Operations on Relations

Lecture 5

Discrete Mathematical Structures

Operations on Relations

- Part I: Basic Operations on Relations
  - Set operations on relations
  - Inverse
  - Composition
  - Closure of Relation
- Part II: Computer Representation and Warshall's Algorithm
  - Representation of Relations in Computer
  - Transitive closure and Warshall's Algorithm

Operations on Relations: (1)

- Relations are sets, so, all the operations on sets are applicable for relations.
  - Examples on the set of natural numbers:
    - $\text{“}<\text{”} \cup \text{“}=\text{”} = \text{“}\leq\text{”}$
    - $\text{“}\leq\text{”} \cap \text{“}\geq\text{”} = \text{“}=\text{”}$
    - $\text{“}<\text{”} \cap \text{“}>\text{”} = \phi$

Operations on Relations (2)

- Inverse
  - $R^{-1} = \{(y,x) | (x,y) \in R\}$
    - Note: if *R* is a relation from *A* to *B*, R^{-1} is a valid relation from *B* to *A*.
  - $(R^{-1})^{-1} = R$
    - Proof: $(R^{-1})^{-1}=\{(x,y)|(y,x) \in R^{-1}\}=\{(x,y)|(x,y) \in R\}$
  - $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$
    - Proof:

$$(x,y) \in (R_1 \cup R_2)^{-1} \Leftrightarrow (y,x) \in R_1 \cup R_2$$

$$\Leftrightarrow (y,x) \in R_1 \text{ or } (y,x) \in R_2 \Leftrightarrow (x,y) \in R_1^{-1} \text{ or } (x,y) \in R_2^{-1}$$

$$\Leftrightarrow (x,y) \in R_1^{-1} \cup R_2^{-1}$$

Operations on Relations (3)

- Composition
  - Rule:

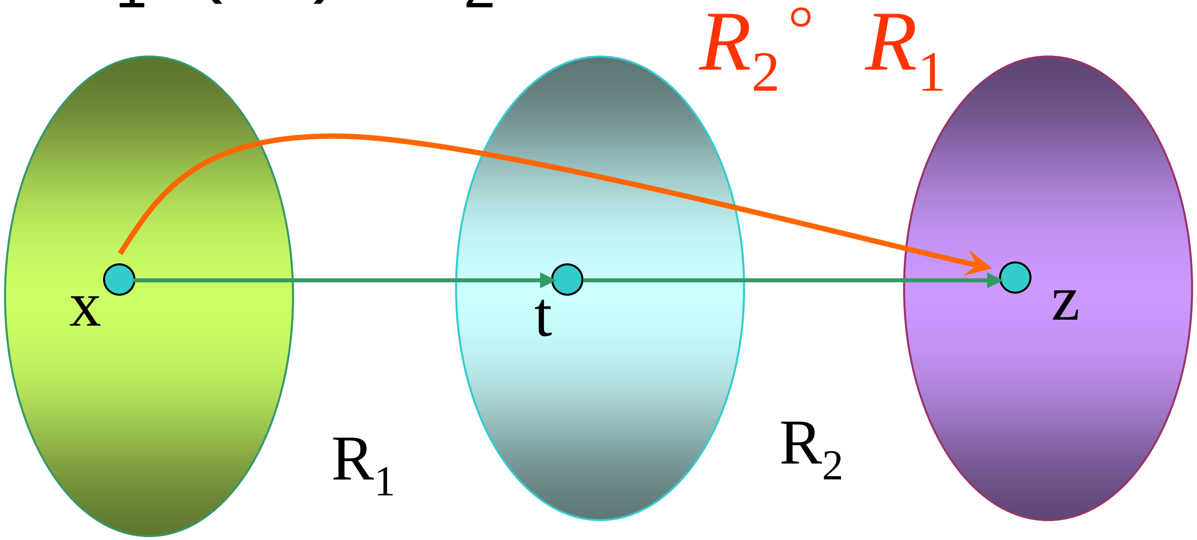
  If $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$, (A,B,C are sets)

  then: the composition of R_1 and R_2, written as $R_2 \circ R_1$ is a relation from A to C, and

  $R_2 \circ R_1 = \{(x,z) | x \in A, z \in C$, and there exists some $y \in B$, such that $(x,y) \in R_1$, $(y,z) \in R_2\}$

Composition of Relation

- $(x,z) \in R_2 \circ R_1$ if and only if $x \in A$, $z \in C$, and there exists some $t \in B$, such that $(x,t) \in R_1$, $(t,z) \in R_2$



