.214)$$

$$v - \theta = \text{const} \quad \text{on} \quad \mathcal{C}_2. \quad (3.215)$$

### 3.8.2 METHOD OF CHARACTERISTICS FOR HOMENTROPIC FLOW

The relations (3.214) and (3.215) are formally identical to the relations (3.47) and (3.48). In Section 3.2.2, we explained how to use the method of characteristics to find the solution (in this case, the variables  $\theta$  and  $U$ ) in certain regions of the  $x, y$  plane under suitable initial conditions, so that we shall dispense with a detailed repeated discussion here.

In plane flow, the first initial-value problem has the following form: On a noncharacteristic curve segment  $A_0B_0$  in the  $x, y$  plane, the velocity vector  $\mathbf{v}$  is prescribed, i.e.,  $U$  and  $\theta$  are prescribed on  $A_0B_0$ . From  $U$  and the stagnation quantities  $h_t$  and  $s_t$  (or two equivalent state variables), which are constant in the entire homentropic, isoenergetic flow field,  $v$  is determined on the curve  $A_0B_0$ . We can then find the solution in the triangle  $A_0B_0P$  (Fig. 83) in the

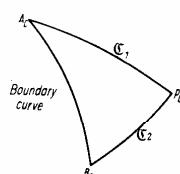


Fig. 83. Diagram for the first initial-value problem.

manner explained in Section 3.2.2. Naturally, as with one-dimensional unsteady flow, it is assumed that no shocks appear in the triangle  $A_0B_0P_0$ ; the occurrence of shocks can be noticed from the intersection of Mach lines of the same family. For continuously differentiable initial values and a smooth curve  $A_0B_0$ , there is always a neighborhood of  $A_0B_0$  contained in  $A_0B_0P_0$  in which a unique continuous solution exists. In favorable cases, this neighborhood fills out the entire triangle  $A_0B_0P_0$ .<sup>45</sup> The approximate solution for finite grid size, obtained from the method described in Section 3.2.2, converges to the exact solution as the grid size decreases to zero. Similarly, the explanation of the second and third initial-value problems given in Section 3.2.2 can be transferred to the present problem of steady supersonic flow without further difficulty.

As in the one-dimensional unsteady flow in Section 3.2.2, there are also many cases of plane steady supersonic flow with regions in which the flow is a simple wave, i.e., in this case, a Prandtl-Meyer wave. This always occurs when the flow field contains a region of uniform state. Since in a uniform region the characteristics are always parallel straight lines, such a region must have the shape of a parallelogram (Fig. 84; unless a portion of the

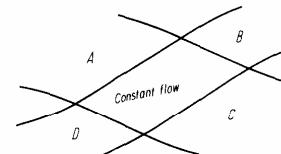


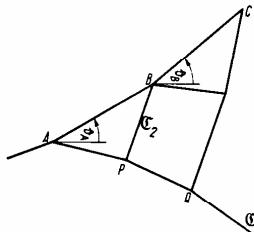
Fig. 84. Region of constant flow state bounded by four simple waves  $A, B, C$ , and  $D$ .

parallelogram ends in a wall or a freestream boundary). In regions  $A, B, C$ , and  $D$  adjacent to the four edges of the parallelogram, simple waves exist; this is readily shown by arguments completely analogous to those used in connection with Fig. 31.

For the solution of concrete flow problems, we must still have a knowledge of the boundary conditions for  $v$  and  $\theta$  which are to be satisfied on solid walls or freestream boundaries (these being the two most important types of

<sup>45</sup> See R. von Mises, "Mathematical Theory of Compressible Fluid Flow," Academic Press, New York, 1958.

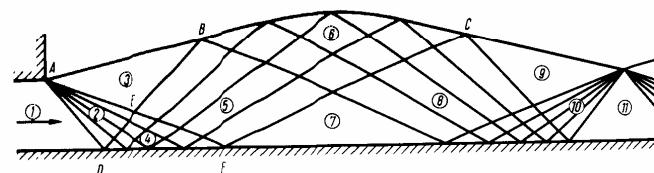
stream boundaries): Since the flow direction must be tangential to the wall,  $\theta$  is given on a solid wall. On a freestream boundary, the pressure of the gas is constant and equal to that of the gas at rest on the other side of the boundary. Fixing the pressure  $p$  also fixes the Mach number on the boundary (see Section 3.1). On the other hand, for constant  $M$ , the quantity  $v(M)$  is also constant and given on the boundary. In contrast to a solid wall, the shape of a freestream boundary is not known a priori, but must be determined piece by piece during the solution process by the method of characteristics. In this case, the situation sketched in Fig. 85 appears: In con-



**Fig. 85.** Construction of the free stream boundary  $ABC \dots$  by the method of characteristics

structing the characteristic net, the quantities  $v$  and  $\theta$  at the points  $A$ ,  $P$ , and  $Q$  will have been determined;  $A$  lies on the free boundary. The next point  $B$  on the free boundary is found from the intersection of the characteristic  $C_2$  from  $P$  (approximated as a straight line segment) and the free boundary (also approximated between  $A$  and  $B$  by a straight line using the flow direction already found at  $A$ ). Since  $v$  has a known constant value  $v_f$  on the free boundary,  $\theta_B$  results from the relation valid along  $PB$ :  $\theta_B = (\theta_p - v_p) + v_f$ . In this way, the broken line  $ABC$  is constructed as an approximation to the free boundary.

As an example, we consider the efflux of a supersonic stream from a nozzle when the external pressure is lower than that of the stream (see Section 3.1, p.82, case 8:  $p_3 > p_e$ ). The flow is sketched in Fig. 86; since the flow is symmetric with respect to the center line of the jet, we can replace the center line by a solid wall. The incoming parallel flow 1 will expand to the external pressure through a Prandtl-Meyer wave 2. The free boundary starting from the corner  $A$  is a straight line up to the point  $B$ , and it is



**Fig. 86.** Supersonic jet into an under-pressure region.

directed at an angle  $\theta$  with respect to the initial flow,  $\theta$  being the deflection produced by the Prandtl-Meyer flow 2. Region 3 is another parallel flow of constant state. Region 4 may be regarded as the intersection of the Prandtl-Meyer wave 2 and its reflected wave from the flat wall. To compute the flow in 4, we must solve the third initial-value problem: On the characteristic  $DE$ ,  $v$  and  $\theta$  are both known, while on  $DF$  only  $\theta$  is known. By the constancy of the flow variables in region 3, region 5 must again be a Prandtl-Meyer wave (it may be regarded as the reflection of wave 2 from the wall). This wave is determined when  $v$  and  $\theta$  on  $EF$  are determined. In region 6, the wave 5 intersects its reflected wave from the free boundary, which then continues outside the region of interaction as a Prandtl-Meyer flow 8. In region 6, the third initial-value problem must again be solved, with the added complication that the free boundary between  $B$  and  $C$  must be constructed together with the solution. Region 7 is another uniform region, imbedded between the two simple waves 5 and 8. In contrast to the expansion waves 2 and 5, wave 8 is a compression wave. Region 9 is again a region of constant flow state, and has a straight free boundary, while 10 is a compression wave. In 11, the flow is exactly the same as that in 1, and the entire flow pattern periodically repeats itself downstream. The numerical method of characteristics for the pointwise construction of the solution is, in this case, only necessary for the regions 4, 6, etc.; in all other regions, the flow is determined either as a simple wave or as a uniform parallel flow.

For the sake of completeness, we should also sketch the corresponding flow pattern for the case when the pressure of the parallel flow at the nozzle exit is below the external pressure (Section 3.1, p. 82, case 6:  $p_2 > p_e > p_3$ ). Instead of the Prandtl-Meyer wave 2, there now appears a shock  $S_1$ , which then reflects on the wall as shock  $S_2$  (under certain conditions, a Mach reflection may occur, see Section 3.5.1). This shock  $S_2$  will in turn reflect

from the free boundary as a centered Prandtl-Meyer expansion wave, which just compensates for the pressure increase in  $S_2$ . Naturally, downstream of  $S_2$ , we have exactly the same flow as an underexpanded nozzle flow (Fig. 87).

We now come back to the problem, already treated in another way in Section 3.6.2, of a constant parallel flow of Mach number  $M_1 > 1$  expanding to another such flow of  $M_2 > M_1$  (Fig. 88). The contour of the wall between

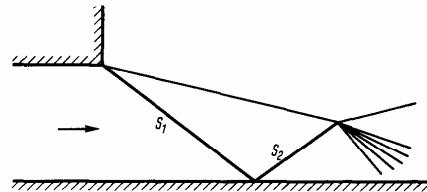


Fig. 87. Supersonic jet into an over-pressure region.

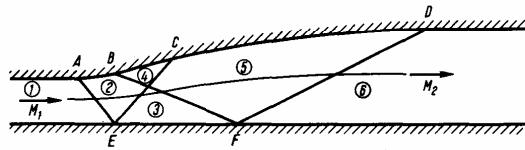


Fig. 88. Expansion of a parallel flow of Mach number  $M_1 > 1$  into a parallel flow of Mach number  $M_2 > M_1$  in the same direction.

$A$  and  $B$  can be selected arbitrarily. The Prandtl-Meyer wave 2 is determined by the data on  $AB$ . Downstream of  $B$ , the upper wall will first be extended as a straight line in the direction attained at  $B$ . In 3, we must again solve the third initial-value problem. Wave 5 is a Prandtl-Meyer wave, which adjoins the two uniform regions 4 and 6. The wall contour  $CD$  is determined from the wave 5. Downstream of  $D$ , the upper wall again runs parallel to the lower wall. Reflecting the flow pattern with respect to the lower wall, we get a symmetric nozzle flow. If the Mach numbers  $M_1$  and  $M_2$  are given and the problem consists of finding a suitable wall contour  $ABCD$ , then we proceed as follows: We select the wall contour starting from  $A$  up to a point  $B$  with wall angle  $\theta$  monotonically increasing in the flow direction. We can then calculate the flow up to the characteristic  $BF$  and find the Mach number

$M$  at  $F$ . If this Mach number  $M(F)$  agrees with the desired given value  $M_2$ , then we continue the upper wall from  $C$  to  $D$  in the manner explained before. In some cases, the parallel flow 4 and the straight portion of the wall  $BC$  do not appear at all. This can be illuminated by considering the fact that any streamline can be used to replace a solid wall, including, for example, the streamline drawn in Fig. 88, which does not penetrate the region 4.

The construction of flow fields as just explained is important in the determination of the wall contours of symmetric Laval nozzles for the production of disturbance-free parallel gas streams (see Section 3.6.2). The gas flows through the convergent part of the nozzle with subsonic velocity and attains sonic velocity in the neighborhood of the throat. An initial characteristic for the method of characteristics, corresponding to the Mach line  $AE$ , is in general found from an analytic calculation of the flow field in the neighborhood of the throat (transonic flow,  $M \approx 1$ ).

*Supplementary Remarks.* 1. The families of characteristics  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are not fixed given curves in the  $x, y$  plane; they depend on the flow field in the  $x, y$  plane, and must therefore be determined together with the solution to a concrete problem. This is different if we consider the  $u, v$  plane, i.e., velocity or hodograph plane, instead of the flow plane, or  $x, y$  plane. We imagine a point in this plane specified by the polar coordinates  $U$  and  $\theta$ . Each point in the flow plane corresponds to exactly one point in the hodograph plane. The converse does not hold in general; for example, in a parallel flow with constant velocity, all points of the flow plane will be transformed into one single point in the hodograph plane,—namely, the endpoint of the constant velocity vector. The characteristics  $\mathcal{C}_1$  of the flow plane form images in the hodograph plane as curves  $\mathcal{C}_1^*$  on which  $v + \theta = \text{const}$ , in accordance with Eq. (3.214), and the characteristics  $\mathcal{C}_2$  form images as curves  $\mathcal{C}_2^*$  on which  $v - \theta = \text{const}$ . Since for a given stagnation state,  $v$  in a homentropic, isoenergetic flow field depends only on  $M$ , i.e., only on  $U$ , we can draw these curves in the hodograph plane. For a calorically ideal gas, these are epicycloids, obtained by rolling a circle of diameter  $U_{\max} - a_*$  outside a circle of radius  $a_*$ . Figure 89 shows a segment of a  $\mathcal{C}_1^*$  curve in the hodograph plane. The Mach angle  $\mu$  can be read off immediately from this characteristic diagram in the hodograph plane: Since  $\overline{AP}^* = -U d\theta$  and  $\overline{AQ}^* = dU$ , we get the following relation for the angle  $AP^*Q^*$ :

$$\tan(AP^*Q^*) = \overline{AQ}^*/\overline{AP}^* = -dU/U d\theta.$$

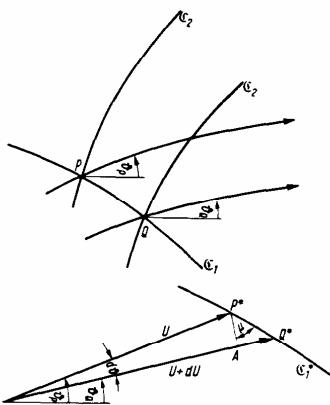


Fig. 89. Characteristics  $C_1$  and  $C_2$  in the flow plane (top) and characteristic  $C_1^*$  in the hodograph plane (below).

Now, however, on  $C_1^*$ , we have by Eq. (3.210),

$$dU = -U(M^2 - 1)^{-\frac{1}{2}} d\theta,$$

so that

$$\tan(AP^*Q^*) = (M^2 - 1)^{-\frac{1}{2}} = \tan \mu.$$

The angle  $AP^*Q^*$  is thus the same as the Mach angle  $\mu$ .

We can apply the geometric properties of the characteristics in the hodograph plane to the graphical construction of plane supersonic flows. If shocks occur in the flow field, it is convenient to use a *shock polar* together with the characteristic diagram in the hodograph plane. A shock polar is the geometric locus of the endpoints of all vectors  $v_2$  in the hodograph plane which are the possible velocities downstream of a shock for a given upstream velocity  $v_1$  ( $|v_1| > a_1$ ). Figure 90 shows a shock polar. At the initial point  $v_1$  the polar is tangent to the characteristic through this point, since for infinitesimal-strength shocks the shock angle  $\zeta$  coincides with the Mach angle  $\mu$ . The construction of the shock angle  $\zeta$  shown in Fig. 90 is made on the basis that the velocity component tangential to the shock is unchanged through the shock.

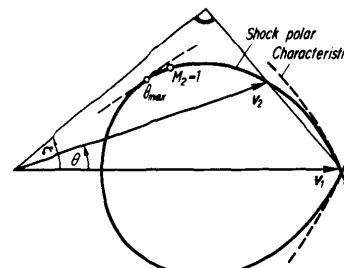


Fig. 90. Shock polar.

2. Using the method of characteristics explained above, the flow quantities at the corner points of the characteristic net (which is approximated by straight-line segments) are determined. Thus, we sometimes call this a *net-point* or *grid-point* method. However, we can just as well imagine the flow state to be constant in each mesh of the characteristic net, which covers the entire plane. The values of the flow quantities will jump from one mesh to the next across each mesh boundary; the mesh boundaries thus play the role of the wavefronts of the Mach waves. If we cross the boundary of a mesh which is a left-running wavefront, then the jumps in  $v$  and  $\theta$  must satisfy the relation  $\Delta v + \Delta \theta = 0$ , which is valid along a right-running characteristic  $C_1$ ; similarly, if we cross a right-running characteristic which is a mesh boundary, then  $\Delta v - \Delta \theta = 0$ . In this manner, we can easily develop a numerical scheme which works in a manner similar to the net-point method and which will be called the *field-line* method. In this method, we can uniquely define the directions of the piecewise linear approximating wavefronts by identifying them with the characteristic directions in the mesh immediately upstream of the wave front. Figure 70a can now be interpreted as the approximation of a continuous Prandtl-Meyer flow by the field-line method. Since, due to the especially simple structure of a Prandtl-Meyer wave, it is not necessary to draw in the characteristics  $C_2$ , the meshes of the characteristic net degenerate in this case to the strip-like regions of Fig. 70a.

3. The method of characteristics, derived for plane supersonic flows, can also be carried over after some minor modifications to axisymmetric supersonic flows with no azimuthal velocity components. It suffices in these flows

to consider only the flow in the meridian plane. Let the  $x$  axis coincide with the axis of symmetry, and let  $y$  denote the perpendicular distance from this axis. Let the velocity components in the  $x$  and  $y$  directions be  $u$  and  $v$ , respectively. The magnitude of the velocity is  $U$ , and  $\theta$  denotes the angle between the velocity vector and the  $x$  axis. The starting point is now Eq. (3.203). In cylindrical coordinates  $x, y$  (by axisymmetry, all quantities are independent of the third, or azimuthal, coordinate) the expression for  $\operatorname{div} \mathbf{v}$  is  $\operatorname{div} \mathbf{v} = u_x + v_y + (v/y)$ , while the expression for  $Dv/Dt$  has the same form as in plane flow; we thus obtain from (3.203) the following relation in place of (3.204):

$$u_x \left(1 - \frac{u^2}{a^2}\right) + v_y \left(1 - \frac{v^2}{a^2}\right) - \frac{uv}{a^2}(u_y + v_x) + \frac{v}{y} = 0. \quad (3.204^*)$$

This differs from (3.204) for  $w = 0$  only in the term  $v/y$ . After transformation to the variables  $U$  and  $\theta$  according to (3.205), there results from (3.204\*) a relation corresponding to (3.206) and differing from it only in the presence on the left side of an additional term  $-(U \sin \theta)/y$ , which came from  $v/y$ . Equation (3.207), the irrotationality condition, is unchanged. Carrying out on this equation the transformation explained more precisely in connection with Eq. (3.207), we obtain an equation corresponding to Eq. (3.208) and differing from it only in the presence of the additional term  $-(\sin \theta)/y$  on the left side. From this equation, we finally obtain, after a short transformation, the following in place of (3.211) for the characteristics  $\mathfrak{C}_1$  defined by (3.209):

$$dv + d\theta = \frac{\sin \theta}{yM \cos(\theta - \mu)} dx \quad \text{along } \mathfrak{C}_1, \quad (3.211^*)$$

and, in an entirely analogous manner,

$$dv - d\theta = \frac{\sin \theta}{yM \cos(\theta + \mu)} dx \quad \text{along } \mathfrak{C}_2. \quad (3.213^*)$$

The quantity  $dx$  appearing on the right refers to the change of the coordinate  $x$  when advancing along the particular characteristic.

In the same manner as explained before for plane flow, the equations (3.211\*) and (3.213\*) can be made the starting point of a numerical method for calculating axisymmetric supersonic flows. A difficulty arises in the use of this method of characteristics when one starts advancing along a charac-

teristic from an initial point  $y = 0$  on the axis of rotation, since, for such a case, the values of  $y$  and  $\theta$  are both zero, and the value of  $\sin \theta/y$  is at first indeterminate. This difficulty can be overcome if we approximate these values by the corresponding values at the next grid point away from the axis.

### 3.8.3 METHOD OF CHARACTERISTICS FOR NONHOMENTROPIC FLOW

In the flow past a profile discussed in Section 3.6.2, the flow downstream of the curved shock wave from the nose of the profile is indeed isoenergetic but no longer homentropic: The specific entropy varies from streamline to streamline. Thus, by Crocco's theorem, the flow is no longer irrotational. To calculate such flow fields, the method of characteristics must be generalized to account for the entropy gradients. As was already established, Eq. (3.204) [and thus, Eq. (3.206)] still holds in plane, nonhomentropic flow; but Eq. (3.207), asserting the irrotationality of a homentropic flow field, is no longer valid. If we mark at each point in the flow field the direction of the velocity by a unit vector  $\mathbf{t}$  and the direction perpendicular to it by a unit vector  $\mathbf{n}$ , as shown in Fig. 91, then we have

$$\operatorname{grad} s = \mathbf{n} \partial s / \partial n$$

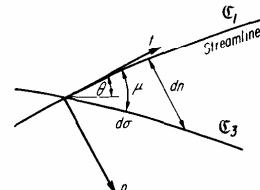


Fig. 91. Diagram for the derivation of Eq. (3.216).

and

$$\mathbf{w} \equiv \operatorname{curl} \mathbf{v} = (v_x - u_y) \mathbf{n} \times \mathbf{t},$$

where  $\partial/\partial n$  denotes differentiation in the direction of  $\mathbf{n}$ . Furthermore,

$$-\mathbf{v} \times \mathbf{w} = -U(v_x - u_y) \mathbf{t} \times (\mathbf{n} \times \mathbf{t}) = -U(v_x - u_y) \mathbf{n}.$$

Substituting these expressions into Eq. (2.78) and replacing  $u$  and  $v$  by  $U$  and

$\theta$  [according to (3.205)], then we obtain the following relation in place of Eq. (3.207):

$$U_x \sin \theta - U_y \cos \theta + U \theta_x \cos \theta + U \theta_y \sin \theta = - \frac{T}{U} \frac{\partial s}{\partial n}. \quad (3.216)$$

Multiplying this equation by  $U^{-1}(M^2 - 1)^{1/2}$  and multiplying Eq. (3.206) by  $U^{-1}$  and then adding them, we get an equation nearly the same as (3.208), except that the right-hand side has the expression

$$- \frac{T}{U^2} (M^2 - 1)^{\frac{1}{2}} \frac{\partial s}{\partial n} \quad (3.217)$$

instead of zero. If we denote for the moment the arc length along a right-running characteristic  $\mathfrak{C}_1$  by  $\sigma$  and differentiation in the direction of this characteristic by  $\partial/\partial\sigma$ , then, for advancing along  $\mathfrak{C}_1$  (Fig. 91), we have

$$\frac{\partial}{\partial\sigma} = \cos(\theta - \mu) \frac{\partial}{\partial x} + \sin(\theta - \mu) \frac{\partial}{\partial y}. \quad (3.218)$$

Dividing Eq. (3.208) (with the right-hand side given by (3.217) instead of zero) by  $M$ , taking into account (3.218), and using the relations

$$\cos(\theta - \mu) = M^{-1} [(M^2 - 1)^{\frac{1}{2}} \cos \theta + \sin \theta]$$

and

$$\sin(\theta - \mu) = M^{-1} [(M^2 - 1)^{\frac{1}{2}} \sin \theta - \cos \theta],$$

we get:

$$\frac{(M^2 - 1)^{\frac{1}{2}}}{U} \frac{\partial U}{\partial \sigma} + \frac{\partial \theta}{\partial \sigma} = - \frac{T}{U^2} \frac{(M^2 - 1)^{\frac{1}{2}}}{M} \frac{\partial s}{\partial n},$$

or, since  $M^{-1} d\sigma = \sin \mu d\sigma = dn$ , with  $dn$  the element of length along the direction perpendicular to the streamline (see Fig. 91),

$$\frac{(M^2 - 1)^{\frac{1}{2}}}{U} dU + d\theta = - \frac{T}{U^2} (M^2 - 1)^{\frac{1}{2}} ds, \quad (3.219)$$

i.e.

$$dv + d\theta = - \frac{T}{U^2} (M^2 - 1)^{\frac{1}{2}} ds \quad \text{along } \mathfrak{C}_1. \quad (3.220)$$

In a similar way, we also obtain

$$dv - d\theta = - \frac{T}{U^2} (M^2 - 1)^{\frac{1}{2}} ds \quad \text{along } \mathfrak{C}_2. \quad (3.221)$$

Finally, if we define the streamlines whose direction at every point is given by  $dy/dx = \tan \theta$  as the third family of characteristics, then we can write the conservation of specific entropy along streamlines in the form

$$ds = 0 \quad \text{along } \mathfrak{C}_3. \quad (3.222)$$

The relations hold for isoenergetic ( $h_t = \text{const}$ ), plane, inviscid, steady supersonic flows. If, in addition, the flow is homentropic, then  $ds = 0$  not only along  $\mathfrak{C}_3$ , but also along  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ ; then the relations (3.220) and (3.221) revert to (3.211) and (3.213). In the numerical calculation of flow fields, we must note that in the relation between  $U$  and  $M$  in steady flow (Section 3.1), the stagnation enthalpy  $h_t$  as well as the entropy (which in general varies from streamline to streamline, in contrast to the case in Section 3.8.2) both enter. The same is true for the relation between  $T$  and  $M$ . An exception is the case of a calorically ideal gas, for which the relation between  $U$  (or  $T$ ) and  $M$  depends only on  $h_t$  and not on  $s_t$ . For the calorically ideal gas, moreover, the right side of Eqs. (3.220) and (3.221) can be further simplified: Using  $a^2 = \gamma RT$  [Eq. (1.135)] and  $M^2 = U^2/a^2$ , we have

$$\frac{T}{U^2} (M^2 - 1)^{\frac{1}{2}} ds = \frac{(M^2 - 1)^{\frac{1}{2}}}{\gamma M^2} d\left(\frac{s}{R}\right). \quad (3.223)$$

The first initial-value problem now has the following form: On a non-characteristic curve  $A_0B_0$  (nowhere is it tangent to the characteristics  $\mathfrak{C}_1$ ,  $\mathfrak{C}_2$ , or  $\mathfrak{C}_3$ ), the velocity vector  $v$  and the entropy  $s$  are prescribed. From  $v$ , both  $U = |v|$  and  $\theta$  are determined. From  $s$  and  $h = h_t - \frac{1}{2}U^2$ , we calculate  $a(h; s)$  on  $A_0B_0$ , and from this, the Mach number  $M = U/a$  and the quantity  $v(M; h_t, s_t)$ . Similarly, we can calculate the temperature  $T = T(h, s)$  on  $A_0B_0$ , or we can read it off a Mollier diagram; for a calorically ideal gas, the temperature need not be determined, since by Eq. (3.223),  $T$  can be eliminated from Eqs. (3.220) and (3.221). We calculate the quantities  $v$ ,  $\theta$ , and  $s$  for a point not lying on  $A_0B_0$  as follows: First, we subdivide  $A_0B_0$ , and let  $AB$  be a segment of the subdivision. Then we find a point  $P$  as the intersection of the Mach lines  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  issuing from  $A$  and  $B$ , each approximated by a straight line. Through  $P$  we draw the straight line  $PC$  in the streamline direction backwards to determine the point  $C$  on the segment  $AB$ . The streamline direction at  $P$  is determined by bisecting the angle made by the two Mach lines at  $P$ . Eq. (3.220) holds along  $AP$ , Eq. (3.221) along  $BP$ , and Eq. (3.222) along  $CP$ , so that  $v_p$ ,  $\theta_p$ , and  $s_p$  can be found from the initial values at  $A$ ,  $B$ ,  $C$ . We can also improve these values by iterating, as

explained at the end of Section 3.2.5. This method simplifies significantly for a calorically ideal gas: By prescribing  $M$ ,  $\theta$ , and  $s$  on  $A_0B_0$  and using expression (3.170) for  $v(M)$  and relation (3.223), we can find the quantities  $M$ ,  $\theta$ , and  $s$  without the use of any other relations by the method of characteristics just described.

In the flow past a profile discussed in Section 3.6.2, the shock wave is not determined a priori, but must be constructed piece by piece together with the calculation of the flow field by the method of characteristics. Let us explain the principle for a shock issuing from the nose of a profile (Fig. 92):

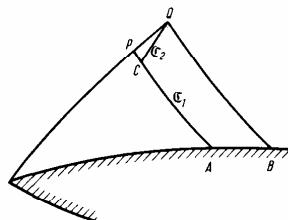


Fig. 92. Construction of the shock  $PQ\dots$  by the method of characteristics.

In front of the shock we have a homentropic parallel flow of Mach number  $M_\infty$ . We assume that the shock is already known up to a point  $P$ , and that the flow is already determined behind the shock up to the  $C_1$  characteristic  $PA$  through the point  $P$ . We first extend the shock by a segment of a straight line, using the inclination of the shock  $\zeta_p$  previously calculated at  $P$ . In addition, we select a point  $C$  near  $P$  on the characteristic  $PA$ . The characteristic  $C_2$  through  $C$  will be approximated by a straight-line segment until it intersects the shock at  $Q$ . Since the flow in front of the shock is given, all the flow variables behind the shock at  $Q$  depend on one parameter, which we can select to be the flow direction  $\theta_Q$ . Thus,  $v_Q$ ,  $M_Q$ , and  $s_Q$  are, by the shock relations, known functions of  $\theta_Q$ . We substitute this function into relation (3.221), valid along  $CQ$ , approximate  $dv$  by  $v_Q - v_C$ , etc., and calculate  $\theta_Q$ . But then, the flow state at  $Q$  immediately behind the shock is known. We can then use the method of characteristics and calculate the flow states at the grid points on the  $C_1$  characteristic  $QB$ , and finally extend the shock in the direction  $\zeta_Q$  calculated for  $Q$ . This process can then be repeated.

