Lecture 3: Counting

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At the Last Class

1. Propositional and Predicate Logic
   - Logical operations and truth tables
   - Quantifiers
   - Logic statement
2. Methods of Proof
   - Rules of inference
   - Indirect method of proof
   - Proof by contradiction
   - Disproving by counterexamples

Overview

1. Countable and Comparison
   - Countable Set
   - Comparing the size of infinite set
   - Infinite Sets – Larger and Smaller
   - Pigeonhole principles
2. Some Techniques for Analysis
   - Elements of probability
   - Recurrence relations

Countable Set

A set A is countable if and only if we can arrange all of its elements in a linear list in a definite order.

- "Definite" means that we can specify the first, second, third element, and so on.
- If the list ended and with the n^{th} element as its last element, it is finite.
- If the list goes on forever, it is infinite.

Proof of Countability

The set of all integers is countable.

- We can arrange all integer in a linear list as follows:

$$0, -1,\ 1, -2,\ 2, -3,\ 3, \cdots$$

that is , positive k is the $(2k+1)^{\text{th}}$ element, and negative k is the $(2k)^{\text{th}}$ element.

Set of Ordered Pairs

The set of all objects with the form $\langle i, j\rangle$ is countable, where i, j are nonnegative integers.

$$
\begin{array}{cccccc}
\langle 0,0\rangle & \langle 1,0\rangle & \langle 2,0\rangle & \langle 3,0\rangle & \langle 4,0\rangle & \cdots \\
\langle 0,1\rangle & \langle 1,1\rangle & \langle 2,1\rangle & \langle 3,1\rangle & \cdots & \\
\langle 0,2\rangle & \langle 1,2\rangle & \langle 2,2\rangle & \cdots & & \\
\langle 0,3\rangle & \langle 1,3\rangle & \cdots & & & \\
\langle 0,4\rangle & \cdots & & & &
\end{array}
$$

Set of Ordered Pairs

The set of all objects with the form $\langle i, j \rangle$ is countable, where i, j are nonnegative integers.

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\langle 0,3\rangle & \langle 1,3\rangle & \cdots & & & \\
\langle 0,4\rangle & \cdots & & & &
\end{array}$$

The order: $\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 0,2\rangle, \langle 1,1\rangle, \langle 2,0\rangle, \langle 0,3\rangle, \cdots$

$$I(m,n) = \sum_{i=1}^{m+n} i + (m+1) = \frac{(m+n)(m+n+1)}{2} + (m+1)$$

Real Number Set Is Not Countable

$(0, 1)$ is not a countable set.

Proof.

"diagonal proof"

Assuming that elements in $(0, 1)$ can be listed linearly

$0.b_{11} b_{12} b_{13} b_{14} \cdots$
$0.b_{21} b_{22} b_{23} b_{24} \cdots$
$0.b_{31} b_{32} b_{33} b_{34} \cdots$
$0.b_{41} b_{42} b_{43} b_{44} \cdots$
$\vdots$

then $0.b_1 b_2 b_3 b_4 \cdots b_i \neq b_{ii}$ can't be in the above list.

So, it is impossible to arrange all real number in a linear list. □

Finite and Infinite

Full?

No Problem!

I'll have the guest in Room No.1 moved to No.2, . . . , and the guest in No.k moved to No.k+1, . . . , and you can stay in Room No.1.

Done!

Finite and Infinite

BigFoot, he had many sons, but he never counted beyond 3.

Cantor, he had many "numbers", however, he didn't know anything for which he had to use more than 3.

Proving by Counting

Black? White?
What's the next?

Pigeonhole Principle

If n pigeons are assigned to m pigeonholes, and $m < n$, then at least one pigeonhole contains two or more pigeons.

Proof.

Proof by contradiction:
Suppose each pigeonhole contains at most 1 pigeon. Then at most m pigeons have been assigned. Since $m < n$, so $n - m > 0$, there are $(n - m)$ pigeons have not been assigned. Its a contradiction. $\square$

Pigeonhole by Odd Factor

Problem: show that if any 11 numbers are chosen from the set $\{1, 2, \cdots, 20\}$, then one of them will be a multiple of another.

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Solution:
- Observation: every natural number n can be represented as $2^k m$, where m is the largest odd factor of n, k is a nonnegative integer.
- Let each odd number in $\{1, 2, \cdots, 20\}$ correspond to a pigeonhole, then there are 10.
- Each element in $\{1, 2, \cdots, 20\}$ corresponds to a pigeon, and there are 20.
- If n_1, n_2 are in one pigeonhole, then one of them must be the multiple of another.

Shaking Hands at a Gathering

Situation: at a gathering of n people, everyone shook hands with at least one person, and no one shook hand more than once with the same person.

Problem: show that there must have been at least two of them who had the same number of handshaking.

Solution:

Pigeon: the n participants.
Pigeonhole: different number between 1 and $n-1$.

Extended Pigeonhole Principle

If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

Proof.

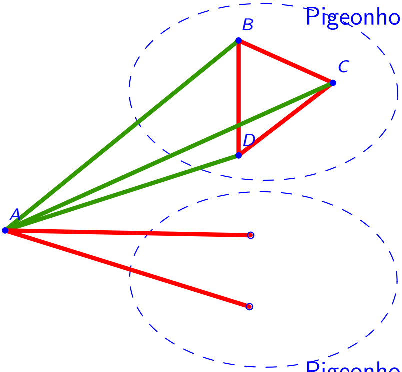
Proof by contradiction:

If each pigeonhole contains no more than $\lfloor (n-1)/m \rfloor$, then there are at most $m\lfloor \dfrac{n-1}{m} \rfloor \leq n-1$ pigeons at all.

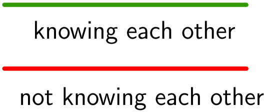
Its a contradiction. $\square$

Knowing Each Other or Not

Problem: show that among any 6 persons, there are 3 who know each other, or there are 3 who don't know any two others.



There must be at least 3 elements which fall into one of the two pigeonhole.



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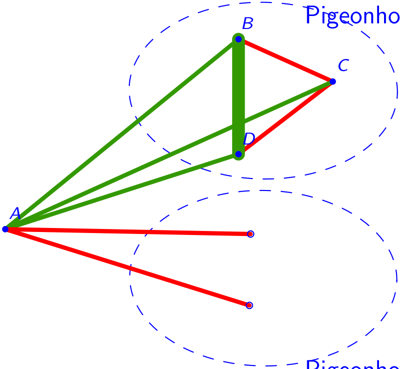

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knowing each other

not knowing each other

Hidden Pigeons and Invisible Pigeonholes

Scheduling the Practice Games

Situation: A chess player wants to prepare for a championship match by playing some practice games in 77 days. She wants to play at least one game a day but no more than 132 games altogether.

Problem: show that no matter how she schedules the games there is a period of consecutive days within which she plays exactly 21 games.

Scheduling the Practice Games: Solution

Let a_i denote the total number of games she plays up through the i^{th} day. Then, $a_1, a_2, a_3, \cdots, a_{76}, a_{77}$ is a monotonically increasing sequence, with $a_1 \geq 1$, and $a_{77} \leq 132$.

Considering the sequence

$$a_1, a_2, a_3, \cdots, a_{76}, a_{77}, a_1 + 21, a_2 + 21, a_3 + 21, \cdots, a_{76} + 21, a_{77} + 21$$

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The least element in the sequence is 1, and the largest is 153. However, there are 154 elements in the sequence, so, there must be at least two elements having the same value.

Note that both the first and second half sequences are monotonically increasing, so, it is impossible for the two elements having the same value to be within one half sequence, that is, we have $a_i + 21 = a_j$ (which means the player plays 21 games during the day $i+1$, up through j).

Probabilistic Event

Experiment: throwing two dices

Probabilistic Event

Sample spaces

What you want to record

Number pattern:

$\{(i,j) | 1 \leq i,j \leq 6\}$

Sum of numbers:

$\{2, 3, 4, \cdots, 11, 12\}$

Two one's:

$\{yes, no\}$

11 different outcomes

An event, for example:

"no less than 8"

Experiment: throwing two dices

Probability of an Event

Probability of an event E is a number, denoted as $p(E)$, reflecting ones assessment of the likelihood that the event will occur.

