Exercise 2

1 Convex optimization

Consider the optimization problem

$$\begin{aligned} \min_{x_1, x_2} \quad & f(x_1, x_2), \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 1, \\ & x_1 + 3x_2 \leq 1, \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value (adapted from Exercise 4.1 in Convex Optimization by Stephen Boyd and Lieven Vandenberghe).

1. $f(x_1, x_2) = x_1 + x_2$.
2. $f(x_1, x_2) = -x_1 - x_2$.
3. $f(x_1, x_2) = x_1$.
4. $f(x_1, x_2) = \max\{x_1, x_2\}$.
5. $f(x_1, x_2) = x_1^2 + 9x_2^2$.

2 Logarithmic barrier interior-point method

Consider the following Quadratically Constrained Quadratic Program (QCQP)

$$\begin{aligned} \min_{x} \quad & x^\top A x + c^\top x, \\ \text{s.t.} \quad & x^\top B x \leq \alpha, \end{aligned}$$

1. Give the condition for B and α such that the problem will always have an optimal solution? Describe the feasible set if these conditions are satisfied.
2. If both A and B are symmetric, write down the augmented objective function for the logarithmic barrier interior-point method.
3. Calculate the gradient and Hessian of the augmented objective function.
4. Assume that A is positive definite, compute the Newton search direction, and prove that it is a descent direction if your condition given in 1 is satisfied.

1 Convex optimization

Consider the optimization problem

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\min_{x_1,x_2} \quad & f(x_1,x_2), \\
\text{s.t.} \quad & 2x_1 + x_2 \leq 1, \\
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Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value (adapted from Exercise 4.1 in Convex Optimization by Stephen Boyd and Lieven Vandenberghe).

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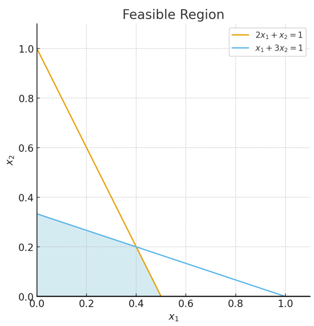
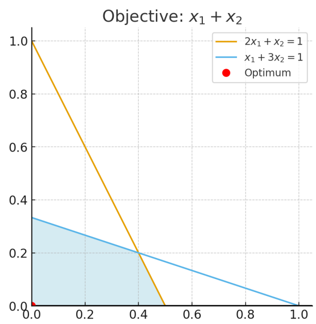
2. $f(x_1,x_2) = -x_1 - x_2$.
3. $f(x_1,x_2) = x_1$.
4. $f(x_1,x_2) = \max\{x_1,x_2\}$.
5. $f(x_1,x_2) = x_1^2 + 9x_2^2$.

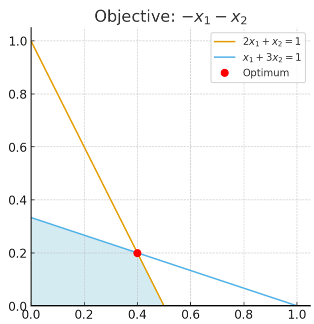
See the Fig. 1



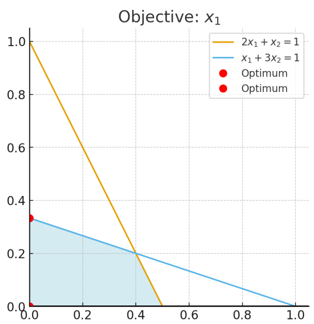
Feasible region.



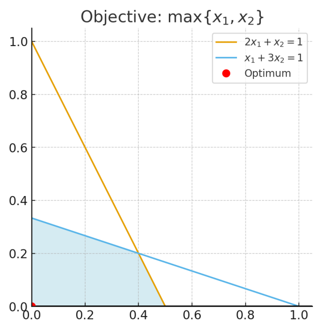
Objective 1.



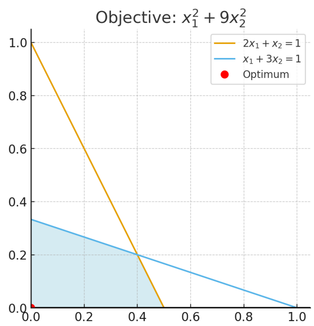
Objective 2.



Objective 3.



Objective 4.



Objective 5

Figure 1: Solution of problem one.

2 Logarithmic barrier interior-point method

Consider the following Quadratically Constrained Quadratic Program (QCQP)

$$\begin{aligned} \min_{x} \quad & x^\top A x + c^\top x, \\ \text{s.t.} \quad & x^\top B x \leq \alpha, \end{aligned}$$

1. Give the condition for B and α such that the problem will always have an optimal solution? Describe the feasible set if these conditions are satisfied.

   If B is positive definte, i.e., $B \succ 0$ and $\alpha \geq 0$. The problem will always have a solution. Since in this case the feasible set is compact (bounded and closed), and $x^\top A x + c^\top x$ is bounded from below on a compact set. The feasible set is an ellipsoid.

2. If both A and B are symmetric, write down the augmented objective function for the logarithmic barrier interior-point method.

   $f(x) = x^\top A x + c^\top x - \kappa \log(-x^\top B x + \alpha)$

3. Calculate the gradient and Hessian of the augmented objective function.

   Gradient: $\nabla f(x) = 2Ax + c + \frac{2\kappa B x}{\alpha - x^\top B x}$.
   Hessian: $\nabla^2 f(x) = 2A + \frac{2\kappa B(\alpha - x^\top B x) + 4\kappa B x (Bx)^\top}{(\alpha - x^\top B x)^2}$

4. Assume that A is positive definite, compute the Newton search direction, and prove that it is a descent direction if your condition given in 1 is satisfied.

   If d is a descent direction, $\nabla f(x)^\top d \leq 0$. We need to check if our Newton direction satisfies this condition. Netwon direction: $d = -(\nabla^2 f(x))^{-1} \nabla f(x) = -\big(2A + \frac{2\kappa B(\alpha - x^\top B x) + 4\kappa B x (Bx)^\top}{(\alpha - x^\top B x)^2}\big)^{-1} \big(2Ax + c + \frac{2\kappa B x}{\alpha - x^\top B x}\big)$. Since A is positive definite, $\big(2A + \frac{2\kappa B(\alpha - x^\top B x) + 4\kappa B x (Bx)^\top}{(\alpha - x^\top B x)^2}\big)^{-1}$ is also positive definite. Hence $y^\top (\nabla^2 f(x))^{-1} y \geq 0, \forall y$. This means for the Newton direction d, $\nabla f(x)^\top d \leq 0$. This shows that d is indeed a descent direction.