Exercise 4

1 Stable and recursively feasible MPC

Recall the following definitions

$$
\begin{aligned}
X \oplus Y &:= \{x + y | x \in X, y \in Y\} && \text{Minkowski sum of } X \text{ and } Y, \\
\alpha X &:= \{\alpha x | x \in X\} && \text{scaling of a set,} \\
\text{if } X \text{ is convex: } x_1,\ x_2 \in X \to \alpha x_1 + (1-\alpha) x_2 &\in X, \forall \alpha \in [0,1] && \text{convex set.}
\end{aligned}
$$

1. Let X_1 and X_2 be convex invariant sets for the system $x_{k+1} = Ax_k$. Show that $\alpha X_1 \oplus (1-\alpha)X_2$ is also an invariant set for any $\alpha \in [0,1]$.
2. Let $X_1 \subseteq \mathbb{X}$ and $X_2 \subseteq \mathbb{X}$, where X_1, X_2 and $\mathbb{X}$ are convex sets. Show that $\alpha X_1 \oplus (1-\alpha)X_2 \subseteq \mathbb{X}$ for any $\alpha \in [0,1]$.
3. Let $V_i(x) := x^\top P_i x$ be a Lyapunov function for the system $x_{k+1} = Ax_k$ for $i = 1,2$, with a rate of decrease of $x^\top \Gamma x$, i.e.: $V_i(x_{k+1}) - V_i(x_k) \leq -x^\top \Gamma x$. Show that $V(x) = \alpha V_1(x) + (1-\alpha)V_2(x)$ is also a Lyapunov function with a rate of decrease of $x^\top \Gamma x$ for any $\alpha \in [0,1]$.
4. Let K be a stabilizing controller for the system $x_{k+1} = Ax_k + Bu_k$, and $X_i \subset \mathbb{X}$ be a convex invariant set for the system $x_{k+1} = (A+BK)x_k$, with $KX_i \subset \mathbb{U}$ for each $i = 1,2$. $V_i(x) = x^\top P_i x$ are Lyapunov functions for the system $x_{k+1} = (A+BK)x_k$ with a rate of decrease of $Q + K^\top RK$, for some $Q = Q^\top \succ 0$ and $R = R^\top \succ 0$.

$$
\begin{aligned}
J^*(x(t)) = \min_u \quad & \sum_{k=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i + \alpha V_1(x_N) + (1-\alpha)V_2(x_N), \\
\text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k, \forall k = 0, \dots, N-1, \\
& x_k \in \mathbb{X}, \forall k = 1, \dots, N, \\
& u_k \in \mathbb{U}, \forall k = 0, \dots, N-1, \\
& x_N \in \alpha X_1 \oplus (1-\alpha)X_2, \\
& x_0 = x(t).
\end{aligned}
$$

Prove that this MPC controller is stabilizing and recursively feasible for any $\alpha \in [0,1]$ by listing sufficient conditions for stability and proving them. You can use the result of the previous three questions.

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1 Stable and recursively feasible MPC

Recall the following definitions

$$\begin{aligned}
X \oplus Y &:= \{x + y | x \in X, y \in Y\} && \text{Minkowski sum of } X \text{ and } Y,\\
\alpha X &:= \{\alpha x | x \in X\} && \text{scaling of a set,}\\
\text{if } X \text{ is convex: } x_1,\, x_2 \in X &\rightarrow \alpha x_1 + (1-\alpha)x_2 \in X, \forall \alpha \in [0,1] && \text{convex set.}
\end{aligned}$$

1. Let X_1 and X_2 be convex invariant sets for the system $x_{k+1} = Ax_k$. Show that $\alpha X_1 \oplus (1-\alpha)X_2$ is also an invariant set for any $\alpha \in [0,1]$.

   We want to show that if $x_k \in \alpha X_1 \oplus (1-\alpha)X_2$, $x_{k+1} \in \alpha X_1 \oplus (1-\alpha)X_2$. Suppose that $x_k \in \alpha X_1 \oplus (1-\alpha)X_2$, which means that $\exists x_1 \in X_1, x_2 \in X_2$. such that $x_k = \alpha x_1 + (1-\alpha)x_2$. $x_{k+1} = Ax_k = \alpha Ax_1 + (1-\alpha)Ax_2$. Since X_1, X_2 are invariant sets. $Ax_1 \in X_1, Ax_2 \in X_2$. That means $x_{k+1} \in \alpha X_1 \oplus (1-\alpha)X_2$.

2. Let $X_1 \subseteq \mathbb{X}$ and $X_2 \subseteq \mathbb{X}$, where X_1, X_2 and $\mathbb{X}$ are convex sets. Show that $\alpha X_1 \oplus (1-\alpha)X_2 \subseteq \mathbb{X}$ for any $\alpha \in [0,1]$.