If the event E has occurred n_E times after n trials of the underlying experiment, we call $f_E = \frac{n_E}{n}$ the **frequency of occurrence** of E in n trials.

If we believe that the fraction f_E will tend ever closer to a certain number as n becomes larger, $p(E)$ is the number.

$$p(E) = \lim_{n \to \infty} f_E = \lim_{n \to \infty} \frac{n_E}{n}$$

Axioms for a probability space

Some properties the assigned probability should satisfy: (let A be the sample space)

- P1: $0 \leq p(E) \leq 1$ for every event E in A.
- P2: $p(A) = 1$ and $p(\emptyset) = 0$
- P3: $p(E_1 \cup E_2 \cup \cdots \cup E_k) = p(E_1) + p(E_2) + \cdots + p(E_k)$ whenever the events are mutually exclusive.

For a given experiment and a specific sample space, if the probabilities are assigned to all events with P1-P3 satisfied, we get a probability space.

Finite Probability Space

- There are only finite outcomes.
- Each outcomes individually consists an elementary event. Eg., for one coin toss, there are two outcomes – head and tail. "Head" is an elementary event.
- The probability of an elementary event corresponds a specific outcome.
- If all outcomes are equally likely, then the probability of an event E can be computed as:

$$p(E) = \frac{|E|}{|A|} = \frac{\text{total number of outcomes in } E}{\text{total number of outcomes}}$$

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- So, the result is $126/216 = 7/12$

Principle of Inclusion and Exclusion

For a whole set of N elements, $A_1, A_2, \cdots, A_n$ are the corresponding subset of n different "properties". Then the number of elements which satisfies none of the n properties is:

$$N(\bar{A}_1 \bar{A}_2 \cdots \bar{A}_n) = N - S_1 + S_2 - S_3 + \cdots + (-1)^k S_k + \cdots + (-1)^n S_n$$

where $S_k = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$, $k = 1, 2, \cdots, n$.

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Example (The formulae for 4 subsets)

$$\begin{aligned}
N &- (|S_1| + |S_2| + |S_3| + |S_4|) \\
&+ (|S_1 \cap S_2| + |S_1 \cap S_3| + |S_1 \cap S_4| + |S_2 \cap S_3| + |S_2 \cap S_4| + |S_3 \cap S_4|) \\
&- (|S_1 \cap S_2 \cap S_3| + |S_1 \cap S_2 \cap S_4| + |S_1 \cap S_3 \cap S_4| + |S_2 \cap S_3 \cap S_4|) \\
&+ |S_1 \cap S_2 \cap S_3 \cap S_4|
\end{aligned}$$

Hatcheck Problem

- A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hat. When customers return for their hats, the checker gives them back at random from the remaining hats. What is the probability that no one receives the correct hat?

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- Mathematical model: arrange $1, 2, 3, \cdots, n$ randomly, resulting in a new sequence $i_1, i_2, i_3, \cdots, i_n$. What is the probability that for any $k (1 \le k \le n)$, $i_k \ne k$?

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- Mathematical model: arrange $1, 2, 3, \cdots, n$ randomly, resulting in a new sequence $i_1, i_2, i_3, \cdots, i_n$. What is the probability that for any $k (1 \leq k \leq n), i_k \neq k$?
- The resulting sequence is called a **derangement**.

Number of Derangement

Define $i_k = k$ as Property A_k, and A_k is used for the subset of all permutations satisfying property A_k.

Example (The number of derangement)

$$N(\bar{A}_1\bar{A}_2\cdots\bar{A}_n) = N - S_1 + S_2 - S_3 + \cdots + (-1)^k S_k + \cdots + (-1)^n S_n$$

where $N = n!$, $S_k = \sum_{1\le i_1\le i_2\le\cdots\le i_k\le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$.

Note: S_k is the number of permutations keeping exactly k elements in their original positions, and the other $n-k$ elements as any possible permutation. So:

$$S_1 = \binom{n}{1}(n-1)!; S_2 = \binom{n}{2}(n-2)!; \cdots; S_k = \binom{n}{k}(n-k)! = \frac{n!}{k!}$$

Example (The Probability of Derangement)

We have known that the number of derangement is

$$N(\bar{A}_1\bar{A}_2\cdots\bar{A}_n) = N - S_1 + S_2 - S_3 + \cdots + (-1)^k S_k + \cdots + (-1)^n S_n$$

where $N = n!$, $S_k = \binom{n}{k}(n-k)! = \frac{n!}{k!}(k = 1, 2, \cdots, n)$.

$\therefore N(\bar{A}_1\bar{A}_2\cdots\bar{A}_n) = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$; and the probability $y : \sum_{k=0}^{n}\frac{(-1)^k}{k!}$

Since $\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = e^{-1}$, the difference between the probability y and $e^{-1} = 0.367879\cdots$ is less than $\frac{1}{n!}$, which means the probability y is about 0.38, independent of n, except for very small n.

Average Behavior of an Algorithm

Sequential search a list of n items for K

- Assuming no same entries in the list, and K does occur in the list
- Look all inputs with K in the i^{th} location as one input (so, inputs totaling n)
- Each input occurs with equal probability (i.e. $1/n$)

$$
\begin{aligned}
A(n) &= \sum_{i=0}^{n-1} [p(K \text{ is at position } i) \cdot (i+1)] \\
&= \sum_{i=0}^{n-1} \left[\frac{1}{n} \cdot (i+1) \right] \\
&= \frac{n+1}{2}
\end{aligned}
$$

Probabilistic Paradox

- For a family of four children, is it most likely there are two boys and two girls?
  (It is assumed that each child has a equal chance to be male or female at his/her birth)

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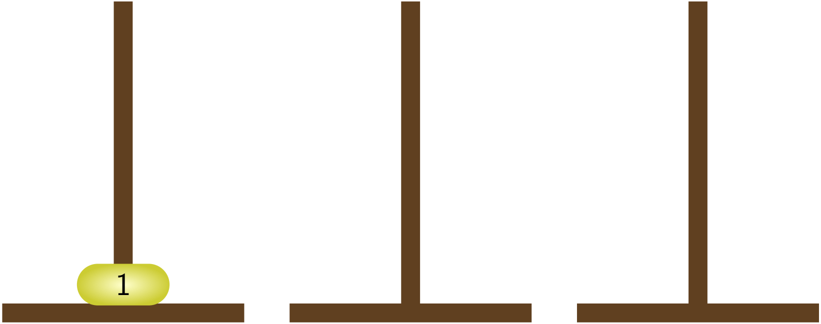
  (It is assumed that each child has a equal chance to be male or female at his/her birth)
- The probability of the event of 2-2 is $\frac{6}{16}$, and the probability of the event of 1-3 is $\frac{8}{16}$.

