Assignment 7

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1 Problem

GATE 2019 (ST) , Q.49 (Statistics section)

Let X be a random variable with characteristic function $\phi_X(\cdot)$ such that $\phi_X(2\pi) = 1$.Let $\mathbb{Z}$ denote the set of integers.Then $P(X \in \mathbb{Z})$ is equal to ...

2 Solution

General solution for the characteristic solution which is consistent with our condition is,

$$\phi_X(t) = \sum_{z=-\infty}^{\infty} \alpha_z e^{itz} \quad \text{where} \quad z \in \mathbb{Z} \quad \text{and} \tag{2.0.1}$$

$$\sum_{z=-\infty}^{\infty} \alpha_z = 1 \tag{2.0.2}$$

Using Gil-Pelaez formula for probability density function,

$$f_X(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{itx}\phi(-t) + e^{-itx}\phi(t)\ dt \tag{2.0.3}$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{itx} \sum_{z=-\infty}^{\infty} \alpha_z e^{itz} + \frac{1}{4\pi} e^{-itx} \sum_{z=-\infty}^{\infty} \alpha_z e^{itz}\ dt \tag{2.0.4}$$

$$= \frac{1}{4\pi} \sum_{z=-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha_z e^{-it(x-z)} dt + \frac{1}{4\pi} \sum_{z=-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha_z e^{it(x+z)} dt \tag{2.0.5}$$

We know that,

$$\int_{-\infty}^{\infty} e^{\pm ik(x-x_0)} dk = 2\pi\ \delta(x - x_0) \tag{2.0.6}$$

Using (2.0.6) we get,

$$f_X(x) = \frac{1}{2} \sum_{z=-\infty}^{\infty} \alpha_z \delta(x - z) + \frac{1}{2} \sum_{z=-\infty}^{\infty} \alpha_z \delta(x + z) \tag{2.0.7}$$

$$= \frac{1}{2} \sum_{z=-\infty}^{\infty} (\alpha_z + \alpha_{-z})\delta(x - z) \tag{2.0.8}$$

Taking $\beta_{|z|} = \dfrac{\alpha_z + \alpha_{-z}}{2}$,

$$f_X(x) = \sum_{z=-\infty}^{\infty} \beta_{|z|}\delta(x - z) \tag{2.0.9}$$

We know that,

$$\Pr(X = z_0 | z_0 \in \mathbb{Z}) = \int_{-\infty}^{\infty} f_X(z_0)\ dx \tag{2.0.10}$$

$$= \int_{-\infty}^{\infty} \sum_{z=-\infty}^{\infty} \beta_{|z|}\delta(z_0 - z)\ dx \tag{2.0.11}$$

$$= \beta_{|z_0|} \tag{2.0.12}$$

$$P(X \in \mathbb{Z}) = \sum_{z_0=-\infty}^{\infty} \Pr(X = z_0) \quad \text{where} \quad z_0 \in \mathbb{Z} \tag{2.0.13}$$

$$= \sum_{z_0=-\infty}^{\infty} \beta_{|z_0|} \tag{2.0.14}$$

$$= \sum_{z_0=-\infty}^{\infty} \frac{\alpha_{z_0} + \alpha_{-z_0}}{2} \tag{2.0.15}$$

$$= \frac{\sum_{z_0=-\infty}^{\infty} \alpha_{z_0} + \sum_{z_0=-\infty}^{\infty} \alpha_{-z_0}}{2} \tag{2.0.16}$$

Using (2.0.2) we get,

$$P(X \in \mathbb{Z}) = 1$$