Lecture 5: Operations on Relations

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At the Last Class

1. Relations and Digraphs
   - Product sets and partitions
   - Binary relations and their digraphic form
   - Paths in relations
   - Representing relations
2. Equivalence Relation
   - Properties of relations
   - Equivalence relations and equivalence classes
   - Equivalence relations and partitions

Overview

1. Basic Operations on Relations
   - Set operations on relations
   - Inverse
   - Composition
   - Closure of Relation
2. Computer Representation and Warshall's Algorithm
   - Representation of Relations in Computer
   - Transitive closure and Warshall's Algorithm

Operations on Relations: Set Operations

Set operations

Relations are sets, so all the operations on sets are applicable for relations.

Example (Operations on relations on $\mathbb{N}$)

- $\text{“}<\text{”} \cup \text{“}=\text{”} \quad = \quad \text{“}\leq\text{”}$
- $\text{“}\leq\text{”} \cap \text{“}\geq\text{”} \quad = \quad \text{“}=\text{”}$
- $\text{“}<\text{”} \cap \text{“}>\text{”} \quad = \quad \text{“}\emptyset\text{”}$

Operations on Relations: Inverse

$R^{-1} = \{(y, x) | (x, y) \in R)\}$

Note: if R is a relation from A to B, R^{-1} is a valid relation from B to A.

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$\left(R^{-1}\right)^{-1} = R$

Proof:
$\left(R^{-1}\right)^{-1} = \{(x, y) | (y, x) \in R^{-1})\} = \{(x, y) | (x, y) \in R)\} = R$

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Proof:
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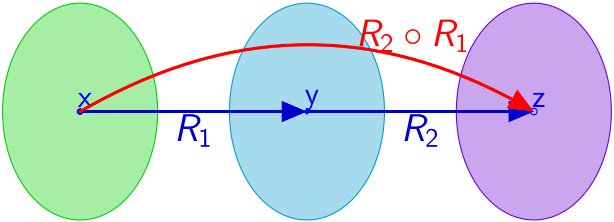
$(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$

Proof: $(x,y) \in (R_1 \cup R_2)^{-1} \Leftrightarrow (y,x) \in R_1 \cup R_2$
$\Leftrightarrow (y,x) \in R_1$ or $(y,x) \in R_2 \Leftrightarrow (x,y) \in R_1^{-1}$ or $(x,y) \in R_2^{-1}$
$\Leftrightarrow (x,y) \in R_1^{-1} \cup R_2^{-1}$

Operations on Relations: Composition

If $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$, (A, B, C are sets).
then: the composition of R_1 and R_2, written as $R_2 \circ R_1$, is a relation from A to C , and $R_2 \circ R_1 =$

$$\{(x, z) | x \in A \wedge z \in C \wedge \exists y(\; y \in B \wedge (x, y) \in R_1 \wedge (y, z) \in R_2 \;)\}$$



Operations on Relations: Composition

Example

Let $A = \{a, b, c, d\}$, R_1, R_2 are relations on A:

- $R_1 = \{(a, a), (a, b), (b, d)\}$
- $R_2 = \{(a, d), (b, c), (b, d), (c, b)\}$

then:

- $R_2 \circ R_1 = \{(a, d), (a, c)\}$
- $R_1 \circ R_2 = \{(c, d)\}$
- $R_1 \circ R_1 = \{(a, a), (a, b), (a, d)\}$
- $(R_1 \circ R_1) \circ R_1 = \{(a, a), (a, b), (a, d)\}$

Power of Composition

$$\begin{cases} R^0 & = I_A \\ R^{n+1} & = R \circ R^n \qquad (n \in \mathbb{N}) \end{cases}$$

R^n corresponds to the relation defined by the path of length n in Digraph of R.

Properties of Relation Composition: Association

Theorem (Associative Law)

$$(R_3 \circ R_2) \circ R_1 = R_3 \circ (R_2 \circ R_1)$$

where $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$, $R_3 \subseteq C \times D$.

