Lecture 4: Relations and Digraphs

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At the Last Class

1. Basics of Combinatorics
   - Permutations
   - Combinations
   - Pigeonhole principles
2. Some Techniques for Analysis
   - Elements of probability
   - Recurrence relations

Overview

1. Relations and Digraphs
   - Product sets and partitions
   - Binary relations and their digraphic form
   - Paths in relations
   - Representing relations
   - Properties of relations
2. Equivalence Relation
   - Equivalence relations and partitions
   - Equivalence relations and equivalence classes

为“关系”建立数学模型

可以将“大学在籍”看成某个个人与某个大学之间的关系。我们能够如何描述这个关系呢?

Ordered Pair and Cartesian Product

Cartesian Product

For any sets A, B

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

is called the **Cartesian Product** of A and B.

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Example

$\{1, 2, 3\} \times \{a, b\} = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$

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$\{1, 2, 3\} \times \{a, b\} = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$

For finite A, B, $|A \times B| = |A| \times |B|$

Generalized Cartesian Product

Cartesian product of m nonempty sets:

$$A_1 \times A_2 \times \cdots A_m = \{(a_1, a_2, \ldots, a_m) | a_i \in A_i, i = 1, 2..., m\}$$

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Example (Describing the attributes of objects)

A computer program can be characterized by 3 attributes:

$$\begin{aligned} \text{Language} &= \{\text{C}(c), \text{Java}(j), \text{Fortran}(f), \text{Pascal}(p), \text{Lisp}(l)\} \\ \text{Memory} &= \{2 \text{ meg}(2), 4 \text{ meg}(4), 8 \text{ meg}(8)\} \\ \text{OS} &= \{\text{UNIX}(u), \text{Windows}(w), \text{Linux}(l)\} \end{aligned}$$

Then, any object in Language $\times$ Memory $\times$ OS can be assigned to a specific program to characterized it.

Properties of Cartesian Product

- $A \times \emptyset = \emptyset \times A = \emptyset$
- $A \times B = B \times A \Leftrightarrow A = B \vee A = \emptyset \vee B = \emptyset$
  Proof:

  $\Leftarrow$ Note that for any set S, $S \times \emptyset = \{(x, y) | x \in S, y \in \emptyset\}$, since no such y exists, so $A \times \emptyset = \emptyset$, and $\emptyset \times S = \emptyset$ as well.

  $\Rightarrow$ If $A \neq B$ and $A \neq \emptyset$, we can prove that $B = \emptyset$ by contradiction.
  Assume that $B \neq \emptyset$, since $A \neq B$, let $a \in A$, but $a \notin B$; let b be any element in B (may be in A or not), then $(a, b) \in A \times B$, but $(a, b) \notin B \times A$, contradiction.

Properties of Cartesian Product

For any set A, B and C

$$\begin{aligned} A \times (B \cup C) &= \{(x, y) | x \in A, y \in B \text{ or } y \in C\} \\ &= \{(x, y) | x \in A, y \in B \text{ or } x \in A, y \in C\} \\ &= \{(x, y) | (x, y) \in A \times B \text{ or } (x, y) \in A \times C\} \\ &= (A \times B) \cup (A \times C) \end{aligned}$$

Easy to see:

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Relation as a Set

Let A and B be nonempty sets. A **relation** R **from** A **to** B is a subset of $A \times B$.
If $a \in A, b \in B$, then "a is related to b by R" is written as:

$$(a, b) \in R, \quad \text{or} \quad aRb$$

R is **a relation on** A, if $R \subseteq A \times A$

Example (Relations)

- $A = \{1,2,3\}$, $B = \{r,s\}$, $R = \{(1,r),(2,s),(3,r)\}$, then R is a relation from A to B.
- $A = \{1,2,3,4\}$, $R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$, then R is a relation on A, i.e., "not larger than".
- $\mathbb{N}$ is the set of all natural numbers (starting from 1), defining a relation R on $\mathbb{N}$, such that, for any $m,n \in \mathbb{N}$, $(m,n) \in R$ if and only if m divides n. So, $R \subseteq \mathbb{N} \times \mathbb{N}$, and R contains $(3,6),(5,25),(7,21)$, etc.

Special Binary Relations

- Empty relation on (any) set A.
  It is just a empty set.

