QF620-G1-Stochastic Modelling in Finance

Group Project

Group 7

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Part I: Analytical Option Formulae

1. Black-Scholes Model

Vanilla Call/Put

The Black-Scholes formula for a call option is given by:

$$C(S,K,r,\sigma,T) = S_0\,\Phi(d_1) - K\,e^{-rT}\,\Phi(d_2)$$

$$d_1 = \frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

The Black-Scholes formula for a put option is given by:

$$P(S,K,r,\sigma,T) = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$

Digital Cash-or-Nothing Call/Put

The Black-Scholes formula for a digital cash-or-nothing call option is given by:

$$C(S,K,r,\sigma,T) = e^{-rT}\Phi(d_1)$$

$$d_1 = \frac{\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

The Black-Scholes formula for a digital cash-or-nothing put option is given by:

$$P(S,K,r,\sigma,T) = e^{-rT}\Phi(-d_1)$$

Digital Asset-or-Nothing Call/Put

The Black-Scholes formula for a digital asset-or-nothing call option is given by:

$$C(S,K,r,\sigma,T) = S_0\Phi(d_1)$$

$$d_1 = \frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

The Black-Scholes formula for a digital asset-or-nothing put option is given by:

$$P(S,K,r,\sigma,T) = S_0\Phi(-d_1)$$

2. Bachelier Model

Vanilla Call/Put

The Bachelier formula for a call option is given by:

$$C(S,K,r,\sigma,T) = e^{-rT}[(\mathrm{S}_0 - K)\Phi(d_1) + \sigma\sqrt{T}\phi(d_1)]$$

$$d_1 = \frac{S_0 - K}{\sigma\sqrt{T}}$$

The Bachelier formula for a put option is given by:

$$P(S,K,r,\sigma,T) = e^{-rT}[(K - \mathrm{S}_0)\Phi(-d_1) + \sigma\sqrt{T}\phi(d_1)]$$

Digital Cash-or-Nothing Call/Put

The Bachelier formula for a digital cash-or-nothing call option is given by:

$$C(S,K,r,\sigma,T) \ = \ e^{-rT}\Phi(d_1)$$

$$d_1 \ = \frac{S_0 - K}{\sigma\sqrt{T}}$$

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$$P(S,K,r,\sigma,T) = e^{-rT}[\mathrm{S}_0\Phi(-d_1) + \sigma\sqrt{T}\phi(d_1)$$

3. Black Model

Vanilla Call/Put

The Black formula for a call option is given by:

$$C(S,K,r,\sigma,T) \ = \ e^{-rT}[F\Phi(d_1) - K\Phi(d_2)]$$

$$d_1 \ = \frac{\log\frac{F}{K} + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

The Black formula for a put option is given by:

$$P(S,K,r,\sigma,T) = e^{-rT}[K\Phi(-d_2) - F\Phi(-d_1)]$$

Digital Cash-or-Nothing Call/Put

The Black formula for a digital cash-or-nothing call option is given by:

$$C(S,K,r,\sigma,T) \ = \ e^{-rT}[F\Phi(d_1) - K\Phi(d_2)]$$

$$d_1 \ = \frac{\log\frac{F}{K} + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

The Black formula for a digital cash-or-nothing put option is given by:

$$P(S,K,r,\sigma,T) = e^{-r} \ [K\Phi(-d_2) - F\Phi(-d_1)]$$

Digital Asset-or-Nothing Call/Put

The Black formula for a digital asset-or-nothing call option is given by:

$$C(S,K,r,\sigma,T) \ = \ e^{-r} \ [F\Phi(d_1)]$$

$$d_1 = \frac{\log\frac{F}{K} + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

