Partial Order and Lattices

Lecture 7

Discrete Mathematical Structures

Partial Order and Lattices

- Part I: Partial Order
  - Order relations and Hasse Diagrams
  - Extremal elements in partially ordered sets
- Part II: Lattices
  - Lattices as a mathematical structure
  - Isomorphic lattices
  - Properties of lattices

Partial Order

- Reflexive, anti-symmetric and transitive
  - Generalization of "less than or equal to"
- Denotation: $\preccurlyeq$

- Example 1: set containment
  - Note: not any two of sets are "comparable"
- Example 2: divisibility on Z^+

Partially Ordered Set

- A partially ordered set (poset) is a set with a partial order defined on it.
- Denotation: $(A, \preccurlyeq)$
- Examples
  - $(Z, \leq)$ or $(Z, \geq)$
  - $(Z^+, \mid)$
  - $(2^A, \subseteq)$

Product Partial Order

- Given two posets, $(A, \preccurlyeq_A)$ and $(B, \preccurlyeq_B)$, we can define a new partial order $\preccurlyeq$ on $A \times B$:

  $(a,b) \preccurlyeq (a',b')$ iff. $a \preccurlyeq_A a'$ in A and $b \preccurlyeq_B b'$ in B

- It is easy to prove that $(A \times B, \preccurlyeq)$ is a poset
  - Reflexive
  - anti-symmetric
  - transitive

Lexicographic Order

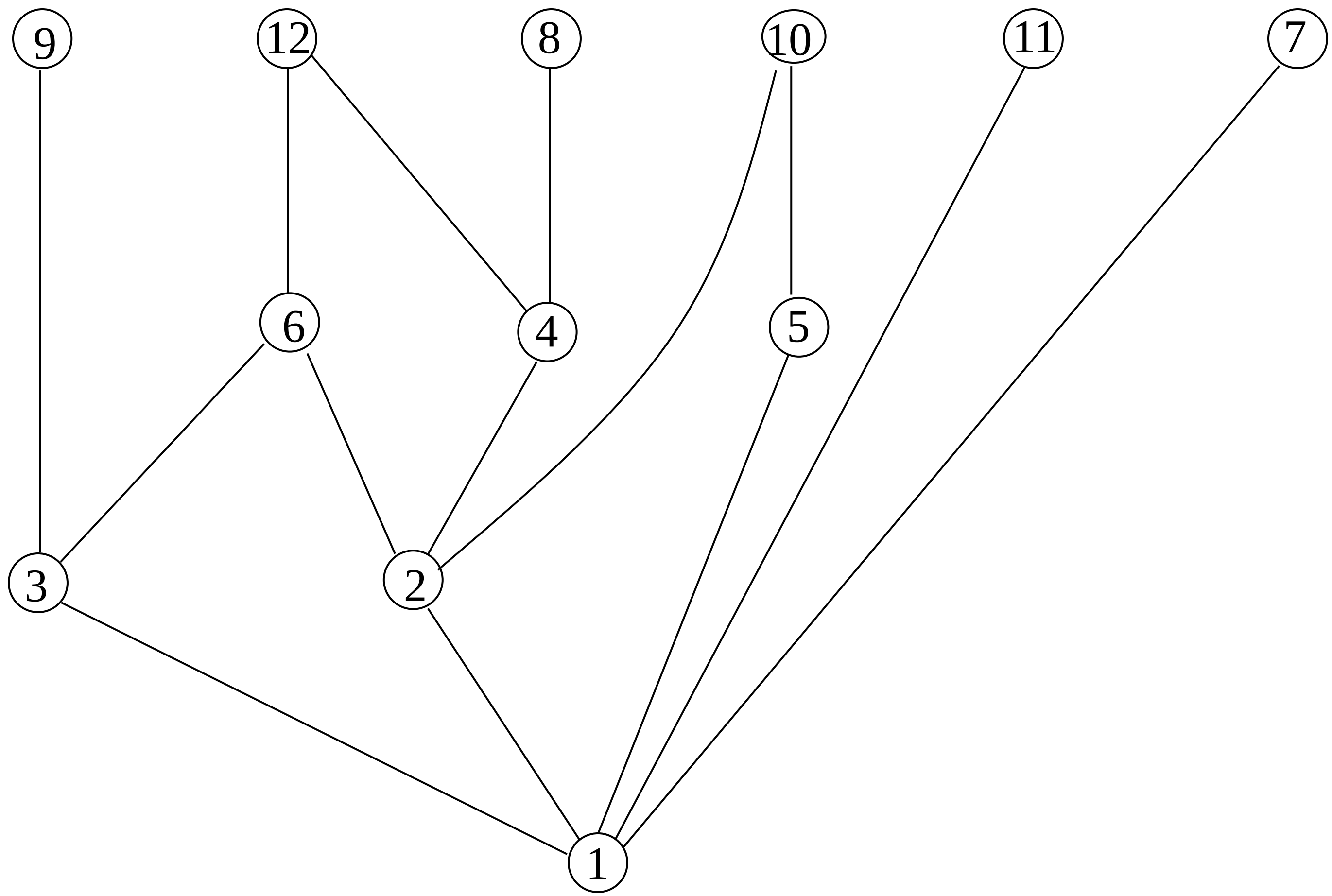
- Lexicographic order, as simplified:
  - Given a partial order on a alphabet a, then $\preccurlyeq$ is a simplified "dictionary" order:

$(a,b)\preccurlyeq(a',b')$ iff. $\underbrace{a\preccurlyeq a' \text{ and } a\neq a'}_{\text{Denoted as } a\prec a'}$ or $a=a'$ and $b\preccurlyeq b'$

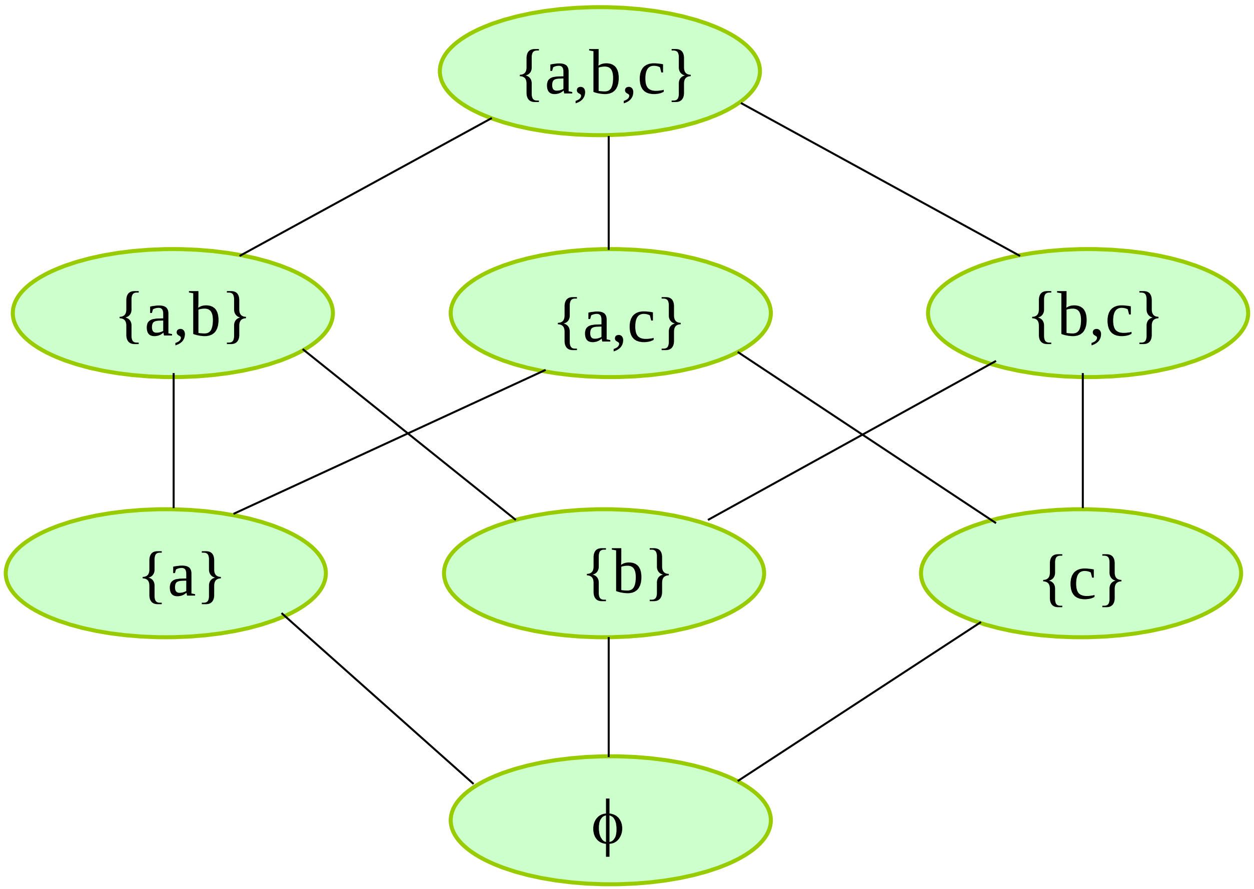
Hasse Diagrams

- Partial order can be represented by common relation diagram
- The special properties of partial order can be used to simplifying the diagram
  - Reflexivity: ring everywhere, so no need
  - Antisymmetric: no cycle, location dependent
  - Transitivity: there is a path, there is a edge