### 3.9 Theory of Small Perturbations (Acoustic Approximation)

#### 3.9.1 LINEAR WAVE EQUATION

We continue the discussion of Section 3.2.1: Let there be given a body of gas initially at rest and then set in motion by a small disturbance. In contrast to Section 3.2.1, we now no longer limit ourselves to the one-dimensional motion of the gas in the  $x$  direction, but shall consider general motion in space. Finally, for pure convenience, we shall somewhat alter the notation of Section 3.2.1 and use the subscript “ $\infty$ ” instead of “ $0$ ” to denote the unperturbed initial state of the gas; the perturbation quantities will be denoted again by  $p'$ ,  $\varrho'$ , and velocity  $\mathbf{v}'$ . As in Section 3.2.1, we linearize the continuity equation (2.33) and the momentum equation (2.51) for the perturbation quantities:

$$\partial \varrho' / \partial t + \varrho_\infty \operatorname{div} \mathbf{v}' = 0, \quad (3.224)$$

$$\varrho_\infty \partial \mathbf{v}' / \partial t = -\operatorname{grad} p', \quad (3.225)$$

or, since  $dp = a^2 d\varrho$  (see p. 86),

$$\varrho_\infty \partial \mathbf{v}' / \partial t = -a_\infty^{-2} \operatorname{grad} \varrho'. \quad (3.226)$$

Applying the operations  $a_\infty^{-2} \operatorname{grad}$  to Eq. (3.224) and  $\partial / \partial t$  to Eq. (3.226) and subtracting the resulting equations from each other, we eliminate  $\varrho'$  and obtain for the velocity  $\mathbf{v}'$

$$\partial^2 \mathbf{v}' / \partial t^2 = a_\infty^{-2} \operatorname{grad} \operatorname{div} \mathbf{v}'. \quad (3.227)$$

On the other hand, it follows from applying the operation  $\operatorname{curl}$  to Eq. (3.225) that  $(\partial / \partial t) \operatorname{curl} \mathbf{v}' = 0$ . If the gas at some arbitrary instant is at rest everywhere and therefore irrotational, then this states that the gas will remain irrotational for all time. This also follows from the results of Section 2.5, and is the reason why irrotational flow is of such great importance. We can then let the velocity field  $\mathbf{v}'$  to be the gradient of a scalar potential  $\phi$ :<sup>46</sup>  $\mathbf{v}' = \operatorname{grad} \phi$ . Equation (3.227) then becomes

$$\operatorname{grad}(\phi_n - a_\infty^{-2} \Delta \phi) = 0.$$

<sup>46</sup> If  $\operatorname{curl} \mathbf{v}' \equiv 0$ , then there always exists a velocity potential  $\phi$ , although not necessarily a unique one.

From this, it follows that the quantity  $\phi_{tt} - a_\infty^2 \Delta\phi$  depends only on time and not on the space variables  $x, y, z$ . Thus, without loss of generality, we may set

$$\phi_{tt} - a_\infty^2 \Delta\phi = 0, \quad (3.228)$$

since we can add an arbitrary function of time to  $\phi$  without changing the perturbation velocity  $v' = \nabla\phi$  and therefore also any of the other perturbation quantities ( $p', q'$ , etc.). From Eq. (3.225), we get

$$\nabla(p' + \rho_\infty \phi_t) = 0.$$

From this, we have for the perturbation pressure  $p'$

$$p' = -\rho_\infty \phi_t. \quad (3.229)$$

Here we have again set to zero a possible additive function of time on the right side. The justification for this is that in the problems being considered here, the perturbed variables, including  $p'$ , vanish identically for all time in some regions of space (e.g., far in front of a slender body moving through a gas at rest). Since we can then without loss of generality set  $\phi = 0$  there, Eq. (3.229) immediately follows.

In the following discussion, we shall essentially fix our attention to a slender body in steady flow. It is then convenient to introduce a coordinate system  $\bar{x}, \bar{y}, \bar{z}$ , which moves in the negative  $x$  direction with respect to the  $x, y, z$  system at a constant velocity  $u_\infty$  (later on we shall identify  $u_\infty$  with the velocity of the body moving relative to the gas at rest). The independent variables are thus transformed as follows:

$$\bar{x} = x + u_\infty t; \quad \bar{y} = y; \quad \bar{z} = z; \quad \bar{t} = t. \quad (3.230)$$

For the space and time derivatives appearing in Eq. (3.228), we have

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial \bar{x}^2}; & \frac{\partial^2}{\partial y^2} &= \frac{\partial^2}{\partial \bar{y}^2}; & \frac{\partial^2}{\partial z^2} &= \frac{\partial^2}{\partial \bar{z}^2}; \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial \bar{t}} + u_\infty \frac{\partial}{\partial \bar{x}}; & \frac{\partial^2}{\partial t^2} &= \frac{\partial^2}{\partial \bar{t}^2} + 2u_\infty \frac{\partial^2}{\partial \bar{t} \partial \bar{x}} + u_\infty^2 \frac{\partial^2}{\partial \bar{x}^2}. \end{aligned} \quad (3.231)$$

Since from now on we shall always use this new coordinate system, we can for the sake of simplicity drop the bars and obtain, after transforming (3.228) to this new coordinate system and using the notation  $M_\infty = u_\infty/a_\infty$ ,

$$(1 - M_\infty^2) \phi_{xx} + \phi_{yy} + \phi_{zz} = (2M_\infty/a_\infty) \phi_{xt} + (1/a_\infty^2) \phi_{tt}. \quad (3.232)$$

For the perturbation pressure  $p'$ , we have from Eq. (3.229) with (3.231)

$$p' = -\rho_\infty (\phi_t + u_\infty \phi_x). \quad (3.233)$$

The velocity of the gas relative to the new coordinate system is

$$v = u_\infty e_x + \nabla\phi = u_\infty e_x + v', \quad (3.234)$$

where  $e_x$  is the unit vector in the  $x$  direction.

These formulas will now be specialized to flows which are stationary in the new coordinate system. This is, for example, the case if the motion of the gas is created by a body moving at constant velocity  $u_\infty$  in the negative  $x$  direction in a gas at rest. In the coordinate system moving with the body, we then have a steady flow: The undisturbed flow past the body has a velocity  $u_\infty$  in the  $x$  direction. The assumption of small perturbations (i.e., that perturbations to the uniform flow velocity  $u_\infty$  and to the variables  $p_\infty, \rho_\infty$ , etc., be small) without which linearization of the equation is unjustified, is then satisfied if the body is so slender that the disturbance velocities generated by it are small compared to  $u_\infty$ . If the surface of the body is given by the equation  $F(x, y, z) = 0$ , then the necessary condition of slenderness is the relation

$$|F_x| \ll (F_x^2 + F_y^2 + F_z^2)^{1/2}, \quad (3.235)$$

which must be satisfied everywhere on the body surface; in other words, the unit vector normal to the body surface must be everywhere almost perpendicular to the  $x$  axis.<sup>47</sup> As indicated in the discussion in Section 3.7, we must here remark that both in the transonic and hypersonic flow regimes, the slenderness condition (3.235) is no longer sufficient to justify linearization of the gas dynamical equations. The flow processes in these regimes can, in general, only be described in a meaningful way by taking the nonlinear effects into account.

By the assumption of steady flow, the time derivatives in Eqs. (3.232) and (3.233) drop out, and we get

$$(1 - M_\infty^2) \phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad (3.236)$$

$$p' = -\rho_\infty u_\infty \phi_x = -\rho_\infty u_\infty v', \quad (3.237)$$

<sup>47</sup> In subsonic flow past an airfoil ( $M_\infty > 1$ ), the leading edge is generally rounded to prevent separation of the flow at this edge. The slenderness condition (3.235) is then no longer fulfilled near the leading edge, and the actual flow in this region is different from that calculated from linear theory.

where  $u'$  is the  $x$  component of the perturbation velocity  $\mathbf{v}'$ ; [also see formula (3.154)]. In place of  $p'$  it is convenient to introduce the pressure coefficient  $c_p = (p - p_\infty)/(\frac{1}{2}\rho_\infty u_\infty^2) = p'/( \frac{1}{2}\rho_\infty u_\infty^2)$  [see Eq. (3.172)]:

$$c_p = -2\phi_x/u_\infty = -2u'/u_\infty. \quad (3.238)$$

This relation for the pressure coefficient holds for the plane flow past a thin profile and also for the three-dimensional flow past a wing-shaped body occupying a certain region in the  $x, y$  plane and having only a slight thickness in the  $z$  direction (Fig. 94); for such bodies, the results are reliably valid in the entire flow field. In the flow past a spindle-shaped body, e.g., a slender body of revolution, however, Eq. (3.238) can lead to wrong results on the surface and in the immediate vicinity of the body, and a correction is required. It turns out that in the immediate neighborhood of the body, the components  $v'$  and  $w'$  of the perturbation velocity perpendicular to the flow direction can be much greater than the component  $u'$  along the flow direction, although all components are still small compared to  $u_\infty$ . Thus,  $(v'^2 + w'^2)/u_\infty^2$  can reach the same order of magnitude as  $u'/u_\infty$ . Detailed investigations have shown that in this case we can still use the linear equation (3.236) to calculate  $u'$ ,  $v'$ , and  $w'$ , but the relation

$$c_p = -2 \frac{u'}{u_\infty} - \frac{v'^2 + w'^2}{u_\infty^2} \quad (3.238^*)$$

gives a better approximation for the pressure coefficient than (3.238). In what follows, whenever we speak of the pressure coefficient, we shall confine ourselves to cases in which the relation (3.238) is applicable.

### 3.9.2 FLOW PAST A WAVY WALL

As a first application of these ideas, we shall consider the plane flow along a wavy wall. The uniform parallel flow with velocity  $u_\infty$  will be distorted by the waviness of the wall. Let the wall contour be given by

$$y_w = A \cos kx \quad (3.239)$$

(Fig. 93), in which, by Eq. (3.235), the condition  $|Ak| \ll 1$  must be satisfied, i.e., the amplitude  $A$  of the waviness must be much smaller than the wavelength  $\lambda = 2\pi k^{-1}$ . The perturbation potential  $\phi$  satisfies Eq. (2.236), in which the term  $\phi_{zz}$  drops out because of independence of  $z$ . At the wall, the flow

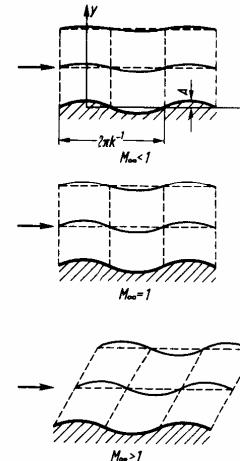


Fig. 93. Flow past a wavy wall in the linear approximation.

must be tangential to the wall. Thus,  $\phi$  must satisfy the boundary condition

$$\frac{\phi_y}{u_\infty + \phi_x} = \frac{dy_w}{dx} = -Ak \sin kx$$

at the wall. In the linear approximation, we neglect  $\phi_x$  when compared to  $u_\infty$  in the denominator on the left side, and, moreover, the boundary condition is not to be satisfied at the wall itself but at the mean wall contour, namely, the straight line  $y = 0$ . In other words, we seek a solution of Eq. (3.236) which satisfies the boundary condition

$$\phi_y = -u_\infty Ak \sin kx \quad (3.240)$$

on  $y = 0$ . Because of the periodicity of the wall, we expect the solution  $\phi$  to be periodic with the same wavelength in the  $x$  direction, and thus use a trial solution

$$\phi = [B \cos k(x - \varepsilon y) + C \sin k(x - \varepsilon y)] \exp(-k\delta y) \quad (3.241)$$

with real constants  $\varepsilon$ ,  $\delta$ ,  $B$ , and  $C$ . Upon substituting (3.241) into Eq. (3.236),

we obtain the following relations for  $\varepsilon$  and  $\delta$ :

$$(1 - M_\infty^2) + \varepsilon^2 - \delta^2 = 0, \quad (3.242)$$

$$\varepsilon\delta = 0. \quad (3.243)$$

By Eq. (3.243), at least one of the quantities  $\varepsilon$  and  $\delta$  must vanish. Since, on the other hand,  $\varepsilon$  and  $\delta$  must by assumption be real, it results from Eq. (3.242) that

$$\left. \begin{aligned} \varepsilon = 0, & \quad \delta = + (1 - M_\infty^2)^{\frac{1}{2}} \quad \text{for } M_\infty < 1, \\ \varepsilon = + (M_\infty^2 - 1)^{\frac{1}{2}}, & \quad \delta = 0 \quad \text{for } M_\infty > 1. \end{aligned} \right\} \quad (3.244)$$

In both cases, the positive sign is chosen for the square root sign, although for different reasons: For  $M_\infty < 1$ , this sign is chosen from the requirement that the disturbances in the parallel stream must decay and not increase without bound with increasing distance from the wall. For  $M_\infty > 1$ , this sign is chosen on the basis explained in Section 3.5.2: The disturbances from the wall should propagate downstream and not upstream along Mach lines (Fig. 93). The constants  $B$  and  $C$  are determined by substituting the trial solution (3.241) into the boundary condition (3.240). This boundary condition is satisfied when  $B$  and  $C$  satisfy the following equations:

$$\left. \begin{aligned} B\varepsilon - C\delta &= -u_\infty A, \\ B\delta + C\varepsilon &= 0. \end{aligned} \right\} \quad (3.245)$$

The final result may be written in the following form:

$$\left. \begin{aligned} \phi &= \frac{u_\infty A}{(1 - M_\infty^2)^{\frac{1}{2}}} (\sin kx) \exp[-ky(1 - M_\infty^2)^{\frac{1}{2}}] \quad \text{for } M_\infty < 1, \\ \phi &= \frac{-u_\infty A}{(M_\infty^2 - 1)^{\frac{1}{2}}} \cos[k[x - y(M_\infty^2 - 1)^{\frac{1}{2}}]] \quad \text{for } M_\infty > 1. \end{aligned} \right\} \quad (3.246)$$

The result for  $M_\infty > 1$  (supersonic flow) need not be discussed any further, since it is a special case of the Ackeret solution treated in Section 3.5.2. The disturbances originating from the wall propagate with undiminished strength along the Mach lines  $x - y(M_\infty^2 - 1)^{1/2} = \text{const}$ . In subsonic flow ( $M_\infty < 1$ ), on the other hand, the disturbances decay exponentially with distance from the wall. This decay diminishes as  $M_\infty$  gets closer to the value 1. This behavior is clearly seen from the streamline pattern (Fig. 93). From Eq. (3.237) we can calculate the perturbation pressure  $p'$  by which the flow pressure differs

from the unperturbed pressure  $p_\infty$ . The pressure directly on the wall ( $y = 0$ ) is given by

$$p'_w = -\frac{\rho_\infty u_\infty^2 A k}{(|1 - M_\infty^2|)^{\frac{1}{2}}} \cdot \begin{cases} \cos kx & \text{for } M_\infty < 1 \\ \sin kx & \text{for } M_\infty > 1. \end{cases} \quad (3.247)$$

$p'_w$  is thus  $180^\circ$  out of phase from the waviness of the wall for  $M_\infty < 1$ , and  $90^\circ$  for  $M_\infty > 1$ . This leads to the result that in supersonic flow the pressure force acting on the wall has a component in the flow direction, which results in a finite drag force  $D$  in supersonic flow (for a wavelength  $\lambda = 2\pi k^{-1}$  in the  $x$  direction and unit width in the  $z$  direction). For the drag coefficient found from  $D$ , we get

$$c_D = \frac{D}{\frac{1}{2} \rho_\infty u_\infty^2 2\pi k^{-1}} = \frac{A^2 k^2}{(M_\infty^2 - 1)^{\frac{1}{2}}}. \quad (3.248)$$

This result can also be derived immediately from Eq. (3.163) if we identify  $l$  there with the wavelength  $2\pi k^{-1}$  and set  $y_u = y_w = A \cos kx$ ,  $y_l = 0$ , and  $\alpha = 0$ .

### 3.9.3 SUBSONIC FLOW PAST SLENDER BODIES

Limiting ourselves to subsonic flow, we introduce for short  $\beta^2 = 1 - M_\infty^2 > 0$ . Equation (2.236) now reads

$$\beta^2 \phi_{xx} + \phi_{yy} + \phi_{zz} = 0. \quad (3.249)$$

We consider a slender body, say, an airfoil (Fig. 94), in a steady flow with velocity  $u_\infty$ . The perturbation potential  $\phi$  satisfies Eq. (3.249); as  $M_\infty \rightarrow 0$ , i.e.,  $\beta \rightarrow 1$ , this equation turns into the Laplace equation  $\Delta\phi = 0$  satisfied by incompressible flow. The solutions of this equation for various flow problems are well known, as are the methods for obtaining such solutions. Consequently, it is of great theoretical and practical significance that we can transform Eq. (3.249) into the equation  $\Delta\phi = 0$  by a simple coordinate transformation, and that therefore by a corresponding transformation of the boundary conditions each compressible flow past a slender body (for  $M_\infty < 1$ ) can be transformed into an equivalent incompressible flow, as will be further explained below.

Let the surface of the body be given by  $F(x, y, z) = 0$ . The components of the velocity vector are  $u_\infty + \phi_x, \phi_y, \phi_z$ . Considering the fact that the

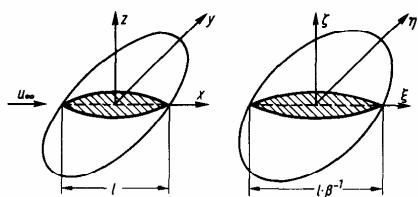


Fig. 94. Affine transformation of an airfoil-shaped body according to Goethert's rule.

vector with components  $F_x, F_y, F_z$  on the surface  $F = 0$  is everywhere normal to it, the condition of tangential flow direction on this surface may be expressed in the form  $(u_\infty + \phi_x) F_x + \phi_y F_y + \phi_z F_z = 0$ . With the assumption that for sufficiently slender bodies  $|\phi_x| \ll u_\infty$  (this is an assumption for the validity of linear theory), we neglect  $\phi_x$  in comparison with  $u_\infty$  in the first term and obtain the following boundary condition for  $\phi$ :

$$u_\infty F_x + \phi_y F_y + \phi_z F_z = 0. \quad (3.250)$$

In addition, there is the boundary condition  $\text{grad } \phi \rightarrow 0$  for sufficiently large distances from the body. We now transform the variables as follows:<sup>48</sup>

$$\left. \begin{aligned} x &= \beta \xi; & y &= \eta; & z &= \zeta; \\ F(x, y, z) &= F(\beta \xi, \eta, \zeta) = \tilde{F}(\xi, \eta, \zeta); \\ \phi(x, y, z) &= \phi(\beta \xi, \eta, \zeta) = \tilde{\phi}(\xi, \eta, \zeta). \end{aligned} \right\} \quad (3.251)$$

We have then  $\phi_x = \beta^{-1} \tilde{\phi}_\xi$ ,  $\phi_{xx} = \beta^{-2} \tilde{\phi}_{\xi\xi}$ ,  $\phi_{yy} = \tilde{\phi}_{\eta\eta}$ , etc., and Eq. (3.249) becomes

$$\tilde{\phi}_{\xi\xi} + \tilde{\phi}_{\eta\eta} + \tilde{\phi}_{\zeta\zeta} = 0. \quad (3.252)$$

In a similar way, the boundary condition (3.250) becomes

$$u_\infty \tilde{F}_\xi + \beta \tilde{\phi}_\eta \tilde{F}_\eta + \beta \tilde{\phi}_\zeta \tilde{F}_\zeta = 0. \quad (3.253)$$

The boundary condition  $\text{grad } \phi \rightarrow 0$  at sufficiently large distances from the body goes over unchanged into the corresponding boundary condition  $\text{grad } \tilde{\phi} \rightarrow 0$ , where the gradient is now taken with respect to the variables  $\xi, \eta, \zeta$ . Finally, setting  $\beta \tilde{\phi} = \phi_{\text{inc}}(\xi, \eta, \zeta)$ , then, on the basis of Eq. (3.252),

<sup>48</sup> The tilde (~) here of course has nothing to do with unconstrained thermodynamic equilibrium.

which can naturally also be written as

$$\left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2} \right) \phi_{\text{inc}} = 0,$$

and on the basis of the boundary condition (3.253), we can interpret the function  $\phi_{\text{inc}}$  as the perturbation potential of an incompressible flow at speed  $u_\infty$  past a body with a surface  $\tilde{F}(\xi, \eta, \zeta) = 0$  in the  $\xi, \eta, \zeta$ -space. Compared to the body  $F(x, y, z) = 0$ , this new body has been stretched by a factor  $\beta^{-1}$  in the flow direction (Fig. 94). If we know the solution of the incompressible flow problem in the  $\xi, \eta, \zeta$ -space, and in particular, the perturbation velocities  $u'_{\text{inc}} = \partial \phi_{\text{inc}} / \partial \xi$ ,  $v'_{\text{inc}} = \partial \phi_{\text{inc}} / \partial \eta$ , and  $w'_{\text{inc}} = \partial \phi_{\text{inc}} / \partial \zeta$ , then we can obtain the perturbation velocities in the compressible problem as follows:

$$\left. \begin{aligned} u' &= \phi_x = \tilde{\phi}_\xi \xi_x = \beta^{-2} u'_{\text{inc}}, \\ v' &= \phi_y = \tilde{\phi}_\eta \eta_y = \beta^{-1} v'_{\text{inc}}, \\ w' &= \phi_z = \tilde{\phi}_\zeta \zeta_z = \beta^{-1} w'_{\text{inc}}. \end{aligned} \right\} \quad (3.254)$$

For the perturbation pressure  $p'$ , we have

$$p' = -\rho_\infty u_\infty \phi_x = -\beta^{-2} \rho_\infty u_\infty u'_{\text{inc}} = \beta^{-2} p'_{\text{inc}}, \quad (3.255)$$

or, upon introducing the pressure coefficient  $c_p$  [Eq. (3.238)],

$$c_p = \beta^{-2} c_{p,\text{inc}}. \quad (3.256)$$

Thus, from every *incompressible* flow past a slender body with upstream velocity  $u_\infty$ , we can deduce from formulas (3.254)–(3.256) a *compressible* flow by shortening the body by a factor  $\beta$  in the flow direction. These formulas embody Goethert's rule.<sup>49</sup>

We now apply this rule to the plane flow past a profile at subsonic velocity (Fig. 95). In practice, the most interesting quantity is the pressure or pressure coefficient on the surface of the profile, since this determines the force exerted on the profile, which in subsonic flow has only a component perpendicular to the flow direction (lift). The horizontal double arrow in Fig. 95 indicates the relation between the pressure coefficients at affinely related points on the profile for the compressible subsonic flow and for the corresponding equivalent incompressible flow based on Goethert's rule. The

<sup>49</sup> B. Goethert, Ebene und räumliche Strömung bei hohen Unterschallgeschwindigkeiten, *Jb. d. dtsh. Luftfahrtforschung* I, 156–157, (1941).

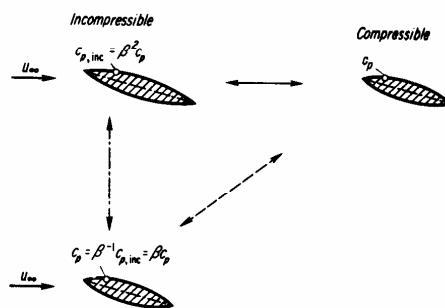


Fig. 95. Diagram for the derivation of the Prandtl-Glauert rule for the plane flow past a thin profile.

vertical double arrow connects two profiles placed in the same incompressible flow and differing in shape by the shortening factor  $\beta$ . As is known from the theory of incompressible flow past slender profiles, the pressure coefficients at affinely related points on the contour are connected by the relation  $c_{p0} = \beta^{-1} c_{p,inc}$ . The last step of our discussion is to produce the connection indicated by the dotted double arrow: Here we have the same profile (same shape, same angle of attack) in incompressible flow in one case and in compressible flow in the other. Hence  $c_{p0}$  denotes the limiting value of  $c_p$  for  $M_\infty = 0$ , and we obtain the relation between the pressure coefficients:

$$c_p = c_{p0}(1 - M_\infty^2)^{-\frac{1}{2}}; \quad (3.257)$$

this is known as the Prandtl-Glauert rule.<sup>50</sup> By this rule, the dependence of the pressure coefficient on the Mach number  $M_\infty$  in the linear approximation is defined. Further statements on the dependence of the entire flow field on  $M_\infty$  may easily be deduced from Goethert's rule in the manner indicated.

There are several methods for calculating the incompressible flow past a profile, and, in particular, the pressure on the surface, but we cannot enter into further details here.<sup>51</sup> The Prandtl-Glauert rule permits the generalization of the results found by these methods to the entire subsonic regime. The

<sup>50</sup> H. Glauert, The effect of compressibility on the lift of airfoils, *Proc. Roy. Soc. A* **118** 113, (1927).

<sup>51</sup> See footnote 36.

analogous result for supersonic flow is provided by the Ackeret theory (Section 3.5.2). Thus, we have found the lift force on the profile as a function of the Mach number  $M_\infty$  within the framework of linear theory (drag exists in inviscid flow only for  $M_\infty > 1$ ). A flat plate with an angle of attack  $\alpha$ , for example, has a lift coefficient  $2\pi\alpha$  in incompressible flow, so that  $c_L = 2\pi\alpha(1 - M_\infty^2)^{-1/2}$  for  $M_\infty < 1$ , while  $c_L = 4\alpha(M_\infty^2 - 1)^{-1/2}$  for  $M_\infty > 1$ , as was given by Eq. (3.160).

The fact that according to both (3.257) and the Ackeret theory  $c_L$  increases without bound as  $M_\infty \rightarrow 1$  indicates that the linear theory breaks down for transonic flow, i.e., flow with  $M_\infty$  near 1 (also see Section 3.7). In Fig. 96,

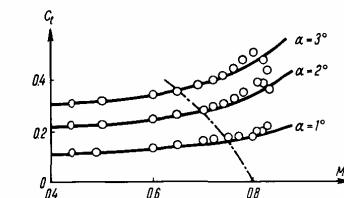


Fig. 96. Comparison of the Prandtl-Glauert rule with measured results on a symmetric profile with thickness ratio 0.09. (From A. Lippisch and W. Beuschausen, Deutsche Luftfahrtforschung. Forschungsbericht # 1669, 1942.)

the Prandtl-Glauert rule is compared with measured results. The dot-dash line corresponds to the particular Mach number  $M_\infty$  at which, for each given angle of attack  $\alpha$ , the local Mach number  $M = 1$  was first attained (according to measurements) at some point of the profile surface; this freestream Mach number  $M_\infty$  is called the critical Mach number. The agreement between experimental values and the Prandtl-Glauert rule is rather good up to Mach numbers  $M_\infty$  slightly above the critical, and then quickly becomes poor: The lift coefficient finally decreases with further increase in  $M_\infty$ .

Measurement of the drag coefficient  $c_D$  on a profile of the type used in Fig. 96 shows that in the regime of validity of the Prandtl-Glauert rule (and for a Reynolds number of the order of  $10^5$ ),  $c_D$  has a value of around 0.01.<sup>52</sup> Inviscid theory gives  $c_D = 0$ , as already mentioned; the finite drag is due to

<sup>52</sup> See Section 4.3 for the definition of Reynolds number.

the viscous friction of the gas that occurs in actuality and that leads to the formation of a boundary layer on the profile surface (see Section 4.3). This boundary layer causes both a *friction drag*, due to the shear stresses acting on the profile, and a *pressure drag*. Because of the presence of the boundary layer, the pressure in actuality is changed from the inviscid pressure, and produces a finite drag force when integrated over the entire profile. This change in the pressure is a consequence of the displacement effect of the boundary layer, which changes the effective shape of the profile seen by the flow outside the boundary layer (see Section 4.3.3). As  $M_\infty = 1$  is approached, the drag increases, and, in fact,  $c_D$  rapidly rises by at least an order of magnitude when we reach the Mach number regime where the drop in  $c_L$  occurs.

Both effects, the increase in drag and the decrease in lift, are connected with the appearance of shock waves in local, bounded regions of supersonic flow around the profile. This phenomenon cannot be given by linear theory, which, for  $M_\infty < 1$ , gives a continuous flow field independently of  $M_\infty$ .

When a shock wave appears, the flow with  $M_\infty < 1$  past a symmetric (fore and aft) profile becomes asymmetric, and, together with this, the pressure distribution on the profile becomes asymmetric (fore and aft) even with viscosity and boundary layers neglected, so that a finite drag results.<sup>53</sup> At the same time, the shock wave interacts with the boundary layer which occurs in reality on the profile contour. In many cases, this interaction leads to the separation of the boundary layer from the profile, which is still another cause for the increase in drag. Apart from this very complicated process in the boundary layer, we can also easily see the inadequacy of the linear approximation near  $M_\infty = 1$  as follows: The linear equation (3.236) is elliptic for  $M_\infty < 1$  and hyperbolic for  $M_\infty > 1$ . Thus, the type of the differential equation depends only on the freestream Mach number  $M_\infty$ . However, actual experience shows that in transonic flow an elliptic region with local Mach number  $M < 1$  exists side by side with a hyperbolic region with  $M > 1$ . These phenomena, which result in significant complication, can only be theoretically understood when the nonlinearity of the gasdynamic equations is taken into account. In Fig. 97, we show in a schematic manner how the flow field around a symmetric profile in inviscid flow changes with changes in the freestream Mach number  $M_\infty$ . The sonic line (shown dashed) and part of the shock

<sup>53</sup> The existence of this drag can also be deduced from purely thermodynamic arguments concerning the increase of entropy in the shock, using a theorem of K. Oswatitsch.