Composition: Examples

- Let $A=\{a,b,c,d\}$, R_1, R_2 are relations on A：

  $R_1 = \{(a,a),(a,b),(b,d)\}$

  $R_2 = \{(a,d),(b,c),(b,d),(c,b)\}$

  then:

  $R_2 \circ R_1 = \{(a,d),(a,c)\}$

  $R_1 \circ R_2 = \{(c,d)\}$

  $R_1 \circ R_1 = \{(a,a),(a,b),(a,d)\}$

  $(R_1 \circ R_1) \circ R_1 = \{(a,a), (a,b), (a,d)\}$

Power of Composition

$$\begin{cases} R^0 = I_A \\ R^{n+1} = R \circ R^n \end{cases}$$

R^n corresponds the relation defined by the path of length n in Digraph of R.

Properties of Relation Composition(1)

- Associative Law

$$(R_3 \circ R_2) \circ R_1 = R_3 \circ (R_2 \circ R_1)$$

(where, $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$, $R_3 \subseteq C \times D$)

  - Proof
  - $(x,y) \in (R_3 \circ R_2) \circ R_1 \Leftrightarrow x \in A$, $y \in D$, and there exists $s \in B$, such that xR_1s and $s(R_3 \circ R_2)y \Leftrightarrow$ there exist $t \in C$, such that xR_1s, sR_2t, $tR_3y \Leftrightarrow$ $x(R_2 \circ R_1)t$, $tR_3y \Leftrightarrow (x,y) \in R_3 \circ (R_2 \circ R_1)$

Properties of Relation Composition(2)

- Inverse of composition
  - $(R_2 \circ R_1)^{-1} = R_1^{-1} \circ R_2^{-1}$ (where $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$)
  - Proof
  - $(x,y) \in (R_2 \circ R_1)^{-1} \Leftrightarrow (y,x) \in R_2 \circ R_1 \Leftrightarrow$ there exists some $t \in B$, such that yR_1t and $tR_2x \Leftrightarrow xR_2^{-1}t$ and $tR_1^{-1}y \Leftrightarrow (x,y) \in R_1^{-1} \circ R_2^{-1}$

Properties of Relation Composition(3)

- Distribution Law
  - $(G \cup H) \circ F = G \circ F \cup H \circ F$

    (where $F \subseteq A \times B$, and H, $G \subseteq B \times C$)
  - $(G \cap H) \circ F \subseteq G \circ F \cap H \circ F$
    - Why the equality doesn't hold?
    - A wrong proof: $G \circ F \cap H \circ F \subseteq (G \cap H) \circ F$
      - if $(x,y) \in G \circ F \cap H \circ F$, then $(x,y) \in G \circ F$, $(x,y) \in H \circ F$. So, there exists some t, such that $(x,t) \in F$, and $(t,y) \in G$, $(t,y) \in H$. So, $(t,y) \in G \cap H$, it follows that $(x,y) \in (G \cap H) \circ F$

Operations and Relation Matrix

$$M_{R\cap S} = M_R \wedge M_S$$

$$M_{R\cup S} = M_R \vee M_S$$

$$M_{R^{-1}} = (M_R)^T$$

Suppose that $M_R = \lfloor r_{ij} \rfloor, M_S = \lfloor s_{ij} \rfloor, M_{S\circ R} = \lfloor t_{ij} \rfloor$

then, $t_{ij} = 1$ if and only if

$(i,t) \in R, (t,j) \in S$ for some $t \in B$,

so, $r_{it} = 1$, $s_{tj} = 1$, which results in $M_R \otimes M_S[i,j] = 1$

Let $A = \{a_1,\ldots,a_n\}, B = \{b_1,\ldots,b_p\}, C = \{c_1,\ldots,c_m\}$

$R \subseteq A\times B, S \subseteq B\times C$, then $M_{S\circ R} = M_R \otimes M_S$

Connectivity Relation

Connectivity relation, R^∞ on some set A is defined as:

$\forall x, y \in A, (x, y) \in R^\infty$ if and only if

there is some path in R from x to y

Note: $R^\infty = R \cup R^2 \cup R^3 \cup \cdots = \bigcup_{n=1}^{\infty} R^n$

So, $M_{R^\infty} = M_R \vee M_{R^2} \vee M_{R^3} \vee \cdots$

$$= M_R \vee \left(M_R\right)^2_\otimes \vee \left(M_R\right)^3_\otimes \vee \cdots$$

