Exercise 1

1 Understanding LQR

Consider the following LQR formulation

$$\begin{aligned} \min_{u} \quad & \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k + x_N^\top P x_N, \\ \text{s.t.} \quad & x_{k+1} = A x_k + B u_k, \forall k \in [N-1]. \end{aligned}$$

1. Explain how the controller will behave when 1) $Q = 0, P = 0$. 2), $R = 0$.
2. Explain the difference between finite horizon LQR and infinite horizon LQR ($N = \infty$). Why is the infinite horizon controller time invariant?

2 Dynamic programming

Consider the following optimal control problem

$$\begin{aligned} V^*(x_0) = \min_{u} \quad & \gamma x_N^2 + \sum_{k=0}^{N-1} u_k^2, \\ \text{s.t.} \quad & x_{k+1} = x_k + u_k, \end{aligned}$$

where only the cost of the final state is penalized with $\gamma \geq 0$. Your goal is to solve this problem with dynamic programming.

1. Assume that $V_{k+1}(x_{k+1}) = h x_{k+1}^2$ and derive $V_k(x_k)$.
2. What is $V_N(x_N)$?
3. Compute the optimal value function $V^*(x_0)$ and the control law $\kappa^*(x_0)$ for a finite horizon N.
4. Compute the closed-loop pole of your system. Will your system converge to zero for all finite N and γ?
5. What is your control law when $N \to \infty$. Will you system converge to zero for infinite N and finite γ?

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Consider the following LQR formulation

$$\begin{aligned} \min_u \quad & \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k + x_N^\top P x_N, \\ \text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k, \forall k \in [N-1]. \end{aligned}$$

1. Explain how the controller will behave when 1) $Q = 0, P = 0$. 2), $R = 0$.

   If $Q = 0$, $P = 0$, the cost function becomes $\min_u \quad \sum_{k=0}^{N-1} u_k^\top R u_k$, which is only related to the inputs. The optimal solution to this problem is $u_k = 0, \forall k \in [N-1]$.

   If $R = 0$, the objective becomes $\min_u \quad \sum_{k=0}^{N-1} x_k^\top Q x_k + x_N^N P x_N$, which is not related to the control inputs u. The system will settle to $x = 0$ as fast as possible.

2. Explain the difference between finite horizon LQR and infinite horizon LQR ($N = \infty$). Why is the infinite horizon controller time invariant?

   For finite horizon LQR, the controller is $u_k = -K_k x_k$, where K_k is obtained by solving the discrete riccati equation. For infinite horizon LQR, $lim_{k\to\infty} K_k = K$. The controller becomes $u_k = -Kx_k$ with K being a constant matrix. Hence, the infinite horizon controller is time-invariant.

2 Dynamic programming

Consider the following optimal control problem

$$\begin{aligned} V^*(x_0) = \min_u \quad & \gamma x_N^2 + \sum_{k=0}^{N-1} u_k^2, \\ \text{s.t.} \quad & x_{k+1} = x_k + u_k, \end{aligned}$$

where only the cost of the final state is penalized with $\gamma \geq 0$. Your goal is to solve this problem with dynamic programming.

1. Assume that $V_{k+1}(x_{k+1}) = hx_{k+1}^2$ and derive $V_k(x_k)$.

   The Bellman recursion is

   $$V_k(x_k) = \min_{u_k} u_k^2 + V_{k+1}(x_{k+1}) = \min_{u_k} u_k^2 + h(x_k + u_k)^2.$$

   Take the derivative of $V_k(x_k)$ with respect to u_k, we have

   $$u_k + h(x_k + u_k) = 0,$$

   and

   $$u_k = -\frac{h}{1+h} x_k.$$

   Plug u_k back in $V_k(x_k)$, we obtain

   $$\begin{aligned} V_k(x_k) &= \frac{h^2}{(1+h)^2} x_k^2 + \frac{h}{(1+h)^2} x_k^2 \\ &= \frac{h}{1+h} x_k^2. \end{aligned}$$

2. What is $V_N(x_N)$?

   $V_N(x_N) = \gamma x_N^2$

3. Compute the optimal value function $V^*(x_0)$ and the control law $\kappa^*(x_0)$ for a finite horizon N.

   Starting from $V_N(x_N)$, we know

$$V_{N-1}(x_{N-1}) = \frac{\gamma}{\gamma+1}x_{N-1}^2,$$
$$V_{N-2}(x_{N-2}) = \frac{\gamma}{2\gamma+1}x_{N-2}^2,$$
$$V_{N-3}(x_{N-3}) = \frac{\gamma}{3\gamma+1}x_{N-3}^2.$$

   Hence, $V^*(x_0) = \frac{\gamma}{N\gamma+1}x_0^2$. The optimal control law $\kappa^*(x_0) = -\frac{h_1}{h_1+1}x_0 = -\frac{\gamma/((N-1)\gamma+1)}{\gamma/((N-1)\gamma+1)+1}x_0 = -\frac{\gamma}{N\gamma+1}x_0$.

4. Compute the closed-loop pole of your system. Will your system converge to zero for all finite N and γ?

   The closed-loop system is

$$x_{k+1} = x_k + \kappa(x_k) = \frac{(N-1)\gamma+1}{N\gamma+1}x_k.$$

   Hence, the closed-loop pole is $\frac{(N-1)\gamma+1}{N\gamma+1}$. The system will converge to zero for all finite N and γ since the pole is smaller than one.

5. What is your control law when $N \to \infty$. Will you system converge to zero for infinite N and finite γ?

   $\lim_{N\to\infty} \kappa(x_k) = 0$. That means the control inputs will always be zero. The pole of the closed-loop system converges to one. Hence, the system will not move.