“离散数学”课堂练习

（2014 年 11 月 14 日）

参考答案

1. 试定义谓词，符号化下面的命题并推证其结论：

“每个大学生不是文科生就是理科生，有的大学生是优等生，小张不是理科生，但他是优等生，因而如果小张是大学生，他就是文科生。”

参考解答： 1、定义谓词：$G(x)$: x是大学生，$S(x)$: x是理科生，$L(x)$: x是文科生，

$E(x)$: x是优等生. 则上述描述可符号化如下子句（c代表小张）：

$$(\forall x)\big(G(x) \to L(x) \vee S(x)\big), (\exists x)\big(G(x) \wedge E(x)\big), \neg S(c) \vdash E(c) \Rightarrow G(c) \to L(c)$$

2、推理过程如下：

(1) $G(c)$ (Premise)

(2) $(\forall x)(G(x) \to L(x) \vee S(x))$ (Premise)

(3) $G(c) \to L(c) \vee S(c)$ (UI from 2)

(4) $L(c) \vee S(c)$ (MP from 3)

(5) $\neg S(c)$ (Premise)

(6) $L(c)$ (DS from 4)

(7) $G(c) \to L(c)$ (from 4,5)

2. 试证明下式永真：$(\boldsymbol{p} \to \neg \boldsymbol{q}) \wedge (\boldsymbol{r} \to \boldsymbol{q}) \to (\boldsymbol{r} \to \neg \boldsymbol{p})$.

证明：

方法一：真值表：

$A=(p\to\neg q)\land(r\to q)\to(r\to\neg p)$ 的真值表

p	q	r	$p\to\neg q$	$r\to q$	$r\to\neg p$	A
0	0	0	1	1	1	1
0	0	1	1	0	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	1	1	1
1	0	1	1	0	0	1
1	1	0	0	1	1	1
1	1	1	0	1	0	1

方法二：命题逻辑等值演算

$$
\begin{aligned}
&(p\to\neg q)\land(r\to q)\to(r\to\neg p)\\
\Leftrightarrow&(\neg p\lor\neg q)\land(q\lor\neg r)\to(\neg p\lor\neg r)\\
\Leftrightarrow&(p\land q)\lor(\neg q\land r)\lor\neg p\lor\neg r\\
\Leftrightarrow&((p\land q)\lor\neg p)\lor((\neg q\land r)\lor\neg r)\\
\Leftrightarrow&(\neg p\lor q)\lor(\neg q\lor\neg r)\\
\Leftrightarrow&\neg p\lor(q\lor\neg q)\lor\neg r\\
\Leftrightarrow&1
\end{aligned}
$$

方法三：主析取范式法

$$
\begin{aligned}
&(p\to\neg q)\land(r\to q)\to(r\to\neg p)\\
\Leftrightarrow&(\neg p\lor\neg q)\land(q\lor\neg r)\to(\neg p\lor\neg r)\\
\Leftrightarrow&(p\land q)\lor(\neg q\land r)\lor\neg p\lor\neg r\\
\Leftrightarrow&(m_6\lor m_7)\lor(m_1\lor m_5)\lor(m_0\lor m_1\lor m_2\lor m_3)\lor(m_0\lor m_2\lor m_4\lor m_6)\\
\Leftrightarrow&m_0\lor m_1\lor m_2\lor m_3\lor m_4\lor m_5\lor m_6\lor m_7
\end{aligned}
$$

由于原式的主析取范式含全部 8 个极小项，为重言式，故推理正确.

3. 设R是集合A上任意自反且传递的二元关系，试证明：$R\circ R=R$.

证明：因为R是A上的传递关系，故$R\circ R\subseteq R$，只需证$R\subseteq R\circ R$即可：

$$\forall(a,b)\in R\Longrightarrow\ (a,a)\in R\ \land\ (a,b)\in R\ \Longrightarrow\ (a,b)\in R\circ R$$

证毕. 注意该命题的逆命题并不成立.

