Table 10.11 Price of a European call option with stochastic interest rate and stochastic volatility

Call option						
Parameters			VAS		CIR	
$S(0)$ \$	K \$	T years	Price \$	Error \$	Price \$	Error \$
90	100	1	4.61	0.07	4.69	0.08
90	100	2	9.37	0.13	9.48	0.17
100	100	1	10.31	0.08	10.33	0.09
100	100	2	15.87	0.17	16.10	0.17
110	100	1	17.79	0.10	17.88	0.08
110	100	2	23.56	0.11	23.79	0.19

```
%Calculation of the prices of the European call and put options
PrixCall = mean(max(0,S-Strike).*exp(-rAct));
PrixPut = mean(max(0,K-Strike).*exp(-rAct));
end

function Rep=ReQuadratic(Sample, MoyTheo,CovTheo)
%Function performing the quadratic resampling
%Sample: Simulated sample
%MoyTheo: Theoretical mean of the variables
%CovTheo: Theoretical covariance matrix of the variables
%Calculation of the parameters of the sample distribution
CovEmp=cov(Sample');
MoyEmp=mean(Sample,2);
LEmp=chol(CovEmp)';
%Resampling based on the theoretical covariance matrix
LTheo=chol(CovTheo)';
Sample=LTheo*inv(LEmp)*(Sample-...
               repmat(MoyEmp,1,size(Sample,2)))+...
               repmat(MoyTheo,1,size(Sample,2));
Rep=Sample;
end
```

This program is used to estimate European call and put option prices. Tables 10.11 and 10.12 present the simulations results. The error is measured by the standard deviation of the result vector of the 20 batches.

As in the section above, the reader can use antithetic and control variables to obtain more accurate results. One suggested control variable is the Black-Scholes formula. We leave this as an additional practice exercise for the reader.

10.2 AMERICAN OPTIONS

The feature of an American option is that it can be exercised at any time before its maturity. Given the fact that these options offer more flexibility to their holders, their price should be higher than the price of their equivalent counterpart European options. However, for a call

Table 10.12 Price of a European put option with stochastic interest rate and stochastic volatility

Put option						
Parameters			VAS		CIR	
$S(0)$ \$	K \$	T years	Price \$	Error \$	Price \$	Error \$
90	100	1	9.77	0.08	9.81	0.06
90	100	2	10.06	0.10	9.98	0.09
100	100	1	5.46	0.06	5.49	0.08
100	100	2	5.60	0.09	6.55	0.10
110	100	1	2.98	0.09	3.01	0.07
110	100	2	4.33	0.13	4.29	0.10

option on a non-dividend-paying stock, the American and European options are worth the same; there's no advantage in early exercise of a call in this case.

There exists no simple formula to price American options. One may use binomial trees and examine at each node if it is optimal to exercise the option immediately or not. This simple approach can take a considerable amount of calculation time to obtain an adequate precision for options on several underlying assets.

We present next two approaches to price American options. These two approaches use Monte Carlo simulations instead of binomial trees. The approaches are the Least-Squares Method of Longstaff and Schwartz (2001) and the Dynamic Programming Technique with Stratified States Aggregation of Barraquand and Martineau (1995).

10.2.1 Simulations Using The Least-Squares Method of Longstaff and Schwartz (2001)

When dealing with American options, it is not always easy to decide whether to exercise them immediately or keep them in the portfolio until their expiration date. The optimal strategy would be to exercise the option if the immediate payment is larger than the expected future payments, otherwise it should be kept. Let's assume that the price of the stock is $S(t)$ and the payoff function is $\widetilde{P}(S(t), t)$. Then the optimal strategy at time $t = t_1$ is

$$\text{Exercise} = \begin{cases} \text{Yes if } \widetilde{P}(S(t_1), t_1) > E_{t_1}\left[\widetilde{P}(S(t_2), t_2)|\mathcal{F}_{t_1}\right] \\ \text{No if } \widetilde{P}(S(t_1), t_1) < E_{t_1}\left[\widetilde{P}(S(t_2), t_2)|\mathcal{F}_{t_1}\right] \end{cases}, \tag{10.17}$$

where $E_{t_1}[\widetilde{P}(S(t_2), t_2)|\mathcal{F}_{t_1}]$ is the expectation of future payments (of time $t_2 > t_1$) at time $t = t_1$, and $\mathcal{F}_{t_1}$ represents the available information set at $t = t_1$. It is therefore essential to "know" the expected value of future payments in order to make an accurate evaluation of an American option. The Least-Squares method is a technique that enables us to perform this exercise.

