Relations and Digraphs

Lecture 4

Discrete Mathematical Structures

Relations and Digraphs

- Part I: Relations and Digraphs
  - Product sets and partitions
  - Binary relations and their digraphic form
  - Paths in relations
  - Representing relations
- Part II: Equivalence Relation
  - Properties of relations
  - Equivalence relations and equivalence classes
  - Equivalence relations and partitions

为“关系”建立数学模型

可以将“大学在籍”看成某个个人与某个大学之间的关系。

我们能够如何描述这个关系呢？

Ordered Pair and Cartesian Product

- For any sets A,B

$$A \times B = \{(a,b) | a \in A,\ b \in B\}$$

  is called ***Cartesian Product*** of A and B

- Example：

  $\{1,2,3\} \times \{a,b\} = \{(1,a), (2,a), (3,a), (1,b),(2,b), (3,b)\}$

- For finite A， B， $|A \times B| = |A| \times |B|$

Generalized Cartesian Product

- Cartesian product of *m* nonempty sets:

  $$A_1 \times A_2 \times \ldots \times A_m = \{(a_1, a_2, \ldots, a_m) \mid a_i \in A_i, i=1,2\ldots,m\}$$

- Describing the attributes of objects using Cartesian product:

A computer program can be characterized by 3 attributes:

Language={C(c), Java(j), Fortran(f), Pascal(p), Lisp(l)}

Memory={2 meg(2), 4 meg(4), 8 meg(8)}

OS={UNIX(u), Windows(w), Linus(l)}

Then, any object in Language×Memory×OS can be assigned to a specific program to characterized it.

Properties of Cartesian Product

- $A\times\phi=\phi\times A=\phi$
- $A\times B=B\times A \Leftrightarrow A=B \vee A=\phi \vee B=\phi$
  - Proof
  - $\Leftarrow$
    - If $A=\phi$, $A\times B = \phi\times B=\phi$, $B\times A=\phi$, So $A\times B=B\times A$
    - If $B=\phi$, $A\times B=B\times A$
    - If $A=B$, $A\times B=B\times A$
  - $\Rightarrow$ By contradiction
  - If $A\neq B$, $A\neq\phi$, and $B\neq\phi$, there must an a such that either $a\in A$, but $a\notin B$; or $a\notin A$, but $a\in B$.
    - CASE 1: let b be any element in B, then $(a,b)\in A\times B$, but $(a,b)\notin B\times A$, $A\times B \neq B\times A$. Contradiction!
    - CASE 2:…

Properties of Cartesian Product

$$For\ any\ sets\ A, B\ and\ C$$

$$\begin{aligned}
A \times (B \cup C) &= \{(x,y) \mid x \in A,\ y \in B\ or\ y \in C\} \\
&= \{(x,y) \mid x \in A,\ y \in B\ \ or\ \ x \in A,\ y \in C\} \\
&= \{(x,y) \mid (x,y) \in A \times B\ \ or\ \ (x,y) \in A \times C\} \\
&= (A \times B) \cup (A \times C)
\end{aligned}$$

$$Easy\ to\ see:$$

These are called the first distribution laws

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Relation as a Set

- Let A and B be nonempty sets. A **relation *R* from *A* to *B*** is a subset of $A \times B$.
  - If $a \in A$, $b \in B$, then "a is related to b by R" is written as:

    $$(a,b) \in R, \text{ or, } aRb$$

- R is **a relation on A**, if $R \subseteq A \times A$

Some Relations as Examples

- A={1,2,3}, B={r,s}, R={(1,r),(2,s),(3,r)}, then R is a relation from A to B.
- A={1,2,3,4}, R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}
  - then R is a relation on A, $\{(x,y) \mid x<=y,\ x\in A,\ y\in A\}$
- N is the set of all natural numbers (starting from 1), defining a relation R on N, such that, for any $m,n\in N$, $(m,n)\in R$ if and only if m divides n. So, $R\subseteq N\times N$, and R contains (3,6), (5,25), (7,21), etc.
  - $\{(m,n) \mid m\in N, n\in N, m|n\}$