Inverse Keeping Properties of Relation

- **Reflexivity**: $\forall x$, $(x,x) \in R_1 \Leftrightarrow (x,x) \in R_1^{-1}$
- **Irreflexivity**: $\forall x$, $(x,x) \notin R_1 \Leftrightarrow (x,x) \notin R_1^{-1}$
- **Symmetry**: $\forall x$,y, if $(x,y) \in R_1^{-1}$, then $(y,x) \in R_1$, since R_1 is symmetric，$(x,y) \in R_1$, $\therefore (y,x) \in R_1^{-1}$
- **Antisymmetry**: $\forall x$,y, if $(x,y) \in R_1^{-1}$, $(y,x) \in R_1^{-1}$, then $(y,x) \in R_1$, $(x,y) \in R_1$, since R_1 is antisymmetric，x=y.
- **Transitivity**: $\forall x,y,z$, if $(x,y) \in R_1^{-1}$, $(y,z) \in R_1^{-1}$,

  then $(y,x) \in R_1$, $(z,y) \in R_1$, since R_1 is transitive，$(z,x) \in R_1$, $\therefore (x,z) \in R_1^{-1}$

Composition Keeping Properties of Relation

- **Reflexivity**: $\forall x$, $\because (x,x)\in R_1$ and $(x,x)\in R_2$, $\therefore (x,x)\in R_2 \circ R_1$
- **Irreflexivity**: counterexample: $R_1=\{(a,b)\}$, $R_2=\{(b,a)\}$, then $R_2\circ R_1=\{(a,a)\}$
- **Symmetry**: counterexample: $R_1=\{(c,b),(b,c)\}$, $R_2=\{(c,d),(d,c)\}$, then $R_2\circ R_1=\{(b,d)\}$
- **Antisymmetry**: counterexample: $R_1=\{(a,b)\}$, $R_2,=\{(b,a)\}$, then $R_2\circ R_1=\{(a,a)\}$
- **Transitivity**: counterexample: $R_1=\{(x,t),(y,s)\}$, $R_2=\{(t,y),(s,z)\}$, then $R_2\circ R_1=\{(x,y),(y,z)\}$

Summary of Keeping Properties

	reflexivity	irreflexivity	symmetry	anti-symmetry	transitivity
R_1^{-1}	✓	✓	✓	✓	✓
$R_1 \cap R_2$	✓	✓	✓	✓	✓
$R_1 \cup R_2$	✓	✓	✓	✗	✗
$R_1 \circ R_2$	✓	✗	✗	✗	✗

Closure – the Idea

an object

The orange circle：

１． circle（property）

２． Contain the object

３． If there is a green circle which satisfies above 1,2, then it must contain the orange circle.

The purple square：

１． square（property）

２． Contain the object

３． Any square contain the object contain the purple square as well

Closure: the Generic Definition

- Let R be a relation on A, $\mathcal{P}$ is some property, R_1 is called **$\mathcal{P}$ closure** if:
  - R_1 has property $\mathcal{P}$ is
  - $R \subseteq R_1$
  - If there is some relation R' on A has property $\mathcal{P}$ and includes R as well, then $R_1 \subseteq R'$

Reflexive Closure

- Reflexive closure of R is $R \cup I_A$
  - For any $x \in A$, $(x,x) \in I_A$, so, $(x,x) \in R \cup I_A$
  - $R \subseteq R \cup I_A$
  - Let R' is a reflexive relation on A, and $R \subseteq R'$, then, for any $(x,y) \in R \cup I_A$, $(x,y) \in R$, or $(x,y) \in I_A$. In both cases, $(x,y) \in R'$, so, $R \cup I_A \subseteq R'$

Symmetric Closure

- Symmetric closure of R is $R \cup R^{-1}$
  - For any $x,y \in A$, if $(x,y) \in R \cup R^{-1}$, then $(x,y) \in R$ or $(x,y) \in R^{-1}$, it follows that $(y,x) \in R^{-1}$, or $(y,x) \in R$, then $(y,x) \in R \cup R^{-1}$
  - $R \subseteq R \cup R^{-1}$
  - Let R' is a symmetric relation on A, and $R \subseteq R'$, then, for any $(x,y) \in R \cup R^{-1}$, $(x,y) \in R$, or $(x,y) \in R^{-1}$.
    - Case 1: $(x,y) \in R$, then $(x,y) \in R'$
    - Case 2: $(x,y) \in R^{-1}$, then $(y,x) \in R$, then $(y,x) \in R'$. Since R' is symmetric, $(x,y) \in R'$

