# Final Exam: Financial Econometrics

*MSFT: Financial Engineering, 2014-2015*

***Instructions:***

- You have two and half hours.
- No notes are allowed.
- Answer all questions with clarity.

**A.** Let $\{\epsilon_0, \epsilon_1, \ldots\}$ be i.i.d with mean zero and variance $\sigma_\epsilon^2$. Consider a process $\{y_0, y_1, \ldots\}$ generated by, for $y_0 = 0$,

$$y_t \quad = \quad \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}. \tag{1}$$

1. Show that $\{y_1, y_2, \ldots\}$ is covariance stationary. Derive the expressions for the autocovariances $\gamma_j = E(y_t, y_{t-j})$ for $j = 1, 2, \cdots$.

2. Let

$$\begin{aligned} r_{tj} \quad \equiv \quad & E\left[y_t | y_{t-j}, y_{t-j-1}, \ldots, y_0, y_{-1}\right] \\ & -E\left[y_t | y_{t-j-1}, y_{t-j-2}, \ldots, y_0, y_{-1}\right] \end{aligned} \tag{2}$$

   where $t = j, j+1, \ldots$ and $j = 0, 1, \ldots$. Prove

$$r_{t0} = \epsilon_t, \;\; r_{t1} = \theta_1 \epsilon_{t-1}, \;\; r_{t2} = \theta_2 \epsilon_{t-2}, \;\; r_{t3} = 0, \;\; r_{t4} = 0, \ldots$$

3. Let

$$\bar{y}_n \equiv \frac{1}{n} \sum_{i=1}^{n} y_i. \tag{3}$$

   Compute the mean and the variance of $\sqrt{n}\bar{y}_n$.

**B.** Consider the standard GARCH(1, 1) model

$$\begin{aligned} r_t \quad &= \quad \mu + h_t^{\frac{1}{2}} z_t, \quad z_t \to i.i.d \;\; N(0,1); & (4) \\ h_t \quad &= \quad \omega + \alpha(r_{t-1} - \mu)^2 + \beta h_{t-1} & (5) \end{aligned}$$

where $\omega > 0$, $\alpha > 0$, $\beta \geq 0$ and $\alpha + \beta < 1$.

1. Compute the unconditional mean, variance, skewness, and kurtosis of the return distribution. Discuss which stylized facts of asset returns and volatility can be captured by the $GARCH(1,1)$ model and which cannot?

2. Define $v_t = \epsilon_t^2 - h_t$. Show that $v_t$ is a white noise and the $r_t - \mu$ sequence acts as an ARMA(1, 1) process.

**C.** We consider the monthly returns of the S&P 500 index starting from 1926 for 792 observations. Firstly, we use a AR(3)-GARCH(1, 1) model with the normal errors and have the following fitted model

$$r_t = 0.0078 + 0.032r_{t-1} - 0.029r_{t-2} - 0.008r_{t-3} + e_t, \tag{6}$$
$$h_t = 0.000084 + 0.1213e_{t-1}^2 + 0.8523h_{t-1}, \tag{7}$$

and the maximized log likelihood function $LLF = 501.6$. We also run the standard GARCH(1, 1) model and obtain

$$r_t = 0.0076 + e_t, \tag{8}$$
$$h_t = 0.000086 + 0.1216e_{t-1}^2 + 0.8511h_{t-1}, \tag{9}$$

and the maximized log likelihood function $LLF = 498.9$.

1. Which model should be selected at the 5% significance level? Does your result change at the 10% level? (See attached table of critical values of $\chi^2$ distribution.)?

2. What is the unconditional variance of returns under your preferred model?

3. In Figure 1, we present the estimated volatility process, the standardized residuals, the sample ACF of the standardized residuals, and the sample ACF of the squared standardized residuals for the GARCH(1, 1) model. Discuss whether the GARCH(1, 1) model is adequate to model our data?

4. If you are the model builder, what further diagnosis should you conduct to refine the model?

**D.** Figure 2 plots the national average home sales and the national average home rental price in the US from 1975 through 2007. Prices are normalized so that both averages are 100 in 1975. Since both prices seem to follow random walk processes, and they are determined by similar factors such as policies, earning incomes and so on, they could not get too far apart.

1. Explain in detail, step by step, how you would test the hypothesis that these two prices are cointegrated assuming you know the cointegrating relationship.