4. 证明：对于所有集合A，B，C，有：

$$(A \cap B) \cup C = A \cap (B \cup C) \text{ 当且仅当} C \subseteq A.$$

证明：

必要性：$\forall x \in C \Rightarrow x \in (A \cap B) \cup C \quad \Rightarrow x \in A \cap (B \cup C) \Rightarrow x \in A \Rightarrow C \subseteq A$;

充分性：$C \subseteq A \Rightarrow A \cup C = A \quad \Rightarrow (A \cap B) \cup C = (A \cup C) \cap (B \cup C) = A \cap (B \cup C)$.

证毕.

5. 试求定义在一个n元集合上的不同二元关系的总数.

解：

方法一：将每个二元关系看做一个关系矩阵，因为无任何限制，因此，关系矩阵的个数即为二元关系的个数。关系矩阵为$n \times n$个元素，每个元素只能取0或者1，因此关系矩阵的总数为2^{n^2}个，这表明定义在n元集上的二元关系总数为2^{n^2}.

方法二：设n元集为A，因为任何二元关系皆为$A \times A$之子集，而$|A \times A| = n^2$故所有定义在n元集A上的二元关系的总数等于$A \times A$之子集的总数，即$|P(A \times A)| = 2^{n^2}$.

6. 请给出集合$\{a, b, c, d, e\}$上包含关系$\{(a, b), (a, c), (d, e)\}$的最小等价关系（*i.e.* 上述关系的“等价闭包”）.

解：这个最小等价关系为：$\{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (d, e), (e, d), (a, a), (b, b), (c, c), (d, d), (e, e)\}$

7. 设B为布尔代数，试证明：

(1) $(\forall a_1, a_2, \cdots, a_n \in B)((a_1 \vee a_2 \vee \cdots \vee a_n)' = a_1' \wedge a_2' \wedge \cdots \wedge a_n')$

(2) $(\forall x, y \in B)(x \preccurlyeq y \Leftrightarrow y' \preccurlyeq x')$

其中x'表示x之补元.

(1) 证明：

对n实施数学归纳法。当$n=2$时，等式成立（德摩根律），

I.H. $n=k$时命题成立，则

I.S.

$$(a_1 \vee a_2 \vee \cdots \vee a_{k+1})' = ((a_1 \vee a_2 \vee \cdots \vee a_k) \vee a_{k+1})'$$
$$= (a_1 \vee a_2 \vee \cdots \vee a_k)' \wedge a_{k+1}'$$
$$= (a_1' \wedge a_2' \wedge \cdots \wedge a_k') \wedge a_{k+1}'$$
$$= a_1' \wedge a_2' \wedge \cdots \wedge a_k' \wedge a_{k+1}'$$

(2) 证法一：

证：

$$x \preccurlyeq y \Leftrightarrow x \wedge y = x \Leftrightarrow x' \vee y' = x' (\text{de Morgen}) \Leftrightarrow y' \preccurlyeq x'$$

证法二：$(\forall x, y \in B) x \preccurlyeq y \Leftrightarrow x \wedge y = x \Leftrightarrow (x\square y)' = x \Leftrightarrow x' \vee y' = x' \Leftrightarrow y' \preccurlyeq x'$. □

8. 一逻辑学家误入某部落，被拘于牢狱，部落酋长是一位逻辑爱好者，他对逻辑学家说:“今有两门，一为自由，二为死亡，你可以任意开启一门。今加派两名卫兵负责解答你所提的任何问题。遗憾的是，此两卫兵中一名天性诚实，一名总是说谎。你的生死就看你的智慧了。”逻辑学家沉思片刻，即向一卫兵问了一个问题，得到答案后从容开门而去。该逻辑学家问了一个怎样的问题？请简要分析.

解：

逻辑学家手指某一门问其中一名战士说:“这扇门是死亡之门，另外一名战士将回答‘是’，对吗？”. 当被问战士回答“对”，则逻辑学家开启所指的门离去。当被问战士回答“否”，则逻辑学家开启另一门离去.简要分析如下：

设命题变元 P:被问战士是诚实人；

Q:被问战士的回答是“是”；

R:另一战士的回答是“是”；

S:这扇门是死亡门.

故有真值表:

P	Q	R	S
T	T	T	F
T	F	F	T
F	T	F	F
F	F	T	T

观察真值表可知: $S \Leftrightarrow \neg Q$。即被问人回答“是”时,此门不是死亡之门;否则是死亡之门.