Special Binary Relations

- Empty relation on (any) set A.
  - It is just a empty set.
- Universal relation on set A: E_A
  - $E_A = A \times A$
- Equality: I_A
  - $I_A = \{(x,x) | x \in A\}$

Domain and Range of Relations

- Let $R \subseteq A \times B$, then
  - The domain of R, Dom(R) is defined as: $\{x | x \in A$, and exists some $y \in B$, such that $xRy\}$
  - The range of R, Ran(R) is defined as: $\{y | y \in B$, and exists some $x \in A$, such that $xRy\}$
  - Note: Dom(R)$\subseteq A$, and, Ran(R)$\subseteq B$

R-relative Set

- If R is a relation from set A to B
  - For any $x \in A$, R-relative set of x, $R(x)$ is:

    $\{y \mid y \in B, xRy\}$ (this is a subset of B)
  - For any $A_1 \subseteq A$, R-relative set of $R(A_1)$ is:

    $\{y \mid y \in B$, there exists some $x \in A_1$ such that $xRy\}$
  - Note that: $R(A_1) = \bigcup_{x \in A_1} R(x)$

Properties of R-relative Sets

- Let R be a relation from A to B, A_1, A_2 be subsets of A, then:

$$(a)\quad A_1 \subseteq A_2 \Rightarrow R(A_1) \subseteq R(A_2)$$
$$(b)\quad R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$$
$$(c)\quad R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$$

Proof of (c): for any $y \in R(A_1 \cap A_2)$, then there exists some x in $A_1 \cap A_2$ such that xRy. So, $x \in A_1$, $x \in A_2$. It follows that $y \in R(A_1)$ and $y \in R(A_2)$, thus $y \in R(A_1) \cap R(A_2)$.

Equality doesn't hold. Counterexample: considering relation "≤" on Z, $A_1=\{0,1,2\}$, $A_2=\{9,13\}$, $R(A_1)$ is the set of all nonnegative integers, and $R(A_2)$ is the set of integers not less than 9, so, $R(A_1) \cap R(A_2)=\{9,10,11,12,...\}$, but $A_1 \cap A_2=\phi$, which results $R(A_1 \cap A_2)=\phi$.

Representing Relations as Matrices

$A=\{a_1, a_2, a_3\}$

$B=\{b_1, b_2, b_3, b_4\}$

$R=\{(a_1, b_1), (a_1, b_4),$
$(a_2, b_2), (a_2, b_3),$
$(a_3, b_1), (a_3, b_3)\}$

$$\mathbf{M} = \begin{array}{cc} \begin{matrix} \;\;b_1 & \;b_2 & \;b_3 & \;b_4 \end{matrix} & \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} & \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} \end{array}$$

$(a_i, b_j) \in R$ if and only if:

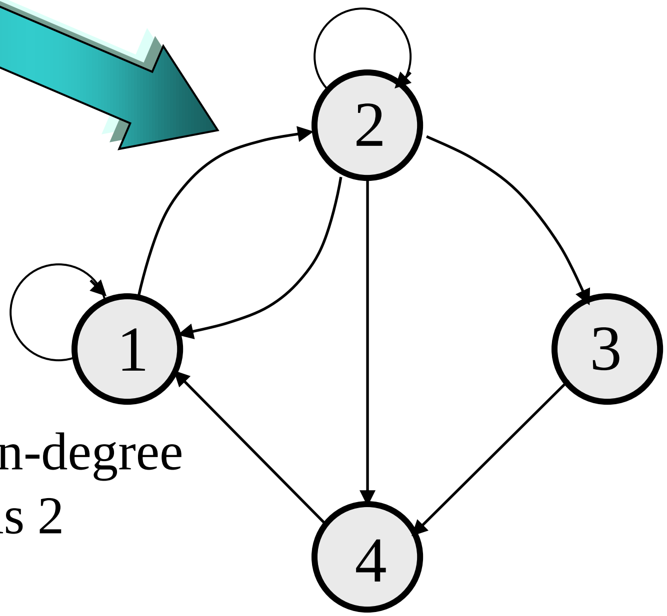
$\boldsymbol{m_{i,j}=1}$

Representing Relations as Digraphs

Digraph representation is used only for relations on one set.

