Counting

Lecture 3

Discrete Mathematical Structures

At the Last Class …

- Part I: Propositional and Predicate Logic
  - Logical operations and truth tables
  - Quantifiers
  - Logic statements
- Part II: Methods of Proof
  - Rules of inference
  - Indirect method of proof
  - Proof by contradiction
  - Disproving by counterexamples

Counting

- Part I: Countable and Comparison
  - Countable Set
  - Comparing the size of infinite set
  - Pigeonhole principles
- Part II: Some Techniques for Analysis
  - Elements of probability
  - Recurrence relations

Countable Set

- A set *A* is countable if and only if we can arrange all of its elements in a linear list in a definite order.
  - "Definite" means that we can specify the first, second, third element, and so on.
  - If the list ended and with the *n*th element as its last element, it is finite.
  - If the list goes on forever, it is infinite.

Proof of Countability

- The set of all integers is countable.
  - We can arrange all integer in a linear list as follows:

$$0,-1,1,-2,2,-3,3,...$$

that is: positive k is the $(2k+1)$th element, and negative k is the $2k$th element in the list.

Set of Ordered Pairs

- The set of all objects with the form <*i*,*j*> is countable, where *i*,*j* are nonnegative integers.

<0,0>	<1,0>	<2,0>	<3,0>	<4,0>	•••
<0,1>	<1,1>	<2,1>	<3,1>	•••	
<0,2>	<1,2>	<2,2>	•••		
<0,3>	<1,3>	•••			
<0,4>					

<0,0>, <0,1>, <1,0>, <0,2>, <1,1>, <2,0>, <0,3>, ...

$$l(m,n)=\frac{1}{2}\sum_{i=1}^{m+n} i+(m+1)=\frac{(m+n)(m+n+1)}{2}+(m+1)$$

Real Number Set Is Not Countable

- (0,1) is not a countable set
  - "diagonal proof"

  Assuming that elements in (0,1) can be listed linearly：

  $0.b_{11}b_{12}b_{13}b_{14}\ldots$

  $0.b_{21}b_{22}b_{23}b_{24}\ldots$

  $0.b_{31}b_{32}b_{33}b_{34}\ldots$

  $0.b_{41}b_{42}b_{43}b_{44}\ldots$

  $\vdots$

  then $0.\ b_1b_2b_3b_4\ldots$ （$b_i \neq b_{ii}$） can't be in the above list.
- So, it is impossible to arrange all real number in a linear list.

Finite and Infinite

Full?

No problem!
I’ll have the guest in Room No.1 moved to No.2, … , and the guest in No.k moved to No.k+1, …, and you can stay in Room No.1.

Done!

Proving by Counting

Pigeonhole Principle

- If n pigeons are assigned to m pigeonholes, and $m<n$, then at least one pigeonhole contains two or more pigeons.
  - Proof by contradiction:

    Suppose each pigeonhole contains at most 1 pigeon. Then at most m pigeons have been assigned. Since $m<n$, so $n-m>0$, there are $(n-m)$ pigeons have not been assigned. It's a contradiction.

Pigeonhole by Odd Factor

- Problem: show that if any 11 numbers are chosen from the set {1,2,...,20}, then one of them will be a multiple of another.
- Solution:
  - Observation: every natural number n can be represented as $2^k m$, where m is the largest odd factor of n, k is a nonnegative integer.
  - Let each odd number in {1,2,...,20} correspond to a pigeonhole, then there are 10.
  - Each element in {1,2,...,20} corresponds to a pigeon, and there are 20.
  - If n_1, n_2 are in one pigeonhole, then one of them must be the multiple of another.

Shaking Hands at a Gathering

- **Situation**: at a gathering of *n* people, everyone shaked hands with at least one person, and no one shaked hand more than once with the same person.
- **Problem**: show that there must have been at least two of them who had the same number of handshaking.
- **Solution**:
  - Pigeon: the *n* participants
  - Pigeonhole: different number between 1 and *n*-1.