Tower of Hanoi – 1 Disc



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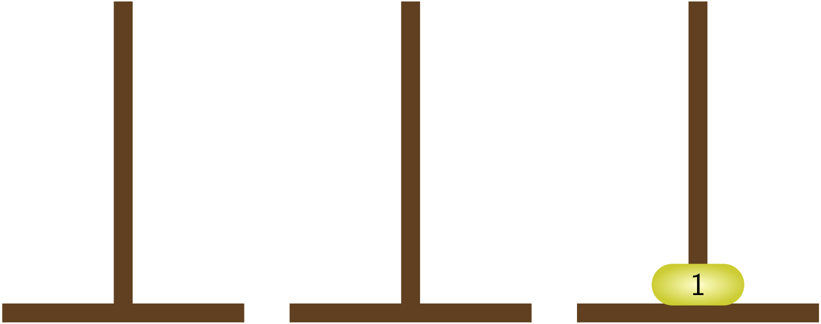
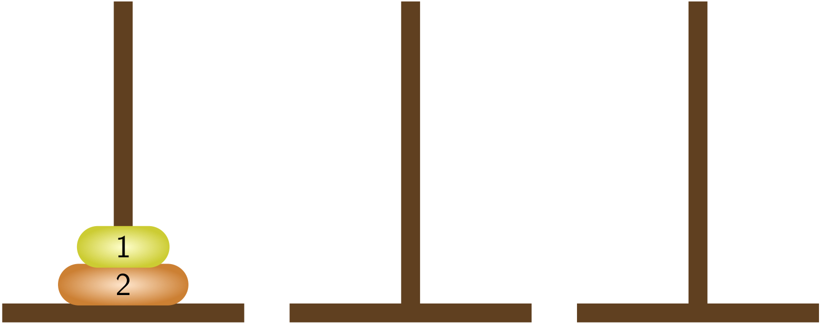


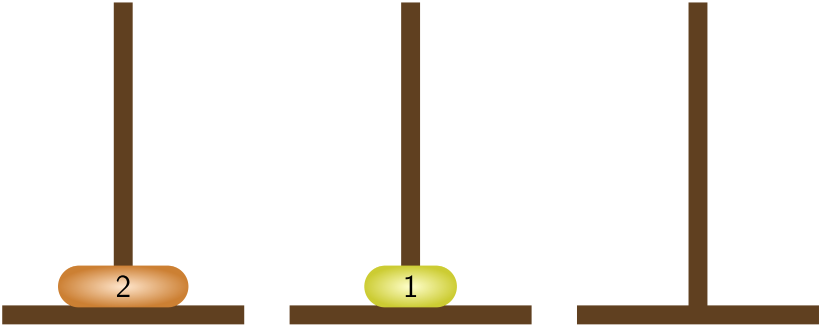
Moved disc from pole 1 to pole 3.

Tower of Hanoi – 1 Disc

Tower of Hanoi – 2 Discs

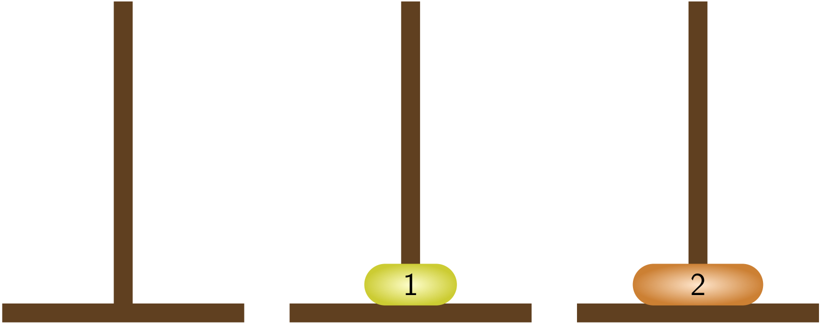


Tower of Hanoi – 2 Discs



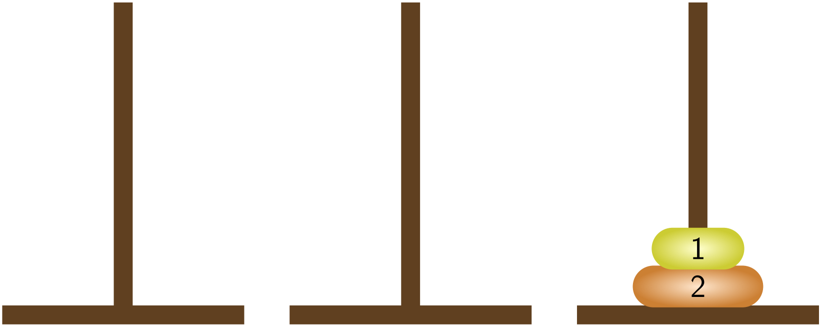
Moved disc from pole 1 to pole 2.

Tower of Hanoi – 2 Discs



Moved disc from pole 1 to pole 3.

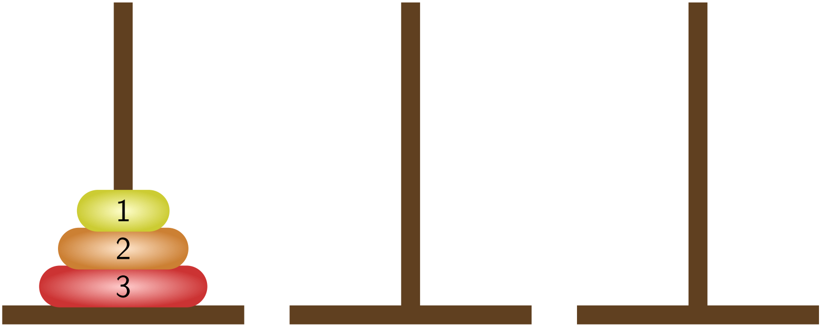
Tower of Hanoi – 2 Discs



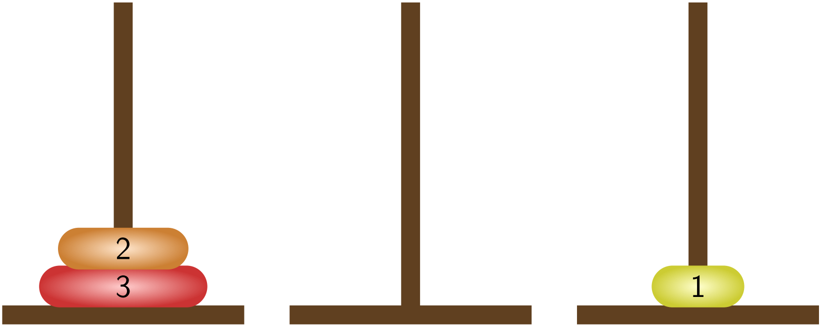
Moved disc from pole 2 to pole 3.

Tower of Hanoi – 2 Discs

Tower of Hanoi – 3 Discs

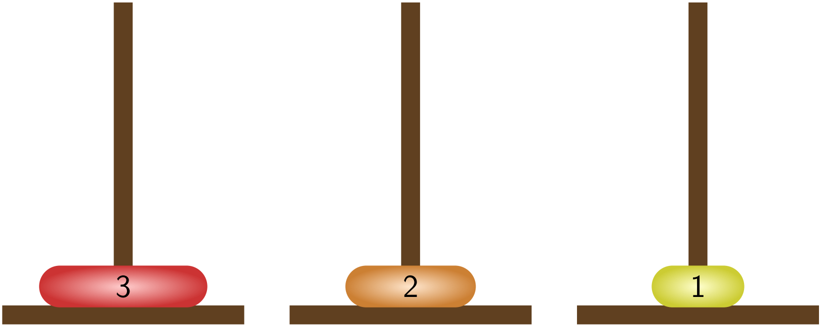


Tower of Hanoi – 3 Discs



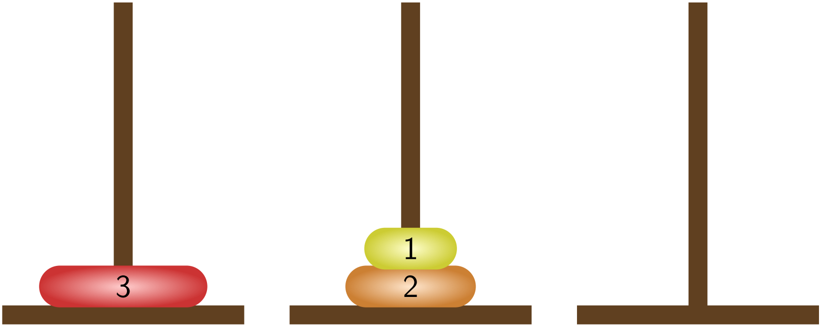
Moved disc from pole 1 to pole 3.

