What are the Differences between White, Blue and Pink Noise?

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Outline

1. Motivation
2. Spectral Density
3. White, Blue and Pink Noise
4. Conclusion

Sound That We Are Familiar With...

What is that?

Process of Audio Snippet

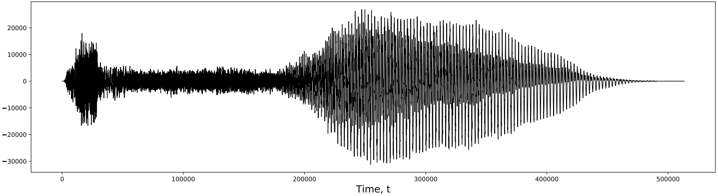


Figure 1: HBO Opening Audio Signal

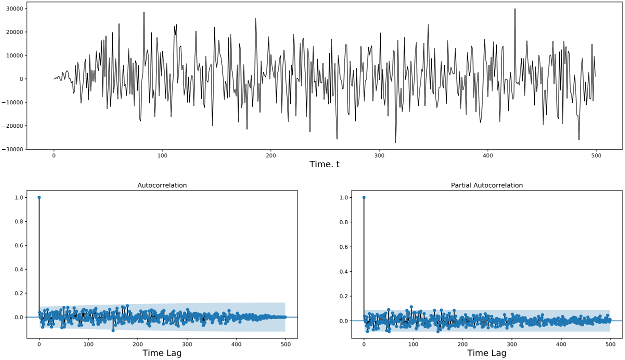
WBPNoise

- Time domain representation of HBO opening audio process $\{X_t\}_{t=0}^{\infty}$
- Periodic, stationarity

What do White, Blue and Pink Noise sound like?

White Noise, *Blue Noise* And *Pink Noise*

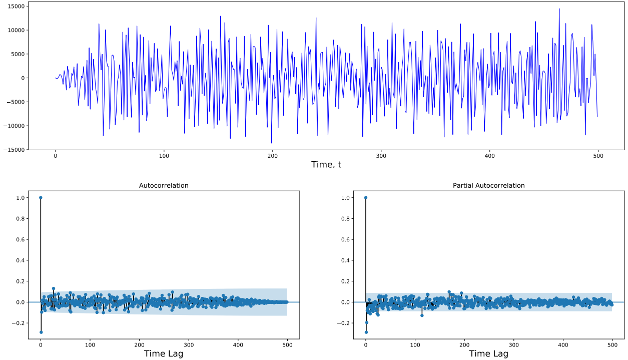
Time Domain of White Noise Sample



WBPNoise

- $\{X_t\}_{t=0}^{500} \sim \mathrm{N}(0, \sigma^2)$, stationary process, $AR\,(0)$ process

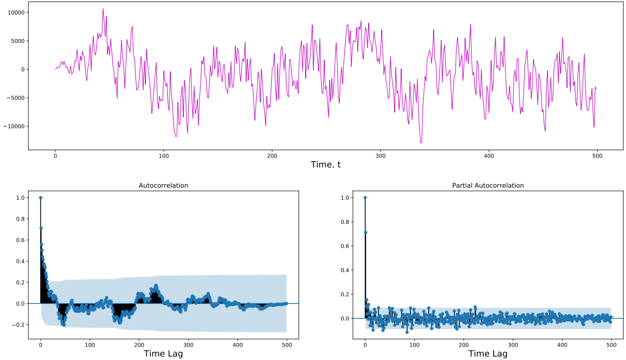
Time Domain of Blue Noise



WBPNoise

- Non-normal distributed, stationary process, $AR\,(p)$ process $(p > 0)$

Time Domain of Pink Noise



WBPNoise

- Non-normal distributed, stationary process, $AR\,(p)$ process $(p > 0)$, long memory

Decomposition of Time Series

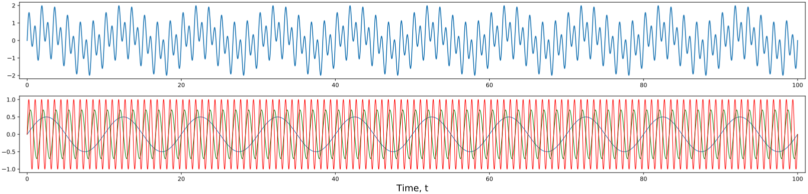


Figure 2: Time series $\{X_t\}_{t=0}^{100}$ (upper panel) can be decomposed into sinusoids with frequencies $\omega = 1.2$, $\omega = 0.6$ and $\omega = 0.1$ (lower panel)

WBPNoise

Fourier Transform

- Fourier analysis: approximate a function by weighted sums of trigonometric functions
- A stationary process $x(t)$ can be written as
  - function of time time $x(t)$ (time domain)
  - function of frequency $S(\omega)$ (frequency domain)

Jean Baptiste Joseph Fourier on BBI: 

Fourier Transform ctd

- Conversion of time function into frequency domain function and vice versa

$$S(\omega) = \int_{-\infty}^{\infty} x(t) e^{-2\pi\omega i t}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} S(\omega) e^{2\pi\omega i t}\, d\omega$$

- ω = frequency

Spectral Density

- Stationary process X_t with autocovariance $\gamma(\tau)$, $(\tau \in \mathbb{Z})$

$$\gamma(\tau) = \mathrm{E}[(x_{t+\tau} - \mu)(x_t - \mu)], \sum_{\tau=-\infty}^{\infty} |\gamma(\tau)| < \infty \tag{1}$$

