Lecture 11: Transport Network and Matching

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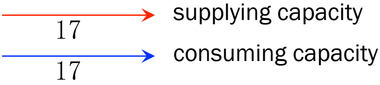
At the Last Class

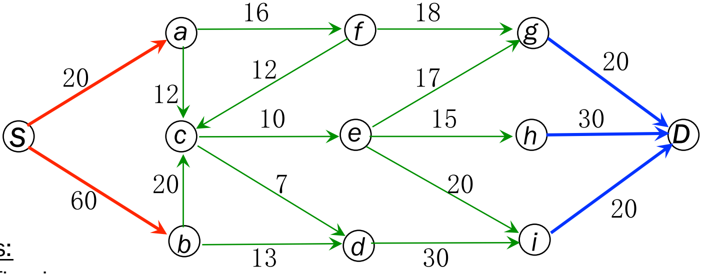
1. Graph as a Model
    - Paths and circuits in a graph
    - Subgraph and quotient graph
    - Graph as a model for problem solving
2. Traversal in a Graph
    - Euler paths and circuits
    - Hamiltonian paths and circuits

Overview

1. Network and Flow
   - Transport networks and Maximum flow
   - Labeling algorithm
   - Matching in a bipartite graph
   - Existence Condition of Perfect Matching
2. Graph Coloring
   - Vertex Coloring of Graph
   - Four-color and Five-color Theorem

A Model of Oil Supply



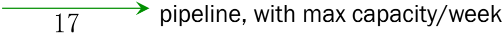


Vertices:

a, *b*: refineries

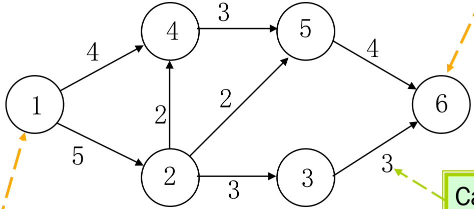
g, *h*, *i*: markets

others: relays



Transport Networks

The unique node with out-degree 0
The **sink**



Capacity of edge, $C_{i,j}$

The unique node with in-degree 0
The **source**

It is assume that all edges are in one direction.

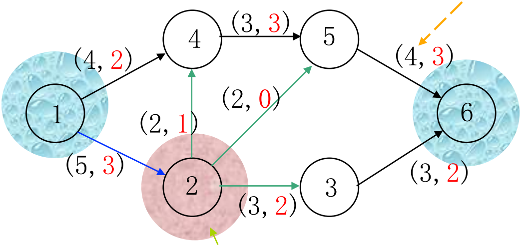
Transport Network: the Definition

A **transport network** is an ordered pair (G, k), where:

- G is a weakly connected directed graph containing no loops.
- k is a nonnegative real function defined on E_G
- There are two distinguished vertices S and D in V_G , and usually S has in-degree 0, called **source**, D has out-degree 0, called **sink**.

Flows

$(C_{i,j}, F_{i,j})$,
For any edge, $F_{i,j} \le C_{i,j}$



Value of the flow is 5

Conservation of flow:
Here: 3=1+0+2

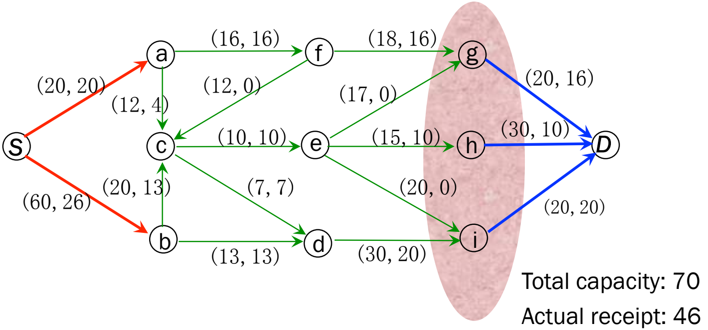
Flow: the Definition

Let (G, k) be a transport network with source S and sink D. Assume the capacity function k is defined on the edges of G. A **flow** in G is a nonnegative real-valued function F defined on the edges of G such that:

- **Capacity constraint** $0 \leq F(e) \leq k(e)$ for each edge $e \in E_G$.
- **Conversation equation** $$\sum_{y \in A(x)} F(xy) = \sum_{z \in B(x)} F(zx)$$ for every $x \in V_G - \{S, D\}$, where $A(x) = \{y | xy \in E_G\}$ and $B(x) = \{z | zx \in E_G\}$.

Flow: One More Example

The actual transported amount on any road cannot exceed the edge capacity. For any vertex, the total input amount must be equal to the output total.



