4-3 Isomorphism

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TJ 9-16

Find the order of each of the following elements.

(a) $(3, 4)$ in $\mathbb{Z}_4 \times \mathbb{Z}_6$

(b) $(6, 15, 4)$ in $\mathbb{Z}_{30} \times \mathbb{Z}_{45} \times \mathbb{Z}_{24}$

(c) $(5, 10, 15)$ in $\mathbb{Z}_{25} \times \mathbb{Z}_{25} \times \mathbb{Z}_{25}$

(d) $(8, 8, 8)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{24} \times \mathbb{Z}_{80}$

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(d) $(8, 8, 8)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{24} \times \mathbb{Z}_{80}$

Theorem 9.17. *Let $(g, h) \in G \times H$. If g and h have finite orders r and s respectively, then the order of (g, h) in $G \times H$ is the least common multiple of r and s.*

Corollary 9.18. *Let $(g_1, \ldots, g_n) \in \prod G_i$. If g_i has finite order r_i in G_i, then the order of $(g_1, \ldots, g_n)$ in $\prod G_i$ is the least common multiple of $r_1, \ldots, r_n$.*

TJ 9-23

Prove or disprove the following assertion.
Let G, H, and K be groups. If $G \times K \cong H \times K$, then $G \cong H$.

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证明.

- **Case 1:** K is infinite. An anti-example:

$$G = \mathbb{Z} \ncong \mathbb{Z} \times \mathbb{Z} = H$$

$$K = \mathbb{Z} \times \mathbb{Z} \times \cdots = \Pi_{i \in \mathbb{N}} \mathbb{Z}$$

- **Case 2:** K is finite.
  Assume $G \ncong H$, let $K = \{e\}$, then $G \times K \ncong H \times K$

$\square$

TJ 10-1(a,c)

For each of the following groups G, determine whether H is a normal subgroup of G. If H is a normal subgroup, write out a Cayley table for the factor group G/H.

(a) $G = S_4$ and $H = A_4$.

(c) $G = S_4$ and $H = D_4$.

(a) $G = S_4$ and $H = A_4$

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- S_4: 所有 4 阶置换的群
- A_4: 所有 4 阶偶置换的群
- 任意给定 $g \in G = S_4$
  - g 是偶置换，则 $gA_4 = A_4 g = A_4$
  - g 是奇置换，则 $gA_4 = A_4 g = S_4 - A_4$

(c) $G = S_4$ and $H = D_4$

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- S_4: 所有 4 阶置换的群
- D_4:$\{1, r, r^2, r^3, s, sr, sr^2, sr3\} =$
  $\{(1), (1234), (13)(24), (1432), (24), (12)(34), (13), (14)(23)\}$
- $(14)D_4 = \{(14), (123), (12)(34), (324), (142), (1243), (134), (23)\}$
- $D_4(14) = \{(14), (234), (1243), (132), (124), (1342), (143), (23)\}$

TJ 10-11

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TJ 10-11

If a group G has exactly one subgroup H of order k, prove that H is normal in G.

Theorem 10.3. *Let G be a group and N be a subgroup of G. Then the following statements are equivalent.*

1. *The subgroup N is normal in G.*
2. *For all $g \in G$, $gNg^{-1} \subset N$.*
3. *For all $g \in G$, $gNg^{-1} = N$.*

- Suppose H is not normal

$$\exists g_0 \in G, \exists h_0 \in H, g_0 h_0 g_0^{-1} \notin H$$

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  - $K \neq \emptyset$, since $e = g_0 e g_0^{-1} \in K$

$$\forall k_1, k_2 \in K, \exists h_1, h_2 \in H, \text{ s.t. } k_1 = g_0 h_1 g_0^{-1}, k_2 = g_0 h_2 g_0^{-1}$$

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  - So, K is a subgroup of G
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  So, f is one to one.

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  So, the assumption is wrong.

TJ 10-12

Define the ***centralizer*** of an element g in a group G to be the set

$$C(g) = \{x \in G : xg = gx\}$$

Show that $C(g)$ is a subgroup of G. If g generates a normal subgroup of G, prove that $C(g)$ is normal in G.

