

Calculus:

The Method of First & Ultimate Ratios of Evanescent Quantities

(ie becoming Vanishingly small)

(...in the Principia: Book 1(The Motion of Bodies), Section 1)
(with different Notation)

$x(t)$ we call a Function, like $y=f(x)$, Newton called it a
Fluent Quantity (trajectories were traced out by Particles)
 dx/dt we call a derivative (time rate of change of x), Newton called a
Fluxion of the Fluent- x

The Power Law, as an example:
 dx/dt where $x = t^n$

Let Δt be the increment in t
(Newton called it a Moment of Time after it was imagined to decrease
to less than any preassigned value,
ie less than what you can think of, an infinitesimal dt .)

so the corresponding Increment in Position is:

$$\Delta x = x(t + \Delta t) - x(t) = (t + \Delta t)^n - t^n$$

Binomial Theorem: Newton's First Original Work in 1665 @ age 23.
Proved it for Rational # = n (integers included)
It is on the inside cover of your Text.

$$(1 + x)^n = [1 + nx/1! + n(n-1)x^2/2! + n(n-1)(n-2)x^3/3! + \dots];$$

This Converges for $x^2 < 1$

$$(t + \Delta t)^n = t^n(1 + \Delta t/t)^n = t^n[1 + n(\Delta t/t) + n(n-1)(\Delta t/t)^2/2 + \dots]$$

$$\begin{aligned}\Delta x &= t^n[1 + n(\Delta t/t) + n(n-1)(\Delta t/t)^2/2 + \dots] - t^n \\ &= n(\Delta t/t)t^n + n(n-1)(\Delta t/t)^2t^n/2 + \dots \\ &= n(\Delta t)t^{n-1} + n(n-1)(\Delta t)^2t^{n-2}/2 + \dots\end{aligned}$$

$$\Delta x/\Delta t = nt^{n-1} + n(n-1)(\Delta t)t^{n-2}/2 + \dots \quad \leftarrow \text{First Ratio}$$

$$dx/dt = \lim_{\Delta t \rightarrow 0} (\Delta x/\Delta t) = nt^{n-1} \quad \leftarrow \text{Ultimate Ratio}$$

The Ultimate Ratio of Evanescent Quantities: Δx & Δt is dx/dt .

The derivative exists because Particles do have Instantaneous Velocities