Divisability on {1,2,3,…12}



Containment on ρ({a,b,c})



Isomophism

- Let $(A,\preccurlyeq)$ and $(A', \preccurlyeq')$ be posets and let $f:A\rightarrow A'$ be a one-to-one correspondence between A and A'. The function f is called an isomophism from $(A,\preccurlyeq)$ to $(A', \preccurlyeq')$ if for any a and b in A,

$$a\preccurlyeq b \text{ iff. } f(a)\preccurlyeq' f(b).$$

  The two posets are called isomophic posets.
- Example: Z^+ and the set of positive even number are isomophic under "$\leq$"
- Principle of Correspondence.

Maximal and Minimal Elements

- An element $a \in A$ is called a **maximal element** of A if there is no element c in A such that $a \prec c$.

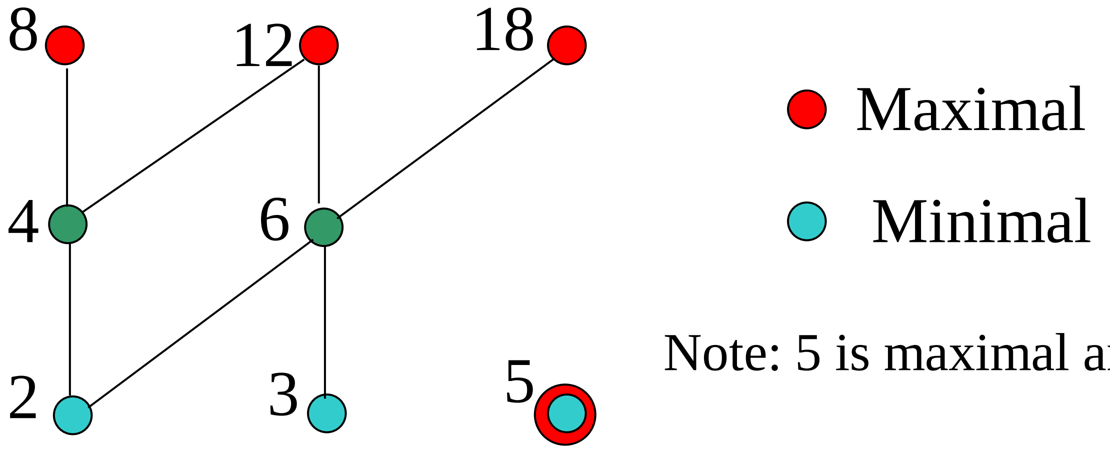
- An element $b \in A$ is called a **minimal element** of A if there is no element c in A such that $c \prec b$.

Existence of Maximal/Minimal Elements

- Given a poset $(A,\preccurlyeq)$, if A is finite, then there is at least one maximal element and at least one minimal element.
- Proof:
  - Let a be any element of A.
  - If a is not maximal, there must be some a_1 such that $a \prec a_1$.
  - If a_1 is not maximal either, there must be some a_2, such that $a_1 \prec a_2$. …
  - Since A is finite, we can't continue this procedure indefinitely, and find some a_k, which is maximal.
  - Same for the minimal element.

Examples of Maximal/Minimal Elements

- Divisibility on {2,3,4,5,6,8,12,18}
  - Maximal elements: 5, 8, 12, 18
  - Minimal elements: 2, 3, 5



Note: 5 is maximal and minimal

Greatest and Least Elements

- An element $a \in A$ is called a **greatest element** of a if $x \preccurlyeq a$ for all $x \in A$.

- An element $a \in A$ is called a **least element** of a if $a \preccurlyeq x$ for all $x \in A$.

Examples of Greatest and Least Elements

- Containment on $\rho(\{a,b,c\})$
  - Greatest element: $\{a,b,c\}$
  - Least element: ϕ
- Divisibility on $\{2,3,4, 6,12\}$
  - Greatest element: 12
  - Least element: none (Note: there are two minimal elements: 2 and 3)

Uniqueness of Largest Element

- A poset has **at most** one greatest element.
- Proof

  Suppose that a,b are greatest elements. By the definition of the greatest elements, we have $a \preccurlyeq b$, and $b \preccurlyeq a$. So, $a=b$, by the antisymmetry property

- It is same for the least element.