- Spectrum $S(\omega)$ as Fourier transform of $\gamma(\tau)$:

$$S(\omega) = \sum_{\tau=-\infty}^{\infty} \gamma(\tau) e^{-2\pi\omega i\tau} \tag{2}$$

- $S(\omega) \in \mathbb{R}$, $S(\omega) \geq 0$

Properties

- Symmetric of S

$$S(-\omega) = S(\omega) \tag{3}$$

- Periodicity of S

$$S(\omega + 2\pi t) = S(\omega) \tag{4}$$

for every $t \in \mathbb{Z}$

- Inverse Fourier transform

$$\gamma(\tau) = \int_{-0.5}^{0.5} e^{2\pi\omega i\tau} S(\omega) d(\omega) \tag{5}$$

Periodogram

- Periodogram $P(\omega)$ for realizations $\{x_t\}$:

$$P(\omega) = \frac{1}{N} \left| \sum_{t=0}^{N-1} x_t e^{-2\pi\omega i t} \right|^2, \quad 0 \leq \omega \leq \frac{1}{2} \tag{6}$$

- Unbiased (but inconsistent) estimator of spectral density

Spectrum: $\sum_{i=1}^{3} \gamma_i sin\,(2\pi\omega_i t)$

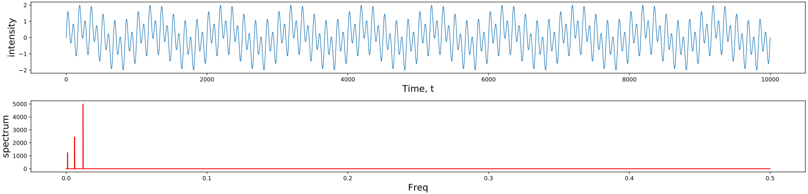


Figure 3: Spectrum of sine waves

WBPNoise

- $\gamma_i \in \{1, 0.7, 0.5\}\,, \omega_i \in \{1.2, 0.6, 0.1\}$ (upper panel)
- Periodogram detects the exact frequency of sine waves (lower panel)

Spectrum: White Noise

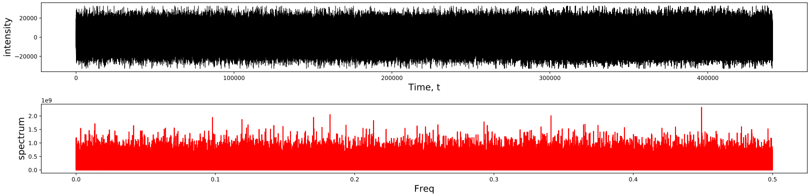


Figure 4: Spectrum of white noise

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- Periodgram (lower panel) shows that white noise dominate in whole spectrum
- Theoretically, $S(\omega) = \sigma^2/2\pi$

Spectrum: Blue Noise

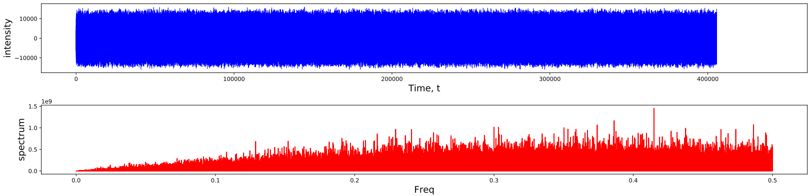

Figure 5: Spectrum of blue noise

WBPNoise

- Periodgram (lower panel) shows that blue noise has more energy in higher frequency domain
- Theoretically, $S(\omega) = \eta\omega, \eta \in \mathbb{R}$

Spectrum: Pink Noise

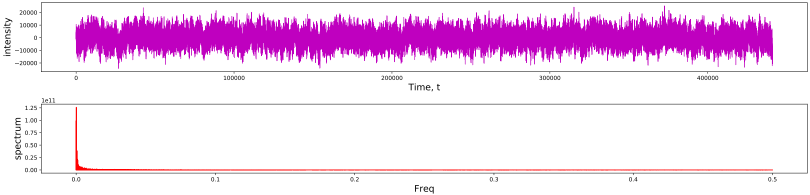

Figure 6: Spectrum density of pink noise

WBPNoise

- Periodgram (lower panel) shows clearly that pink noise is low frequency signal with no high frequency energy
- Theoretically, $S(\omega) = \eta\,(1/\omega)\,, \eta \in \mathbb{R}$

Conclusion

- Different charactristics on time series perspective:
  - White noise are shown to be $AR(0)$ process
  - Pink noise tends to have long memory
- Spectral density perspective
  - White noise is uniformly distributed in whole spectrum
  - Blue noise has more enery in higher freqency domain
  - Pink noise is purly low freqency sound

Bibliography

- Oppenheim, A.V., Willsky, A.S. and Hamid, S., 1996
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  Pearson Press
- Tsay, R. S., 2016
  Analysis of financial time series, 3^{rd} Edition
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Normality Test

Using the omnibus normality test in $DAgostino, R.B.(1971, 1973)$, we have following result:

Signal	Statistics	P-Value
HBO audio	4.48×10^{4}	0.00
White noise	5.60×10^{-2}	0.97
Blue noise	21.60	0.00
Pink noise	5.22	0.07