Queens College, CUNY, Department of Computer Science
Computational Finance
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due Friday October 27, 2017 at 11.59 pm

5 Homework: Options 1

Note: Continuous interest rate compounding is used in all questions.

5.1 Option price: arbitrage 1

- Suppose the market price of a stock is S_0 at time t_0.
- The stock does not pay dividends.
- The interest rate is $r > 0$ (a constant).
- The following options all have strike $K > 0$ and expiration time $T > t_0$.
  1. European call c_{Eur}.
  2. American call C_{Am}.
  3. European put p_{Eur}.
  4. American put P_{Am}.
- **Question: For each case below, either formulate an arbitrage strategy to take advantage of the option price, else explain why an arbitrage strategy is not possible for that situation.**
  1. Market price $c_{\text{Eur}} = S_0 + 1.5$.
  2. Market price $C_{\text{Am}} = S_0 + 2.5$.
  3. Market price $p_{\text{Eur}} = e^{-r(T-t_0)}K + 0.75$.
  4. Market price $P_{\text{Am}} = K + 1.75$.

5.2 Option price: arbitrage 1, solution for European call

- **Sec. 5.1 will be solved in class on Monday Oct 23, 2017.**
- In most textbooks, the most common point of view is that the investor (= you) holds the option. Therefore you (= holder) decide whether to exercise the option or not.
- However, it is different when you are the **writer (= seller)** of an option. The arbitrage strategies for the call options require you to take a **short position** in the call option.
- ***You must be taught what happens and what to do, if you sell an option and it is exercised against you.***
- The perspective of the option writer is unfamiliar and not obvious.
  1. Recall that an option writer has **obligations** but **no rights** (see Lecture 6).
  2. This is an important fact and you must be taught the consequences.
  3. An option is exercised when it is in the money, *i.e. the option holder makes a profit.*
  4. This means the option writer will **suffer a loss** when the option is exercised.
  5. For a call, the option writer must **deliver the stock and accept cash** $= K$ **(= strike),** *even though the stock price is* $S_T \geq K$ *and the option writer will suffer a loss.*
  6. We shall discuss put options later. Otherwise it is too confusing.
- We begin with the arbitrage strategy for the European call option.
  1. At time t_0, you **sell** the call option and receive cash $= S_0 + 1.5$.
  2. You use some of the money to buy one share of stock, cost $= S_0$.
  3. Therefore you have "extra cash" $= 1.5$. This cash is saved in a bank.
- At the time t_0, your portfolio is: short one European call option, long one share of stock, cash in bank. The total value of your portfolio is zero.
- At the expiration time T, the money in the bank compounds to $1.5e^{r(T-t_0)}$.
- **However, we do not know the final value of the stock price S_T at the expiration time T.**
- Hence we must analyze all cases, i.e. all values of S_T.
- We must prove that in all cases, the profit is positive (or zero),
  but ***never negative (= loss).***
- 1. We do this because S_T is the only random variable in the problem.
  2. If there were two random variables, for example stock price and interest rate, then we must analyze all the possible values of both random variables.
  3. It gets complicated when there are multiple random variables.
- There are two cases (i) $S_T \geq K$ or (ii) $S_T < K$.

5.2.1 Case $S_T \geq K$

- In this situation the option holder exercises the call option.
- The holder pays the writer (= you) cash K (= strike price).
- The writer (= you) delivers the stock to the holder and receives cash $= K$.
- The option writer (= you) **must obey** because the option writer has **no rights, only obligations.**
- However, remember that you already bought the stock at the time t_0.
- Hence the writer (= you) delivers the stock (which you already have).
- So after the option is exercised, you have cash $= K$ and *no more stock.*
- *The stock cancels out:* you bought it at time t_0 and delivered it at time T.
- But ... remember that you also have cash in the bank $= 1.5e^{r(T-t_0)}$.
- Hence your total money is $K + 1.5e^{r(T-t_0)}$.
- This is your profit $= K + 1.5e^{r(T-t_0)}$.
- You started with zero and you have a positive amount of money at the end.
- You have no more stock and no more option, **only cash.**

5.2.2 Case $S_T < K$

- The holder does not exercise because the call option is out of the money.
- So you throw away the option (= not exercised).
- You receive nothing and you deliver nothing (= not exercised).
- So you have stock $= S_T$ plus cash $= 1.5e^{r(T-t_0)}$.
- Hence you **sell the stock,** because you do not want to lose money in case the stock price goes down some more.
- Remember that you want a ***guaranteed profit*** (not a random number).
- Hence you sell the stock and receive *cash* $= S_T$.
- Hence your total money is $S_T + 1.5e^{r(T-t_0)}$.
- This is your profit $= S_T + 1.5e^{r(T-t_0)}$.
- You started with zero and you have a positive amount of money at the end.
- You have no more stock and no more option, **only cash.**

