Exercise 7

1 Tube MPC

Consider the following scalar linear discrete-time system

$$x^+ = 1.1x + u + w$$

under the constraints $x \in \mathbb{X} = [-2, 2]$ and $u \in \mathbb{U} = [-0.8, 0.8]$ and subject to a bounded disturbance $w \in \mathbb{W} = [-0.2, 0.1]$. Design a tube-based MPC controller for this system with the following steps:

1. Compute the minimal robust invariant set $\mathcal{E}$ for this system under the control law $u = Kx$ with $K = -0.3$. Hint: $[a, b] \oplus [c, d] = [a + c, b + d]$ and $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ if $|x| < 1$.
2. Compute the tightened constraints $\mathbb{X} \ominus \mathcal{E}$ and $\mathbb{U} \ominus K\mathcal{E}$.
3. Verify terminal set feasibility. Check if $\mathcal{X}_f = [-1, 1.5]$ ensures recursive feasibility for terminal control law $u = -0.3x$.
4. Write out the formulation of a tube-MPC controller with all ingredients above.

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   $x^+ = (A + BK)x = 0.8x$ is the closed-loop system.

$$\begin{aligned}
\Omega_0 &= [0, 0] \\
\Omega_1 &= [-0.2, 0.1] \\
\Omega_2 &= 0.8 \cdot [-0.2, 0.1] \oplus [-0.2, 0.1] \\
\Omega_3 &= 0.8^2 \cdot [-0.2, 0.1] \oplus 0.8 \cdot [-0.2, 0.1] \oplus [-0.2, 0.1] \\
&\vdots \\
\mathcal{E} &= \left[-0.2 \cdot \sum_{i=0}^{\infty} 0.8^i, 0.1 \cdot \sum_{i=0}^{\infty} 0.8^i \right] \\
&= [-0.2 \cdot 5, 0.1 \cdot 5] \\
&= [-1, 0.5]
\end{aligned}$$

2. Compute the tightened constraints $\mathbb{X} \ominus \mathcal{E}$ and $\mathbb{U} \ominus K\mathcal{E}$.

$$\mathbb{X} \ominus \mathcal{E} = \tilde{\mathbb{X}} = [-2, 2] \ominus [-1, 0.5] = [-1, 1.5]$$
$$\mathbb{U} \ominus K\mathcal{E} = \tilde{\mathbb{U}} = [-0.8, 0.8] \ominus -0.3 \cdot [-1, 0.5] = [-0.8, 0.8] \ominus [-0.15, 0.3] = [-0.65, 0.5]$$

3. Verify terminal set feasibility. Check if $\mathcal{X}_f = [-1, 1.5]$ ensures recursive feasibility for terminal control law $u = -0.3x$.

   The set $\mathcal{X}_f$ must be invariant for the nominal system subject to the tightened constraints.
   Invariance:

$$(A + BK)\mathcal{X}_f = 0.8 \cdot [-1, 1.5] = [-0.8, 1.2] \subset \mathcal{X}_f$$

   Constraint satisfaction:

$$\begin{aligned}
\mathcal{X}_f \subseteq \tilde{\mathbb{X}} &\Leftrightarrow [-1, 1.5] \subseteq [-1, 1.5] \\
K\mathcal{X}_f \subseteq \tilde{\mathbb{U}} &\Leftrightarrow -0.3 \cdot [-1, 1.5] \subseteq [-0.65, 0.5] \\
&\Leftrightarrow [-0.45, 0.3] \subseteq [-0.65, 0.5]
\end{aligned}$$

4. Write out the formulation of a tube-MPC controller with all ingredients above.

Recall that a tube-MPC controller solves the following optimization problem

$$
\begin{aligned}
\min_{z,v} \quad & \sum_{k=0}^{N-1} z_k^\top Q z_k + v_k^\top R v_k + z_N^\top P z_N \\
\text{s.t.} \quad & z_{k+1} = A z_k + B v_k \quad \forall k \in [N] \\
& z_k \in \mathbb{X} \ominus \mathcal{E} \quad \forall k \in [N-1] \\
& v_k \in \mathbb{U} \ominus K\mathcal{E} \quad \forall k \in [N-1] \\
& z_N \in \mathcal{X}_f \\
& x_0 \in z_0 \oplus \mathcal{E}
\end{aligned}
$$

collect the trajectory $\{z_k^*, v_k^*\}$ and use the control law: $\mu_{\text{tube}}(x) = K(x - z_0^*) + v_0^*$. The stabilizing controller K_s is chosen apriori. The terminal cost $V_f(z) = z^\top P z$ and the terminal constraint $\mathcal{X}_f$ must satisfy certain conditions (slide 35) under the local control law $\kappa_f(z) = K_f z$. In general, K_s can be different from K_f. In our case, the minimal robust invariant set $\mathcal{E}$, the tightened constraints $\tilde{\mathbb{X}}, \tilde{\mathbb{U}}$, and the terminal constraints $\mathcal{X}_f$ are all calculated based on $K = -0.3$. The algebraic Riccati equation and the coresponding feedback matrix are:

$$
\begin{aligned}
P &= Q + A^\top P A - A^\top P B (R + B^\top P B)^{-1} B^\top P A \\
&= Q + K^\top R K + (A + BK)^\top P (A + BK) \\
K &= -(R + B^\top P B)^{-1} B^\top P A
\end{aligned}
$$

In our case, $A = 1.1, B = 1, K = -0.3$. So,

$$
\begin{aligned}
K &= -(R + P)^{-1} P \cdot 1.1 = -0.3 \\
P &= Q + 0.3^2 R + 0.8^2 P
\end{aligned}
$$

which makes $P = \frac{3}{8} R$ and $Q = \frac{9}{200} R$. The tube-MPC now becomes

$$
\begin{aligned}
\min_{z,v} \quad & \sum_{k=0}^{N-1} \frac{9}{200} z_k^2 + v_k^2 + \frac{3}{8} z_N^2 \\
\text{s.t.} \quad & z_{k+1} = 1.1 z_k + v_k \quad \forall k \in [N] \\
& z_k \in [-1, 1.5] \quad \forall k \in [N-1] \\
& v_k \in [-0.65, 0.5] \quad \forall k \in [N-1] \\
& z_N \in [-1, 1.5] \\
& x_0 \in z_0 \oplus [-1, 0.5]
\end{aligned}
$$