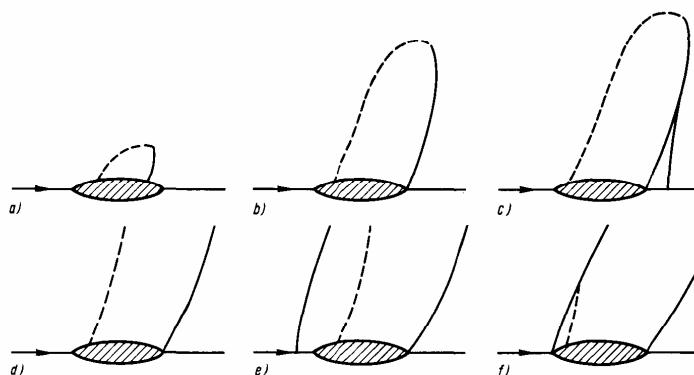


Fig. 97. Transonic flow past a thin symmetric profile (schematic); — shock; --- sonic line.

waves separate the subsonic and supersonic regions. The Mach number  $M_\infty$  increases from part (a) of the figure to part (f), and  $M_\infty = 1$  in part (d). The Prandtl-Glauert rule is only valid in some degree when the flow is continuous, i.e., when the Mach number  $M_\infty$  is still below that corresponding to the flow of Fig. 97a. On the other hand, the range of validity of the Ackeret theory of supersonic flow starts from Fig. 97f onward. The transonic regime in between is defined not only by the Mach number  $M_\infty$ , but also depends on the profile shape and the angle of attack; the thinner the airfoil and the smaller the angle of attack, then the smaller this transonic interval will be around  $M_\infty = 1$ . This is plausible when we think of the discussion in Section 3.7, where we found for a special example the condition for the validity of linear theory to be  $K \gg 1$ , where  $K$  is the transonic similarity parameter defined by Eq. (3.185); this equation can be generalized to the flow past a slender profile if we now regard  $\theta$  to be a typical angle of inclination between the profile surface and the flow direction (e.g., ratio of profile thickness to chord length). The transonic flow regime is then characterized by the condition  $-1 \lesssim K \lesssim +1$ .

*Supplementary Remarks.* 1. Equation (3.236) can be derived from Eq. (3.204) by linearization. If we set into (3.204)  $u = u_\infty + u'$ ,  $v = v'$ ,  $w = w'$ , and  $a = a_\infty + a'$  and keep only linear terms in the perturbation quantities

denoted by  $(')$ , we transform (3.204) into

$$u_x'(1 - M_\infty^2) + v_y' + w_z' = 0. \quad (3.258)$$

Introducing the potential  $\phi$  of the perturbation velocity  $v'$ , Eq. (3.236) follows immediately.

2. The Prandtl lifting-line theory gives, for a wing of elliptical planform and large aspect ratio  $A_{\text{inc}} \gg 1$  in incompressible flow, the lift coefficient

$$c_{L, \text{inc}} = \frac{c' \alpha_{\text{inc}}}{1 + (c'/\pi A_{\text{inc}})}, \quad (3.259)$$

where  $c'$  is a constant depending on the airfoil section of the wing, equal to  $2\pi$  for a profile of infinitesimal thickness. By Goethert's rule, the equivalent wing in compressible flow has the aspect ratio  $A = A_{\text{inc}} \beta^{-1}$ . The angle of attack will be  $\alpha = \alpha_{\text{inc}} \beta^{-1}$  and the lift coefficient  $c_L = c_{L, \text{inc}} \beta^{-2}$ . From Eq. (3.259), this gives

$$c_L = \frac{c' \alpha}{(1 - M_\infty^2)^{\frac{1}{2}} + (c'/\pi A)}. \quad (3.260)$$

With  $c' = 2\pi$  and  $A = \infty$ , we obtain the previous result for a flat plate in plane flow.

3. Tsien has given, by theoretical methods, an improvement of the formula (3.257) for the dependence of the pressure coefficient at the surface of a thin profile on the freestream Mach number.<sup>54</sup> Using the notation introduced above, Tsien's result is

$$c_p = \frac{c_{p0}}{(1 - M_\infty^2)^{\frac{1}{2}} + \frac{1}{2} c_{p0} [1 - (1 - M_\infty^2)^{\frac{1}{2}}]}. \quad (3.261)$$

This formula is in better agreement with experimental values than Eq. (3.257).

### 3.10\* Relaxation Processes in Steady Flow

#### 3.10.1\* LINEAR WAVE EQUATION<sup>55</sup>

In what follows we shall repeat the discussions of Section 3.9.1 for a gas which does not experience unconstrained thermodynamic equilibrium during

<sup>54</sup> H.S. Tsien, Two-dimensional subsonic flow of compressible fluids, *J. Aero Sci.* **6**, 399, (1939).

<sup>55</sup> W.G. Vincenti, Nonequilibrium flow over a wavy wall, *J. Fluid Mech.* **6**, 481–496,

the flow, but instead experiences a constrained equilibrium that can be described by a single additional thermodynamic variable  $\xi$  (see Section 1.6). Section 3.3 contains the corresponding discussion for one-dimensional wave propagation in the linear approximation, which we shall now extend to three-dimensional wave propagation. We first again consider a gas initially at rest (density  $\rho_\infty$ , pressure  $p_\infty$ , and sound velocities  $a_\infty$  and  $b_\infty$ ; see Section 1.11). The momentum equation (3.225) remains unchanged, as does its consequence, that  $\operatorname{curl} v' \equiv 0$  obtains for all motions originating from rest. We can thus again introduce a velocity potential:  $v' = \operatorname{grad} \phi$ . If, as in Section 3.9.1, the gas is assumed to be always in unconstrained thermodynamic equilibrium,  $\phi$  will satisfy the wave equation (3.228); this results from the one-dimensional wave equation (3.31), with  $\partial^2/\partial x^2$  replaced by the Laplace operator  $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$  (and  $\phi$  replacing  $u$  as the dependent variable). In a completely similar way, the method of elimination, explained in Section 3.3 for one-dimensional wave propagation with relaxation, will lead to the following equation [corresponding to (3.94)] for three-dimensional wave propagation:

$$\tau \frac{\partial}{\partial t} (\phi_{tt} - b_\infty^2 \Delta \phi) + \phi_{tt} - a_\infty^2 \Delta \phi = 0. \quad (3.262)$$

Formula (3.229) for the perturbation pressure  $p'$ , which was derived from the momentum equation (3.225), continues to hold.

Now we again go to a coordinate system which moves with velocity  $u_\infty$  in the negative  $x$  direction. The corresponding transformation formulas are given by formulas (3.230) and (3.231). We confine ourselves to the case when the motion of the gas in the new coordinate system is steady. Then, all the time derivatives vanish in this system, and, in particular, we have the transformation formula  $\partial^n/\partial t^n = u_\infty^n \partial^n/\partial \tilde{x}^n$  for  $n = 1, 2, \dots$ . Since from now on we shall only use this coordinate system, we can again drop the bars on the new variables. Under this transformation, Eq. (3.262) becomes

$$\lambda \frac{\partial}{\partial x} [(1 - M_{b\infty}^2) \phi_{xx} + \phi_{yy} + \phi_{zz}] + (1 - M_{a\infty}^2) \phi_{xx} + \phi_{yy} + \phi_{zz} = 0. \quad (3.263)$$

(1959). F.K. Moore and W.E. Gibson, Propagation of weak disturbances in gas subject to relaxation effects, *J. Aero Space Sci.* **27**, 117–127, (1960). J.F. Clarke, The linearized flow of a dissociating gas, *J. Fluid Mech.* **7**, 577–595, (1960). I.P. Stakhanov and E.V. Stupochenko, The structure of Mach lines in relaxing media. *Sov. Phys. Doklady* **5**, 964–968, (1961). [English transl. of: *Doklady Akademii Nauk SSSR* **134**, 1044–1047, (1960)].

Two Mach numbers appear here,  $M_{a\infty} = u_\infty/a_\infty$  and  $M_{b\infty} = u_\infty/b_\infty < M_{a\infty}$ , as well as the relaxation length  $\lambda = u_\infty t b_\infty^2/a_\infty^2$ . The relaxation length  $\lambda$  is a measure of the distance which a particle must traverse in the flow field in order for its thermodynamic state to attain a new unconstrained equilibrium following a sudden change of its original unconstrained equilibrium. For  $\lambda \rightarrow 0$ , Eq. (3.263) formally changes to Eq. (3.236), with  $M_\infty$  replaced by  $M_{a\infty}$ ; for  $\lambda \rightarrow \infty$ , Eq. (3.263) when divided by  $\lambda$  again changes to Eq. (3.236), but with  $M_\infty$  replaced by  $M_{b\infty}$ . In the next section, we shall study these limiting transitions for a special solution of Eq. (3.263). Formulas (3.237) and (3.238) for the perturbation pressure  $p'$  and the pressure coefficient  $c_p$  are unchanged.

### 3.10.2\* FLOW PAST A WAVY WALL<sup>56</sup>

A simple but very instructive solution of Eq. (3.263) arises for the flow past a wavy wall. Let the wall contour again be given by Eq. (3.239). The boundary condition (3.240) remains unchanged; in addition, we have the condition that  $\phi$  should not increase without bounds with increasing distance from the wall. The trial solution (3.241) with real constants  $\varepsilon$ ,  $\delta$ ,  $B$ , and  $C$  will be used without change. Substitution into Eq. (3.263) leads to the following relations for  $\varepsilon$  and  $\delta$ :

$$\lambda k [(1 - M_{b\infty}^2) + \varepsilon^2 - \delta^2] - 2\varepsilon\delta = 0, \quad (3.264)$$

$$(1 - M_{a\infty}^2) + \varepsilon^2 - \delta^2 + 2\lambda k \varepsilon \delta = 0. \quad (3.265)$$

For  $\lambda k \rightarrow 0$ , i.e., if the relaxation length  $\lambda$  is very small compared to the wall wave length  $2\pi k^{-1}$ , which is the limiting case of equilibrium flow, Eqs. (3.264) and (3.265) become (3.242) and (3.243). Here,  $M_{a\infty}$  appears in place of  $M_\infty$ . Correspondingly, for  $\lambda k \rightarrow \infty$  (frozen flow), we again obtain the system (3.242), (3.243) but with  $M_\infty$  replaced by  $M_{b\infty}$ . In both limiting cases, the solution discussed in Section 3.9.2 will thus still hold, except with  $M_\infty$  interpreted differently in each case. The dependence of  $\varepsilon$  and  $\delta$  on the Mach number  $M_{a\infty}$  for these two limiting cases are shown in Fig. 98. It is assumed there that the ratio  $b_\infty/a_\infty$  of the two sound speeds in the gas being considered has a value of 1.1.<sup>57</sup>

<sup>56</sup> W.G. Vincenti, footnote 55; F.K. Moore and W.E. Gibson, footnote 55.

<sup>57</sup> For a diatomic gas, the speed of sound for completely excited molecular vibrations ( $\gamma = 9/7$ ) is given by  $a^2 = (9/7) RT$ , and for completely unexcited vibrations by  $b^2 = (7/5) RT$ . Thus,  $b^2/a^2 = 49/45$ , or  $b/a = 1.042$ . The value of 1.1 assumed above is somewhat higher than this, but still has the right order of magnitude for processes of this kind.

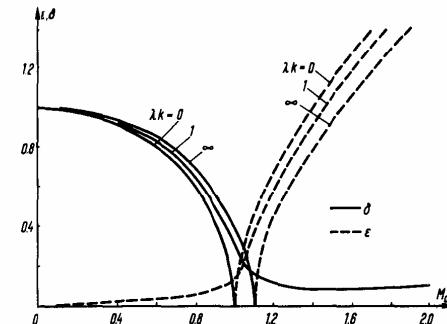


Fig. 98. The quantities  $\delta$  and  $\varepsilon$  for the flow past a wavy wall. (From W.G. Vincenti, Non-equilibrium flow over a wavy wall. J. Fluid Mech. 6 (1957), 481–496.)

Equations (3.264) and (3.265) have real solutions  $\varepsilon$  and  $\delta$  for all positive values of  $\lambda k$ . The somewhat unmanageable formulas for  $\varepsilon$  and  $\delta$  as functions of  $k$ ,  $M_{a\infty}$ , and  $M_{b\infty}$  will not be explicitly given. Instead, the dependence of  $\varepsilon$  and  $\delta$  on the Mach number is shown in Fig. 98 for  $\lambda k = 1$ .

The constants  $B$  and  $C$  are obtained by substituting the solution (3.241) in the boundary condition (3.240); this leads to the equations (3.245) for  $B$  and  $C$ . The final result is

$$\phi = \frac{u_\infty A}{\varepsilon^2 + \delta^2} [\delta \sin k(x - \varepsilon y) - \varepsilon \cos k(x - \varepsilon y)] \exp(-k\delta y), \quad (3.266)$$

with  $\varepsilon$  and  $\delta$  given by (3.264) and (3.265). This representation is valid for all values of  $M_{a\infty}$ ,  $M_{b\infty}$ , and  $\lambda k$ , and, in particular, contains the solution (3.246) as the limiting cases  $\lambda k = 0$ ,  $M_\infty \hat{=} M_{a\infty}$ , or  $\lambda k = \infty$ ,  $M_\infty \hat{=} M_{b\infty}$ . An important characteristic feature of the solution (3.266) is that for finite values of  $\lambda k$ , i.e., whenever the relaxation process plays a role and the flow is neither in unconstrained thermodynamic equilibrium ( $\lambda k = 0$ ) nor completely frozen ( $\lambda k = \infty$ ),  $\phi$  will decay exponentially with increasing distance  $y$  from the wall (since  $\delta$  is always  $> 0$ ). The disturbances originating at the wall thus also decay when  $M_{b\infty} > 1$  (pure supersonic flow). On the other hand, in a pure subsonic flow ( $M_{a\infty} < 1$ ), we have  $\varepsilon > 0$ , so that with increasing distance from

<sup>58</sup> W.G. Vincenti, footnote 55.

the wall the phase shift of the streamlines from that of the waviness of the wall increases; in the solution found in Section 3.9.2, this phase shift is always zero (Fig. 93). Further details on the flow field can be found in the original work of Vincenti.<sup>58</sup>

The perturbation pressure  $p'$  can be calculated from (3.237). If we denote by  $D$  the streamwise component of the pressure force exerted on the wall (per wavelength in  $x$  direction, unit length in  $z$  direction), and define the drag coefficient  $c_D$  by (3.248), then

$$c_D = A^2 k^2 \epsilon / (\epsilon^2 + \delta^2). \quad (3.267)$$

This formula contains formula (3.248) as special case for equilibrium flow or frozen flow. It should be remarked that for finite values of  $\lambda k$ , there is always a finite drag force ( $c_D \neq 0$ ),<sup>59</sup> even in pure subsonic flow. The relaxation process in the gas and the related entropy generation (see Section 2.4) thus cause a flow resistance. That this is true in complete generality and not only for our special example can be shown from purely thermodynamic considerations: Every entropy source in a flow field results in flow resistance.

### 3.10.3\* FLOW PAST A WALL WITH CORNER<sup>60</sup>

In this section, we study, in the linear approximation, the flow along a straight wall which at the point  $x = 0$  has a sudden bend through a small angle  $\delta\theta$  (Fig. 99). Upstream of the corner, there is uniform parallel flow with velocity  $u_\infty$ ; the gas is in unconstrained thermodynamic equilibrium there. We shall assume that  $u_\infty > b_\infty$ , i.e.,  $M_{b\infty} > 1$ , where  $b_\infty$  is the frozen sound speed of the gas upstream of the corner. Then, since  $a_\infty < b_\infty$ , we also have  $M_{a\infty} = (u_\infty/a_\infty) > 1$ . Under this assumption, there is a far-reaching analogy between the steady plane flow past a corner, which we are considering now, and the unsteady one-dimensional flow generated by an impulsively-accelerated piston in a gas-filled tube (see Section 3.3.2).

The starting point of our consideration is Eq. (3.263) for the perturbation potential  $\phi$ . Instead of the real representation (3.241) for the particular solutions of this equation, we now use for convenience complex represen-

<sup>59</sup> J. Ackeret, Über Widerstände, die durch gasdynamische Relaxation hervorgerufen werden, *Z. Flugwiss.* **4**, 14–17 (1956).

<sup>60</sup> J. F. Clarke, footnote 55; I. P. Stakhanov and E. V. Stupochenko, footnote 55.

tation and take particular solutions in the form

$$\phi = A \exp [ik(x - y)], \quad (3.268)$$

where  $A$  is an amplitude constant and the wave number  $k$  and phase velocity  $c$  are also constants, only one of which can be chosen arbitrarily, since there is a relation between the two that can be obtained by substituting (3.268) into (3.263). We get

$$c = \left( \frac{i\lambda k \beta_b^2 + \beta_a^2}{i\lambda k + 1} \right)^{\frac{1}{2}}, \quad (3.269)$$

where we have introduced  $\beta_b^2 = M_{b\infty}^2 - 1 > 0$  and  $\beta_a^2 = M_{a\infty}^2 - 1 > 0$  for conciseness. For the flow along the wall with a corner,  $\phi$  must satisfy the following boundary conditions:

$$\begin{aligned} v' &= \partial\phi/\partial y = 0 && \text{for } y = 0 \quad \text{and} \quad x < 0, \\ v' &= \partial\phi/\partial y = u_\infty \delta\theta && \text{for } y = 0 \quad \text{and} \quad x > 0. \end{aligned}$$

The latter condition is actually to be satisfied on the wall behind the corner; however, in the framework of the linear approximation, it may be satisfied on the straight line  $y = 0$  by the assumption of the smallness of the angle  $\delta\theta$ . Using the step function (3.99), we can write the boundary condition in the form

$$v' \Big|_{y=0} = \frac{\partial\phi}{\partial y} \Big|_{y=0} = \frac{u_\infty \delta\theta}{2\pi i} \int_{-\infty}^0 \frac{\exp(ikx)}{k} dk. \quad (3.270)$$

Here the integration is carried out over the real axis of the complex  $k$  plane with the neighborhood of  $k = 0$  excepted; the point  $k = 0$  is bypassed by an arc going below it. For  $x = 0$ , the integral has the value  $\frac{1}{2}u_\infty \delta\theta$ .

We now construct a solution of the differential equation (3.263) which is compatible with the boundary condition on the wall by superposition of particular solutions of the form (3.268):

$$\phi(x, y) = \int A(k) \exp [ik(x - cy)] dk. \quad (3.271)$$

$\phi$  is a solution of the differential equation (3.263) whenever  $c$  and  $k$  satisfy the frequency equation (3.269). Moreover, the integral can be taken over an arbitrary path in the complex  $k$  plane. To satisfy the boundary condition, we substitute (3.271) into (3.270) and choose the same path of integration. Then

we get

$$-\int_{\text{--}\curvearrowleft} A(k) ikc \exp(ikx) dk = \frac{u_\infty \delta\theta}{2\pi i} \int_{\text{--}\curvearrowleft} \frac{\exp(ikx)}{k} dk.$$

By the uniqueness of the Fourier representation, we can conclude from this that  $A(k) = u_\infty \delta\theta / 2\pi k^2 c$ . Substituting this expression into (3.271), we have for the potential

$$\phi = \frac{u_\infty \delta\theta}{2\pi} \int_{\text{--}\curvearrowleft} \frac{\exp[ik(x - cy)]}{k^2 c} dk. \quad (3.272)$$

This gives the solution, but not yet uniquely. Indeed, there are two solutions  $c(k)$  to the frequency equation (3.269), which differ in the sign. The solution will reach the unperturbed flow at  $x \rightarrow -\infty$ , if we choose from (3.269) the root with positive real part (i.e., the root which becomes positive and real in the limits of  $\lambda=0$  and  $\lambda=\infty$ ). This branch corresponds to disturbances  $y=0$  that propagate downstream. The other branch leads to solutions with upstream-propagating disturbances (see Sections 3.5.2. and 3.10.2).

A detailed investigation of the integral (3.272) gives the following: If  $x - y(M_{b\infty}^2 - 1)^{1/2} < 0$ , i.e., if the point  $x, y$  lies in the region upstream of the frozen Mach line issuing from the corner (which is inclined at an angle  $\mu_{b\infty} = \arcsin M_{b\infty}^{-1}$  against the flow direction), then we can complete the path of integration into a closed path in the lower half of the complex plane without altering the value of the integral. Since this closed path does not enclose any singularities inside it, the residue theorem gives  $\phi = 0$ . Hence, the disturbances produced by the corner on the free stream cannot be noticed in front of the frozen Mach line from the corner. The frozen Mach lines  $x \pm y(M_{b\infty}^2 - 1)^{1/2} = \text{const}$  are the characteristics of the differential equation (3.263). As is known, discontinuities in the derivatives of the solution  $\phi$  propagate along the characteristics. In this case, the boundary condition (3.270) generates a discontinuity in  $\phi_x$  and  $\phi_y$  which originates from the point  $x = y = 0$  and propagates along the left-running characteristic  $x - y(M_{b\infty}^2 - 1)^{1/2} = 0$ . The strength of this discontinuity, however, decays with increasing distance from the wall. This can be shown as follows:<sup>61</sup> We

<sup>61</sup> F.K. Moore and W.E. Gibson, footnote 55. P.P. Wegener and J.D. Cole, Experiments on propagation of weak disturbances in stationary supersonic nozzle flow of chemically reacting gas mixtures, *Eight Internat. Symp. Combustion*, Baltimore, Maryland, p. 348-359, 1962.

introduce characteristic coordinates denoted by  $\sigma$  and  $\tau$ :

$$\sigma = y + (x/\beta_b); \quad \tau = y - (x/\beta_b).$$

The right- or left-running characteristics are the lines  $\sigma = \text{const}$  or  $\tau = \text{const}$ , respectively; the line  $\tau = 0$  is the left-running characteristic through the corner. By replacing the derivatives of  $\phi$  according to

$$\frac{\partial}{\partial x} = \frac{1}{\beta_b} \left( \frac{\partial}{\partial \sigma} - \frac{\partial}{\partial \tau} \right); \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau},$$

we transform (3.263) to the following equation for  $\phi$  as a function of  $\sigma$  and  $\tau$ :

$$-\frac{4\lambda}{\beta_b} \left( \frac{\partial}{\partial \sigma} - \frac{\partial}{\partial \tau} \right) \phi_{\sigma\tau} - 2 \left( \frac{\beta_a^2}{\beta_b^2} + 1 \right) \phi_{\sigma\tau} + \left( \frac{\beta_a^2}{\beta_b^2} - 1 \right) (\phi_{\sigma\sigma} + \phi_{\tau\tau}) = 0. \quad (3.273)$$

Since  $\phi$  is continuous on the characteristic  $\tau = 0$ , so are  $\phi_\sigma$  and  $\phi_{\sigma\sigma}$ ; a discontinuity appears only in  $\phi_\tau$ . If we integrate (3.273) with fixed  $\sigma$  over  $\tau$  from  $\tau = -\varepsilon$  to  $\tau = +\varepsilon$ , i.e., across the characteristic  $\tau = 0$ , and then pass to the limit  $\varepsilon \rightarrow 0$ , we shall obtain for the jump of  $[\phi_\tau]$ , defined as

$$[\phi_\tau] = \lim_{\varepsilon \rightarrow 0} [\phi_\tau(\sigma, +\varepsilon) - \phi_\tau(\sigma, -\varepsilon)], \quad (3.274)$$

the following ordinary differential equation:

$$\frac{4\lambda}{\beta_b} \frac{d[\phi_\tau]}{d\sigma} + \left( \frac{\beta_a^2}{\beta_b^2} - 1 \right) [\phi_\tau] = 0. \quad (3.275)$$

From this, we obtain by integration

$$[\phi_\tau] = [\phi_\tau]_{\sigma=0} \exp \left( -\frac{M_{a\infty}^2 - M_{b\infty}^2}{4\lambda\beta_b} \sigma \right), \quad (3.276)$$

where  $[\phi_\tau]_{\sigma=0}$  denotes the value of the jump at the point  $\sigma = 0$ , i.e., at the corner of the wall. Since, on the characteristic  $\tau = 0$  we have  $y = \frac{1}{2}\sigma$ , we can then write (3.276) as follows:

$$[\phi_\tau] = [\phi_\tau]_{y=0} \exp \left( -\frac{M_{a\infty}^2 - M_{b\infty}^2}{2\lambda\beta_b} y \right). \quad (3.277)$$

In the same manner as by this law for the discontinuity of  $\phi_\tau$ , the discontinuities in all the flow variables (pressure, velocity, etc.) decay with increasing distance from the corner along the frozen Mach line from the corner.

We now turn to the derivation of an approximate formula for the component  $v'$  of the perturbation velocity at large distances from the wall. From (2.372), we have

$$v' = \phi_y = \frac{u_\infty \delta\theta}{2\pi i} \int_{\text{arc}} \frac{\exp[ik(x - cy)]}{k} dk. \quad (3.278)$$

If we think of the arc on the path of integration which bypasses the singular point  $k = 0$  as a semicircle  $K_\epsilon$  with radius  $\epsilon$  about  $k = 0$ , then the integral can be split up as follows:

$$v' = u_\infty \delta\theta \left( \frac{1}{2\pi i} \int_{K_\epsilon} \dots dk + \frac{1}{2\pi i} \int_{-\infty}^{-\epsilon} \dots dk + \frac{1}{2\pi i} \int_{+\epsilon}^{+\infty} \dots dk \right). \quad (3.279)$$

Now,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{K_\epsilon} \dots dk = \frac{1}{2},$$

as we can show by introducing the polar angle  $\Theta$ , with  $k = \epsilon \exp i\Theta$ , as the integration variable. The remaining two integrals are integrals over the real  $k$  axis. The integrand contains the real factor  $\exp[i\lambda k \cdot \text{Im}(c) \cdot y/\lambda]$ , which is 1 for  $y = 0$  and decreases exponentially with increasing  $y$ , since, as said before, we must choose the branch of  $c(k)$  with positive real part; but for real  $k$ ,  $k \cdot \text{Im}(c) < 0$  for this branch. If we confine our attention to distances so great from the wall that  $y/\lambda \gg 1$ , then only the neighborhood of  $k = 0$  is important in the value of the integral. For this neighborhood we can expand  $c$  from Eq. (3.269) in  $i\lambda k$ :

$$c = \beta_a - \frac{M_{a\infty}^2 - M_{b\infty}^2}{\beta_a} \frac{i\lambda k}{2} + \dots. \quad (3.280)$$

If we use the first two terms of the expansion in the integral and pass to the limit  $\epsilon \rightarrow 0$ , we get, after a short computation,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \left( \int_{-\infty}^{-\epsilon} \dots dk + \int_{+\epsilon}^{+\infty} \dots dk \right) \\ = \frac{1}{\pi} \int_0^\infty \frac{\sin k(x - \beta_a y)}{k} \exp\left(-\frac{M_{a\infty}^2 - M_{b\infty}^2}{2\beta_a} y \lambda k^2\right) dk. \end{aligned} \quad (3.281)$$

In (3.281) we recognize an integral representation of the error function  $\Phi$  [see Eq. (3.105)] with the argument<sup>62</sup>

$$(x - \beta_a y) \left( \frac{M_{a\infty}^2 - M_{b\infty}^2}{\beta_a} 2\lambda y \right)^{-1/2}.$$

Using this, we can write the solution for  $v'$  as

$$v' = \frac{u_\infty \delta\theta}{2} \left[ 1 + \Phi\left(\frac{x - \beta_a y}{l}\right) \right], \quad (3.282)$$

in which

$$l = \sqrt{\frac{M_{a\infty}^2 - M_{b\infty}^2}{\beta_a} 2\lambda y}. \quad (3.283)$$

Accordingly, at a sufficiently large distance from the wall,  $v'$  increases in a region of width  $l$  which is centered about the equilibrium Mach line  $x - y(M_{a\infty}^2 - 1)^{1/2} = 0$  from the corner, from a value of zero upstream continuously to a value  $u_\infty \delta\theta$  downstream. The distribution of  $v'$  in the entire flow field is shown qualitatively in Fig. 99, as described in the present discussion. We can also contrast this with the analogous Fig. 39 for the flow generated by an impulsively-accelerated piston.

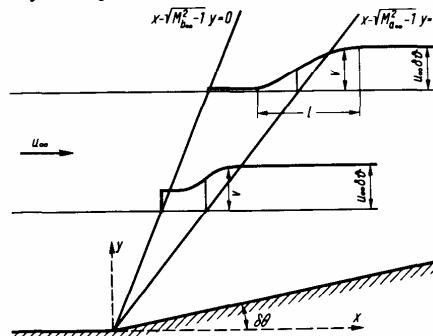


Fig. 99. Distribution of the vertical velocity  $v'$  at various distances from a wall with a corner for flow with relaxation (schematic).

<sup>62</sup> See W. Magnus and F. Oberhettinger, "Formulas and Theorems for the Special Functions of Mathematical Physics." Chelsea, New York, 1949.

As was already explained in Section 3.3.2, the width of the region given by (3.283), which increases with  $y$  and across which  $v'$  and all the other flow variables change from their upstream values to their downstream values, will eventually reach an asymptotic value if the bend in the wall is concave. This asymptotic width cannot be found from linear theory. It results from the balance between the widening tendency as given by linear theory and the nonlinear steepening tendency, which, in a flow without relaxation, finally leads to a shock wave (Section 3.3.2).

