24MA05028

Q1 (a) $\int_0^1\int_0^{y^2} 3y^2 e^{xy}\,dx\,dy$

$\downarrow$

$$3y^3\int_0^{y^2}(ey)^x\,dx = 3y^3\frac{(ey)^x}{\ln(ey)}\Big|_0^{y^2}$$

$$= 3y^2(ey)^x\Big|_0^{y^2} = 3y^2(e^{y^3}-e^0)$$

$$= 3y^2e^{y^3} - 3y^2$$

$$\int_0^1(3y^2e^{y^3}-3y^2)\,dy$$

$\hookrightarrow$ $= e^{y^3} - y^3\Big|_0^1 = (e^1-1)-(e^0-0)$

$= (e-2)$

(b) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}(xyz)\,dz\,dy\,dx$

$\downarrow$

$$\int_0^{\sqrt{1-x^2-y^2}}(xyz)\,dz = (yx)(z^2/2)\Big|_0^{\sqrt{1-x^2-y^2}}$$

$$= \frac{(yx)}{2}\left[(1-x^2-y^2)-0\right]$$

$$= \frac{xy}{2} - \frac{x^3y}{2} - \frac{xy^3}{2}$$

$$\int_0^{\sqrt{1-x^2}}\left(\frac{xy}{2}-\frac{x^3y}{2}-\frac{xy^3}{2}\right)dy = x/2\,(y^2/2)\Big|_0^{\sqrt{1-x^2}} - x^3/2\,(y^2/2)\Big|_0^{\sqrt{1-x^2}}$$

$$-x/2\,(y^4/4)\Big|_0^{\sqrt{1-x^2}}$$

$$= x/4\,(1-x^2) - x^3/4\,(1-x^2) - x/8\,(1-x^2)^2$$

$\downarrow$

$$-x/8\,(1+x^4-2x^2)$$

$$x/4 - x^{3/4} - x^{3/4} + x^{5/4} - x/8 - x^{5/8} + x^{3/4}$$

$$\int_0^1 \left(x/8 - x^{3/4} + x^{5/4}\right)dx = x^2/16 - x^4/16 + x^6/48 \Big|_0^1$$

$$= \left(\frac{1}{16} - \frac{1}{16} + \frac{1}{48}\right) - (0)$$

$$= 1/48$$

(c) $$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy\, dx$$

$$\downarrow$$

$$3/2 \int_0^{\sqrt{x}} e^{y/\sqrt{x}}\, dy = 3/2 \int_0^{\sqrt{x}} \left(e^{1/\sqrt{x}}\right)^y dy = 3/2 \frac{\left(e^{1/\sqrt{x}}\right)^y}{\ln\left(e^{1/\sqrt{x}}\right)}$$

$$= 3/2 \frac{\left(e^{1/\sqrt{x}}\right)^y}{1/\sqrt{x}}$$

$$3/2 (\sqrt{x})\left[e^1 - e^0\right) \longleftarrow = 3/2 \sqrt{x} \left(e^{y/\sqrt{x}}\right)\Big|_0^{\sqrt{x}}$$

$$= 3/2 \sqrt{x}(e-1)$$

$$(e-1)\, 3/2 \int_1^4 \sqrt{x} = (e-1)(x)^{3/2}\Big|_1^4 = (e-1)[8-1] = 7(e-1)$$

(d) $$\int_0^{2\pi} \int_0^1 \int_{-1/2}^{1/2} \left(r^2 \sin^2\theta + z^2\right)\left(dz\, r\, dr\, d\theta\right)$$

$$\downarrow$$

$$r^2 \sin^2\theta \left\{ \int_{-1/2}^{1/2} dz \right\} + \int_{-1/2}^{1/2} z^2\, dz = r^2 \sin^2\theta (z)\Big|_{-1/2}^{1/2} = r^2 \sin^2\theta (1) + \frac{1}{3}\left(\frac{1}{8} + \frac{1}{8}\right)$$

$$+ z^3/3 \Big|_{-1/2}^{1/2} = r^2 \sin^2\theta + \frac{1}{12}$$

$$\int_0^1 \left(r^2 \sin 2\theta + \frac{1}{12}\right) r\,dr = \sin 2\theta \; r^3/3 \Big|_0^1 + 1/12 \; r^2/2 \Big|_0^1$$