A={1,2,3,4}

R={(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4), (4,1)}

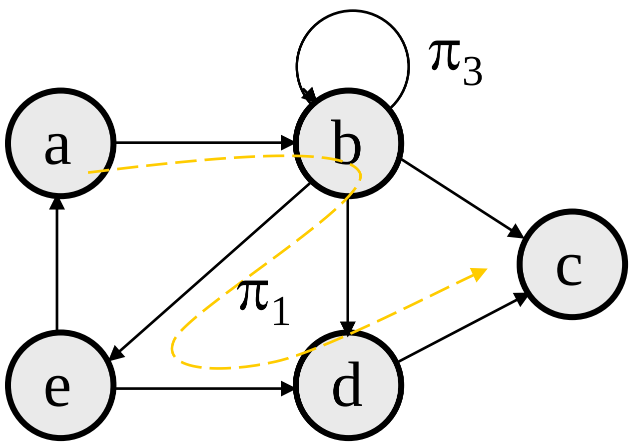


For node 1, the in-degree is 3, out-degree is 2

Path in Digraph

- A path of length n in R from a to b is a finite sequence π: a, x_1, x_2,..., x_{n-1}, b, such that: aRx_1, $x_{n-1}Rb$, and x_iRx_{i+1} for i=1,...n-2
- A path in R corresponds to a succession of edges in the digraph representation of the relation, which consists of n edges.
- It is not required that all elements in a, x_1, x_2,..., x_{n-1}, b are distinct.

New relations defined using Paths



$$aR^4c$$

$$aR^2d \quad (a,b,d)$$

$$aR^{k+2}d \quad (a,\underbrace{b,b,\ldots,b}_{k+1\ b\text{'s}},d)$$

$$bRb$$

$$bR^3b \quad (b,e,a,b)$$

π_1: a, b, e, d, c length: 4

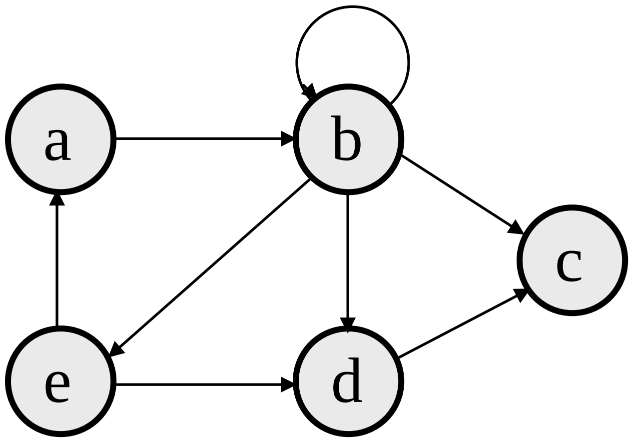
π_2: a, b, e, a length: 3(cycle)

π_3: b, b length: 1(ring)

Generalized(connctivity):

xR^{∞}y if there is a path of any length from x to y.

