ESSEC

Master in Finance

Advanced Master in Financial Engineering (MSTF)

FINM32227

Financial Risk Management

CLASS HANDOUTS

SESSION 1

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Introduction: Greek Letters, Value at Risk and Expected Shortfall

Outline

- Introduction
- Individual Risk Measure: Greek Letters
- Aggregate Risk Measure: The VaR Measure
- Aggregate Risk Measure: The ES Measure
- Coherent Risk Measure
- Choice of Parameters
- Marginal, Incremental and Component Risk Measure
- Back-Testing

- The risks facing banks:
  - Market risk: arises primarily from the bank's trading operations. It is related to the possibility that instruments in the bank's trading book (all the contracts the bank enters into as part of its trading operations, usually marked to market) will decline in value.
  - Credit risk: is the risk that counterparties in loan transactions and derivatives transactions will default. This has traditionally been the greatest risk facing a bank and is usually the one for which the most regulatory capital is required.
  - Operational risk: is the risk that losses are made because internal systems fail to work as intended or because of external events.
  - Liquidity Risk: is the risk that a company cannot raise enough cash when needed.
  - Model Risk: is the risk that the model will give the wrong prices or the wrong measures for hedging.
- There are two broad risk management strategies open to a financial institution or any other organization:
  - Risk decomposition: identify and handle risks one by one

- Risk aggregation: risks are combined and risk diversification are taken into account
- In practice banks use both approaches

$$\Delta P = \frac{\partial P}{\partial S}\Delta S + \frac{\partial P}{\partial \sigma}\Delta\sigma + \frac{\partial P}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 P}{\partial S^2}(\Delta S)^2 + \frac{1}{2}\frac{\partial^2 P}{\partial \sigma^2}(\Delta\sigma)^2 + \ldots$$

- Delta of a portfolio is the partial derivative of a portfolio with respect to the price of the underlying asset
  - Delta can be changed by taking a position in the underlying
  - When the price of a product is linearly dependent on the price of an underlying asset a ``hedge and forget'' strategy can be used
  - Non-linear products require the hedge to be rebalanced to preserve delta neutrality
- Gamma is the rate of change of delta with respect to the price of the underlying asset
- Vega is the rate of change of the value of a derivatives portfolio with respect to volatility
  - Both gamma and vega tends to be greatest for options that are close to the money
  - Gamma and vega can be changed by trading options or other derivatives written on the underlying asset
- Theta of a derivative (or portfolio of derivatives) is the rate of change of the value with respect to the passage of time

- Rho is the partial derivative with respect to a parallel shift in all interest rates in a particular country
- Traders usually ensure that their portfolios are delta-neutral at least once a day

  Whenever the opportunity arises, they improve gamma and vega

  Hedging can be very expensive. As portfolio becomes larger hedging becomes less expensive
- A scenario analysis involves testing the effect on the value of a portfolio of different assumptions concerning asset prices and their volatilities

- VaR (**Value at risk**) is trying to answer the following question:

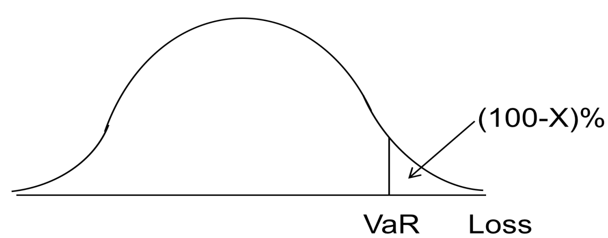
  "What loss level is such that we are *X*% confident it will not be exceeded in *N* business days?" Or

  "We are X percent sure that we will not lose more than V dolloars in time T."

