ESSEC

Master in Finance

Advanced Master in Financial Engineering (MSTF)

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Financial Risk Management

CLASS HANDOUTS

SESSION 4

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Market Risk: The Model-Building Approach

Outline

- The Methodology
- Estimation of Covariance Matrix
- Handling Interest Rates
- Linear Model
- Quadratic Model
- Monte Carlo Simulation
- Non-normal Assumption
- Model Building vs. Historical Simulation Approach

- Model Building approach (also called *variance-covariance approach*) involves assuming a model for the joint distribution of changes in market variables and using historical data to estimate the model parameter
- A simple example:

  We have a position worth $10 million in Microsoft shares

  The volatility of Microsoft is 2% per day (about 32% per year)

  Assume the expected daily return of Microsoft share is 0 and the return follows a normal distribution. What is the 10 day 99% VaR and ES?

  Consider a position of $5 million in AT&T, the daily volatility of AT&T is 1% (approx 16% per year). What is the 10 day 99% VaR and ES?

– Two-Asset Case:

Now consider a portfolio consisting of both \$10 million of Microsoft shares and \$5 million of AT&T shares. Suppose that the returns on the two shares have a bivariate normal distribution with a correlation of 0.3. What is the 10 day 99% VaR and ES of the portfolio?

– Generalization:

Suppose we have a portfolio worth P consisting of n assets with an amount α_i of being invested in asset i $(1 \leq i \leq n)$. Define Δx_i as the return on asset i in one day. Then the dollar change of the portfolio in one day is

$$\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i$$

If we assume that the Δx_i are multivariate normal, then ΔP is normally distributed.

If we assume $E(\Delta x_i) = 0 \; \forall \; i$, then $E(\Delta P) = 0$

If we define σ_i as the daily volatility of the ith asset, cov_{ij} as the covariance between returns on asset i and asset j, and ρ_{ij} as the correlation coefficient between returns on asset i and asset j, then the variance of ΔP, σ_P^2, is given by

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} cov_{ij}\alpha_i\alpha_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\alpha_i\alpha_j \, \sigma_i\sigma_j$$

Using matrix notation:

$$\sigma_P^2 = \boldsymbol{\alpha}^T C \boldsymbol{\alpha}$$

where $\boldsymbol{\alpha}$ is the column vector whose ith element is α_i, $\boldsymbol{\alpha}^T$ is its transpose, and C is the variance-covariance matrix

The portfolio return in one day is $\Delta P/P$ ($\sum_{i=1}^{n} w_i \Delta\, x_i$) and its variance is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} w_i w_j \, \sigma_i\sigma_j = \sum_{i=1}^{n}\sum_{j=1}^{n} cov_{ij} w_i w_j = \boldsymbol{w}^T C \boldsymbol{w}$$

where $w_i = \alpha_i/P$ is the weight of the ith investment in the portfolio.

– The variance and covariance matrix are generally calculated from historical data.
  - Giving equal weight to all data
  - Using EWMA method: the same λ should be used for all calculations to ensure a positive-semidefinite matrix.
  - Multivariate GARCH model can also be used
– Giving equal weight to all data:

Example (from last session): Suppose on September 25, 2008, an investors owns the following portfolio

DJIA Index	FTSE 100	CAC 40	Nikkei 225	**Total**
$4,000,000	$3,000,000	$1,000,000	$2,000,000	**$10,000,000**

The past 501 days of historical data can be downloaded at

www-2.rotman.utoronto.ca/~hull/RMFI/VARExample

Day	Date	DJIA	FTSE 100	CAC 40	Nikkei 225
0	8/7/2006	11219.38	11131.84	6373.89	131.77
1	8/8/2006	11173.59	11096.28	6378.16	134.38
2	8/9/2006	11076.18	11185.35	6474.04	135.94
3	8/10/2006	11124.37	11016.71	6357.49	135.44
…	…	…	…	…	…
500	9/25/2008	11022.06	9599.90	6200.40	112.82