The Black formula for a digital asset-or-nothing put option is given by:

$$P(S, K, r, \sigma, T) = e^{-rT}[F\Phi(-d_1)]$$

4. Displaced-Diffusion Model

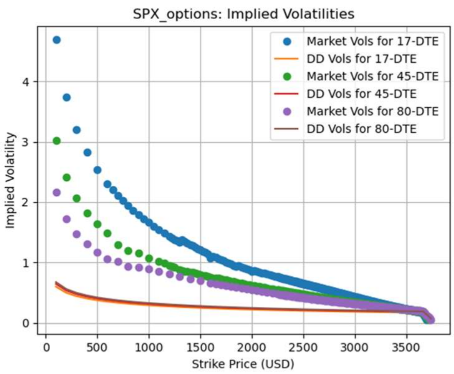
The option price for all three types under the displaced-diffusion model is given by:

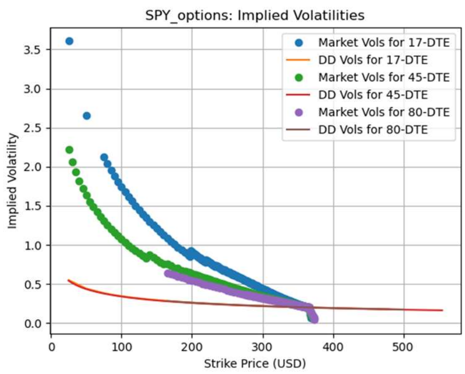
$$for\ \beta > 0, \qquad Displaced - Diffusion = Black\left(\frac{F_0}{\beta}, K + \frac{1-\beta}{\beta}F_0, \sigma\beta, T\right)$$