5.2.3 Review

- You must also understand these facts:
  1. Your total profit ***depends on the value of S_T.***
  2. Your total profit ***depends whether or not the option is exercised.***
  3. It is a different amount of profit in case (i) $S_T \geq K$ or case (ii) $S_T < K$.
  4. But the profit is ***always positive.***
  5. There is ***never a loss.***
  6. The arbitrage strategy gives a *guaranteed profit (always positive).*
  7. But ***you do not know how much profit.***
  8. It is not a constant number.
  9. And the profit is more than $1.5e^{r(T-t_0)}$.
- Do you also understand that you do not need to know the strike price of the call option, to formulate your arbitrage strategy?
- It does not matter if $S_0 \geq K$ or $S_0 < K$.
- That is why the value of K was not stated in the question.

5.3 Option price: arbitrage 1, solution for European put

- If the option holder exercises a put option, the holder **sells the stock and receives cash $= K$ (= strike).**
- Hence for a put, the option writer must **buy the stock and pay cash $= K$ (= strike),** *even though the stock price is $S_T \le K$ and the option writer will suffer a loss.*
- This is the arbitrage strategy for the European put option.
  1. At time t_0, you **sell** the put option and receive cash $= e^{-r(T-t_0)}K + 0.75$.
  2. You save **all the money** in a bank.
  3. The arbitrage strategy for a put **does not involve buying/selling stock.**
  4. It is important to understand that an arbitrage strategy does not always have to involve the stock.
- At the time t_0, your portfolio is: short one European put option, cash in bank.
  The total value of your portfolio is zero.
- At the expiration time T, the money in the bank compounds to $K + 0.75e^{r(T-t_0)}$.
- **However, we do not know the final value of the stock price S_T at the expiration time T.**
- Hence we must analyze all cases, i.e. all values of S_T.
- We must prove that in all cases, the profit is positive (or zero),
  but ***never negative (= loss).***
- There are two cases (i) $S_T \le K$ or (ii) $S_T > K$.

5.3.1 Case $S_T \le K$

- In this situation the option holder exercises the put option.
- The holder delivers the stock (value $= S_T$) to the writer (= you) and receives cash K (= strike price).
- The writer (= you) receives the stock (value $= S_T$) and pays cash $= K$.
- The option writer (= you) **must obey** because the option writer has **no rights, only obligations.**
- However, remember that you already have enough cash in the bank $K + 0.75e^{r(T-t_0)}$.
- Hence the writer (= you) pays K of your cash to the holder.
- So after the option is exercised, you have cash $= 0.75e^{r(T-t_0)}$ and *one share of stock.*
- You **sell the stock** and receive cash $(= S_T)$, because want a ***guaranteed profit,*** not a random number.

- Hence your total money at the end is $S_T + 0.75e^{r(T-t_0)}$.
- This is your profit $S_T + 0.75e^{r(T-t_0)}$.
- You started with zero and you have a positive amount of money at the end.
- You have no more stock and no more option, **only cash.**

5.3.2 Case $S_T > K$

- The holder does not exercise because the put option is out of the money.
- So you throw away the option (= not exercised).
- You receive nothing and you deliver nothing (= not exercised).
- So you have only cash $= K + 0.75e^{r(T-t_0)}$.
- This is your profit $= K + 0.75e^{r(T-t_0)}$.
- You started with zero and you have a positive amount of money at the end.
- You have no stock and no option, **only cash.**

5.3.3 Review

- You must also understand these facts:
  1. Your total profit ***depends on the value of*** S_T***.***
  2. Your total profit ***depends whether or not the option is exercised.***
  3. It is a different amount of profit in case (i) $S_T \leq K$ or case (ii) $S_T > K$.
  4. But the profit is ***always positive.***
  5. There is ***never a loss.***
  6. The arbitrage strategy gives a *guaranteed profit (always positive).*
  7. But ***you do not know how much profit.***
  8. It is not a constant number.
  9. And the profit is more than $0.75e^{r(T-t_0)}$.
- Do you also understand that in this case, you need to know the strike price of the put option, *but you do not need to know the stock price,* to formulate your arbitrage strategy?
- It does not matter if $S_0 \leq K$ or $S_0 > K$.
- That is why the value of S_0 was not stated in the question for the put option.