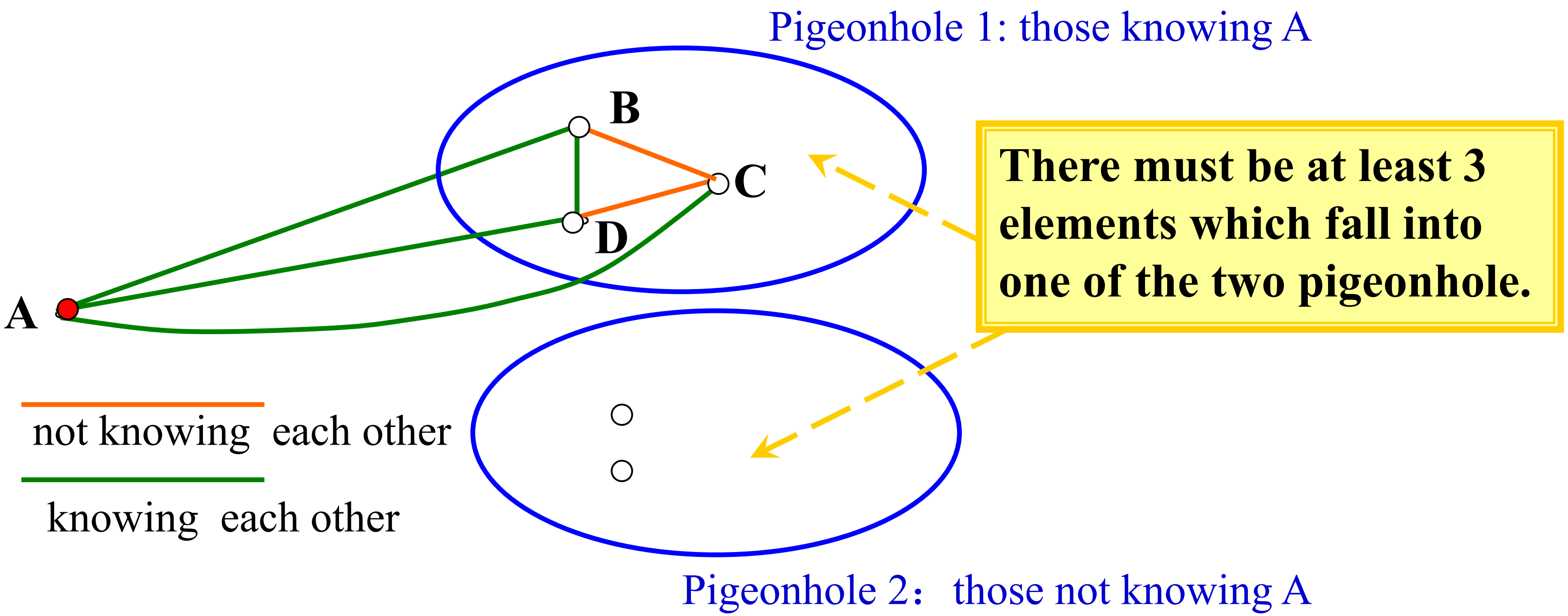
Extended Pigeonhole Principle

- If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.
  - Proof by contradiction

    If each pigeonhole contains no more than $\lfloor (n-1)/m \rfloor$, then there are at most $m \lfloor (n-1)/m \rfloor \le n-1$ pigeons at all. It's a contradiction.

Knowing Each Other or Not

Problem: show that among any 6 persons, there are 3 who know each other, or there are 3 who don't know any two others.



Hidden Pigeons and Invisible Pigeonholes

- **Situation**: A chess player wants to prepare for a championship match by playing some practice games in 77 days. She wants to play at least one game a day but no more than 132 games altogether.
- **Problem**: show that no matter how she schedules the games there is a period of consecutive days within which she plays exactly 21 games.

Scheduling the Practice Games: Solution

Let $\boldsymbol{a_i}$ denote the ***total*** number of games she plays ***up through the ith day***. Then, a_1, a_2, a_3,..., a_{76}, a_{77} is a monotonically increasing sequence, with $a_1 \geq 1$, and $a_{77} \leq 132$.

Note: if a_i+21= a_j then the player plays 21 games during the days i+1, i+2, up through j .

Considering the sequence:

$$a_1, a_2, a_3, \ldots, a_{76}, a_{77}, a_1+21, a_2+21, a_3+21, \ldots, a_{76}+21\ a_{77}+21$$

The least element in the sequence is 1, and the largest is 153. However, there are 154 elements in the sequence, so, there must be at least two elements having the same value.

Note that both the first and second half sequences are monotonically increasing, so, it is impossible for the two elements having the same value to be within one half sequence, that is, we have $\boldsymbol{a_i+21= a_j}$

Probabilistic Event

Sample spaces:

what you want to record

number pattern:

$\{(i,j) \mid 1 \le i,j \le 6\}$

sum of number:

$\{2,3,4,\ldots,11,12\}$

Two one's:

{yes, no}

11 different outcomes

An **event**, for example:

"no less than 8"

Experiment: throwing two dices

Probability of an Event

- Probability of an event E is a number, denoted as $p(E)$, reflecting one's assessment of the likelihood that the event will occur.
- If the event E has occurred n_E times after n trials of the underlying experiment. Then $f_E=n_E/n$ is called the frequency of occurrence of E in n trials.
- If we believe that f_E will tend ever closer to a certain number as n becomes larger, $p(E)$ is the number.