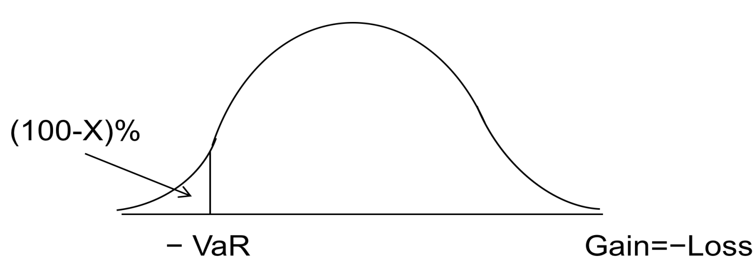
  V is the VaR of the portfolio. It depends on the time horizon, T, and the confidence level, X percent.

- Calculation of VaR:

  "There is (100-X) percent of probability that the loss will be larger than VaR dollars in time T"



  "There is (100-X) percent of probability that the gain will be less than -VaR dollars in time T"



Example 1:

The gain from a portfolio during six month is normally distributed with mean $2 million and standard deviation $10 million. What is the VaR for the portfolio with a six month time horizon and a 99% confidence level?

Example 2:

All outcomes between a loss of $50 million and a gain of $50 million are equally likely for a one-year project. What is the VaR for a one-year time horizon and a 99% confidence level?

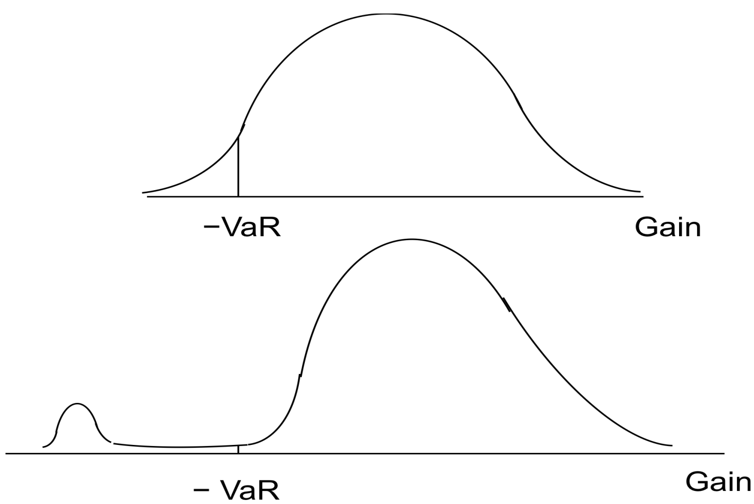
Example 3:

A one-year project has a 98% chance of leading to a gain of $2 million, a 1.5% chance of a loss of $4 million, and a 0.5% chance of a loss of $10 million. What is the VaR for a one-year time horizon with a 99% confidence level? What if the confidence level is 99.9%? What if it is 99.5%?

- When VaR is used in an attempt to limit the risks taken by a trader, it can lead to undesirable results.
- Expected shortfall is defined as the expected loss given that the loss is greater than the VaR level (also called Conditional VaR, conditional tail expectation, or Tail Loss)

  It is also a function of T and X.

  Example: Distributions with the Same VaR but Different Expected Shortfalls



VaR answers "How bad can things get?"

Conditional VaR answers "If things do gets bad, what is the expected loss"

- VaR is used by regulators and financial institutions to determine the amount of capital they should keep.

  Is VaR the best measure for this?

  Properties of coherent risk measure:

  - Monotonicity: If one portfolio always produces a worse outcome than another its risk measure should be greater
  - Translation invariance: If we add an amount of cash *K* to a portfolio its risk measure should go down by *K*
  - Homogeneity: Changing the size of a portfolio by λ should result in the risk measure being multiplied by λ
  - Subadditivity: The risk measures for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged

  VaR satisfies the first three conditions, but not the fourth one, while the conditional VaR measure satisfies all of them.

Example 1:

Each of two independent projects has a probability 0.98 of a loss of $1 million and 0.02 probability of a loss of $10 million during a one-year period.

What is the 97.5% VaR for each project?

What is the 97.5% expected shortfall for each project?