$$for\ \beta = 0, \qquad Displaced - Diffusion = Black\ Scholes\ Forumla$$

Part II: Model Calibration

Displaced Diffusion Model

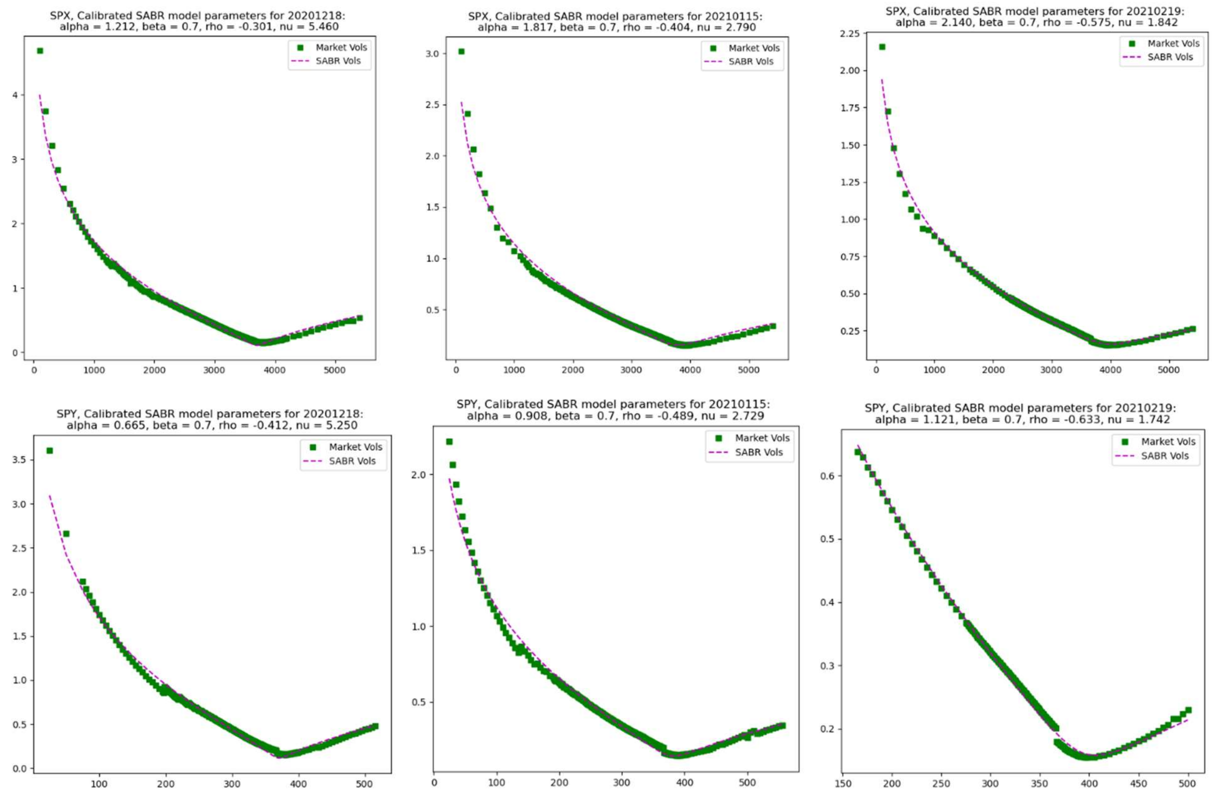




Index	Date	Sigma	Beta
SPX	20201218	0.171	0.1
	20210115	0.183	0.1
	20210219	0.192	0.1
SPY	20201218	0.200	0.1
	20210115	0.197	0.1
	20210219	0.200	0.1

The displaced diffusion model assumes the underlying follows a proportional arithmetic-geometric Brownian motion where the beta (β) parameter controls the degree of proportionality. Its value shifts the model from a lognormal world (β=1) to a normal world (β=0), significantly influencing the shape of the volatility skew and smile, and thereby impacting the pricing of options.

SABR Model



Index	**Date**	**Alpha**	**Beta (fixed)**	**Rho**	**Nu**
SPX	20201218	1.212	0.7	-0.301	5.460
	20210115	1.817	0.7	-0.404	2.790
	20210219	2.140	0.7	-0.575	1.842
SPY	20201218	0.665	0.7	-0.412	5.250
	20210115	0.908	0.7	-0.489	2.729
	20210219	1.121	0.7	-0.633	1.742

The SABR (Stochastic Alpha, Beta, Rho) model captures the behavior of implied volatilities across strike prices, and the shape of the smile is influenced by:

1. Volatility of volatility (ν): This parameter controls the randomness of the volatility process and directly affects the steepness and curvature of the volatility smile.
2. Correlation (ρ): This is the correlation between the underlying asset's returns and its volatility.

Higher ν leads to a more pronounced smile. The wings (far out-of-the-money or in-the-money strikes) become steeper because the increased stochasticity of the volatility amplifies the effect on extreme strikes. Lower ν results in a flatter smile, as there is less stochastic movement in the volatility.

Positive ρ skews the smile to the right (favoring higher strikes). This is less common but can occur in markets where upward moves are associated with increased volatility. Negative ρ causes the smile to become skewed to the left (favoring lower strikes). This reflects the market phenomenon where implied volatility increases when the underlying asset price falls (common in equity markets due to the leverage effect).

Part III: Static Replication

Static Replication is a technique that uses a portfolio of standard European options (calls and puts) to replicate the payoff or risk exposure of a more complex derivative, such as a Variance Swap. The benefits from this technique include:

1. Pricing Variance Swaps as they cannot be directly traded in the market but can be priced and hedged using a replicating portfolio of European options.
2. No frequent adjustments needed. Once the replicating portfolio is constructed, it can remain unchanged until maturity, unlike dynamic hedging strategies.

Formulas Used:

1. "Model-free" integrated variance: $\sigma_{MF}^2 T = E\left[\int_0^T \sigma_t^2 \, dt\right]$
2. Carr-Madan: $V_0 = e^{-rT} h(F) + \int_0^F h''(K) P(K) dK + \int_F^\infty h''(K) C(K) dK$
3. Variance Swaps: $E\left[\int_0^T \sigma_t^2 \, dt\right] = 2e^{rT} \int_0^F \frac{P(K)}{K^2} dK + 2e^{rT} \int_F^\infty \frac{C(K)}{K^2} dK$

Working Process:

First, we calculate and set the common parameters as such:

Common Parameters	
S	3662.45
K	3660
T	45/365
r	0.00205107
σ	0.18537188

We used the discount rate interpolation for r, and the average of the implied volatilities of ATM call and put options for sigma.