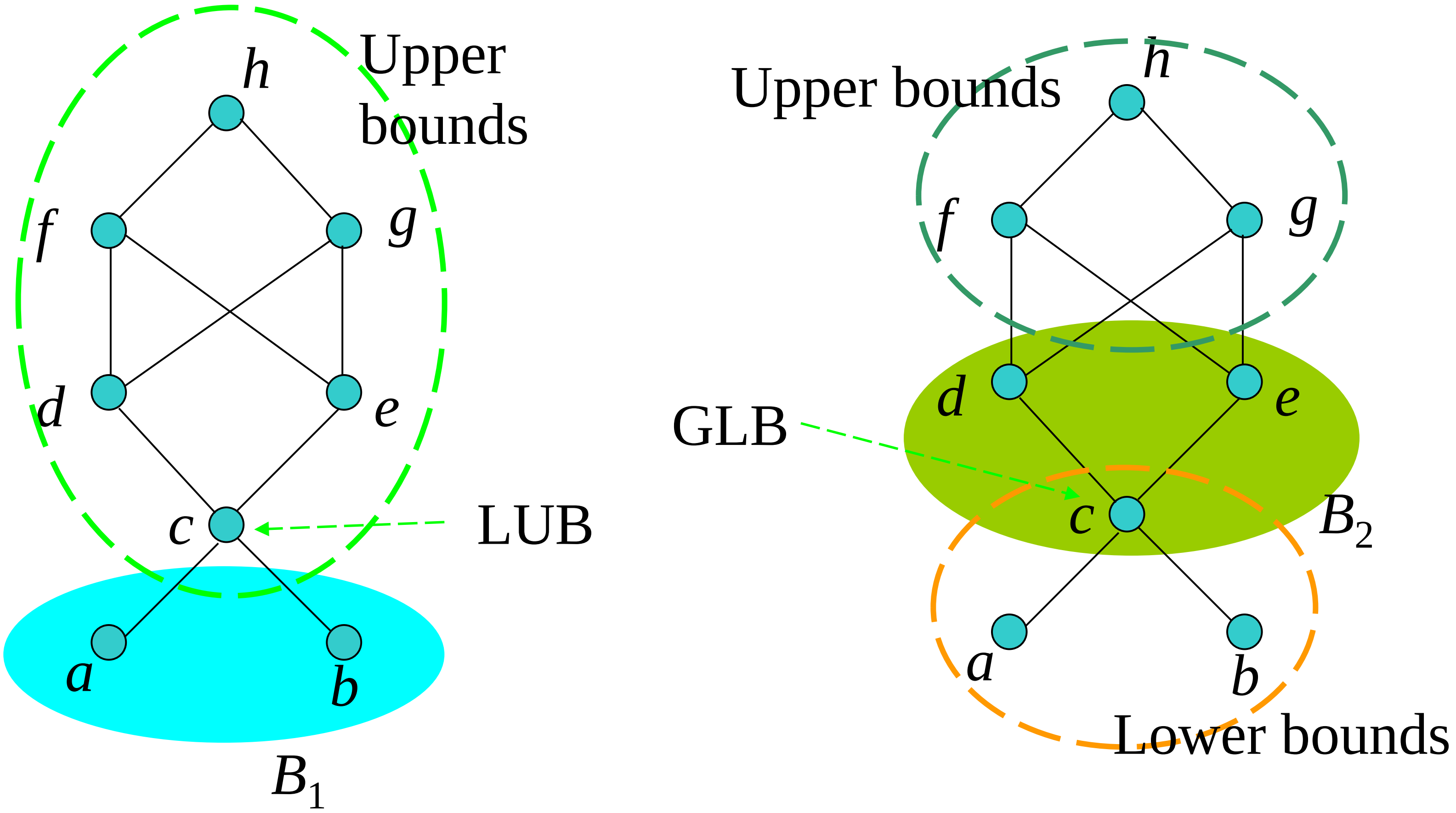
Bounds of Subsets of Poset

- Given a poset A and its subset B, an element $a \in A$ is called an **upper bound** of B if $b \preccurlyeq a$ for all $b \in B$.

- For a given subset B, upper bound may not exist. On the other hand, there may be more than one upper bound. The least element(if existing) of the poset consisting of all upper bounds of B is called the **least upper bound(LUB)**

- **Lower Bound** and **Greatest Lower Bound** can be defined similarily.

Example of Bounds



Linear Ordering and Well-Ordering

- $(A, \preccurlyeq)$ is a poset.
  - Linear-ordering – any two element of A are comparable.
  - Well-ordering - every nonempty subset of A has a least element
- Well-ordering is also Linear-ordering
  - If every nonempty subset of A has a least element, then any two elements of A are comparable.
- But Linear-ordering may not be Well-ordering
  - If any two elements of A are comparable, can it be implied that every nonempty subset of A has a least element?