Axioms for a probability space

- Some properties the assigned probability should satisfy: (let A is the sample space)
  - P1: $0 \leq p(E) \leq 1$ for every event E in A
  - P2: $p(A)=1$ and $p(\varnothing)=0$
  - P3: $p(E_1 \cup E_2 \cup \ldots \cup E_k)=p(E_1)+p(E_2)+\ldots+p(E_k)$ whenever the events are mutually exclusive.
- For a given experiment and a specific sample space, if the probabilities are assigned to all events with P1-P3 satisfied, we get a probability space.

Finite Probability Space

- There are only finite outcomes.
- Each outcomes individually consists an elementary event.
  - For one coin toss, there are two outcomes – head and tail. "Head" is an elementary event.
- The probability of an elementary event corresponds a specific outcome.
- If all outcomes are equally likely, then the probability of an event *E* can be computed as:

$$p(E) = \frac{|E|}{|A|} = \frac{\text{total number of outcomes in } E}{\text{total number of outcomes}}$$

A Fair Six-sided Die

- Toss it 3 times. What is the probability of the event "either all three numbers are equal or none of them is a 4"?
- Equally likely outcome is a reasonable assumption
- The size of the sample space is $6^3=216$
- Let F be the event of "all three numbers are equal", the $|F|=6$ ($F=\{111,222,\ldots,666\}$)
- Let G be the event "none of them is a 4", the $|G|=5^3=125$ (G is the combination of any 3 from $\{1,2,3,5,6\}$)
- $|F\cup G|=|F|+|G|-|F\cap G|=6+125-5=126$
- So, the result is 126/216 = 7/12

Principle of Inclusion and Exclusion

For a whole set of N elements , $A_1, A_2, \ldots, A_n$ are the corresponding subsets for n different "properties".
Then the number of element which satisfies none of the n properties is :

$$N(\overline{A_1}\,\overline{A_2}\ldots\overline{A_n}) = N - S_1 + S_2 - S_3 + \ldots + (-1)^k S_k + \ldots + (-1)^n S_n$$

where , $S_k = \sum_{1 \le i_1 \le i_2 \le \ldots \le i_k \le n} | A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k} | \qquad k = 1,2,\ldots, n$

For an example：the formula for 4 subsets

$N - (|S_1| + |S_2| + |S_3| + |S_4|)$

$+ (|S_1 \cap S_2| + |S_1 \cap S_3| + |S_1 \cap S_4| + |S_2 \cap S_3| + |S_2 \cap S_4| + |S_3 \cap S_4|)$

$- (|S_1 \cap S_2 \cap S_3| + |S_1 \cap S_2 \cap S_4| + |S_1 \cap S_3 \cap S_4| + |S_2 \cap S_3 \cap S_4|)$

$+ |S_1 \cap S_2 \cap S_3 \cap S_4|$

Hatcheck Problem

- A new employee checks the hats of *n* people at a restaurant, forgetting to put claim check numbers on the hat. When customers return for their hats, the checker gives them back at random from the remaining hats. What is the probability that no one receives the correct hat?
- Mathematical model: arrange 1,2,3,…,*n* randomly, resulting in a new sequence i_1, i_2, i_3,…,i_n. What is the probability that for any k $(1 \le k \le n)$, $i_k \ne k$?
- The resulting sequence is called a ***derangement***.

Number of Derangement

- Define $i_k=k$ as Property A_k , and A_k is used for the subset of all permutations satisfying property A_k.

The number of derangement is:

$$N(\overline{A_1}\,\overline{A_2}\,\overline{A_3}\ldots\overline{A_n}) = N - S_1 + S_2 - S_3 + \ldots + (-1)^k S_k + \ldots + (-1)^n S_n$$

where $N = n!$

where S_k is $\sum_{1 \le i_1 \le i_2 \ldots \le i_k \le n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$

Note : S_k is the number of permutations keeping exactly k elements in their original positions, and the other $n-k$ elements as any possible permutation. So :

$$S_1 = \binom{n}{1}(n-1)!;\ S_2 = \binom{n}{2}(n-2)!;\ \ldots,\ S_k = \binom{n}{k}(n-k)! = \frac{n!}{k!}$$

The Probability of Derangement

We have known that the number of derangement is

$$N(\overline{A_1}\,\overline{A_2}\,\overline{A_3}\ldots\overline{A_n}) = N - S_1 + S_2 - S_3 + \ldots + (-1)^k S_k + \ldots + (-1)^n S_n$$

where $N = n!$, and $S_k = \binom{n}{k}(n-k)! = \frac{n!}{k!}\ (k = 1,2,3,\ldots,n)$

$\therefore N(\overline{A_1}\,\overline{A_2}\,\overline{A_3}\ldots\overline{A_n}) = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$; and the probability: $\sum_{k=0}^{n}\frac{(-1)^k}{k!}$

Since $\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = e^{-1}$, the difference between the probability

and $e^{-1} \approx 0.367879\ldots$ is less than $\frac{1}{n!}$, which means that the

probability is about 0.368, independent of n, except for very small n.

