Assignment 7

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1 Problem

GATE 2019 (ST) , Q.49 (Statistics section)
Let X be a random variable with characteristic function $\phi_X(\cdot)$ such that $\phi_X(2\pi) = 1$.Let $\mathbb{Z}$ denote the set of integers.Then $P(X \in \mathbb{Z})$ is equal to ...

2 Solution

We know that characteristic function,

$$\phi_X(t) = E[e^{itX}] \tag{2.0.1}$$
$$\phi_X(2\pi) = E[e^{i2\pi X}] \tag{2.0.2}$$
$$E[\cos(2\pi X)] + iE[\sin(2\pi X)] = 1 \tag{2.0.3}$$
$$\tag{2.0.4}$$

Then,

$$E[\cos(2\pi X)] = 1 \text{ and} \tag{2.0.5}$$
$$E[\sin(2\pi X)] = 0 \tag{2.0.6}$$

In order to maintain expected value at 1,$\cos(2\pi X) \nless 1$.As,if $\cos(2\pi X) \nless 1$ then there must exist a value such that $\cos(2\pi X) > 1$ in order to compensate the latter.But $\cos(2\pi X) > 1$ is not possible.Hence $\cos(2\pi X) \nless 1$.
Here,

$$\Pr(\cos(2\pi X) > 1) = 0 \tag{2.0.7}$$
$$\Pr(\cos(2\pi X) < 1) = 0 \tag{2.0.8}$$

Then there is only one value which can be satisfiable,$\cos(2\pi X) = 1$.
Using simple trigonometry we can get,

$$\cos(2\pi X) = 1 \implies X \in \mathbb{Z} \tag{2.0.9}$$

$X \in \mathbb{Z}$ also satisfy (2.0.6)

$$\sin(2\pi X | X \in \mathbb{Z}) = 0 \tag{2.0.10}$$
$$\implies E[\sin(2\pi X) | X \in \mathbb{Z}] = 0 \tag{2.0.11}$$

Hence,

$$\Pr(\cos(2\pi X) = 1) = 1 \tag{2.0.12}$$
$$\Pr(X \in \mathbb{Z}) = 1 \tag{2.0.13}$$

$$\therefore P(X \in \mathbb{Z}) = 1$$