Flow: One More Example

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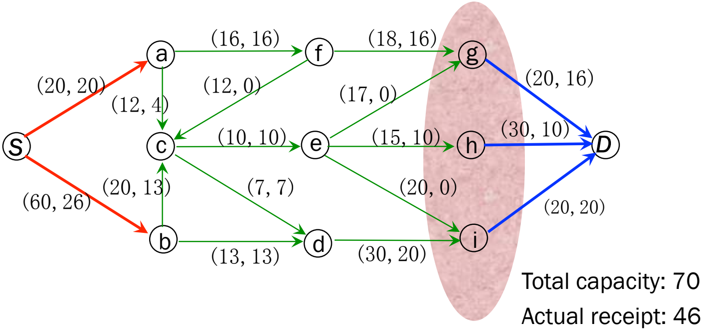
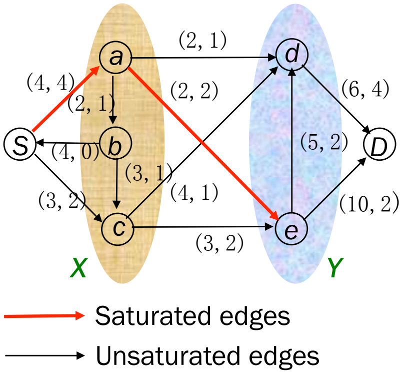


Problem: Can we send more on the network?

Edge Sets in a Flow



Excess capacity(ce)=k(ce)-F(ce)=1

$(X, Y) = \{ad, ae, cd, ce\}$

$k(X, Y) = 2+2+4+3 = 11$

$F(X, Y) = 1+2+1+2 = 6$

Note: $k(Y, X) = 0$, since $(Y, X) = \phi$

$(X, V_G) = \{bS, ad, ae, cd, ce\}$

$(V_G, X) = \{Sa, Sc\}$

Value of Flow

Let S and D be the source and sink, respectively, of a network (G, k). Let F be a flow. Then $F(S, V_G) = F(V_G, D)$

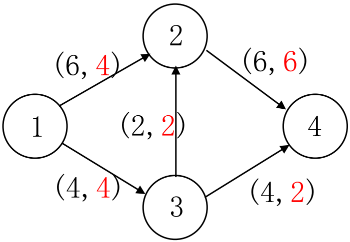
Proof.

$$F(V_G, V_G) = \sum_{x \in V_G} F(x, V_G) = \sum_{x \in V_G} F(V_G, x).$$

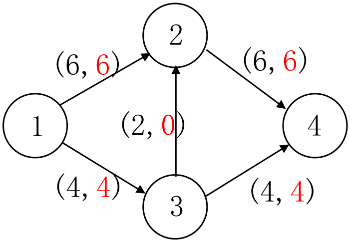
However, $F(x, V_G) = F(V_G, x)$ for each $x \in V_G - \{S, D\}$.
So $F(S, V_G) + F(D, V_G) = F(V_G, S) + F(V_G, D)$.
Deleting the terms equal to 0, we have $F(S, V_G) = F(V_G, D)$. $\square$

The **value of a flow** F (denoted as $|F|$) is defined as the value of $F(S, V_G)$ (or, $F(V_G, D)$).

Flows with Different Value



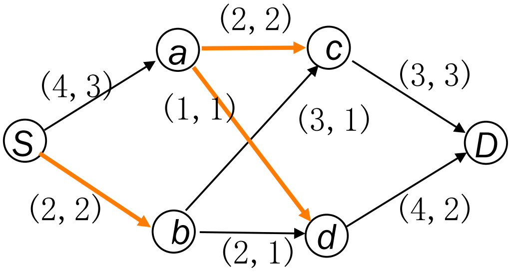
Value of flow: 8



Value of flow: 10

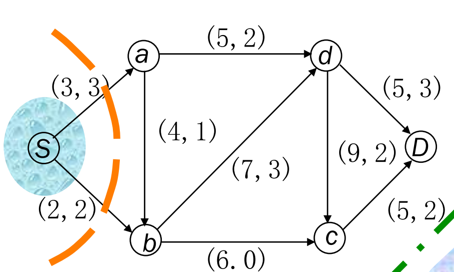
Maximum Flow

A flow F in a network (G, k) is a **maximum flow** if $|F| \geq |F'|$ for every flow F' in (G, k).