Properties of Relation Composition: Association

Theorem (Associative Law)

$$(R_3 \circ R_2) \circ R_1 = R_3 \circ (R_2 \circ R_1)$$

where $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$, $R_3 \subseteq C \times D$.

Proof.

$$
\begin{aligned}
&(x, y) \in (R_3 \circ R_2) \circ R_1 \\
\Leftrightarrow\ & x \in A \wedge y \in D \wedge \exists s(s \in B \wedge xR_1s \wedge s(R_3 \circ R_2)y) \\
\Leftrightarrow\ & x \in A \wedge y \in D \wedge \exists s(s \in B \wedge xR_1s \wedge \exists t(t \in C \wedge sR_2t \wedge tR_3y)) \\
\Leftrightarrow\ & x \in A \wedge y \in D \wedge \exists t(t \in C \wedge tR_3y \wedge \exists s(s \in B \wedge xR_1s \wedge sR_2t)) \\
\Leftrightarrow\ & x \in A \wedge y \in D \wedge \exists t(t \in C \wedge tR_3y \wedge x(R_2 \circ R_1)t) \\
\Leftrightarrow\ & (x, y) \in R_3 \circ (R_2 \circ R_1)
\end{aligned}
$$

$\square$

Properties of Relation Composition: Inverse

Theorem (Inverse of composition)

$$(R_2 \circ R_1)^{-1} = R_1^{-1} \circ R_2^{-1}$$

where $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$.

Properties of Relation Composition: Inverse

Theorem (Inverse of composition)

$$(R_2 \circ R_1)^{-1} = R_1^{-1} \circ R_2^{-1}$$

where $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$.

Proof.

$$\begin{aligned}
& (x, y) \in (R_2 \circ R_1)^{-1} \\
\Leftrightarrow\ & (y, x) \in (R_2 \circ R_1) \\
\Leftrightarrow\ & y \in A \wedge x \in C \wedge \exists t(t \in B \wedge yR_1t \wedge tR_2x) \\
\Leftrightarrow\ & y \in A \wedge x \in C \wedge \exists t(t \in B \wedge tR_1^{-1}y \wedge xR_2^{-1}t) \\
\Leftrightarrow\ & (x, y) \in R_1^{-1} \circ R_2^{-1}
\end{aligned}$$

□

Properties of Relation Composition: Distribution

Theorem (Distribution Law)

$$(G \cup H) \circ F = G \circ F \cup H \circ F$$
$$(G \cap H) \circ F \subseteq G \circ F \cap H \circ F$$

where $F \subseteq A \times B$, $H, G \subseteq B \times C$.

Properties of Relation Composition: Distribution

Theorem (Distribution Law)

$$(G \cup H) \circ F = G \circ F \cup H \circ F$$
$$(G \cap H) \circ F \subseteq G \circ F \cap H \circ F$$

where $F \subseteq A \times B$, $H, G \subseteq B \times C$.

Why the equality doesn't hold?

A wrong proof:

$$
\begin{aligned}
& (x, y) \in G \circ F \cap H \circ F \\
\Leftrightarrow\ & (x, y) \in G \circ F \wedge (x, y) \in H \circ F \\
\Leftrightarrow\ & x \in A \wedge y \in C \wedge \exists t(t \in B \wedge xFt \wedge tGy) \wedge \exists t(t \in B \wedge xFt \wedge tHy) \\
\Leftrightarrow\ & y \in A \wedge x \in C \wedge \exists t(t \in B \wedge xFt \wedge t(G \cap H)y) \\
\Leftrightarrow\ & (x, y) \in (G \cap H) \circ F
\end{aligned}
$$

Perations and Relation Matrix

Let $M_R = |r_{ij}|$, where $r_{ij} = 1$ if $(i, j) \in R$, and 0 otherwise.

$$M_{R\cap S} = M_R \wedge M_S$$
$$M_{R\cup S} = M_R \vee M_S$$
$$M_{R^{-1}} = (M_R)^T$$