II. Estimation of Covariance Matrix

Daily returns are calculated as

Day	Date	DJIA	FTSE 100	CAC 40	Nikkei 225
1	8/8/2006	-0.0040813	-0.0031946	0.0006695	0.0197867
2	8/9/2006	-0.0087179	0.0080270	0.0150323	0.0116197
3	8/10/2006	0.0043508	-0.0150771	-0.0180033	-0.0037163
4	8/11/2006	-0.0032676	0.0021805	0.0011449	-0.009878
…	…	…	…	…	…
500	9/25/2008	0.0181882	0.0170914	0.027588	-0.0125877

Variance-Covariance Matrix of daily return is calculated as

0.0001227	0.0000768	0.0000767	-0.0000095
0.0000768	0.0002010	0.0001817	0.0000394
0.0000767	0.0001817	0.0001950	0.0000407
-0.0000095	0.0000394	0.0000407	0.0001909

and the correlation matrix of daily return is calculated as

1			
0.489105943	1		
0.495709627	0.918108253	1	
-0.061899208	0.200942213	0.210950956	1

What is the one-day 99% VaR and ES of this portfolio?

– Use of EWMA:

Example:

Daily returns are calculated as

Day	Date	DJIA	FTSE 100	CAC 40	Nikkei 225
1	8/8/2006	-0.0040813	-0.0031946	0.0006695	0.0197867
2	8/9/2006	-0.0087179	0.0080270	0.0150323	0.0116197
3	8/10/2006	0.0043508	-0.0150771	-0.0180033	-0.0037163
4	8/11/2006	-0.0032676	0.0021805	0.0011449	-0.009878
…	…	…	…	…	…
500	9/25/2008	0.0181882	0.0170914	0.027588	-0.0125877

Assume $\lambda = 0.94$. The variance is:

Day	DJIA	FTSE 100	CAC 40	Nikkei 225
1	0.0001227	0.000200995	0.00019496	0.00019093
2	0.0001163	0.000189547	0.000183289	0.000202965
3	0.0001139	0.00018204	0.00018585	0.000198889
4	0.0001082	0.000184757	0.000194146	0.000187784
…	…	…	…	…
500	0.0004896	0.001078564	0.00096583	0.000260215
501	0.0004801	0.001031377	0.000953544	0.000254109

The covariance is

	DJIA/ FTSE	DJIA/ CAC	DJIA/ Nikkei	FTSE/ CAC	FTSE/ Nikkei	CAC/ Nikkei
1	7.68E-05	7.667E-05	-9.48E-06	0.0001817	3.94E-05	4.07E-05
2	7.30E-05	7.19E-05	-1.38E-05	0.0001707	3.32E-05	3.91E-05
3	6.44E-05	5.97E-05	-1.90E-05	0.0001677	3.68E-05	4.72E-05
4	5.66E-05	5.14E-05	-1.88E-05	0.0001739	3.80E-05	4.84E-05
…	…					
500	0.00044	0.00042080	-2.76E-05	0.0009944	0.000237	0.000201
501	0.00043	0.00042566	-3.96E-05	0.0009630	0.000210	0.000168

The variance-covariance matrix is:

0.0004801	0.0004303	0.0004257	-0.0000396
0.0004303	0.0010314	0.0009630	0.0002095
0.0004257	0.0009630	0.0009535	0.0001681
-0.0000396	0.0002095	0.0001681	0.0002541

The correlation matrix is:

1.000	0.611	0.629	-0.113
0.611	1.000	0.971	0.409
0.629	0.971	1.000	0.342
-0.113	0.409	0.342	1.000

What is the one-day 99% VaR and ES?

- Alternatives for Handling Interest Rates:
  - Duration approach
  - Cash flow mapping
  - Principal components analysis
- Duration Approach:

  Assume that only parallel shift in the yield curve occur.

  Only one market variable: the size of the parallel change

  The change in the value of bond portfolio is approximately

$$\Delta P = -DP\Delta y$$

  Linear relation between ΔP and Δy but does not usually give enough accuracy
- Cash flow mapping

  Market Variables are zero-coupon bond prices with standard maturities: 1 month, 3 months, 6 months, 1 year, 2 years, 5 years, 7 years, 10 years and 30 years.

  To calculate VaR, the cash flows from instruments in the portfolio are mapped into cash flows occurring on the standard maturity dates.