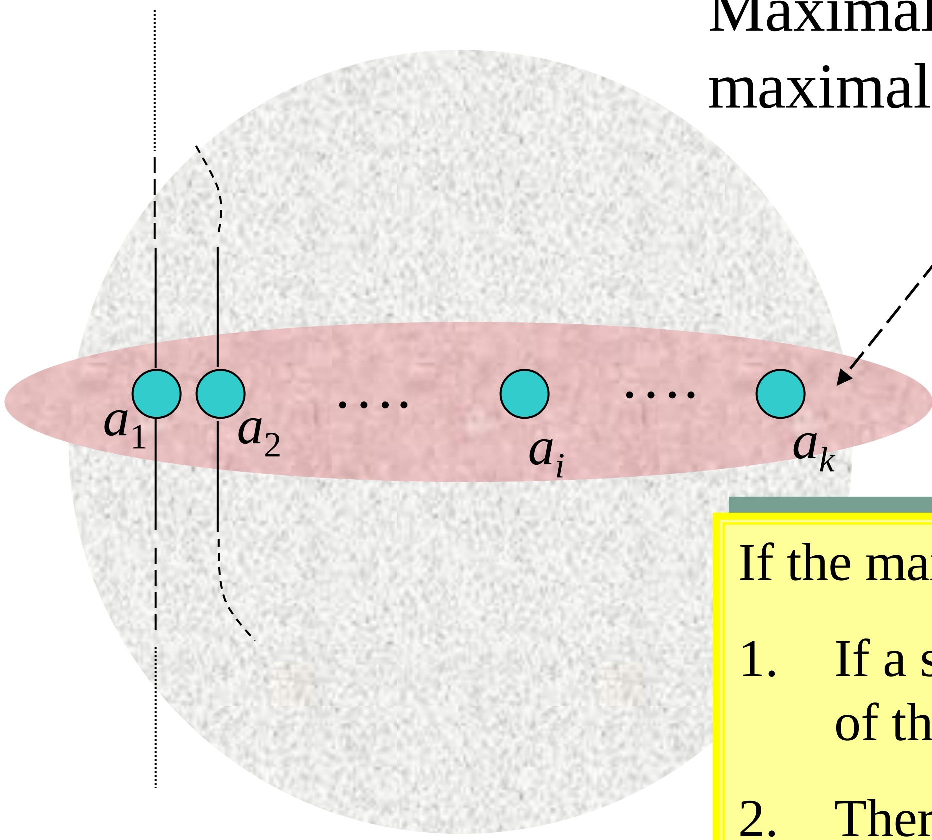
  No, for infinite A.

Chain and Anti-chain

- Given a POSet $(A, \leq)$
  - A subset B of A is said to be a chain, iff for any two elements x, y in B, either $x \leq y$ or $y \leq x$
  - A subset C of A is said to be an antichain, iff for any two elements x, y in C, neither $x \leq y$ nor $y \leq x$

Chain and Anti-chain (Dilworth）

Maximal anti-chain: anti-chain with with maximal number of elements.



If the maximal anti-chain has K elements

1. If a set of chains $L_1,\ldots, L_m$ is a partition of the POSet. m >= k
2. There are a set of k chains which is a partition of the POSet.

Order in Disorder

- In any permutation of natural numbers $1,2,3,\ldots,n^2+1$, there must be a strictly increasing or decreasing sequence with length not less than $n+1$.
- Proof:
  - Given a permutation, labeling each number using a pair (p, q), where p is the length of the largest increasing sequence ending at the number, and q is the length of the largest decreasing sequence ending at the number.
  - Note, each number has a unique label (Why?). If p and q are both not larger than n, there are only n^2 possible label value.

Order in Disorder: PO Model

- In any permutation of natural numbers 1,2,3,…,n^2+1, there must be a strictly increasing or decreasing sequence with length not less than n+1.
- The model of partial order:
  - Set: $A=\{ <i,v_i> \mid i=1,2,\ldots,n^2+1$, each v_i has an unique value in $1,2,\ldots,n^2+1\}$
  - Two partial orderings
    - R_1: $<i,v_i>R_1<j,v_j>$ iff. $i<j$ and $v_i<v_j$ or i==j and $v_i == v_j$
    - R_2: $<i,v_i>R_2<j,v_j>$ iff. $i<j$ and $v_i>v_j$ or i==j and $v_i == v_j$
- Problem: Prove that there must a subset of A with no less than $n+1$ elements, which is a chain of R_1 or R_2.
  - Note: a chain of R_1 is an anti-chain of R_2, and vice versa.