We start by generating M paths for the underlying stock price S. Next, for each path, we need to regress the future payoffs on basis functions F_i, which depend on the stock price S. Let Y be the vector of future payoffs for the M paths and $1, F_1(S)$ and $F_2(S)$ be the basis functions. We regress Y on these basis functions, which yields the expression

$$E[Y|S] = \alpha + \beta F_1(S) + \gamma F_2(S). \tag{10.18}$$

Table 10.13 Simulation paths

Number	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.07	1.53	1.95
2	1.00	0.76	0.78	0.71
3	1.00	0.85	0.69	0.76
4	1.00	0.96	1.01	0.97
5	1.00	0.95	1.06	1.28
6	1.00	1.59	1.26	1.07
7	1.00	1.28	1.23	0.97
8	1.00	1.11	1.57	1.89

This expression gives us an estimation of the expected value of future payoffs as a function of S. This expected value is effectively the value of holding on to the option. From this expression, we can decide if it is preferable to exercise the option immediately or to wait one more period. This procedure is reproduced backward from the maturity date to time $t = 0$. For each path, we find the optimal exercise date of the option. The price of the option is therefore the average of all discounted payoffs.

To better illustrate the method, we use it to price an American put option. We assume $S(0) = \$\,1.0$, the exercise price is $K = \$\ 1.10$, the time to maturity of the option is 3 years and the option's potential exercise dates are 1, 2 and 3 years. If $S(0) \neq \$\ 1.0$, it is better to normalize the initial price to $S(0) = \$\ 1.0$ as the exercise price becomes $K/S(0)$ to reduce the estimation errors in the regressions. We also assume the risk free interest rate to be $r = 6\%$ and $M = 8$ paths for simplification. Table 10.13 presents the simulation values of S.

To use the Least-Squares Method, we need to start at maturity and go backward until we reach the initial time $t = 0$. At time $t = 3$, the holder of the option exercises it only if he gains from it. The payoffs matrix at time $t = 3$ is given by Table 10.14.

This matrix shows the realized payoffs of an equivalent European option. Now we need to determine for which paths it is preferable to exercise the option at date $t = 2$. For $t = 2$, only 4 paths must be considered since the other 4 paths would result in zero payoffs. The results are presented in Table 10.15 (Y represents the discounted expected payoffs).

We use $1, S$ and S^2 as the basis functions. We therefore need to regress Y on these functions, which yields:

$$E[Y|S] = -2.71 + 7.81S - 4.96S^2. \tag{10.19}$$

Table 10.14 Payoffs at $t = 3$

Number	$t = 1$	$t = 2$	$t = 3$
1	–	–	0.00
2	–	–	0.39
3	–	–	0.34
4	–	–	0.13
5	–	–	0.00
6	–	–	0.03
7	–	–	0.13
8	–	–	0.00

Table 10.15 Variables for the first regression

Number	S	Y
1	–	–
2	0.78	0.39*0.9418 = 0.3673
3	0.69	0.34*0.9418 = 0.3202
4	1.01	0.13*0.9418 = 0.1224
5	1.06	0.00*0.9418 = 0.0000
6	–	–
7	–	–
8	–	–

To obtain the regression coefficients, we only need to compute the following matrix product

$$(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top Y = (-2.7077, 7.8084, -4.9566)^\top, \tag{10.20}$$

where

$$\mathbf{X} = \begin{pmatrix} 1 & 0.78 & 0.78^2 \\ 1 & 0.69 & 0.69^2 \\ 1 & 1.01 & 1.01^2 \\ 1 & 1.06 & 1.06^2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0.3673 \\ 0.3202 \\ 0.1224 \\ 0 \end{pmatrix}. \tag{10.21}$$