Example:

Consider a $1 million position in a Treasury bond lasting 0.8 year that pays a coupon of 10% semiannually. ($50,000 is paid in 0.3 years and $1,050,000 is paid in 0.8 years). The following data is available:

	3 months	6 months	1 year
Zero Rate (annual compounding)	5.5%	6%	7%
Zero-coupon Bond price volatility per day	0.06%	0.1%	0.2%

Correlation between daily returns

	3-month bond	6-mo bond	1-yr bond
3-month bond	1	0.9	0.6
6-month bond	0.9	1	0.7
1-year bond	0.6	0.7	1

– Principal component analysis

For any given portfolio, we can convert a set of delta exposures into a delta exposure to the first PCA factor, the second PCA factor, and so on.

Principal component analysis takes historical data on movements in the market variables and attempts to define a set of components or factors that explain the movements

Example:

Factor loadings for US Treasury data produced using 1,543 daily observations between 1989 and 1995

Maturity	PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8	PC9	PC10
3m	0.21	-0.57	0.5	0.47	-0.39	-0.02	0.01	0	0.01	0
6m	0.26	-0.49	0.23	-0.37	0.7	0.01	-0.04	-0.02	-0.01	0
1y	0.32	-0.32	-0.37	-0.58	-0.52	-0.23	-0.04	-0.05	0	0.01
2y	0.35	-0.1	-0.38	0.17	0.04	0.59	0.56	0.12	-0.12	-0.05
3y	0.36	0.02	-0.3	0.27	0.07	0.24	-0.79	0	-0.09	0
4y	0.36	0.14	-0.12	0.25	0.16	-0.63	0.15	0.55	-0.14	-0.08
5y	0.36	0.17	-0.04	0.14	0.08	-0.1	0.09	-0.26	0.71	0.48
7y	0.34	0.27	0.15	0.01	0	-0.12	0.13	-0.54	0	-0.68
10y	0.31	0.3	0.28	-0.1	-0.06	0.01	0.03	-0.23	-0.63	0.52
30y	0.25	0.33	0.46	-0.34	-0.18	0.33	-0.09	0.52	0.26	-0.13

Standard deviation of factor score (in basis points)

PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8	PC9	PC10
17.49	6.05	3.10	2.17	1.97	1.69	1.27	1.24	0.8	0.79

Suppose we have portfolio with the exposures to interest rate:

Changes in portfolio value ($ millions) for 1-basis-point rate move					
Rate	1-year	2-year	3-year	4-year	5-year
Changes	+10	+4	-8	-7	+2

- Linear model can be used for
  - Portfolio of stocks, bonds, foreign exchange and commodities
  - Forward contract on foreign currency

    This can be regarded as a long position in a foreign zero-coupon bond (valued in domestic currency) and a short position in a domestic zero-coupon bond

    Example:

    Some time ago, a company entered into a forward contract to buy £1 million for $1.5 million. The contract now has 6 months to maturity. The daily volatility of a 6-month zero-coupon sterling bond (when its price is translated into dollars) is 0.06% and the daily volatility of a 6-month zero coupon dollar bond is 0.05%. The correlation between returns from the two bonds is 0.8. The current exchange rate is 1.53. Calculate the standard deviation of the change in the dollar value of the forward contract in 1 day. What is the 10-day 99% VaR? Assume that the 6-month interest rate in both sterling and dollars is 5% per annum with continuous compounding.

- Interest-rate swap

  This can be regarded as the exchange of a floating-rate bond to fixed-rate bond.

  The fixed rate bond is a regular coupon-paying bond.

  The floating bond is worth par just after the next payment date. So it can be regarded as a zero coupon bond with a maturity date equal to the next payment date.