What is the 97.5% VaR for the portfolio?

What is the 97.5% expected shortfall for the portfolio?

Example 2:

Consider two $10 million one-year loans. Each has a 1.25% chance of defaulting. All recoveries between 0 and 100% are equally likely. If there is no default the loan leads to a profit of $0.2 million. If one loan defaults it is certain that the other one will not default.

What is the 99% VaR and expected shortfall of each project?

What is the 99% VaR and expected shortfall for the portfolio?

- VaR assigns all weight to *X*th quantile of the loss distribution.

  Expected shortfall assigns equal weight to all quantiles greater than the *X*th quantile and zero weights to all quantiles below the Xth quantile.

  A risk measure can be characterized by the weights it assigns to quantiles of the loss distribution. This type of risk measure is known as *spectral risk measure*

  In general, for a risk measure to be coherent, weights assigned to the *q*th quantile of a loss function must be a non-decreasing function of q

- For VaR and ES calculation, the simplest assumption (not very good) is that the change in the value of the portfolio value at the time horizon is normally distributed.
- When the loss in the portfolio value has a mean of μ and a standard deviation of σ,

$$VaR = \mu + \sigma N^{-1}(X)$$

$$ES = \mu + \sigma \frac{e^{\frac{-[N^{-1}(X)]^2}{2}}}{\sqrt{2\pi}(1-X)}$$

where X is the confidence level, $N^{-1}(\cdot)$ *is the inverse cumulative normal distribution and can be calculated using NORMSINV in Excel.*

Example:

- If daily gains/losses are assumed to be <u>normally distributed</u> and <u>independent</u> with <u>mean zero and standard deviation</u> σ. Then

$$The\ 1-day\ VaR\ = \sigma N^{-1}(X)$$

$$The\ 1-day\ ES\ = \sigma \frac{e^{\frac{-[N^{-1}(X)]^2}{2}}}{\sqrt{2\pi}(1-X)}$$

- When daily changes in a portfolio (ΔP_t) are identically distributed and independent, the variance over *T* days is *T* times the variance over one day. The sum of independent normal variable is normal. Therefore,

$$The\ T-day\ VaR\ = \sqrt{T} \times\ the\ 1-day\ VaR\ =\ \sqrt{T}\sigma N^{-1}(X)$$

$$The\ T-day\ ES\ = \sqrt{T} \times\ the\ 1-day\ ES\ =\ \sqrt{T}\sigma \frac{e^{\frac{-[N^{-1}(X)]^2}{2}}}{\sqrt{2\pi}(1-X)}$$

- In practice, the changes in the value of a portfolio from one day to the next (ΔP_t) are not always totally independent. A simple assumption is first-order autocorrelation, where the correlation between ΔP_t and ΔP_{t-1} is ρ. Suppose the variance of ΔP_t is σ^2 for all t, then

  The *T*-day VaR

  $= \sqrt{T+2(T-1)\rho+2(T-2)\rho^2+2(T-3)\rho^3+\cdots+2\rho^{T-1}}\sigma N^{-1}(X)$

  The *T*-day ES =

$$\sqrt{T+2(T-1)\rho+2(T-2)\rho^2+2(T-3)\rho^3+\cdots+2\rho^{T-1}}\sigma \frac{e^{\frac{-[N^{-1}(X)]^2}{2}}}{\sqrt{2\pi}(1-X)}$$

Note the ratio of the T-day VaR (or ES) and to the one-day VaR (or ES) is independent of σ or confidence level X.

	T=1	T=2	T=5	T=10	T=50	T=250
ρ = 0	1.0	1.41	2.24	3.16	7.07	15.81
ρ = 0.05	1.0	1.45	2.33	3.31	7.43	16.62
ρ = 0.1	1.0	1.48	2.42	3.46	7.80	17.47
ρ = 0.2	1.0	1.55	2.62	3.79	8.62	19.35

– Choice of T:

Time horizon should depend on how quickly portfolio can be unwound.