With  $\phi_x$  from (3.272), the pressure coefficient  $c_p$  from (3.238) is

$$c_p = \frac{\delta\theta}{\pi i} \int_{-\infty}^{\infty} \frac{\exp[ik(x - cy)]}{ck} dk. \quad (3.284)$$

For  $y = 0$ , i.e., the points directly on the wall,  $c_p$  can be expressed as function of  $x$  in terms of Bessel functions.<sup>63</sup> Instead of deriving this somewhat intricate formula, we shall give only a qualitative discussion of the pressure distribution on the wall: Using the substitution  $kx = z$ ,  $c_p$  for  $y = 0$  can be written as

$$c_p(x, 0) = \frac{\delta\theta}{\pi i} \int_{-\infty}^{\infty} \frac{\exp iz}{zc(z)} dz, \quad (3.285)$$

where, by (3.264),  $c(z)$  has the meaning

$$c(z) = \left[ \frac{i\beta_b^2(\lambda/x)z + \beta_a^{-2}}{i(\lambda/x)z + 1} \right]^{\frac{1}{2}}.$$

For  $\lambda/x \rightarrow \infty$ ,  $c(z) \rightarrow \beta_b = (M_{b\infty}^2 - 1)^{1/2}$ . In this case, we get from (3.285):

$$c_p(x, 0) = \frac{2\delta\theta}{(M_{b\infty}^2 - 1)^{\frac{1}{2}}} \quad \text{for } x > 0. \quad (3.286)$$

(For  $x < 0$ , we always have  $c_p = 0$ ). On the other hand, for  $\lambda/x \rightarrow 0$ , the phase velocity  $c(z) \rightarrow \beta_a = (M_{a\infty}^2 - 1)^{1/2}$ , and from (3.285) we get

$$c_p(x, 0) = \frac{2\delta\theta}{(M_{a\infty}^2 - 1)^{\frac{1}{2}}} \quad \text{for } x > 0. \quad (3.287)$$

<sup>63</sup> See J.F. Clarke, footnote 55.

Immediately downstream of the corner, we thus get from (3.286) the pressure corresponding to a completely frozen flow, while at sufficiently large distance from the corner, (3.287) gives the equilibrium pressure. The pressure distribution on the wall is drawn in Fig. 100.

It is evident that we can use the method of superposing particular solutions of Eq. (3.263) in a Fourier integral, shown here for the example of flow past a wall with a corner, to treat arbitrary-shaped walls, provided that the inclination of the wall contour with respect to the flow direction remains everywhere small such that linear theory is applicable. The plane supersonic flow past a thin profile can also be treated in this way.<sup>64</sup> We then get a generalization of the Ackeret theory (see Section 3.5.2) to flows with relaxation. If the inclination of the wall to the flow is too large, then the nonlinear

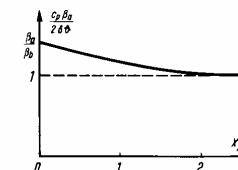


Fig. 100. Pressure coefficient  $c_p$  on the wall for the flow shown in Fig. 99 (schematic).

equations must be used to calculate the flow. In flows without relaxation, the deflection of a flow past a convex wall is achieved through a Prandtl-Meyer wave (see Section 3.6), i.e., through a simple wave, for which the flow states on the Mach lines of one family are constant and these Mach lines are, accordingly, straight lines. In a flow with relaxation, there are no simple waves. Neither are the characteristics of one family straight, nor are the states on them constant. To calculate the flow field, we must return to the method of characteristics. Using the method of characteristics, many authors have treated, among other things, the flow around a convex wall with a

<sup>64</sup> J.F. Clarke, Relaxation effects on the flow over slender bodies, *J. Fluid. Med.* **11**, 577–603, (1961). J.J. Der, Linearized Supersonic Nonequilibrium Flow past an Arbitrary Boundary. NASA TN R-119, 1961. K.C. Wang, Unsteady linearized flow past slender bodies, *Phys. Fluids* **7**, 25–32, (1964).

corner, which is the generalization of the centered Prandtl-Meyer wave.<sup>65</sup> We shall not go into any further discussion of the method of characteristics for plane steady flow with relaxation. A reference to the analogy in Section 3.3.1 should be sufficient here; there, we find the basic Eqs. (3.81)–(3.84) of the method of characteristics for a one-dimensional unsteady flow.

<sup>65</sup> I.I. Glass and H. Kawada, Prandtl-Meyer Flow of Dissociated and Ionized Gases. Univ. of Toronto, Institute of Aerophysics Report No. 85, Toronto, Ontario, (1962). J.P. Appleton, Structure of a Prandtl-Meyer expansion in an ideal dissociating gas, *Phys. Fluids* **6**, 1057–1062 (1963).

## 4 VISCOUS FLOWS

### 4.1 Transport Properties of Gases

#### 4.1.1 VISCOSITY

In Section 2.3, we divided the stress tensor  $\mathbf{S}$  of a flowing gas into a spherically symmetric tensor  $-p\mathbf{E}$  and the viscous stress tensor  $\mathbf{T}$  [Eq. (2.43)]. The pressure  $p$  is a thermodynamic state variable, and is related to the other thermodynamic state variables through the equations of state of the medium (see Sections 1.5 and 1.12). The components of the viscous stress tensor  $\mathbf{T}$  depend on the state of motion of the medium. In a gas at rest, all components of  $\mathbf{T}$  are equal to zero, as is also the case in a body of gas translating or rotating like a solid body. The viscous stresses will only be nonzero when the deformation tensor  $\mathbf{D}$  introduced in Section 2.1 does not vanish identically.

One of the simplest flows in which not all components of  $\mathbf{T}$  vanish is the plane shear flow shown in Fig. 13; the only component of the velocity which is nonzero is  $u$ , and it depends only on the coordinate  $y$ :  $u = u(y)$  (Fig. 13 shows the more special case of  $u = cy$ ). From experience, we know that in such a flow only the components  $\tau_{xy} = \tau_{yx}$  of the viscous stress tensor are nonzero, and

$$\tau_{xy} = \eta du/dy, \quad (4.1)$$

where the proportionality factor  $\eta$  depends only on the thermodynamic state of the gas (e.g., on temperature and pressure) but not on the velocity field, and it is called the *coefficient of shear viscosity*, or, for short, the *viscosity*

of the gas. A medium in which a linear relation (4.1) holds between the shear stress and velocity gradient is called a Newtonian fluid. The Newtonian fluid is an idealization, since in real media the relation between shear stress and velocity gradient is in general not exactly linear. Nevertheless, for almost all practical purposes, it is sufficient to treat a gas as a Newtonian fluid.

As long as the gas does not dissociate, i.e., its temperature is not too high, or it is monatomic, etc., then the viscosity  $\eta$  depends strongly only on temperature but negligibly on pressure. In Fig. 101,  $\eta$  for air is shown in the tem-

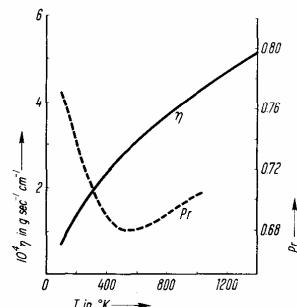


Fig. 101. Viscosity  $\eta$  and Prandtl number  $Pr$  of air as functions of the temperature  $T$  ( $p = 1$  atm). (From J. Hilsenrath *et al.*, Tables of Thermodynamic and Transport Properties of Air, etc. Oxford-London-New York-Paris 1960).

perature range of  $100^\circ\text{K}$  to  $1400^\circ\text{K}$ . In gases, the viscosity increases with temperature (in liquids, it decreases with increase in temperature). This also agrees with the results of the kinetic theory of gases, which (for not too dense gases) gives the following expression for  $\eta$  (with a dimensionless numerical factor):

$$\eta \propto m\bar{v}/\Omega. \quad (4.2)$$

Here,  $m$  is the molecular mass,  $\bar{v}$  the mean thermal speed of the molecules, and  $\Omega$  a cross section for collision between two molecules. For completely hard spherical molecules with diameter  $d$ ,  $\Omega = \pi d^2$  and is thus independent of temperature  $T$ . [In this case, the proportionality factor on the right side of Eq. (4.2) has the value 0.35.<sup>66</sup>] Since, on the other hand, the mean thermal

<sup>66</sup> See, e.g., R.D. Present, "Kinetic Theory of Gases (Chap. 11.2)." McGraw-Hill, New York, 1958.

speed  $\bar{v} \propto \sqrt{T}$ , Eq. (4.2) then gives a direct proportionality between  $\eta$  and  $\sqrt{T}$ . However, this agrees only qualitatively with observations. In many cases, the dependence of viscosity on temperature in practically interesting gases in the important range of temperatures  $10^2$ – $10^3^\circ\text{K}$  is better described by a power law of the form

$$\eta \propto T^\omega, \quad (4.3)$$

with  $\omega = 0.7$  to 0.8 (for vapors—e.g., steam—to  $\omega = 1.0$ ). This departure from the  $T^{1/2}$ -law can be interpreted by taking into account the decrease of effective cross section  $\Omega$  with increase in temperature  $T$ , and hence with increase in thermal velocity and energy of the molecules. A relatively simple formula which takes this effect into account and which is based on the idea that the molecules may be regarded as hard spheres that attract each other only weakly is the Sutherland formula:

$$\eta = C\sqrt{T}\left(1 + \frac{D}{T}\right)^{-1}, \quad (4.4)$$

with two constants  $C$  and  $D$ . By adjusting these constants to experimental results, formula (4.4) can be made to describe very well the dependence of the viscosity  $\eta$  on  $T$  in a wide range of temperatures. When  $T$  is measured in  $^\circ\text{K}$  and  $\eta$  in  $\text{g cm}^{-1} \text{sec}^{-1}$ , the special formula for air is

$$\eta = 1.46 \cdot 10^{-5} \sqrt{T} \left(1 + \frac{112}{T}\right)^{-1}. \quad (4.5)$$

At increased temperatures, Eq. (4.4) becomes the simple law  $\eta = C\sqrt{T}$ , which is to be expected for hard-sphere molecules. The term  $D/T$ , which accounts for the mutual attraction of molecules, is of significance only at low temperatures, hence at low thermal velocities, since only then does the attraction force have important influence on the paths of two interacting molecules, while at high molecular velocities, only the hard centers of the molecules have any significance. For very high temperatures, the Sutherland formula becomes invalid when the gas dissociates or ionizes. The viscosity then depends on pressure in addition to temperature. Figure 102 reproduces the data for air from the calculations of Hansen and Heims;  $\eta_s$  is the Sutherland value corresponding to each temperature as given by Eq. (4.5). Dissociation shows itself in an increase of  $\eta$  over  $\eta_s$ , and ionization in a decrease. This is understandable, since in dissociation the gas becomes

monatomic and the collision cross section of an atom is smaller than that of a molecule, while in ionization, because of the strong electrical interaction between the electrically charged atoms,  $\Omega$  again increases rapidly.

With respect to later applications, we should also note: In many cases, particularly in boundary-layer theory, it is convenient to assume that the

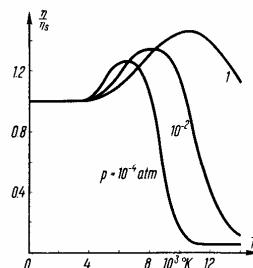


Fig. 102. The ratio of the viscosity  $\eta$  of air to the Sutherland value  $\eta_s$  as a function of temperature  $T$  and pressure  $p$ . (From C.F. Hansen and S.P. Heims, A Review of the Thermodynamic, Transport and Chemical Reaction Rate Properties of High Temperature Air, NACA TN 4359, 1958.)

product  $\eta\varrho$  depends only on the pressure  $p$  and not on the temperature  $T$ . In a thermally ideal gas,  $\eta\varrho = \eta p/(RT)$ , so that this assumption is equivalent to assuming that  $\eta$  be proportional to the absolute temperature  $T$ . Under the assumption  $\eta \propto T^\omega$ , on the other hand,  $\eta\varrho$  decreases with increasing  $T$  at constant pressure if  $\omega < 1$ . If Eq. (4.4) holds,  $\eta\varrho$  increases with  $T$  when  $T < D$ , and decreases with increasing  $T$  when  $T > D$ . Since  $D$  for air and other technically important gases is of the order of  $10^2$  °K, we should expect  $\eta\varrho$  to decrease with increasing  $T$  in applications, where  $T$  is usually above this value. The assumption of  $\eta\varrho = f(p)$  can thus be used in theoretical investigations only if high accuracy is not claimed.

We now generalize the relation (4.1) between shear stress and velocity gradient, which is valid for simple shear flow, to a relation between the components of the viscous stress tensor  $\mathbf{T}$  and the corresponding components of the deformation tensor  $\mathbf{D}$  for arbitrary flow fields. This generalization will be based on the following postulates, which characterize a Newtonian fluid in a completely general way:

1. The components of  $\mathbf{T}$  at a given point in space and a given instant of time depend only on the components of  $\mathbf{D}$  at the same point and same instant.

2. The relation between the components of  $\mathbf{T}$  and  $\mathbf{D}$  is linear and homogeneous (the latter means that the components of  $\mathbf{T}$  will all vanish only when the components of  $\mathbf{D}$  all vanish).

3. With respect to this relation, the medium is isotropic (i.e., there are no preferred directions in the fluid).

In textbooks of continuum mechanics, it is shown that under these assumptions two constants  $\eta$  and  $\eta_b$ , which depend on the thermodynamic state of the medium, will enter this relation; and, indeed, when written in component form, this relation is:<sup>67</sup>

$$\left. \begin{aligned} \sigma_x &= \frac{2}{3}\eta u_x - \frac{2}{3}\eta(v_y + w_z) + \eta_b \operatorname{div} \mathbf{v}, \\ \sigma_y &= \frac{2}{3}\eta v_y - \frac{2}{3}\eta(w_z + u_x) + \eta_b \operatorname{div} \mathbf{v}, \\ \sigma_z &= \frac{2}{3}\eta w_z - \frac{2}{3}\eta(u_x + v_y) + \eta_b \operatorname{div} \mathbf{v}, \\ \tau_{xy} &= \tau_{yx} = \eta(u_y + v_x), \\ \tau_{yz} &= \tau_{zy} = \eta(v_z + w_y), \\ \tau_{zx} &= \tau_{xz} = \eta(w_x + u_z). \end{aligned} \right\} \quad (4.6)$$

This can be summarized in tensorial notation as:

$$\mathbf{T} = 2\eta\mathbf{D} + (\eta_b - \frac{2}{3}\eta)(\operatorname{div} \mathbf{v})\mathbf{E}. \quad (4.7)$$

Specializing formula (4.6) to the simple shear flow  $u = u(y)$ ,  $v = w = 0$ , it immediately follows that  $\eta$  is the shear viscosity discussed before.  $\eta_b$  is called the *coefficient of bulk viscosity*.

We denote by mean normal stress  $\bar{\sigma}$  the arithmetic mean of the diagonal terms of the stress tensor  $\mathbf{S}$  (Section 2.3). This quantity is an invariant of the stress tensor ( $3\bar{\sigma}$  is the *trace* of  $\mathbf{S}$ ), i.e., it does not change under rotation of coordinate system, although the individual stress components do. From Eq. (2.43),

$$\bar{\sigma} = -p + \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = -p + \eta_b \operatorname{div} \mathbf{v}. \quad (4.8)$$

When  $\eta_b \neq 0$ , the mean normal stress  $\bar{\sigma}$  does not equal the negative thermodynamic pressure  $-p$ , as is sometimes incorrectly assumed. This has the

<sup>67</sup> The partial derivatives of the velocity components are denoted by appropriate indices. On the other hand, the subscripts in the stress components do not denote differentiation.

consequence that in a pure isotropic volume change of the gas, work is done not only against the pressure  $p$  but also against the viscous stresses. Let us imagine a spherical body of gas which is isotropically expanded or compressed. Such an expansion can be achieved by a velocity field  $\mathbf{v} = c\mathbf{r}$ , with the constant  $c > 0$  for expansion and  $< 0$  for compression. From Eq. (4.6), it follows that for this case all shear stresses vanish, while the normal stresses are  $\sigma_x = \sigma_y = \sigma_z = 3\eta_b c$ . The stress tensor is thus spherically symmetric:  $\mathbf{S} = (-p + 3\eta_b c)\mathbf{E}$ . The shear viscosity  $\eta$  plays no role in this process, and only the bulk viscosity  $\eta_b$  does. On each surface element of the spherical body of the gas, the normal stress  $-p + 3\eta_b c = \bar{\sigma}$  acts. The kinetic theory of gases gives  $\eta_b = 0$  for hard-sphere molecules, which agrees with measurements on monatomic gases (noble gases). For polyatomic gases, on the other hand, we observe, particularly from the damping of ultrasonic waves, that  $\eta_b \neq 0$ . This is because of the internal degrees of freedom of the molecules (rotational and vibrational degrees of freedom), and is closely connected with the relaxation processes in these degrees of freedom. This connection was already referred to in Section 3.3. Here it suffices to include  $\eta_b$  as an empirically given quantity. In connection with the supplementary remarks to Section 3.3.2, however, we must remember that in certain gasdynamical processes in which the thermodynamic state changes rapidly with time, the assumption of a finite bulk viscosity  $\eta_b$  depending solely on the thermodynamic state must be replaced by a more exact consideration of the relaxation phenomena. Moreover, the concept of shear viscosity (as well as heat conductivity) also becomes shaky when the characteristic time for changes in the flow field becomes so small as to be of the same order of magnitude as the time between two successive collisions of a molecule with others. This time is the relaxation time for the translational degrees of freedom of molecules, which is responsible for the shear viscosity. The methods of continuum theory break down in these cases (see Section 1.12).

#### 4.1.2 THERMAL CONDUCTIVITY

After clarifying the relation between the viscous stress tensor and the velocity field, we must now connect the energy flux vector  $\mathbf{q}$  introduced in Section 2.4 to the state variables of the gas. We shall confine ourselves to the simplest case and take  $\mathbf{q}$  to be identical to the heat flux vector caused by a temperature gradient. Energy transport through diffusion processes (e.g.,

transport of chemical energy) will be neglected. The heat flux vector  $\mathbf{q}$  vanishes when the space-wise temperature gradient  $\text{grad } T$  vanishes. If we assume that the components of  $\mathbf{q}$  are linearly and homogeneously dependent on the components of  $\text{grad } T$  and that the medium is isotropic with respect to heat flow, then Fourier's heat conduction law results:

$$\mathbf{q} = -k \text{ grad } T. \quad (4.9)$$

By inserting the “-” sign,  $k$  becomes a positive quantity. This quantity is dependent on the thermodynamic state of the medium, but not on  $\text{grad } T$ . We call  $k$  the *coefficient of thermal conductivity*, or, for short, the *thermal conductivity*.

Just as with  $\eta$ , the thermal conductivity  $k$  for gases depends on the temperature  $T$  but only slightly on the pressure  $p$ . Figure 103 shows  $k$  for

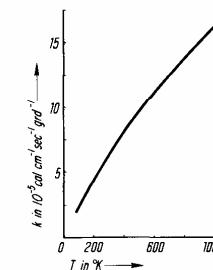


Fig. 103. Thermal conductivity  $k$  for air as a function of the temperature  $T$  ( $p = 1 \text{ atm}$ ). (From J. Hilsenrath *et al.*, Tables of Thermodynamic and Transport Properties of Air, etc. Oxford-London-New York-Paris 1960).

air. Elementary kinetic theory predicts a direct proportionality between  $k$  and  $\eta$ . Since, in the framework of this theory, the specific heat  $c_p$  is also a constant, this means that the dimensionless quantity

$$\text{Pr} = c_p \eta / k \quad (4.10)$$

must be a constant. This is an important quantity in gas dynamics and is called the Prandtl number. For monatomic gases, the Prandtl number is actually very nearly constant. Figure 101 gives the Prandtl number for air. In the temperature range from  $200^\circ\text{K}$  to  $1000^\circ\text{K}$ ,  $\text{Pr}$  varies only slightly from

its mean value of 0.7. By a well confirmed formula of Eucken,  $\text{Pr}$  can be expressed in terms of the adiabatic coefficient  $\gamma$ :

$$\text{Pr} = 4\gamma/(9\gamma - 5). \quad (4.11)$$

The variation of  $\text{Pr}$  with  $T$  can thus be traced back to the variation of  $\gamma$  with  $T$ , resulting from departure from calorically ideal gas behavior. These relations for the heat conductivity will be invalid when the gas dissociates or ionizes. For then, just as with viscosity, the heat conductivity will depend strongly on the pressure.

We can now use relations (4.6) and (4.9) to derive an explicit expression for the entropy generation by viscosity and heat conduction, according to formula (2.69). With the definition (2.66) for the dissipation function  $\Phi$  and by substituting for the viscous stresses in terms of the velocity gradients in accordance with Eq. (4.6), we get for the entropy generated by viscous friction  $\sigma_f$ :

$$\begin{aligned} T\sigma_f = \Phi &= 2\eta[(u_x)^2 + (v_y)^2 + (w_z)^2] + \eta[(u_y + v_x)^2 + (v_z + w_y)^2 \\ &\quad + (w_x + u_z)^2] + (\eta_b - \frac{2}{3}\eta)(u_x + v_y + w_z)^2. \end{aligned} \quad (4.12)$$

We at once recognize this to be the generalization of expression (1.17), which was derived in Section 1.3 for a special example. Since by the second law of thermodynamics,  $\sigma_f$  must be  $\geq 0$  for any velocity field (see Section 2.4), we can conclude, in addition, from Eq. (4.12) that  $\eta \geq 0$  and  $\eta_b \geq 0$  must hold. In an entirely analogous manner, we obtain from Eq. (2.69), using Eq. (4.9), the following for the entropy generated by heat conduction  $\sigma_h$ :

$$\sigma_h = (k/T^2)(\text{grad } T)^2. \quad (4.13)$$

This is the generalization of formula (1.18). By the second law of thermodynamics,  $k$  must be  $\geq 0$ , as was already mentioned.

*Supplementary Remarks.* If we multiply the numerator and denominator of expression (4.2) for  $\eta$  by  $n$ , the number of molecules per unit volume, then the density  $\varrho = nm$  appears in the numerator and the product  $n\Omega$  in the denominator. The quantity  $(n\Omega)^{-1}$  has the dimension of a length. For hard-sphere molecules,  $(n\Omega/\sqrt{2})^{-1}$  has the meaning of mean free path  $l_f$ . We can then also write (4.2) in the form

$$\eta \propto \varrho \bar{v} l_f. \quad (4.14)$$

The kinetic theory of gases gives the value 0.499 for the proportionality factor on the right side.

## 4.2 Flow through a Normal Shock Wave

### 4.2.1 BASIC RELATIONS FOR ARBITRARY SHOCK STRENGTHS

In the following sections we study a normal shock wave into which a gas flows with velocity  $u_1$  in the  $x$  direction; the downstream velocity is  $u_2 < u_1$ . For given upstream quantities (denoted by the subscript "1"), the downstream quantities (denoted by the subscript "2") are determined by the shock relations (3.107)–(3.109). While we have thus far neglected viscosity and thermal conductivity and treated the shock wave as a discontinuity in the flow field, we shall now regard it as a phenomenon occurring over a finite region of space with a continuous distribution of the flow variables (Fig. 104). The

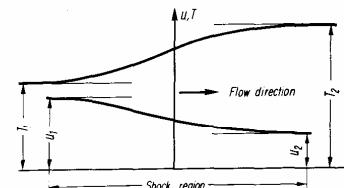


Fig. 104. Velocity and temperature distribution in a shock wave (schematic).

upstream values will be assumed by the flow variables asymptotically for  $x \rightarrow -\infty$ , and the downstream values for  $x \rightarrow +\infty$ . The flow will be assumed to be steady;  $u(x)$  appears as the only velocity component, and all the other flow variables, like  $u$ , are also functions of  $x$  only. Under the fundamental assumption that the relations discussed in Sections 4.1.1 and 4.1.2 for the viscous stress tensor  $T$  and the heat flux vector  $q$  are valid in the shock wave,  $T$  contains, among others, the component  $\sigma_x$ , where<sup>68</sup>

$$\sigma_x = \left(\frac{4}{3}\eta + \eta_b\right) \frac{du}{dx} = \hat{\eta} \frac{du}{dx}, \quad (4.15)$$

For simplification, we have introduced the notation  $\hat{\eta}$ . For the heat flow  $q$ , only the  $x$  component remains:

$$q_x = -k \frac{dT}{dx}. \quad (4.16)$$

<sup>68</sup> In what follows, the other components of the tensor  $T$  play no role, since they drop out of the momentum equation.

The continuity equation (2.33) reduces to

$$\frac{d}{dx}(\rho u) = 0 \quad (4.17)$$

From the momentum equation (2.52), we obtain, with the aid of Eq. (4.15),

$$\rho u \frac{du}{dx} = -\frac{dp}{dx} + \frac{d}{dx} \left( \hat{\eta} \frac{du}{dx} \right). \quad (4.18)$$

Taking (4.17) into account, we can also write this in the form

$$\frac{d}{dx} \left( \rho u^2 + p - \hat{\eta} \frac{du}{dx} \right) = 0. \quad (4.19)$$

The energy equation (2.60), when Eqs. (4.15) and (4.16) are used, becomes

$$\rho u \frac{d}{dx} \left( \frac{u^2}{2} + h \right) = \frac{d}{dx} \left( \hat{\eta} u \frac{du}{dx} + k \frac{dT}{dx} \right). \quad (4.20)$$

Equations (4.17), (4.19), and (4.20) can immediately be integrated once with respect to  $x$ . If we denote the constants of integration by  $\Theta_0$ ,  $p_0$ , and  $h_0$ , (4.17) becomes

$$\rho u = \Theta_0 = \varrho_1 u_1, \quad (4.21)$$

(4.19) becomes

$$\rho u^2 + p - \hat{\eta} \left( \frac{du}{dx} \right) = p_0 = p_1 + \varrho_1 u_1^2, \quad (4.22)$$

and, since  $\rho u = \text{const}$ , (4.20) becomes

$$\frac{u^2}{2} + h - \frac{\hat{\eta}}{\varrho} \frac{du}{dx} - \frac{k}{\varrho u} \frac{dT}{dx} = h_0 = h_1 + \frac{u_1^2}{2}. \quad (4.23)$$

The constants of integration  $\Theta_0$ ,  $p_0$ , and  $h_0$  can be expressed in terms of the given data upstream, since, for  $x \rightarrow -\infty$ , we must have  $\varrho \rightarrow \varrho_1$ ,  $p \rightarrow p_1$ ,  $h \rightarrow h_1$ ,  $u \rightarrow u_1$ ,  $du/dx \rightarrow 0$ , and  $dT/dx \rightarrow 0$ . Similarly, we can also express the constants of integration in terms of the downstream quantities.<sup>69</sup> Equa-

<sup>69</sup> Here it is tacitly assumed that a solution to Eqs. (4.21)–(4.23) exists, which for  $x \rightarrow \pm \infty$  assumes the upstream or downstream values. The question of existence—and also uniqueness—of such a solution is answered positively for the special cases to be treated by explicitly exhibiting the solution. For the general case, see J. Serrin, Mathematical principles of Classical Fluid Mechanics, in "Handbuch der Physik" (S. Flügge, ed.), vol. VIII/1, pp. 125–263. Springer, Berlin, 1960–62.

tions (4.21)–(4.23) are the generalizations of the shock relations (3.107)–(3.109); while these only connect the upstream and downstream quantities, Eqs. (4.21)–(4.23) describe the flow at every point in the shock.

We assume that the gas is in unconstrained thermodynamic equilibrium at every point in the shock; this assumption we shall later analyze critically. In unconstrained thermodynamic equilibrium, two state variables suffice for the unique specification of a thermodynamic state—e.g.,  $T$  and  $\varrho$ . From Eq. (4.21),  $\varrho$  can be expressed in terms of  $u$ :  $\varrho = \Theta_0/u$ , so that we can choose  $T$  and  $u$  as independent thermodynamic variables. If  $T$  and  $u$  are known as functions of  $x$ , then all the other variables are also known as functions of  $x$ . For  $u(x)$  and  $T(x)$  we obtain, from Eqs. (4.22) and (4.23) after a short transformation,

$$\hat{\eta} \frac{du}{dx} = p - p_0 + \Theta_0 u, \quad (4.24)$$

$$\frac{k}{\Theta_0} \frac{dT}{dx} = h - h_0 - \frac{u^2}{2} + \frac{p_0 - p}{\Theta_0} u. \quad (4.25)$$

From the knowledge of the equations of state and the material properties of the gas, i.e., the relations  $p = p(T, \varrho)$ ,  $h = h(T, \varrho)$ ,  $\hat{\eta} = \hat{\eta}(T, \varrho)$ , and  $k = k(T, \varrho)$ , we know the right side of Eqs. (4.24) and (4.25), as well as the factors  $\hat{\eta}$  and  $k$  on the left side, as functions of  $T$  and  $u$ , and we have a system of two differential equations of the first order for  $T(x)$  and  $u(x)$ . Dividing one equation by the other, we next obtain an equation of the form  $dT/du = f(T, u)$ , from which, by integration,  $T$  will result as a function of  $u$ . Substituting the result in Eq. (4.24) and integrating, we then get  $u(x)$ .