A maximal flow with value 5

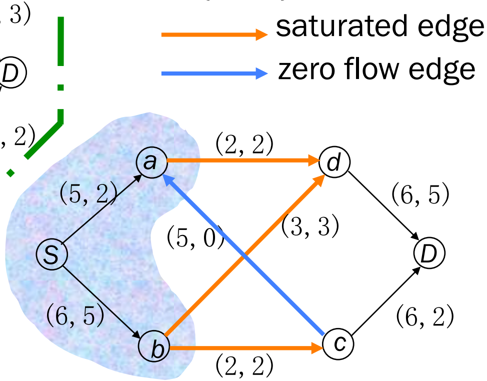
Two Examples of Maximum Flow



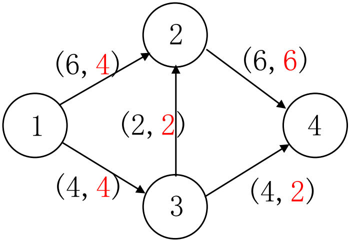
X={S}

Minimal cut={Sa,Sb}

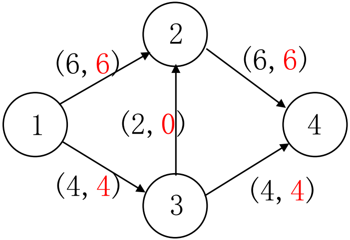
X={S,a,b}



Problem of the Maximum Flows



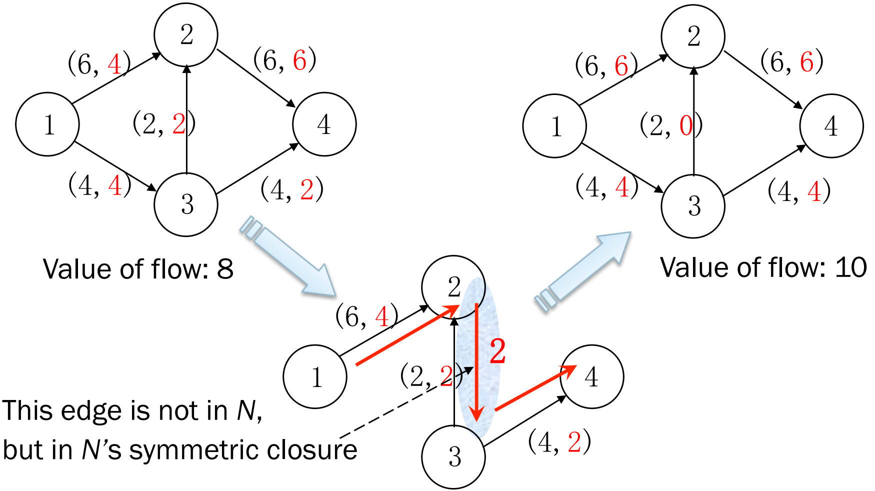
Value of flow: 8



Value of flow: 10

Basic Problems: (1) Largest value of flow? and (2) A flow with the largest value?

Maximum Flows

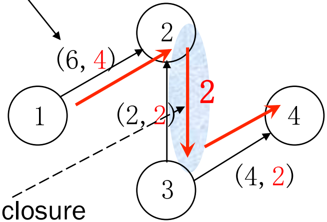


Excess of Capacity

π : 1234 is not a path in N, but in G, the symmetric closure.

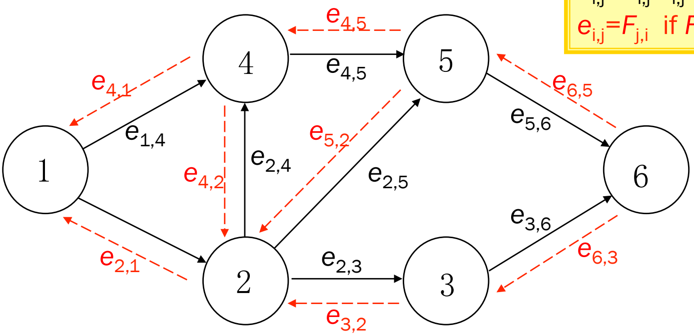
(1, 2) is in N, this edge has a excess capacity 2 (=6-4)

(2, 3) is not in N, this edge has a excess capacity 2



This edge is not in N, but in N's symmetric closure

General Scenario



Excess capacity:

$e_{i,j}=C_{i,j}-F_{i,j}$

$e_{i,j}=F_{j,i}$ if $F_{j,i}>0$

$C_{i,j}$ is the capacity of edge (i,j)

$F_{i,j}$ is the flow on edge (i,j)

edges in N

edges in s(N), but not in N

Labeling Algorithm (Ford & Fulkson)

Initialization: set all flow to 0

Step 1: (1) Identify N_1

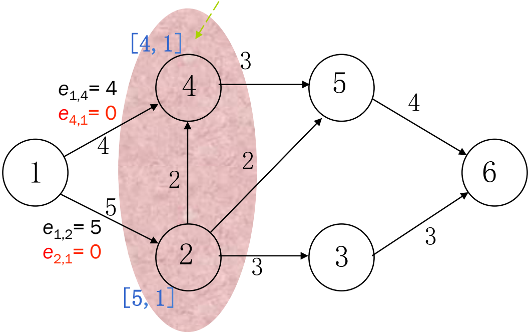
(2) Label nodes in N_1 as follows

Excess capacity of (1,j)

$[E_j, 1]$

Denoting node 1

N_1, all nodes connected to the source by an edge with positive excess capacity



Labeling Algorithm (Ford & Fulkson)

Step 2: (1) Identify $N_2(j)$, based on the node j, with the smallest number, in N_1

(2) Label nodes in $N_2(j)$ as follows

min{E_j, Excess capacity of (j,k)}

$[E_k, j]$

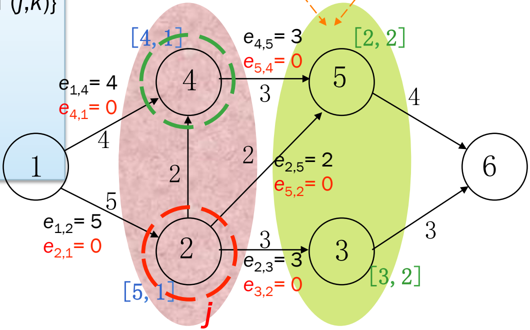
Denoting node j

(3) Do as above for all j in N_1, and let

$$N_2 = \cup_{j \in N_1} N_2(j)$$

$N_2(j)$, all unlabelled nodes connected to node j by an edge with positive excess capacity