Lattices

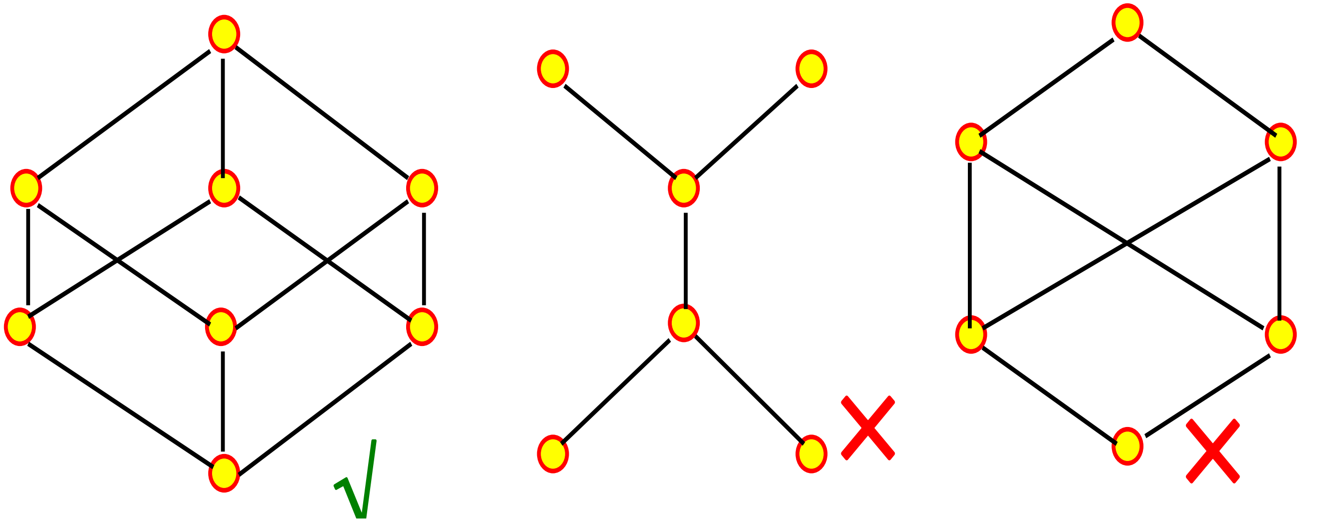
- Definition: $(L,\preccurlyeq)$ is called a lattice if
  - $(L,\preccurlyeq)$ is a poset
  - For any $x,y\in L$, $\{x,y\}$ has a LUB, which is denoted as $x\vee y$ (join)
  - For any $x,y\in L$, $\{x,y\}$ has a GLB, which is denoted as $x\wedge y$ (meet)

Examples of Lattice

- $(Z, \leq)$
  - $x \wedge y = \min\{x,y\}$, $x \vee y = \max\{x,y\}$
- $(\{1,2,3,4,6,8,12,16,24,48\}, |)$
  - $x \wedge y = \gcd(x,y)$, $x \vee y = \mathrm{lcm}(x,y)$
- $\{(true, false), \{(false, true)\}\}$
  - $\wedge$, $\vee$ are boolean operations **and, or** respectively.
- $(\rho(B), \subseteq)$
  - $x \wedge y = x \cap y$, $x \vee y = x \cup y$

Lattice and Hasse Diagram

- The posets represented by the two hasse diagram on the right are **not** lattices.



Basic Formula about Lattices

- By the definitions of LUB and GLB, it is easy to prove that:
  - $a \preccurlyeq a \vee b$, $b \preccurlyeq a \vee b$
  - If $a \preccurlyeq c$, $b \preccurlyeq c$, then $a \vee b \preccurlyeq c$
  - $a \wedge b \preccurlyeq a$, $a \wedge b \preccurlyeq b$
  - If $c \preccurlyeq a$, $c \preccurlyeq b$, then $c \preccurlyeq a \wedge b$

Algebraic Properties of Lattice

- Idempotent properties
  - $a \vee a = a \wedge a = a$
- Commutative properties
  - $a \vee b = b \vee a$; $a \wedge b = b \wedge a$
- Associative properties
  - $a \vee (b \vee c) = (a \vee b) \vee c$; $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- Absorption properties
  - $a \vee (a \wedge b) = \mathrm{a}$; $a \wedge (a \vee b) = a$

More Properties of Lattices

- Let L be a lattice， $\forall a,b,c,d\in L$, If $a\preccurlyeq b$, $c\preccurlyeq d$, then $a\wedge c\preccurlyeq b\wedge d$, $a\vee c\preccurlyeq b\vee d$
  - $\because a\wedge c \preccurlyeq a\preccurlyeq b$, $a\wedge c \preccurlyeq c\preccurlyeq d$, then a∧c is one lower bound of $\{b,d\}$， $\therefore a\wedge c\preccurlyeq b\wedge d$;
  - $\because a\preccurlyeq b\preccurlyeq b\vee d$, $c\preccurlyeq d\preccurlyeq b\vee d$, so, $b\vee d$ is one of the upper bound of {a,c}, $\therefore$ a∨c≼b∨d