Let $A = \{a_1, a_2, \cdots, a_n\}$, $B = \{b_1, b_2, \cdots, b_p\}$, $C = \{c_1, c_2, \cdots, c_n\}$ $R \subseteq A \times B$, $S \subseteq B \times C$, then $M_{S\circ R} = M_R \odot M_S$

Suppose that $M_R = |r_{ij}|$, $M_S = |s_{ij}|$, $M_{S\circ R} = |t_{ij}|$ then $t_{ij} = 1$ if and only if $(i, t) \in R$, $(t, j) \in S$ for some $t \in B$. So, $r_{ij} = 1$, $s_{tj} = 1$, which results in $M_R \odot M_s[i, j] = 1$.

Connectivity Relation

Connectivity relation R^∞ on some set A is defined as:

$\forall x, y \in A, (x, y) \in R^\infty \iff$ there is some path in R from x to y.

Note: $R^\infty = R \cup R^2 \cup R^3 \cup \cdots = \bigcup_{n=1}^{\infty} R^n$

So,

$$\begin{aligned} M_{R^\infty} &= M_R \vee M_{R^2} \vee M_{R^3} \vee \cdots \\ &= M_R \vee (M_R)^2_\odot \vee (M_R)^3_\odot \vee \cdots \end{aligned}$$

Inverse Keeping Properties of Relation

Reflexivity: We have $\forall x(\ (x, x) \in R_1 \Leftrightarrow (x, x) \in R_1^{-1}\ )$

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Irreflexivity: $\forall x(\ (x,x) \notin R_1 \Leftrightarrow (x,x) \notin R_1^{-1}\ )$

Symmetry: $\forall x, y, (x,y) \in R_1^{-1} \Rightarrow (y,x) \in R_1 (R_1 \text{ is symmetric}) \Rightarrow (x,y) \in R_1 \Rightarrow (y,x) \in R_1^{-1}\ )$

Inverse Keeping Properties of Relation

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Symmetry: $\forall x, y, (x,y) \in R_1^{-1} \Rightarrow (y,x) \in R_1$($R_1$ is symmetric) $\Rightarrow$ $(x,y) \in R_1 \Rightarrow (y,x) \in R_1^{-1}\ )$

Antisymmetry: $\forall x, y$, if $(x,y) \in R_1^{-1}, (y,x) \in R_1^{-1}$, then $(y,x) \in R_1, (x,y) \in R_1$, since R_1 is antisymmetric, $x = y$.

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Transitivity: $\forall x, y, z$, if $(x,y) \in R_1^{-1}, (y,z) \in R_1^{-1}$, then $(y,x) \in R_1, (z,y) \in R_1$, since R_1 is transitive, $(z,x) \in R_1$, $\therefore (x,z) \in R_1^{-1}$

Composition Keeping Properties of Relation

Reflexivity: $\forall x$, $\because (x,x) \in R_1$, $(x,x) \in R_2$, $\therefore (x,x) \in R_2 \circ R_1$.

Irreflexivity: ?

Symmetry: ?

Antisymmetry: ?

Transitivity: ?

Composition Keeping Properties of Relation

Reflexivity: $\forall x$, $\because (x, x) \in R_1$, $(x, x) \in R_2$, $\therefore (x, x) \in R_2 \circ R_1$.

Irreflexivity: ? counterexample: $R_1 = \{(a, b)\}, R_2 = \{(b, a)\}$, then $R_2 \circ R_1 = \{(a, a)\}$.

Symmetry: ?

Antisymmetry: ?

Transitivity: ?

Composition Keeping Properties of Relation

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Symmetry: ? counterexample: $R_1 = \{(c,b),(b,c)\}$, $R_2 = \{(c,d),(d,c)\}$, then $R_2 \circ R_1 = \{(b,d)\}$.

Antisymmetry: ?

Transitivity: ?