New relations defined using Paths



Digraph of *R*

R^2={(a,b), (a,c), (a,d), (a,e), (b,a), (b,b), (b,c), (b,d), (b,e), (e,b), (e,c)}

R^3={(a,a), (a,b), (a,c), (a,d), (a,e), (b,a), (b,b), (b,c), (b,d), (b,e), (e,b), (e,e)}

R^∞ ={(a,a), (a,b), (a,c), (a,d), (a,e), (b,a), (b,b), (b,c), (b,d), (b,e), (d,c), (e,a), (e,b), (e,c), (e,d), (e,e)}

R^2 by Matrix Multiplication

If R is a relation on $A=\{a_1,a_2,...,a_n\}$, then $M_{R^2}=M_R \otimes M_R$

Proof:

Let $M_R=[m_{ij}]$ and $M_{R^2}=[n_{ij}]$. Let $M^*=[m^*_{ij}]=M_R \otimes M_R$,

then $m^*_{ij}=1$ if and only if for some $k(1 \le k \le n)$, $m_{ik}=1$ and $m_{kj}=1$.

By definition of relation matrix, $a_i R a_k$, $a_k R a_j$.

Thus $a_i R^2 a_j$, and so $n_{ij}=1$, which means that $m^*_{ij}=1$

if and only if $n_{ij}=1$.

So, $M_R \otimes M_R = M_{R^2}$

R^n by Matrix Multiplication

For $n \geq 2$, and R a relation on a finite set A, we have

$$M_{R^n} = M_R \otimes M_R \otimes \cdots \otimes M_R \quad (n \text{ factors})$$

Let P(n) mean that the statement holds for n >= 2

Base: P(2) holds

Induction: Assuming that P(k) Let $M_{R^{k+1}} = [x_{ij}]$, $M_{R^k} = [y_{ij}]$, and $M_R = [m_{ij}]$.

a) $x_{ij} = 1$ => M_{R^k} Ä $M_R[i,j]=1$.

If $x_{ij} = 1$, these is a (k+1)-path from a_i to a_j. Let a_s be the node next the last node a_j. So, there is a k-path from a_i to a_s, an edge from a_s to a_j,

i.e. $y_{is} = 1$, $m_{sj} = 1$. M_{R^k} Ä $M_R[i,j]=1$.

b) M_{R^k} Ä $M_R[i,j]=1$ => $x_{ij} = 1$

If M_{R^k} Ä $M_R[i,j]=1$, there must be an s, $y_{is} = 1$, and $m_{sj} = 1$. Which means there is a k-path from a_i to a_s, and an edge from a_s to a_j. …

Connectivity Relation

Connectivity relation, R^{∞} on some set A is defined as:

$\forall x, y \in A$, $(x, y) \in R^{\infty}$ if and only if

there is some path in R from x to y

Note: $R^{\infty} = R \cup R^2 \cup R^3 \cup \cdots = \bigcup_{n=1}^{\infty} R^n$

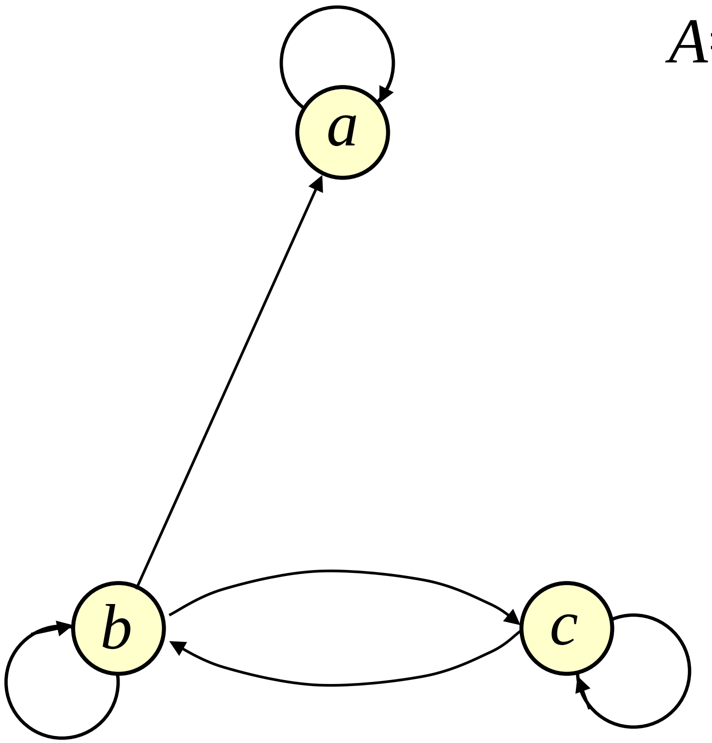
So, $M_{R^{\infty}} = M_R \vee M_{R^2} \vee M_{R^3} \vee \cdots$