- Universal relation on set A: $E_A = A \times A$

- Equality: $I_A = \{(x, x) | x \in A\}$

Domain and Range of Relations

Let $R \subseteq A \times B$, then

- The **domain** of R, $Dom(R)$ is defined as:

$$\{x | x \in A, \text{ and exists some } y \in B, \text{ such that } xRy\}$$

- The **range** of R, $Ran(R)$ is defined as:

$$\{y | y \in B, \text{ and exists some } x \in A, \text{ such that } xRy\}$$

Note: $Dom(R) \subseteq A$, and , $Ran(R) \subseteq B$.

R-relative Set

If R is a relation from set A to B

- For any $x \in A$, R-relative set of x, $R(x)$ is:

$$\{y | y \in B, xRy\} \qquad \text{(this is a subset of B)}$$

- For any $A_1 \subseteq A$, R-relative set of $R(A_1)$ is:

$$\{y | y \in B, \text{ there exists some } x \in A_1 \text{ such that } xRy\}$$

Note that: $R(A_1) = \bigcup_{x \in A_1} R(x)$

Properties of R-relative Sets

Let R be a relation from set A to B, A_1, A_2 be subsets of A, then

(a) $A_1 \subseteq A_2 \Rightarrow R(A_1) \subseteq R(A_2)$

(b) $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

(c) $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

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(c) $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

Proof of (c).

for any $y \in R(A_1 \cap A_2)$, there exists some $x \in A_1 \cap A_2$ such that xRy. So, $x \in A_1 \wedge x \in A_2$. It follows that $y \in R(A_1) \wedge y \in R(A_2)$, thus $y \in R(A_1) \cap R(A_2)$. $\square$

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Equality doesn't hold. Counterexample: considering relation "$\leq$" on $\mathbb{Z}$, $A_1 = \{0, 1, 2\}$, $A_2 = \{9, 13\}$, $R(A_1)$ is the set of all nonnegative integers, and $R(A_2)$ is the set of integers not less than 9, so, $R(A_1) \cap R(A_2) = \{9, 10, 11, 12, \cdots\}$, but $A_1 \cap A_2 = \emptyset$, which results $R(A_1 \cap A_2) = \emptyset$.

Representing Relations as Matrices

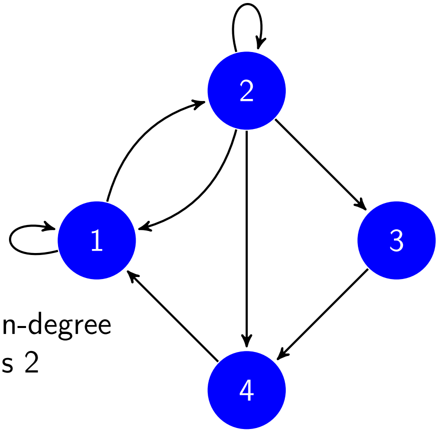
$$A = \{a_1, a_2, a_3\}, \qquad B = \{b_1, b_2, b_3, b_4\}$$
$$R = \{(a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3)\}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$(a_i, b_j) \in R$ *if and only if* $m_{i,j} = 1$

Representing Relations as Digraphs

Digraph representation is used only for relations on one set.
$A = \{1, 2, 3, 4\}$
$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4), (4,1)\}$



For node 1, the in-degree is 2, out-degree is 2

Path in Digraph

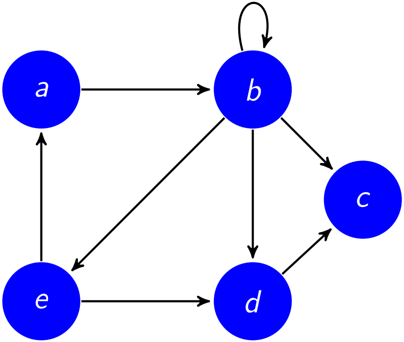
A path of length n in R from a to b is a finite sequence $\pi : a, x_1, x_2, \cdots, x_{n-1}, b$, such that:

$$aRx_1, x_{n-1}Rb, \text{ and } x_iRx_{i+1} \qquad \text{for } i = 1, \cdots n-2$$

A path in R corresponds to a succession of edges in the digraph representation of the relation, which consists of n edges.

It is not required that all elements in $a, x_1, x_2, \cdots, x_{n-1}, b$ are distinct.