Average Behavior of an Algorithm

- Sequential search a list of *n* items for *K*
  - Assuming no same entries in the list, and *K* does occur in the list
  - Look *all* inputs with K in the *i*th location as *one* input (so, inputs totaling *n*)
  - Each input occurs with equal probability (i.e. 1/n)

$$A(n) = \sum_{i=0}^{n-1} [(p(K \text{ is at position } i)) \cdot (i+1)]$$

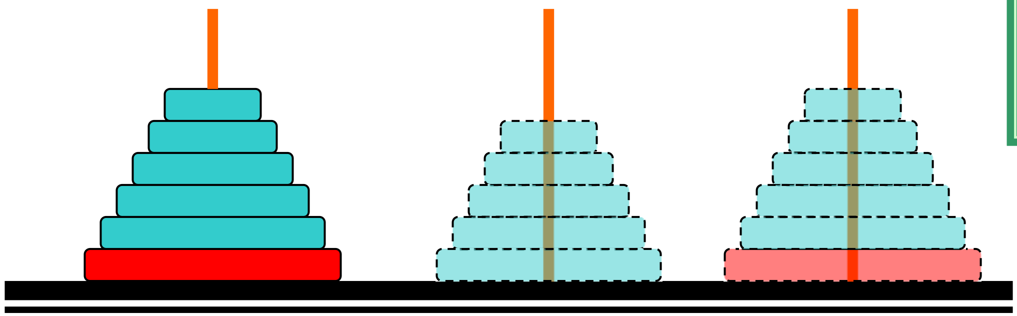
$$= \sum_{i=0}^{n-1} \left[\frac{1}{n} \cdot (i+1) \right] = \frac{n+1}{2}$$

Probabilistic Paradox

- For a family of four children, is it most likely there are two boys and two girls? (It is assumed that each child has a equal chance to be male or female at his/her birth)
- The probability of the event of 2-2 is 6/16, and the probability of the event of 1-3 is 8/16

Thinking Recursively: Problem 1

- Towers of Hanoi
  - How many moves are need to move all the disks to the third peg by moving only one at a time and never placing a disk on top of a smaller one.



$$T(1) = 1$$
$$T(n) = 2T(n-1) + 1$$

Solution of Towers of Hanoi

$$T(n) = 2T(n-1) + 1$$

$$2T(n-1) = 4T(n-2) + 2$$

$$4T(n-2) = 8T(n-3) + 4$$

$$\ldots\ldots.$$

$$2^{n-2}T(2) = 2^{n-1}T(1) + 2^{n-2}$$

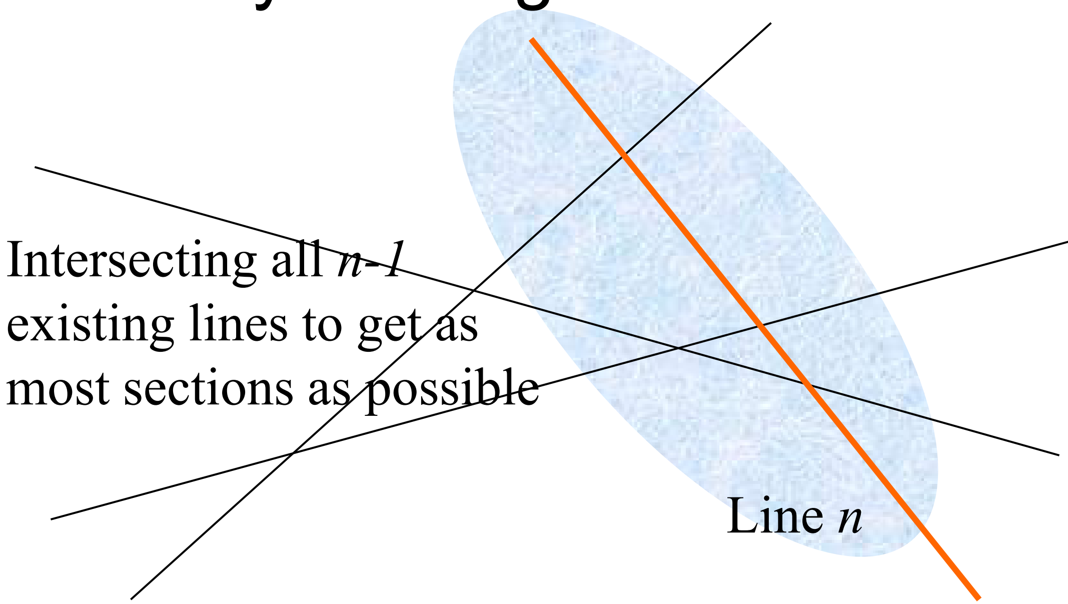
$$\Rightarrow \quad \boldsymbol{T(n) = 2^n - 1}$$

$$T(1) = 1$$
$$T(n) = 2T(n-1) + 1$$

$$T(1) + 1 = 2$$
$$T(n) + 1 = 2(T(n-1) + 1)$$

Thinking Recursively: Problem 2

- Cutting the plane
  - How many sections can be generated at most by *n* straight lines with infinite length.