$$= M_R \vee \left(M_R\right)_{\otimes}^{2} \vee \left(M_R\right)_{\otimes}^{3} \vee \cdots \left(M_R\right)_{\otimes}^{n}$$

Reflexivity

- Relation R on A is
  - **Reflexive** if for all $a \in A$, $(a,a) \in R$
  - **Irreflexive** if for all $(a,a) \notin R$
- Let $A=\{1,2,3\}$, $R \subseteq A \times A$
  - $\{(1,1),(1,3),(2,2),(2,1),(3,3)\}$ is reflexive
  - $\{(1,2),(2,3),(3,1)\}$ is irreflexive
  - $\{(1,2),(2,2),(2,3),(3,1)\}$ is neither reflexvie nor irreflexive.
- R is reflexive relation on A if and only if $I_A \subseteq R$

Visualized Reflexivity



$A=\{a,b,c\}$

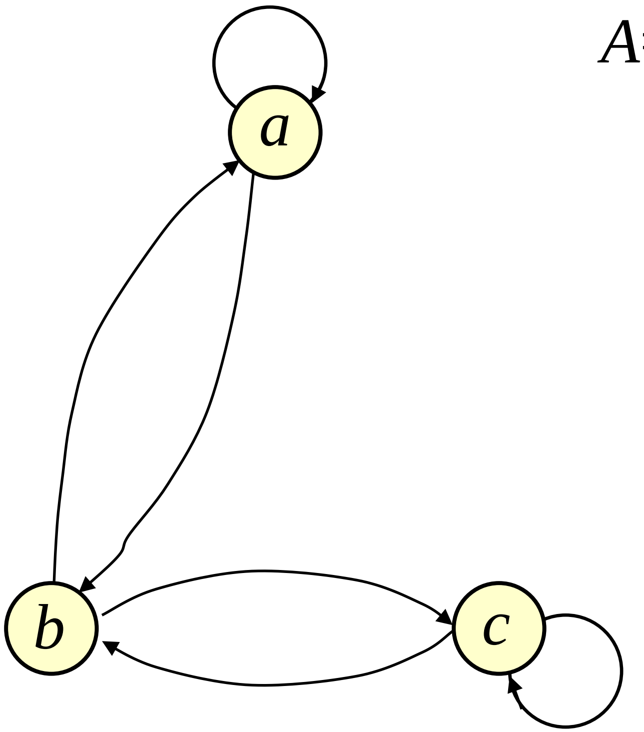
$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Symmetry

- Relation R on A is
  - **Symmetric** whenever $(a,b)\in R$, then $(b,a)\in R$
  - **Antisymmetric** if whenever $(a,b)\in R$ and $(b,a)\in R$ then $a=b$.
  - **Asymmetric** if whenever $(a,b)\in R$ then $(b,a)\notin R$

  (Note: neither anti- nor a-symmetry is the negative of symmetry)
- Let $A=\{1,2,3\}$, $R\subseteq A\times A$
  - $\{(1,1),(1,2),(1,3),(2,1),(3,1),(3,3)\}$ is symmetric.
  - $\{(1,2),(2,3),(2,2),(3,1)\}$ is antisymmetric.
  - $\{(1,2),(2,3),(3,1)\}$ is antisymmetric and asymmetric.
  - $\{(11),(2,2)\}$ is symmetric and antisymmetric.
  - ϕ is symmetric and antisymmetric, and asymmetric!
- R is symmetric relation on A if and only if $R^{-1}=R$