5.4 Option price: arbitrage 1, solution for American call

- This is the arbitrage strategy for the American call option.
  1. At time t_0, you **sell** the call option and receive cash $= S_0 + 2.5$.
  2. You use some of the money to buy one share of stock, cost $= S_0$.
  3. Therefore you have "extra cash" $= 2.5$. This cash is saved in a bank.
- At the time t_0, your portfolio is: short one American call option, long one share of stock, cash in bank. The total value of your portfolio is zero.
- This is the same arbitrage strategy as for the European call option.
- However, there is one important additional detail in the analysis for an American option.
- Because an American option can be exercised at any time, we must prove that our arbitrage strategy yields a guaranteed profit **at any time t, where $t_0 < t \leq T$,** not only at the expiration time T.
- Suppose the option holder exercises the American call at an intermediate time t, where $t_0 < t < T$.
- The stock price at the time t is S_t, and the holder will only exercise if $S_t \geq K$.
- Hence there is only one case to analyze, which is $S_t \geq K$.
- At the time t, the money in the bank compounds to $2.5e^{r(t-t_0)}$.
- The holder exercises the option and pays the writer (= you) cash K (= strike price).
- The writer (= you) delivers the stock to the holder and receives cash $= K$.
- However, remember that you already bought the stock at the time t_0.
- Hence the writer (= you) delivers the stock (which you already have).
- So after the option is exercised, you have cash $= K$ and *no more stock.*
- *The stock cancels out:* you bought it at time t_0 and delivered it at time t (instead of T).
- But remember also that you also have cash in the bank $= 2.5e^{r(t-t_0)}$.
- Hence your total profit is $K + 2.5e^{r(t-t_0)}$ if the holder exercises at the time t.
- You started with zero and you have a positive amount of money after the holder exercises the option.
- You have no more stock and no more option, **only cash.**
- At the expiration time T the analysis is the same as for a European call option, because the terminal payoffs of an American and European call option are the same.
- Therefore the above arbitrage strategy yields a positive guaranteed profit at any time $t_0 < t \leq T$.

5.5 Option price: arbitrage 1, solution for American put

- This is the arbitrage strategy for the American put option.
  1. At time t_0, you **sell** the put option and receive cash $= K + 1.75$.
  2. You save all the money in a bank.
- At the time t_0, your portfolio is: short one American put option, cash in bank. The total value of your portfolio is zero.
- This is the same arbitrage strategy as for the European put option.
- However, notice that the amount of money involved is K, not the present value $e^{-r(T-t_0)}K$.
- This is because an American option can be exercised at any time, as we shall see below.
- Now we must prove that our arbitrage strategy yields a guaranteed profit **at any time t, where $t_0 < t \leq T$,** not only at the expiration time T.
- Suppose the option holder exercises the American put at an intermediate time t, where $t_0 < t < T$.
- The stock price at the time t is S_t, and the holder will only exercise if $S_t \leq K$.
- Hence there is only one case to analyze, which is $S_t \leq K$.
- At the time t, the money in the bank compounds to $(K + 1.75)e^{r(t-t_0)}$.
- The holder exercises the option and delivers the stock to the writer (= you).
- The writer (= you) receives the stock and pays cash $= K$ to the option holder.
- However, you have enough cash available to pay K to the option holder.
- Hence the writer (= you) pays K and has money $K(e^{r(t-t_0)} - 1) + 1.75e^{r(t-t_0)}$ in the bank.
- So after the option is exercised, you have cash $= K(e^{r(t-t_0)} - 1) + 1.75e^{r(t-t_0)}$ and *one share of stock.*
- You **sell the stock,** because you want a guaranteed profit, not a random number.
- Hence your profit is $S_t + K(e^{r(t-t_0)} - 1) + 1.75e^{r(t-t_0)}$ if the holder exercises at the time t.
- You started with zero and you have a positive amount of money after the holder exercises the option.
- You have no stock and no option, **only cash.**
- At the expiration time T the analysis is the same as for a European put option, because the terminal payoffs of an American and European put option are the same.
- Therefore the above arbitrage strategy yields a positive guaranteed profit at any time $t_0 < t \leq T$.

5.6 Option price: arbitrage 2

- Suppose the market price of a stock is S_0 at time t.
- The interest rate is $r > 0$ (a constant).
- The following options all have strike $K > 0$ and expiration time $T > t_0$.
- The stock pays **one dividend**, of amount D_1, at a time t_1, where $t_0 < t_1 < T$.

  1. European call c_{Eur}.
  2. American call C_{Am}.
  3. European put p_{Eur}.
  4. American put P_{Am}.

- **Question: For each case below, either formulate an arbitrage strategy to take advantage of the option price, else explain why an arbitrage strategy is not possible for that situation.**

  1. Market price $c_{\text{Eur}} = S_0 + 1.5$.
  2. Market price $C_{\text{Am}} = S_0 + 2.5$.
  3. Market price $p_{\text{Eur}} = e^{-r(T-t_0)}K + 0.75$.
  4. Market price $P_{\text{Am}} = K + 1.75$.