Composition Keeping Properties of Relation

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Symmetry: ? counterexample: $R_1 = \{(c,b),(b,c)\}$, $R_2 = \{(c,d),(d,c)\}$, then $R_2 \circ R_1 = \{(b,d)\}$.

Antisymmetry: ? counterexample: $R_1 = \{(a,b),(b,c)\}$, $R_2 = \{(c,a),(b,b)\}$, $a \neq b$, then $R_2 \circ R_1 = \{(a,b),(b,a)\}$.

Transitivity: ?

Composition Keeping Properties of Relation

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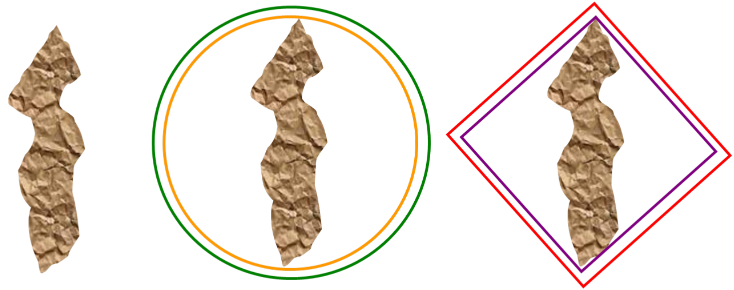
Antisymmetry: ? counterexample: $R_1 = \{(a, b), (b, c)\}$, $R_2 = \{(c, a), (b, b)\}$, $a \neq b$, then $R_2 \circ R_1 = \{(a, b), (b, a)\}$.

Transitivity: ? counterexample:
$R_1 = \{(x, t), (y, s)\}, R_2 = \{(t, y), (s, z)\}$, then $R_2 \circ R_1 = \{(x, y), (y, z)\}$.

Summary of Keeping Properties

	reflexivity	irreflexivity	symmetry	anti-symmetry	transitivity
R^{-1}	✔	✔	✔	✔	✔
$R_1 \cap R_2$	✔	✔	✔	✔	✔
$R_1 \cup R_2$	✔	✔	✔	✘	✘
$R_1 \circ R_2$	✔	✘	✘	✘	✘

Closure – the Idea



an object

The orange circle:
1. circle (property)
2. contain the object
3. if there is a green circle which satisfies above 1 and 2, it must contain the orange

The purple square:
1. square (property)
2. contain the object
3. any square containing the object contains the purple square as well.

Closure: the Generic Definition

Let R be a relation on A, $\mathcal{P}$ is some property, R_1 is called $\mathcal{P}$ **closure** of R if

- R_1 has property $\mathcal{P}$
- $R \subseteq R_1$
- If there is some relation R' on A has property $\mathcal{P}$ and includes R as well, the $R_1 \subseteq R'$

Reflexive Closure

Reflexive closure of R is $R \cup I_A$.

- For any $x \in A, (x, x) \in I_A,$ so $(x, x) \in R \cup I_A$
- $R \subseteq R \cup I_A$
- Let R' be a reflexive relation on A, and $R \subseteq R'$, then for any $(x, y) \in R \cup I_A$, $(x, y) \in R$, or $(x, y) \in I_A$, in both cases $(x, y) \in R'$, so $R \cup I_A \subseteq R'$.

Symmetric Closure

Symmetric closure or R is $R \cup R^{-1}$

- For any $x, y \in A$, if $(x, y) \in R \cup R^{-1}$, then $(x, y) \in R$ or $(x, y) \in R^{-1}$, it follows that $(y, x) \in R^{-1}$ or $(y, x) \in R$, so $(y, x) \in R \cup R^{-1}$
- $R \subseteq R \cup R^{-1}$
- Let R' be a symmetric relation on A and $R \subseteq R'$, then for any $(x, y) \in R \cup R^{-1}$, $(x, y) \in R$ or $(x, y) \in R^{-1}$.
  - case 1: $(x, y) \in R$, then $(x, y) \in R'$
  - case 2: $(x, y) \in R^{-1}$, then $(y, x) \in R$, then $(y, x) \in R'$. Since R' is symmetric, $(x, y) \in R'$.