In general, this integration must be carried out numerically, and only in certain special cases is a solution in closed form possible. We pick out three such special cases:

1. This special case results from a series of increasingly narrow specializations (a to e):

a. The gas is thermally ideal, so that  $h = h(T)$  and  $dh = c_p dT$ . With this, we get from Eq. (4.23)

$$\frac{u^2}{2} + h - h_0 = \frac{\hat{\eta}}{\varrho u} \left( \frac{d}{dx} \left( \frac{u^2}{2} \right) \right) + \frac{k}{c_p \hat{\eta}} \frac{dh}{dx}. \quad (4.26)$$

The combination of material constants appearing in the last term on the right

$$c_p \hat{\eta} / k = \hat{Pr} \quad (4.27)$$

is the Prandtl number characteristic of the flow in the shock wave. When, in particular,  $\eta_b = 0$ , then  $\hat{Pr} = 4\text{Pr}/3$ , where the ordinary Prandtl number  $\text{Pr}$  is defined by Eq. (4.10).

b. We assume  $\text{Pr} = 1$ . Then we get from Eq. (4.26)

$$\frac{u^2}{2} + h - h_0 = \frac{\hat{\eta}}{\varrho u} \frac{d}{dx} \left( \frac{u^2}{2} + h - h_0 \right). \quad (4.28)$$

For  $x \rightarrow -\infty$  and  $x \rightarrow +\infty$ ,  $\frac{1}{2}u^2 + h - h_0$  must vanish, since, by the definition of  $h_0$  in Eq. (4.23):  $\frac{1}{2}u_1^2 + h_1 = \frac{1}{2}u_2^2 + h_2 = h_0$ . The only solution of the differential equation (4.28) for the quantity  $\frac{1}{2}u^2 + h - h_0$  that vanishes at  $x \rightarrow -\infty$  and  $x \rightarrow +\infty$  is the identically vanishing solution. Thus,

$$\frac{1}{2}u^2 + h = h_0. \quad (4.29)$$

In other words, the stagnation enthalpy  $h_t = h + \frac{1}{2}u^2$  of the gas has the same value  $h_0$  everywhere.

c. For further simplification, we now specialize to a calorically ideal gas. We then have from Eq. (1.58) (since only entropy difference plays a part, the constant  $h^*$  may be dropped):

$$h = c_p T = \frac{c_p}{R} \frac{p}{\varrho} = \frac{\gamma}{\gamma - 1} \frac{p}{\varrho} \quad (4.30)$$

or

$$p = \frac{\gamma - 1}{\gamma} h \varrho = \frac{\gamma - 1}{\gamma} \left( h_0 - \frac{u^2}{2} \right) \varrho. \quad (4.31)$$

Substituting into Eq. (4.24) and transforming, we obtain

$$\frac{2\gamma}{\gamma + 1} \frac{\hat{\eta}}{\Theta_0} \frac{du}{dx} = \frac{1}{u} \left( u^2 - \frac{2\gamma}{\gamma + 1} \frac{p_0}{\Theta_0} u + \frac{2(\gamma - 1)}{\gamma + 1} h_0 \right). \quad (4.32)$$

In the brackets on the right is a second-degree polynomial in  $u$ . It has the two roots  $u = u_1$  and  $u = u_2$ , as we can easily establish by calculation. This also immediately follows from the fact that the quantity  $du/dx$  must vanish for  $u = u_1$  and  $u = u_2$ . This permits us to write Eq. (4.32) as follows:

$$\frac{2\gamma}{\gamma + 1} \frac{\hat{\eta}}{\Theta_0} \frac{du}{dx} = \frac{(u - u_1)(u - u_2)}{u}. \quad (4.33)$$

To enable us to find  $u(x)$  by integration from this stage, we must make yet another assumption concerning the dependence of the viscosity  $\hat{\eta}$  on the thermodynamic state variables.

d. We assume that  $\hat{\eta} \sim T^\omega$ . From Eq. (4.29) with  $h = c_p T$  it follows that  $T = (1/c_p)(U_{\max}^2 - u^2)$ , where  $U_{\max} = (2h_0)^{1/2}$  is the maximum speed corresponding to the stagnation enthalpy  $h_t = h_0$  [see Eq. (3.10)]. If we denote by  $\hat{\eta}_0$  the viscosity corresponding to the temperature  $T_0 = h_0/c_p$ , then

$$\hat{\eta} = \hat{\eta}_0 \left[ \frac{U_{\max}^2 - u^2}{U_{\max}^2} \right]^\omega. \quad (4.34)$$

Substituting the expression (4.34) for  $\hat{\eta}$  into Eq. (4.33), and introducing for short

$$\lambda = \frac{2\gamma}{\gamma + 1} \frac{\hat{\eta}_0}{\Theta_0}, \quad (4.35)$$

we obtain the equation

$$\lambda \frac{du}{dx} = \frac{U_{\max}^{2\omega}}{(U_{\max}^2 - u^2)^\omega} \frac{(u - u_1)(u - u_2)}{u}; \quad (4.36)$$

$\lambda$  has the dimension of a length.

Solution of Eq. (4.36) in closed form is particularly simple in the following two cases:

e<sub>1</sub>. Viscosity  $\hat{\eta}$  is independent of the temperature, i.e.,  $\omega = 0$ . In this case, it follows from Eq. (4.36) that

$$\frac{dx}{\lambda} = - \frac{1}{u_1 - u_2} \left( \frac{u_1}{u_1 - u} + \frac{u_2}{u - u_2} \right) du,$$

and by integration,

$$\frac{x}{\lambda} = \frac{u_1 \ln[1 - (u/u_1)] - u_2 \ln[(u/u_2) - 1]}{u_1 - u_2}. \quad (4.37)$$

e<sub>2</sub>. Viscosity  $\hat{\eta}$  is proportional to the absolute temperature, i.e.,  $\omega = 1$ . From Eq. (4.36), it follows after integration that

$$\frac{x}{\lambda} = \frac{u_1 \left( 1 - \frac{u_1^2}{U_{\max}^2} \right) \ln \left( 1 - \frac{u}{u_1} \right) - u_2 \left( 1 - \frac{u_2^2}{U_{\max}^2} \right) \ln \left( \frac{u}{u_2} - 1 \right)}{u_1 - u_2} - \frac{1}{U_{\max}^2} \left[ \frac{u^2}{2} + (u_1 + u_2)u \right]. \quad (4.38)$$

In Eq. (4.37) as well as in (4.38), we have set another integration constant  $x_0$  to be zero without loss of generality, since  $x_0 \neq 0$  in our problem merely corresponds to an unimportant translation of the origin of the  $x$  coordinate. The velocity distributions given by Eqs. (4.37) and (4.38) are qualitatively the same, and correspond to the shape sketched in Fig. 104. The temperature distribution comes out of Eq. (4.29) with  $h = c_p T$  once the velocity distribution is known, and it also corresponds qualitatively to that sketched in Fig. 104.

Before we embark on a discussion of the solution obtained here, we should mention the other two special cases for which we can find a solution just as easily as above:

2. The gas is calorically ideal and its thermal conductivity is  $k=0$ ; for finite viscosity, this implies that  $\hat{\Pr}=\infty$ . From Eq. (4.25), it then follows that

$$h - h_0 - \frac{1}{2}u^2 + (p_0 - p)u/\Theta_0 = 0,$$

and from this, with  $h = [\gamma/\gamma - 1] (p/\varrho)$ , we obtain

$$p = \frac{\gamma - 1}{u} \Theta_0 \left( h_0 + \frac{u^2}{2} - \frac{p_0}{\Theta_0} u \right). \quad (4.39)$$

Substituting into Eq. (4.24) gives

$$\frac{\hat{\eta}}{\Theta_0} \frac{2}{\gamma + 1} \frac{du}{dx} = \frac{(u - u_1)(u - u_2)}{u}. \quad (4.40)$$

Equation (4.40) is the same as Eq. (4.33) except for a factor of  $\gamma$  on the left side. Again assuming  $\hat{\eta} = \hat{\eta}_0 = \text{const}$ , we get the expression (4.37) as the solution of Eq. (4.40), with  $\lambda$  replaced by  $\lambda_1$ , where

$$\lambda_1 = \frac{2}{\gamma + 1} \frac{\hat{\eta}_0}{\Theta_0}; \quad (4.41)$$

$\lambda_1$  is  $\gamma^{-1}$  times the quantity  $\lambda$  defined in (4.35). The velocity distribution and the temperature distribution (and thus the distribution of all the other variables) are thus for  $\hat{\Pr}=\infty$  steeper by a factor of  $\gamma$  than for  $\hat{\Pr}=1$ . Qualitatively, this is also true when  $\hat{\eta}$  depends on the temperature; but then the shock profile for  $\hat{\Pr}=\infty$  is no longer simply obtained from that for  $\hat{\Pr}=1$  by an affine distortion in the  $x$  direction.

3. The limiting case of  $\hat{\eta}=0$ ,  $k \neq 0$ , i.e.,  $\hat{\Pr}=0$ , leads to an interesting

result. From Eq. (4.24) we obtain, by virtue of  $p = p_0 - \Theta_0 u$  and  $p = RT\varrho = RT\Theta_0/u$ ,

$$T = \frac{1}{R} \left( \frac{p_0}{\Theta_0} u - u^2 \right), \quad (4.42)$$

and with it,

$$\frac{dT}{dx} = \frac{1}{R} \left( \frac{p_0}{\Theta_0} - 2u \right) \frac{du}{dx}. \quad (4.43)$$

If we substitute this expression for  $dT/dx$  in the left side of Eq. (4.25) and replace  $h$  on the right side of Eq. (4.25) by  $c_p T$  (under the assumption of a calorically ideal gas), with  $T$  from (4.42), then we obtain, after transforming,

$$\frac{2(\gamma - 1)}{\gamma + 1} \frac{k}{R\Theta_0} \frac{du}{dx} = \frac{(u - u_1)(u - u_2)}{2u - (p_0/\Theta_0)}. \quad (4.44)$$

For simplicity, we now set  $k = k_0 = \text{const}$ . Upon introducing the length  $\lambda_2$  defined by

$$\lambda_2 = \frac{2(\gamma - 1)}{\gamma + 1} \frac{k_0}{R\Theta_0}, \quad (4.45)$$

we get from (4.44), after integration,

$$\frac{x}{\lambda_2} = \frac{\left( 2u_1 - \frac{p_0}{\Theta_0} \right) \ln \left( 1 - \frac{u}{u_1} \right) - \left( 2u_2 - \frac{p_0}{\Theta_0} \right) \ln \left( \frac{u}{u_2} - 1 \right)}{u_1 - u_2}. \quad (4.46)$$

We will get a velocity distribution of the type shown in Fig. 104 from Eq. (4.46) only when  $2u_2 > p_0/\Theta_0$ , i.e., if the shock does not exceed a certain strength. (We always have  $2u_1 > p_0/\Theta_0$ , as we can see from the following:  $2u_1 - (p_0/\Theta_0) = u_1 [1 - (p_1/\varrho_1 u_1^2)] = u_1 [1 - (a_1^2/\gamma u_1^2)] > 0$ , since  $u_1 > a_1$ .) When  $2u_2 < p_0/\Theta_0$ , then the situation shown in Fig. 105 results: The solution  $u(x)$  given by (4.46) and the corresponding temperature  $T(x)$  from Eq. (4.42) are shown in dotted lines. Obviously, we can satisfy the boundary condition  $u \rightarrow u_2$  for  $x \rightarrow +\infty$  only if we assume a discontinuity in the velocity distribution. This discontinuity must appear at the point where the temperature  $T$  reaches the final value  $T_2$ . There is thus an isothermal discontinuity in the shock region. We will obtain more realistic behavior if we take the Prandtl number  $\Pr$  to be different from zero but very small compared to 1; this means that the effect of heat conduction strongly dominates the effect of

viscosity. The discontinuity will then disappear, and we get instead a continuous but very steep velocity distribution.

#### 4.2.2 WEAK SHOCK WAVES

We now confine our attention to weak shock waves. A weak shock is defined here as a shock for which  $\Delta u \ll a_*$ , where  $\Delta u = u_1 - u_2$ . Since as  $\Delta u \rightarrow 0$  the velocities  $u_1$  and  $u_2$  both tend to the critical sound speed  $a_*$ , we have for a weak shock in calorically ideal gas  $u_1 = a_* + \frac{1}{2}\Delta u$ ,  $u_2 = a_* - \frac{1}{2}\Delta u$ ;

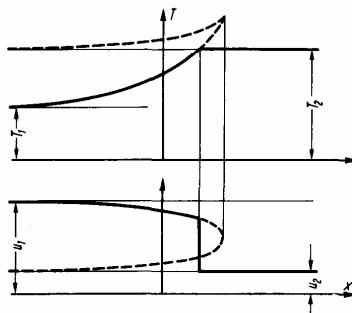


Fig. 105. Velocity and temperature distribution in a sufficiently strong shock for zero viscosity.

these result immediately from the Prandtl relation (3.129) when we neglect the quadratic terms in  $\Delta u/a_*$ . Finally, we should note the obvious fact that in a weak shock all the state variables differ only slightly from the critical values marked with  $a_*$  (see Section 3.1). The state variables will thus be replaced by the critical values in the following when the error so introduced is negligible.

Now we return to Eq. (4.37), which holds for shocks of arbitrary strength under the assumptions (a) through (e<sub>1</sub>) formulated above. For a weak shock, we obtain from Eq. (4.37), using  $u_1 \approx u_2 \approx a_*$ , first

$$\frac{x \Delta u}{a_* \lambda} = \ln \left( \frac{u_1 - u}{u - u_2} \right),$$

and from this, using  $u_1 = a_* + \frac{1}{2}\Delta u$  and  $u_2 = a_* - \frac{1}{2}\Delta u$ ,

$$u = a_* - \frac{\Delta u}{2} \tanh \left( \frac{x \Delta u}{2a_* \lambda} \right), \quad (4.47)$$

where  $\lambda$  has the meaning defined in Eq. (4.35).

We state without proof that the result (4.47) is also correct when the viscosity  $\hat{\eta}$  depends on the temperature and when the Prandtl number  $\hat{\text{Pr}}$  is different from 1. However, we must then replace the quantity  $\lambda$  in Eq. (4.47) by  $\lambda_3$ , where

$$\lambda_3 = \frac{2}{\gamma + 1} \frac{\hat{\eta}_*}{\Theta_0} \left( 1 + \frac{\gamma - 1}{\hat{\text{Pr}}_*} \right), \quad (4.48)$$

with  $\hat{\eta}_*$  the viscosity corresponding to the critical temperature  $T_*$ . For constant viscosity  $\hat{\eta} = \hat{\eta}_0$ , the following holds: For  $\hat{\text{Pr}} = 1$ ,  $\lambda_3$  becomes  $\lambda$ , for  $\hat{\text{Pr}} \rightarrow \infty$ ,  $\lambda_3$  tends to  $\lambda_1$ . For  $\hat{\eta} \rightarrow 0$  and constant heat conductivity  $k = k_0$ , i.e.,  $\hat{\text{Pr}} \rightarrow 0$ , then  $\lambda_3 \rightarrow \lambda_2(\gamma - 1)/\gamma$ . Regarding this last result, we should add that the solution (4.46), when specialized to a weak shock, will become Eq. (4.47) with  $\lambda$  replaced by  $\lambda_2(\gamma - 1)/\gamma$ .

The result (4.47) will now be used to estimate the thickness  $l$  of a weak shock wave. A natural and convenient definition of this width is  $l = \Delta u |du/dx|^{-1}$ , where the derivative  $du/dx$  is taken at the point where  $u = a_*$ , i.e., at the point  $x = 0$  by Eq. (4.47) (Fig. 106). We then get from Eq. (4.47)

$$l = 4a_* \lambda / \Delta u. \quad (4.49)$$

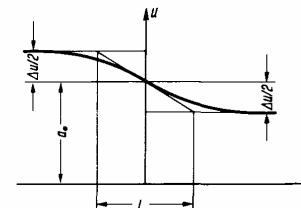


Fig. 106. Definition of shock thickness  $l$ .

For  $\Delta u \rightarrow 0$ , the shock thickness  $l \rightarrow \infty$ , and as the shock strength increases (i.e., increasing  $\Delta u$ ), the thickness decreases. For further discussion, we identify  $\lambda$  with the quantity  $\lambda_3$  defined by Eq. (4.48), and assume for sim-

plicity that  $\hat{Pr} = 1$ , which is approximately valid for many gases. Furthermore, we assume that the bulk viscosity  $\eta_b$  (see Section 4.1.1) vanishes, or is much smaller compared to the shear viscosity  $\eta$  and is hence negligible. Since, in that case,  $\hat{\eta} = 4\eta/3$ , we get from Eq. (4.49) using  $\Theta_0 \approx \varrho_* a_*$ ,

$$l \approx \frac{32\gamma}{3(\gamma+1)} \frac{\eta_*}{\varrho_* \Delta u}. \quad (4.50)$$

If we now substitute  $\eta$  from Eq. (4.14) [with the value 0.499 for the proportionality factor on the right side of (4.14)], then Eq. (4.50) becomes, for  $\gamma = 1.4$ ,

$$l \approx 3\bar{v}l_f/\Delta u. \quad (4.51)$$

The mean thermal velocity of the molecules  $\bar{v}$  is known to be proportional to the sound velocity  $a_*$ , and differs only slightly from it. From Eq. (4.51), it therefore follows that the shock thickness  $l$  decreases toward the order of one mean free path  $l_f$  as the velocity difference  $\Delta u$  approaches the order of the sound speed, i.e., when the shock is no longer very weak. In all problems in which the characteristic lengths of the flow field are very large compared with the mean free path of the gas (and these are the flows that can generally be treated by continuum mechanics, as explained in Section 1.12), we may treat the shock waves—except the very weak shocks—as discontinuities, as we did in Section 3.4. On the other hand, directly from the basic assumption of continuum mechanics that the mean free path must be sufficiently small, doubts arise on the permissibility of treating the flow in a shock wave by continuum mechanics, as was just done in this section; i.e., there are doubts concerning, among other things, the correctness of the forms of the stress  $\sigma_x$  given by (4.15) and of the heat flux  $q_x$  given by (4.16). It is noteworthy, however, that measured velocity and temperature distributions in shock waves show rather good agreement with the results of continuum theory for moderately strong shocks, whereas gas-kinetic calculations of the processes in shocks did not always lead to equally good results.

Even if we succeeded in justifying the continuum-theory treatment of the processes in a shock wave, there remains one assumption made at the beginning which, in view of the thinness of the shock and the resulting shortness of the transit time of a gas particle, remains questionable—namely, the assumption that the gas is everywhere in unconstrained thermodynamic equilibrium and that its state is uniquely specified by two independent vari-

ables. It has been pointed out many times in this book that, for example, on account of the internal degrees of freedom of the gas molecules (vibration, rotation), relaxation times for the establishment of unconstrained thermodynamic equilibrium appear which are much longer than the time needed for the equilibration of the translational degrees of freedom, and are thus also much longer than the transit time through a shock. In many cases, we can, however, introduce additional variables to describe the state of the gas as constrained equilibrium. We shall discuss the consequence of this with regard to the structure of the shock wave in Section 4.2.3.

Starting from Eq. (4.47), we shall now explicitly calculate the entropy generated in a weak shock. The entropy source terms  $\sigma_f$  from Eq. (4.12) and  $\sigma_h$  from (4.13) reduce to the following for the flow in a weak normal shock:

$$\sigma_f = \frac{\hat{\eta}_*}{T_*^2} \left( \frac{du}{dx} \right)^2; \quad \sigma_h = \frac{k_*}{T_*^2} \left( \frac{dT}{dx} \right)^2. \quad (4.52)$$

For further calculations, we now express  $dT/dx$  in terms of  $du/dx$ : were the state changes in a shock isentropic, then the relation  $c_p T + \frac{1}{2} u^2 = \text{const}$  would hold; from this it follows that  $c_p dT/dx = -u du/dx = -a_* du/dx$ . Now, the state changes in a shock are indeed not isentropic, but, by Eq. (3.118), the entropy increase in a weak shock is small enough so that we may use this relation between  $dT/dx$  and  $du/dx$  as an approximation [for  $\hat{Pr} = 1$ , this relation is exact because of Eq. (4.29)!] We get thus

$$\sigma_h = \frac{k_*}{T_*^2} \frac{a_*^2}{c_p^2} \left( \frac{du}{dx} \right)^2 = \frac{(\gamma-1) k_*}{c_p \hat{\eta}_*} \frac{\hat{\eta}_*}{T_*} \left( \frac{du}{dx} \right)^2 = \frac{\gamma-1}{\hat{Pr}} \sigma_f. \quad (4.53)$$

Here we have substituted  $a_*^2 = \gamma R T_* = c_p (\gamma-1) T_*$ . The entropy generated per unit volume per unit time is thus

$$\sigma = \sigma_f + \sigma_h = \left( 1 + \frac{\gamma-1}{\hat{Pr}} \right) \frac{\hat{\eta}_*}{T_*} \left( \frac{du}{dx} \right)^2. \quad (4.54)$$

In a streamtube of unit cross-sectional area, the entropy generated per unit time will thus be:

$$\Delta \dot{S} = \left( 1 + \frac{\gamma-1}{\hat{Pr}} \right) \frac{\hat{\eta}_*}{T_*} \int_{-\infty}^{+\infty} \left( \frac{du}{dx} \right)^2 dx.$$

If we calculate  $du/dx$  from Eq. (4.47) with  $\lambda = \lambda_3$  [Eq. (4.48)], and consider

the fact that  $\int_{-\infty}^{\infty} \cosh^{-4} t dt = \frac{1}{3}$ , then we get

$$\Delta S = \rho_* a_* \frac{\gamma(\gamma+1)}{12} R \left( \frac{\Delta u}{a_*} \right)^3. \quad (4.55)$$

Consequently, as the gas passes through the shock region, the specific entropy rises by an amount

$$s_2 - s_1 = \frac{\Delta S}{\Theta_0} = \frac{\Delta S}{\rho_* a_*} = \frac{\gamma(\gamma+1)}{12} R \left( \frac{\Delta u}{a_*} \right)^3. \quad (4.56)$$

This agrees with the result of Eq. (3.118) up to terms  $O[(\rho_2 - \rho_1)^4]$ , which are negligible for weak shocks. For weak shocks  $\Delta u/a_* = (\rho_2 - \rho_1)/\rho_1$  [see Eq. (3.35)], and for calorically ideal gases,  $A^2$  [of Eq. (3.1)] is equal to  $\gamma(\gamma-1) RT/\rho$  (see Supplementary Remark 2 to Section 3.1).

The fact mentioned earlier that the shock wave for  $\hat{P}_r = \infty$  (i.e., negligible thermal conductivity) is steeper than that for  $\hat{P}_r = 1$  is immediately clear if we consider the fact that the total entropy generated in a shock wave is already determined by the upstream variables independently of viscosity and heat conduction, as was discussed in Section 3.4. If entropy generation by heat conduction is suppressed, the entropy generated by viscosity must be increased correspondingly, which, for the same viscosity, can only be attained by an increase of the velocity gradients. The fact that for vanishing viscosity a continuous solution for the velocity distribution can be found only for moderately strong shocks can be qualitatively interpreted as follows: While for moderately strong shocks the entropy generation from heat conduction is sufficient for the total entropy increase, for strong shocks there must be additional entropy generation. This occurs within the framework of continuum theory in the formation of a nearly discontinuous zone in the velocity distribution, where the velocity gradients are so high that even the slightest viscosity will result in finite entropy generation.

#### 4.2.3\* RELAXATION PROCESSES IN A SHOCK WAVE

As was mentioned before, the thickness of a shock (except the very weak shocks) turns out to be so small that the assumption of thermodynamic equilibrium in a shock is often no longer valid. In what follows, we shall assume that, for such cases, the state of the gas in the shock region can be described as constrained equilibrium by the introduction of an additional thermo-

dynamic variable  $\xi$ . In terms of the notation first introduced in Section 1.11, the specific enthalpy  $h$  will then be a function  $\hat{h}(p, \varrho, \xi)$  of three independent variables  $p$ ,  $\varrho$ , and  $\xi$ . In unconstrained thermodynamic equilibrium,  $\xi$  assumes its equilibrium value  $\tilde{\xi}(p, \varrho)$ , which depends only on  $p$  and  $\varrho$ , and the enthalpy becomes a function of  $p$  and  $\varrho$  only:  $\hat{h}(p, \varrho, \tilde{\xi}(p, \varrho)) = h(p, \varrho)$ . Outside the shock region in which relaxation processes, viscosity, and heat conduction all play a role, the gas is in thermodynamic equilibrium. Its thermodynamic state far ahead of the shock is defined by  $p_1$ ,  $\varrho_1$ , and  $\xi_1 = \tilde{\xi}(p_1, \varrho_1)$ ; the velocity ahead of the shock will as always be denoted by  $u_1$ . The state of the gas far behind the shock ( $p_2$ ,  $\varrho_2$ ,  $\xi_2 = \tilde{\xi}(p_2, \varrho_2)$ ,  $u_2$ ) results from the shock relations (3.107)–(3.109). The relation (3.109), when written out completely, reads

$$\hat{h}(p_2, \varrho_2, \tilde{\xi}(p_2, \varrho_2)) + \frac{1}{2} u_2^2 = \hat{h}(p_1, \varrho_1, \tilde{\xi}(p_1, \varrho_1)) + \frac{1}{2} u_1^2. \quad (4.57)$$

The consequences of the shock relations have been mentioned in Sections 3.4.1 and 3.4.2. A necessary condition for the existence of a shock is  $u_1 > a_1$ , where  $a_1$  is the equilibrium sound speed of the gas ahead of the shock, as defined by Eq. (1.136).

Since in what follows we shall be interested only in the influence of relaxation on the shape of the shock, we shall neglect viscosity and heat conduction, i.e., we set  $\hat{\eta} = 0$  and  $k = 0$ . Under this assumption, we obtain from Eqs. (4.21)–(4.23)

$$\varrho = \Theta_0/u, \quad (4.58)$$

$$p = p_0 - \Theta_0 u, \quad (4.59)$$

$$\hat{h}(p, \varrho, \xi) + \frac{1}{2} u^2 = h_0 = \hat{h}(p_1, \varrho_1, \tilde{\xi}(p_1, \varrho_1)) + \frac{1}{2} u_1^2. \quad (4.60)$$

With the appearance of the new variable  $\xi$ , another equation will be necessary to describe the change of  $\xi$  with changes in the other variables. We obtain this equation from Section 3.3.1, Eq. (3.75):

$$u d\xi/dx = L(p, \varrho, \xi). \quad (4.61)$$

The meaning of the function  $L$  was given in Section 3.3.1. Now, the velocity distribution  $u(x)$  can be calculated as follows: By substituting  $\varrho$  and  $p$  from Eqs. (4.58) and (4.59) into Eq. (4.60) and solving Eq. (4.60) for  $\xi$ , we get  $\xi$  as a function of  $u$ ,  $\xi = \xi(u)$ . Substituting this into Eq. (4.61), we then get

$$\frac{du}{dx} = \frac{L((p(u), \varrho(u), \xi(u)))}{u d\xi/du}, \quad (4.62)$$

and, by integration, the velocity distribution  $u(x)$ .

Without carrying out the integration explicitly, which would require special assumptions on the thermodynamic equations of state of the gas and on  $L$ , we can use these equations to study several important facts. We first investigate the possibility of the appearance of a discontinuity in the velocity distribution. At the point  $x$ , where  $u$  and thus  $p$  and  $\varrho$  are all discontinuous,  $d\xi/dx$  undergoes a jump of the same magnitude as  $L/u$ , according to Eq. (4.61);  $\xi$  itself is thus continuous. In other words, the relaxation is frozen. For fixed  $\xi = \tilde{\xi}(p_1, \varrho_1)$ , however, Eqs. (4.58)–(4.61) formally reduce to the relations for a normal shock without relaxation (in a medium without viscosity and heat conduction) if we replace the equilibrium sound speed  $a$  in these relations everywhere by the frozen sound speed  $b$ . All the consequences of the shock relations hold correspondingly. In particular, in analogy to the condition  $u_1 > a_1$  for the equilibrium shock, the condition  $u_1 > b_1$  is necessary for the occurrence of a frozen shock,  $b_1$  being the frozen sound speed ahead of the shock defined by Eq. (1.140)<sup>70</sup>; conversely, a frozen shock is impossible in a flow with  $a_1 < u_1 < b_1$ .

*Case I:* ( $u_1 > b_1$ ). Here, Eq. (4.60) for the special value of  $\xi = \tilde{\xi}(p_1, \varrho_1)$  [as well as  $\varrho(u)$  and  $p(u)$  from (4.58) and (4.59), respectively], has, in addition to the trivial solution  $u = u_1$ , another solution in the region  $u_1 \leq u \leq u_2$ , which we shall denote by  $u_{2f}$  (Fig. 107; "f" indicates *frozen*). Now let us imagine the function  $\xi(u)$ , which results from solving (4.60) and which is assumed to be continuous, to be drawn in the  $\xi, u$  plane. Then it follows from  $\xi(u_{2f}) = \xi(u_1)$  that  $\xi(u)$  must have at least one extremum. Under rather general assumptions about the thermodynamic behavior of the gas, one can indeed show that it has not more than one maximum.<sup>71</sup> In the vibration relaxation of a thermally ideal gas, for example,  $\xi(u)$  is a parabola in the  $\xi, u$  plane (Fig. 107)<sup>72</sup> if we identify  $\xi$  with the temperature characterizing the vibrational degree of freedom. Integrating Eq. (4.62) with such a function  $\xi(u)$ , we obtain a velocity distribution which, because of the change

<sup>70</sup> We here assume that the gas in unconstrained equilibrium as well as in the frozen state will satisfy relations (3.1) and (3.113), which are the assumptions for the validity of the results derived in Section 3.4.2.