– Linear model and options

- Consider a portfolio of options dependent on a single stock price, S. Define

$$\delta = \frac{\Delta P}{\Delta S}$$

and

$$\Delta x = \frac{\Delta S}{S}$$

As an approximation:

$$\Delta P = \delta \Delta S = S\delta \Delta x$$

When we have a position in several underlying market variables,

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i$$

where δ_i is the delta of the portfolio with respect to the ith asset

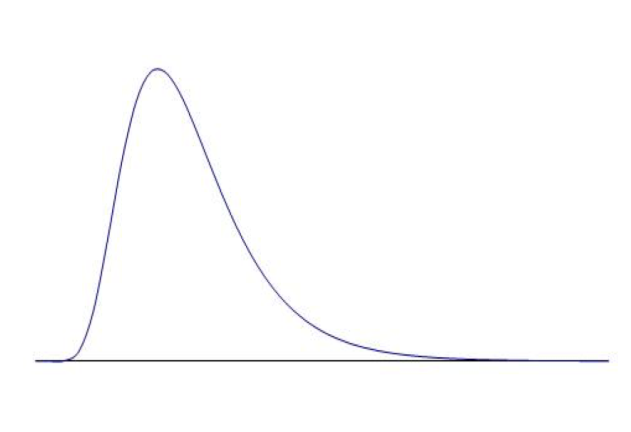
- Example:

Consider an investment in options on Microsoft and AT&T. Suppose the stock prices are \$120 and \$30 respectively and the deltas of the portfolio with respect to the two stock prices are 1,000 and 20,000 respectively. Assuming that the daily volatility of Microsoft is 2% and that of AT&T is 1%, and the correlation between the daily changes is 0.3. What is the 5-day 95% VaR?

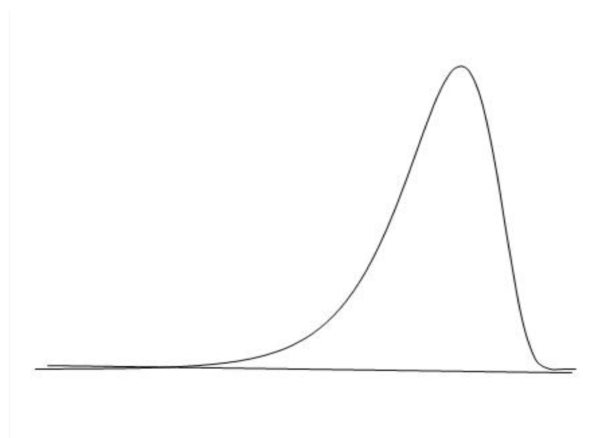
- However, when a portfolio includes options, the linear model is an approximation. It does not take account of the gamma of the portfolio.

  The linear model fails to capture skewness in the probability distribution of the portfolio value.

  When the gamma of the portfolio is positive, the probability distribution tends to be positively skewed. When the gamma of the portfolio is negative, the probability distribution tends to be negatively skewed