  So, $R \cup R^{-1} \subseteq R'$

Transitive Closure

Theorem

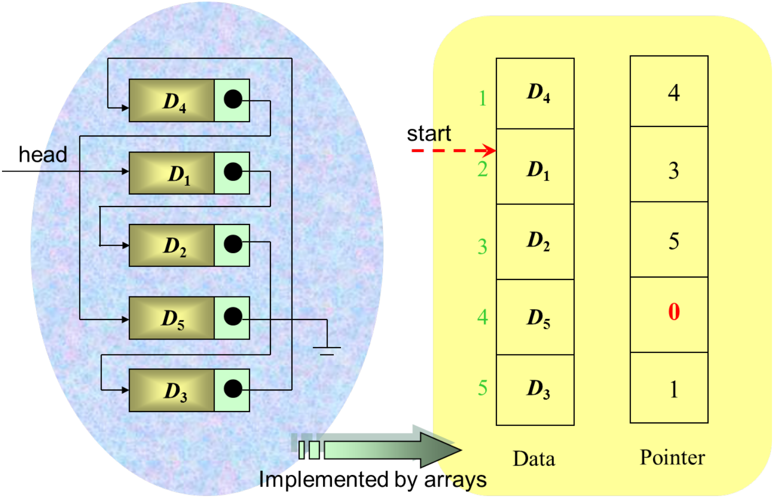
Let R be a relation on a set A, R^∞ is the transitive closure of R.

Proof.

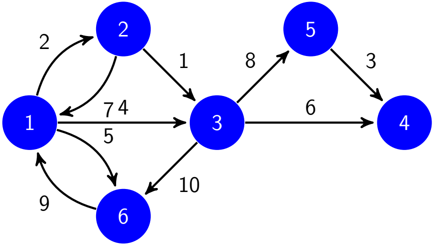
1. if $xR^\infty y$ and $yR^\infty z$, then there exists $s_1, s_2, \cdots, s_j$ and $t_1, t_2, \cdots, t_k$ such that $xRs_1, s_1Rs_2, \cdots, s_jRy$, $yRt_1, t_1Rt_2, \cdots, t_kRz$, so $xR^\infty z$.
2. $R \subseteq R^\infty$
3. Let R' be a transitive relation on A, and $R \subseteq R'$. If $(x, y) \in R^\infty$, then there exits $t_1, t_2, \cdots, t_k$ such that $xRt_1, t_1Rt_2, \cdots, t_kRy$, then $xR't_1, t_1R't_2, \cdots, t_kR'y$. Since R' is transitive, $xR'y$. So $R^\infty \subseteq R'$.

$\square$

Linked List and Its Implementation



Representing a Diagraph as a matrix

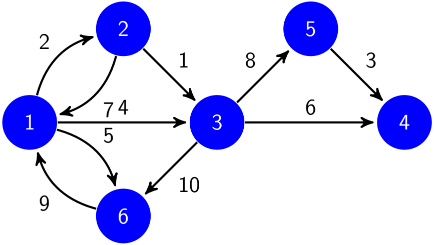


A relation with 10 ordered pairs.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix as 2-dimensional array $A[][]$.

Representing a Diagraph as a Linked List

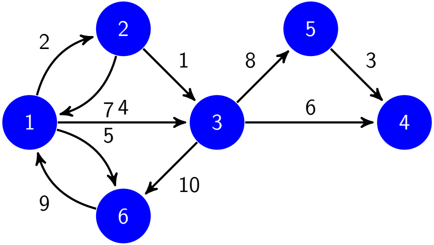


A relation with 10 ordered pairs.

start: $\boxed{2}$

tail	head	next
1	3	9
2	3	10
2	1	4
3	5	8
5	4	1
3	4	3
3	6	0
6	1	7
1	6	6
1	2	5

Indexed by Vertices



A relation with 10 ordered pairs.