Also N_2, in this case



Labeling Algorithm (Ford & Fulkson)

Step 3: Continue as in step 2, forming N_3, N_4, N_5, ..., until:

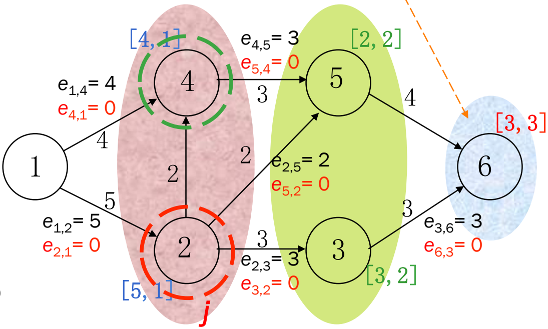
(i) the sink has been labeled, and the total flow is the maximum flow (step 4)

or

(ii) the sink has not been labeled, but no othe nodes can be labeled according to the rules

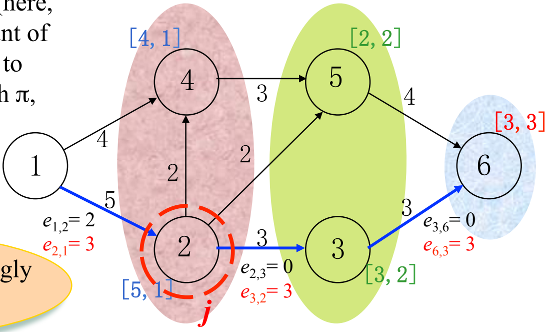
(note: the source is not labeled)

N_3, the sink has been labeled



Situation after one full cycle:

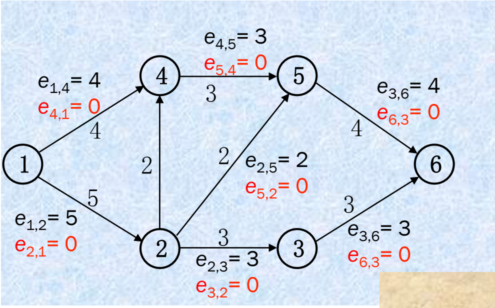
The label of sink is [E_n, m] (here, [3,3]), where E_n is the amount of extra flow that can be made to reach the sink through a path π, and the path can be traced backward by node m



$e_{i,j}$, $e_{j,i}$ are changed accordingly

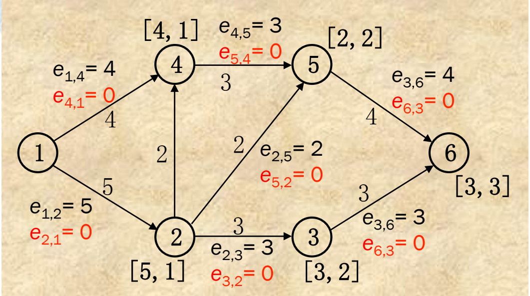
and then return to step 1

Applying Labeling Algorithm



At the beginning, setting all flow to 0

After the first cycle



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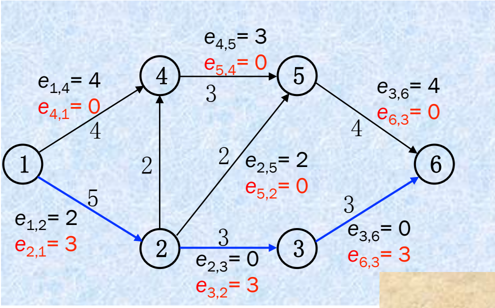
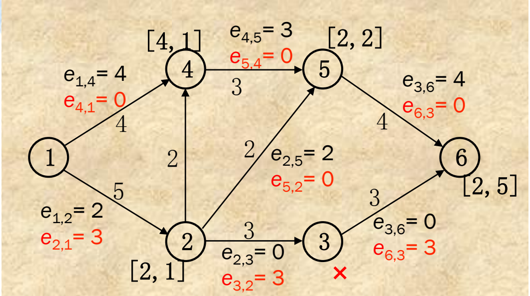
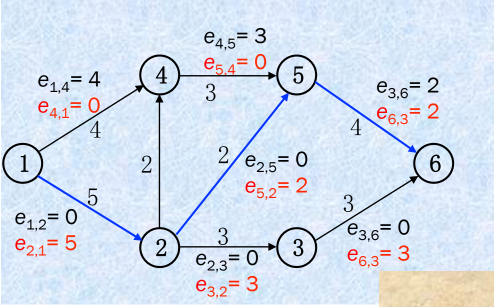