2. Suppose that you determine the prices are cointegrated and estimate the following Error Correction Model (ECM):

$$\Delta y_t = \begin{bmatrix} 6.2 \\ 1.2 \end{bmatrix} + \begin{bmatrix} 0.43 & 0.11 \\ 0.08 & 0.54 \end{bmatrix} \Delta y_{t-1} + \begin{bmatrix} -0.12 \\ 0.02 \end{bmatrix} z_{t-1} + \begin{bmatrix} \epsilon_t \\ \eta_t \end{bmatrix}, \tag{10}$$

where $y_t = \begin{bmatrix} sales_t \\ rent_t \end{bmatrix}$ and $z_t = sales_t - rent_t$. Provide an interpretation of the coefficients on $z_{t-1}$. Do they make sense?

3. In the last period, 2007, $sales = 160$, $rent = 97$, and $\Delta y_t = \begin{bmatrix} -6.1 \\ 3.2 \end{bmatrix}$. Find the one-step ahead forecast for the change in the home price index and the change in the rental price index.

4. Again, in the last period, 2007, $sales = 160$, $rent = 97$, and $\Delta y_t = \begin{bmatrix} -6.1 \\ 3.2 \end{bmatrix}$. Find the two-step ahead forecast for the change in the home price index and the change in the rental price index.

5. Suppose instead that both sales and rent appear to follow random walk processes, but you conclude that the two series are not cointegrated. Explain how you would model the series in this case. Be specific and write down a possible model.