More Properties of Lattice

- Distributive Inequality

$$\forall a,b,c \in L,\ a \vee (b \wedge c) \preccurlyeq (a \vee b) \wedge (a \vee c)$$

- Proof:
  - Since $a \preccurlyeq a$ and $b \wedge c \preccurlyeq b$, we have $a \vee (b \wedge c) \preccurlyeq (a \vee b)$;
  - Since $a \preccurlyeq a$ and $b \wedge c \preccurlyeq c$, we have $a \vee (b \wedge c) \preccurlyeq (a \vee c)$;
  - $a \vee (b \wedge c)$ is a lower bound of $\{(a \vee b), (a \vee c)\}$
  - $\therefore\ a \vee (b \wedge c) \preccurlyeq (a \vee b) \wedge (a \vee c)$

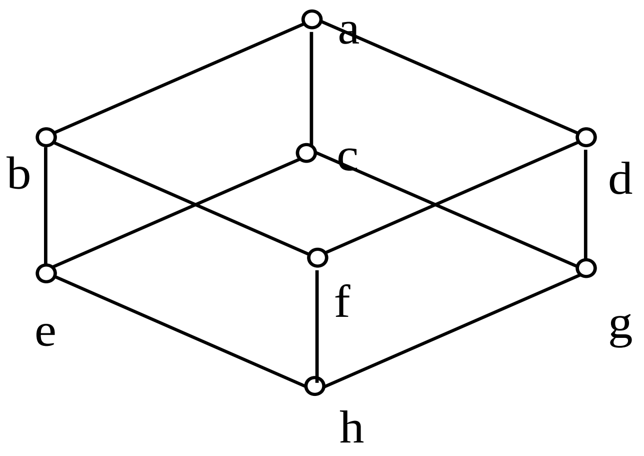
Similarily, it is easy to prove that:

$$(a \wedge b) \vee (a \wedge c) \preccurlyeq a \wedge (b \vee c)$$

Sublattice

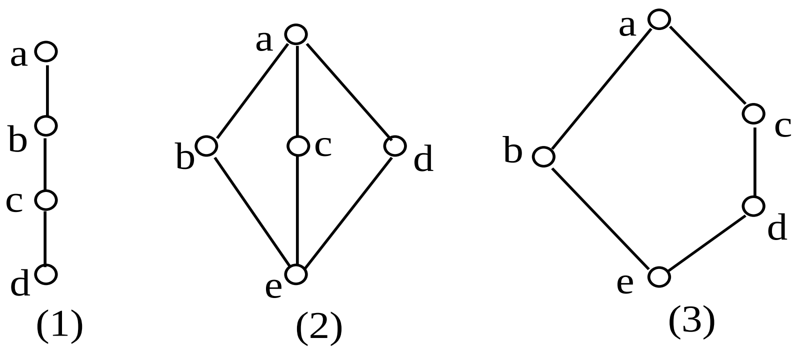
- Let (L,∧,∨) is a lattice, S is a nonempty subset of L. If S is close under the operations ∧ and ∨, then S is a sublattice of L.
- Example:
  - Let S_1={a,b,d,h}; S_2={a,b,d,f}
  - Then S_2 is a sublattice,

    but S_1 is not ($b \wedge d \notin S_1$)



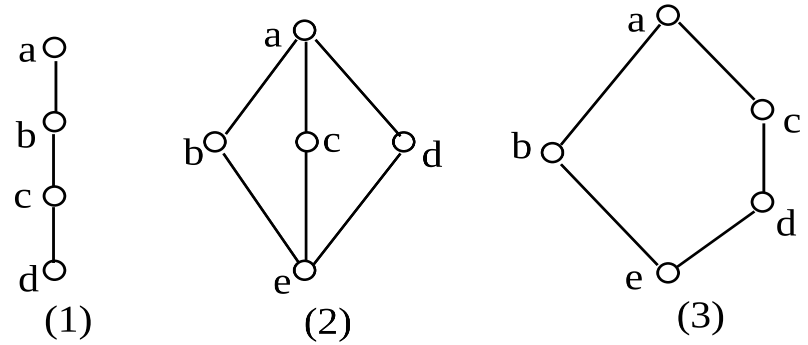
Several Special Lattice

- (1) Chain
- (2) Diamond lattice
  - Note: $b \vee (c \wedge a) = (b \vee c) \wedge a = a$
- (3) Pentagon lattice
  - Note: $c \vee (b \wedge d) = c \vee e = c \neq (c \vee b) \wedge d = a \wedge d = d$