$L(0) = 1$

$L(n) = L(n-1) + n$

Solution of Cutting the Plane

$$L(0) = 1$$
$$L(n) = L(n-1) + n$$

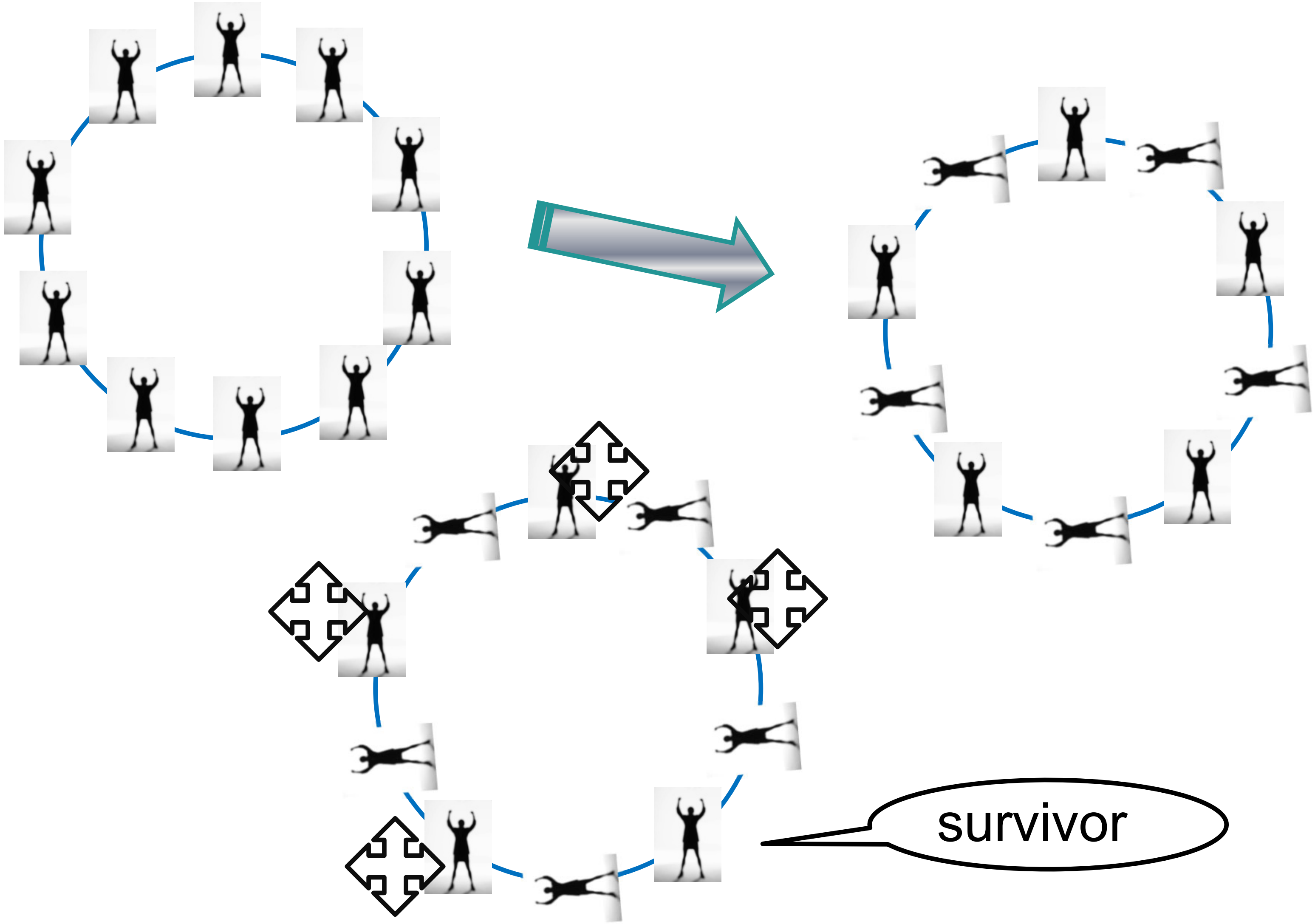
$$
\begin{aligned}
L(n) &= L(n-1)+n \\
&= L(n-2)+(n-1)+n \\
&= L(n-3)+(n-2)+(n-1)+n \\
&= \ldots\ldots \\
&= L(0)+1+2+\ldots\ldots+(n-2)+(n-1)+n
\end{aligned}
$$