Tower of Hanoi – 3 Discs



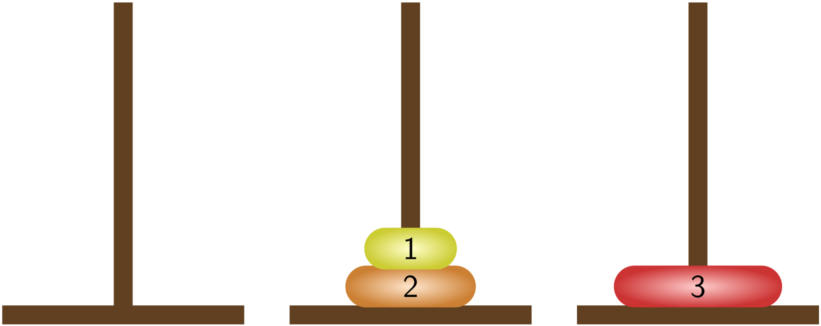
Moved disc from pole 1 to pole 2.

Tower of Hanoi – 3 Discs



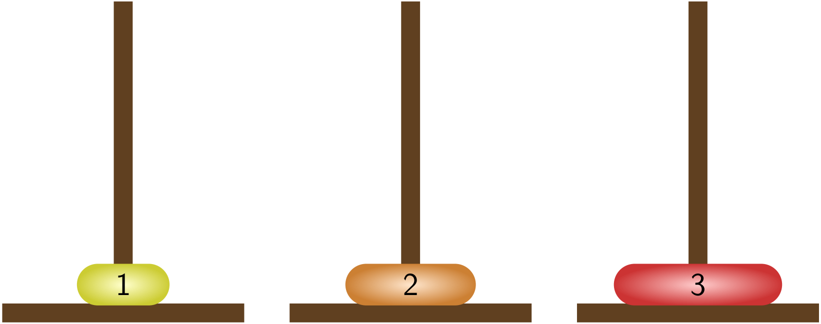
Moved disc from pole 3 to pole 2.

Tower of Hanoi – 3 Discs



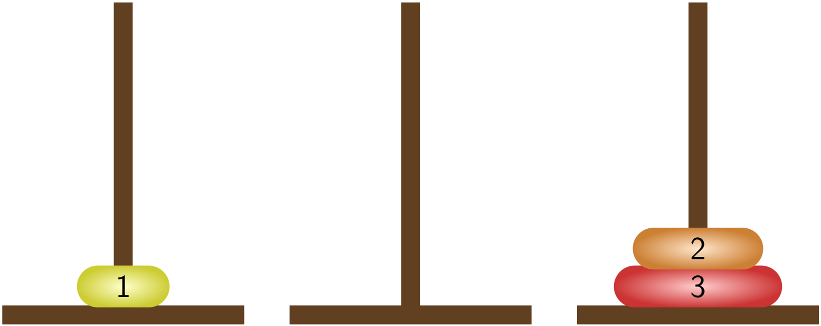
Moved disc from pole 1 to pole 3.

Tower of Hanoi – 3 Discs



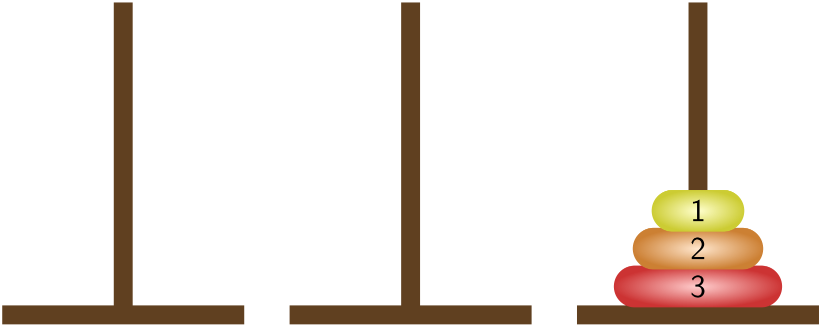
Moved disc from pole 2 to pole 1.

Tower of Hanoi – 3 Discs



Moved disc from pole 2 to pole 3.

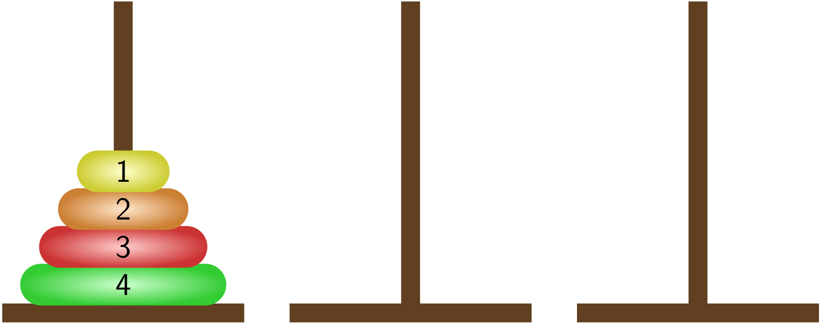
Tower of Hanoi – 3 Discs



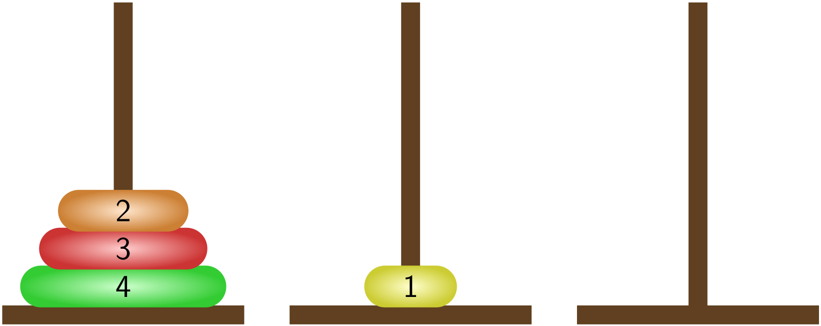
Moved disc from pole 1 to pole 3.

Tower of Hanoi – 3 Discs

Tower of Hanoi – 4 Discs

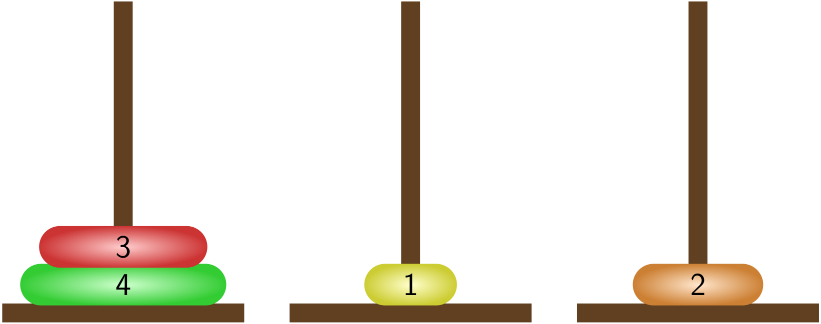


Tower of Hanoi – 4 Discs



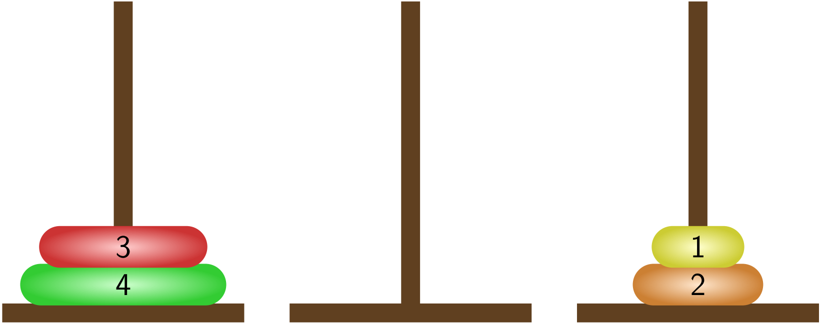
Moved disc from pole 1 to pole 2.