Figure 1

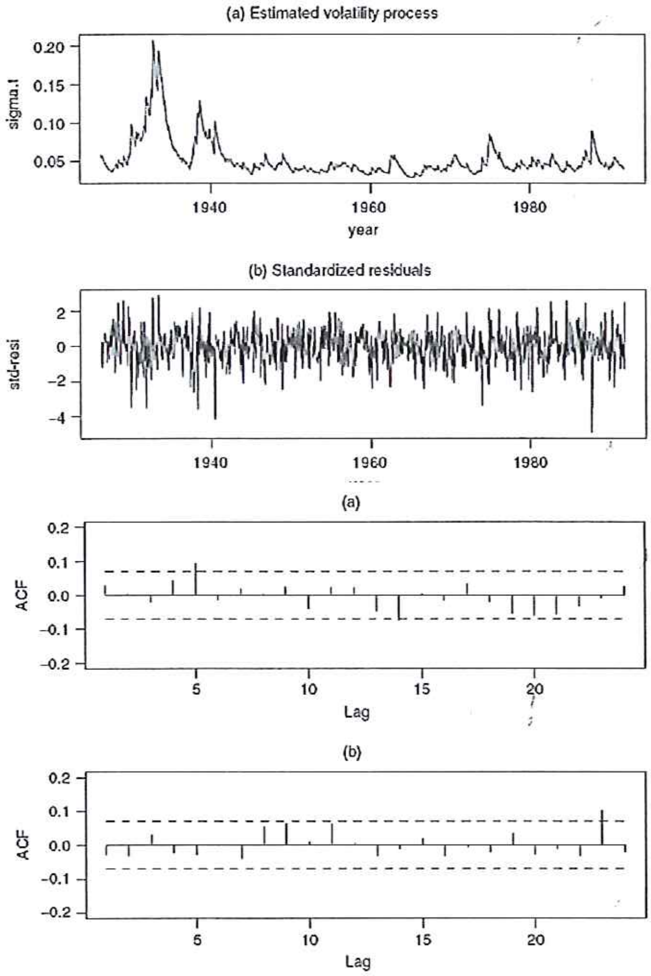


Figure 2

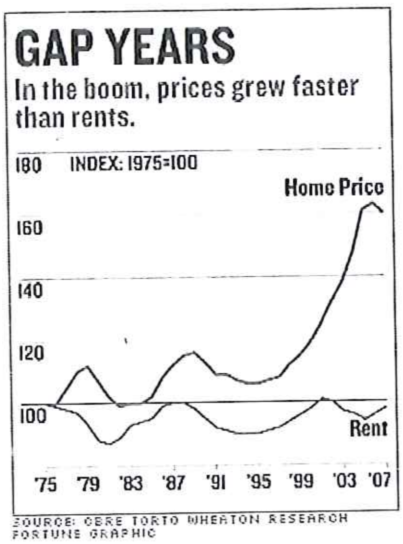

**Table A2.5** Chi-squared critical values for different values of $\alpha$ and degrees of freedom, $\upsilon$

| $\upsilon$ | 0.995 | 0.990 | 0.975 | 0.950 | 0.900 | 0.750 | 0.500 | 0.250 | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00004 | 0.00016 | 0.00098 | 0.00393 | 0.01579 | 0.1015 | 0.4549 | 1.323 | 2.706 | 3.841 | 5.024 | 6.635 | 7.879 |
| 2 | 0.01003 | 0.02010 | 0.05065 | 0.1026 | 0.2107 | 0.5754 | 1.386 | 2.773 | 4.605 | 5.991 | 7.378 | 9.210 | 10.597 |
| 3 | 0.07172 | 0.1148 | 0.2158 | 0.3518 | 0.5844 | 1.213 | 2.366 | 4.108 | 6.251 | 7.815 | 9.348 | 11.345 | 12.838 |
| 4 | 0.2070 | 0.2971 | 0.4844 | 0.7107 | 1.064 | 1.923 | 3.357 | 5.385 | 7.779 | 9.488 | 11.143 | 13.277 | 14.860 |
| 5 | 0.4117 | 0.5543 | 0.8312 | 1.145 | 1.610 | 2.675 | 4.351 | 6.626 | 9.236 | 11.070 | 12.833 | 15.086 | 16.750 |
| 6 | 0.6757 | 0.8721 | 1.237 | 1.635 | 2.204 | 3.455 | 5.348 | 7.841 | 10.645 | 12.592 | 14.449 | 16.812 | 18.548 |
| 7 | 0.9893 | 1.239 | 1.690 | 2.167 | 2.833 | 4.255 | 6.346 | 9.037 | 12.017 | 14.067 | 16.013 | 18.475 | 20.278 |
| 8 | 1.344 | 1.646 | 2.180 | 2.733 | 3.490 | 5.071 | 7.344 | 10.219 | 13.362 | 15.507 | 17.535 | 20.090 | 21.955 |
| 9 | 1.735 | 2.088 | 2.700 | 3.325 | 4.168 | 5.899 | 8.343 | 11.389 | 14.684 | 16.919 | 19.023 | 21.666 | 23.589 |
| 10 | 2.156 | 2.558 | 3.247 | 3.940 | 4.865 | 6.737 | 9.342 | 12.549 | 15.987 | 18.307 | 20.483 | 23.209 | 25.188 |
| 11 | 2.603 | 3.053 | 3.816 | 4.575 | 5.578 | 7.584 | 10.341 | 13.701 | 17.275 | 19.675 | 21.920 | 24.725 | 26.757 |
| 12 | 3.074 | 3.571 | 4.404 | 5.226 | 6.304 | 8.438 | 11.340 | 14.845 | 18.54 | 21.026 | 23.337 | 26.217 | 28.300 |
| 13 | 3.565 | 4.107 | 5.009 | 5.892 | 7.041 | 9.299 | 12.340 | 15.984 | 19.812 | 22.362 | 24.736 | 27.688 | 29.819 |
| 14 | 4.075 | 4.660 | 5.629 | 6.571 | 7.790 | 10.165 | 13.339 | 17.117 | 21.064 | 23.685 | 26.119 | 29.141 | 31.319 |
| 15 | 4.601 | 5.229 | 6.262 | 7.261 | 8.547 | 11.036 | 14.339 | 18.245 | 22.307 | 24.996 | 27.488 | 30.578 | 32.801 |
| 16 | 5.142 | 5.812 | 6.908 | 7.962 | 9.312 | 11.912 | 15.338 | 19.369 | 23.542 | 26.296 | 28.845 | 32.000 | 34.267 |
| 17 | 5.697 | 6.408 | 7.564 | 8.672 | 10.085 | 12.792 | 16.338 | 20.489 | 24.769 | 27.587 | 30.191 | 33.409 | 35.718 |
| 18 | 6.265 | 7.015 | 8.231 | 9.390 | 10.865 | 13.675 | 17.338 | 21.605 | 25.989 | 28.869 | 31.526 | 34.805 | 37.156 |
| 19 | 6.844 | 7.633 | 8.907 | 10.117 | 11.651 | 14.562 | 18.338 | 22.718 | 27.204 | 30.143 | 32.852 | 36.191 | 38.582 |
| 20 | 7.434 | 8.260 | 9.591 | 10.851 | 12.443 | 15.452 | 19.337 | 23.828 | 28.412 | 31.410 | 34.170 | 37.566 | 39.997 |
| 21 | 8.034 | 8.897 | 10.283 | 11.591 | 13.240 | 16.344 | 20.337 | 24.935 | 29.615 | 32.670 | 35.479 | 38.932 | 41.401 |
| 22 | 8.643 | 9.542 | 10.982 | 12.338 | 14.041 | 17.240 | 21.337 | 26.039 | 30.813 | 33.924 | 36.781 | 40.289 | 42.796 |