<sup>71</sup> E. Becker, Verdichtungsstöße in einem Gas mit Relaxation, *ZAMM* **45** (1965), T145; Eindimensionale stationäre Verdichtungsströmungen in einem Gas mit Relaxation. *ZAMM* **46** 363, (1966); Steady one-dimensional flow of a gas with relaxation. Recent Advances in Aerothermochemistry, *AGARD Conference Proc.* **12**, 477, (1967).

<sup>72</sup> L.J.F. Broer, On the influence of acoustic relaxation on compression flow, *Appl. Sci. Res.* **A2**, 447–468, (1950).

of the sign of  $d\xi/du$  in the denominator of (4.62), cannot be a single-valued function of  $x$ . Such a solution will be physically meaningless; to obtain a single-valued velocity distribution, we must assume a jump in  $u$ , which is just a frozen shock by the above discussion. Downstream of the shock, there is now a relaxation zone (Fig. 107) in which the velocity distribution can be calculated according to Eq. (4.62), with  $\xi(u)$  given by the branch  $BC$  of the curve. In this relaxation zone, the thermodynamic state variables change from their frozen values  $p_{2f}, \varrho_{2f}$ , etc. immediately behind the frozen shock to their equilibrium values  $p_2, \varrho_2$ , etc.; the velocity changes from  $u_{2f}$  to  $u_2$ . When viscosity and heat conductivity are taken into account, the frozen shock will

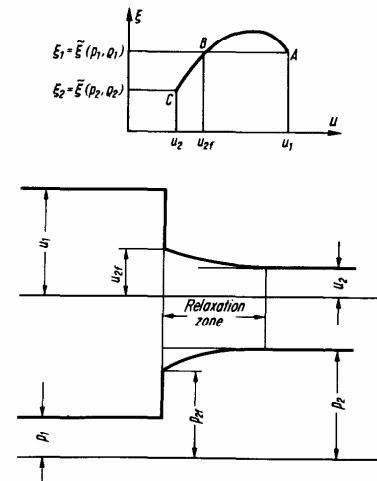


Fig. 107. Shock wave in a gas with relaxation for the case  $u_1 > b_1$ . Top:  $\xi(u)$ . Below: Velocity and pressure distribution.

naturally no longer be an exact discontinuity, but will be "smeared" out into a region of finite width in the manner discussed in Sections 4.2.1 and 4.2.2.

*Case II:* ( $a_1 < u_1 < b_1$ ). Under this assumption, as was mentioned before, no discontinuity is possible, i.e., Eq. (4.60) for the special value of  $\xi = \tilde{\xi}(p_1, \varrho_1)$

has, in the region  $u_1 \leq u \leq u_2$ , only the trivial solution  $u = u_1$ . Since then  $\xi(u)$  is a monotone function, we will get by integrating Eq. (4.62) a continuous velocity distribution (Fig. 108), for which we shall keep the name "shock."

For weak shocks, i.e., the upstream velocity  $u_1$  is only slightly higher than

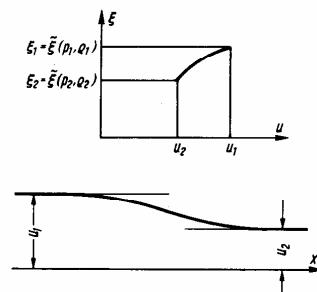


Fig. 108. Shock wave in a gas with relaxation for the case  $b_1 > u_1 > a_1$ . Top:  $\xi(u)$ . Below: Velocity distribution.

$a_1$ , nowhere in the flow field is the thermodynamic state much different from the upstream state, which is an equilibrium state by assumption. In the supplementary remarks to Section 3.3.2, it was pointed out that under this circumstance the effect of relaxation can be described by the assumption of a finite bulk viscosity  $\eta_b$ . In our case,  $\eta_b$  is given by (see the supplementary remarks to Section 3.3.2):

$$\eta_b = \rho_* \tau (b_*^2 - a_*^2), \quad (4.63)$$

where  $\tau$  is the relaxation time given by (3.95). The velocity distribution is obtained from Eq. (4.47), which is valid for weak shocks, in which  $\lambda$  is set equal to

$$\lambda_4 = \frac{2}{\gamma + 1} \frac{\eta_b}{\Theta_0}, \quad (4.64)$$

with  $\eta_b$  taken from Eq. (4.63). (Thus, a thermally ideal gas has been assumed,  $\gamma$  being the equilibrium value of the adiabatic coefficient.) The weak shock is none other than the fully-dispersed wave mentioned in Section 3.3.2 which is established after a long time in the flow in a tube with an impulsively started piston.

### 4.3 Boundary Layer Flow

#### 4.3.1 BASIC CONCEPTS OF BOUNDARY LAYER THEORY

Thus far, when the theory of inviscid gas flows was considered<sup>73</sup> and flow problems having a solid wall as a boundary were solved, we always assumed as boundary condition on the wall the requirement that the gas slide along it tangentially. The necessity of this condition on an impermeable wall is so evident that no discussion is needed. But experience shows that as a gas flows along a wall it adheres to it. At least this is true when the mean free path of the gas is small compared to a characteristic length in the immediate neighborhood of the body, i.e., in the regime of validity of continuum theory according to the remarks of Section 1.12. In the continuum treatment of flow problems, therefore, we must also satisfy as boundary condition on the wall the no-slip condition. The length which is characteristic for this boundary condition on the wall is the thickness of the *boundary layer* on the wall, the meaning of which we shall discuss thoroughly in the following paragraphs.<sup>74</sup>

To illuminate the following discussion, we shall for now imagine as a concrete example a steady, uniform, plane parallel flow past a stationary, solid, two-dimensional body; such flows are of significance in many practical questions. In inviscid theory, as was explained before, a single boundary condition for the velocity is prescribed on the body surface—namely, that the normal component of the velocity vanish there. In addition, there are boundary conditions characterizing the behavior of the flow at large distances from the body. In contrast, the tangential component of the velocity on the body surface cannot be prescribed; it comes out from the solution and will generally violate the no-slip condition. In general, inviscid theory will even yield an infinite manifold of solutions for the above-mentioned boundary condition, none of which satisfy the no-slip condition. (Regarding the non-

<sup>73</sup> We remind the reader that the notion of *inviscid* always also includes *non-heat-conducting*; see Section 2.4.

<sup>74</sup> This statement should be supplemented as follows: In some flows, the mean free path is still so small that the equations of continuum theory are applicable for the flow in the boundary layer, but the boundary condition on the wall must be altered to account for the finite mean free path. In this case, the gas will have a finite slip velocity at the wall (slip flow).

uniqueness of inviscid solutions, also see Supplementary Remark 3 to Section 3.5.) By considering viscosity and thermal conductivity, however, the no-slip condition can be satisfied. By prescribing the normal component of the velocity on the body surface, the tangential component is then not yet determined, so that we can prescribe it so as to satisfy the no-slip condition.<sup>75</sup> Mathematically, this is based on the fact that the differential forms of the momentum and energy equations [(2.51) or (2.60)] contain second derivatives of the velocity components and the temperature when the dissipative processes of viscous friction and heat conduction are considered, but they only contain first derivatives when the dissipative processes are neglected. The increase in the order of the differential equations by 1 necessitates and permits the increase in the number of boundary conditions on the body by 1.

What we have said here on the boundary condition for the velocity is similarly true for the boundary condition for the temperature which must be satisfied by the gas at the body surface: In inviscid theory, the gas temperature on the body surface is determined by the velocity on the body surface, since the velocity and temperature are in this case coupled in the manner explained in Section 3.1. In the framework of this theory, therefore, the gas temperature on the body cannot be arbitrarily prescribed. However, we can always use various methods, such as heating, cooling, etc., to fix the body surface temperature in a way independent of the gas flow, and experience shows that the gas immediately next to the body surface assumes the temperature of this surface. We must therefore demand as a further boundary condition on the body surface that the gas temperature be the same as the arbitrarily prescribed body temperature. (In some cases, this boundary condition will be given in terms of the temperature gradient on the body surface, see Section 4.3.2. This is, however, unimportant for our immediate discussion.) The equality of the gas temperature and wall temperature corresponds to the no-slip condition for the velocity on the wall, and is also only guaranteed under the same assumption on the mean free path of the gas. Prescribing the gas temperature (or equivalently, the temperature gradient) on the wall is mathematically possible only when viscosity and heat con-

<sup>75</sup> To satisfy the no-slip condition, the consideration of viscosity is already enough on a purely formal basis. But viscosity and heat conduction in a gas are so intimately coupled with each other that they should always be considered simultaneously.

duction are considered, just as with the no-slip condition on the velocity.<sup>76</sup>

Under these circumstances, it at first appears very questionable that the inviscid theory of flows is of any practical value at all, since the solutions based on the theory cannot satisfy the boundary conditions on the body demanded by actual experience. It has been shown, however, that despite these defects the inviscid theory yields acceptable results in many cases, particularly in flow past slender bodies, where the actual flow picture is one in which the flow departs significantly from that predicted by inviscid flow only in a thin layer right next to the body surface.

In the following manner, we shall obtain a criterion which must be fulfilled in order for inviscid theory (with the restriction just mentioned) to lead to meaningful results: We compare an inertial force term appearing on the left side of the momentum equation (2.52), such as  $\rho u (\partial u / \partial x)$ , with a viscous force term appearing on the right side, such as  $\partial \tau_{xy} / \partial y$ . If we estimate these quantities in terms of a typical length  $l_c$ , a typical density  $\rho_c$ , a typical velocity  $U_c$ , and a typical value of the viscosity  $\eta_c$ , then the inertial force is of the order of magnitude  $\rho_c U_c^2 / l_c$ , and the viscous force is of the order of magnitude  $\eta_c U_c / l_c^2$  (since  $\tau_{xy}$  is of the order of magnitude  $\eta_c U_c / l_c$ ). Neglecting the viscous term in the momentum equation is certainly only permissible when the inertial force terms on the left side are several orders of magnitude larger than the viscous force terms on the right, or when the dimensionless ratio

$$Rc = \frac{\rho_c U_c^2 / l_c}{\eta_c U_c / l_c^2} = \frac{\rho_c U_c l_c}{\eta_c}, \quad (4.65)$$

the *Reynolds number*, is much greater than 1. In the flow past a slender body of length  $l$ , the typical variables can, for example, be chosen to be those in the free stream, and we then have the Reynolds number based on the free-stream quantities,  $Re_\infty = \rho_\infty U_\infty l / \eta_\infty$ . For air under normal conditions,  $\eta / \rho = 15 \cdot 10^{-6} \text{ m}^2/\text{sec}$ . For velocities of several hundred m/sec and lengths of the order of 1 m, as is typical in aerodynamics, the Reynolds number is so large (about  $10^7$  to  $10^8$ ) that inviscid theory is justified. Nonetheless, for example, this is no longer true for flight at high altitudes, since the density of the atmosphere decreases rapidly with increase in altitude (at 100-km altitude, the density is only  $10^{-7}$  the sea level value), while the viscosity  $\eta$  changes but little. If we consider the flow in the immediate neighborhood of the body

<sup>76</sup> Considering the heat conduction of the gas is sufficient per se. See, however the previous footnote.

surface, however, neglecting viscosity and heat conduction is not permissible even at large Reynolds numbers, since velocity and temperature must change from their values in the inviscid flow to the values required by the boundary conditions on the wall. Thus, a thin boundary layer is formed (Fig. 109) in

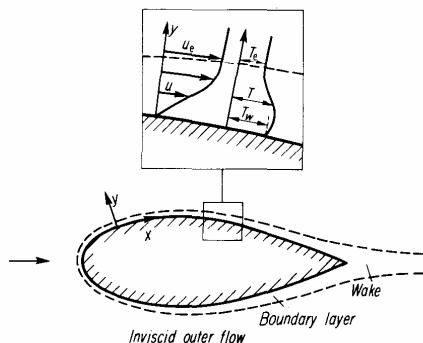


Fig. 109. Plane boundary layer flow.

which the gradients of velocity and temperature perpendicular to the wall assume such large values that the shear stresses and heat flux there can no longer be neglected. The greater the Reynolds number as defined above, the thinner the boundary layer will be, and the smaller the deviation of the flow outside the boundary layer from the inviscid flow calculated with the boundary layer completely neglected will be.

The procedure of boundary layer theory,<sup>77</sup> which was founded by L. Prandtl and on which a considerable part of modern fluid dynamics is based, thus consists of the following: For a given flow problem with sufficiently large Reynolds number, we first neglect the boundary layer to find the inviscid solution, and then calculate the boundary layer corresponding to this inviscid solution on the body surface. Because of the thinness of the boundary layer, certain terms in the momentum and energy equations may be neglected, resulting in a simplification of the boundary layer problem without losing the physically important phenomena. As boundary conditions for the flow

<sup>77</sup> L. Prandtl, Über Flüssigkeitsbewegung bei sehr kleiner Reibung. *Proc. of 3rd Internat. Congress of Math. Heidelberg, Germany*, (1904), p. 484. Teubner, Leipzig, 1905.

in the boundary layer, we have the above-mentioned conditions on the wall together with conditions on the outer edge of the boundary layer to insure a continuous transition of the boundary layer flow to the inviscid external flow. Instead of this outer boundary condition, we do, in some cases, prescribe an initial distribution of the flow quantities in the boundary layer; these flow variables can then be continued downstream. Such a program, when carried out for the flows past a slender body with Mach numbers below the hypersonic regime, leads, in general, to useful results that agree with experiments.

Naturally, this method will not work if the presence of the boundary layer significantly alters the inviscid flow outside the boundary layer from that calculated with the boundary layer neglected. This is, for example, often of decisive significance in hypersonic flow; also, the boundary layer plays an important role if shock waves hit the body surface, as is particularly noticeable in the transonic regime. Furthermore, it cannot be neglected when boundary layer separation occurs. If the pressure on the body surfaces rises in the direction of the flow, it can slow down the flow near the wall so much as to produce, among other things, a reversal in the direction of the flow. When this happens, the boundary layer detaches from the wall, and often a large *wake region* is formed behind the separation point. Without entering into the nature of the flow in such a wake region, we can say that its occurrence will greatly change the flow field from that calculated by inviscid flow which will often no longer be even approximately the same as it. Here we should make a certain reservation: It was already mentioned in Supplementary Remark 3 to Section 3.5 that the inviscid solution is not uniquely determined unless we make certain additional assumptions originating outside the realm of inviscid theory. Figure 69 gives such an example. It is often possible to determine inviscid flows which contain discontinuities in the form of vortex sheets (contact discontinuities) in such a way that the actual flow resulting from boundary layer separation is well approximated by this inviscid flow. Of course, the choice of the correct inviscid flow presumes a knowledge of the processes in the boundary layer, which in turn depends on the inviscid solution. As a result of this coupling, the calculation of flow fields with boundary layer separation is very difficult. If the boundary layer does not separate, then the choice of the correct inviscid solution is also not completely straightforward, but is generally possible on the basis of some empirical facts. The Kutta-Joukowski condition, for example, is one such

empirical postulate, which must be satisfied by the inviscid subsonic flow around a profile with sharp trailing edge; as is well known, it requires the gas to flow off the trailing edge smoothly.

The above complications are beyond the scope of the present work. Similarly, we must also omit from consideration a phenomenon which always appears at high enough Reynolds numbers—namely, the transition of laminar flow in the boundary layer to turbulent flow.<sup>78</sup> While for small to moderately large Reynolds numbers the gas flow is laminar (i.e., neighboring layers of the gas slide along each other without distortion), above a critical Reynolds number which depends on various factors, the flow in the boundary layer changes over into an unsteady turbulent flow, even though the flow outside is steady. The flow variables (velocity, pressure, etc.) then fluctuate statistically, and there is strong eddy formation and mixing of neighboring layers over a macroscopic scale. While in laminar flow an exchange of momentum and energy between neighboring fluid layers is possible only through molecular transport processes, in turbulent flow there is an added exchange of momentum and energy due to the macroscopic mixing. We can regard this as causing a large increase in the effective viscosity and thermal conductivity of the gas in a turbulent boundary layer. The theory of turbulent flows is very complicated, and at present cannot be carried through without the addition of several semiempirical assumptions. The following discussions are thus restricted to laminar boundary layers.

#### 4.3.2 BOUNDARY LAYER EQUATIONS

We consider the steady laminar boundary layer on a plane wall. The  $x$  axis of the Cartesian coordinate system will lie along the wall and the  $y$  axis will be perpendicular to it. All the flow variables shall depend only on  $x$  and  $y$ , and the  $w$  component of the velocity shall be zero (plane flow). For readers not familiar with boundary layer theory, we shall derive the boundary-layer differential equations from the momentum and energy equations heuristically, and refer to the special literature for a more rigorous

<sup>78</sup> See K. Wieghardt, *Theoretische Strömungslehre*. Teubner, Stuttgart, 1965.

<sup>79</sup> See H. Schlichting, *Boundary Layer Theory* (J. Kestin, transl.). McGraw-Hill, New York, 1960; N. Curle, *The Laminar Boundary Layer Equations*. Oxford, Univ. Press, London, 1962; K. Stewartson, *The Theory of Laminar Boundary Layer in Compressible Fluids*. Oxford, Univ. Press, London, 1964.

foundation.<sup>79</sup> We assume that constraints on thermodynamic equilibrium play no role, so that the thermodynamic state of the gas is specified by two variables. The inviscid flow field in the half-space  $y > 0$  will be assumed to be known, and, in particular, the velocity and temperature directly on the wall are known. This flow is to be replaced in the immediate neighborhood of the wall by a boundary layer flow in which the velocity and temperature will change from the values  $u = 0$  and  $T = T_w(x)$  on the wall to the values  $u_e(x)$  and  $T_e(x)$  at the exterior edge of the boundary layer as given by the inviscid flow solution. When the boundary layer is sufficiently thin, as will be assumed in the following, then we can identify  $u_e(x)$  and  $T_e(x)$  with the corresponding values of the inviscid flow on the wall (Fig. 109).

We make the assumption that the boundary layer is "thin" more precise: this means that in the boundary layer the derivatives of the velocity and temperature in the direction perpendicular to the wall are much greater in magnitude than those in the direction parallel to the wall. In accordance with this assumption, we may assume that the direction of the velocity in the boundary layer is only slightly different from the  $x$  direction, and that therefore  $|v| \ll |u|$ . These two assumptions serve as justification for neglecting all the components of the viscous stress tensor  $\mathbf{T}$  except  $\tau_{xy} = \tau_{yx}$ . For this we get from Eq. (4.6)

$$\tau_{yx} = \tau_{xy} = \eta \frac{\partial u}{\partial y}, \quad (4.66)$$

where  $\partial v / \partial x$  is neglected in comparison to  $\partial u / \partial y$ . For the same reason, we shall only keep the normal component  $q_y$  of the heat flux vector  $\mathbf{q}$ , which, by Eq. (4.9), is

$$q_y = -k \frac{\partial T}{\partial y}. \quad (4.67)$$

Since the gas in the boundary layer flows nearly parallel to the wall everywhere—or, in other words, there is not much acceleration in the direction perpendicular to the wall—the pressure gradient perpendicular to the wall remains small. Thus, in a thin boundary layer, the pressure change in the direction perpendicular to the wall is insignificant. Consequently, we are permitted in boundary layer theory to replace the pressure in the entire boundary layer by its inviscid value at the wall  $p_e(x)$ .

With this assumption and ignoring volume forces, the first momentum equation (2.52) becomes

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{dp_e}{dx} + \frac{\partial}{\partial y} \left( \eta \frac{\partial u}{\partial y} \right). \quad (4.68)$$

Although  $|v| \ll |u|$  in the boundary layer, we must still keep the term containing  $v$  on the left side, since the same term also contains the derivative  $\partial u / \partial y$ , which is significantly larger in magnitude than  $\partial u / \partial x$  in the term containing  $u$ . The energy equation will be used in the form (2.60). The term  $\operatorname{div}(\mathbf{v} \cdot \mathbf{T})$  reduces to  $(\partial/\partial y) [\eta u (\partial u / \partial y)]$  after neglecting the terms as described above. We thus get

$$\rho u \frac{\partial h_t}{\partial x} + \rho v \frac{\partial h_t}{\partial y} = \frac{\partial}{\partial y} \left( \eta u \frac{\partial u}{\partial y} + k \frac{\partial T}{\partial y} \right). \quad (4.69)$$

Since  $|v| \ll |u|$ , we can neglect  $v^2$  when compared with  $u^2$  in the definition (2.55) for the total enthalpy; thus, for the flow in the boundary layer, we can set:

$$h_t = h + \frac{1}{2}u^2. \quad (4.70)$$

The continuity equation (2.32) remains unchanged for boundary layer flow:

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0. \quad (4.71)$$

Together with the thermal equation of state  $\Phi(p, \rho, T) = 0$  and the calorical equation of state  $\Psi(h, p, T) = 0$ , Eqs. (4.68), (4.69), and (4.71) suffice for the determination of the five unknowns  $u, v, \rho, T$ , and  $h$  as functions of  $x$  and  $y$  in the boundary layer.

To solve for these unknowns, we naturally need boundary conditions. On the wall, we have

$$y = 0: \quad u = 0, \quad (4.72)$$

$$v = 0, \quad (4.73)$$

$$T = T_w(x). \quad (4.74)$$

(4.72) and (4.73) express the no-slip condition. In boundary condition (4.74),  $T_w(x)$  denotes the prescribed wall temperature. In many cases, however, the temperature has to satisfy an entirely different boundary condition on the wall: If a body of finite heat capacity is in contact with no heat reservoirs other than the flowing gas and has been in the gas flow for a sufficiently long time, then the body temperature will finally reach a steady value which is just large enough so that no further heat is exchanged between the body and the gas. This steady final state is a state often encountered in practice. Since in this state the heat flux  $q_w$  on the wall vanishes,<sup>80</sup> the boundary condition

<sup>80</sup> It will become clear later that this discussion rigorously requires the assumption that the recovery factor be independent of the position on the body surface.

characterizing this is

$$y = 0: \quad \partial T / \partial y = 0. \quad (4.75)$$

Instead of the boundary conditions (4.74) and (4.75) for the temperature, an equivalent boundary condition can be given for the specific enthalpy  $h$ ; for (4.74) we then obtain

$$y = 0: \quad h = h_w(x), \quad (4.74^*)$$

where  $h_w(x)$  denotes the prescribed enthalpy at the wall, which is calculated from the prescribed wall temperature  $T_w$  and the known pressure  $p_e$  through the caloric equation of state. To convert the boundary condition (4.75), we observe that by the independence of the pressure on the  $y$  coordinate in the boundary layer, the following holds:

$$\frac{\partial h}{\partial y} = \left( \frac{\partial h}{\partial T} \right)_p \frac{\partial T}{\partial y} = c_p \frac{\partial T}{\partial y}. \quad (4.76)$$

We thus get, in place of (4.75),

$$y = 0: \quad \partial h / \partial y = 0. \quad (4.75^*)$$

At the exterior edge of the boundary layer, the velocity  $u$  and temperature  $T$  must approach their respective values in inviscid flow. We thus require:

$$\lim_{y \rightarrow \infty} u(x, y) = u_e(x), \quad (4.77)$$

$$\lim_{y \rightarrow \infty} T(x, y) = T_e(x), \quad (4.78)$$

$$\lim_{y \rightarrow \infty} h(x, y) = h_e(x). \quad (4.78^*)$$

In the regime of validity of boundary layer theory, the distance from the wall at which  $u$  and  $T$  practically attain their asymptotic values (for  $y \rightarrow \infty$ ) is so small that these quantities for the inviscid outer flow will coincide with the wall values of the inviscid flow  $u_e$  and  $T_e$  with sufficient accuracy, and (4.77) and (4.78) will guarantee a continuous transition of the boundary layer flow to the outer flow.

Now, boundary condition (4.77) leads to the consequence that

$$\lim_{y \rightarrow \infty} \partial u / \partial y = 0.$$

A transition to the outer flow in which not only  $u$  is continuous, but also

$\partial u / \partial y$ , thus requires that for the inviscid outer flow  $\partial u / \partial y = 0$  on the wall. Since on the wall,  $v = 0$  always holds and thus also  $\partial v / \partial x = 0$ , this condition will be fulfilled when  $\partial u / \partial y - \partial v / \partial x = 0$  near the wall, or when the inviscid flow is irrotational. In many important gasdynamical problems, this assumption is not valid, e.g., when the gas passes through a strong curved shock. When the flow ahead of the shock is irrotational, the flow behind it will be rotational, since the entropy increase is different at different points along the shock (see Section 2.5). The outer boundary condition (4.77) is, in this case, somewhat questionable. The same is also true for the boundary condition (4.78) for the temperature. It insures a continuous transition of  $\partial T / \partial y$  of the boundary layer flow to the inviscid outer flow if the outer flow is not only irrotational but also isoenergetic. In the following paragraphs, we shall only treat those applications for which these assumptions are satisfied, thereby evading these difficulties connected with the outer boundary condition.

The above equations and boundary conditions have been derived for the boundary layer on a plane wall. We can show, however, that they are also valid on a curved wall, and therefore can be applied to the flow past a cylindrical body. In this case, we only have to identify  $x$  with the arc length in the direction of the wall and  $y$  with the direction perpendicular to the wall (Fig. 109). The assumption for the continued validity of the equations is, however, that the boundary layer thickness everywhere be small compared to the local radius of curvature of the wall.

From the energy equation (4.69), we can draw an important conclusion with regard to the total enthalpy  $h_i$  in the boundary layer: If we replace  $\partial T / \partial y$  on the right side of (4.69) by  $\partial h / \partial y$  in accordance with (4.76) and consider the definition (4.10) of the Prandtl number  $Pr$ , then we can write (4.69) in the following form:

$$\rho u \frac{\partial h_i}{\partial x} + \rho v \frac{\partial h_i}{\partial y} = \frac{\partial}{\partial y} \left[ \eta \left( \frac{\partial (u^2/2)}{\partial y} + \frac{1}{Pr} \frac{\partial h}{\partial y} \right) \right]. \quad (4.79)$$

With the additional assumption of  $Pr = 1$ , (4.79) becomes

$$\rho u \frac{\partial h_i}{\partial x} + \rho v \frac{\partial h_i}{\partial y} = \frac{\partial}{\partial y} \left( \eta \frac{\partial h_i}{\partial y} \right). \quad (4.80)$$

A solution of this equation is  $h_i = \text{const}$ . This solution is obviously compatible with boundary condition (4.78\*) at the outer edge of the boundary layer if we identify the constant  $h_i$  with the constant total enthalpy of the

inviscid outer flow:

$$h_i = h + \frac{1}{2} u^2 = h_e + \frac{1}{2} u_e^2. \quad (4.81)$$

In this case,  $h \rightarrow h_e$  for  $u \rightarrow u_e$  at the outer edge of the boundary layer. The solution for the enthalpy found from (4.81) satisfies boundary condition (4.75\*) at the wall. This follows from differentiation of (4.81) by  $y$ :

$$\frac{\partial h}{\partial y} + u \frac{\partial u}{\partial y} = 0. \quad (4.82)$$

Since  $u = 0$  on the wall,  $\partial h / \partial y$  also vanishes there, according to (4.82). Solution (4.81) thus corresponds just to the case in which no heat exchange takes place between the wall and the gas; we shall from now on use the term *heat-insulating wall* for conciseness.

The results of these discussions can be summarized as follows: For a gas with Prandtl number  $Pr = 1$  and a heat-insulating wall, the total enthalpy in the boundary layer is constant and equal to the total enthalpy of the inviscid outer flow. Under these circumstances, the flow in the boundary layer is thus isoenergetic. It therefore behaves exactly like an inviscid flow as far as energy is concerned. In inviscid flow, the energy equation given in Section 2.4 has no  $L_2$  term (representing the work done by the viscous stresses) and no  $L_4$  term (representing the energy flow per unit time due to heat transfer), since both the viscous stresses and heat flux are identically zero. In the boundary layer flow just considered,  $L_2$  and  $L_4$  do not vanish individually, but their sum vanishes, i.e., in each volume of gas, the work done by the viscous stresses per unit time is exactly equal to the energy transported out of this volume per unit time due to heat conduction. This can be understood as follows: The right side of (4.79) can be written in the form  $\partial(u\tau_{xy})/\partial y - \partial q_y/\partial y$ . The first term is the work  $L_2$  done by the viscous stress  $\tau_{xy}$  per unit volume per unit time, and the second term is the energy  $L_4$  transported by heat flow  $q_y$ . Under the assumption of  $Pr = 1$  and a heat-insulating wall, the right side of (4.79) vanishes, since, for  $Pr = 1$ , it is, according to Eq. (4.80), identical to  $(\partial/\partial y)(\eta \partial h_i / \partial y)$ , and in this case  $h_i = \text{const}$ ; we thus have

$$\frac{\partial}{\partial y} (u\tau_{xy}) - \frac{\partial q_y}{\partial y} = 0. \quad (4.83)$$

This equation, however, expresses just the state of affairs described above, that the viscous work and heat flow cancel each other. Since on the edge of the boundary layer  $\tau_{xy}$  and  $q_y$  vanish, Eq. (4.83) becomes, moreover, the

special relation

$$u\tau_{xy} - q_y = 0. \quad (4.84)$$

For the flow past a solid body with freestream velocity  $U_\infty$ , we have

$$h_e + \frac{1}{2}u_e^2 = h_\infty + \frac{1}{2}U_\infty^2. \quad (4.85)$$

If the surface of the body is heat-insulating, at the surface ( $u = 0$ ) the gas will, by Eq. (4.81), assume the enthalpy  $h_r$ :

$$h_r = h_e + \frac{1}{2}u_e^2 = h_\infty + \frac{1}{2}U_\infty^2. \quad (4.86)$$

In thermal steady state, with no further heat exchange between the gas and the body, the gas thus assumes the constant enthalpy  $h_r$  given by (4.86) on the body surface, which is exactly equal to the stagnation enthalpy of the stream. The surface everywhere attains the temperature  $T_r$  corresponding to this enthalpy. This enthalpy  $h_r$  is exactly the same as the enthalpy of the gas at a stagnation point in inviscid theory. What was said in Section 3.4.4 about heating at the stagnation point is thus true here for every point on the body surface, as was already suggested in that section. This is strictly valid only for  $\text{Pr} = 1$ . Since the Prandtl number of actual gases does not vary much from 1, the actual enthalpy attained on the body surface is only slightly different from the value given by (4.86).