Tower of Hanoi – 4 Discs



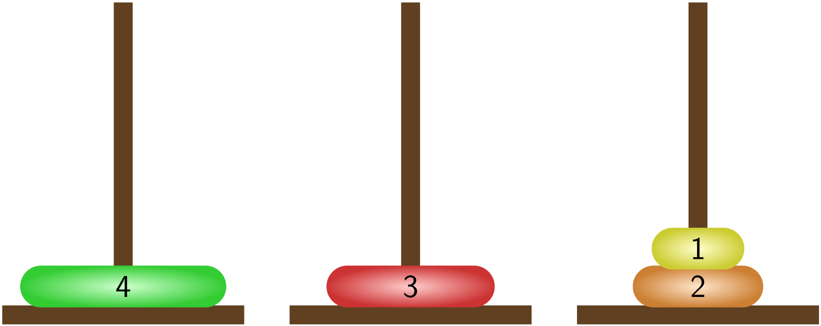
Moved disc from pole 1 to pole 3.

Tower of Hanoi – 4 Discs



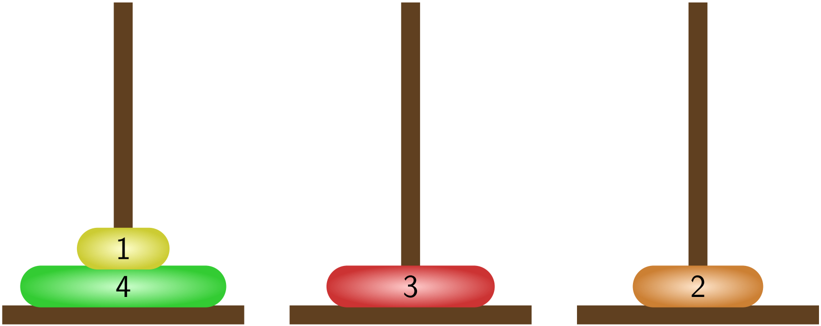
Moved disc from pole 2 to pole 3.

Tower of Hanoi – 4 Discs



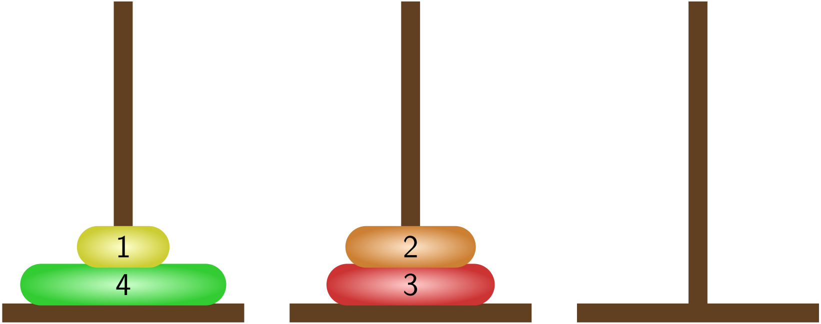
Moved disc from pole 1 to pole 2.

Tower of Hanoi – 4 Discs



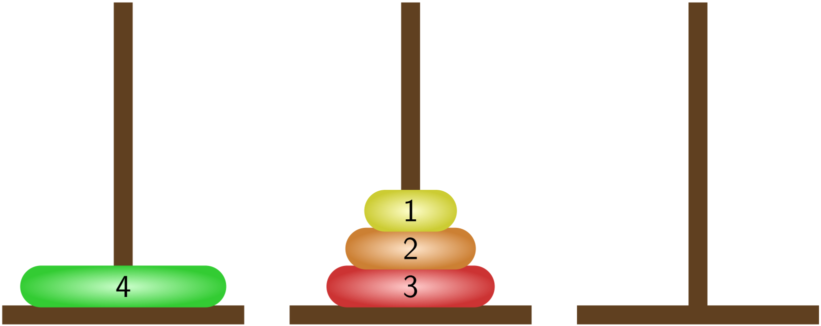
Moved disc from pole 3 to pole 1.

Tower of Hanoi – 4 Discs



Moved disc from pole 3 to pole 2.

Tower of Hanoi – 4 Discs



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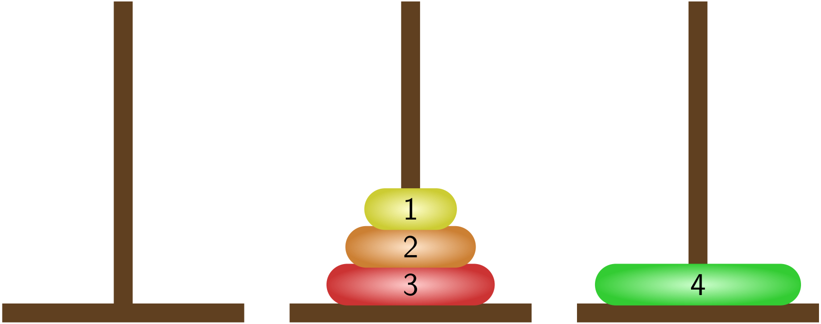

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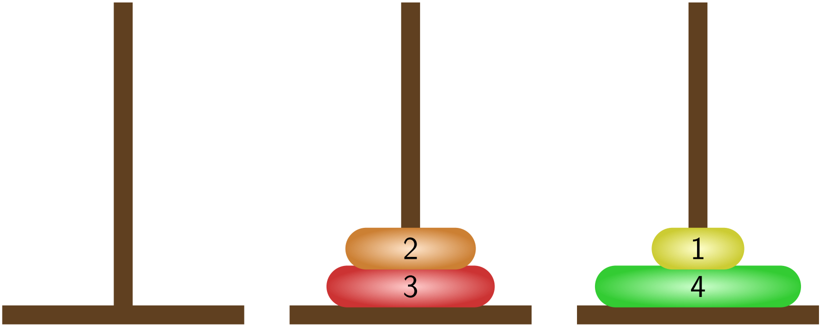
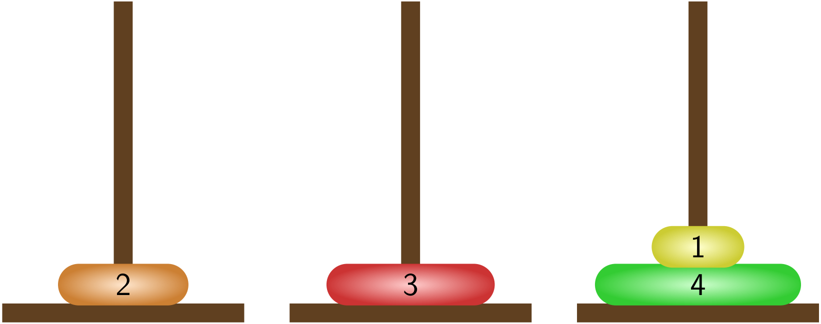
Tower of Hanoi – 4 Discs



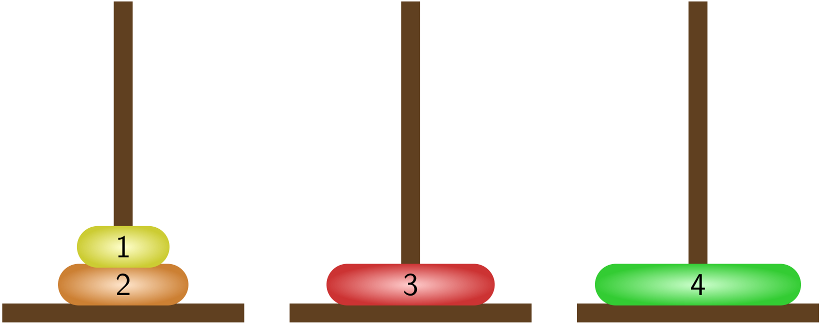
Moved disc from pole 2 to pole 3.

Tower of Hanoi – 4 Discs



Moved disc from pole 2 to pole 1.