After the first cycle

After the second cycle

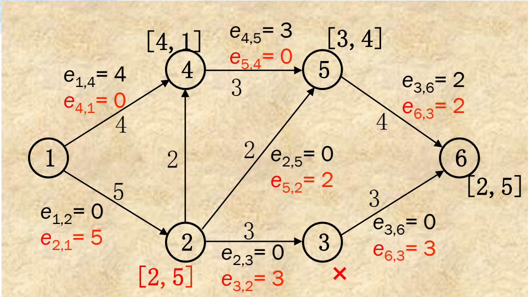


Applying Labeling Algorithm

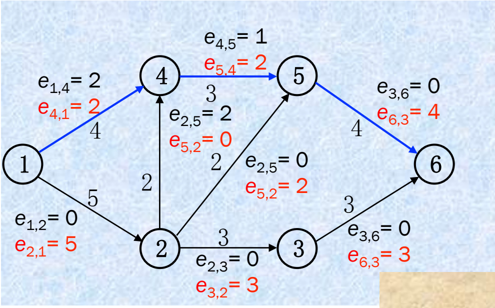


After the second cycle

After the third cycle



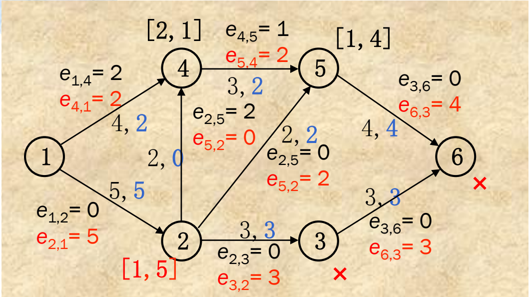
Applying Labeling Algorithm



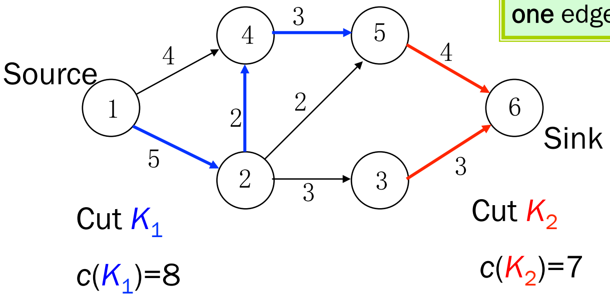
After the third cycle

After the fourth cycle

The sink has not been labeled, so the final result reached



Flow and Cut



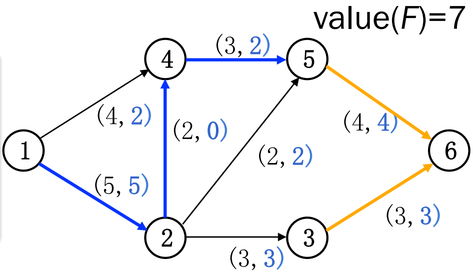
Cut: a set K of edges in a network N, having the property that **every** path from the source to the sink contains **at least one** edge from K.

Max Flow Min Cut Theorem

For any flow F, and any cut K, all parts of F must pass through the edges of K. Since $c(K)$ is the maximum amount that can pass through the edges of K, so, $value(F) \leq c(K)$. If $value(F) = c(K)$, then the flow uses the full capacity of all edges in K, F must be a flow with maximum value, and, on the other hand, K must be a cut with minimum capacity.

Theorem

A maximum flow F in a network has value equal to the capacity of a minimum cut of the network



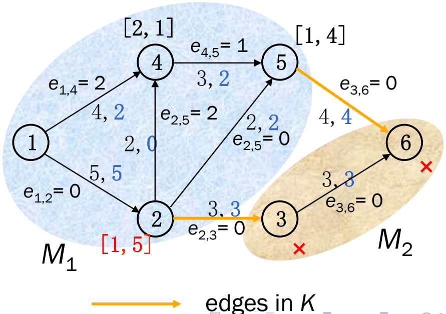
Correctness of Labeling Algorithm

Any path π in N from the source to the sink begins with a node in M_1 and ends with a node in M_2.

Let K consists of all edges in N that connect a node in M_1 with a node in M_2. So, there must be a edge (i, j), with i is the last node in π that belongs to M_1, and j in M_2. So, (i, j) is in K, and K is a cut.

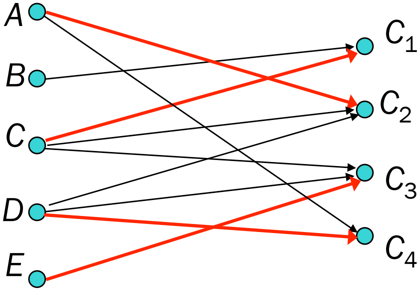
For all such (i, j), the final flow produced by the algorithm must result in (i, j) carrying its full capacity, otherwise, the positive excess capacity will cause j labeled, contradiction.

Algorithm stops at Step 4



"Non-Transport" Transport Network

5 persons A, B, C, D, E belongs to 4 committees C_1, C_2, C_3, C_4, where $C_1 = \{B, C\}$; $C_2 = \{A, C, D\}$; $C_3 = \{C, D, E\}$; $C_4 = \{A, D\}$. Is it possible to select 4 chairperson where no one chairs more than one?



