1. Consider an $\mathsf{ARMA}(1,1)$

$$y_t = \varphi_0 + \varphi_1 y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}.$$

   (a) Precisely describe the two types of stationarity.
   answer: weak and strict, see the lecture notes

   (b) Why is stationarity a useful property?
   answer: it provides the context in which estimation of time series models can be done as with cross sections. Also it's useful for forecasting.

   (c) What is a minimal set of assumptions sufficient to ensure $\{y_t\}$ is covariance stationary if $\{\varepsilon_t\}$ is a white noise sequence?
   answer: the model rewrites, for $\mu = \varphi_0 / (1 - \varphi_1)$

$$(1 - \varphi_1 L)(y_t - \mu) = (1 + \theta_1 L)\varepsilon_t$$

   hence the process is stationary if either:
   (i) $\varphi_1 = -\theta_1$ (common factor) so it simplifies as $y_t = \mu + \varepsilon_t$; or
   (ii) $\varphi_1 \neq -\theta_1$ and $|\varphi_1| < 1$ (stationary AR root)

   (d) What are the values of the following quantities? You may assume that y_t is weakly stationary (for this question only). $\mathcal{I}_t$ denotes the information set available at time t.

      i. $\mathsf{E}[y_{t+1}] = (1 - \varphi_1)^{-1} \varphi_0$
      ii. $\mathsf{E}_t[y_{t+1}] = \mathsf{E}[y_{t+1} | \mathcal{I}_t] = \varphi_0 + \varphi_1 y_t + \theta_1 \varepsilon_t$
      iii. $\mathsf{V}[y_{t+1}] = \left(1 - \varphi_1^2\right)^{-1} \left(1 + 2\varphi_1\theta_1 + \theta_1^2\right) \sigma_\varepsilon^2$
      iv. $\mathsf{V}_t[y_{t+1}] = \mathsf{Var}[y_{t+1} | \mathcal{I}_t] = \sigma_\varepsilon^2$

   (e) Suppose you were trying to differentiate between an $\mathsf{AR}(1)$ and an $\mathsf{MA}(1)$ but could not estimate any regressions. What information would you try to obtain?
   answer: e.g. the ACF and PACF, describre how they can be used.

   (f) Now suppose that $\varphi_1 = 1$. What can you say about $\mathsf{E}_t[y_{t+h}]$ and $\mathsf{V}_t[y_{t+h}]$ for $h > 2$?
   answer:

$$\mathsf{E}_t[y_{t+h}] = h\varphi_0 + y_t + \theta_1 \varepsilon_t$$
$$\mathsf{V}_t[y_{t+h}] = \sigma_\varepsilon^2 + (1 + \theta)^2 (h - 1)$$

2. **The timing of expectations in forward looking models with an emphasis in monetary policy** as presented in Clarida, Galí and Gertler (Journal of Economic Literature, 1999).
   The model of interest is the so-called new Keynesian Phillips curve

$$\pi_t = \beta \pi_{t+1}^e + x_t \tag{1}$$

   where π_t denote inflation at time t, x_t is some measure of activity (say the output gap, unemployment or the share of labor in aggregate output) and π_{t+1}^e denotes a representative agent's expectation of future inflation at time $t+1$. The question of interest here relates to the timing of this expectation. We consider two possibilities, and denote them by $\pi^e_{t+1|t}$ and $\pi^e_{t+1|t-1}$ respectively. These are defined as

$$\begin{aligned} &\text{model } \mathsf{CI}\text{: } \pi^e_{t+1|t} = E\left(\pi_{t+1}|\mathcal{I}_t\right), \text{ and} \\ &\text{model } \mathsf{LI}\text{: } \pi^e_{t+1|t-1} = E\left(\pi_{t+1}|\mathcal{I}_{t-1}\right) \end{aligned}$$

where $E\left(\cdot|\mathcal{I}\right)$ denotes the expectation conditional on information $\mathcal{I}$. Hence $\pi^e_{t+1|t}$ denotes the expectation conditional on contemporanous information (CI) available to the agent at time t and $\pi^e_{t+1|t}$ denotes the expectation conditional on lagged information (LI) available at time $t-1$.We consider the forward solutions to expression (1) both under model CI and LI. We assume first $\beta \in (-1,1)$.
Expression (1) can be solved as

$$\pi_t = E\left[\sum_{j=0}^{\infty} \beta^j x_{t+j} \middle| \mathcal{I}_{t-\delta}\right] + \underbrace{\left[x_t - E\left[x_t|\,\mathcal{I}_{t-\delta}\right]\right]}_{\text{missing in the exam question}} \qquad (2)$$

where $\delta = 0$ for model CI and $\delta = 1$ for model LI.

(a) What assumptions do you need to make on the long run conditional expectations of π_{t+k} as $k \to \infty$ to obtain (2)?
answer: we need to assume the transversality condition holds. Replacing π_{t+1} with $\beta E\left[\pi_{t+2}|\,\mathcal{I}_{t+1-\delta}\right] + x_{t+1}$ in the following, leads to

$$\begin{aligned} \pi_t &= \beta E\left[\pi_{t+1}|\,\mathcal{I}_{t-\delta}\right] + x_t \\ &= \beta^2 E\left[\pi_{t+2}|\,\mathcal{I}_{t-\delta}\right] + \beta E\left[x_{t+1}|\,\mathcal{I}_{t-\delta}\right] + x_t \end{aligned}$$

