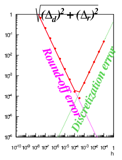
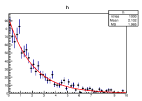
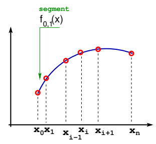
# Computational Physics

*numerical methods with C++ (and UNIX)*







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# Computational Physics
## Monte-Carlo methods

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# Monte-Carlo methods

- ✔ any method using random variables for a numerical calculation
  - ☞ we ask for a statistical answer!
- ✔ founding article:
  "The monte carlo method", N. Metropolis, S. Ulam (1949)
- ✔ applications: physics, engineering, finance, ...
- ✔ aims of the method:
  - ► generate samples of random variables ($\vec{X}$) according to a density probability distribution $p(\vec{X})$
  - ► estimate expectation values (<>) of variables or functions



# Statistical concepts

- ✔ the expected value of a variable $X$ sampled $N$ times $(X_1, X_2, \cdots, X_N)$

$$E(X) = \langle X \rangle = \frac{1}{N}\sum_{i=1}^{N} X_i$$

- ✔ the variance of the sample:

$$Var(X) \equiv \sigma_X^2 \simeq \frac{1}{N}\sum_{i=1}^{N}(X_i - \langle X \rangle)^2 = \left\langle (X - \langle X \rangle)^2 \right\rangle = \left\langle X^2 \right\rangle - \langle X \rangle^2$$

- ✔ the standard deviation of the sample:

$$\sigma_X \simeq \sqrt{\frac{1}{N}\sum_{i=1}^{N}(X_i - \langle X \rangle)^2}$$

# PDFs - prob density distributions

- PDFs: the probability density function $p(X)$ give us the probability of an event (a value $X_i$ in this case) to occur

$$\int_{-\infty}^{+\infty} p(X)\, dX = 1$$

- for a discrete variable $X$, its expectation value is given by:

$$\langle X \rangle = \frac{1}{N} \sum_{i=1}^{N} p(X_i)\, X_i$$

- for a continuous variable $X$ or function $f(X)$, the expectation value is given by:

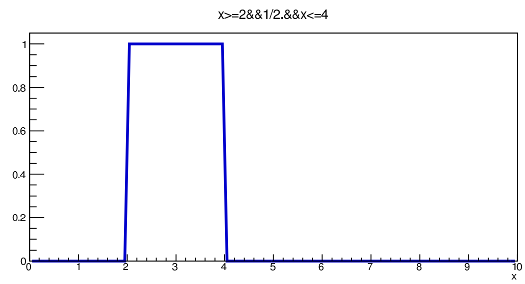
$$\langle X \rangle = \int_{-\infty}^{+\infty} p(X)\, X\, dX$$
$$\langle f \rangle = \int_{-\infty}^{+\infty} p(X)\, f(X)\, dX$$



# Important PDFs

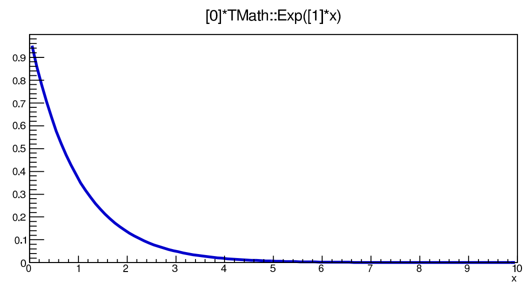
- uniform distribution: $X[a, b]$

$$p(X) = \frac{1}{b-a} H(X-a) H(b-X)$$



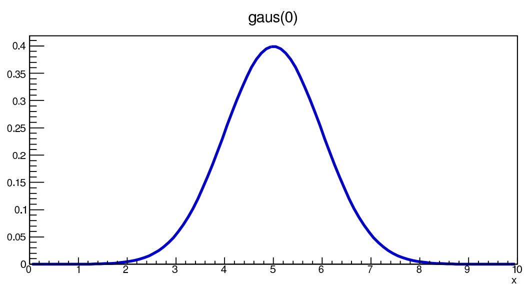

- exponential distribution: $X[0, \infty]$

$$p(X) = \alpha e^{-\alpha X}$$




- normal distribution: $X[-\infty, +\infty]$

$$p(X) = \frac{1}{\sqrt{2\pi}\sigma} \alpha e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$$