New relations defined using Paths



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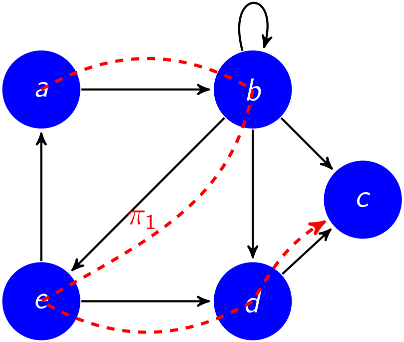
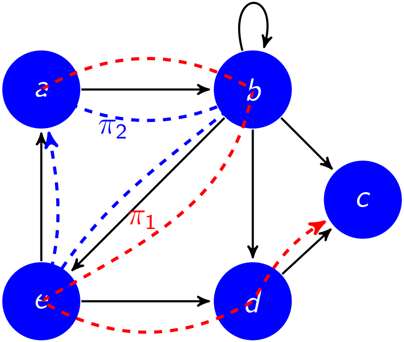


$\pi_1 : a, b, e, d, c$ length: 4

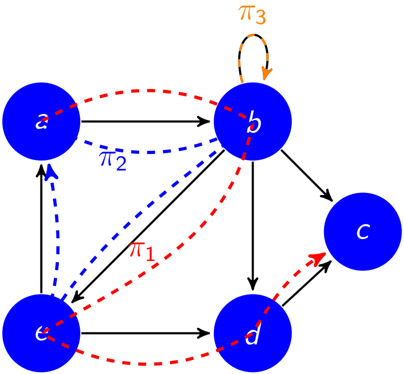
New relations defined using Paths



π_1 : a, b, e, d, c length: 4

π_2 : a, b, e, a length: 3 (cycle)

New relations defined using Paths

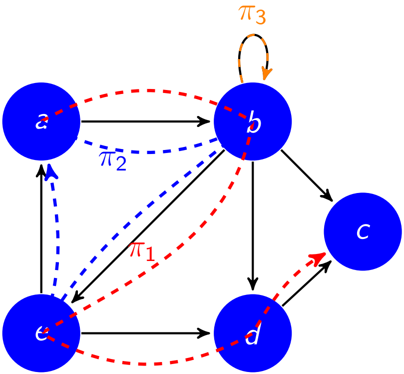


π_1 : a, b, e, d, c length: 4

π_2 : a, b, e, a length: 3 (cycle)

π_2 : b, b length: 1 (ring)

New relations defined using Paths



aR^4c
$aR^2d \quad (a, b, d)$
$aR^{k+2}d \quad (a, b, b, \cdots, d)$ $//k+1$ b's
bRb
bR^3b

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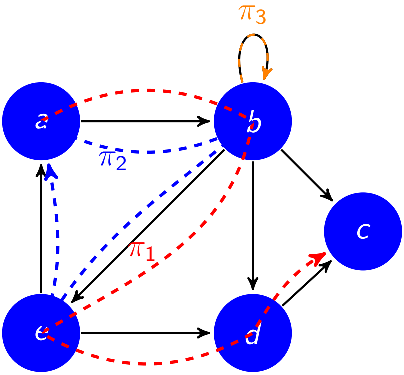

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aR^4c
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$aR^{k+2}d$ $(a, b, b, \cdots, d)$ $//k+1$ b's
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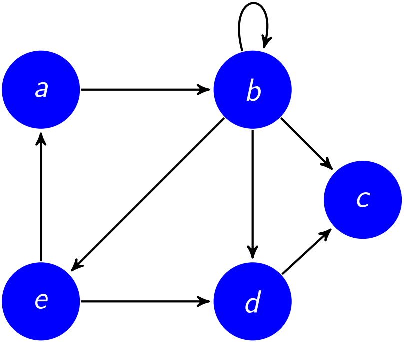
Generalized(connectivity)

$xR^{\infty}y$ if there is a path of any length from x to y.

New relations defined using Paths



Digraph of R

$R^2 = \{(a,b),(a,c),(a,d),(a,e),(b,a),(b,b),(b,c),(b,d),(b,e),(e,b),(e,c)\}$

$R^3 = \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,a),(b,b),(b,c),(b,d),(b,e),(e,b),(e,e)\}$

$R^\infty = \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,a),(b,b),(b,c),(b,d),(b,e),(d,c),(e,a),(e,b),(e,c),(e,d),(e,e)\}$

R^2 by Matrix Multiplication

If R is a relation on $A = \{a_1, a_2, \cdots, A_n\}$,
$M_{R^2} = M_R \odot M_R$

Proof.