1. Black Scholes Model

With our payoff function, using Black-Scholes model, we can derive a valuation formula:

$$S_0^{\frac{1}{3}} \cdot e^{\left(\frac{1}{3}r - \frac{1}{9}\sigma^2\right)T} + 1.5 \cdot \left(\ln(S_0) + \left(r - \frac{1}{2}\sigma^2\right)T\right) + 10$$

Black-Scholes Model Pricing: 37.714381258

The expected integrated variance is: 0.004236501

2. **Bachelier Model**

With our payoff function, using Bachelier model, we can derive a valuation formula:

$$\left(S_0 + \sigma \cdot T^{\frac{1}{2}} \cdot x\right)^{\frac{1}{3}} + 1.5 \cdot \ln\left(S_0 + \sigma \cdot T^{\frac{1}{2}} \cdot x\right) + 10$$

$$V_0 = e^{-rT} \cdot \sqrt{\frac{1}{2\pi}} \cdot \int bachelier(x)dx$$

Bachelier Model Derivative Pricing: 37.713596817

Bachelier Model Integrated Variance: 0.004263876

3. **Static-Replication of European Payoff**

Using the parameters in Part II for SPX – 20210115:

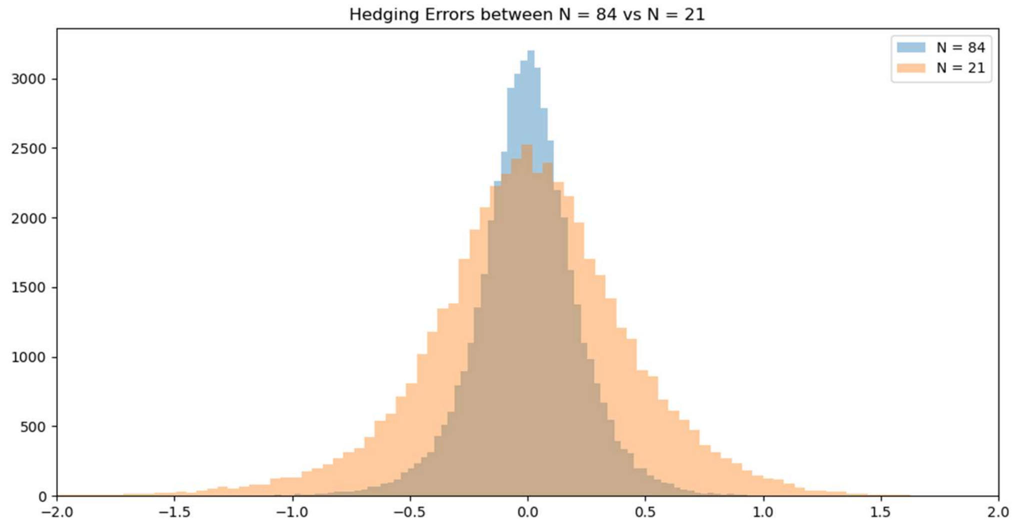
Parameters	
Alpha	1.817
Beta	0.7
Rho	-0.404
Nu	2.790

Black-Scholes Model Derivative Pricing (SABR): 37.709209373

Bachelier Model Derivative Pricing (SABR): 37.713596817

SABR Model Integrated Variance: 0.006337324

Part IV: Dynamic Hedging



Hedging Error for N = 21

For N = 21, we rebalance the portfolio every trading day, and we can see from the graph that the hedging errors have a wider distribution ranging from 1.5 to -1.5, centered around the 0 mean.

Hedging Error for N = 84

For N = 84, we rebalance the portfolio 4 times per trading day, and we can see from the graph that the hedging errors have a narrower distribution ranging from 0.75 to -0.75, centered around the 0 mean.

In conclusion, we observe that frequent rebalancing reduces hedging errors as the distribution of errors is halved when you rebalance 4 times more often. However, this is not practical in reality as it does not factor in transaction costs and market impacts must be balanced against this benefit. For volatile environments, maintaining a margin in collected premiums to cover replication errors is critical.