ESSEC

Master in Finance

Advanced Master in Financial Engineering (MSTF)

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Financial Risk Management

CLASS HANDOUTS

SESSION 3

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Market Risk: The Historical Simulation Approach

Outline

- The Methodology
- Accuracy
- Extensions
- Extreme Value Theory

I. The Methodology

- Historical simulation involves using past data as a guide to what will happen in the future
- Suppose calculate the one-day market risk VaR with 99%
  - Identify the market variables (e.g., exchange rates, equity prices, interest rates and commodity prices) affecting the portfolio
  - Collect data on the daily movements in all market variables
  - The first simulation trial assumes that the percentage changes in all market variables are as on the first day
  - The second simulation trial assumes that the percentage changes in all market variables are as on the second day
  - and so on
  - For each scenario, the dollar change in the value of the portfolio between today and tomorrow is calculated
  - The one-day VaR with 99% confidence level, assuming n=500, is the 5^{th} worst loss.
- Suppose we use $n+1$ days of historical data and today is day n
  - Let v_i be the value of a variable on day i ($i = 0 \cdots n$)

- There are n simulation trials (e.g., $n = 500$)
- The ith trial assumes that the value of the market variable tomorrow (i.e., on day $n + 1$) is

$$v_n \frac{v_i}{v_{i-1}}$$

– Example:

Suppose on September 25, 2008, an investors owns the following portfolio

DJIA Index	FTSE 100	CAC 40	Nikkei 225	**Total**
\$4,000,000	\$3,000,000	\$1,000,000	\$2,000,000	**\$10,000,000**

The past 501 days of historical data can be downloaded at

www-2.rotman.utoronto.ca/~hull/RMFI/VARExample

Day	Date	DJIA	FTSE 100	CAC 40	Nikkei 225
0	8/7/2006	11219.38	11131.84	6373.89	131.77
1	8/8/2006	11173.59	11096.28	6378.16	134.38
2	8/9/2006	11076.18	11185.35	6474.04	135.94
3	8/10/2006	11124.37	11016.71	6357.49	135.44
…	…	…	…	…	…
499	9/24/2008	10825.17	9438.58	6033.93	114.26
500	9/25/2008	11022.06	9599.90	6200.40	112.82

Scenarios generated for September 26, 2008 are

Scenario number	DJIA $\frac{v_i}{v_{i-1}}$	FTSE 100 $\frac{v_i}{v_{i-1}}$	CAC 40 $\frac{v_i}{v_{i-1}}$	Nikkei 225 $\frac{v_i}{v_{i-1}}$	Portfolio Value ($000s)	**Loss ($000s)**
1	0.9959	0.9968	1.0007	1.0198	10014.33	**-14.33**
2	0.9913	1.0080	1.0150	1.0116	10027.48	**-27.48**
3	1.0044	0.9849	0.9820	0.9963	9946.74	**53.26**
…	…	…	…	…		
499	0.9827	0.9775	0.9761	1.0091	9857.46	**142.54**
500	1.0182	1.0171	1.0276	0.9874	10126.44	**-126.44**

Then rank the loss for the 500 different scenarios:

Loss ($000s)	Scenario number
477.841	494
345.435	339
282.204	349
277.041	329
253.385	487
217.974	227
202.256	131
201.389	238
191.269	473
191.050	306
185.127	477
184.450	495
182.707	376
180.105	237
…	…

So the one-day VaR with 99% confidence level can be estimated as the 5th worst loss, \$253,385.

One-day 99% ES is

(477,841+345,453+282,204+277,041+253,385)/5=\$327,181

- Each day the VaR estimate is updated using the most recent days of data, based on the portfolio value observed on that day
- The VaR on any given day is calculated on the assumption that the portfolio will remain unchanged over the next business day.
- The market variables to be considered include exchange rates, interest rates, equity prices and commodity prices.

  For interest rates, a bank needs to consider treasury and LIBOR/swap term structure of zero-coupon interest rate in a number of currencies. The market variable considered are the ones used to calculate the term structure. There might be up to 10 market variables for each zero curve the bank is exposed.

- Suppose that x is the qth quantile of the loss distribution when it is estimated from n observations. The standard error of x is

$$\frac{1}{f(x)}\sqrt{\frac{(1-q)q}{n}}$$

  where $f(x)$ is an estimate of the probability density of the loss at the qth quantile calculated by assuming a probability distribution for the loss

- Example 1:

  We estimate the 0.99-quantile from 500 observations as $25 million. We estimate *f*(*x*) by approximating the actual empirical distribution with a normal distribution mean zero and standard deviation $10 million. The 0.99 quantile of the normal distribution is NORMINV(0.99,0,10) = 23.26 and the value of *f*(*x*) is NORMDIST(23.26,0,10,FALSE)=0.0027. The standard error of the estimate is therefore

$$\frac{1}{0.0027}\sqrt{\frac{0.01\times 0.99}{500}} = 1.67$$

So the 95% confidence interval of the 0.99 quantile is

$25 + N^{-1}(2.5\%)\times 1.67$ to $25 + N^{-1}(97.5\%)\times 1.67$, i.e., \$21.7 to \$28.3 million

- Note: the standard error decreases when q decreases (when q>50%) or n increases

  Also, the historical simulation assumes the joint distribution of daily changes in market variable is stationary through time. This is not exactly true and creates additional uncertainty about VaR.