Assignment Problem

- An organisation has n positions to fill and m applicants. Each applicant has a list of qualifications which make him suitable for certain positions.

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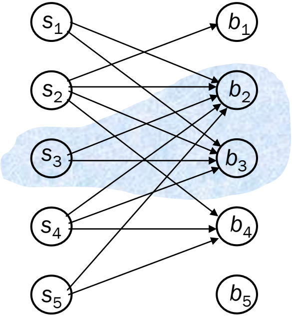
Assignment Problem

- An organisation has n positions to fill and m applicants. Each applicant has a list of qualifications which make him suitable for certain positions.
- Is it possible to assign each applicant to a position to which he or she is suitable?
- If not, what is the largest number of people that can be assigned to the positions?
- How should these assignments be made?

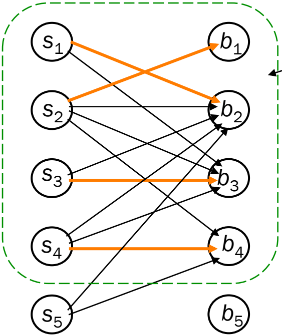
Matching



Relation R

Note: $R(s_3)= b_2$, and $R(s_3)= b_3$,

$R(\{s_3\})=\{ b_2, b_3\}$

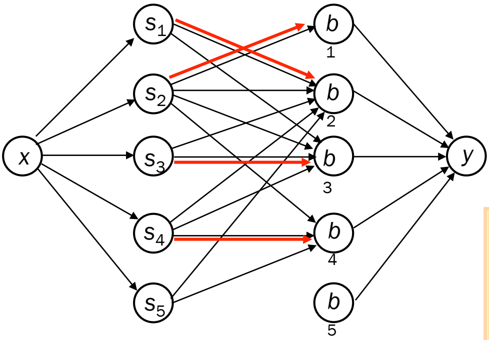


If this is a relation R', then M is a complete matching compatible to R'

Matching function M, compatible to R

A maximal matching, but not complete

Matching and Flow in Network



with each capacity set to 1

Labeling algorithm for max flow is used in the network to compute the matching

Hall's Marriage Theorem

Theorem

Let R be a relation from A to B. Then there exists a complete matching M if and only if for each $X \subseteq A$, $|X| \leq |R(X)|$.

Hall's Marriage Theorem

Theorem

Let R be a relation from A to B. Then there exists a complete matching M if and only if for each $X \subseteq A$, $|X| \leq |R(X)|$.

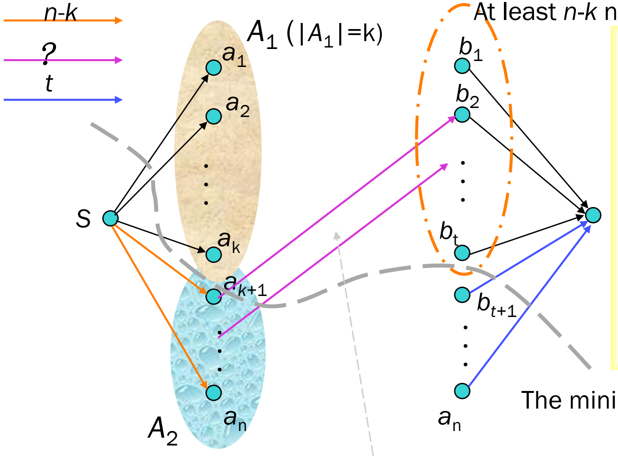
Proof.

$\Rightarrow$: Obvious.

$\Leftarrow$: Show that the minimum cut in N has value $n = |A|$. Suppose K is a minimal cut. We can consider all edges in K as in three sets: S_1: those begin at supersource; S_2: those correspond to pairs in R; S_3: those end at supersink. Considering the situation of removing the three sets one by one, we can see that K contains at least n edges.

$\square$

Proof of Hall's Theorem



Note:

If there are r purple edges, there are at least (n-k)-r nodes in B_2 still connected from S, since removing one edge from K_2 disconnecting one nodes in B_2 at most.

So, there are at least (n-k)-r blue edges. So, the $|K|$ =k+r+(n-k)-r=n

Chromatic Number of Graph

Definition

Coloring each vertex in a graph without rings, if no adjacent vertices has the same color, the coloring is called a **proper coloring**.

Definition

The smallest number of colors needed to produce a proper coloring of a graph G is called the chromatic number of the graph, denoted by $\chi(G)$.

"Commonsense" about $\chi(G)$

- $\chi(G) \leq |V_G|$, and only when $G = K_n$ the equality holds.

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- If H is a subgraph of G, if $\chi(H) = k$, then $\chi(G) \geq k$.

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- If H is a subgraph of G, if $\chi(H) = k$, then $\chi(G) \geq k$.
- If $d(v) = k$, then all the vertices adjacent to v can be properly colored in at most k colors.