9. (选人问题)

a) n个人排成一列,从中选出k个人$(n > 2k)$,使得选出的人在原队列中不相邻,共有多少种选法?
b) n个人坐成一圈,从中选出k个人$(n > 2k)$,使得选出的人在原圈子中不相邻共有多少种选法?

解:

(a) 可反过来想:把 k 个人插回到 n-k 个人的队列中,要使得插回的人不相邻,则共有 n-k+1 个合法位置,于是共有$\binom{n-k+1}{k}$种选法。

(b) 有多种方法,结果可写成下列任何一种形式:

$$\begin{aligned}\frac{n}{k}\binom{n-k-1}{k-1} &= \binom{n-k+1}{k} - \binom{n-k-1}{k-2} \\ &= \binom{n-k-1}{k-1} + \binom{n-k}{k} \\ &= 2\binom{n-k-1}{k-1} + \binom{n-k-1}{k}\end{aligned}$$

- *Solution 1:*
  We pick one random person to be the 'root', and we choose that guy to be one of the k *people.*
  For the remaining part, it is equal to choose $k-1$ *non-adjacent people from a list of* $n-3$ *people (as we cannot choose the 'root' and its neighbors), which have* $\binom{(n-3)-(k-1)+1}{k-1} = \binom{n-k-1}{k-1}$ *choices.*
  Picking different person to be the 'root' construct a different combination, so we have n *possibilities.*
  For each combination, picking those k *people to be the 'root' create a different answer above, but still the same combination, so it is a* $k-1$ *mapping.*
  The answer is $\frac{n}{k}\binom{n-k-1}{k-1}$

- *Solution 2:*
  Let treat the round table as a list, so we know the answer of choosing k *non-adjacent people is* $\binom{n-k+1}{k}$
  We find that it is invalid in this problem if and only if we choose the first guy and the last guy at the same time.
  If we choose the first and the last person, then the remaining part is equal to choosing $k-2$ *non-adjacent person from a list of* $n-4$ *person(as we cannot choose the* $1st$*,* $2nd$*,* $(n-1)$*-th and* n*-th), number of choices is* $\binom{n-4-(k-2)+1}{k-2} = \binom{n-k-1}{k-2}$.
  The answer is $\binom{n-k+1}{k} - \binom{n-k-1}{k-2}$

- *Solution 3:*
  Consider the $1st$ *person.*
  If we choose the $1st$ *person, then we cannot choose the* $2nd$ *and the* n*-th person, and the remaining part is equal to choosing* $k-1$ *non-adjacent person from a list of* $n-3$ *person, which have* $\binom{(n-3)-(k-1)+1}{k-1} = \binom{n-k-1}{k-1}$ *choices*
  If we do not choose the $1st$ *person, the remaining part is equal to choosing* k *non-adjacent person from a list of* $n-1$ *person, which have* $\binom{(n-1)-k+1}{k} = \binom{n-k}{k}$ *choices.*
  The answer is $\binom{n-k-1}{k-1} + \binom{n-k}{k}$

- *Solution 4:*
  There are three possible cases:
  If we choose the $1st$ *person and not the* n*-th person, then we cannot choose the* $2nd$*, and the remaining part is equal to choosing* $k-1$ *non-adjacent person from a list of* $n-2$ *person, which have* $\binom{(n-3)-(k-1)+1}{k-1} = \binom{n-k-1}{k-1}$ *choices.*
  If we choose the n*-th person and not the* $1st$ *person, then we cannot choose the* $n-1$*-th, and the remaining part is equal to choosing* $k-1$ *non-adjacent person from a list of* $n-2$ *person, which have* $\binom{(n-3)-(k-1)+1}{k-1} = \binom{n-k-1}{k-1}$ *choices.*
  If we choose neither the n*-th person nor the* $1st$ *person, the remaining part is equal to choosing* k *non-adjacent person from a list of* $n-2$ *person, which have* $\binom{(n-2)-k+1}{k} = \binom{n-k-1}{k}$ *choices.*
  The answer is $2\binom{n-k-1}{k-1} + \binom{n-k-1}{k}$