vert
10
2
4
0
5
8

tail	head	next
1	2	0
2	3	3
2	1	0
3	5	6
5	4	0
3	4	7
3	6	0
6	1	0
1	6	1
1	3	9

Adding a New Edge: A Comparison

Adding a pair (i, j) to a relation R:

Using matrix:
Simply: $MAT[i, j] \leftarrow 1$

Using linked list:
$P \leftarrow P + 1$ TAIL[P]$\leftarrow$ I
HEAD[P]$\leftarrow$ J
NEXT[P]$\leftarrow$VERT[I]
VERT[I]$\leftarrow$ P

(insert the new inter in front of the list of vertex i)

Checking Transitivity Using Matrix

Determine whether a relation with p ordered pairs is transitive or not.

$RESULT \leftarrow \mathbf{T}$
for $I \leftarrow 1$ **to** N **do**
  for $J \leftarrow 1$ **to** N **do**
    if $\boxed{MAT[I,J]} = 1$ **then** N^2 different MAT$[I,J]$, among which P are "1"
      for $K \leftarrow 1$ **to** N **do** executes P times at most
        if $MAT[J,K] = 1$ *and* $MAT[I,K] = 0$ **then**
          $RESULT = \mathbf{F}$
        end
      end
    end
  end
end

Checking Transitivity Using Matrix

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    if $\boxed{MAT[I, J]} = 1$ **then** N^2 different MAT[I, J], among which P are "1"
      for $K \leftarrow 1$ **to** N **do** executes P times at most
        if $MAT[J, K] = 1$ *and* $MAT[I, K] = 0$ **then**
          $RESULT = \mathbf{F}$
        end
      end
    end
  end
end

So, the total steps executed $T_A = PN + (N^2 - P)$.
Let $P = kN^2$, then $T_A = kN^3 + (1 - k)N^2$

Checking Transitivity Using Linked List

```
RESULT ← T
for I ← 1 to N do                              N times
    X ← VERT[I]
    while X ≠ 0 do                             about D = P/N times
        J ← HEAD[X], Y ← VERT[J]
        while Y ≠ 0 do                         about D times
            K ← HEAD[Y], TEST ← EDGE[I, K]     EDGE takes ∼ D steps
            if TEST = 1 then
                Y ← NEXT[Y]
            else
                RESULT ← F, Y ← NEXT[Y]
            end
        end
        X ← NEXT[X]
    end
end
```

Checking Transitivity Using Linked List

$RESULT \leftarrow \mathbf{T}$
for $I \leftarrow 1$ **to** N **do** N times
    $X \leftarrow VERT[I]$
    while $X \neq 0$ **do** about $D = P/N$ times
        $J \leftarrow HEAD[X]$, $Y \leftarrow VERT[J]$
        while $Y \neq 0$ **do** about D times
            $K \leftarrow HEAD[Y]$, $TEST \leftarrow EDGE[I, K]$ EDGE takes $\sim D$ steps
            if $TEST = 1$ **then**
                $Y \leftarrow NEXT[Y]$
            else
                $RESULT \leftarrow \mathbf{F}$, $Y \leftarrow NEXT[Y]$
            end
        end
        $X \leftarrow NEXT[X]$
    end
end

The total steps is ND^3 averagely.
Assuming $P = kN^2 (0 \leq k \leq 1)$,

$$T_L = N\left(\frac{kN^2}{N}\right)^3 = k^3 N^4$$

Transitive Closure on Finite Set

if $A = n$, then the transitive closure of R is

$$\bigcup_{i=1}^{n} R^i = R \cup R^2 \cup \cdots \cup R^n$$

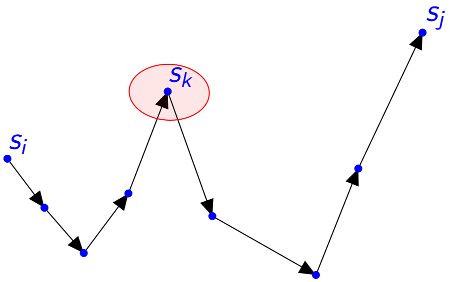
Since the total of elements in A is n, if there is a path of length m from x to y, and $m > n - 1$, then all the nodes on the path cannot be distinct. The segment between two identical nodes can be deleted, which means that: if $xR^{\infty}y$, then for some k, $1 \leq k \leq n$, such that $xR^k y$.