using the fact that $E\left[E\left[\pi_{t+2}|\,\mathcal{I}_{t+1-\delta}\right]|\,\mathcal{I}_{t-\delta}\right] = E\left[\pi_{t+2}|\,\mathcal{I}_{t-\delta}\right]$ by the law of iterated expectations. Hence doing the same again and again, for any $k \geq 1$

$$\pi_t = \beta^k E\left[\pi_{t+k}|\,\mathcal{I}_{t-\delta}\right] + E\left[\sum_{j=1}^{k-1} \beta^j x_{t+j} \middle| \mathcal{I}_{t-\delta}\right] + x_t$$

and hence letting $k \to \infty$, if

$$\beta^k E\left[\pi_{t+k}|\,\mathcal{I}_{t-\delta}\right] = \frac{E\left[\pi_{t+k}|\,\mathcal{I}_{t-\delta}\right]}{(1/\beta)^k} \to 0$$

i.e. agents do not expect inflation to balloon away at a growth rate larger than $1/\beta$, then

$$\pi_t = E\left[\sum_{j=0}^{\infty} \beta^j x_{t+j} \middle| \mathcal{I}_{t-\delta}\right] + \left[x_t - E\left[x_t|\,\mathcal{I}_{t-\delta}\right]\right]$$

where $x_t - E\left[x_t|\,\mathcal{I}_{t-\delta}\right] = 0$ when $\delta = 0$ and constitutes a martingale difference sequence otherwise (it was missing in the original exam paper).

(b) We make in turn the following assumptions on the dynamics of x_t :

$$\begin{aligned} (i) \quad & x_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} \\ (ii) \quad & x_t = \rho x_{t-1} + \varepsilon_t \end{aligned}$$

where $\mu \neq 0$, $\theta \in (-1, 1)$, $\rho \in \mathbb{R}$, and $\varepsilon_t \sim \mathsf{NID}\left(0, \sigma_\varepsilon^2\right)$. We assume that $\mathcal{I}_t$ consists of all the information available at time t regarding the processes $\pi_{t'}, x_{t'}$ and $\varepsilon_{t'}$ for $t' \leq t$; $\mathcal{I}_{t-1}$ is similarly defined.

i. For each of (i) and (ii) above, compute the conditional expectations of future $x_{t'}$ for $t' \geq t$ and solve expression (2) for π_t under models CI and LI.
answer:
(i)

$$\begin{aligned} \text{model } \mathsf{CI}\text{: } & E\left[x_{t+1}|\mathcal{I}_t\right] = \mu + \theta\varepsilon_t, \\ \text{model } \mathsf{LI}\text{: } & E\left[x_{t+1}|\mathcal{I}_{t-1}\right] = \mu \end{aligned}$$

model CI:

$$\pi_t = \sum_{j=0}^{\infty} \beta^j E\left[x_{t+j}|\mathcal{I}_t\right] = \frac{\mu}{1-\beta} + \theta\varepsilon_t$$

model LI:

$$\begin{aligned} \pi_t &= \sum_{j=0}^{\infty} \beta^j E\left[x_{t+j}|\mathcal{I}_{t-1}\right] + \left[x_t - \mu - \theta\varepsilon_{t-1}\right] \\ &= \frac{\mu}{1-\beta} + \underbrace{[\varepsilon_t]}_{\text{absent in the exam}} \end{aligned}$$

the variance of $\theta\varepsilon_t$ – and hence of the inflation π_t under model CI – is lower than that ε_t (and π_t under model LI) since $|\theta| < 1$ in stationary and invertible MA models. Here, if agents use contemporaneous expectations, the variance of inflation is lower.
In the exam since the error term was missing in model LI the difference between models was starker since π_t becomes constant (nonstochastic) under model LI.
(ii)

$$\begin{aligned} \text{model } \mathsf{CI}\text{: } & E\left[x_{t+1}|\mathcal{I}_t\right] = \rho x_t, \\ \text{model } \mathsf{LI}\text{: } & E\left[x_{t+1}|\mathcal{I}_{t-1}\right] = \rho x_{t-1} \end{aligned}$$

model CI

$$\pi_t = x_t = \frac{x_t}{1-\beta\rho}$$

to model LI

$$\begin{aligned} \pi_t &= \rho\sum_{j=0}^{\infty} (\beta\rho)^j x_t + \underbrace{[\varepsilon_t]}_{\text{absent in the exam}} \\ &= \frac{\rho x_t}{1-\beta\rho} + \varepsilon_t \end{aligned}$$

In both models CI and LI, inflation is proportional (with an error in the case of lagged information). We see that if x_t is a measure which is inversely proportional to the employment slack in the economy (measured e.g. as Unemployment) then we obtain the famed Phillips curve which shows a tradeoff between inflation and unemployment. A typical measure for x_t is the output gap (difference between observed GDP growth and its potential). The output gap is a measure of the economic cycle which is inversely proportional to unemployment.

ii. What conditions do β and ρ need to satisfy for (2) to provide a valid solution to expression (1)?
answer: well we need $|\beta\rho| < 1$ so $\sum_{j=0}^{\infty} (\beta\rho)^j$ is not infinite.

iii. Under what conditions are π_t and x_t integrated and cointegrated?
answer: for the variables to be integrated, we need $\rho = 1$, since the MA(1) model is always stationary. Then $x_t \sim \mathsf{I}(1)$ and so is π_t. In model CI trivial cointegration results since the processes are multiples of each other. In model LI there is cointegration since

$$\pi_t - \frac{\rho}{1-\beta\rho} x_t \sim \mathsf{I}(0)$$

i.e. stationary.