A5

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1 Number of different binary trees

(From *CLRS* 12-4)

Let b_n denote the number of different binary trees with n nodes. In this problem, you will find a formula for b_n, as well as an asymptotic estimate.

1. Show that $b_0 = 1$ and that, for $n \geq 1$,

$$b_n = \sum_{k=0}^{n-1} b_k b_{n-1-k}$$

2. Let $B(x)$ be the generating function

$$B(x) = \sum_{n=0}^{\infty} b_n x^n$$

   Show that $B(x) = xB(x)^2 + 1$, and hence one way to express $B(x)$ in closed form is

$$B(x) = \frac{1}{2x}(1 - \sqrt{1-4x})$$

3. Show that

$$b_n = \frac{1}{n+1}\binom{2n}{n}$$

   (the nth **Catalan number**) by using the Taylor expansion of $\sqrt{1-4x}$ around $x = 0$.

   p.s. If you wish, instead of using the Taylor expansion, you may use the generalization of the binomial theorem (where n can be any real number) to noninteger exponents.

4. Show that

$$b_n = \frac{4^n}{\sqrt{\pi} n^{3/2}}(1 + O(1/n))$$

2 AVL trees

(From CLRS 13-3)

An **AVL tree** is a binary search tree that is **height balanced**: for each node x, the heights of the left and right subtrees of differ by at most 1. To implement an AVL tree, maintain an extra attribute h in each node such that $x.h$ is the height of node x. As for any other binary search tree T, assume that node $T.root$ points to the root node. Prove that an AVL tree with n nodes has height $O(\lg n)$.

(*Hint:* Prove that an AVL tree of height h has at least F_h nodes, where F_h is the hth Fibonacci number.)