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$$ag = ga \Rightarrow g = a^{-1}ga \Rightarrow ga^{-1} = a^{-1}g$$

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So, $a^{-1} \in C(g)$

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$$\begin{aligned} C(g) \text{ is nomal} \quad &\Leftrightarrow aca^{-1} \in C(g) \\ &\Leftrightarrow gaca^{-1} = aca^{-1}g \end{aligned}$$

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Let $x \in G$

$$\begin{aligned} \langle g \rangle \text{ is nomal} \quad &\Rightarrow x \langle g \rangle = \langle g \rangle x \\ &\Rightarrow \exists k, k'(x = g^k x g^{-1}, x = g^{-1} x g^{k'}) \end{aligned}$$

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So,

$$\begin{aligned} aca^{-1}g \quad & = aca^{-1}g \\ & = (g^{k_1} a^{-1} g^{-1})^{-1} c (g^{k_1} a^{-1} g^{-1}) g \\ & = gag^{-k_1} c g^{k_1} a^{-1} \\ & = gag^{-k_1} g^{k_1} c a^{-1} \\ & = gaca^{-1} \end{aligned}$$

TJ 11-5

Describe all of the homomorphisms from $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$.

TJ 11-5

Describe all of the homomorphisms from $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$.

$$P = \{\phi_k : k \in 3\mathbb{Z}\}, \text{ where}$$

$$\phi_k(x) = kx \mod 18$$

P is the set of all homomorphisms from $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$ (H)

$P \supseteq H$

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- Let $\phi \in H$, with $\phi(1) = k$, then

$$\begin{aligned} \forall x \in \mathbb{Z}_{24}, \phi(x) &= \phi(1 + 1 + \cdots + 1) \\ &= x\phi(1) \mod 18 \\ &= xk \mod 18 \end{aligned}$$

$P \supseteq H$

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\end{aligned}$$

- Then,

$$\phi(0) \quad = \phi(24) = 24k \mod 18$$

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- Then,

$$\phi(0) = \phi(24) = 24k \mod 18, ✗$$

$$\begin{aligned} 0 = \phi(0) &= \phi(1 + 23 \mod 24) \\ &= \phi(1) + \phi(23) \mod 18 \\ &= 24k \mod 18 \end{aligned}$$

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- Let $\phi \in H$, with $\phi(1) = k$, then

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$$\begin{aligned} 0 = \phi(0) &= \phi(1 + 23 \mod 24) \\ &= \phi(1) + \phi(23) \mod 18 \\ &= 24k \mod 18 \end{aligned}$$

So, k must be a multiple of 3, and $H \subseteq P$.

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- For any $\phi_k \in P$, for some $k = 3z, z \in \mathbb{Z}$, then

$$
\begin{aligned}
&\forall a, b \in \mathbb{Z}_{24}, \phi_k(a + b \mod 24)\\
&= 3k(a + b \mod 24) \mod 18\\
&= 3k(a \mod 24 + b \mod 24) \mod 18\\
&= 3k(a \mod 24) \mod 18 + 3(b \mod 24) \mod 18\\
&= \phi(a) + \phi(b) \mod 18
\end{aligned}
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$P \subseteq H$

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$$\begin{aligned}
&\forall a, b \in \mathbb{Z}_{24}, \phi_k(a+b \mod 24) \\
&= 3k(a+b \mod 24) \mod 18 \\
&= 3k(a \mod 24 + b \mod 24) \mod 18 \\
&= 3k(a \mod 24) \mod 18 + 3(b \mod 24) \mod 18 \\
&= \phi(a) + \phi(b) \mod 18
\end{aligned}$$

So, $\phi_k \in H$, and $P \subseteq H$.

TJ 11-2(b,d,e)

Which of the following maps are homomorphisms? If the map is a homomorphism, what is the kernel?

(b) $\phi : \mathbb{R} \to GL_2(\mathbb{R})$ defined by

$$\phi(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

(d) $\phi : GL_2(\mathbb{R}) \to \mathbb{R}^*$ defined by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$$

(e) $\phi : \mathbb{M}_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = b,$$

(b)

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$$\phi(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

- ϕ is well-defined.
- Let $a, b \in \mathbb{R}$, then

$$\begin{aligned} \phi(a+b) &= \begin{pmatrix} 1 & 0 \\ a+b & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \end{aligned}$$

- Kernel $\{0\}$

(d)

(d) $\phi : GL_2(\mathbb{R}) \to \mathbb{R}^*$ defined by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$$

- ϕ is well defined.
- Let $M_1, M_2 \in GL_2(\mathbb{R})$, $M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M_2 = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$, then

$$\begin{aligned}
\phi(M_1M_2) &= \phi\left(\begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix}\right) = \det\left(\begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix}\right) \\
&= (aw+by)(cx+dz) - (ax+bz)(cw+dy) \\
&= (acwx+adwz+bcxy+bdyz) - (acwx+adxy+bcwz+bdyz) \\
&= adwz - adxy + bcxy - bcwz \\
&= (ad-bc)(wz-xy) \\
&= \phi(M_1)\phi(M_2)
\end{aligned}$$