  So, $R \cup R^{-1} \subseteq R'$

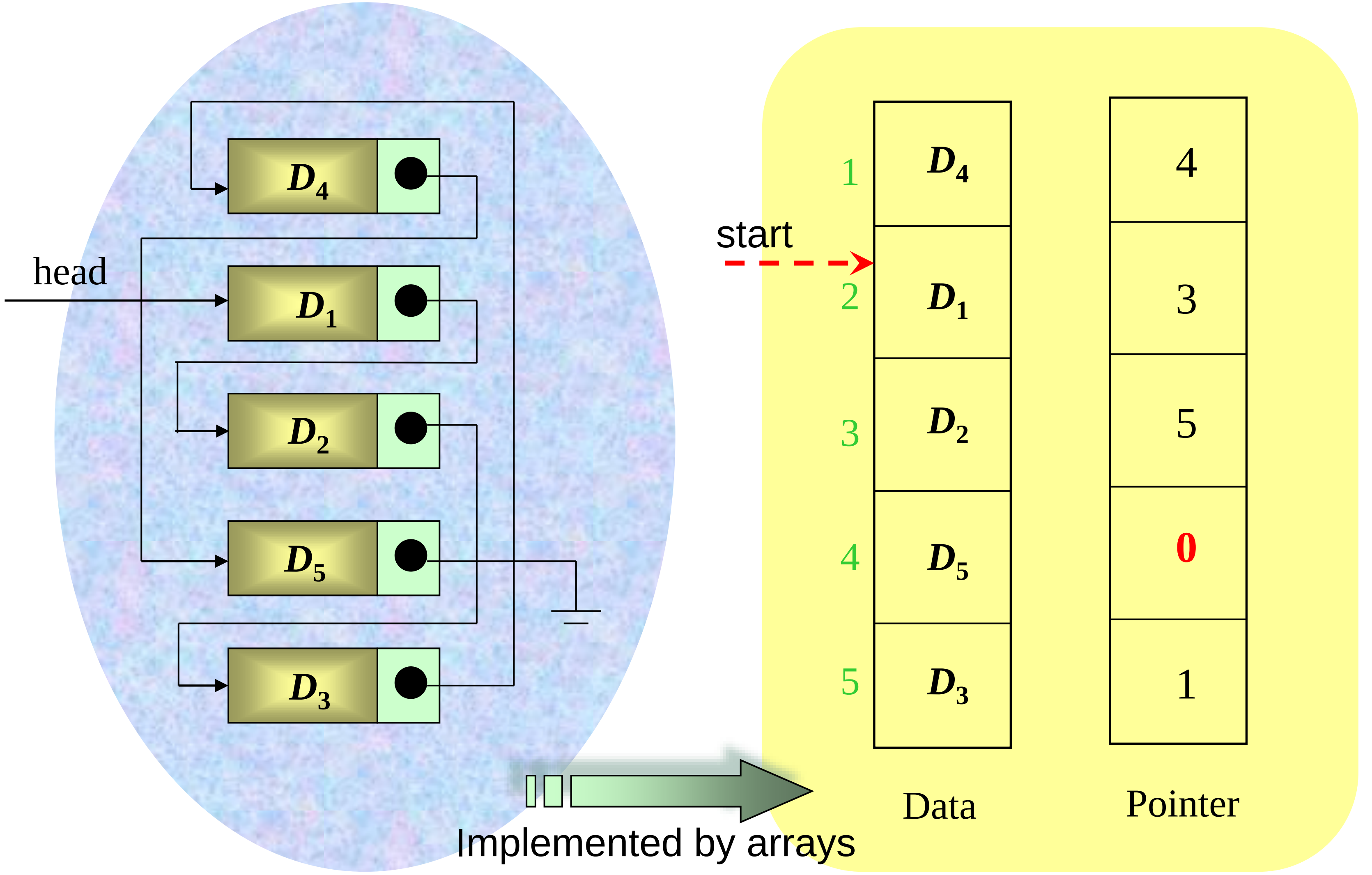
Transitive Closure

Let R be a relation on a set A. Then R^{∞} is the transitive closure of R.

