Assignment 8

Gorantla Pranav Sai- CS20BTECH11018

Download all python codes from

```
https://github.com/pranav−159/
    ai1103_Probability_and_Random_variables/
    blob/main/Assignment_8/codes/
    experimental_verification_Assignment8.py
```

1 Problem

GATE 2021 (ME-SET1), Q.42 (ME section)
Consider a single machine workstation to which jobs arrive according to a Poisson distribution with a mean arrival rate of 12 jobs/hour. The process time of the workstation is exponentially distributed with a mean of 4 minutes. The expected number of jobs at the workstation at any given point of time is …(*round off to the nearest integer*).

2 Solution

In a Poisson process,

$$\Pr(X = x) = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^x}{x\,!} \tag{2.0.1}$$

In a small interval Δt approximate probability that only one Poisson job occur is,

$$\Pr(X = 1) = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^1}{1\,!} \tag{2.0.2}$$

Here $e^{-\lambda\Delta t} \to 1$ as $\Delta t \to 1$

$$\Pr(X = 1) = \lambda\Delta t \tag{2.0.3}$$

For job arrival,
As it is a Poisson distribution its Rate parameter λ=12 jobs/hour.
Using (2.0.3),Probability that a single job arrives in a small interval $\Delta t = \lambda\Delta t$.
Process time is distributed exponentially with mean of 4 minutes then we can assume that job completions are distributed as Poisson distribution with rate parameter $\mu = 15$ jobs/hour
Once again using (2.0.3), Probability that a single job will be completed in a small interval $\Delta t = \mu\Delta t$

Here After we will assume either one job is created,one job is completed or nothing happens in the time interval Δt as probabilities for occurrence of more jobs is extremely low as Δt becomes very small.
Initial no.of jobs at workstation is 0.Let $P_j(t)$ denote the probability of having j jobs waiting at the workstation at the time t with initial number of jobs at workstation as 0.
As time goes probability of having j jobs becomes steady as an example in the case of $j = 4$ initially it is 0 but increases and finally becomes constant.Let us call steady state probability of having j jobs as π_j.
Condition which ensures that steady state is reached is

$$\frac{\mathrm{d}P_j(t)}{\mathrm{d}t} = 0 \tag{2.0.4}$$

$$\lim_{\Delta t\to 0}\frac{P_j(t+\Delta t) - P_j(t)}{\Delta t} = 0 \tag{2.0.5}$$

We can reach a state of j jobs at time $t+\Delta t$ from a state of $j-1$ jobs,$j+1$ jobs and of course j jobs at time t with a new job occurring in the time interval Δt,job completed in the time interval Δt and nothing being happened respectively.The above relation can be written as probability equation as below.

$$\begin{aligned} P_j(t+\Delta t) &= P_{j-1}(t)\lambda\Delta t + P_{j+1}(t)\mu\Delta t \\ &\quad + P_j(t)(1-\lambda\Delta t - \mu\Delta t) \end{aligned} \tag{2.0.6}$$

$$\begin{aligned} \frac{P_j(t+\Delta t) - P_j(t)}{\Delta t} &= P_{j-1}(t)\lambda + P_{j+1}(t)\mu \\ &\quad - P_j(t)\lambda - P_j(t)\mu \end{aligned} \tag{2.0.7}$$

Using (2.0.5) we get,

$$\implies P_{j-1}(t)\lambda + P_{j+1}(t)\mu = P_j(t)\lambda + P_j(t)\mu \tag{2.0.8}$$

$$\pi_{j-1}\lambda + \pi_{j+1}\mu = \pi_j\lambda + \pi_j\mu \tag{2.0.9}$$

Note that the above equations are for $j \geq 1$.
For j=0 jobs at time $t+\Delta t$ we can reach it from j=1 job at time t with a job completion in Δt or else

stay at j=0 at time t and do nothing.

$$P_0(t+\Delta t) = P_1(t)\mu\Delta t + P_0(t)(1-\lambda\Delta t) \tag{2.0.10}$$

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = P_1(t)\mu\Delta t - P_0(t)\lambda\Delta t \tag{2.0.11}$$

Once again using (2.0.5),we will get,

$$P_0(t)\lambda\Delta t = P_1(t)\mu\Delta t \tag{2.0.12}$$

$$P_0(t)\lambda = P_1(t)\mu \tag{2.0.13}$$

$$\pi_0\lambda = \pi_1\mu \tag{2.0.14}$$

Solving (2.0.14) and (2.0.9) with appropriate j recursively,we will get P_j in terms of P_0 as

$$P_j = \left(\frac{\lambda}{\mu}\right)^j P_0 \tag{2.0.15}$$

consider $\sigma = \frac{\lambda}{\mu}$.

$$P_j = \sigma^j P_0 \tag{2.0.16}$$

We can prove that (2.0.16) is indeed the solution of recursion equation (2.0.9) by using mathematical induction.
Assuming $\sigma < 1$,let us calculate P_0 in terms of σ

$$\sum_{j=0}^{\infty} P_j = 1 \tag{2.0.17}$$

$$\sum_{j=0}^{\infty} \sigma^j P_0 = 1 \tag{2.0.18}$$

$$\frac{P_0}{1-\sigma} = 1 \tag{2.0.19}$$

$$P_0 = 1-\sigma \tag{2.0.20}$$

This yields,

$$P_j = \sigma^j(1-\sigma) \tag{2.0.21}$$

Let us calculate expected value of jobs waiting at workstation.

$$E(j) = \sum_{j=0}^{\infty} jP_j \tag{2.0.22}$$

$$E(j) = (1-\sigma)\sum_{j=0}^{\infty} j\sigma^j \tag{2.0.23}$$

$$\sigma E(j) = (1-\sigma)\sum_{j=0}^{\infty} j\sigma^{j+1} \tag{2.0.24}$$

$$\sigma E(j) = (1-\sigma)\sum_{j=1}^{\infty} (j-1)\sigma^j \tag{2.0.25}$$

Subtracting (2.0.25) from (2.0.23).we get,

$$(1-\sigma)E(j) = (1-\sigma)\sum_{j=1}^{\infty} \sigma^j \tag{2.0.26}$$

$$E(j) = \sum_{j=1}^{\infty} \sigma^j \tag{2.0.27}$$

$$E(j) = \frac{\sigma}{1-\sigma} \tag{2.0.28}$$

In our case $\sigma = \frac{\lambda}{\mu} = \frac{12}{15} = \frac{4}{5}$.

Substituting it in the (2.0.28) we get,

$$E(j) = 4 \tag{2.0.29}$$

$\therefore$ Expected no.of jobs at workstation is 4.