- Suppose the stock pays two dividends in the time interval from t_0 to T.
  The dividend amounts are D_1 and D_2, paid at times t_1 and t_2, where $t_0 < t_1 < t_2 < T$.

  Question: Describe your arbitrage strategy (if it exists) in this scenario, for each case above.

5.7 Call option spreads: arbitrage

- Suppose the market price of a stock is S at time t.
- The stock does not pay dividends.
- The interest rate is $r > 0$ (a constant).
- All the options below have the same expiration time T (where $T > t$).

5.7.1 European options

- We are given two European call options on the stock:
  c_1 with strike K_1 and c_2 with strike K_2 (where $K_2 > K_1 > 0$).
- Recall that European options can be exercised only at the the expiration time T.
- For simplicity, denote the prices of the options by c_1 and c_2, respectively.
  Create a bull call spread with price $c_1 - c_2$.
- **Question: Show that at expiration, the option prices must satisfy the following inequality:**

$$c_1(T) - c_2(T) \le K_2 - K_1 \qquad \text{(at expiration)}\,. \tag{5.7.1}$$

- **Question: Formulate an arbitrage strategy to show that before expiration (time $t < T$), the option prices must satisfy the following inequality:**

$$c_1(t) - c_2(t) \le \mathrm{PV}(K_2 - K_1) = e^{-r(T-t)}(K_2 - K_1) \qquad (t < T)\,. \tag{5.7.2}$$

5.7.2 American options

- Next we are given two American call options on the stock:
  C_1 with strike K_1 and C_2 with strike K_2 (where $K_2 > K_1 > 0$).
- Recall that American options can be exercised at any time $t \le T$.
- For simplicity, denote the prices of the options by C_1 and C_2, respectively.
  Create a bull call spread with price $C_2 - C_1$.
- **Question: Show that at expiration, the option prices must satisfy the following inequality:**

$$C_1(T) - C_2(T) \le K_2 - K_1 \qquad \text{(at expiration)}\,. \tag{5.7.3}$$

- **Question: Formulate an arbitrage strategy to show that before expiration (time $t < T$), the American option prices must satisfy the following inequality:**

$$C_1(t) - C_2(t) \le K_2 - K_1 \qquad (t < T)\,. \tag{5.7.4}$$

5.7.3 Stock dividends

- **(Optional question)**
  Suppose the stock pays n dividends D_i at times t_i, $i = 1, 2, \ldots, n$ during the lifetime of the options. Explain how the inequalities in eqs. (5.7.2) and (5.7.4) would be modified.

5.8 Put option spreads: arbitrage

- Suppose the market price of a stock is S at time t.
- The stock does not pay dividends.
- The interest rate is $r > 0$ (a constant).
- All the options below have the same expiration time T (where $T > t$).

5.8.1 European options

- We are given two European put options on the stock:
  p_1 with strike K_1 and p_2 with strike K_2 (where $K_2 > K_1 > 0$).
- Recall that European options can be exercised only at the the expiration time T.
- For simplicity, denote the prices of the options by p_1 and p_2, respectively.
  Create a bear put spread with price $p_2 - p_1$.
- **Question: Show that at expiration, the option prices must satisfy the following inequality:**

$$p_2(T) - p_1(T) \le K_2 - K_1 \qquad \text{(at expiration)}\,. \tag{5.8.1}$$

- **Question: Formulate an arbitrage strategy to show that before expiration (time $t < T$), the option prices must satisfy the following inequality:**

$$p_2(t) - p_1(t) \le \mathrm{PV}(K_2 - K_1) = e^{-r(T-t)}(K_2 - K_1) \qquad (t < T)\,. \tag{5.8.2}$$

5.8.2 American options

- Next we are given two American put options on the stock:
  P_1 with strike K_1 and P_2 with strike K_2 (where $K_2 > K_1 > 0$).
- Recall that American options can be exercised at any time $t \le T$.
- For simplicity, denote the prices of the options by P_1 and P_2, respectively.
  Create a bull put spread with price $P_2 - P_1$.
- **Question: Show that at expiration, the option prices must satisfy the following inequality:**

$$P_2(T) - P_1(T) \le K_2 - K_1 \qquad \text{(at expiration)}\,. \tag{5.8.3}$$

- **Question: Formulate an arbitrage strategy to show that before expiration (time $t < T$), the American option prices must satisfy the following inequality:**

$$P_2(t) - P_1(t) \le K_2 - K_1 \qquad (t < T)\,. \tag{5.8.4}$$

5.8.3 Stock dividends

- **(Optional question)**
  Suppose the stock pays n dividends D_i at times t_i, $i = 1, 2, \dots, n$ during the lifetime of the options. Explain how the inequalities in eqs. (5.8.2) and (5.8.4) would be modified.