Visualized Symmetry



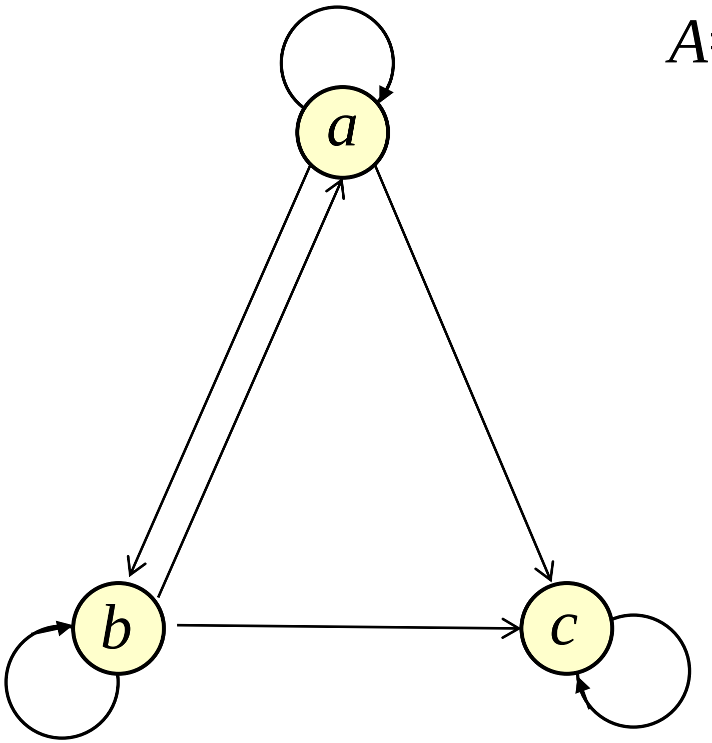
$$A=\{a,b,c\}$$

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Transitivity

- Relation R on A is
  - **Transitivity** if whenever $(a,b) \in R$, $(b,c) \in R$, then $(a,c) \in R$
- Let $A=\{1,2,3\}$, $R \subseteq A \times A$
  - $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,3)\}$ is transitive
  - $\{(1,2),(2,3),(3,1)\}$ is not transitive.
  - Both $\{(1,3)\}$ and ϕ are transitive.
- R is transitive relation on A if and only if $R^n \subseteq R$ for all $n \geq 1$

Visualized Transitivity



$A=\{a,b,c\}$

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Some Often Used Relations

	=	≤	<	|	$\equiv_3$	ϕ	E
reflexivity	✓	✓	✗	✓	✓	✗	✓
irreflexivity	✗	✗	✓	✗	✗	✓	✗
symmetry	✓	✗	✗	✗	✓	✓	✓
antisymmetry	✓	✓	✓	✓	✗	✓	✗
transitivity	✓	✓	✓	✓	✓	✓	✓

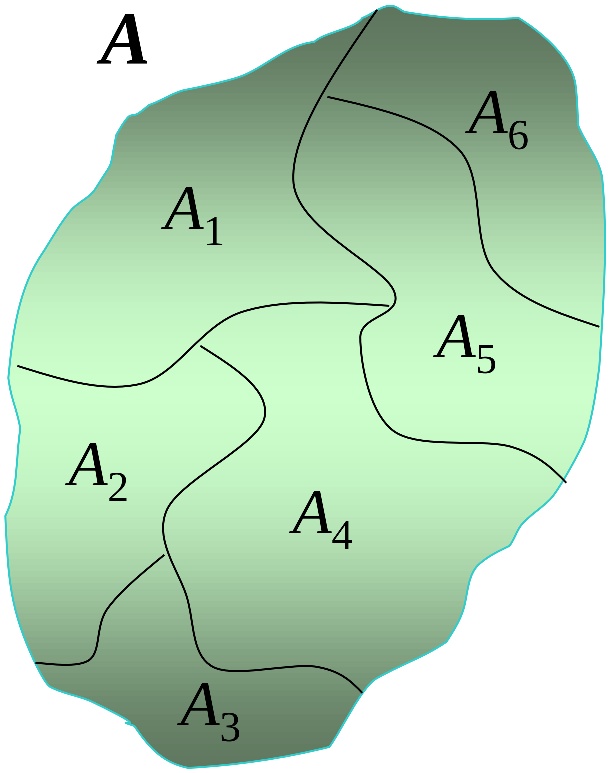
What's Wrong?