Positive Gamma

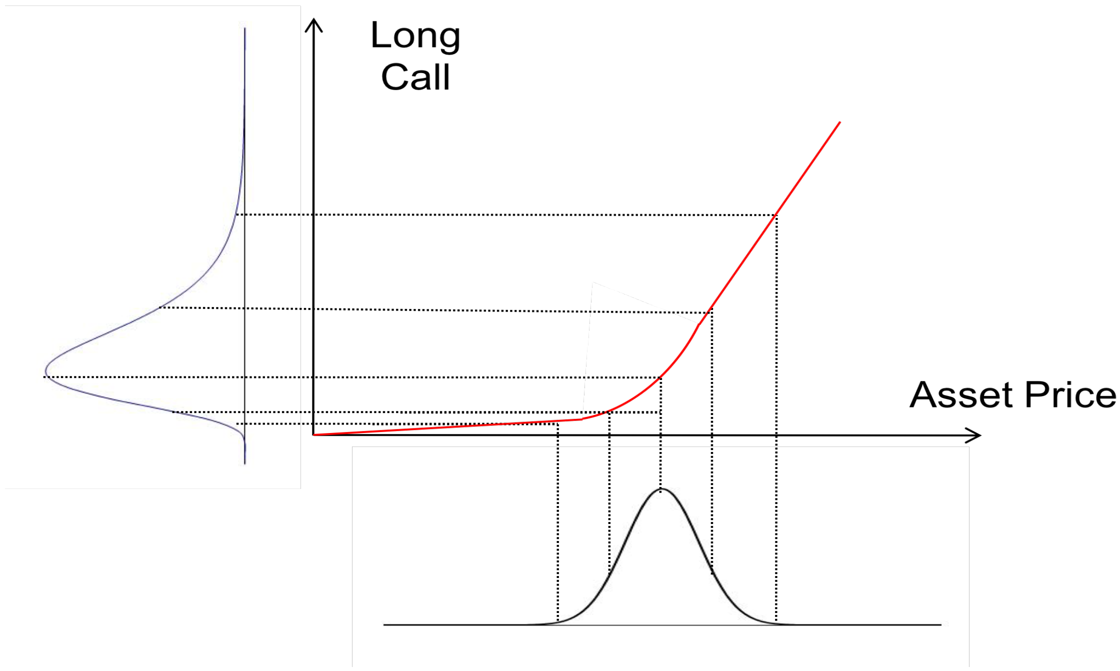


Negative Gamma

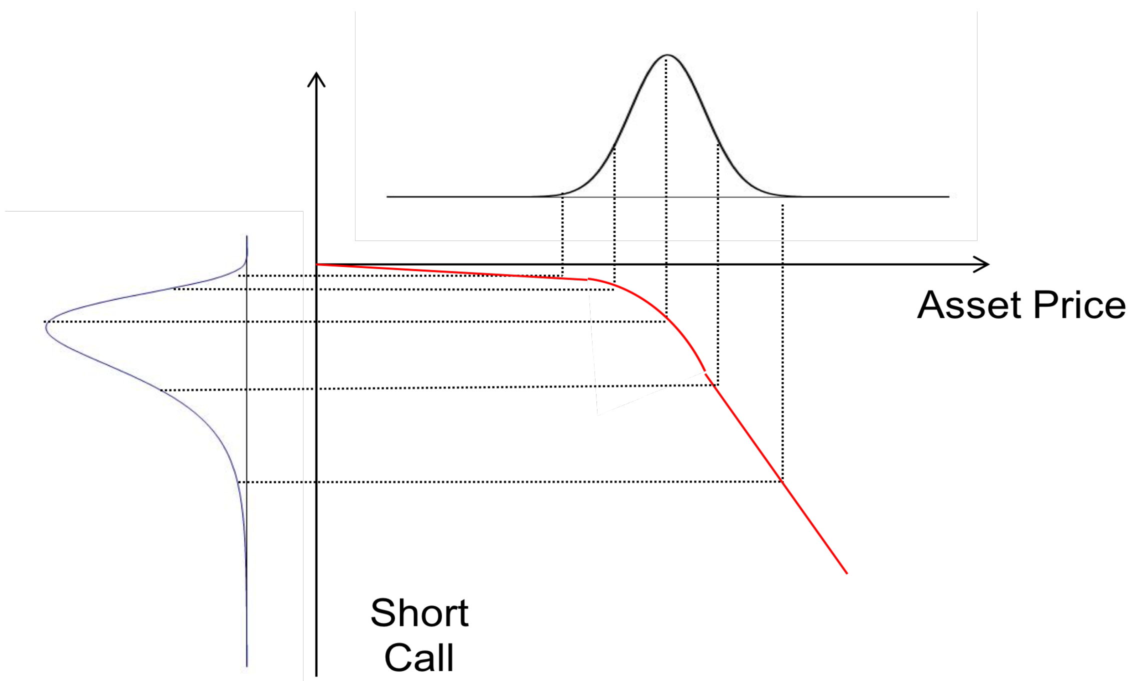
A positive gamma tends to have a less heavy left tail than the normal distribution. So the VaR calculated assuming normal distribution tends to be too high. The opposite is true for negative gamma.

Example:

A long position in call option has positive gamma



A short position in call option has negative gamma



$\bar{E}^2(\Delta x^2) = \sigma^4$

$E(\Delta x^4) = 3\sigma^4$

– For a more accurate estimate of VaR, both delta and gamma measures can be used.