Tower of Hanoi – 4 Discs



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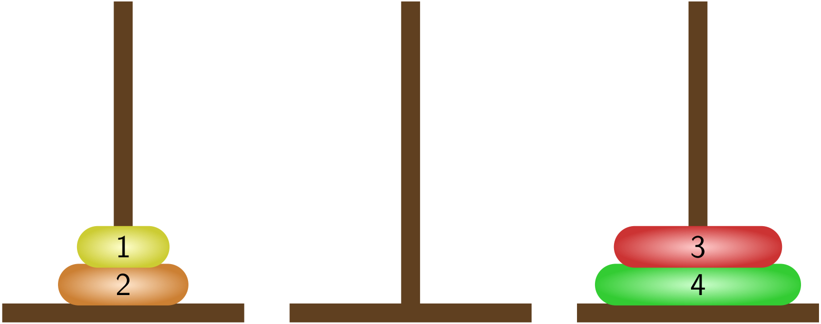
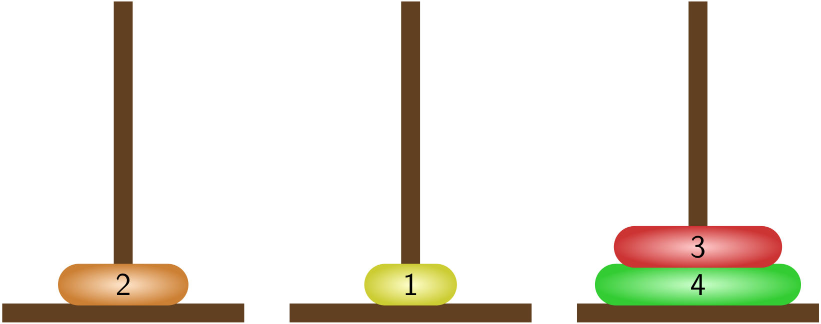


Moved disc from pole 2 to pole 3.

Tower of Hanoi – 4 Discs



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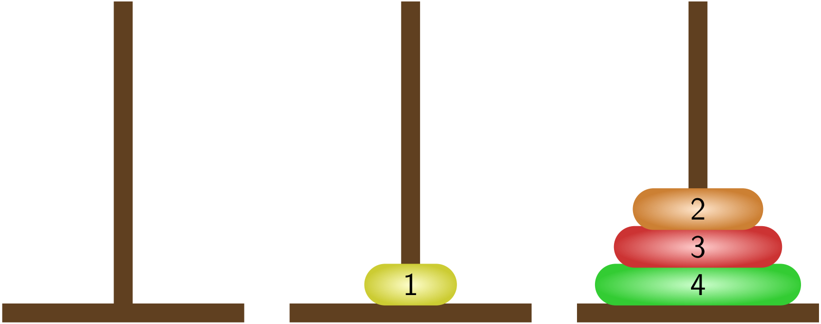
Tower of Hanoi – 4 Discs



Moved disc from pole 1 to pole 3.

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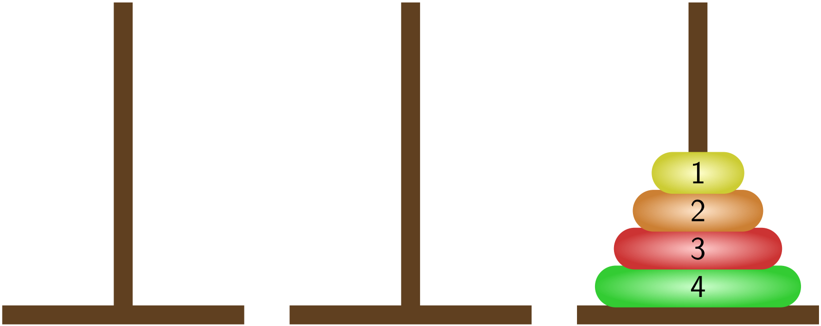
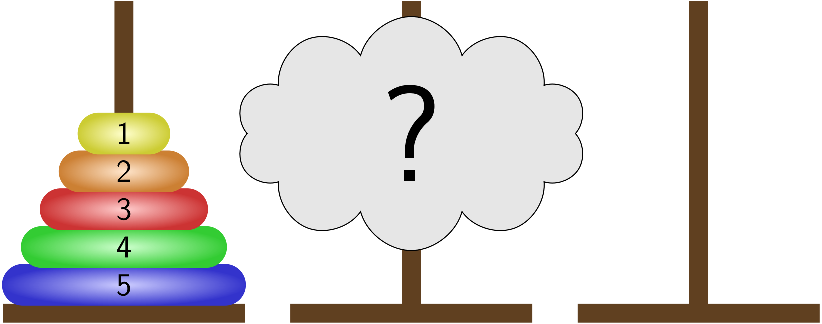


Moved disc from pole 2 to pole 3.

Tower of Hanoi – 4 Discs

Tower of Hanoi – 5 Disc



Thinking Recursively: Problem 1

Example (Tower of Hanoi)

How many moves are need to move all the disks to the third peg by moving only one at a time and never placing a disk on top of a smaller one.

$$T(1) = 1$$
$$T(n) = 2T(n-1) + 1$$

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$$\begin{aligned} T(1) &= 1 \\ T(n) &= 2T(n-1) + 1 \end{aligned}$$

$$\begin{aligned} 2T(n-1) &= 4T(n-2) + 2 \\ 4T(n-2) &= 8T(n-3) + 4 \\ &\vdots \\ 2^{n-2}T(2) &= 2^{n-1}T(1) + 2^{n-2} \end{aligned}$$

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&\vdots \\
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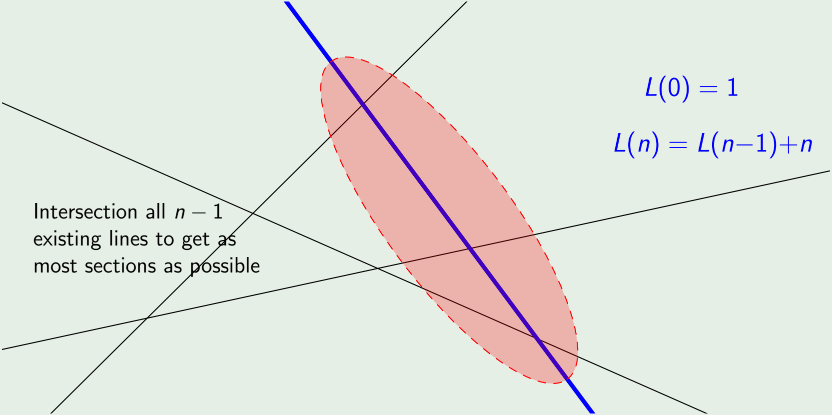
\end{aligned}
$$

$$T(n) = 2^n - 1$$

Thinking Recursively: Problem 2

Example (Cutting the plane)

How many sections can be generated at most by n straight lines with infinite length.



$$L(0) = 1$$

$$L(n) = L(n-1) + n$$

Example (Cutting the plane)

How many sections can be generated at most by n straight lines with infinite length.

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In
ex
m

$$\begin{aligned} L(n) &= L(n-1) + n \\ &= L(n-2) + (n-1) + n \\ &= \vdots \\ &= L(0) + 1 + 2 + \cdots + (n-1) + n \\ &= \frac{n(n+1)}{2} + 1 \end{aligned}$$

Example (Josephus Problem)

Live or die, it's a problem!

Legend has it that Josephus wouldn't have lived to become famous without his mathematical talents. During the Jewish Roman war, he was among a band of 41 Jewish rebels trapped in a cave by the Romans. Preferring suicide to capture, the rebels decided to form a circle and, proceeding around it, to kill every third remaining person until no one was left. But Josephus, along with an unindicted co-conspirator, wanted none of this suicide nonsense; so he quickly calculated where he and his friend should stand in the vicious circle.