We call the enthalpy attained on a heat-insulating wall the recovery enthalpy  $h_r$ , and the corresponding temperature the recovery temperature  $T_r$ . If  $\text{Pr} \neq 1$ , we can generalize (4.86) to

$$h_r = h_e + \frac{1}{2}ru_e^2 = h_\infty + \frac{1}{2}(U_\infty^2 - u_e^2) + \frac{1}{2}ru_e^2. \quad (4.87)$$

This defines the *recovery factor*  $r$ . In general, the recovery factor is a function of the location on the surface; this function contains the Prandtl number as an important parameter, but also depends to a small extent on the viscosity law  $\eta(T)$ . For  $\text{Pr} = 1$ , the recovery factor is everywhere  $r = 1$ ; for  $\text{Pr} \neq 1$ ,  $r \neq 1$  also, but the difference from 1 is everywhere small if the Prandtl number is not too different from 1. We shall return to this in further detail in Section 4.3.3.

### 4.3.3 BOUNDARY LAYER ON A FLAT PLATE

We now specialize our consideration to an outer flow for which the wall pressure  $p_e$  is independent of  $x$ . An example is the flow past a flat plate in

the direction parallel to the plate, with the leading edge of the plate at  $x = 0$  (Fig. 110). In this flow,  $p_e = p_\infty$  and  $u_e = U_\infty$ .

Before we investigate the steady boundary layer on the flat plate we shall first consider a very simple unsteady flow of a fluid with constant density  $\rho$

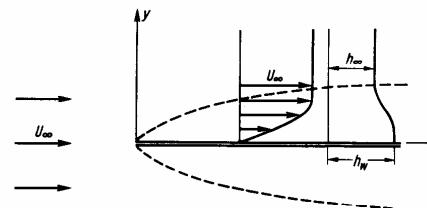


Fig. 110. Boundary layer on a semi-infinite flat plate.

and constant viscosity, which permits us to study in a simple way several essential features of boundary layer flow. This flow consists of the following: An infinite plane wall  $y = 0$  at rest for time  $t < 0$  is in contact with a fluid which fills the half-space  $y > 0$  and is at rest for time  $t < 0$  (Fig. 111). At

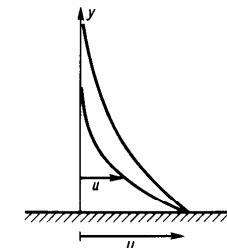


Fig. 111. Boundary layer on an impulsively started plane wall ("Rayleigh boundary layer").

time  $t = 0$ , the wall is brought to velocity  $U$  impulsively in its own plane, and remains moving at the same speed for  $t > 0$  (the "Rayleigh problem"). Then, an increasingly-wider layer of fluid will be carried along by the wall

(Fig. 111). The only nonzero component of velocity is the component  $u$ , which depends only on  $y$  and  $t$ . From the momentum equation (2.51), we get for  $u$ , with nothing neglected

$$\frac{\partial u}{\partial t} = v \frac{\partial^2 u}{\partial y^2}, \quad (4.88)$$

where the kinematic viscosity  $v = \eta/\rho$  has been introduced for the sake of conciseness. Equation (4.88) has the form of a one-dimensional heat conduction equation, where the velocity corresponds to the temperature and the kinematic viscosity to the temperature diffusivity. The solution of (4.88), which on the one hand satisfies the boundary condition  $u = U$  on the wall  $y = 0$  for all  $t > 0$ , and on the other hand satisfies the initial condition  $u = 0$  at  $t = 0$  for all  $y > 0$ , is well known from heat-conduction theory to be

$$u(y, t) = U \left[ 1 - \Phi \left( \frac{y}{2(vt)^{1/2}} \right) \right], \quad (4.89)$$

where  $\Phi$  is the error function defined in connection with Eq. (3.105). If we define the thickness  $\delta$  of the boundary layer carried along by the wall to be the distance in which  $u$  has dropped to a certain fraction of  $U$  (e.g., 1%), then we conclude from (4.89) that

$$\delta \sim (vt)^{1/2}. \quad (4.90)$$

The boundary layer thickness grows as  $\sqrt{t}$ ; for a given time, the smaller the kinematic viscosity  $v$ , the thinner the boundary layer. For  $t \rightarrow 0$ , it contracts to a discontinuity immediately on the wall, which can be regarded as a vortex sheet across which the velocity jumps from the value 0 in the fluid to the value  $U$  on the wall. With increasing time, this vortex sheet *diffuses* by constantly widening into the fluid, as a result of the viscosity.

We can transfer these results qualitatively to a steady flow past a flat plate with the leading edge at  $x = 0$  (Fig. 110). The presence of the plate is completely unnoticed in the region  $x < 0$ , in any case, since within the realm of boundary layer approximation, the disturbances to the simple parallel flow originating from the plate can only propagate in two interrelated ways:

(1) In the direction perpendicular to the wall, they spread out because of the diffusion due to viscosity, as described above. (The diffusion in the direction parallel to the wall is neglected in boundary layer theory, since, in the viscous stresses, all  $x$  derivatives are neglected; the same holds for heat conduction.)

(2) In the direction parallel to the wall, the disturbances spread by convection with the flow velocity.

These two processes, however, can only act downstream from the leading edge of the plate. With complete generality, the region of influence of a point with the abscissa  $x = x_0$  is the region  $x > x_0$ . From this, it also follows that the flow upstream of the point  $x = x_0$  will not change at all if we cut the plate there, i.e., make  $x = x_0$  the trailing edge of the plate.

From a mathematical standpoint, this is a consequence of the fact that the boundary layer equations, with the parallel components (to the wall) of the velocity and temperature gradients neglected in the viscous stresses and the heat flux, i.e., in the terms with the highest derivatives, have become parabolic equations. Without neglecting these terms, the equations of steady viscous flow are elliptic. When the complete elliptic equations are used to describe the flow, the region of influence of the flat plate also contains the upstream region  $x < 0$ .

Since a fluid particle located at the outer edge of the boundary layer at  $x$  has come from the location  $x = 0$  during the time interval  $t = x/U_\infty$  and has entered the region of influence of the plate at the beginning of that time interval, it is reasonable to assume that the thickness of the boundary layer at the point  $x$  is approximately given by formula (4.90), in which  $t = x/U_\infty$ . Thus, we have

$$\delta \sim \left( \frac{vx}{U_\infty} \right)^{1/2} = \frac{x}{\sqrt{Re}}. \quad (4.91)$$

The thickness of the boundary layer thus increases as the square root of the distance from the leading edge of the plate. In formula (4.91) we have introduced the Reynolds number based on  $x$ ,  $Re = U_\infty x/v$ . At a fixed location  $x$ , the larger the Reynolds number  $Re$ , the thinner the boundary layer. The fundamental assumption of boundary layer theory—that changes in the flow variables in the  $x$  direction are small compared to the changes in the  $y$  direction—is certainly only assured if the boundary layer thickness does not change much over a distance in the  $x$  direction comparable to  $\delta$ , i.e., if

$$\frac{d\delta}{dx} \sim \frac{1}{2\sqrt{Re}} \ll 1. \quad (4.92)$$

This condition is satisfied for sufficiently large Reynolds numbers  $Re$ . On the other hand,  $Re \sim x$ , so that in the immediate neighborhood of the leading edge  $x = 0$ , condition (4.92) and the basic assumption of boundary

layer theory are not satisfied. If, however, the Reynolds number based on the total length of the plate  $l$ ,  $U_\infty l/v$ , is sufficiently large, this expected error will be confined to a region near the leading edge which is small compared to the total length of the plate, so that in many cases it can be neglected.

With these preparatory remarks we return to the equations for the compressible boundary layer on a plate. With  $dp_e/dx = 0$ , the momentum equation (4.68) becomes

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left( \eta \frac{\partial u}{\partial y} \right). \quad (4.93)$$

Comparing this with Eq. (4.80), which is valid for  $\text{Pr} = 1$ , we get the following theorem: If  $u(x, y)$  is a solution of (4.93), then

$$h_t = a + bu,$$

with the constants  $a$  and  $b$ , is a solution of Eq. (4.80). This solution can also be written as Crocco's integral:

$$h = a + bu - \frac{1}{2}u^2. \quad (4.94)$$

We can adjust the solution (4.94) to fit the boundary condition  $h = h_w = \text{const}$  on the wall ( $y = 0; u = 0$ ) and  $h = h_\infty$  at the outer edge of the boundary layer ( $u = U_\infty$ ) by selecting the constants  $a$  and  $b$  accordingly. We then get the following relation between the specific enthalpy  $h$  and the velocity  $u$  in the boundary layer of the plate for  $\text{Pr} = 1$ :

$$h = h_\infty + \left[ h_w - \left( h_\infty + \frac{U_\infty^2}{2} \right) \right] \left( 1 - \frac{u}{U_\infty} \right) + \frac{U_\infty^2}{2} \left( 1 - \frac{u^2}{U_\infty^2} \right). \quad (4.95)$$

For a calorically ideal gas with  $h = h^* + c_p T$ , we can immediately rewrite all formulas given in specific enthalpy in terms of temperature. From (4.95), for example,

$$T = T_\infty + \left[ T_w - \left( T_\infty + \frac{U_\infty^2}{2c_p} \right) \right] \left( 1 - \frac{u}{U_\infty} \right) + \frac{U_\infty^2}{2c_p} \left( 1 - \frac{u^2}{U_\infty^2} \right). \quad (4.96)$$

If we take it for granted that the velocity distribution  $u$  in the boundary layer is of the form shown on the left side of Fig. 112, then the enthalpy will have the distribution shown on the right side of Fig. 112. For a calorically ideal gas, the temperature distribution has the same shape, by Eq. (4.96). When, in particular,  $h_w = h_\infty + \frac{1}{2}U_\infty^2$ , then we get, from Eq. (4.95),  $h = h_\infty + \frac{1}{2}U_\infty^2 - \frac{1}{2}u^2$ , i.e., the total enthalpy  $h + \frac{1}{2}u^2$  is constant. In this case,

the heat transferred to the wall vanishes, since it follows from Eq. (4.95) that  $\partial h/\partial y = 0$  for  $u = 0$ . The recovery enthalpy is thus  $h_r = h_\infty + \frac{1}{2}U_\infty^2$ . This is a special case of the state of affairs established in general form in Section 4.3.2.

The heat flux  $q_w$  on the wall is not zero when the wall temperature does

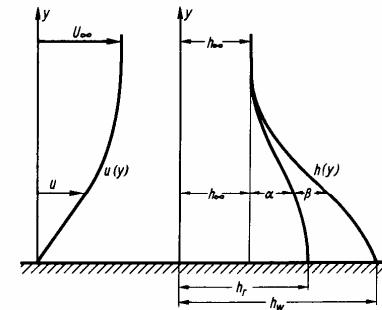


Fig. 112. Velocity and enthalpy distribution in the flat-plate boundary layer (schematic);  $\alpha = \frac{1}{2} U_\infty^2 [1 - (u^2/U_\infty^2)]$ ;  $\beta = (h_w - h_r) [1 - (u/U_\infty)]$ .

not equal the recovery temperature, or, equivalently, the enthalpy  $h_w$  of the gas right on the wall is different from the recovery enthalpy  $h_r$ . We find an expression for the heat flux  $q_w$  in the following manner (the subscript "w" denotes the value of the quantity at the wall  $y = 0$ ):

$$q_w = -k_w \frac{\partial T}{\partial y} \Big|_{y=0} = -\frac{k_w}{c_{pw}} \frac{\partial h}{\partial y} \Big|_{y=0} = -\frac{\eta_w}{\text{Pr}_w} \frac{\partial h}{\partial y} \Big|_{y=0}. \quad (4.97)$$

The derivative  $\partial h/\partial y$  is found by differentiation of Eq. (4.95); if we set the result into Eq. (4.97) and use the notation

$$\tau_w = \eta_w \frac{\partial u}{\partial y} \Big|_{y=0}, \quad (4.98)$$

we get, for  $\text{Pr}_w = \text{Pr} = 1$ , the expression

$$q_w = \frac{h_w - (h_\infty + \frac{1}{2}U_\infty^2)}{U_\infty} \tau_w = \frac{h_w - h_r}{U_\infty} \tau_w, \quad (4.99)$$

where  $\tau_w = \tau_{xy}(y = 0)$  denotes the wall shear stress. Equation (4.99) shows

that everywhere on the wall the heat flux  $q_w$  is proportional to the local wall shear stress  $\tau_w$ . For a given wall shear stress,  $q_w$  is proportional to the difference between the wall enthalpy and the recovery enthalpy. When  $h_w > h_r$ , heat flows from wall to the gas ( $q_w > 0$ ); when  $h_w < h_r$ , the gas gives heat to the wall ( $q_w < 0$ ). The proportionality between the heat flux and the wall shear stress is called the Reynolds analogy between these two quantities.

It is usual to introduce dimensionless quantities instead of the dimensional quantities  $\tau_w$  and  $q_w$ . We define the friction coefficient  $c_f$  as a dimensionless measure for the wall shear stress:

$$c_f = \frac{\tau_w}{\frac{1}{2} \rho_e u_e^2} = \frac{\tau_w}{\frac{1}{2} \rho_\infty U_\infty^2}. \quad (4.100)$$

The first equation is entirely general, while the second only holds for the flat plate, for which  $u_e = U_\infty$ , and  $\rho_e = \rho_\infty$ . In a similar way, we define as a dimensionless measure for the heat flux on the wall  $q_w$  the Stanton number

$$St = \frac{q_w}{(h_w - h_r) \rho_e u_e} = \frac{q_w}{(h_w - h_r) \rho_\infty U_\infty}, \quad (4.101)$$

where the second equation again holds only for the flat plate. The ratio

$$s = 2St/c_f. \quad (4.102)$$

is called the Reynolds analogy factor. From Eq. (4.99), the Reynolds analogy factor  $s = 1$  for the flow past a flat plate with  $Pr = 1$ . For arbitrary boundary layer flows,  $s$  is a function of the position on the surface of the body, just as is the recovery factor  $r$ . This function contains as essential parameter the Prandtl number  $Pr$  (assumed constant). As we shall further show below, in the special case of flow past a flat plate, which we consider here,  $r$  and  $s$  are independent of position, i.e., of  $x$ . If  $Pr$  is not much different from 1, we can, in good approximation, set for the flat plate

$$r = Pr^{1/2}, \quad (4.103)$$

$$s = Pr^{-2/3}. \quad (4.104)$$

These approximate formulas have been obtained from numerical calculations. While the viscosity law  $\eta(T)$  has indeed a certain influence on  $r$  and  $s$ , this effect is negligible for practical purposes. The approximate formulas (4.103) and (4.104) can, moreover, be applied to boundary layer flow with a pressure gradient if the pressure gradient is kept within moderate limits.

An interesting conclusion can be drawn from Eq. (4.99): Imagine a flat plate of finite extent moving with velocity  $U_\infty$  through a gas at rest. The wall temperature will be kept at the temperature of the unperturbed gas at rest, i.e.,  $h_w = h_\infty$ ; we speak of this as the *cold wall* case. From Eq. (4.99), it follows that

$$q_w = -\frac{1}{2} U_\infty \tau_w. \quad (4.105)$$

Integrating the heat flux  $q_w$  over the entire surface of the plate, we get

$$\frac{dQ}{dt} = - \iint q_w dA = + \frac{U_\infty}{2} \iint \tau_w dA. \quad (4.106)$$

Here,  $dQ/dt$  is the heat flow per unit time from the gas to the plate. On the other hand,  $D_f = \iint \tau_w dA$  is the drag force exerted on the plate, the *friction drag*. Thus,

$$dQ/dt = \frac{1}{2} U_\infty D_f. \quad (4.107)$$

Thus, half of the work  $U_\infty D_f$  done by the plate moving with constant velocity  $U_\infty$  against the friction drag  $D_f$  per unit time flows on to the plate as heat. The other half of this work goes into the kinetic energy as well as into the heating of the gas particles in the boundary layer, and causes a *wake* to form downstream of the flat plate; the temperature and velocity profiles in the wake widen continuously and level off, and the kinetic energy is continuously transformed into heat, so that far downstream the wake asymptotically becomes the undisturbed gas at rest again. For a heat-insulating plate, i.e., when the plate temperature equals the recovery temperature, the entire work done against the friction drag goes into the kinetic energy and heating of the boundary layer, since for this case no heat can flow on to the plate.

If a flat plate with initial velocity  $U_0$  and a cold wall is brought into a gas at rest, then, in the absence of any propulsive force to overcome the friction drag  $D_f$ , the plate will be slowed down by  $D_f$  and finally come to rest. The entire initial kinetic energy  $E_0$  of the plate is then changed into heat or transformed into the kinetic energy of the gas particles in the boundary layer (which, however, also dissipates as heat after sufficiently long time). During the process, half the energy,  $E_0/2$ , flows into the plate as heat. Thereby we assume that during the unsteady braking process, Eq. (4.99) is valid at each instant. This assumption is not very critical; in any case, it is permissible when the flow in the boundary layer at each instant is not too far different from a steady boundary layer flow, as we have considered here (quasisteady flow). Moreover, during the braking, the plate must not heat up so much

that its temperature  $T_w$  greatly exceeds  $T_\infty$ . For initial velocities  $U_0$  of the order of magnitude of reentry velocities for spacecraft into the earth's atmosphere (satellite velocity 7.9 km/sec, escape velocity 11.2 km/sec), the kinetic energy  $E_0$  of a solid body exceeds the heat energy required for its vaporization. If such a spacecraft were to be slowed down like a flat plate with a cold wall, the energy  $E_0/2$  would be transferred to the craft, and it would be completely destroyed. In actuality, the heat transferred is smaller than  $E_0/2$ , and, in fact, considerably so, on account of the following reasons:

1. The surface temperature  $T_w$  of the spacecraft increases during the braking process and can reach values considerably above  $T_\infty$ . Thus, the heat transferred from the gas to the spacecraft is smaller than if it were a cold body. Moreover, at high wall temperatures, the energy radiated from the wall into space plays a role; according to the Stefan-Boltzmann law, it increases as the 4th power of the wall temperature  $T_w$ , and it partially or totally compensates for the heat flux  $q_w$  from the gas to the wall. This *radiative cooling effect* can be increased by selecting a high-emissivity material for the surface of the spacecraft.

2. The flat plate, for which the total drag  $D$  is identical to the friction drag  $D_f$ , is not a realistic body shape for spacecraft. For bodies blunted in the direction of flight, a considerable portion of the total drag  $D$  is pressure drag. (A pressure drag results when the integral of the pressure forces over the surface of the body has a component in the flow direction; examples of pressure drag have been seen in Sections 3.5.2 and 3.6.2.) The decrease of the kinetic energy during braking is given by

$$\frac{dE}{dt} = -DU_\infty, \quad (4.108)$$

where  $U_\infty$  denotes the instantaneous velocity. If we again take Eq. (4.107) as valid for the heat transferred to the body, we obtain from a combination of (4.107) and (4.108)

$$\frac{dQ}{dE} = -\frac{1}{2} \frac{D_f}{D}. \quad (4.109)$$

If during the braking process the ratio  $D_f/D$  of the friction drag to the total drag does not change very much, then we get, by integrating (4.109) for deceleration to velocity 0, the total heat transferred to the body:

$$Q = \frac{1}{2} \frac{D_f}{D} E_0. \quad (4.110)$$

The smaller the friction drag relative to the total drag, the smaller the fraction of total energy transferred as heat to the body [here, the effect explained in (1) above is completely ignored]. More heat will then be transferred to the gas, since the shock waves, which cause a large pressure drag, are also the agents of that energy transfer.

Finally, we should mention that in the reentry of space vehicles into the earth's atmosphere at velocities of the order of the escape velocity (11.2 km/sec), the gas in the boundary layer in the shock around the body becomes so hot that its self-radiation can no longer be ignored, and it constitutes, among other things, an important contribution to the heat transferred to the spacecraft. Unable to enter further into these processes here, we shall only mention that such studies have started to develop a new branch of gas dynamics, which is called *radiation gas dynamics*.

#### 4.3.4 VELOCITY AND ENTHALPY DISTRIBUTION IN THE FLAT-PLATE BOUNDARY LAYER

In Sections 4.3.2 and 4.3.3, we studied several important properties of boundary layer flow without having to solve the boundary layer equations (4.68), (4.69), and (4.71) explicitly. However, numerical results, particularly for the physically important quantities of friction coefficient  $c_f$ , Stanton number  $St$ , recovery factor  $r$ , and Reynolds analogy factor  $s$ , can only be obtained by calculating the velocity and enthalpy or temperature distribution in the boundary layer explicitly. For  $Pr = 1$ , the enthalpy calculation can be spared, since the enthalpy is coupled to the velocity according to Eq. (4.95). We always assume in the following that the wall enthalpy  $h_w = \text{const}$  [otherwise, Eq. (4.95) will not hold, and the discussion following Eq. (4.121) will no longer be valid].

The calculation of the velocity and enthalpy in the boundary layer will be simplified by the introduction of a stream function. From the continuity equation (4.71) there follows the existence of a stream function  $\Psi(x, y)$  such that

$$u = \frac{\varrho_\infty}{\varrho} \frac{\partial \Psi}{\partial y}, \quad (4.111)$$

$$v = -\frac{\varrho_\infty}{\varrho} \frac{\partial \Psi}{\partial x}. \quad (4.112)$$

It is easily seen that the continuity equation is satisfied with Eqs. (4.111) and (4.112). Moreover, it is convenient to transform the coordinate  $y$  perpendicular to the wall. We denote the new independent variables by  $\bar{x}$  and  $\bar{y}$ , and set

$$\bar{x} = x, \quad (4.113)$$

$$\bar{y} = \int_0^y \frac{\varrho(x, \lambda)}{\varrho_\infty} d\lambda. \quad (4.114)$$

$\bar{y}$  is a function of  $x$  and  $y$  which will be known only when the density  $\varrho(x, y)$  in the boundary layer is known. In the following discussion,  $\Psi$  and  $h$  will be regarded as functions of the new variables  $\bar{x}$  and  $\bar{y}$ ; for the sake of simplicity, we retain the same notation  $\Psi$  and  $h$ . Derivatives with respect to  $x$  and  $y$  are now converted to derivatives with respect to  $\bar{x}$  and  $\bar{y}$  in the following manner:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{y}}{\partial x} \frac{\partial}{\partial \bar{y}}, \\ \frac{\partial}{\partial y} &= \frac{\partial \bar{y}}{\partial y} \frac{\partial}{\partial \bar{y}} = \frac{\varrho}{\varrho_\infty} \frac{\partial}{\partial \bar{y}}. \end{aligned}$$

Using this, we obtain from Eqs. (4.111) and (4.112)

$$u = \frac{\partial \Psi(\bar{x}, \bar{y})}{\partial \bar{y}}, \quad (4.115)$$

$$v = -\frac{\varrho_\infty}{\varrho} \left[ \frac{\partial \Psi(\bar{x}, \bar{y})}{\partial \bar{x}} + \frac{\partial \Psi}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial \bar{x}} \right]. \quad (4.116)$$

Putting these into the momentum equation (4.93), we obtain, after some transformation,

$$\Psi_{\bar{y}} \Psi_{\bar{x}\bar{y}} - \Psi_{\bar{x}} \Psi_{\bar{y}\bar{y}} = \frac{\partial}{\partial \bar{y}} \left( \frac{\eta \varrho}{\varrho_\infty^2} \Psi_{\bar{y}\bar{y}} \right) \quad (4.117)$$

(The subscripts denote partial derivatives). The energy equation (4.69) can be brought into the following form if we consider the definition (4.70) for  $h$ , the relation (4.76), and the fact that the momentum equation (4.93) is satisfied:

$$\varrho u \frac{\partial h}{\partial x} + \varrho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\eta}{\text{Pr}} \frac{\partial h}{\partial y} \right) + \eta \left( \frac{\partial u}{\partial y} \right)^2. \quad (4.118)$$

Using (4.115) and (4.116) and transforming the variables to  $\bar{x}$  and  $\bar{y}$ , we obtain

$$\Psi_{\bar{y}} h_{\bar{x}} - \Psi_{\bar{x}} h_{\bar{y}} = \frac{\partial}{\partial \bar{y}} \left( \frac{\eta \varrho}{\varrho_\infty^2} h_{\bar{y}} \right) + \frac{\eta \varrho}{\varrho_\infty^2} (\Psi_{\bar{y}\bar{y}})^2. \quad (4.119)$$

Equations (4.117) and (4.119) are to be solved under the following boundary conditions: On the wall,

$$\bar{y} = 0: \quad \Psi = 0, \Psi_{\bar{y}} = 0, \quad (4.120)$$

$$h = h_w = \text{const}. \quad (4.121)$$

Condition (4.120) results from (4.72) and (4.73). Strictly, by Eq. (4.73) we must require  $\Psi_{\bar{x}} = 0$ , or  $\Psi = \text{const}$ , but we can set the constant zero without loss of generality. Condition (4.121) corresponds to condition (4.74) under the additional assumption that the wall enthalpy is constant. At the exterior edge of the boundary layer, we must have

$$\lim_{\bar{y} \rightarrow \infty} \Psi_{\bar{y}} = U_\infty, \quad (4.122)$$

$$\lim_{\bar{y} \rightarrow \infty} h = h_\infty. \quad (4.123)$$

Finally, the boundary layer thickness must be zero at the leading edge of the plate, i.e., we must have

$$u = \Psi_{\bar{y}} = U_\infty \quad \text{and} \quad h = h_\infty \quad \text{for} \quad x = 0, y > 0. \quad (4.124)$$

We find a solution of (4.117) and (4.119) that satisfies boundary conditions (4.120) to (4.124) by tentatively assuming

$$\Psi = (2v_\infty U_\infty \bar{x})^{1/2} f(\zeta), \quad (4.125)$$

$$h = h_\infty g(\zeta), \quad (4.126)$$

where the variable  $\zeta$  has the following meaning:

$$\zeta = \bar{y} \left( \frac{U_\infty}{2v_\infty \bar{x}} \right)^{1/2}. \quad (4.127)$$

From (4.125) and (4.127), it follows in particular that

$$u = \Psi_{\bar{y}} = U_\infty f'(\zeta) \quad (4.128)$$

(Here, the prime denotes differentiation with respect to  $\zeta$ ). Provided that (4.125) is valid, the velocity [Eq. (4.128)] and the enthalpy [Eq. (4.126)]

depend only on a single variable  $\zeta$  (and by the constancy of the pressure, so will all the other thermodynamic state variables). The distribution of these quantities in the  $\bar{y}$  direction at a fixed location  $x_1$  differs from that at another location  $x_2$  only by an  $x$ -dependent stretching in the direction perpendicular to the wall. Such solutions, which instead of depending on two variables  $x$  and  $y$  depend only on one variable and for which therefore the boundary layer profiles at various locations  $x$  are related to each other by an affine transformation, are called similar solutions of the boundary layer equations, or, for short, similar boundary layers. Substituting (4.125) and (4.126) into Eqs. (4.117) and (4.119), we obtain, after some transformation,

$$\left( \frac{\eta\varrho}{\eta_\infty\varrho_\infty} f'' \right)' + ff'' = 0, \quad (4.129)$$

$$\left( \frac{\eta\varrho}{Pr\eta_\infty\varrho_\infty} g' \right)' + fg' = - \frac{\eta\varrho}{\eta_\infty\varrho_\infty} \frac{U_\infty^2}{h_\infty} (f'')^2, \quad (4.130)$$

with the boundary conditions following from (4.120)–(4.124):

$$\zeta = 0: \quad f = f' = 0, \quad (4.131)$$

$$g = h_w/h_\infty = \text{const}, \quad (4.132)$$

and

$$\lim_{\zeta \rightarrow \infty} f' = \lim_{\zeta \rightarrow \infty} g = 1. \quad (4.133)$$

The solution of this system of equations for the functions  $f$  and  $g$  will be particularly simple if we assume that the product  $\eta\varrho$  is temperature independent. Since, moreover, the pressure in the flat-plate boundary layer is constant, then  $\eta\varrho$  is equal to a constant:

$$\eta\varrho = \eta_\infty\varrho_\infty. \quad (4.134)$$

We shall assume in the following that Eq. (4.134) holds. The permissibility of this assumption has already been discussed in Section 4.1.1. Under (4.134), Eq. (4.129) becomes

$$f''' + ff'' = 0. \quad (4.135)$$

This equation can be solved completely independently of the equation for  $g$ . Moreover, the same equation is encountered for the flat-plate boundary layer for a constant-density, constant-viscosity fluid. This boundary layer was studied by H. Blasius in 1908 in one of the first works on boundary layer

theory,<sup>81</sup> in which he gave a numerical solution of (4.135). Existence and uniqueness of the solution of (4.135) with the boundary conditions (4.134) and (4.133) were proved by H. Weyl.<sup>82</sup> This existence proof yields at the same time an iteration procedure which is particularly suited for a numerical calculation of the solution. Figure 113 shows the result of the numerical

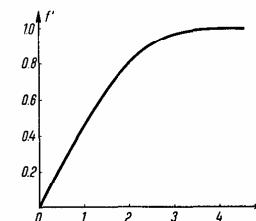


Fig. 113. Blasius function  $f'(\zeta)$ ; dimensionless velocity distribution in the flat-plate boundary layer.

calculation. Knowing  $f(\zeta)$ , we can calculate the wall shear stress  $\tau_w$ :

$$\tau_w = \eta_w \frac{\partial u}{\partial y} \Big|_{y=0} = \eta_w \left( \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \frac{\partial y}{\partial x} \right)_{y=0}.$$

From this, it follows with  $\eta_w\varrho_w = \eta_\infty\varrho_\infty$  that

$$\tau_w = \eta_\infty U_\infty (U_\infty/2vx)^{1/2} f''(0). \quad (4.136)$$

The friction coefficient defined by (4.100) will then be

$$c_f = \frac{\sqrt{2} f''(0)}{(U_\infty x/v_\infty)^{1/2}} = \frac{0.664}{\sqrt{Re}}. \quad (4.137)$$

Here,  $Re = U_\infty x/v_\infty$  denotes the Reynolds number based on the freestream quantities  $U_\infty$  and  $v_\infty$  and the distance  $x$  from the leading edge of the plate. The friction coefficient decreases with increasing distance as  $1/\sqrt{x}$ ; this is caused by the growth in boundary layer thickness as  $\sqrt{x}$ , as was concluded earlier by analogy [Eq. (4.91)]. The product  $c_f \sqrt{Re}$  is, by (4.137), a constant.