# Random numbers

✔ the most common uniform random number generators are based on Linear Congruential relations

$$N_i = (aN_{i-1} + c)\,\%m$$

☞ For example, with the parameters: a=6, c=7, m=5
and a seed: $N_0 = 2$
we get a **period 5** generator: 4, 1, 3, 0, 2, 4, 1, 3, 0, 2, ...

✔ a good uniform random number generator:

☞ produces a uniform distribution in the all the generation range

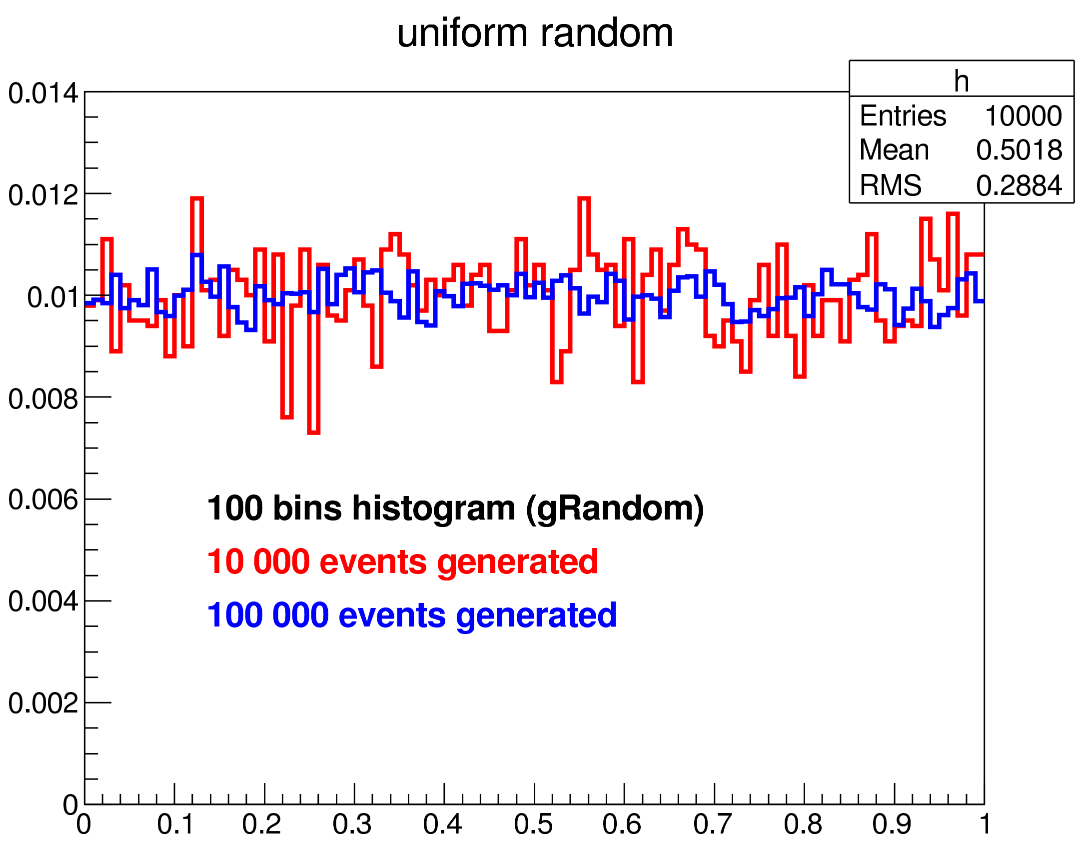
☞ shows no correlations between random numbers