   If $x \in \alpha X_1 \oplus (1-\alpha)X_2$, $\exists x_1 \in X_1, x_2 \in X_2$ such that $x = \alpha x_1 + (1-\alpha)x_2$. Since both $x_1, x_2 \in \mathbb{X}$ and $\mathbb{X}$ is a convex set. By convexity, we conclude $x \in \mathbb{X}$.

3. Let $V_i(x) := x^\top P_i x$ be a Lyapunov function for the system $x_{k+1} = Ax_k$ for $i = 1,2$, with a rate of decrease of $x^\top \Gamma x$, i.e.: $V_i(x_{k+1}) - V_i(x_k) \leq -x^\top \Gamma x$. Show that $V(x) = \alpha V_1(x) + (1-\alpha)V_2(x)$ is also a Lyapunov function with a rate of decrease of $x^\top \Gamma x$ for any $\alpha \in [0,1]$.

   To show $V(x)$ is a Lyapunov function with the same decreasing rate, we need to show that $V(x_{k+1}) - V(x_k) \leq -x^\top \Gamma x$. $V(x_{k+1}) = \alpha V_1(x_{k+1}) + (1-\alpha)V_2(x_{k+1}) = \alpha x_k^\top A^\top P_1 A x_k + (1-\alpha)x_k^\top A^\top P_2 A x_k$. $V(x_{k+1}) - V(x_k) = \alpha x_k^\top (A^\top P_1 A - P_1)x_k + (1-\alpha)x_k^\top (A^\top P_2 A - P_2)x_k \leq -\alpha x_k^\top \Gamma x_k - (1-\alpha)x_k^\top \Gamma x_k = -x_k^\top \Gamma x_k$.

4. Let K be a stabilizing controller for the system $x_{k+1} = Ax_k + Bu_k$, and $X_i \subset \mathbb{X}$ be a convex invariant set for the system $x_{k+1} = (A + BK)x_k$, with $KX_i \subset \mathbb{U}$ for each $i = 1,2$. $V_i(x) = x^\top P_i x$ are Lyapunov functions for the system $x_{k+1} = (A+BK)x_k$ with a rate of decrease of $Q + K^\top RK$, for some $Q = Q^\top \succ 0$ and $R = R^\top \succ 0$.

$$\begin{aligned}
J^*(x(t)) = \min_u \quad & \sum_{k=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i + \alpha V_1(x_N) + (1-\alpha)V_2(x_N),\\
\text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k, \forall k = 0, \dots, N-1,\\
& x_k \in \mathbb{X}, \forall k = 1, \dots, N,\\
& u_k \in \mathbb{U}, \forall k = 0, \dots, N-1,\\
& x_N \in \alpha X_1 \oplus (1-\alpha)X_2,\\
& x_0 = x(t).
\end{aligned}$$

   Prove that this MPC controller is stabilizing and recursively feasible for any $\alpha \in [0,1]$ by listing sufficient conditions for stability and proving them. You can use the result of the previous three questions.

   Sufficient conditions:

   - stage cost is positive definite,
   - the terminal set is invariant under the local control law (proved by question 1.1),
   - all state and input constraints are satisfied in the terminal set (proved by question 1.2),
   - the terminal cost is a Lyapunov function in the terminal set (proved by question 1.3).

The original terminal set $Hx \leq h$ is an invariant set for the closed-loop system $x_{k+1} = (A - BK)x_k$. The terminal control law for both the original system and the delta formulation is the same. Hence, it is also an invariant set for $\Delta x_{k+1} = (A - BK)\Delta x_k$. The second choice is larger. Since it is the maximum control invariant set with the terminal control law K and the new constraints. The first choice is just an invariant set. If the assumptions are not met, it will even be rescaled to a smaller set.

4. Let $\mathcal{X}_s := \{x | H' \Delta x_N \leq h'\}$. An unknown constant disturbance d perturbs the LTI system, which makes the system $x_{k+1} = Ax_k + Bu_k + d$. Assume your state estimator returns a disturbance estimation $\hat{d}$ and state estimation $\hat{x}(0)$. Write down the delta formulation that tracks x_{ref} while accounting for such a disturbance. Will $\hat{d}$ appear in your problem formulation?

$$
\begin{aligned}
\min_{u} \quad & \sum_{k=0}^{N-1} \Delta x_k^\top Q \Delta x_k + \Delta u_k^\top R \Delta u_k + \Delta x_N^\top P \Delta x_N, \\
\text{s.t.} \quad & \Delta x_{k+1} = A\Delta x_k + B\Delta u_k, \quad \forall k \in [N], \\
& F\Delta x_k \leq f - Fx_s, \quad \forall k \in [N-1], \\
& G\Delta u_k \leq g - Gu_s, \quad \forall k \in [N-1], \\
& H'\Delta x_N \leq h', \\
& \Delta x_0 = \hat{x}(0) - x_s.
\end{aligned}
$$

No, $\hat{d}$ will not appear in the problem formulation. The disturbance estimate cancels out in the delta formulation since at each time step, the reference is calculated via $x_s = Ax_s + Bu_s + \hat{d}$.