From the regression equation, we evaluate the function $E[Y|S]$ for different values of S at time $t = 2$.

$$\begin{aligned} E[Y|S = 0.78] &= 0.3672 \\ E[Y|S = 0.69] &= 0.3202 \\ E[Y|S = 1.01] &= 0.1225 \\ E[Y|S = 1.06] &= -0.0001. \end{aligned}$$

We then compare these values to the payoffs resulting from an immediate exercise of the option (see Table 10.16).

We note that it is preferable to exercise the option at date 2 for paths 3 and 5. For simulations 2 and 4, the expected payoffs are higher when the option is not exercised. Then, taking into

Table 10.16 Exercise decision at time $t = 2$

Number	Exercise	Continuation
1	–	–
2	0.32	0.36
3	0.41	0.32
4	0.09	0.12
5	0.04	0.00
6	–	–
7	–	–
8	–	–

Table 10.17 Payoffs at date $t = 2$

Number	$t = 1$	$t = 2$	$t = 3$
1	–	0.00	0.00
2	–	0.00	0.39
3	–	0.41	0.00
4	–	0.00	0.13
5	–	0.04	0.00
6	–	0.00	0.03
7	–	0.00	0.13
8	–	0.00	0.00

Table 10.18 Variables for the second regression

Number	S	Y
1	1.07	0.00
2	0.76	$0.39*0.9418^2 = 0.3459$
3	0.85	$0.41*0.9418 = 0.3861$
4	0.96	$0.13*0.9418^2 = 0.1153$
5	0.95	$0.04*0.9418 = 0.0377$
6	–	–
7	–	–
8	–	–

Table 10.19 Exercise decision at date $t = 1$

Number	Exercise	Continuation
1	0.03	−0.04
2	0.34	0.38
3	0.25	0.26
4	0.14	0.11
5	0.15	0.13
6	–	–
7	–	–
8	–	–

account the payoffs of paths 3 and 5 at time 2, we obtain the following payoff matrix in Table 10.17.

We need to follow the same steps to obtain the payoffs at time $t = 1$. For $t = 1$, only 3 paths provide values of S greater than 1.10 (number 6, 7 and 8). We then need to consider all other 5 paths in the regression, which yields Table 10.18.

The regression gives:

$$E[Y|S] = 1.38 - 1.28S - 0.04S^2. \tag{10.22}$$

Then we have to compare the immediate payoffs if the option is exercised and the expected payoffs if not. Table 10.19 presents the results.

Here we observe that it is preferable to exercise the option immediately for paths 1, 4 and 5. We can therefore complete the option exercise decision table. Table 10.20 presents the results.

Table 10.20 Payoffs at $t = 1$

Number	$t = 1$	$t = 2$	$t = 3$
1	0.03	0.00	0.00
2	0.00	0.00	0.39
3	0.00	0.41	0.00
4	0.14	0.00	0.00
5	0.15	0.00	0.00
6	0.00	0.00	0.03
7	0.00	0.00	0.13
8	0.00	0.00	0.00

The price of the American put option is obtained by taking the average value of all discounted payoffs. We obtain an option price of 0.1406. If the option was European, its price would have been 0.1065. The Least-Squares Method allows us to price the early exercise flexibility feature associated with American options; the value of that flexibility is $0.1406 - 0.1065 = 0.0341$.

This method can be used in many other option pricing cases with early exercise possibility. In addition, it is possible to use many basis functions, which improve the precision. For example, Longstaff and Schwartz (2001) suggest using the Laguerre, Hermite, Legendre, Chebychev polynomials among others.