☞ the period of sequence repetition is as large as possible

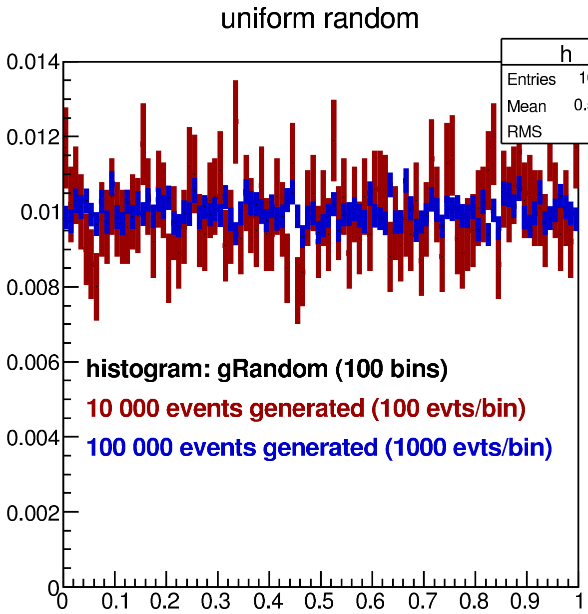
☞ the generation algorithm shall be fast

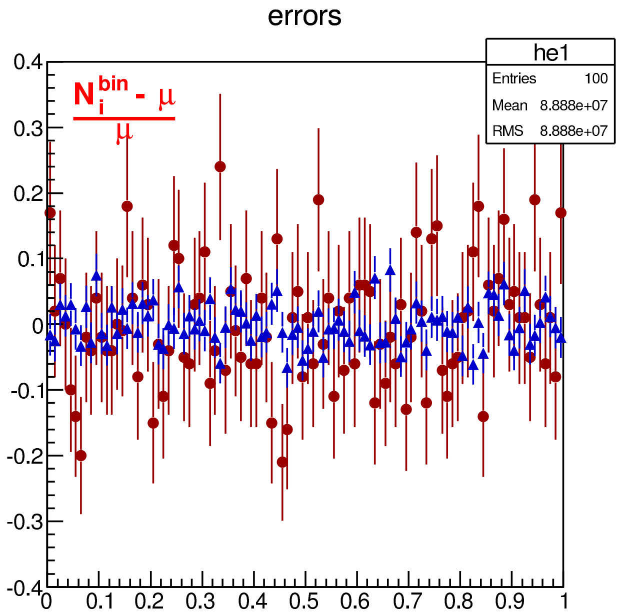


# Random numbers



# random numbers

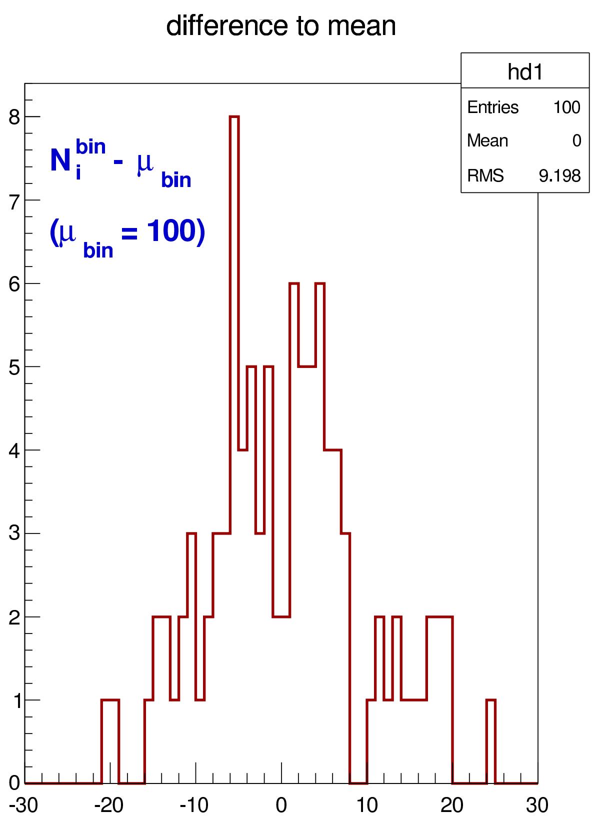



- ✔ The number of random numbers per bin fluctuates wrt to the expected nb of events per bin ($Ngen/Nbins = \frac{10000}{100} = 100$)
- ✔ The deviation of the number of events per bin wrt to the expected mean of events per bin ($\mu$) in percentage: $\frac{N_i^{bin} - \mu}{\mu}$ error: $\frac{\sigma_N}{\mu} \sim \frac{1}{\sqrt{N}} \sim \frac{1}{\sqrt{100}}$



# random numbers




The differences of every bin statistics to the expected mean ($\mu$) per bin

$N_i^{bin} - \mu$

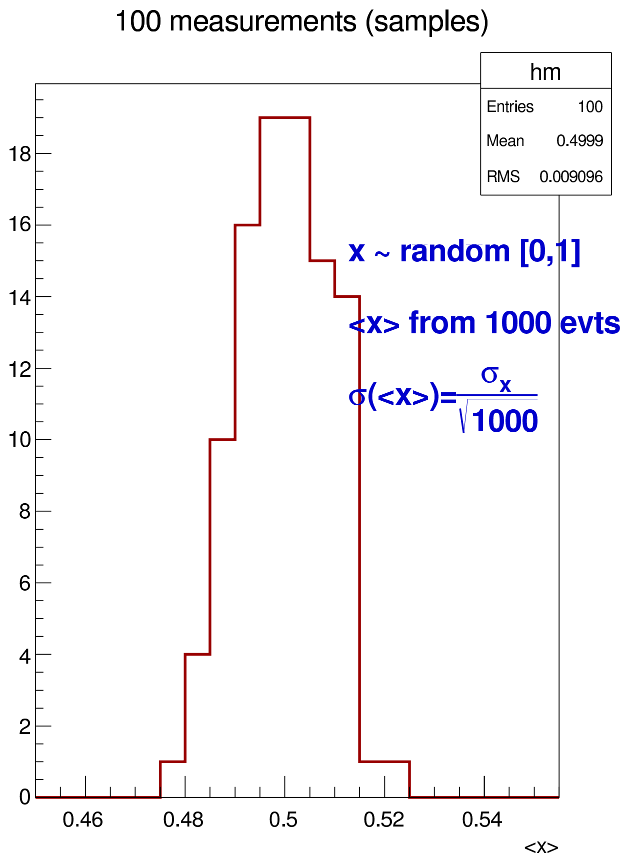
distributes according to a **normal (gaussian) distribution** with

mean: $\mu$

width: $\sigma \sim \sqrt{\mu}$

Consequence of the central limit theorem: the statistics accumulated in every bin

# random numbers




Suppose we have N=100 data samples (experiments) and in every sample we throw n=1000 random numbers (measurement)

The mean of every sample $< x >$ is distributed in the plot

$< x >\sim 0.5 \qquad \sigma(< x >) = \frac{\sigma_x}{\sqrt{n}} \sim \frac{0.3}{33} \sim 0.01$

The distribution is gaussian (central limit theorem) with a mean

$\mu = \frac{<x_1>+<x_2>+\cdots+<x_N>}{N}$

and a width

$\sigma < x > \sim \frac{\sigma_x}{\sqrt{n}} \sim 0.01$

the average of our 100 measurements is very precise

$\sigma_\mu = \frac{\sigma(<x>)}{\sqrt{100}} \sim \frac{0.01}{10} = 0.001$

This is equivalent to have 100*1000 measurements!