- Let $M_R = [m_{ij}]$, and $M_{R^2} = [n_{ij}]$.
- Let $M^* = \left[m^*_{ij}\right] = M_R \odot M_R$, then $m^*_{ij} = 1$ if and only if for some $k(1 \leq k \leq n)$, $m_{ik} = 1$ and $m_{kj} = 1$.
- By definition of relation matrix, $a_i R a_k$, $a_k R a_j$.
- Thus $a_i R^2 a_j$, and so $n_{ij} = 1$, which means that $m^*_{ij} = 1$ if and $n_{ij} = 1$.
- So, $M_R \odot M_R = M_{R^2}$.

$\square$

R^n by Matrix Multiplication

For $n \geq 2$, and R a relation on a finite set A, we have
$M_{R^n} = M_R \odot M_R \odot \cdots \odot M_R$ (n factors).

Proof.

Proof by induction:
Let $P(n)$ mean that the statement holds for an integer $n \geq 2$.
$P(2)$ has been proved.
Let $M_{R^{k+1}} = [x_{ij}]$, $M_{R^k} = [y_{ij}]$, and $M_R = [m_{ij}]$. Let the node next to the last a_j is a_s, then there is a path of length k from a_i to a_s, and an edge from a_s to a_j. So $y_{is} = 1, m_{sj} = 1$, so $M_{R^k} \odot M_R[i,j] = 1$. On the other hand, if $M_{R^k} \odot M_R[i,j] = 1$, we have $x_{ij} = 1$. So $M_{R^{k+1}} = M_{R^k} \odot M_R$, by inductive hypothesis, $P(k+1)$. $\square$

Connectivity Relation

Connectivity relation R^∞ on some set A is defined as:

$\forall x, y \in A$, $(x, y) \in R^\infty \iff$ there is some path in R from x to y.

Note: $R^\infty = R \cup R^2 \cup R^3 \cup \cdots = \bigcup_{n=1}^{\infty} R^n$

So,

$$\begin{aligned} M_{R^\infty} &= M_R \vee M_{R^2} \vee M_{R^3} \vee \cdots \\ &= M_R \vee (M_R)^2_\odot \vee (M_R)^3_\odot \vee \cdots \end{aligned}$$

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So,

$$\begin{aligned} M_{R^\infty} &= M_R \vee M_{R^2} \vee M_{R^3} \vee \cdots \\ &= M_R \vee (M_R)^2_\odot \vee (M_R)^3_\odot \vee \cdots \end{aligned}$$

if A_1 is a subset of A, what is $R^\infty(A_1)$?

Reflexivity

Relation R on A is

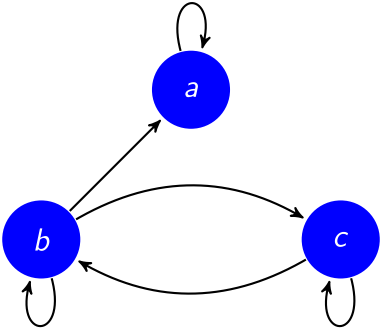
- **Reflexive** if for all $a \in A$, $(a, a) \in R$.
- **Irreflexive** if for all $a \in A$, $(a, a) \notin R$.

Let $A = \{1, 2, 3\}$, $R \subseteq A \times A$

- $\{(1,1), (1,3), (2,2),(2,1),(3,3)\}$ is reflexive
- $\{(1,2),(2,3),(3,1)\}$ is irreflexive
- $\{(1,2),(2,2),(2,3),(3,1)\}$ is neither reflexive nor irreflexive.

R is reflexive relation on A if and only if $I_A \subseteq R$.

Visualize reflexivity



$$A = \{a, b, c\}$$

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetry

Relation R on A is

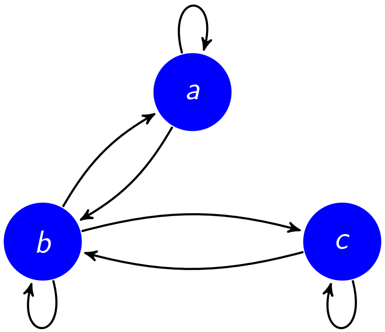
- **Symmetric** whenever $(a, b) \in R$, then $(b, a) \in R$.
- **Antisymmetric** if whenever $(a, b) \in R \wedge (b, a) \in R$ then $a = b$.
- **Asymmetric** if whenever $(a, b) \in R$ then $(b, a) \notin R$
  (Note: neither anti- nor a-symmetry is the negative of symmetry)

Let $A = \{1, 2, 3\}$, $R \subseteq A \times A$

- $\{(1,1),(1,2),(1,3),(2,1),(3,1),(3,3)\}$ is symmetric.
- $\{(1,2),(2,3),(2,2),(3,1)\}$ is antisymmetric.
- $\{(1,2),(2,3),(3,1)\}$ is antisymmetric and asymmetric.
- $\{(11),(2,2)\}$ is symmetric and antisymmetric.
- $\emptyset$ is symmetric and antisymmetric, and asymmetric!