- Example 2:

  Suppose the loss distribution computed in last section can be approximated by a normal distribution. The loss is measured in \$000s. The mean of the loss function is \$0.87 and the standard deviation is \$93.698. What is the standard deviation of the estimate of VaR? What is 95% confidence interval of VaR?

- Extension 1: Weighting of observations
  - Let weights assigned to observations decline exponentially as we go back in time. The weight given to scenario i is

$$\frac{\lambda^{n-i}(1-\lambda)}{1-\lambda^n}$$

  - Rank observations from worst to best
  - Starting at worst observation sum weights until the required quantile is reached
- Example:

Assume $\lambda = 0.995$

Scenarios generated for September 26, 2008 are

Scenario number	Portfolio Value ($000s)	Loss ($000s)	**Weight**
1	10014.33	-14.33	0.000446
2	10027.48	-27.48	0.000449
3	9946.74	53.26	0.000451
…			
499	9857.46	142.54	0.005417
500	10126.44	-126.44	0.005444

Then rank the loss for the 500 different scenarios:

Loss ($000s)	Scenario number	Weight	Cumulative Weight
477.841	494	0.00528279	0.00528
345.435	339	0.002429074	0.00771
282.204	349	0.002553936	0.01027
277.041	329	0.002310317	0.01258
253.385	487	0.005100642	0.01768
217.974	227	0.001385562	0.01906
202.256	131	0.000856331	0.01992
201.389	238	0.001464104	0.02138
191.269	473	0.004754972	0.02614
191.050	306	0.002058745	0.02820
…	…		

So the one day 99% VaR is $282,204

One day 99% ES is

[0.00528×477,841+0.00243×345,435+(0.01−0.00528 − 0.00243)×282,204]/0.01 = $400,483

- The parameter λ can be chosen by experimenting to see which value back-tests best

– Extension 2: Incorporating Volatility Updating

- Use a volatility updating scheme and adjust the percentage change observed on day *i* for a market variable for the differences between volatility on day *i* and current volatility
- Value of market variable under *i*th scenario becomes

$$v_n(1 + \frac{v_i - v_{i-1}}{v_{i-1}} \frac{\sigma_{n+1}}{\sigma_i})$$

- Each market variable is handled in the same way
- Example: Assume we use EWMA model with $\lambda = 0.94$ to update the volatility

Day	Date	DJIA	DJIA Return	DJIA Variance	DJIA Vol
0	8/7/2006	11219.38			
1	8/8/2006	11173.59	-0.0040813	0.000122952	1.11%
2	8/9/2006	11076.18	-0.0087179	0.000116575	1.08%
3	8/10/2006	11124.37	0.00435078	0.00011414	1.07%
4	8/11/2006	11088.02	-0.0032676	0.000108428	1.04%
5	8/14/2006	11097.87	0.00088835	0.000102563	1.01%
…	…	…	…	…	…
499	9/24/2008	10825.17	-0.0172953	0.000501763	2.24%
500	9/25/2008	11022.06	0.01818817	0.000489605	2.21%
501				0.000480077	2.19%

Other variables are also handled the same way

Day/ Scenario	DJIA Vol	FTSE Vol	CAC Vol	Nikkei Vol	Value ($000s)	Loss ($000)
1	1.11%	1.42%	1.40%	1.38%	9993.140	6.860
2	1.08%	1.38%	1.36%	1.43%	10045.583	-45.583
3	1.07%	1.35%	1.36%	1.41%	9878.998	121.002
…	…					
500	2.21%	3.28%	3.11%	1.61%	10124.715	-124.715
501	2.19%	3.21%	3.09%	1.59%		

Rank the 500 scenario:

Scenario	Loss ('000s)
131	1082.969
494	715.512
227	687.720
98	661.221
329	602.968
339	546.540
74	492.764
…	…

So the one-day 99% VaR is $602,968.

One-day 99% ES =

(1,082,969+715,512+687,720+661,221+602,968)/5 =

$750,078

- A Simper Approach to Adjusting for Volatility Changes:
  - Monitor variance of simulated losses on the portfolio using EWMA
  - If current standard deviation of losses is b times the standard deviation of simulated losses on Day i, multiply the ith loss given by the standard approach by b

Scenario	Loss ('000s)	variance	SD	SD Ratio	Adj Loss
1	-14.33384585	8779.391675	93.698	2.161	-30.974
2	-27.48131312	8264.955723	90.912	2.227	-61.205
3	53.26405916	7814.371733	88.399	2.290	121.999
…	…	…	…	…	…
500	-126.4389672	42592.08081	206.378	0.981	-124.047
501		40995.76471	202.474		

Scenario	Adj Loss
131	874.5386408
494	749.3678094
227	743.0265196
339	684.4065991
98	616.0365694
329	598.2784055
283	513.3596669
…	…

  - This approach gives VaR and ES as $616.037 and $733,475, respectively

- Extension 3: Bootstrap Method
  - Suppose there are 500 daily changes
  - Calculate a 95% confidence interval for VaR by sampling many (e.g. 500,000) times with replacement from daily changes to obtain 1000 sets of changes over 500 days
  - Calcuate VaR for each set and rank the VaR
  - calculate a 95% confidence interval as the range between the 25th largest VaR and the 975th largest VaR

– Extreme value theory can be used to investigate the properties of the right tail of the empirical distribution of a variable $v.$ (If we interested in the left tail we consider the variable $-v$.)