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- If H is a subgraph of G, if $\chi(H) = k$, then $\chi(G) \geq k$.
- If $d(v) = k$, then all the vertices adjacent to v can be properly colored in at most k colors.
- The chromatic number of G is the chromatic number of the largest component of G.

Chromatic Number: Examples

- $\chi(G) = 1$ iff. there is no edges in the graph.

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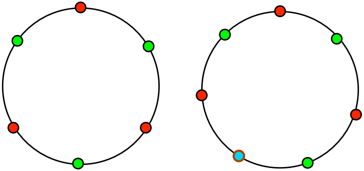
- $\chi(K_n) = n$

Chromatic Number: Examples

- $\chi(G) = 1$ iff. there is no edges in the graph.
- $\chi(K_n) = n$
- if G is a cycle:

$$\chi(G) = \begin{cases} 2 & \text{if } |V_G| = 2k \\ 3 & \text{if } |V_G| = 2k+1 \end{cases}$$



Chromatic Number: Examples

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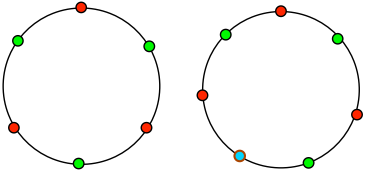
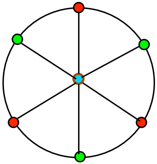


$$\chi(G) = \begin{cases} 2 & \text{if } |V_G| = 2k \\ 3 & \text{if } |V_G| = 2k+1 \end{cases}$$

- if G is a wheel:



$$\chi(G) = \begin{cases} 4 & \text{if } |V_G| = 2k \\ 3 & \text{if } |V_G| = 2k+1 \end{cases}$$

Chromatic Number of Bipartite Graph

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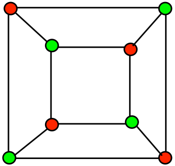
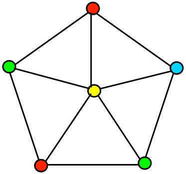
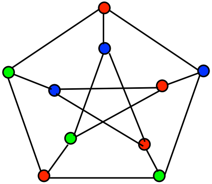
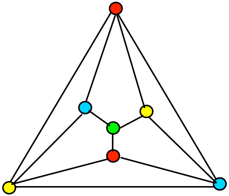
  $\Rightarrow$: if $\chi(G) = 2$, then there is no odd cycle in G;
  $\Leftarrow$: if G is a bipartite graph, then only one color is needed for one set in the partition.

Chromatic Number: Examples









Upper left:

a bipartite graph

Upper right: wheel

Lower left

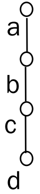
Δ=3， χ=3

Lower right

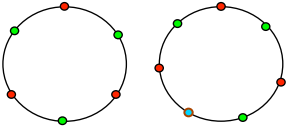
Δ=4， χ>3

Chromatic Polynomial

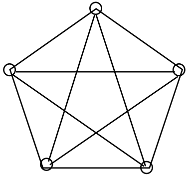
Give a graph G and a set of "color" ($\{c_1, c_2, \cdots, c_n\}$), the number of different coloring for G is a function of n. The function is a polynomial function, called the **chromatic polynomial**.



$P_G(x)=x(x-1)^3$



$P_G(x)=(x-1)^n + (-1)^n(x-1)$



$P_G(x)=x(x-1)(x-2)(x-3)(x-4)$

Recursive Formula

Theorem

$P_G(x) = P_{G_e}(x) - P_{G^e}(x)$, *where* $G_e = G - \{e\}$, G^e *is the graph by merging the edge* e *in* G.

Recursive Formula

Theorem

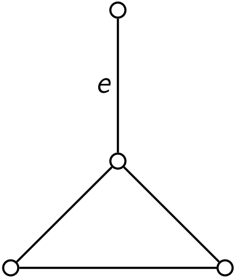
$P_G(x) = P_{G_e}(x) - P_{G^e}(x)$, *where* $G_e = G - \{e\}$, G^e *is the graph by merging the edge* e *in* G.

Proof.

Compare the coloring of G and G_e. For G_e, any coloring in which the two endpoints of e are the same color is not in the coloring of G. Howerver, the number of such coloring scheme is just that of G^e.

□

An Example



G^e is K_3, so, the polynomial is:

$$x(x-1)(x-2)$$

G_e has two component, one a single vertex, and the other, K_3. So, the polynomial is:

$$x(x(x-1)(x-2)) = x^2(x-1)(x-2)$$

So,

$$P_G(x) = x^2(x-1)(x-2) - x(x-1)(x-2) = x(x-1)^2(x-2)$$

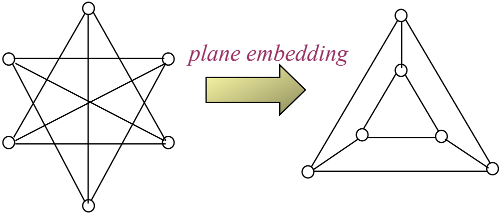
$$\chi(G) = 3$$

Concept of Planar Graph

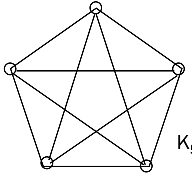
- Embedding a graph on a plane: Drawing a digram for a graph, such that no two edges cross unless on the endpoint.
- A graph is a **planar graph** if it has plane embedding.