R is symmetric relation on A if and only if $R^{-1} = R$

Visualized Symmetry



$$A = \{a, b, c\}$$

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Transitivity

Relation R on A is

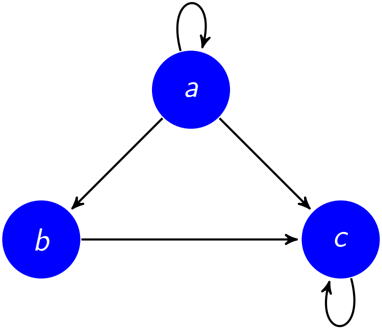
- **Transitive** whenever $(a, b) \in R$, $(b, c) \in R$ then $(a, c) \in R$.

Let $A = \{1, 2, 3\}$, $R \subseteq A \times A$

- $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 3)\}$ is transitive.
- $\{(1, 2), (2, 3), (3, 1)\}$ is not transitive.
- Both $\{(1, 3)\}$ and $\emptyset$ are transitive.

R is transitive relation on A if and only if $R^n \subseteq R$ for all $n \geq 1$.

Visualized Transitivity



$$A = \{a, b, c\}$$

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Some Often Used Relations

	$=$	$\leq$	$<$	$\mid$	$\equiv_3$	$\emptyset$	E
reflexivity	✔	✔	✘	✔	✔	✘	✔
irreflexivity	✘	✘	✔	✘	✘	✔	✘
symmetry	✔	✘	✘	✘	✔	✔	✔
antisymmetry	✔	✔	✔	✔	✘	✔	✘
transitivity	✔	✔	✔	✔	✔	✔	✔

What's Wrong?

A wrong proof: if R is a symmetric and transitive relation on A, then R must be reflexive.

Proof:

For any $a, b \in A$, if $(a, b) \in R$, by the symmetry of R, $(b, a) \in R$; since R is transitive, $(a, a) \in R$. So, R is reflexive.

Equivalence Relation

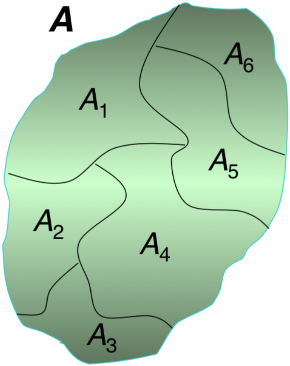
Relation R on A is an **equivalence relation** if and only if it is reflexive, symmetric and transitive.

"Equility" is a special case of equivalence relation.

An example:

- $R \subseteq \mathbb{Z} \times \mathbb{Z}$, $(x, y) \in R$ if and only if $\dfrac{|x - y|}{3} \in \mathbb{Z}$, i.e., $x \equiv_3 y$

Partition of a Set



A **partition** of a set A, π, is a set of the nonempty subsets of A, i.e., $\pi \subseteq P(A)$, satisfying:

1. For any $x \in A$, there is some $A_i \in \pi$, such that $x \in A_i$. That is,

$$\bigcup_i A_i = A$$

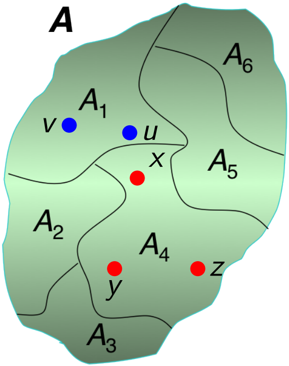
2. for any $A_i, A_j \in \pi$, if $i \neq j$, then

$$A_i \cap A_j = \emptyset$$

Partition Generated by Equivalence

- **Equivalence class**: Let R is a equivalence relation on A, then given $a \in A$, $R(a)$ is a equivalence class induced by R.
- **Quotient set**:
  $Q = \{R(x) | x \in A, \text{ and } R \text{ is a equivalence on } A\}$
- Quotient set is a partition:
  - For any $a \in A$, $a \in R(a)$ (remember that R is reflexible)
  - For any $a, b \in A$
    $(a, b) \in R$ if and only if $R(a) = R(b)$, and
    $(a, b) \notin R$ if and only if $R(a) \cap R(b) = \emptyset$

Equivalence Induced by Partition



Given a partition of A, we can define a relation R on A as following:

- $\forall x, y \in A, (x, y) \in R$ if and only if x, y belong to a same block.
- Ex. $(x, y) \in R$, $(y, z) \in R$, $(x, z) \in R$, $(x, x) \in R$, $(u, v) \in R$, $(u, x) \notin R$, etc.