Warshall's Algorithm

```
CLOSURE ← MAT
for K ← 1 to N do //K is the intermediate vertex between I, J
    for I ← 1 to N do
        for J ← 1 to N do
            CLOSURE[I, J] ← CLOSURE[I, J] ∨
                            (CLOSURE[I, K] ∧ CLOSURE[K, J])
        end
    end
end
```

Highest-Numbered Intermediate Vertex

We assume that there is a total order for all vertices (e.g., numbered from 1 to N). s_k is called the **highest-numbered intermediate vertex** in a simple path p from s_i to s_j if it is greater than any other intermediate vertex in p (note that neither s_i nor s_j is an intermediate vertex).



Correctness of Washall's Algorithm

Notation: the value of r_{ij} changes during the execution of the body of the "**for** K" loop

- After initialisations: $r_{ij}^{(0)}$
- After the k-th time of execution: $r_{ij}^{(k)}$

Correctness of Washall's Algorithm

If there is a simple path from s_i to s_j $(i \neq j)$ for which the highest-numbered intermediate vertex is s_k, then $r_{ij}^{(k)} = 1 \quad (\mathbf{T})$.

Correctness of Washall's Algorithm

If there is a simple path from s_i to s_j $(i \neq j)$ for which the highest-numbered intermediate vertex is s_k, then $r_{ij}^{(k)} = 1 \quad (\mathbf{T})$.

Proof by induction.

Base case: $r_{ij}^{(0)} = 1$ if and only if $s_i R s_j$.

Hypothesis: the conclusion holds for $h < k$ $(h \geq 0)$

Induction: the simple $s_i s_j$-path can be divided into $s_i s_k$-path and $s_k s_j$-path, with the indices h_1, h_2 as their highest-numbered intermediate vertices, respectively. Because $h_1 < k$ and $h_2 < k$, $r_{ik}^{(h_1)} = 1$ and $r_{kj}^{(h_2)} = 1$, then $r_{ik}^{(k-1)} = 1$ and $r_{kj}^{(k-1)} = 1$ (Remember, false to true can not be reversed), So $r_{ij}^{(k)} = 1$.

□

Correctness of Washall's Algorithm

If there is no path from s_i to s_j $(i \neq j)$, then $r_{ij} = 0$ $\quad$ (**F**).

Correctness of Washall's Algorithm

If there is no path from s_i to s_j $(i \neq j)$, then $r_{ij} = 0$ $\quad$ (**F**).

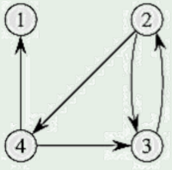
Proof.

if $r_{ij} = 1$, then there are only two cases:

- r_{ij} is set by initialisation, then $s_i R s_j$
- Otherwise, r_{ij} is set during the k-th execution of "**for** K" when $r_{ik}^{(k-1)} = 1$ and $r_{kj}^{(k-1)} = 1$, which, recursively, leads to the conclusion of the existence of a $s_i s_j$-path. (Note, if a $s_i s_j$-path exists, there exist a simple $s_i s_j$-path)

$\square$

Example



$$T^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$T^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad T^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Source: http://integrator-crimea.com/ddu0157.html

Home Assignments

To be checked

Ex 4.6: 2,3,4,6,8,12

Ex 4.7: 7, 8, 12, 14, 19, 20, 23-24, 26-28, 30-31, 36-37

Ex 4.8: 8,10,12,14,18, 20, 23-25

The End