$$L(n) = n(n+1)/2 + 1$$

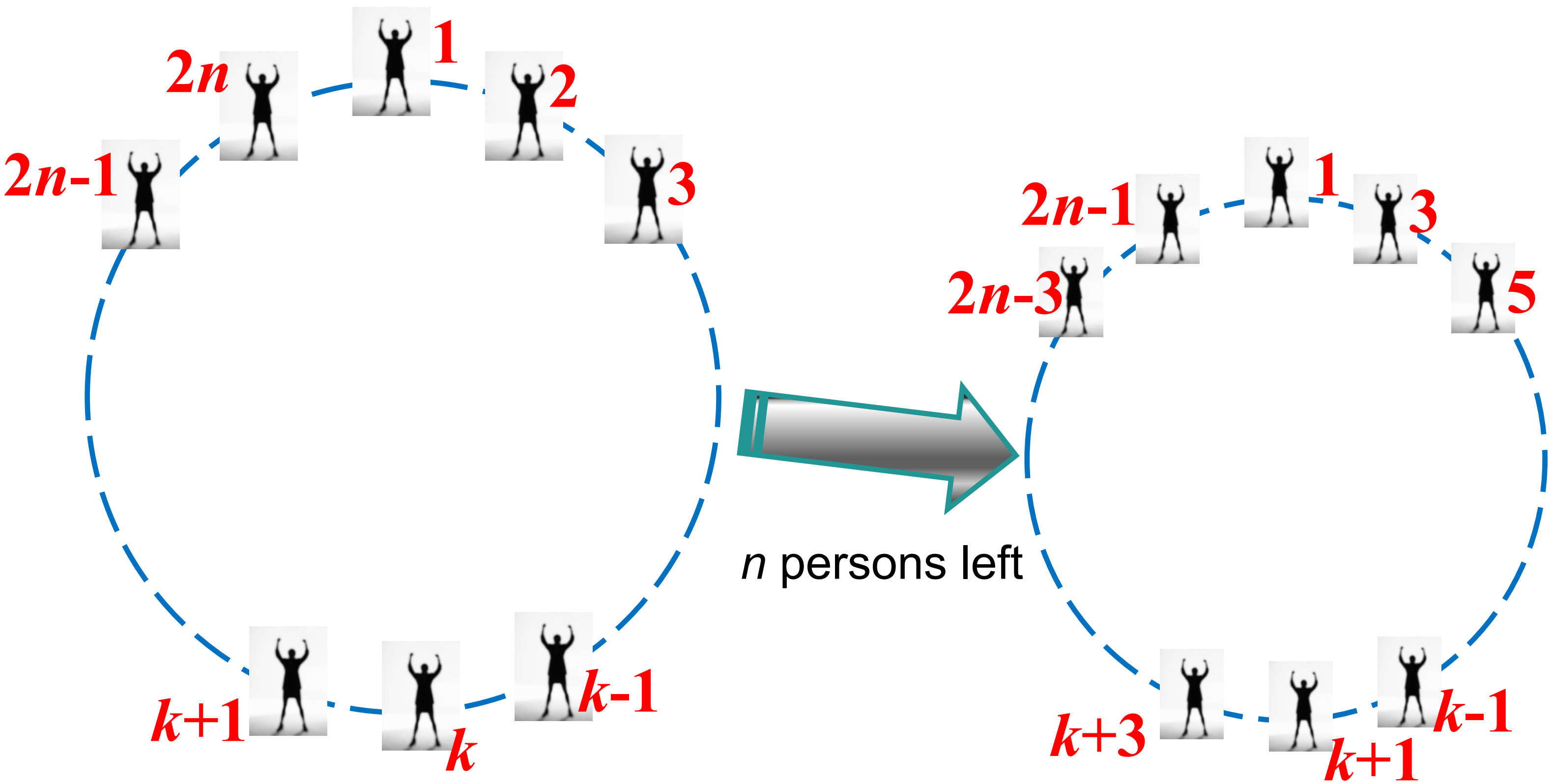
Josephus Problem

- Live or die, it’s a problem!
- Legend has it that Josephus wouldn't have lived to become famous without his mathematical talents. During the Jewish Roman war, he was among a band of 41 Jewish rebels trapped in a cave by the Romans. Preferring suicide to capture, the rebels decided to form a circle and, proceeding around it, to kill every third remaining person until no one was left. But Josephus, along with an unindicted co-conspirator, wanted none of this suicide nonsense; so he quickly calculated where he and his friend should stand in the vicious circle.

