第一类换元法

(1) $\int (x-1)^2 dx$

$= \int (x-1)^2 d(x-1)$

$= \frac{1}{3}(x-1)^3 + C$

(2) $\int x^2\sqrt{x^3+1}\, dx$

$= \frac{1}{3}\int \sqrt{x^3+1}\ d(x^3+1)$

$= \frac{2(x^3+1)^{\frac{3}{2}}}{9} + C$

(3) $\int \frac{x^4}{(x^5+1)^2} dx$

$= \frac{1}{5}\int \frac{1}{(x^5+1)^2} d(x^5+1)$

$= \frac{1}{5(x^5+1)} + C$

(4) $\int \frac{dx}{(x^2+1)^{\frac{3}{2}}}$ $(x>0)$.

$= \int \frac{dx}{x^3(1+\frac{1}{x^2})^{\frac{3}{2}}}$

$= -\frac{1}{2}\int \frac{d(1+\frac{1}{x^2})}{(1+\frac{1}{x^2})^{\frac{3}{2}}}$

$= (1+\frac{1}{x^2})^{-\frac{1}{2}} + C$

(5) $\int \frac{\cos\sqrt{t}}{\sqrt{t}} dt$

$= 2\int \cos\sqrt{t}\ d\sqrt{t}$

$= 2\sin\sqrt{t} + C$

(6) $\int \frac{(\ln x)^2}{x} dx$

$= \int \cancel{(\ln x)^2}(\ln x^2)\ d\ln x$

$= \cancel{\frac{\ln x^3}{3} + C}\ \frac{\ln x^3}{3} + C$

(7) $\int e^{e^x+x} dx$

$= \int e^{e^x}\cdot e^x dx$

$= \int e^{e^x} de^x$

$= e^{e^x} + C$

(8) $\int \frac{dx}{a^2-x^2}$ $(a\neq 0)$.

$= \frac{1}{a^2}\int \frac{dx}{1-(\frac{x}{a})^2}$ $(a\neq 0)$

$= \int \frac{dx}{(a+x)(a-x)}$

$= \frac{1}{2a}\int (\frac{1}{a-x} + \frac{1}{a+x})\, dx$

$\cancel{= \frac{1}{2a}(\ln(a-x}$

$= \frac{1}{2a}\int -\frac{d(a-x)}{a-x} + \frac{d(a+x)}{a+x} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

(9) $\int \frac{x^3}{\sqrt{1+x^2}}dx$

$=\frac{1}{2}\int \frac{x^2}{\sqrt{1+x^2}}d(1+x^2)$

$=\frac{1}{2}\int \frac{x^2+1-1}{\sqrt{1+x^2}}d(1+x^2)$

$=\frac{1}{2}\int (\frac{x^2+1}{\sqrt{1+x^2}}-\frac{1}{\sqrt{1+x^2}})d(1+x^2)$

$=\frac{1}{2}\int (\sqrt{1+x^2}-\frac{1}{\sqrt{1+x^2}})d(1+x^2)$

$=\frac{1}{3}(1+x^2)^{\frac{3}{2}}-(1+x^2)^{\frac{1}{2}}+C$

(10) $\int \frac{dx}{2^x(1+4^x)}$

$=\frac{1}{\ln 2}\int \frac{d2^x}{4^x(1+4^x)}$

$=\frac{1}{\ln 2}(\int \frac{1}{4^x}d2^x-\int \frac{1}{1+4^x}d2^x)$

$=\frac{1}{\ln 2}(-\frac{1}{2^x}-\arctan 2^x)+C$

(11) $\int \frac{x^2+1}{x^4+1}dx$

$=\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$ 令 $dx \to d(x-\frac{1}{x})$

