1 Laboratory Nr 1: Simulating sequences of i.i.d. random variables

We have learned about two simulation methods:

1. **Inversion method**: Simulation is based on the following observation: If X is uniformly distributed on $<0,1>$, and $F(x)$ is some invertible c.d.f. then $Y = F^{-1}(X)$ has c.d.f equal to F. Hence for example remembering that $F(x) = 1 - \exp(-\lambda x);\ x \geq 0$ is the cumulative distribution function of exponential distribution we see that (using language of package Mathematica) $Y = -\frac{1}{\lambda} Log[Random[]]$ has exponential distribution with parameter λ.

2. **Rejection method:** Based on two facts:

   (a) For a given function $g(x) \geq 0$ being a density on $<0,a>$ two dimensional random variables (X,Y), where X is distributed as g, $Y = Random[] \times g(X)$, have uniform distribution on $\{(x,y) : 0 \leq x \leq a, 0 \leq y \leq g(x)\}$. In particular Y has density equal to $g(x) / \int_0^a g(y)\,dy$.

   (b) If (X,Y) is uniformly distributed over the area $D = \{(x,y) : 0 \leq x \leq a, 0 \leq y \leq g(x)\}$ and we have function $h(x) \leq g(x)$, then the the pair constructed in the following way: Take a pair (X,Y) uniformly distributed over D. If $Y \leq h(X)$ then accept this pair and set $(S,T) = (X,Y)$. Otherwise generate new pair (X,Y) and check the condition. One shows that pair (S,T) has uniform distribution on $D' = \{(x,y) : 0 \leq x \leq a, 0 \leq y \leq h(x)\}$. In particular T has density proportional to h.

   (c) **The rejection method is thus as follows.** Suppose we want to generate variable S having density h. Suppose there exists another density $f(x)$ (such that variables with this density are easy to generate) and such that $Cf(x) \geq h(x)$ for all $x \in <0,a>$.

   (d) If $h(x)$, $f(x)$ and C are respectively two densities and a constant $C > 1$ such that $Cf(x) \geq h(x)$ for all $x \in <0,a>$, then the procedure:

      i. generate a pair (X,Y) such that X has density $f(x)$ and U is independent of X random variable uniformly distributed on $<0,1>$. (another words obtained by call of $Random[]$.)

      ii. Accept this pair if $UCf(X)/h(X) \leq 1$. Then set $S = X$. Otherwise if $UCf(X)/h(X) > 1$ return to point i.

There is one more way of simulating variables namely by noting that the desired one has distribution that is marginal of some easy generated 2−dimensional distribution. Example is below:

More complicated is generation of Normal distribution (its c.d.f. cannot be expressed in concise form and hence is difficult to invert). Normal variables are generated from pairs of independent Uniform distributions on <0,1> i.e. calls of $Random[]$ function. Namely we have: $X_1 = Random[]$and $X_2 = Random[]$ be two independent uniformly distributed r.v.'s. Then

$$\begin{aligned} Y_1 &= \sqrt{-Log[X_1]}Cos[2\pi X_2], \\ Y_2 &= \sqrt{-Log[X_1]}Sin[2\pi X_2] \end{aligned}$$

are independent identically distributed random variables having normal $N(0,1)$ distribution.

Using Mathematica:

1. Generate a sequence of $N = 1000$ independent observations of random variable with distribution:

   (a) Uniform on $< 0, 1 >$

   (b) Uniform on $< -2, 3 >$

   (c) Exponential with parameter $\lambda = 1$, by inversion method

   (d) Exponential with parameter $\lambda = 10$ by inversion method

   (e) Uniformly on the unit circle $\left\{(x, y) : x^2 + y^2 \leq 1\right\}$

   (f) with the density $f(x) = \frac{2}{\pi}\sqrt{1 - x^2}$. as a marginal distribution of pairs considered in point e.

   (g) Normal with parameters $m = 0$, $\sigma^2 = 1$ by rejection method

   (h) Normal with parameters $m = 0$, $\sigma^2 = 1$, as distributed above

   (i) Discrete random variable X with distribution

-1.5	2.3	3.0	4.8	5.7	5.9
.2	.15	.1	.05	.45	.05

2. Present graphically obtained sequences(except for those generated in point e) i.e. e.g.

   (a) i. plot in the coordinates (No. obs., value of the obs)

   ii. plot in the coordinates (obs No n, obs. No $n + i$) for $i = 1, 2, 3$.

   iii. plot so called *covariance function* for some values. i.e. find averages:

   $$K(\tau) = \frac{1}{N - \tau} \sum_{i=1}^{N-\tau} (obs_i - sr) * (obs_{i+\tau} - sr),$$

   where $sr = \frac{1}{N} \sum_{i=1}^{N} obs_i$, for some values of τ.

3. Generate 10 realizations of random walk i.e. discrete random process of the form: $S_0 = 0$, $S_n = S_{n-1} + X_n$; $n \geq 1$, where $\{X_i\}_{i \geq 1}$ is a sequence of i.i.d. random variables.

   (a) simple symmetric one i.e. $P(X_1 = -1) = P(X_1 = 1) = 1/2$

(b) simple non-symmetric one i.e. $P(X_1 = -1) = 1 - P(X_1 = 1) = \varepsilon$, where $\varepsilon \in (0, .5)$

(c) symmetric one with say $X_1 \sim U(-1, 1)$.

Write short description of what have you done and observed and submit it in pdf format by E-mail: pjsz@hotmail.com or in case of a trouble pszablowski@elka.pw.edu.pl .