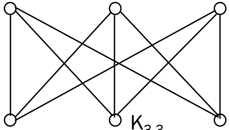
Notes:

- Acyclic graph is planar.
- A graph is non-planar, if any of its subgraph is.
- Unconnected graph can be considered by components.

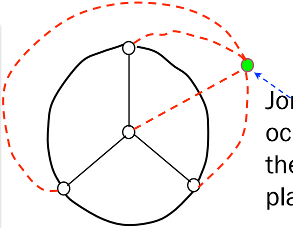


Typical Non-planar Graph





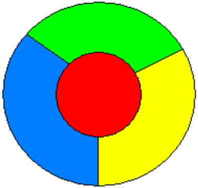
Jordan theorem: *a closed curve C divides the plane into 2 parts: the inner and the outer. The line connecting two vertices located in the two sections respectively must intersection with C*.



Jordan condition ocuurs, wherever the vertex is placed

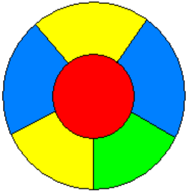
Francis Guthrie's Conjecture

Coloring areas in a map, such that no two areas with common border have the same color, four colors is enough.



3 colors can't do.

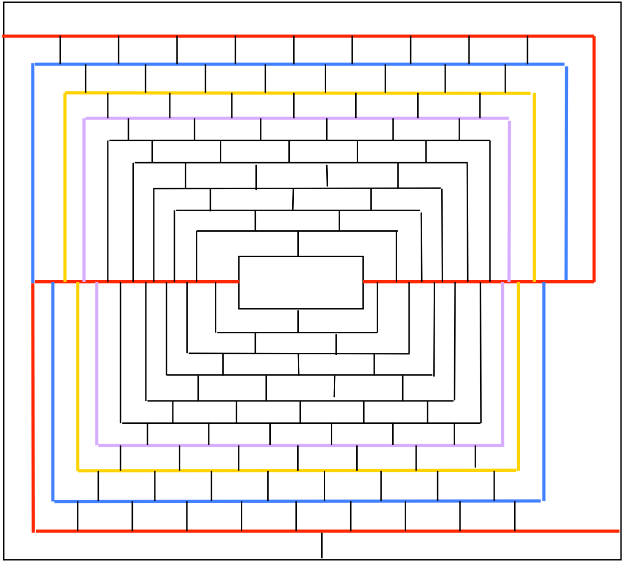
-- Guthrie, de Morgan



It can be proved that no such pattern exits: 5 sections, with each adjacent to all other 4.

But ...

Martin Gardner's Gift for Fool's Day

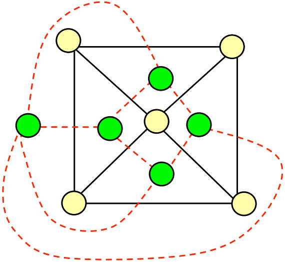


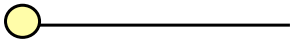
A counterexample for four-color conjecture

?

Scientific American
Fool's Day of 1975

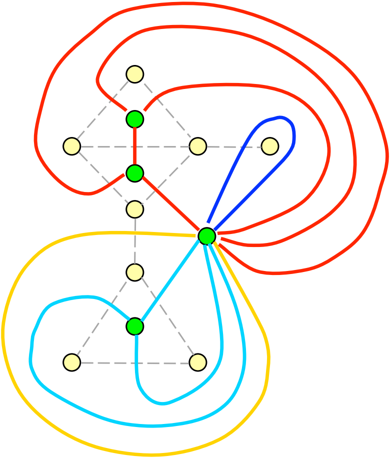
Dual Graph of a Planar Graph





Vertices and edges in G

Vertices and edges in the dual graph of G



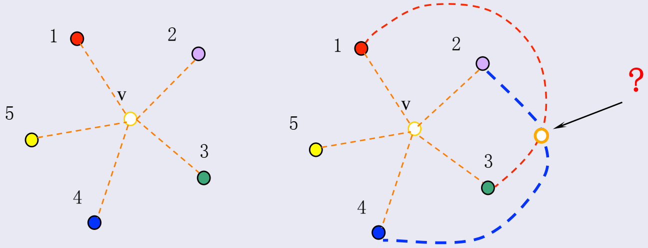
Five Color Theorem

Theorem

Coloring a planar map, with different colors for adjacent areas, five colors is enough.

Proof.

Sketch: There must be a 5-degree (or smaller) vertex in any simple planar. Induction on the number of vertices (if $n \leq 5$, obviously). induction as follows:



Home Assignments

To be checked

- Ex.8.4: 5-11, 14, 19-21
- Ex.8.5: 4, 5, 8, 10, 14-19
- Ex.8.6: 15, 16, 19, 23, 26, 27

Self tests

The End