Distributive Lattice

- Definition: L is a lattice, if for all $a,b,c\in L$, $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$, then L is called a distributive lattice.
  - Note: $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$ iff. $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$
  - Because $(a\vee b)\wedge(a\vee c) = ((a\vee b)\wedge a)\vee((a\vee b)\wedge c)$
    $= a\vee(b\wedge a)\vee(a\wedge c)\vee(b\wedge c) = a\vee(b\wedge c)$
- Diamond(2) and pentagon(3) are not distributive lattices.
  - In (2), $b\wedge(c\vee d)=b$, but $(b\wedge c)\vee(b\wedge d)=e$
  - In (3), $d\vee(b\wedge c)=d$, but $(d\vee b)\wedge(d\vee c)=c$



Characteristics of Distributive Lattices

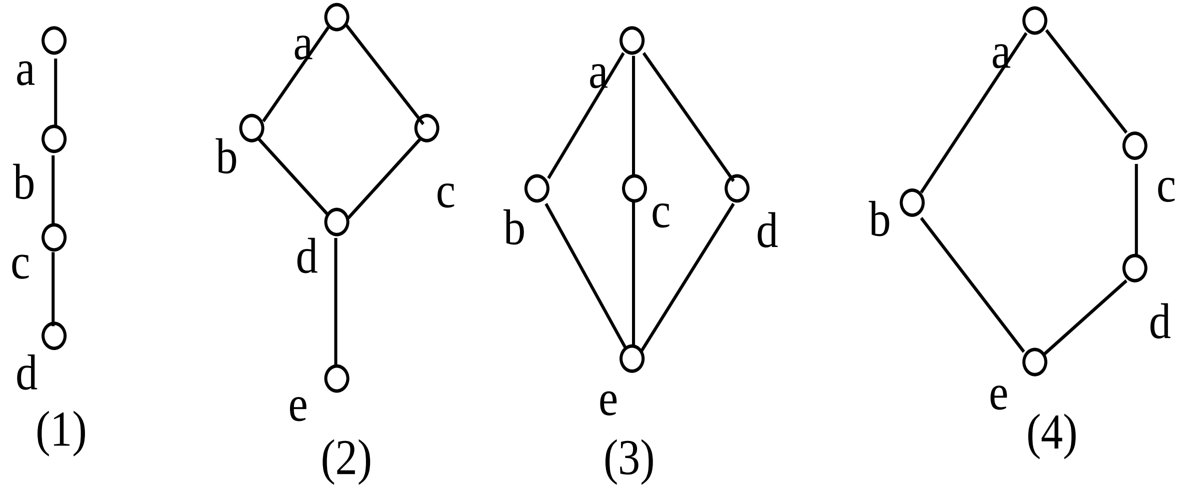
- Lattice L is a distributive lattice if and only if it does not contain sublattice isomophic to diamond lattice or pentagon lattice.

Bounded Lattice

- A lattice *L* is bounded if *L* has both a greatest element *I* and a least element *0*.
- Finite lattice is bounded lattice
  - *I* is $a_1 \vee a_2 \vee \ldots \vee a_n$
  - *0* is $a_1 \wedge a_2 \wedge \ldots \wedge a_n$。
- If *L* is a bounded lattice, then for all *x* in *L*
  - $I \wedge x = x$; $I \vee x = I$
  - $0 \wedge x = 0$; $0 \vee x = x$

Complement

- Let L is a bounded lattice. For any given element a in L, if there exists some b in L, such that a⋁b=1 **and** a⋀b=0, then b is called the complement of a.



Note: 0 and 1 are complement of each other.

Uniqueness of Complement

- Let L be a bounded distributive lattice. If a complement exists, it is unique.
- Proof
  - Suppose that b and c are both complements of a, i.e. $a\vee b=1$, $a\wedge b=0$; $a\vee c=1$, $a\wedge c=0$, then:
  - $b=b\vee 0=b\vee(a\wedge c)=(b\vee a)\wedge(b\vee c)=(b\vee c)$
  - Also, $c=c\vee 0=c\vee(a\wedge b)=(c\vee a)\wedge(c\vee b)=(b\vee c)$
  - So, $b=c$

Home Assignments

- To be checked
  - Ex. 6.1: 10, 13, 14, 16, 18, 26-28, 29, 30, 34-36, 38, 40
  - Ex. 6.2: 6, 8, 12, 14, 17-19, 20, 22, 23-26, 32, 33, 35-38
  - Ex. 6.3: 1-6, 13-15, 18-20, 22, 24-26, 27, 29, 34, 37-40