| 23 | 9.260 | 10.196 | 11.688 | 13.090 | 14.848 | 18.137 | 22.337 | 27.141 | 32.007 | 35.172 | 38.076 | 41.638 | 44.181 |
| 24 | 9.886 | 10.856 | 12.401 | 13.848 | 15.659 | 19.037 | 23.337 | 28.241 | 33.196 | 36.415 | 39.364 | 42.080 | 45.558 |
| 25 | 10.520 | 11.524 | 13.120 | 14.611 | 16.473 | 19.939 | 24.337 | 29.339 | 34.382 | 37.652 | 40.646 | 44.314 | 46.928 |
| 26 | 11.160 | 12.198 | 13.844 | 15.379 | 17.292 | 20.843 | 25.336 | 30.434 | 35.563 | 38.885 | 41.923 | 45.642 | 48.290 |
| 27 | 11.808 | 12.879 | 14.573 | 16.151 | 18.114 | 21.749 | 26.336 | 31.528 | 36.741 | 40.113 | 43.194 | 46.963 | 49.645 |
| 28 | 12.461 | 13.565 | 15.308 | 16.928 | 18.939 | 22.657 | 27.336 | 32.620 | 37.916 | 41.337 | 44.461 | 48.278 | 50.993 |
| 29 | 13.121 | 14.256 | 16.047 | 17.708 | 19.768 | 23.567 | 28.336 | 33.711 | 39.087 | 42.557 | 45.722 | 49.588 | 52.336 |
| 30 | 13.787 | 14.954 | 16.791 | 18.493 | 20.599 | 24.478 | 29.336 | 34.800 | 40.256 | 43.773 | 46.979 | 50.892 | 53.672 |
| 35 | 17.192 | 18.509 | 20.569 | 22.465 | 24.797 | 29.054 | 34.336 | 40.223 | 46.059 | 49.802 | 53.203 | 57.342 | 60.275 |
| 40 | 20.707 | 22.164 | 24.433 | 26.509 | 29.050 | 33.660 | 39.335 | 45.616 | 51.805 | 55.758 | 59.342 | 63.691 | 66.766 |
| 45 | 24.311 | 25.901 | 28.366 | 30.612 | 33.350 | 38.291 | 44.335 | 50.985 | 57.505 | 61.656 | 65.410 | 69.957 | 73.166 |
| 50 | 27.991 | 29.707 | 32.357 | 34.764 | 37.689 | 42.942 | 49.335 | 56.334 | 63.167 | 67.505 | 71.420 | 76.154 | 79.490 |
| 55 | 31.735 | 33.571 | 36.398 | 38.958 | 42.060 | 47.611 | 54.335 | 61.665 | 68.796 | 73.311 | 77.381 | 82.292 | 85.749 |
| 60 | 35.535 | 37.485 | 40.482 | 43.158 | 46.459 | 52.294 | 59.335 | 66.981 | 74.397 | 79.082 | 83.298 | 85.379 | 91.952 |
| 70 | 43.275 | 45.442 | 48.758 | 51.739 | 55.329 | 61.698 | 69.334 | 77.577 | 85.527 | 90.531 | 95.023 | 100.425 | 104.215 |
| 80 | 51.172 | 53.540 | 57.153 | 60.391 | 64.278 | 71.144 | 79.334 | 88.130 | 96.578 | 101.879 | 106.629 | 112.329 | 116.321 |
| 90 | 59.196 | 61.754 | 65.647 | 69.126 | 73.291 | 80.625 | 89.334 | 98.650 | 107.565 | 113.145 | 118.136 | 124.116 | 128.299 |
| 100 | 67.328 | 70.065 | 74.222 | 77.929 | 82.358 | 90.133 | 99.334 | 109.141 | 118.498 | 124.342 | 129.561 | 135.807 | 140.169 |
| 120 | 83.829 | 86.909 | 91.568 | 95.705 | 100.627 | 109.224 | 119.335 | 130.051 | 140.228 | 146.565 | 152.214 | 158.963 | 163.670 |
| 150 | 109.122 | 112.655 | 117.980 | 122.692 | 126.278 | 137.987 | 149.334 | 161.258 | 172.577 | 179.579 | 185.803 | 193.219 | 198.380 |
| 200 | 152.224 | 156.421 | 162.724 | 168.279 | 174.825 | 156.175 | 199.334 | 213.099 | 226.018 | 233.993 | 241.060 | 249.455 | 255.281 |
| 250 | 196.145 | 200.929 | 208.095 | 214.392 | 221.809 | 234.580 | 249.334 | 264.694 | 279.947 | 287.889 | 295.691 | 304.948 | 311.361 |

Source: Biometrika Tables for Statisticians (1966), [illegible]