Make a Try: for n=10



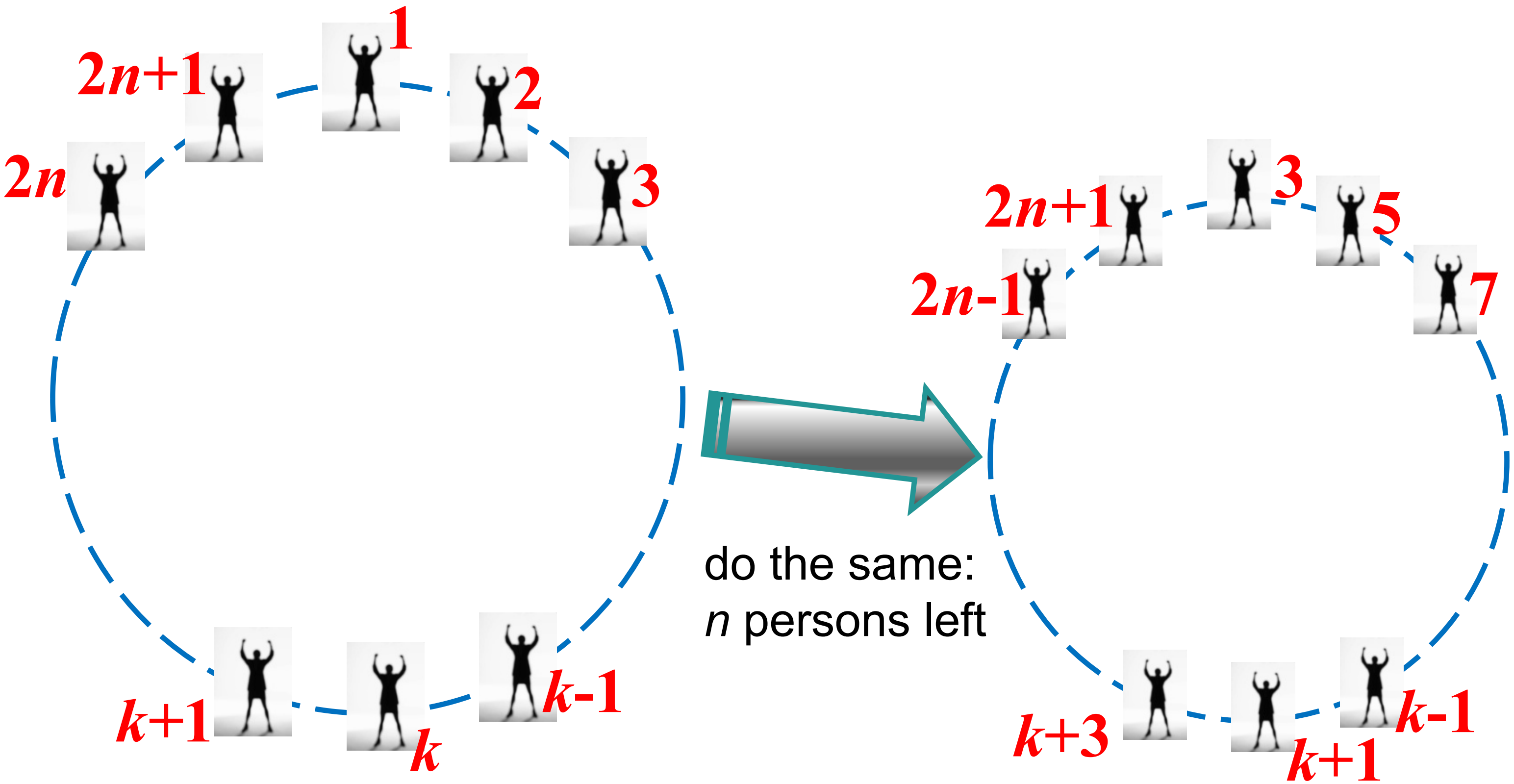
For 2n Persons (n=1,2,3,...)



The solution is: newnumber (J(n))

And the newnumber(k) is **2k-1**

And What about 2n+1 Persons (n=1,2,3,...)



The solution is: newnumber (J(n))

And for the time, the newnumber(k) is **2k+1**

Solution in Recursive Equations

$$
\begin{aligned}
J(1) &= 1; \\
J(2n) &= 2J(n) - 1, \qquad \text{for } n \geq 1; \\
J(2n+1) &= 2J(n) + 1, \qquad \text{for } n \geq 1.
\end{aligned}
$$

Explicit Solution for small *n*'s

n	1	2 3	4 5 6 7	8 9 10 11 12 13 14 15	16
J(n)	1	1 3	1 3 5 7	1 3 5 7 9 11 13 15	1

Look carefully ...

and, find the pattern...

and, prove it!