It is straightforward to prove that R is reflexive, symmetric and transitive, so, it is an equivalence relation.

Product of Equivalence

R_1, R_2 are equivalences defined respectively on sets X_1 and X_2. Define relation S on $X_1 \times X_2$ as follows:

$$\langle x_1, x_2 \rangle S \langle y_1, y_2 \rangle \iff x_1 R_1 y_1 \land x_2 R_2 y_2$$

Then, S is also a equivalence, defined on $X_1 \times X_2$.

Reflexivity for any $\langle x, y \rangle \in X_1 \times X_2$, since both R_1, R_2 are reflexive, $\langle x, x \rangle \in R_1$, $\langle y, y \rangle \in R_2$; so, $\langle x, y \rangle S \langle x, y \rangle$;

Symmetry assume that $\langle x_1, x_2 \rangle S \langle y_1, y_2 \rangle$, which means that $x_1 R_1 y_1$ and $x_2 R_2 y_2$, so, $y_1 R_1 x_1$ and $y_2 R_2 x_2$, because of the symmetry of R_1 and R_2. So, $\langle y_1, y_2 \rangle S \langle x, x \rangle$;

Transitivity assume that $\langle x_1, x_2 \rangle S \langle y_1, y_2 \rangle$, and $\langle y_1, y_2 \rangle S \langle z_1, z_2 \rangle$, then $x_1 R_1 y_1$ and $y_1 R_1 z_1$, $x_2 R_2 y_2$ and $y_2 R_2 z_2$. Since both R_1 and R_2 are transitive, we have $x_1 R_1 z_1$ and $x_2 R_2 z_2$, so, $\langle x_1, x_2 \rangle S \langle z_1, z_2 \rangle$.

Example (An Example with Geometry)

For (x, y) and (u, v) in $\mathbb{R}^2$, define:

$$(x, y) \sim (u, v) \iff x^2 + y^2 = u^2 + v^2$$

Prove that $\sim$ defines an equivalence relation on $\mathbb{R}^2$ and interpret the equivalence classes geometrically.

Example (Another example, revisited)

Among any 1001 difference numbers randomly selected from the subset of natural numbers $\{1, 2, ..., 2000\}$ must be two, x, y, satisfying $\frac{x}{y} = 2^k$. (k is an integer)

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Proof.

Create 1000 sets, each contain a unique odd integer between 1 and 2000, along with its multiplication of 2^k not greater than 2000. Prove that the set of the 1000 sets is a partition of the set $\{1, 2, \cdots, 2000\}$.

Note that, for any positive integer, x, y between 1 and 2000, they belong to the same set iff. $\frac{x}{y} = 2^k$. (k is an integer).

Define a relation R on $\{1, 2, 3, ..., 2000\}$: for any x, y in the set, xRy if and only if $\frac{x}{y} = 2^k$. It is easy to prove that it is an equivalence, and the associated quotient set if the partition above. $\square$

等价关系用于计数

用英语单词“hello”中的 5 个字母可以造出多少个不同的“词”？

- 可以先假设两个“l”一个是大写，一个是小写，显然可以造出5!个“词”。在这些“词”的集合上定义关系 R, aRb 当且仅当忽略大小写，a,b 完全一样。可以证明这是等价关系，我们要求的结果恰是等价类的个数。

如果是用英语单词“aardvark”代替上述例子中的“hello”，结果是多少呢?

Home Assignments

To be checked

- Ex 4.1: 16; 18; 24; 30-31, 33-40
- Ex 4.2: 20; 25-26; 28, 32, 34; 36
- Ex 4.3: 18-21; 27-28; 30-33
- Ex 4.4: 14, 16, 18, 20, 22, 31-36; 38; 40
- Ex 4.5: 19-20, 22-24, 27-29

The End