- Identity of $(\mathbb{R}^*, \cdot)$ is 1, $\ker\phi = \{M \in GL_2(\mathbb{R}) : \det M = 1)\}$

(e)

(e) $\phi : \mathbb{M}_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = b,$$

- ϕ is well defined.
- Let $M_1, M_2 \in \mathbb{M}_2(\mathbb{R})$, $M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M_2 = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$, then

$$\begin{aligned} \phi(M_1 + M_2) &= b + x \\ &= \phi(M_1) + \phi(M_2) \end{aligned}$$

- Identity of $(\mathbb{R}, +)$ is 0, $\ker\phi = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{R}) : b = 0\}$

TJ 11-17

Let $\phi : G_1 \to G_2$ be a surjective group homomorphism. Let H_1 be a normal subgroup of G_1 and suppose that $\phi(H_1) = H_2$. Prove or disprove that $G_1/H_1 \cong G_2/H_2$.

TJ 11-17

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Proposition 11.4. *Let $\phi : G_1 \to G_2$ be a homomorphism of groups. Then*

1. *If e is the identity of G_1, then $\phi(e)$ is the identity of G_2;*
2. *For any element $g \in G_1$, $\phi(g^{-1}) = [\phi(g)]^{-1}$;*
3. *If H_1 is a subgroup of G_1, then $\phi(H_1)$ is a subgroup of G_2;*
4. *If H_2 is a subgroup of G_2, then $\phi^{-1}(H_2) = \{g \in G_1 : \phi(g) \in H_2\}$ is a subgroup of G_1. Furthermore, if H_2 is normal in G_2, then $\phi^{-1}(H_2)$ is normal in G_1.*

A anti-example

$$G_1 = \mathbb{Z}_2 \times \mathbb{Z}_2, G_2 = \mathbb{Z}_2$$

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Consider

$$H_1 = \{(0, 0)\} \text{ (normal)}, H_2 = \{0\} \Rightarrow f(H_1) = H_2$$

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$$G_1 = \mathbb{Z}_2 \times \mathbb{Z}_2, G_2 = \mathbb{Z}_2$$

Consider a **surjective homomorphism** $\phi : G_1 \to G_2$:

$$\phi((a, b)) = a$$

$$\phi((a, b) + (c, d)) = \phi((a + c, b + d)) = a + c = \phi((a, b)) + \phi((c, d))$$

Consider

$$H_1 = \{(0, 0)\} \text{ (normal)}, H_2 = \{0\} \Rightarrow f(H_1) = H_2$$

$$|G_1/H_1| = [G_1 : H_1] = 4, |G_2/H_2| = [G_2 : H_2] = 2$$

Non-isomorphic groups of order 6

https://groupprops.subwiki.org/wiki/Groups_of_order_6

Non-isomorphic groups of order 6

$$\mathbb{Z}_6, S_3$$

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TJ 9-11

Find five non-isomorphic groups of order 8.

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Find five non-isomorphic groups of order 8.

$$\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2^3$$

TJ 9-11

Find five non-isomorphic groups of order 8.

$$\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2^3$$

$$D_4 = \{1, r^1, r^2, r^3, s, sr^1, sr^2, sr^3\}$$

TJ 9-11

Find five non-isomorphic groups of order 8.

$$\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2^3$$

$$D_4 = \{1, r^1, r^2, r^3, s, sr^1, sr^2, sr^3\}$$

$$Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$$

Example 3.15. Let

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

where $i^2 = -1$. Then the relations $I^2 = J^2 = K^2 = -1$, $IJ = K$, $JK = I$, $KI = J$, $JI = -K$, $KJ = -I$, and $IK = -J$ hold. The set $Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$ is a group called the ***quaternion group***. Notice that Q_8 is noncommutative.

- D_4, Q_8: non-abelian. $D_4 \ncong Q_8$
    - D_4: at least 5 elements with order 2 $\{r^2, s, sr, sr^2, sr^3\}$
    - Q_8: at most 4 elements with order 2, as $|1| = 1, |I| = |J| = |K| = 4$
- $\mathbb{Z}_8$: cyclic group
- Z_2^3: $\forall a \in \mathbb{Z}_2^3, |a| \leq 2$

https://groupprops.subwiki.org/wiki/Groups_of_order_8


Thank You!