Assignment 7

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1 Problem

GATE 2019 (ST) , Q.49 (Statistics section)
Let X be a random variable with characteristic function $\phi_X(\cdot)$ such that $\phi_X(2\pi) = 1$.Let $\mathbb{Z}$ denote the set of integers.Then $P(X \in \mathbb{Z})$ is equal to ...

2 Solution

General solution for the characteristic solution which is consistent with our condition is,

$$\phi(t) = \int_{-\infty}^{\infty} \sum_{z=-\infty}^{\infty} \alpha_z \delta(x' - z) e^{itx'} \ dx' \quad , z \in \mathbb{Z} \tag{2.0.1}$$

$$\text{and} \quad \sum_{z=-\infty}^{\infty} \alpha_z = 1 \quad , \quad \alpha_z \in \mathbb{R} \tag{2.0.2}$$

Using Gil-Pelaez formula for probability density function,

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t) \ dt \tag{2.0.3}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \int_{-\infty}^{\infty} \sum_{z=-\infty}^{\infty} \alpha_z \delta(x' - z) e^{itx'} \ dx' dt \tag{2.0.4}$$

$$= \frac{1}{2\pi} \sum_{z=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha_z \delta(x' - z) e^{-it(x-x')} dt \ dx' \tag{2.0.5}$$

We know that,

$$\int_{-\infty}^{\infty} e^{\pm ik(x-x_0)} dk = 2\pi \ \delta(x - x_0) \tag{2.0.6}$$

Using (2.0.6) we get,

$$f_X(x) = \sum_{z=-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha_z \delta(x' - z) \delta(x - x') \ dx' \tag{2.0.7}$$

$$= \sum_{z=-\infty}^{\infty} \alpha_z \delta(x - z) \tag{2.0.8}$$

We know that,

$$\Pr(X = z_0) = \lim_{\epsilon \to 0} \int_{z_0-\epsilon}^{z_0+\epsilon} f_X(z_0) \ dx \tag{2.0.9}$$

$$= \lim_{\epsilon \to 0} \int_{z_0-\epsilon}^{z_0+\epsilon} \sum_{z=-\infty}^{\infty} \alpha_z \delta(z_0 - z) \ dx \tag{2.0.10}$$

if $z_0 \in \mathbb{Z}$

$$\Pr(X = z_0) = \alpha_{z_0} \tag{2.0.11}$$

else

$$\Pr(X = z_0) = 0 \tag{2.0.12}$$

$$P(X \in \mathbb{Z}) = \sum_{z_0=-\infty}^{\infty} \Pr(X = z_0) \quad \text{where} \quad z_0 \in \mathbb{Z} \tag{2.0.13}$$

$$= \sum_{z_0=-\infty}^{\infty} \alpha_{z_0} \tag{2.0.14}$$

Using (2.0.2) we get,

$$P(X \in \mathbb{Z}) = 1$$