Eureka!

If we write n in the form $n = 2^{\mathrm{m}} + l$,
(where 2^{m} is the largest power of 2 not exceeding n and where l is what's left),
the solution to our recurrence seems to be:

$$J(2^m + l) = 2l + 1, \qquad \text{for } m \geq 0 \text{ and } 0 \leq l < 2^m.$$

As an example: J(100) = J(64+36) = 36*2+1 = 73

Binary Representation

- Suppose *n*'s binary expansion is :
  - $n = (b_m b_{m-1} \ldots b_1 b_0)_2$, where $b_m = 1$
  - $l = b_{m-1} \ldots b_1 b_0$
  - $J(n) = 2l + 1 = b_{m-1} \ldots b_1 b_0 1$

  - $2n = (b_m b_{m-1} \ldots b_1 b_0 0)_2$
  - $J(2n) = 2(J(n)) - 1 = b_{m-1} \ldots b_1 b_0 01$

  - $2n+1 = (b_m b_{m-1} \ldots b_1 b_0 1)_2$
  - $J(2n+1) = 2(J(n))+1 = = b_{m-1} \ldots b_1 b_0 11$

Another Solution for Josephus Problem

- If it begins with 2^m persons, the 1st one survives
  - $J(2^m) = 1$
- If it begins with $2^m + l$ persons
  - After l persons killed, 2^m persons left.
  - Now the current 1st person is originally numbered as $2l + 1$.
  - So $J(2^m+l) = 2l + 1$.

Linear Homogeneous Relation

$$a_n = r_1 \quad a_{n-1} + r_2 a_{n-2} + 6 \quad + r_m a_{n-k}$$

is called linear homogeneous relation of degree k.

$$c_n = (-2)c_{n-1}$$

$$f_n = f_{n-1} + f_{n-2}$$

\ hv

$$a_n = a_{n-1} + 3$$

$$g_n = g_{n-1}^2 + g_{n-2}$$

Q r

Characteristic Equation

- For a linear homogeneous recurrence relation of degree *k*

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + 6 + r_k a_{n-k}$$

the polynomial of degree *k*

$$x^k = r_1 x^{k-1} + r_2 x^{k-2} + 6 + r_k$$

is called its characteristic equation.

- The characteristic equation of linear homogeneous recurrence relation of degree 2 is:

$$\boxed{x^2 - r_1 x - r_2 = 0}$$

Solution of Recurrence Relation

- If the characteristic equation $x^2 - r_1 x - r_2 = 0$ of the recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ has two distinct roots s_1 and s_2, then

$$a_n = u s_1^n + v s_2^n$$

  where u and v depend on the initial conditions, is the explicit formula for the sequence.

- If the equation has a single root s, then, both s_1 and s_2 in the formula above are replaced by s

Proof of the Solution

Remember the equation : $x^2 - r_1 x - r_2 = 0$

We need prove that : $us_1^n + vs_2^n = r_1 a_{n-1} + r_2 a_{n-2}$

$$
\begin{aligned}
& us_1^n + vs_2^n = us_1^{n-2} s_1^2 + vs_2^{n-2} s_2^2 \\
& = us_1^{n-2}(r_1 s_1 + r_2) + vs_2^{n-2}(r_1 s_2 + r_2) \\
& = r_1 us_1^{n-1} + r_2 us_1^{n-2} + r_1 vs_2^{n-1} + r_2 vs_2^{n-2} \\
& = r_1(us_1^{n-1} + vs_2^{n-1}) + r_2(us_1^{n-2} + vs_2^{n-2}) \\
& = r_1 a_{n-1} + r_2 a_{n-2}
\end{aligned}
$$

Fibonacci Sequence

$f_1 = 1$

$f_2 = 1$

$f_n = f_{n-1} + f_{n-2}$

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Explicit formula for Fibonacci Sequence

The characteristic equation is x^2-x-1=0, which has roots:

$$s_1 = \frac{1+\sqrt{5}}{2} \quad and \quad s_2 = \frac{1-\sqrt{5}}{2}$$

Note: (by initial conditions) $f_1 = us_1 + vs_2 = 1 \;\; and \;\; f_2 = us_1^{\,2} + vs_2^{\,2} = 1$

which results:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Home Assignments

- To be checked
  - 1.3 Ex. : 23-24
  - 3.1 Ex. : 25-26, 29, 34
  - 3.2 Ex. : 19, 23, 27, 32
  - 3.3 Ex. : 10, 12, 17-19, 21-24
  - 3.4 Ex. : 34, 37-41
  - 3.5 Ex. : 14, 18, 26, 28, 34, 36