Banks often use a short time-horizon, while fund managers often use longer time horizon.

Bank regulators in effect use 10-day VaR for market risk and 1-year for credit/operational risk. When they move to use ES for market risk, bank regulators plan to relate time horizons to the liquidity of the relevant asset.

- Choice of X:

  Confidence level depends on objectives.

  Regulators use 99% for market risk VaR and 99.9% for credit/operational risk. When they move to use ES for market risk, they propose 97.5% confidence level.

  A bank wanting to maintain a AA credit rating will often use confidence levels as high as 99.97% for internal calculations.

- Analysts often calculate additional measures in order to understand VaR and ES.
- Consider a portfolio with a number of subportfolios where the investment in the ith subportfolio is x_i.
  - The *marginal* VaR is defined as $\frac{\partial VaR}{\partial x_i}$
  - The Euler's theorem shows that $VaR = \sum_{i=1}^{N} \frac{\partial VaR}{\partial x_i} x_i$
  - The *component* VaR for the ith subportfolio is $C_i = \frac{\partial VaR}{\partial x_i} x_i$

    The total VaR is the sum of the component VaR's.

    The component VaR therefore provides a sensible way of allocating VaR to different activities.
  - The *incremental* VaR is the incremental effect of the ith component on VaR, i.e., the difference between VaR with and without subportfolio i

    The incremental VaR is approximately the same as the component VaR

- Marginal ES, incremental ES, and component ES can be defined similarly to Marginal VaR, incremental VaR, and component VaR, respectively.

  Euler's theorem similarly applies to ES.

- Euler's decomposition is a useful tool in determining risk in *risk budgeting*. If Euler's decomposition shows that an unacceptable percentage of the risk is attributable to a particular component, the portfolio can be rebalanced.

VIII. Back-Testing

- Back-testing is an important reality check for a risk measure.
- It is much more difficulty to back-test ES than VaR.

  As the regulators are moving from VaR to ES for market risk, their future plans involve using ES to determine regulatory capital, but back-testing using VaR estimates.
- Back-testing a VaR calculation methodology involves looking at how often exceptions (loss > VaR) occur.

  One issue in back-testing a one-day VaR is whether we consider changes made in the portfolio during a day. Two alternatives:

  a) Compare VaR with actual change in portfolio value

  b) compare VaR with change in portfolio value assuming no change in portfolio composition

  In practice, risk managers and regulators use both.
- Formal statistical test:

  Suppose the theoretical probability of an exception is p (=1−X).

  - Suppose we observe m exceptions in n days, and $m/n > p$

    H_0: the probability of an exception on any given day = p

    H_1: the probability of an exception on any given day > p

From the properties of the binominal distribution, the probability of *m* or more exceptions in *n* days is

$$\sum_{k=m}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

This can be calculated using the BINOMDIST function in Excel. An often-used confidence level in statistical test is 95%. If the probability of the VaR level being exceeded on *m* or more days is less than 5%, then we reject the null hypothesis.

- Suppose we observe m exceptions in n days, and $\frac{m}{n} < p$

  H_0: the probability of an exception on any given day = *p*

  H_1: the probability of an exception on any given day < *p*

  The probability of *m* or less exceptions in *n* days is

$$\sum_{k=0}^{m} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

  Again, this is compared with the 5% threshold.

- Two-tail test:

  H_0: the probability of an exception on any given day = *p*

  H_1: the probability of an exception on any given day ≠ *p*

If the probability of an exception under the VaR model is *p* and *m* exceptions are observed in n trials, then

$$-2\ln[(1-p)^{n-m}p^m] + 2\ln\left[\left(1-\frac{m}{n}\right)^{n-m}(\frac{m}{n})^m\right]$$

should have a chi-square distribution with one degree of freedom.

Reject the model whenever the expression above is greater than 3.84.