- A wrong proof: if R is a symmetric and transitive relation on A, then R must be reflexive.
- Proof:
  - For any a,b$\in A$, if $(a,b)\in R$, by the symmetry of R, $(b,a)\in R$; since R is transitive, $(a,a)\in R$. So, R is reflexive.

Equivalence Relation

- Relation R on A is an equivalence relation if and only if it is reflexible, symmetric and transitive.
- "Equility" is a special case of equivalence relation.
- An example:
  - $R \subseteq Z \times Z$, $(x,y) \in R$ if and only if $\frac{|x-y|}{3}$ is integer, i.e. $x \equiv y \pmod 3$

Partition of a Set



A **partition** of a set A, π, is a set of the nonempty subsets of A. (so, $\pi \subseteq \rho(A)$), satisfying:

1. For any $x \in A$, there is some $A_i \in \pi$, such that $x \in A_i$.

$$\text{i.e. } \bigcup_i A_i = A$$

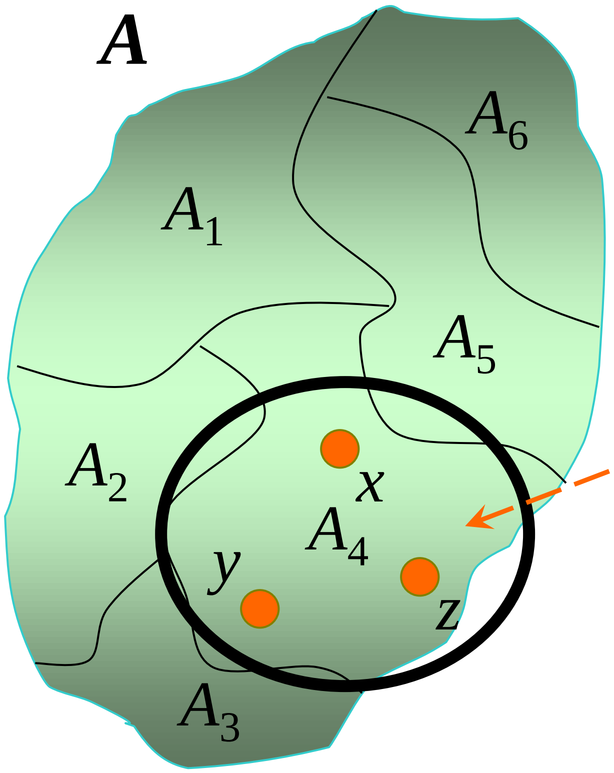
2. For any $A_i, A_j \in \pi$, if $i \neq j$, then:

$$A_i \cap A_j = \phi$$

Partition Generated by Equivalence

- **Equivalence class**:
  - Let R is a equivalence relation on A, then given $a \in A$, $R(a)$ is a equivalence class induced by R.
- **Quotient set**:
  - $Q=\{R(x)|x \in A$, and R is a equivalence on $A\}$
- Quotient set is a partition:
  - For any $a \in A$, $a \in R(\mathrm{a})$ (remember that R is reflexible)
  - For any $a,b \in A$
    - $(a,b) \in R$ if and only if $R(a)=R(b)$, and
    - $(a,b) \notin R$ if and only if $R(a) \cap R(b)=\phi$

Equivalence induced by Partition



Given a partition on A, we can define a relation R on A as following:

$\forall x,y \in A$, $(x,y) \in R$ if and only if:

x,y belong to the same block.