– Suppose the cumulative distribution function of v is $F(v)$

The probability distribution that v lies between u and $u+y$ conditional that it is greater than u is

$$F_u(y) = \frac{F(u+y) - F(u)}{1 - F(u)}$$

– Gnedenko's result shows that for a wide class of distributions as u increases the probability distribution that v lies between u and $u+y$ conditional that it is greater than u tends to a generalized Pareto distribution

– The Pareto cumulative distribution (conditional) is

$$G_{\xi,\beta}(y) = 1 - (1 + \xi\frac{y}{\beta})^{-\frac{1}{\xi}}$$

– The two parameters ξ (the shape parameter) and β (the scale parameter) can be estimated from the data using MLE

The (conditional) probability density function is

$$g_{\xi,\beta}(y) = \frac{1}{\beta}(1 + \xi \frac{y}{\beta})^{-\frac{1}{\xi}-1}$$

MLE:

- First choose a value for u (e.g. 95% percentile point in the empirical distribution)
- Then rank the observations on v from the highest to the lowest and focus on the observations with $v > u$.

  Suppose there are n_u such observations and they are $v_i \ (1 \leq i \leq n_u)$
- The likelihood function (assuming $\xi \neq 0$) is

$$\prod_{i=1}^{n_u} \frac{1}{\beta}(1 + \xi \frac{(v_i - u)}{\beta})^{-\frac{1}{\xi}-1}$$

- The MLE estimate of ξ and β maximize

$$\sum_{i=1}^{n_u} ln\left[\frac{1}{\beta}(1 + \xi \frac{(v_i - u)}{\beta})^{-\frac{1}{\xi}-1}\right]$$

– The unconditional probability that $v > x$ ($x > u$) is

$$[1 - F(u)][1 - G_{\xi,\beta}(x - u)]$$

$$or \qquad Prob(v > x) = \frac{n_u}{n}\left[1 - G_{\xi,\beta}(x - u)\right] = \frac{n_u}{n}(1 + \xi \frac{x - u}{\beta})^{-\frac{1}{\xi}}$$

Note: if we set $u = \frac{\beta}{\xi}$, $K = \frac{n_u}{n}(\frac{\xi}{\beta})^{-\frac{1}{\xi}}$ *and* $\alpha = \frac{1}{\xi}$, *then* $Prob(v > x) = Kx^{-\alpha}$, *which is referred to as the power law*

– If v is the loss on a portfolio during time horizon T, by definition of VaR with a confidence level q, $Prob(v < VaR) = q$

$$q = 1 - \frac{n_u}{n}(1 + \xi\frac{VaR - u}{\beta})^{-\frac{1}{\xi}}$$

and

$$VaR = u + \frac{\beta}{\xi}\left[(\frac{n}{n_u}(1 - q))^{-\xi} - 1\right]$$

$$Expected\ shortfall = \frac{VaR + \beta - \xi u}{1 - \xi}$$

– Example:

Consider the loss of the portfolio on Page 5:

Scenario number	Loss ($000s)	Rank
494	477.841	1
339	345.435	2
349	282.204	3
…		
304	160.778	22

Set u=160 then $n_u = 22$

MLE estimate for ξ and β are 0.436 and 32.532

Note: β is scale sensitive

What's the probability that the portfolio loss between September 25 and September 26, 2008, will be more than $300,000? What is the 99% one-day VaR? What is the expected shortfall with 99% confidence level and 1-day time horizon? What if the confidence level is 99.9%?

- EVT can be used in conjunction with
  - the weighting of observations procedure

    In this case, for MLE, the loglikelihood function must be multiplied by the weights applicable to the underlying observations
  - the volatility updating procedure

- EVT can be used to refine the confidence interval

$$f(x) = \frac{n_u}{n}\frac{1}{\beta}(1 + \xi\frac{x-u}{\beta})^{-\frac{1}{\xi}-1}$$

- Choice of u:

  It's often found that the estimate of ξ and β do depend on u, but $F(x)$ is roughly the same.

  u should be sufficiently high that we are truly investigating the tail of the distribution, but sufficiently low that we have not too low number of data in MLE estimation.

  A rule of thumb is u is approximately 95th percentile of the empirical distribution.

  In search optimal value of ξ and β, both parameters should be constrained to be positive. Otherwise, it's likely a sign that the tail distribution is not heavier than the normal distribution or u is inappropriate.