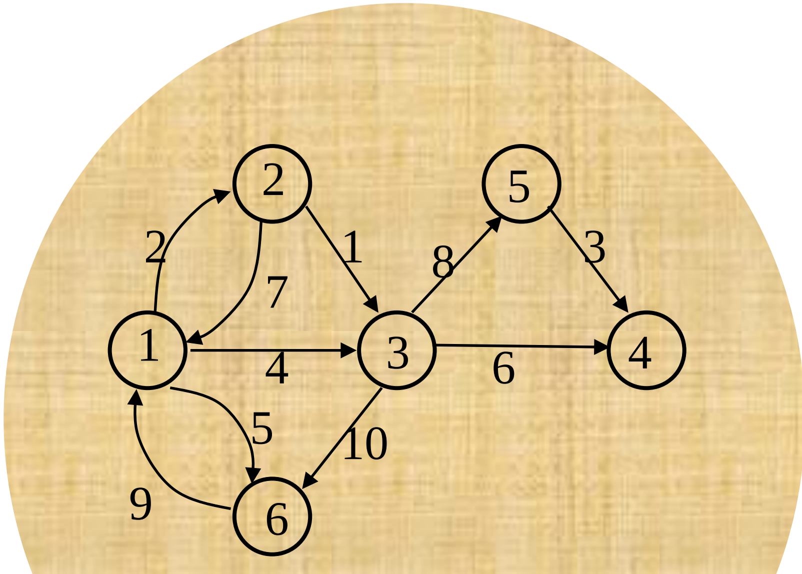
Proof:

1. If $(x,y)\in R^{\infty}$, $(y,z)\in R^{\infty}$, then there exist $s_1,s_2,\ldots,s_j$ and $t_1,t_2,\ldots,t_k$, such that $(x,s_1),(s_1,s_2),\cdots,(s_j,y),(y,t_1),(t_1,t_2),\cdots,(t_k,z)\in R$, so, $(x,z)\in R^{\infty}$.
2. $R\subseteq R^{\infty}$
3. Let R' is a transitive relation on A, and includes R as well. If $(x,y)\in R^{\infty}$, then there exist $t_1,t_2,\ldots,t_k$, such that $(x,t_1),(t_1,t_2),\cdots,(t_k,y)\in R$, then $(x,t_1),(t_1,t_2),\cdots,(t_k,y)\in R'$ however, R' is transitive, so, $(x,y)\in R'$.

Linked List and Its Implementation



Representing a Digraph as a Matrix

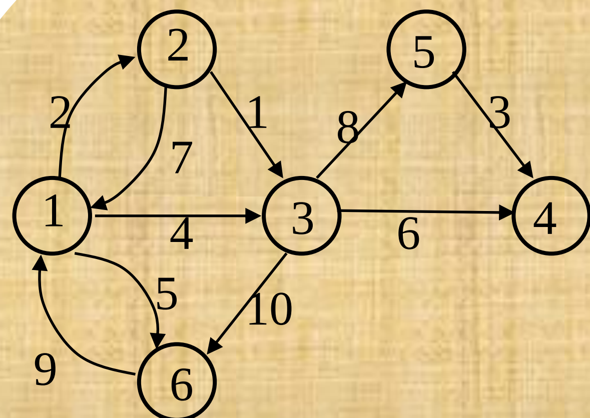


A relation with 10 ordered pairs

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix as a 2-dimensional array *A*[ ][ ]

Representing a Digraph as a Linked List

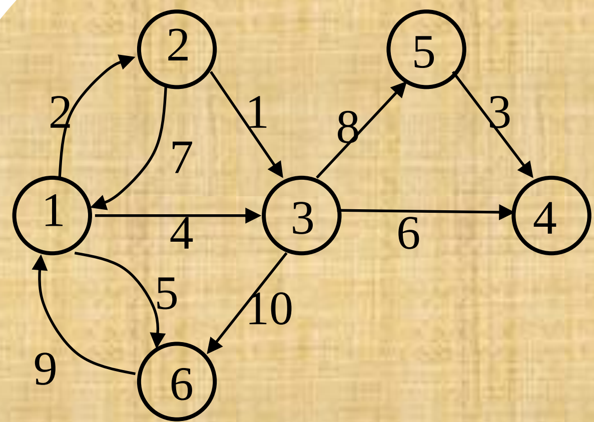


A relation with 10 ordered pairs

start: 2

tail	head	next
1	3	9
2	3	10
2	1	4
3	5	8
5	4	1
3	4	3
3	6	0
6	1	7
1	6	6
1	2	5

Indexed by Vertices



A relation with 10 ordered pairs

vert
10
2
4
0
5
8

tail	head	next
1	2	0
2	3	3
2	1	0
3	5	6
5	4	0
3	4	7
3	6	0
6	1	0
1	6	1
1	3	9

Adding a New Edge: a Comparison

Adding a pair (i, j) to a relation R

Using matrix:

