

Assignment 7

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1 PROBLEM

gov/stats/2015/statistics-I(1), Q.3(C)

Three points are chosen on the line of unit length. Find the probability that each of the 3 line segments have length greater than $\frac{1}{4}$.

2 SOLUTION

Let $X, Y \in \{0, 1\}$ be the random variables which represent the position of two points on the line of unit length.

Conditions which should be satisfied to have three line segments with length greater than $\frac{1}{4}$ are given

Event	Condition
A	$\frac{1}{4} < X < \frac{3}{4}$
B	$\frac{1}{4} < Y < \frac{3}{4}$
C	$\frac{1}{4} < X - Y $

TABLE 0: Events and their conditions

in the below table.

Then the required event which solves the problem is ABC .

As A and B are independent events.

$$\Pr(ABC) = \Pr(A) \Pr(B) \Pr(C|AB) \quad (2.0.1)$$

As X and Y are uniformly distributed between 0 and 1.

$$\Pr(A) = \Pr\left(\frac{1}{4} < X < \frac{3}{4}\right) = \frac{1}{2} \quad (2.0.2)$$

$$\Pr(B) = \Pr\left(\frac{1}{4} < Y < \frac{3}{4}\right) = \frac{1}{2} \quad (2.0.3)$$

When event AB is known to occur, X and Y are uniformly distributed between $\frac{1}{4}$ and $\frac{3}{4}$

Conditional probability density functions of X and Y with condition AB are

$$f_{X|AB}(x) = \begin{cases} 2 & x \in \left(\frac{1}{4}, \frac{3}{4}\right) \\ 0 & \text{otherwise} \end{cases} \quad (2.0.4)$$

$$f_{Y|AB}(y) = \begin{cases} 2 & y \in \left(\frac{1}{4}, \frac{3}{4}\right) \\ 0 & \text{otherwise} \end{cases} \quad (2.0.5)$$

Let $Z = X - Y$. Then pdf of Z with condition AB can be given as

$$f_{Z|AB}(z) = \int_{-\infty}^{\infty} f_{X|AB}(y+z) f_{Y|AB}(y) dy \quad (2.0.6)$$

$$f_{Z|AB}(z) = \int_{\frac{1}{4}}^{\frac{3}{4}} f_{X|AB}(y+z) f_{Y|AB}(y) dy \quad (2.0.7)$$

if $z \in \left[0, \frac{1}{2}\right]$,

$$f_{Z|AB}(z) = \int_{\frac{1}{4}}^{\frac{3}{4}-z} f_{X|AB}(y+z) f_{Y|AB}(y) dy \quad (2.0.8)$$

$$f_{Z|AB}(z) = 4 \left(\frac{1}{2} - z\right) \quad (2.0.9)$$

$$f_{Z|AB}(z) = 2 - 4z \quad (2.0.10)$$

if $z \in \left[-\frac{1}{2}, 0\right]$,

$$f_{Z|AB}(z) = \int_{\frac{1}{4}-z}^{\frac{3}{4}} f_{X|AB}(y+z) f_{Y|AB}(y) dy \quad (2.0.11)$$

$$f_{Z|AB}(z) = 4 \left(\frac{1}{2} + z\right) \quad (2.0.12)$$

$$f_{Z|AB}(z) = 2 + 4z \quad (2.0.13)$$

if $z \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$,

$$f_{Z|AB}(z) = 0 \quad (2.0.14)$$

Interval	$f_{X AB}$
$z \in \left[0, \frac{1}{2}\right]$	$2 - 4z$
$z \in \left[-\frac{1}{2}, 0\right]$	$2 + 4z$
$z \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$	0

TABLE 0: Pdf of Z in different intervals

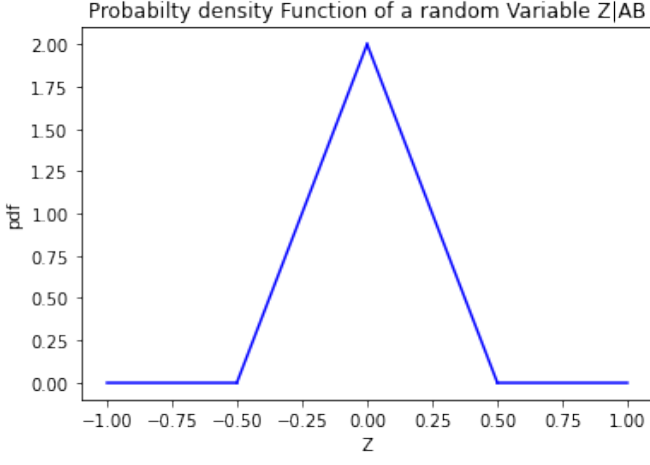


Fig. 0: pdf Z with AB already occurred

$$\Pr(C|AB) = \int_{|z| > \frac{1}{4}} f_{z|AB} dz \quad (2.0.15)$$

$$= \int_{-\frac{1}{2}}^{-\frac{1}{4}} f_{z|AB} dz + \int_{\frac{1}{4}}^{\frac{1}{2}} f_{z|AB} dz \quad (2.0.16)$$

$$= \frac{1}{8} + \frac{1}{8} \quad (2.0.17)$$

$$= \frac{1}{4} \quad (2.0.18)$$

$$\Pr(ABC) = \Pr(A) \Pr(B) \Pr(C|AB) \quad (2.0.19)$$

$$\Pr(ABC) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{4} \quad (2.0.20)$$

$$\Pr(ABC) = \frac{1}{16} \quad (2.0.21)$$

\therefore probability that each of the three line segments have length greater than $\frac{1}{4}$ is $\frac{1}{16}$.