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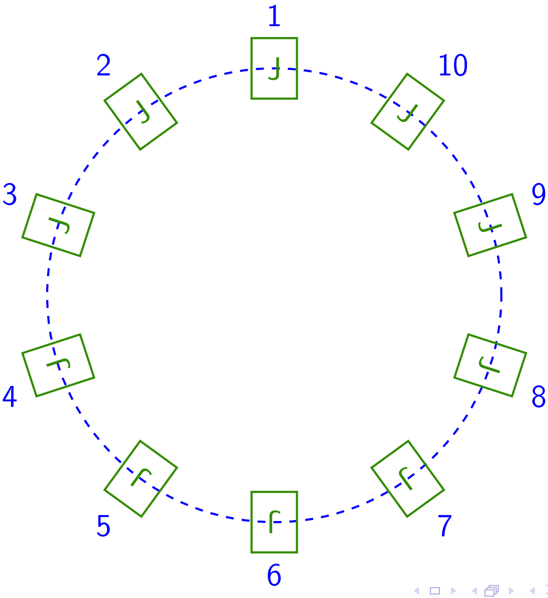
We use a simpler version: "every second ... "

Make a Try: for $n = 10$



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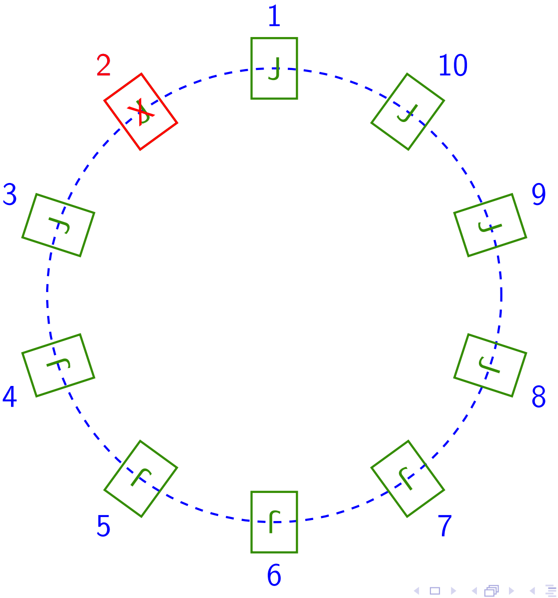

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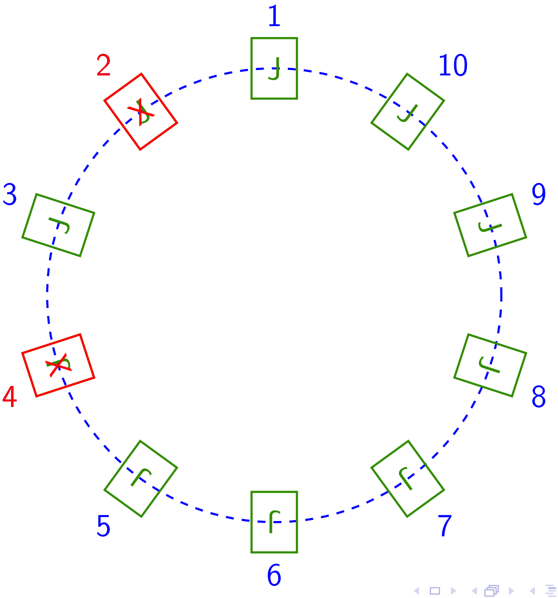

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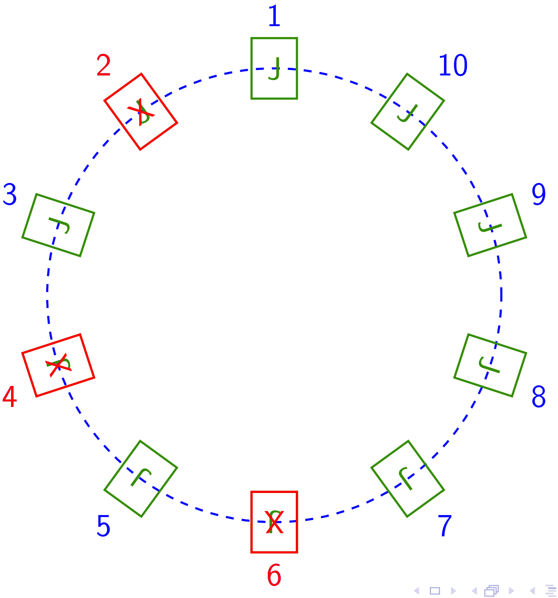

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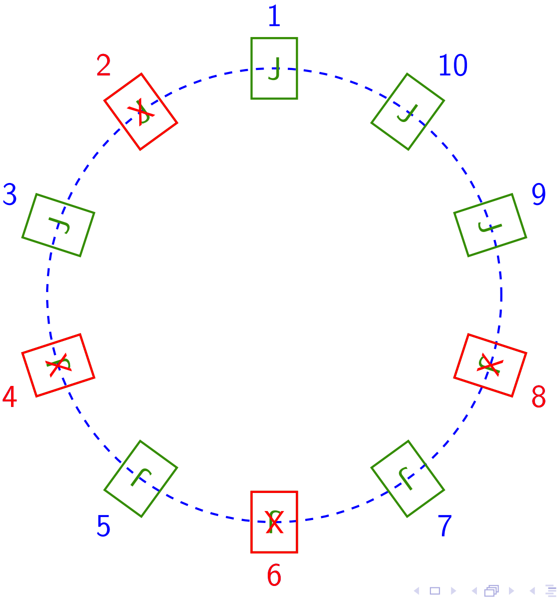

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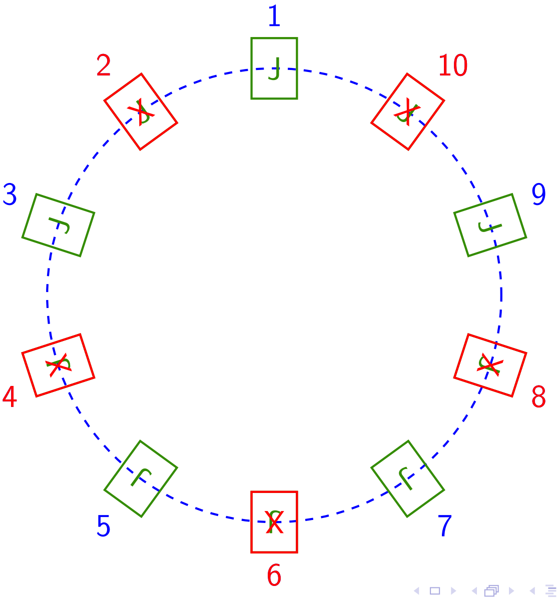

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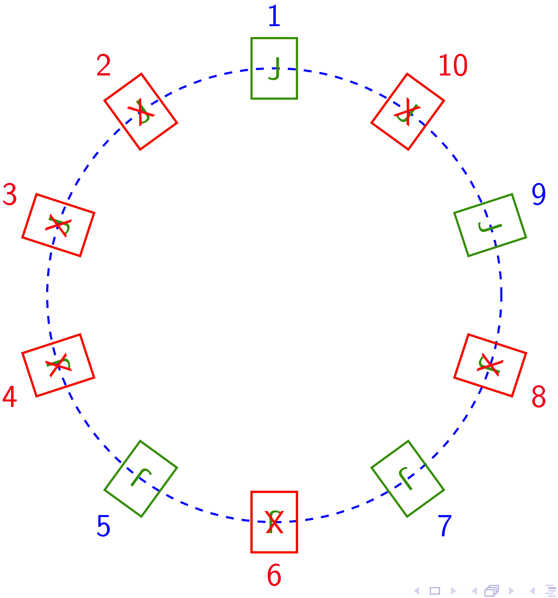

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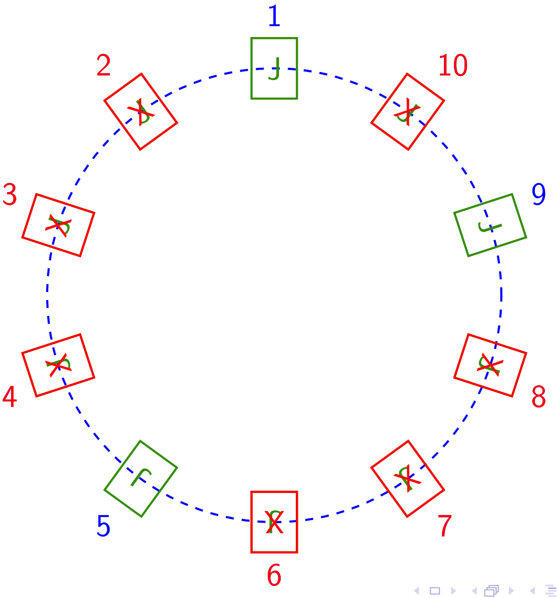