Simply: MAX[i,j]$\leftarrow$1

Using linked list:

$P \leftarrow P+1$

TAIL[P]$\leftarrow I$

HEAD[P]$\leftarrow J$

NEXT[P]$\leftarrow$VERT[I]

VERT[I]$\leftarrow P$

Insert the new item in front of the list of vertex *i*

Checking Transitivity Using Matrix

Determine whether a relation with p ordered pairs is transitive or not

```
RESULT←T
FOR I=1 THRU N
    FOR J=1 THRU N
        IF (MAT[I,J]=1) THEN
            FOR K=1 THRU N
                IF (MAT[J,K]=1 and MAT[I,K]=0) THEN
                    RESULT=F
```

N^2 different MAT[I,J], among which P are "1"

Execute P times at most

So, the total steps executed $T_A=PN+(N^2-P)$.

Let $P=kN^2$, then $T_A=kN^3+(1-k)N^2$.

Transitivity Using Linked List

Determine whether a relation with p ordered pairs is transitive or not

```
RESULT←T
FOR I=1 THRU N
    X←VERT[I]
    WHILE (X≠0)
        J←HEAD[X]
        Y←VERT[J]
        WHILE (Y≠0)
            K←HEAD[Y]
            TEST←EDGE[I,K]
            IF (TEST) THEN Y←NEXT[Y]
                      ELSE RESULT←F
                           Y←NEXT[Y]
        X←NEXT[X]
```

N D D

Averagely, *P*/*N*=*D* edges begin at a vertex, so, the function EDGE takes about *D* steps.

The total steps executed is ND^3 averagely. As before, we assume that $P=KN^2$ ($0\le k\le 1$), so:

$$T_L = N\left(\frac{kN^2}{N}\right)^3 = k^3N^4$$

Transitive Closure on Finite Set

If $|A| = n$, then the transitive closure of R is

$$\bigcup_{i=1}^{n} R^i = R \cup R^2 \cup \cdots \cup R^n$$

Since the total of elements in *A* is *n*, if there is a path of length *m* from *x* to *y*, and *m*>*n*-1, then all the nodes on the path cannot be distinct. The segment between two identical nodes can be deleted, which means that: **if $xR^{\infty}y$, then for some *k*, $1 \le k \le n$, such that $xR^{n}y$**.

