Assignment 8

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Download all python codes from

```
https://github.com/pranav−159/
    ai1103_Probability_and_Random_variables/
    blob/main/Assignment_8/codes/
    experimental_verification_Assignment8.py
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1 Problem

GATE 2021 (ME-SET1), Q.42 (ME section)
Consider a single machine workstation to which jobs arrive according to a Poisson distribution with a mean arrival rate of 12 jobs/hour. The process time of the workstation is exponentially distributed with a mean of 4 minutes. The expected number of jobs at the workstation at any given point of time is . . . (*round off to the nearest integer*).

2 Solution

In a Poisson process,

$$\Pr(X = x) = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^x}{x\,!} \tag{2.0.1}$$

If $\Delta t \to 0$ then probability of having only one Poisson job is

$$\Pr(X = 1) = \lambda\Delta t \tag{2.0.2}$$

Some assumptions:
In time interval Δt,

- Exactly one job is arrived
- or Exactly one job is completed
- or Nothing happens

Assumptions seem quite reasonable as Δt is very small then the probability of occurrence of more than one poisson job is very low.
For job arrival,

- It is distributed according to Poisson distribution.
- Its Rate parameter λ=12 jobs/hour.
- Using (2.0.2),Probability that a single job arrives in a small interval $\Delta t = \lambda\Delta t$.

For Job completions,

- Job completion time is distributed exponentially with mean of 4 minutes
- Then we can assume that no. of job completions are distributed as Poisson distribution with rate parameter μ = 15 jobs/hour
- Once again using (2.0.2), Probability that a single job will be completed in a small interval $\Delta t = \mu\Delta t$

Some notations,

Parameter	Definition
λ	Poisson rate parameter for the arrival of jobs
μ	Poisson rate parameter for the completion of jobs
$\lambda\Delta t$	Probability that a single job arrives in a small interval Δt
$\mu\Delta t$	Probability that a single job will be completed in a small interval Δt
$P_j(t)$	probability of having j jobs at workstation at time t
π_j	steady probability of having j jobs at workstation

TABLE 0: Parameters and their definitions used in the problem

- Initial no.of jobs at workstation is 0.
- Let $P_j(t)$ denote the probability of having j jobs waiting at the workstation at the time t for this initial case.
- After a long time,probability of having j jobs becomes steady.
- Let us denote steady state probability of having j jobs as π_j.

Condition which ensures that steady state is reached is

$$\frac{\mathrm{d}P_j(t)}{\mathrm{d}t} = 0 \tag{2.0.3}$$

$$\lim_{\Delta t\to 0}\frac{P_j(t+\Delta t) - P_j(t)}{\Delta t} = 0 \tag{2.0.4}$$

We can reach a state of j jobs at time $t + \Delta t$ from

- A state of $j-1$ jobs at time t with a new job arriving in the next Δt
- A state of $j+1$ jobs at time t with a job completing in the next Δt
- A state of j jobs at time t and nothing happening in the next Δt

Assuming time t is long enough for the occurrence of steady state.The above relations can be shown in probability equations as:

$$P_j(t+\Delta t) = P_{j-1}(t)\lambda\Delta t + P_{j+1}(t)\mu\Delta t + P_j(t)(1-\lambda\Delta t-\mu\Delta t) \quad (2.0.5)$$

$$\frac{P_j(t+\Delta t)-P_j(t)}{\Delta t} = P_{j-1}(t)\lambda + P_{j+1}(t)\mu - P_j(t)\lambda - P_j(t)\mu \quad (2.0.6)$$

Using (2.0.4) we get,

$$\implies P_{j-1}(t)\lambda + P_{j+1}(t)\mu = P_j(t)\lambda + P_j(t)\mu \quad (2.0.7)$$

$$\pi_{j-1}\lambda + \pi_{j+1}\mu = \pi_j\lambda + \pi_j\mu \quad (2.0.8)$$

Note that the above equations are for $j \geq 1$.
For j=0 jobs at time $t+\Delta t$ we can reach it from j=1 job at time t with a job completion in the next Δt or else stay at j=0 at time t and do nothing the next Δt

$$P_0(t+\Delta t) = P_1(t)\mu\Delta t + P_0(t)(1-\lambda\Delta t) \quad (2.0.9)$$

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = P_1(t)\mu\Delta t - P_0(t)\lambda\Delta t \quad (2.0.10)$$

Once again using (2.0.4),we will get,

$$P_0(t)\lambda\Delta t = P_1(t)\mu\Delta t \quad (2.0.11)$$

$$P_0(t)\lambda = P_1(t)\mu \quad (2.0.12)$$

$$\pi_0\lambda = \pi_1\mu \quad (2.0.13)$$

Solving (2.0.13) and (2.0.8) with appropriate j one by one,we will get P_j in terms of P_0 as

$$P_j = \left(\frac{\lambda}{\mu}\right)^j P_0 \quad (2.0.14)$$

consider $\rho = \frac{\lambda}{\mu}$.

$$P_j = \rho^j P_0 \quad (2.0.15)$$

We can prove that (2.0.15) is indeed the solution of recursion equation (2.0.8) by using mathematical induction.

Parameter	Definition
$E(j)$	Expected no. of jobs at workstation
ρ	$\frac{\lambda}{\mu}$

TABLE 0: Parameters and their definitions used in the problem

Assuming $\rho < 1$,let us calculate P_0 in terms of ρ

$$\sum_{j=0}^{\infty} P_j = 1 \quad (2.0.16)$$

$$\sum_{j=0}^{\infty} \rho^j P_0 = 1 \quad (2.0.17)$$

$$\frac{P_0}{1-\rho} = 1 \quad (2.0.18)$$

$$P_0 = 1-\rho \quad (2.0.19)$$

This yields,

$$P_j = \rho^j(1-\rho) \quad (2.0.20)$$

Let us calculate expected value of jobs waiting at workstation.

$$E(j) = \sum_{j=0}^{\infty} jP_j \quad (2.0.21)$$

$$E(j) = (1-\rho)\sum_{j=0}^{\infty} j\rho^j \quad (2.0.22)$$

$$\rho E(j) = (1-\rho)\sum_{j=0}^{\infty} j\rho^{j+1} \quad (2.0.23)$$

$$\rho E(j) = (1-\rho)\sum_{j=1}^{\infty} (j-1)\rho^j \quad (2.0.24)$$

Subtracting (2.0.24) from (2.0.22).we get,

$$(1-\rho)E(j) = (1-\rho)\sum_{j=1}^{\infty} \rho^j \quad (2.0.25)$$

$$E(j) = \sum_{j=1}^{\infty} \rho^j \quad (2.0.26)$$

$$E(j) = \frac{\rho}{1-\rho} \quad (2.0.27)$$

In our case $\rho = \frac{\lambda}{\mu} = \frac{12}{15} = \frac{4}{5}$.

Substituting it in the (2.0.27) we get,

$$E(j) = 4 \tag{2.0.28}$$

$\therefore$ Expected no.of jobs at workstation is 4.