CSIR UGC NET EXAM(Dec 2016), Q.107

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Question

Let X be a random variable with a certain non-degenerate distribution. Then identify the correct statements

1. If X has an exponential distribution then $median(X) < E(X)$
2. If X has a uniform distribution on an interval $[a, b]$, then $E(X) < median(X)$
3. If X has a Binomial distribution then $V(X) < E(X)$
4. If X has a normal distribution, then $E(X) < V(X)$

Pre-requisites

- Expected value
- Variance
- Median
- Exponential Distribution
- Uniform Distribution
- Binomial Distribution
- Normal Distribution

Expected value

It is nothing but weighted mean
Notation - $E(X)$, μ

Formula

$$E(X) = \mu = \sum k \Pr(X = k) \tag{1}$$

where $\Pr(X)$ is the probability function of a distribution.

Median

It is the value separating the higher half from the lower half of a data sample

Notation - $median(X)$

Formula

$$\int_{lowerlimit}^{median(X)} f(X)dx = \frac{1}{2} = \int_{median(X)}^{upperlimit} f(X)dx \tag{2}$$

We can find $median(X)$ by solving this equation,

where $f(X)$ is the probability function.

Variance

It is the expectation of the squared deviation of a random variable from its mean.

Notation - $V(X)$, σ^2

Formula

$$V(X) = \sigma^2 = E((X - E(X))^2) = E(X^2) - (E(X))^2 \tag{3}$$

Exponential Distribution

It is a distribution in which events occur continuously and independently at a constant average rate

Notation - $Exp(\lambda)$, where λ is the rate parameter

Probability function

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{4}$$

where X is the random variable of the distribution.

Option 1

If X has an exponential distribution then $median(X) < E(X)$

Solution(Option 1)

Let X has exponential distribution
The expected value of $X \sim Exp(\lambda)$,

$$E(X) = \int_0^\infty x\lambda e^{-\lambda x} dx \tag{5}$$

$$E(X) = \frac{1}{\lambda} \tag{6}$$

The median of $X \sim Exp(\lambda)$,

$$\frac{1}{2} = \int_0^{median(X)} \lambda e^{-\lambda x} dx \tag{7}$$

$$median(X) = \frac{\ln 2}{\lambda} \tag{8}$$

Solution(Option 1) Contd.

$$\ln 2 < 1 \tag{9}$$

$$\frac{\ln 2}{\lambda} < \frac{1}{\lambda} \tag{10}$$

$$median(X) < E(X) \tag{11}$$

Hence, option 1 is correct.

Uniform Distribution

It is a distribution that describes an experiment where there is an arbitrary outcome that lies between certain bounds($[a, b]$)
Notation - $U(a, b)$, where $[a, b]$ is the bounded region

Probability function

$$f_X(k) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & x < a, x > b \end{cases} \tag{12}$$

where X is the random variable of the distribution.

Option 2

If X has a uniform distribution on an interval $[a, b]$, then $E(X) < median(X)$

Solution(Option 2)

Let X has Uniform distribution
The expected value of $X \sim U(a, b)$,

$$E(X) = \int_a^b x \frac{1}{b-a} dx \tag{13}$$

$$E(X) = \frac{1}{2}(a+b) \tag{14}$$

The median of $X \sim U(a, b)$,

$$\frac{1}{2} = \int_a^{median(X)} \frac{1}{b-a} dx \tag{15}$$

$$median(X) = \frac{a+b}{2} \tag{16}$$

Solution(Option 2) Contd.

$$median(X) = E(X) \tag{17}$$

Hence, option 2 is incorrect.

Binomial Distribution

It is a distribution of the possible number of successful outcomes in a given number of trials in each of which there is the same probability of success. Notation - $B(n,p)$, where n = no. of trails, p = success parameter

Probability function

$$f_X(k) = \begin{cases} {}^nC_k p^k (1-p)^{n-k} & 0 \leq k \leq n \\ 0 & otherwise \end{cases} \tag{18}$$

where X is the random variable of the distribution.

Option 3

X has a Binomial distribution then $V(X) < E(X)$

Solution(Option 3)

Let X has Binomial distribution
The expected value of $X \sim B(n, p)$,

$$E(X) = \sum_{k=0}^{n} k.^{n}C_k p^k (1-p)^{n-k} \tag{19}$$

$$E(X) = np \tag{20}$$

The variance of $X \sim B(n, p)$,

$$V(X) = \sigma^2 = E(X^2) - (E(X))^2 \tag{21}$$

$$V(X) = \sum_{k=0}^{n} k^2.^{n}C_k p^k (1-p)^{n-k} \tag{22}$$

$$V(X) = np(1-p) \tag{23}$$

Solution(Option 3) Contd.

$$1 - p \leq 1 \tag{24}$$
$$np(1 - p) \leq np \tag{25}$$
$$V(X) \leq E(X) \tag{26}$$

Hence, option 3 is incorrect.

Normal Distribution

It is a distribution that is symmetric about the mean, showing that data near the mean are more frequent in occurrence than data far from the mean.
It is the more frequently used distribution.
Notation - $N(\mu, \sigma^2)$, where $\mu = mean$, $\sigma^2 = variance$

Probability function

$$f_X(k) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(\frac{x-\mu}{2\sigma})^2} \tag{27}$$

where X is the random variable of the distribution.

Option 4

X has a normal distribution, then $E(X) < V(X)$

Solution(Option 4)

Let X has Normal distribution
The expected value of $X \sim N(\mu, \sigma^2)$,

$$E(X) = \mu \tag{28}$$

The variance of $X \sim N(\mu, \sigma^2)$,

$$V(X) = \sigma^2 \tag{29}$$

These are user defined values.

Solution(Option 4) Contd.

So, $E(X)$ and $V(X)$ can take any value independent of each other.
Hence, option 4 is incorrect.