MATLAB Program

The MATLAB program to implement the algorithm of the Least-Squares Method is given below. This program includes a function to generate the paths and another function to perform the regression and price the option in a backward manner (one step at a time).

```
function LSM
%Function to calculate the price of an American option
%using the Least-Squares method.

T=1; %Maturity
TimePresent=T; %Time to maturity
NbStep=50; %Number of steps.
K=40; %Exercise price
sigma=0.2; %Volatility of the asset
NbTraj=50000; %Number of paths
DeltaT=T/NbStep;
SqDeltaT=sqrt(DeltaT);
r=0.06;
SBegin=36; %Initial price of the asset

%To increase the precision of the regression,
%we use an initial price of 1
%and we decrease the exercise price consequently
S0=1; %
Strike=K/SBegin;
```

```
%We generate the paths of the asset price
S=GeneratePaths(NbTraj,NbStep,DeltaT,SqDeltaT,r,sigma,S0);

%Payoff is a vector formed of the largest cash flows
%using the optimal strategy
Payoff=zeros(2*NbTraj,1);
Payoff(:,1)=max(0,Strike-S(:,NbStep+1));

%Loop for backwardation by step from the maturity
for cptStep=1:NbStep-1
   %Discounting of the payoff vector
   Payoff=exp(-r*DeltaT)*Payoff;

   TimePresent=TimePresent-DeltaT;

   %Calculation of the new payoff vector by deciding if it is
    optimal
   %to exercise the option immediately.
   Payoff=BackwardStep(Payoff,Strike,TimePresent,NbTraj,r,DeltaT,...
                        S(:,NbStep+1-cptStep));
end

%Calculation of the option price
Price=mean(exp(-r*DeltaT)*Payoff);

%Calculation of the option price with the initial price of the
 asset
disp(Price*SBegin);

end

function S=GeneratePaths(NbTraj,NbStep,DeltaT,SqDeltaT,r,sigma,S0);
%Function to simulate the paths of the asset price

   dW=SqDeltaT*randn(NbTraj,NbStep);
   dW=cat(1,dW,-dW);

   Increments=(r-(sigma^2)/2)*DeltaT+sigma*dW;
   LogPaths=cumsum([log(S0)*ones(2*NbTraj,1),Increments],2);
   S=exp(LogPaths);
end

function Payoff=BackwardStep(Payoff,Strike, TimePresent,
                              NbTraj,r, ... DeltaT,S)
%Function using the Least-Squares method to determine
%if it is preferable to exercise immediately the option.
%Payoff: Old vector of Payoff at time t+1
%Strike: Exercise price of the option
%TimePresent: time t
```

```
%NbTraj: Number of paths
%r: interest rate
%DeltaT
%S: Vector of asset prices at time t

   %Vector containing the paths where the price of the
    asset is lower than
   %the exercise price (otherwise it is preferable to exercise)
   SelectedPaths=(Strike>S);

   %Matrix of regressors
   X=[ones(2*NbTraj,1).*SelectedPaths,(S.*SelectedPaths),...
           (S.*SelectedPaths).^2];

   %Vector of expected payoffs
   Y=Payoff.*SelectedPaths;

   %Regression to determine the coefficients
   A=inv(X'*X)*X'*Y;

   %Calculation of the payoffs values when the option is not
    exercised
   Continuation=(X*A).*SelectedPaths;

   %Values when the option is exercised immediately
   Exercise=max(0,Strike-S);

   %Paths, exercise decision and update of the Payoff vector.
   for i=1:(2*NbTraj)
       if ((Exercise(i,1)>0)&(Exercise(i,1)>Continuation(i,1)))
           Payoff(i,1)=Exercise(i,1);
       end
   end

end
```

We use this program to calculate the value of an American put option. We use 100 000 paths including 50 000 antithetic, 50 time steps per year (hence 50 possible exercise dates per year), a risk-free rate of 6% as well as an exercise price of 40$ and an initial price $S(0)$ in $\{38, 40, 42\}$. In Table 10.21, we show the results and compare them to those of the finite difference method presented in Longstaff and Schwartz (2001).

Least-Squares Method with Multiple Underlying Assets

The previous section introduced the Least-Squares Method for the case of one dimension. With only one underlying asset, it is relatively easy to understand the approach. However, this method is also useful for cases featuring several underlying assets and complex cash flows. We use the Least-Squares Method to price an American call option written on three underlying