Ex. $(x,y)\in R$ $(y,z)\in R$ $(x,z)\in R$ $(x,x)\in R$ etc.

It is straight to prove that R is reflexible, symmetric and transitive, so, it is an equivalence relation.

Product of Equivalence

- R_1,R_2 are equivalences defined respectively on sets X_1 and X_2. Define relation S on $X_1 \times X_2$ as follows:

  $$\langle x_1,x_2\rangle S\langle y_1,y_2\rangle \text{ if and only if } x_1R_1y_1 \text{ 且 } x_2R_2y_2$$

- Then, S is also a equivalence, defined on $X_1 \times X_2$.
  - Reflexivity: for any $\langle x,y\rangle \in X_1 \times X_2$, since both R_1,R_2 are reflexive, $\langle x,x\rangle \in R_1$, $\langle y,y\rangle \in R_2$; $\therefore \langle x,y\rangle S\langle x,y\rangle$;
  - Symmetry: assume that $\langle x_1,x_2\rangle S\langle y_1,y_2\rangle$, which means that $x_1R_1y_1$ and $x_2R_2y_2$, so, $y_1R_1x_1$ and $y_2R_2x_2$, because of the symmetry of R_1 and R_2. So, $\langle y_1,y_2\rangle S\langle x_1,x_2\rangle$;
  - Transitivity: assume that $\langle x_1,x_2\rangle S\langle y_1,y_2\rangle$, and $\langle y_1,y_2\rangle S\langle z_1,z_2\rangle$, then $x_1R_1y_1$, $y_1R_1z_1$, $x_2R_2y_2$, $y_2R_2z_2$. Since both R_1 and R_2 are transitive, we have: $x_1R_1z_1$, and $x_2R_2z_2$, so, $\langle x_1,x_2\rangle S\langle z_1,z_2\rangle$.

An Example with Geometry

- For (x, y) and (u, v) *in* R^2, define:

  $(x, y) \sim (u, v)$ iff. $x^2 + y^2 = u^2 + v^2$.
- Prove that ~ defines an equivalence relation on R^2 and interpret the equivalence classes geometrically.

Another example, revisited

- Prove:

  Among any 1001 different numbers randomly selected from the subset of natural numbers {1,2,...,2000} must be two, x,y， satisfying $x/y=2^k$.

  (k is an integer)

The Proof

- Create 1000 sets, each contains a unique odd integer between 1 and 2000, along with its multiplication of 2^k not greater than 2000.
  - A relation R: for any x,y in the sett, x*R*y if and only if both can be represented as $p2^{k1}$ $p2^{k2}$ for same p and some k1,k2 (k1,k2>=1).
  - It is easy to prove that it is an equivalence,
  - x R y implies $x/y = 2^m$ or $y/x = 2^m$, for some m

- The **Quotient** has 1000 elements.

等价关系用于计数

- 用英语单词“hello”中的5个字母可以造出多少个不同的“词”？
  - 可以先假设两个“l”一个是大写，一个是小写，显然可以造出5！个“词”。在这些“词”的集合上定义关系*R*, *a R b*当且仅当忽略大小写，*a*,*b*完全一样。可以证明这是等价关系，我们要求的结果恰是等价类的个数。
- 如果是用英语单词“aardvark”代替上述例子中的“hello”，结果是多少呢？

Home Assignments

- To be checked
  - Ex 4.1: 16; 18; 24; 30-31, 33-40
  - Ex 4.2: 20; 25-26; 28, 32, 34; 36
  - Ex 4.3: 18-21; 27-28; 30-33
  - Ex 4.4: 14, 16, 18, 20, 22, 31-36; 38; 40
  - Ex 4.5: 19-20, 22-24, 27-29