$d(x-\frac{1}{x})=(1+\frac{1}{x^2})dx$

$=\int \frac{d(x-\frac{1}{x})}{x^2+\frac{1}{x^2}} \cancel{d(x-\frac{1}{x})}$

$=\int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2}$

$=\frac{1}{2}\int \frac{d(x-\frac{1}{x})}{(\frac{x-\frac{1}{x}}{\sqrt{2}})^2+1}$

$=\frac{1}{\sqrt{2}}\int \frac{d(\frac{x-\frac{1}{x}}{\sqrt{2}})}{(\frac{x-\frac{1}{x}}{\sqrt{2}})^2+1}$

$=\frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x}+C$

(12) $\int \frac{1}{1+\sin x} dx$

$= \int \frac{1-\sin x}{1-\sin^2 x} dx$

$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx$

$= \int \sec^2 x \, dx - \int \tan x \sec x \, dx$

$= \tan x - \sec x + C$

(13) $\int \frac{\sin 2x}{\sqrt{3-\cos^4 x}} dx$

$= \int \frac{2\sin x \cos x}{\sqrt{3-\cos^4 x}} dx$

$= -\int \frac{d\cos^2 x}{\sqrt{3-\cos^4 x}}$

$= -\frac{1}{\sqrt{3}} \int \frac{d\cos^2 x}{\sqrt{1-\frac{\cos^4 x}{3}}}$

$= -\int \frac{d\frac{\cos^2 x}{\sqrt{3}}}{\sqrt{1-\frac{\cos^4 x}{3}}}$

$= -\arcsin \frac{\cos^2 x}{\sqrt{3}} + C$

(14) $\int \tan^3 x \, dx$ $\quad 1+\tan^2 x = \sec^2 x$

$= \cancel{\int \tan x(1-\sec^2 x)}$

$= \int \tan x(1+\tan^2 x) dx - \int \tan x \, dx$

$= \int \cos^2 x \sin x(1+\tan^2 x) d\tan x - \int \tan x \, dx$

$= \int \tan x \, d\tan x - \int \tan x \, dx$

$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$

(15) $\int \frac{\sin x - \cos x}{\sin x + 2\cos x} dx$

$= a(\sin x + 2\cos x) + b(\cos x - 2\sin x) = \sin x - \cos x$

$\begin{cases} a-2b=1 \\ 2a+b=-1 \end{cases} \quad a=-\frac{1}{5} \quad b=-\frac{3}{5}$

$\therefore \int \frac{-\frac{1}{5}(\sin x+2\cos x) - \frac{3}{5}(\cos x - 2\sin x)}{\sin x + 2\cos x} dx$

$= \int -\frac{1}{5} dx - \frac{3}{5}\int \frac{d(\sin x+2\cos x)}{\sin x+2\cos x} = -\frac{x}{5} - \frac{3}{5}\ln|\sin x+2\cos x| + C$

(16) $\int \frac{1}{\sin^2 x + 2\cos^2 x}\, dx$

$= \int \frac{1}{\cos^2 x(\tan^2 x + 2)}\, dx \qquad d\frac{\tan x}{\sqrt{2}} = \frac{1}{\sqrt{2}}\, d\tan x$

$= \int \frac{1}{\tan^2 x + 2}\, d\tan x$

$= \int \frac{1}{2\left(\frac{\tan x}{\sqrt{2}}\right)^2 + 1}\, d\tan x$

$= \frac{1}{\sqrt{2}} \arctan \frac{\tan x}{\sqrt{2}} + C$

①去根号

第二类换元

(1) $\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$

令 $x = \tan t$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\therefore \int \frac{1}{\tan^2 t \cdot \sec t}\, d\tan t \qquad d\tan t = \sec^2 t\, dt$

$= \int \frac{1}{\tan^2 t \cdot \sec t} \cdot \sec^2 t\, dt$

$= \int \frac{\sec t}{\tan^2 t}\, dt$

$= \int \frac{\cos t}{\sin^2 t}\, dt = \int \frac{d\sin t}{\sin^2 t} = -\frac{1}{\sin t} + C = -\frac{\sqrt{x^2+1}}{x} + C.$