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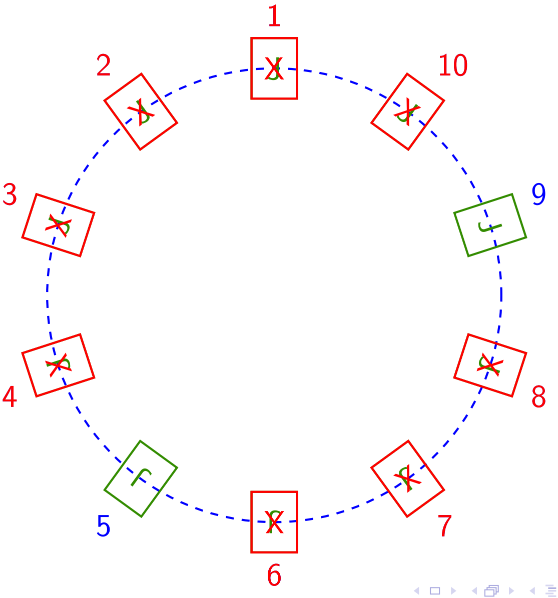

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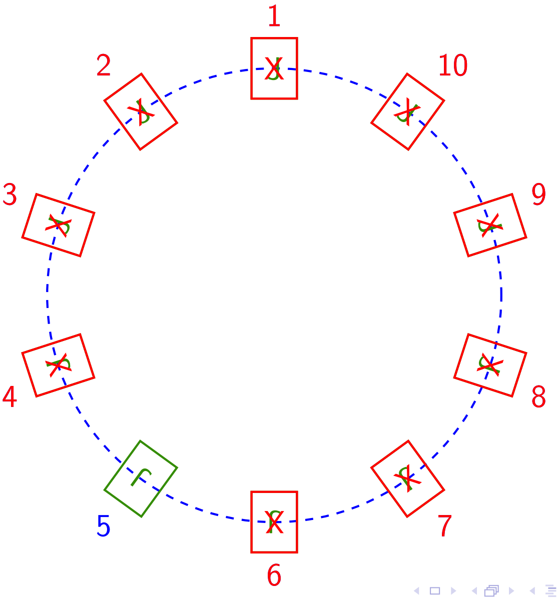

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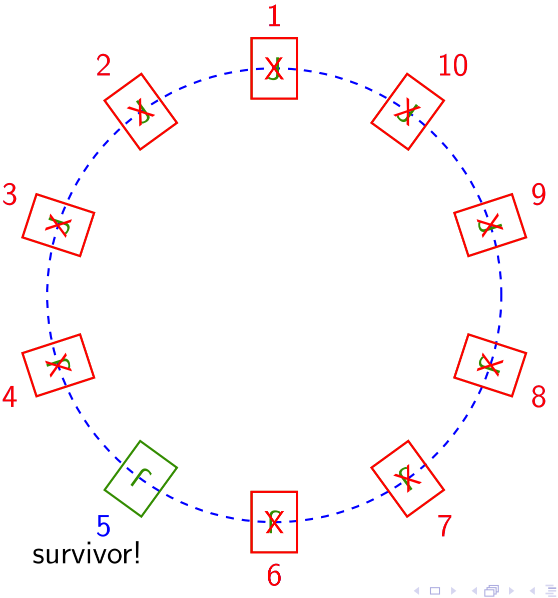

Solution in Recursive Equations

$$
\begin{aligned}
J(1) &= 1 \\
J(2n) &= 2J(n) - 1, && \text{for } n \geq 1; \\
J(2n+1) &= 2J(n) + 1, && \text{for } n \geq 1.
\end{aligned}
$$

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Explicit solution for small n's

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

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$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Look carefully, . . .
and, find the pattern, . . .
and, prove it!

Eureka!

If we write n in the form

$$n = 2^m + l$$

where 2^m is the largest power of 2 not exceeding n and where l is what's left,

the solution to our recurrence seems to be:

$$J(2^m + l) = 2l + 1, \text{ for } m \geq 0 \text{ and } 0 \leq l < 2^m$$

As an example, $J(100) = J(64 + 36) = 36 \times 2 + 1 = 73$.

Binary Representation

Suppose n's binary expansion is

$$n = (b_m b_{m-1} \cdots b_1 b_0)_2$$

then

$$\begin{aligned} n &= (1 b_{m-1} b_{m-2} \cdots b_1 b_0)_2 \\ l &= (0 b_{m-1} b_{m-2} \cdots b_1 b_0)_2 \\ 2l &= (b_{m-1} b_{m-2} \cdots b_1 b_0 0)_2 \\ 2l + 1 &= (b_{m-1} b_{m-2} \cdots b_1 b_0 1)_2 \\ J(n) &= (b_{m-1} b_{m-2} \cdots b_1 b_0 b_m)_2 \end{aligned}$$

Linear Homogeneous Relation

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

is called linear homogeneous relation of degree k.

$$c_n = (-2)c_{n-1}$$

$$f_n = f_{n-1} + f_{n-2}$$

$$a_n = a_{n-1} + \ 3$$

$$g_n = \ g_{n-1}^2 \ + g_{n-2}$$

yes!

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yes!

$$a_n = a_{n-1} + 3$$
$$g_n = g_{n-1}^2 + g_{n-2}$$

no!

Characteristic Equation

For a linear homogeneous recurrence relation of degree k

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

the polynomial of degree k

$$x^k = r_1 x^{k-1} + r_2 x^{k-2} + \cdots + r_k$$

is called its characteristic equation.

The characteristic equation of linear homogeneous recurrence relation of degree 2 is:

$$x^2 - r_1 x - r_2 = 0$$

Solution of Recurrence Relation

If the characteristic equation $x^2 - r_1 x - r_2 = 0$ of the recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ has two distinct roots s_1 and s_2, then

$$a_n = \mu s_1^n + \nu s_2^n$$

where μ and ν depend on the initial conditions, is the explicit formula for the sequence.

If the equation has a single root s, then, both s_1 and s_2 in the formula above are replaced by s.

Proof of the solution

Remember the equation $x^2 - r_1 x - r_2 = 0$, we need to prove that $\mu s_1^n + \nu s_2^n = r_1 a_{n-1} + r_2 a_{n-2}$.

$$
\begin{aligned}
\mu s_1^n + \nu s_2^n &= \mu s_1^{n-2} s_1^2 + \nu s_2^{n-2} s_2^2 \\
&= \mu s_1^{n-2}(r_1 s_1 + r_2) + \nu s_2^{n-2}(r_1 s_2 + r_2) \\
&= r_1 \mu s_1^{n-1} + r_2 \mu s_1^{n-2} + r_1 \nu s_2^{n-1} + r_2 \nu s_2^{n-2} \\
&= r_1(\mu s_1^{n-1} + \nu s_2^{n-1}) + r_2(\mu s_1^{n-2} + \nu s_2^{n-2}) \\
&= r_1 a_{n-1} + r_2 a_{n-2}
\end{aligned}
$$

Fibonacci Sequence

$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$, that is,

$$1, 1, 2, 3, 5, 8, 13, 21, \cdots$$

Explicit formula for Fibonacci Sequence:

- the characteristic equation is $x^2 - x - 1 = 0$, which has roots:

$$s_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad s_2 = \frac{1-\sqrt{5}}{2}$$

- by initial conditions, $f_1 = \mu s_1 + \nu s_2 = 1$, $f_2 = \mu s_1^2 + \nu s_2^2 = 1$, which results

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Home Assignments

To be checked

1.3 Ex. : 23-24

3.1 Ex. : 25-26, 29, 34

3.2 Ex. : 19, 23, 27, 32

3.3 Ex. : 10, 12, 17-19, 21-24

3.4 Ex. : 34, 37-41

3.5 Ex. : 14, 18, 26, 28, 34, 36

The End