$D(\Delta x^2) = 3\sigma^4 - \sigma^4$

$= 2\sigma^4$

– For a portfolio dependent on a single stock price it is approximately true that

0

$D(\Delta P) = S^2\delta^2\sigma^2 + \frac{1}{4}S^4\gamma^2\sigma^4 + 2Cov(S\delta\Delta x, \frac{1}{2}S^2\gamma\Delta x^2)$

$E(\Delta P^2) =$

$$\Delta P = \delta\Delta S + \frac{1}{2}\gamma(\Delta S)^2 = S\delta\Delta x + \frac{1}{2}S^2\gamma(\Delta x)^2$$

– If Δx is assumed to be normal with volatility σ, we have

$$E(\Delta P) = 0.5S^2\gamma\sigma^2$$

$$E(\Delta P^2) = S^2\delta^2\sigma^2 + 0.75S^4\gamma^2\sigma^4$$

$$E(\Delta P^3) = 4.5S^4\delta^2\gamma\sigma^4 + 1.875S^6\gamma^3\sigma^6$$

– With many market variables and each instrument dependent on only one

$$\Delta P = \sum_{i=1}^{n} S_i\delta_i\Delta x_i + \sum_{i=1}^{n}\frac{1}{2}S_i^2\gamma_i(\Delta x_i)^2$$

where S_i is the value of the ith market variable, and δ_i and γ_i are the delta and gamma of the portfolio with respect to the ith market variable.

– When some of the instruments in the portfolio are dependent on more than one market variables

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2} S_i S_j \gamma_{ij} \Delta x_i \Delta x_j$$

where γ_{ij} is a cross gamma defined as

$$\gamma_{ij} = \frac{\partial^2 P}{\partial S_i S_j}$$

– Cornish-Fisher Expansion:

Define $\mu_P = E(\Delta P)$, $\sigma_P^2 = E[(\Delta P)^2] - [E(\Delta P)]^2$, and

$$\xi_P = \frac{1}{\sigma_P^3} E[(\Delta P - \mu_P)^3] = \frac{E[(\Delta P)^3] - 3E[(\Delta P)^2]\mu_P + 2\mu_P^3}{\sigma_P^3}$$

The Cornish-Fisher Expansion estimate the q-quantile of the distribution of ΔP as

$$\mu_P + w_q \sigma_P$$

where

$$w_q = z_q + \frac{1}{6}(z_q^2 - 1)\xi_P$$

and z_q is the q-quantile of the standard normal distribution

– Example:

Suppose for a certain portfolio we calculate $\mu_P = -0.2$, $\sigma_P = 2.2$ and $\xi_P = -0.4$. What is the 1-day 99%VaR if we assume the change of the portfolio follows a normal distribution? What if we use Cornish-Fisher expansion?

- To calculate VaR using MC simulation we

  1. Value portfolio today
  2. Sample once from the multivariate distributions of the Δx_i
  3. Use the Δx_i to determine the value of market variables at end of one day
  4. Revalue the portfolio at the end of day
  5. Calculate ΔP
  6. Repeat many times to build up a probability distribution for ΔP

  VaR is the appropriate percentile of the distribution

- Speeding up Calculations with the Partial Simulation Approach
  - Use the approximate delta/gamma relationship between ΔP and the Δx_i(the formula at the top of Page 19) and jump from step 2 to step 5 to calculate the change in value of the portfolio
  - This can also be used to speed up the historical simulation approach

- In a Monte Carlo simulation we can assume non-normal distributions for the Δx_i (e.g., a multivariate t-distribution)
- Can also assume any set of distribution for the Δx_i in conjunction with a Gaussian or other copula model

Suppose we transform Δx_i on a percentile to percentile basis to normally distributed variables u_i

We can follow the 6 steps given earlier except step 2 should be changed to

2. Sample once from the multivariate probability distribution for the u_i and transform each to Δx_ion a percentile to percentile basis

The marginal distribution of Δx_i can be calculated by fitting a more general distribution to empirical data

- Model building approach:
  - Can produce results very fast and easily incorporate volatility and covariance updating scheme
  - However, it assume the market variable have a multivariate normal distribution.
  - Usually used for investment portfolios. It is less popular for the trading portfolios of financial institutions because it does not work well when the deltas are low.
- Historical Simulation approach:
  - The historical data determine the joint distribution of the market variables. It requires no assumption about probability distribution and correlation.
  - It is easier to handle interest rate using historical simulation.
  - It is computationally much slower
  - It incorporate volatility and covariance updating scheme in a rather artificial way