(2) $\int \frac{1}{(1-x)\sqrt{1-x^2}}\,dx$

令 $x = \cos t$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$dx = -\sin t\, dt \qquad \frac{\cos t}{\sin t} = \frac{x}{\sqrt{1-x^2}} \quad +.\ \frac{1}{\sqrt{1-x^2}}.$

$\therefore \int \frac{1}{(1-\cos t)\sin t}\, dx$

$= \int \frac{1}{\cos t - 1}\, dt$

$= -\int \frac{\cos t + 1}{\sin^2 t}\, dt$

$= -\int \csc^2 t\, dt - \int \cot t \cdot \cancel{\cos}^{\csc} t\, dt$

$= \cot t + \cancel{\cos}^{\csc} t + C$

$= \frac{x+1}{\sqrt{1-x^2}} + C.$

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③ $\int \frac{dx}{x^4(1+x^2)}$

令 $x = \frac{1}{t}$

t^2+1

$d\frac{1}{t} = -\frac{1}{t^2}\,dt$

$= \int \frac{t^4}{1+\frac{1}{t^2}}\,d\frac{1}{t}$

$= \int \frac{t^6}{t^2+1}\,d\frac{1}{t}$

$= -\int \frac{t^4}{t^2+1}\,dt$

$= -\int \frac{t^4-1+1}{t^2+1}\,dt$

$= -\int (t^2-1)\,dt - \int \frac{1}{t^2+1}\,dt$

$= -\frac{t^3}{3} + t - \arctan t + C$

$= -\frac{1}{3x^3} + \frac{1}{x} - \arctan\frac{1}{x} + C$

④ $\int \frac{1}{\sqrt{b^2-x^2}}\,dx \quad (b>0)$

常用积分表

(1) $\int \tan x\,dx = -\ln|\cos x| + C$　(2) $\int \cot x\,dx = \ln|\sin x| + C$

(3) $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

(4) $\int \csc x\,dx = \ln|\csc x - \cot x| + C$

(1). $$\int \frac{dx}{x+\sqrt{a^2-x^2}}$$

令 $x = a\sin t$, $t = \arcsin\frac{x}{a}$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$dx = a\cos t\,dt$

$$原式 = \int \frac{dx}{x + a\cos t}$$

$$= \int \frac{a\cos t\,dt}{a\sin t + a\cos t}$$

$$= \int \frac{\cos t - \sin t}{\cos t + \sin t}dt + \int \frac{\sin t}{\cos t + \sin t}dt$$

$$= \int \frac{\cos t - \sin t}{\cos t + \sin t}dt + \int \frac{\sin t + \cos t}{\sin t + \cos t}dt - \int \frac{\cos t}{\sin t + \cos t}dt$$

$$\therefore \int \frac{\cos t}{\sin t + \cos t}dt = \frac{1}{2}\int \frac{d(\cos t + \sin t)}{\cos t + \sin t} + \frac{1}{2}t$$

$$= \frac{1}{2}\ln|\cos t + \sin t| + \frac{1}{2}t + C$$

$$\therefore 原式 = \frac{1}{2}\left(\ln\left|\frac{x}{a} + \cancel{\frac{\sqrt{x^2-a^2}}{a}}\frac{\sqrt{a^2-x^2}}{a}\right| + \arcsin\frac{x}{a}\right) + C$$

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要讨论x与a的大小。

12) $\int \frac{\sqrt{x^2-a^2}}{x} dx \quad (a>0)$.

设 当 $x>a$ 时 令 $x=a\sec t$, $t\in(0,\frac{\pi}{2})$

$$\int \frac{a\tan t}{a\sec t} dx$$

$$= \int \frac{\tan t}{\sec t} a\sec t\cdot\tan t \, dt$$

$$= \int a\tan^2 t \, dt$$

$$= \int a(\sec^2 t-1) \, dt$$

$$= a(\tan t - t)+C = \sqrt{x^2-a^2} - a\arccos\frac{a}{x} + C$$

当 $x<-a$ 时 令 $x=a\sec t$, $t\in(\frac{\pi}{2},\pi)$