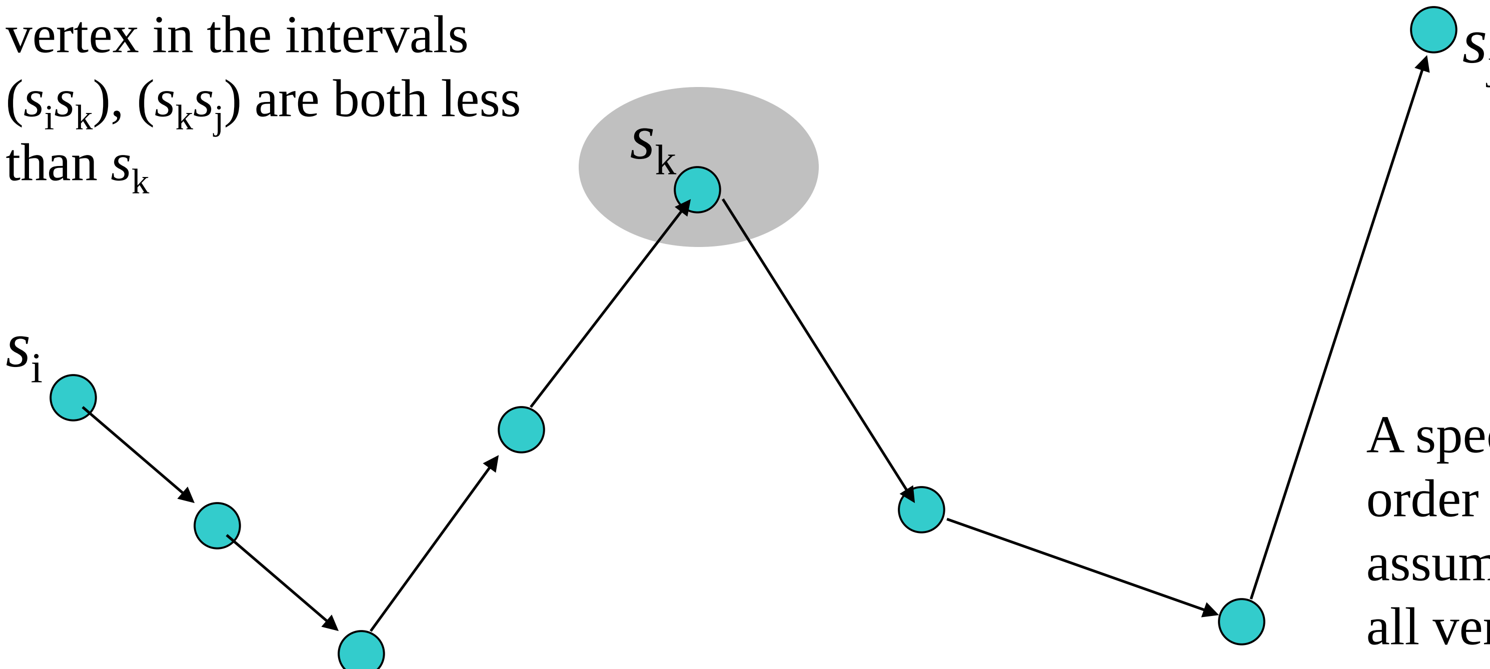
Warshall's Algorithm

- ALGORITHM MARSHALL
- 1. CLOSURE ← MAT
- 2. **FOR** K=1 **THRU** N
-     a. **FOR** I=1 **THRU** N
-         1. **FOR** J=1 **THRU** N
-             a. CLOSURE[I,J] ← CLOSURE[I,J] ∨ (CLOSURE[I,K]∧CLOSURE[K,J])
- END OF ALGORITHM WARSHALL

K is the intermediate vertex between I , J.

Highest-numbered intermediate vertex

The highest intermediate vertex in the intervals $(s_i s_k)$, $(s_k s_j)$ are both less than s_k



A specific order is assumed for all vertices

Vertical position of vertices reflect their vertex numbers

Correctness of Washall's Algorithm

- Notation:
  - The value of r_{ij} changes during the execution of the body of the "**for** *k*…" loop
    - After initializations: $\boldsymbol{r_{ij}^{(0)}}$
    - After the *k*th time of execution: $\boldsymbol{r_{ij}^{(k)}}$

Correctness of Washall's Algorithm

- If there is a simple path from s_i to s_j($i \neq j$) for which the highest-numbered intermediate vertex is s_k, then $r_{ij}^{(k)}$=true.
- Proof by induction:
  - Base case: $r_{ij}^{(0)}$=true if and only if $s_i s_j \in E$
  - Hypothesis: the conclusion holds for h<k($h \geq 0$)
  - Induction: the simple $s_i s_j$-path can be looked as $s_i s_k$-path+$s_k s_j$-path, with the indices h_1, h_2 of the highest-numbered intermediate vertices of both segment **strictly**(simple path) less than k. So, $r_{ij}^{(h1)}$=true, $r_{ij}^{(h2)}$=true, then $r_{ij}^{(k-1)}$=true, $r_{ij}^{(k-1)}$=true(Remember, false to true can not be reversed). So, $r_{ij}^{(k)}$=true

Correctness of Washall's Algorithm

- If there is ***no*** path from s_i to s_j, then r_{ij}=*false*.
- Proof
  - If r_{ij}=*true*, then only two cases:
  - r_{ij} is set by initialization, then $s_i s_j \in E$
  - Otherwise, r_{ij} is set during the *k*th execution of (**for** *k*…) when $r_{ij}^{(k-1)}$=true, $r_{ij}^{(k-1)}$=true, which, recursively, leads to the conclusion of the existence of a $s_i s_j$-path. (Note: If a $s_i s_j$-path exists, there exists a simple $s_i s_j$-path)

Home Assignments

- To be checked
  - Ex 4.6: 2,3,4,6,8,12
  - Ex 4.7: 7, 8, 12, 14, 19, 20, 23-24, 26-28, 30-31, 36-37
  - Ex 4.8: 8,10,12,14,18, 20, 23-25