<sup>81</sup> H. Blasius, Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Z. Math. Phys.* **56** 1–37, (1908).

<sup>82</sup> H. Weyl, On the differential equations of the simplest boundary layers problems. *Ann. Math.* **43** 381–407, (1942).

It thus depends neither on the Mach number  $M_\infty = U_\infty/a_\infty$  nor on the wall temperature  $T_w$ . This remarkable result is an immediate consequence of the assumption (4.134). Only when this assumption is correct can we solve Eq. (4.129) independently of (4.130), and only then will the wall shear stress be independent of the enthalpy or temperature distribution in the boundary layer. We shall later find an obvious interpretation for this. As soon as relation (4.134) is violated,  $c_f \sqrt{\text{Re}}$  will be a function of the Mach number  $M_\infty$ , the Prandtl number  $\text{Pr}$ , the viscosity law  $\eta(T)$  [or the exponent  $\omega$  if we use (4.3)], and the ratio  $T_w/T_\infty$  of the wall temperature to the freestream temperature. Numerical results for the calorically ideal gas are given in Fig. 115.

To calculate the specific enthalpy  $h$  we assume in addition to (4.134) that the Prandtl number  $\text{Pr}$  is constant. Then, with the function  $f(\zeta)$  now known, Eq. (4.130) becomes

$$\frac{1}{\text{Pr}} g'' + f g' = - \frac{U_\infty^2}{h_\infty} (f'')^2. \quad (4.138)$$

This equation for  $g(\zeta)$  is linear, and its solution can thus be written as the sum of a solution to the homogeneous equation and a particular solution, as follows:

$$h_\infty g \equiv h = h_\infty + [h_w - h_\infty - \frac{1}{2} U_\infty^2 \text{Pr} R(0)] S(\zeta) + \frac{1}{2} U_\infty^2 \text{Pr} R(\zeta). \quad (4.139)$$

Here,  $S(\zeta)$  is the solution of the homogeneous equation corresponding to (4.138), which satisfies the boundary conditions  $S(0) = 1$  and  $S(\infty) = 0$ .  $(U_\infty^2 \text{Pr}/2h_\infty) R(\zeta)$  is the solution to the inhomogeneous equation (4.138), which satisfies the boundary conditions  $R'(0) = 0$  and  $R(\infty) = 0$ . Under the assumptions on  $R$  and  $S$ , the boundary conditions (4.132) and (4.133) for  $g$  are satisfied, i.e.,  $h = h_w$  on the wall and  $h = h_\infty$  at the outer edge of the boundary layer. We can confirm by substitution in (4.138) that the following functions  $R$  and  $S$  satisfy the differential equations and boundary conditions in question:

$$R(\zeta) = 2 \int_{\zeta}^{\infty} [f''(\lambda)]^{\text{Pr}} \int_0^{\lambda} [f''(\mu)]^{2-\text{Pr}} d\mu d\lambda, \quad (4.140)$$

$$S(\zeta) = \int_{\zeta}^{\infty} [f''(\lambda)]^{\text{Pr}} d\lambda / \int_0^{\infty} [f''(\lambda)]^{\text{Pr}} d\lambda. \quad (4.141)$$

Thus, the expression for  $g$  given by (4.139) also satisfies Eq. (4.138). Since by (4.139) the derivative  $dh/d\zeta$  on the wall will just vanish when

$$h_w = h_r = h_\infty + \frac{1}{2} U_\infty^2 \text{Pr} R(0) \quad (4.142)$$

[because  $R'(0) = 0$  while  $S'(0) \neq 0$ ], we get for the recovery factor

$$r = \text{Pr} R(0) = 2\text{Pr} \int_0^{\infty} [f''(\lambda)]^{\text{Pr}} \int_0^{\lambda} [f''(\mu)]^{2-\text{Pr}} d\mu d\lambda. \quad (4.143)$$

The recovery factor thus depends only on the Prandtl number. If  $\text{Pr}$  does not differ much from 1, the numerical evaluation of (4.143) is well approximated by the result given in Eq. (4.103).

When  $h_w \neq h_r$ , there is a heat flow  $q_w$  at the wall given by

$$q_w = - \frac{\eta_w}{\text{Pr}} \left( \frac{dh}{d\zeta} \frac{\partial \zeta}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial y} \right)_{y=0}. \quad (4.144)$$

Using (4.139) and the definition (4.101) of the Stanton number, we get from (4.144)

$$\text{St} = - \frac{v_\infty}{U_\infty \text{Pr}} \left( \frac{U_\infty}{2v_\infty x} \right)^{1/2} S'(0). \quad (4.145)$$

In combination with the result for the friction coefficient (4.137) and taking (4.141) into account, we find the following for the Reynolds analogy factor defined by (4.102):

$$s = \frac{[f''(0)]^{\text{Pr}-1}}{\text{Pr} \int_0^{\infty} [f''(\lambda)]^{\text{Pr}} d\lambda}. \quad (4.146)$$

Just as the recovery factor, the Reynolds analogy factor depends only on the Prandtl number. If  $\text{Pr}$  is not much different from 1, formula (4.104) gives a good approximation.

In the special case of  $\text{Pr} = 1$ , we get from (4.140) and (4.141)

$$R(\zeta) = 1 - f'^2 = 1 - (u^2/U_\infty^2), \quad (4.147)$$

$$S(\zeta) = 1 - f' = 1 - (u/U_\infty). \quad (4.148)$$

With this, Eq. (4.139) becomes the previously derived formula (4.95) for  $\text{Pr} = 1$ . The enthalpy portions  $\alpha$  and  $\beta$  introduced in Fig. 112 thus correspond to the terms  $\frac{1}{2} U_\infty^2 \text{Pr} R$  and  $(h_w - h_r) S$ , respectively in, Eq. (4.139). We call

$\alpha$  the *heat of friction*, while for  $\beta$  the expression *conducted heat* is perhaps suitable. The former depends only on the freestream velocity  $U_\infty$  and the thermodynamic properties of the gas, while the latter depends essentially on the wall enthalpy  $h_w$ .

In order to obtain the distribution of the velocity  $u$  and the thermodynamic state variables with the distance  $y$  from the wall, we must finally transform back from the variable  $\zeta$  to the variable  $y$ . From relations (4.127), (4.113), and (4.114), it follows that

$$\frac{\partial \zeta}{\partial y} = \left( \frac{U_\infty}{2\nu_\infty x} \right)^{1/2} \frac{\varrho}{\varrho_\infty}. \quad (4.149)$$

$\varrho$  will be determined from  $h(p, \varrho) = h(\zeta)$  with  $p = p_e$ , and is thus a known function of  $\zeta$ . Integrating (4.149) with the initial condition  $y = 0$  for  $\zeta = 0$ , we get

$$\left( \frac{U_\infty}{2\nu_\infty x} \right)^{1/2} y = \int_0^\zeta \frac{\varrho_\infty}{\varrho} d\zeta. \quad (4.150)$$

Thus,  $\zeta(x, y)$  is known; substituting this function into the state variables we get these quantities as functions of  $x$  and  $y$ .

As an example, we consider the calorically ideal gas with  $\text{Pr} = 1$  and a heat-insulating wall. For this case,

$$\frac{\varrho_\infty}{\varrho} = \frac{T}{T_\infty} = 1 + \frac{U_\infty^2}{2c_p T_\infty} (1 - f'^2) \quad (4.151)$$

[see formula (4.96)]. Substituting this into (4.150) and in addition setting  $c_p T_\infty = a_\infty^2 / (\gamma - 1)$ , we get

$$\left( \frac{U_\infty}{2\nu_\infty x} \right)^{1/2} y = \zeta + \frac{\gamma - 1}{2} M_\infty^2 \int_0^\zeta (1 - f'^2) d\zeta. \quad (4.152)$$

According to (4.152), at a fixed location  $x$  and for fixed value of  $U_\infty/\nu_\infty$ , the greater the Mach number  $M_\infty$ , the greater will be the value of  $y$  corresponding to a given  $\zeta$  and thus to a given  $f'(\zeta) = u/U_\infty < 1$ . The velocity profile, i.e., the velocity as function of the distance  $y$ , will become flatter as the Mach number  $M_\infty$  increases, as can plainly be seen in the numerical results plotted in Fig. 114. In the same way, the profiles of the thermodynamic variables will also become flatter. This thickening of the boundary layer with increasing Mach number is qualitatively easy to understand if we realize that because of the

heat of friction the gas is heated more at higher Mach numbers, so that its density must be less. This expansion of the gas causes an inflation of the boundary layer. It will now also be qualitatively understandable that under the assumption  $\eta\varrho = \text{const}$  the friction coefficient  $c_f$  depends only on the Reynolds

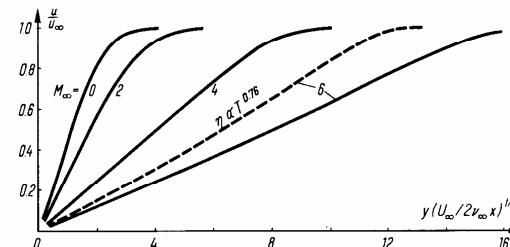


Fig. 114. Velocity profiles for the flat-plate boundary layer of a heat-insulating wall;  $\gamma = 1.4$ ,  $\text{Pr} = 1$ . (From K. Stewartson, The Theory of Laminar Boundary Layers in Compressible Fluids. Oxford, England, 1964.)

number  $\text{Re} = U_\infty x / \nu_\infty$  and not on the Mach number  $M_\infty$  or wall temperature  $T_w$ , as was established earlier. With the density decrease in the boundary layer as a result of the heat of friction or the conducted heat, the boundary layer reacts with an increase in its thickness. Thus, all the gradients of the flow variables in the direction perpendicular to the wall will decrease, while the viscosity  $\eta$  increases, since  $\eta\varrho = \text{const}$ . The increase of  $\eta$  will now just be compensated by the decrease of  $\partial u / \partial y$  on the wall, so that  $\tau_w = \eta_w (\partial u / \partial y)|_{y=0}$  does not change its value. An analogous effect is present when the density in the boundary layer increases. In Section 4.1.1, it was indicated that the product  $\eta\varrho$  at constant pressure generally decreases with increasing temperature. This means that at constant pressure the product  $\eta\varrho$  decreases together with  $\varrho$ . As a boundary layer widens due to a density decrease, the viscosity does not increase fast enough to be able to compensate for the decrease in the velocity gradient at the wall, so that the friction coefficient cannot remain constant. For this reason the friction coefficient decreases with increasing Mach number  $M_\infty$ , for example, for a heat-insulating wall, as can be seen in Fig. 115. This effect is even more pronounced for a turbulent boundary layer than for a laminar boundary layer. The effective viscosity in a turbulent boundary layer,

which is mainly generated by the macroscopic mixing process between neighboring gas layers, is dependent on the temperature to a much lesser extent than is the molecular viscosity.

*Supplementary Remarks.* The starting point for many approximate methods

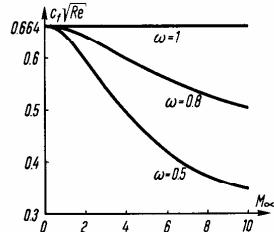


Fig. 115. Pressure coefficient of the flat-plate boundary layer as a function of the free-stream Mach number. Heat-insulating wall, calorically ideal gas with  $\gamma = 1.4$ ,  $\text{Pr} = 1$ ,  $\eta \propto T^{\omega}$ . (From W. Hantzsche and H. Wendt, Zum Kompressibilitätsseinsfluß bei der laminaren Grenzschicht der ebenen Platte, *Jahrbuch der deutschen Luftfahrtforschung I*, 517–521, (1940).

of calculating boundary layers is the use of the *integral theorems* of boundary layer theory. These are equations derived by integration of the momentum equation (4.68) and the energy equation (4.69) in the  $y$  direction. As an example, we shall derive such an integral theorem from the energy equation. Using the notation  $\eta \frac{\partial u}{\partial y} = \tau$  and  $-k \frac{\partial T}{\partial y} = q$ , we can write Eq. (4.69) in the following form:

$$\varrho u \frac{\partial h_t}{\partial x} + \varrho v \frac{\partial h_t}{\partial y} = \frac{\partial}{\partial y} (u\tau - q). \quad (4.153)$$

We now integrate this equation with respect to  $y$  between the fixed limits 0 and  $\Delta$ :

$$\int_0^\Delta \left( \varrho u \frac{\partial h_t}{\partial x} + \varrho v \frac{\partial h_t}{\partial y} \right) dy = (u\tau - q) \Big|_0^\Delta. \quad (4.154)$$

From integration by parts, we form:

$$\int_0^\Delta \varrho v \frac{\partial h_t}{\partial y} dy = \varrho v h_t \Big|_0^\Delta - \int_0^\Delta h_t \frac{\partial \varrho v}{\partial y} dy. \quad (4.155)$$

By the continuity equation,  $\partial \varrho v / \partial y = -\partial \varrho u / \partial x$ , so that

$$\varrho v h_t \Big|_0^\Delta = -h_t(\Delta) \int_0^\Delta \frac{\partial \varrho u}{\partial x} dy.$$

With this, Eq. (4.155) becomes

$$\int_0^\Delta \varrho v \frac{\partial h_t}{\partial y} dy = \int_0^\Delta [h_t - h_t(\Delta)] \frac{\partial \varrho u}{\partial x} dy. \quad (4.156)$$

The specific total enthalpy at the edge of the boundary layer will be denoted by  $h_{te}$ ; we have  $\partial h_{te}/\partial x = 0$ , since in the isoenergetic inviscid outer flow,  $h_t$  is constant. Thus, the following identity holds:

$$\varrho u \frac{\partial h_t}{\partial x} + [h_t - h_t(\Delta)] \frac{\partial \varrho u}{\partial x} = \frac{\partial}{\partial x} [\varrho u (h_t - h_{te})] + [h_{te} - h_t(\Delta)] \frac{\partial \varrho u}{\partial x}. \quad (4.157)$$

Taking (4.156) and (4.157) into consideration, we can now write Eq. (4.154) in the following form:

$$\frac{d}{dx} \int_0^\Delta \varrho u (h_t - h_{te}) dy + [h_{te} - h_t(\Delta)] \int_0^\Delta \frac{\partial \varrho u}{\partial x} dy = (u\tau - q) \Big|_0^\Delta. \quad (4.158)$$

Here the differentiation with respect to  $x$  in the first integral could be pulled

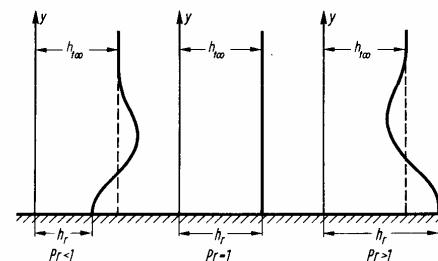


Fig. 116. Distribution of total enthalpy in the flat-plate boundary layer for a heat-insulating wall for various Prandtl numbers (schematic).

in front of the integral sign, since the upper limit  $A$  is, by assumption, independent of  $x$ . We now assume the existence of the improper integral  $\int_0^\infty \varrho u (h_t - h_{te}) dy$ . Obviously, this integral will exist when the difference  $h_t(A) - h_{te}$  goes to zero faster than  $A^{-1}$  as  $A \rightarrow \infty$ . Experience indicates that this assumption on the asymptotic behavior at the outer boundary layer edge is always fulfilled. Since, on the other hand  $\int_0^A \partial \varrho u / \partial x \, dy$  increases at most proportionally to  $A$  as  $A \rightarrow \infty$ , the two terms on the left side of Eq. (4.158) drop out in the limit of  $A \rightarrow \infty$ . By this limiting process we obtain

$$(d/dx)(\varrho_e u_e h_{te} \theta) = -q_w, \quad (4.159)$$

where the quantity  $\theta$  has the dimension of length and is defined as follows:

$$\theta = \int_0^\infty \frac{\varrho u}{\varrho_e u_e} \left(1 - \frac{h_t}{h_{te}}\right) dy. \quad (4.160)$$

We call this quantity  $\theta$  the enthalpy thickness.

For a heat-insulating wall, i.e.,  $q_w = 0$ , it follows from Eq. (4.159) that  $\varrho_e u_e h_{te} \theta$  must be a constant, or  $\varrho_e u_e \theta$  is a constant, since  $h_{te} = \text{const}$ . For the flat-plate boundary layer considered above, the boundary layer thickness is zero at the leading edge of the plate, so that  $\theta = 0$  there. However, this implies that for the heat-insulating flat plate,  $\theta = 0$  must hold everywhere. When the Prandtl number  $\text{Pr}$  and, accordingly, the recovery factor  $r$  both have the value 1, then  $h_t = h_{te}$  everywhere in the boundary layer (see Section 4.3.2) and the integral (4.160) thus vanishes. If  $\text{Pr} < 1$ , then we also have  $r < 1$ , i.e., on the wall  $h_t = h_r < h_{te}$ . At some distance from the wall,  $h_t > h_{te}$  must then hold, or else the integral (4.160) cannot vanish ( $u$  is everywhere positive in the flat-plate boundary layer, and  $\varrho$  is always positive). When  $\text{Pr} > 1$ , then  $r > 1$ , and thus  $h_r > h_{te}$ . Then,  $h_t < h_{te}$  must occur at some distance from the wall. Thus, the different enthalpy distributions for different Prandtl numbers sketched in Fig. 116 can be qualitatively understood.

In closing, we should mention that if we integrate the momentum equation (4.68) with respect to  $y$ , then the following integral theorem can be derived in exactly the same manner as was done in deriving Eq. (4.159) from the energy equation (4.69):

$$\frac{d}{dx} (\varrho_e u_e^2 \Theta) + \varrho_e u_e \delta^* \frac{du_e}{dx} = \tau_w. \quad (4.161)$$

Here,  $\Theta$  and  $\delta^*$  are two quantities with the dimension of a length:

$$\Theta = \int_0^\infty \frac{\varrho u}{\varrho_e u_e} \left(1 - \frac{u}{u_e}\right) dy, \quad (4.162)$$

$$\delta^* = \int_0^\infty \left(1 - \frac{\varrho u}{\varrho_e u_e}\right) dy. \quad (4.163)$$

We call  $\Theta$  the momentum thickness and  $\delta^*$  the displacement thickness of the boundary layer. Both quantities are natural, physically meaningful measures for the thickness of the boundary layer. The meaning of the displacement thickness can be clarified by the following: In an inviscid flow without boundary layer, the mass  $\varrho_e u_e b$  flows at the wall through a streamtube of width  $b$  and depth 1 perpendicular to the flow plane. Here it is assumed that  $b$  is so small that the density and velocity in the streamtube practically coincide with the values at the wall. If now a boundary layer exists on the wall, then the mass flow through the same stream tube will be  $\int_0^b \varrho u \, dy$ . The difference between the inviscid mass flow and the mass flow in the presence of the boundary layer is  $\int_0^b (\varrho_e u_e - \varrho u) \, dy$ . If the boundary layer thickness is small compared to  $b$ , we can also substitute  $b$  for the upper limit in the definition (4.163) for the displacement thickness  $\delta^*$ . The difference of the two mass flows is then  $\int_0^b (\varrho_e u_e - \varrho u) \, dy - \varrho_e u_e \delta^*$ . From this we can conclude that the presence of the boundary layer affects the inviscid outer flow to first approximation exactly as if the wall had been displaced outward through a distance  $\delta^*$  in the direction perpendicular to the wall. The momentum thickness has the following meaning: The momentum  $\int_0^\infty \varrho u^2 \, dy$  flows per unit time through the streamtube of width  $b$ . If the entire mass flowing through this streamtube were to have the velocity of the inviscid flow  $u_e$ , then the momentum flow per unit time would be  $\int_0^b \varrho u u_e \, dy$ . The momentum deficiency, i.e., the difference between these two momentum flows, is  $\int_0^b \varrho u (u_e - u) \, dy = \varrho_e u_e^2 \Theta$ .

#### 4.3.5 BOUNDARY LAYERS WITH SUCTION OR BLOWING AT THE WALL

In the previous sections, it was assumed that the wall on which the boundary layer built up is impermeable to gas, and the gas must satisfy the

no-slip condition on the wall. For many practical applications, a permeable wall is of significance, through which either the gas from the stream is sucked out or additional gas is blown in into the stream. To achieve such *boundary layer control*, the wall must be porous, or be provided with holes, slits, etc. A detailed knowledge of the condition of the wall is not needed below; it suffices to assume that the properties of the wall permit a sufficiently continuous distribution of suction or blowing velocities. For walls with holes or slits, this will be the case when the dimensions of the holes or slits as well as the distances between neighboring holes or slits are very small compared with the boundary layer thickness. The boundary layer equations for the flow in the immediate neighborhood of the wall remain valid with suction or blowing if the suction and blowing velocities are sufficiently small. However, one of the boundary conditions on the wall is changed: While we still assume  $u = 0$  on the wall, we now assume a finite velocity  $v_w$  on the wall in the direction perpendicular to it. For suction,  $v_w < 0$ , and for blowing,  $v_w > 0$ . We admit only small velocities  $v_w$  such that the basic assumption of the boundary layer equations,  $|v| \ll |u|$ , is still satisfied (except in the boundary region immediately next to the wall, where this assumption and the condition  $u = 0$  cannot be satisfied at the same time; the width of this region, however, remains small compared to the thickness of the boundary layer if  $v_w$  is sufficiently small). The theory shows that even very small wall velocities  $v_w$ , for which the boundary layer approximations are still valid, can already have considerable influence on the boundary layer. In the example which we shall treat below, we shall in addition assume for the case of blowing that the gas blown into the boundary layer is chemically the same as the gas in the stream, thereby avoiding the complications arising from the mixing of two gases.

As an example, we again consider the flat-plate boundary layer, with a specially prescribed velocity  $v_w$  on the wall, namely

$$\varrho_w v_w = \pm \varrho_\infty U_\infty (\lambda/x)^{1/2}. \quad (4.164)$$

Here,  $\lambda$  is a constant having the dimension of a length, the “+” sign corresponds to blowing, and the “−” sign to suction. For this special choice of  $v_w$ , the flat-plate boundary layer remains a *similar* boundary layer in the sense of Section 4.3.4, so that the theoretical treatment is greatly simplified. For  $x \rightarrow 0$ , by (4.164), the prescribed velocity  $v_w$  becomes unbounded, so that near the leading edge, i.e., for small  $x$ , the condition  $|v| \ll |u|$  cannot be satisfied.

However, we established in Section 4.3.3 that the boundary layer equations for  $v_w = 0$  also begin to be valid only at some distance from the leading edge; the singular behavior of  $v_w$  at the leading edge is thus no worse to the theoretical treatment of flow with suction or blowing than are the defects in the boundary layer assumptions in the neighborhood of the leading edge in flows without boundary layer control.

With the same assumptions on the material properties of the gas as were introduced in Section 4.3.4 for Eq. (4.135) and using the same notation introduced there, we again have Eq. (4.135), also for the case  $v_w \neq 0$ . The boundary condition (4.133) also remains unchanged. From the two conditions (4.131),  $f'(0) = 0$  still holds, since this condition follows from  $u = 0$ . In contrast,  $f(0) = 0$  no longer holds. The present condition for  $f$  results from (4.164) with the use of (4.116):

$$\varrho_w v_w = -\varrho_\infty \left[ \Psi_{\bar{x}} + \Psi_{\bar{y}} \frac{\partial \bar{y}}{\partial \bar{x}} \right]_{y=0} = \pm \varrho_\infty U_\infty \left( \frac{\lambda}{x} \right)^{1/2}. \quad (4.165)$$

Since on the wall ( $y = 0$ )  $u = \Psi_{\bar{y}} = 0$ , this implies that

$$\Psi_{\bar{x}} = \mp U_\infty (\lambda/x)^{1/2}. \quad (4.166)$$

However, it follows from (4.125) that

$$\Psi_{\bar{x}} = \frac{1}{2} (2v_\infty U_\infty / x)^{1/2} (f - \zeta f'). \quad (4.167)$$

On the wall,  $\zeta = 0$ , and we obtain, by substituting expression (4.167) for

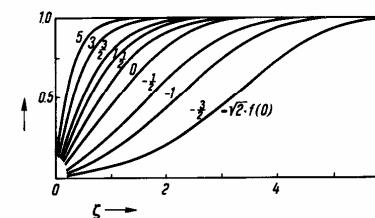


Fig. 117. Dimensionless velocity distribution  $f'(\zeta)$  for the flat-plate boundary layer with suction or blowing. (From H. Schlichting and K. Bussmann, Exakte Lösungen für die laminare Reibungsschicht mit Absaugung und Ausblasen, *Schr. d. dt. Akademie d. Luftfahrtforschung* 7B, Nr. 2, 1943.)

$\Psi_x$  into (4.166),

$$f(0) = \mp (2U_\infty \lambda / v_\infty)^{1/2}, \quad (4.168)$$

where the upper sign holds for blowing.

Numerical solutions of Eq. (4.135) which satisfy the boundary condition (4.168) and the other boundary conditions  $f'(0)=0$  and  $f'(\infty)=1$  are shown in Fig. 117. From this figure, we draw the obvious conclusion that in suction, for otherwise unchanged flow conditions, the boundary layer thickness is decreased, while in blowing it is increased. In addition, the velocity profile  $u/U_\infty = f'(\zeta)$  contains an inflection point for blowing, while for suction the profile becomes fuller in form and its curvature has the same sign everywhere. The decrease in the boundary layer thickness and the change in the profile shape due to suction has a stabilizing influence on the laminar boundary layer, i.e., it changes to turbulent flow at higher Reynolds numbers than in the case without suction. This effect is utilized in aerodynamics to reduce the high friction coefficient connected with turbulent boundary layers. Moreover, for boundary layers with a pressure increase in the direction of the flow (for which flows suction has qualitatively the same effect as in boundary layers without pressure gradient) suction deters the separation of the boundary layer.

In high-speed aerodynamics, boundary layer control by blowing is of particular significance, since this is an effective means of reducing the heat transfer from the hot gas in the boundary layer of a hypersonic body to the body surface. As can be seen from Fig. 117, for otherwise unchanged flow conditions, blowing reduces the velocity gradients on the wall. This has the consequence of decreasing the wall shear stress  $\tau_w$ , and with it the conduction heat transfer  $q_w$  between a unit surface of the wall and the gas per unit time (see Section 4.3.3). This effect is called *heat blocking* by blowing. In addition to heat blocking, blowing produces yet another cooling effect: the gas blown through the porous wall or the wall with holes absorbs the heat from the wall and transports it away from the body; this is called the *heat sink effect*. The cooling capacity of the heat sink effect can be greatly increased if, instead of a gas, a fluid is used, which vaporizes as it is blown through the hot wall. This vapor issuing from the surface will have absorbed its entire heat of vaporization from the wall. The effect of heat blocking is still present, since the reduction of the gradients at the wall occurs regardless of the type of gas blown in, and by no means requires the gas to be chem-

ically the same as the gas in the flow. Both the heat blocking and heat sink effect also occur when instead of providing blowing through the wall, the wall temperature is allowed to become so high that the wall material vaporizes from the surface. The vaporized material enters the boundary layer and works just as a gas blown through the wall would. In general, there will be a layer of melted wall material inside the gas boundary layer. However, there are also materials, e.g., certain plastics, which directly pass from the solid state to the vapor state by depolymerization. Such processes, collectively known as ablation, play a great role in the reentry of space vehicles and the entry of meteorites into the earth's atmosphere, and have been studied in detail both theoretically and experimentally in the last decade.

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