$$= (\sin 2\theta)(1/3) + (1/24)$$

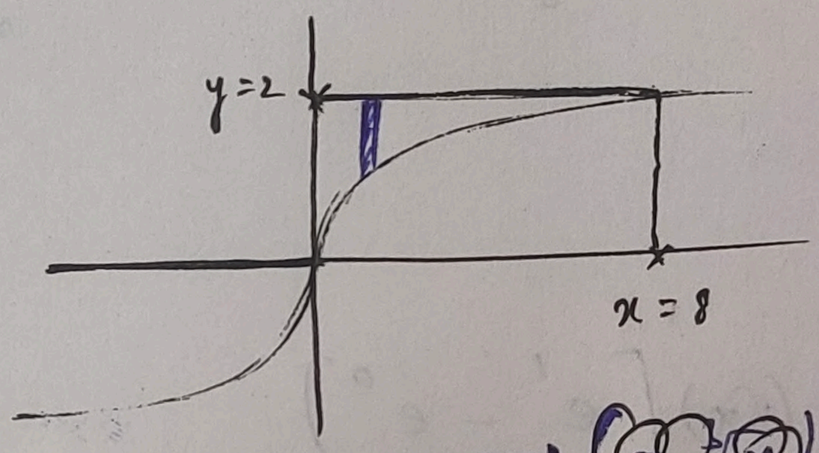
$$\int_0^{2\pi} \frac{\sin 2\theta}{3}\,d\theta + \frac{1}{24}\int_0^{2\pi} d\theta = \left(\frac{1}{3}\right)(\pi/4)(4) + \frac{1}{24}(2\pi)$$

$$= \left(\pi/3 + \pi/12\right) = 5\pi/12$$

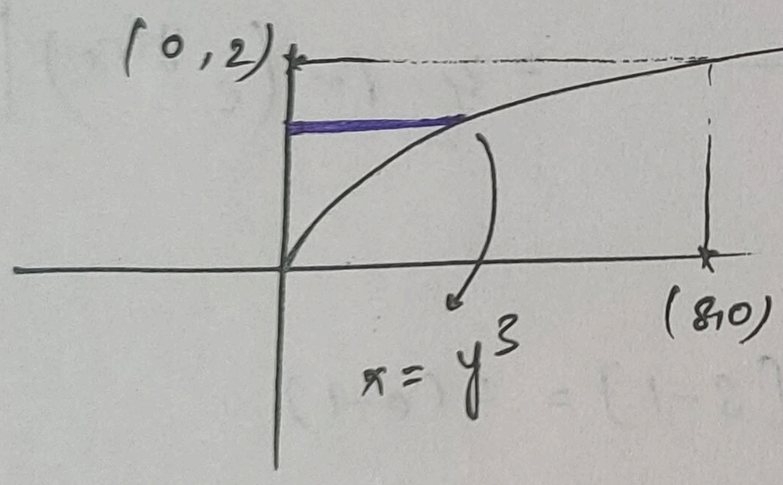
<u>Ques 2</u> :- (a) $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{dy\,dx}{y^4+1}$

inner limit : of dy

outer limit : of dx.



current limits make a vertical strip.



inner limit : of dx $\quad (0, y^3)$

outer limit : of dy.

$\downarrow$

$(0,2)$

$$\int_0^2 \int_0^{y^3} \frac{dx\,dy}{(y^4+1)} \longrightarrow \int_0^{y^3} \frac{dx}{y^4+1} = \frac{1}{y^4+1}(x)\Big|_0^{y^3}$$

$$= y^3/y^4+1$$

$$\int_0^2 \frac{y^3}{y^4+1}\,dy$$

$y^4+1 = t$

$dy\; 4y^3 = dt$

$y^3\,dy = dt/4$

$y=0$, $t=1$ $\qquad$ $y=2$, $t=17$

$$\int_1^{17} \frac{dt/4}{t} = \frac{\ln(t)}{4}\Big|_1^{17}$$

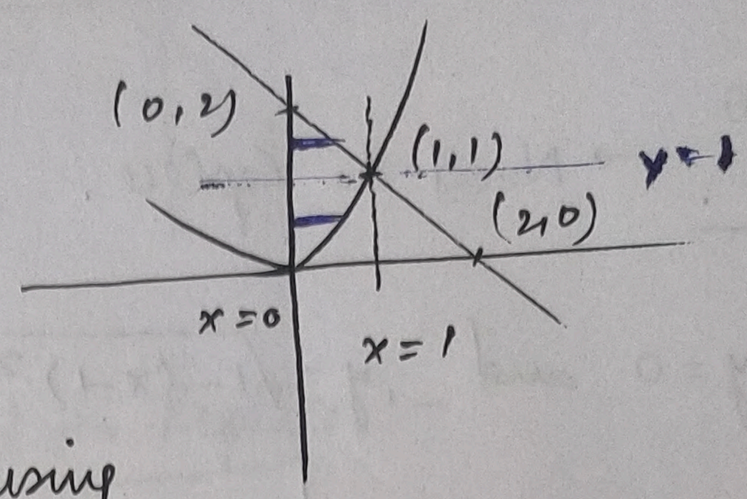
$$= \left(\frac{\ln 17}{4}\right)$$

(b) $\int_0^1 \int_{x^2}^{2-x} xy \, dx\,dy$

inner limit of dy

outer limit of dx.

Region :



current region evaluated using vertical strip.

~~changing~~ changing the limit using Horizontal strip. This needs 2 functions :-

$$\int_0^1 \int_0^{\sqrt{y}} xy \, dx\, dy + \int_1^2 \int_0^{2-y} xy \, dx\, dy$$

$$y \, x^2/2 \Big|_0^{\sqrt{y}}$$

$$= y\left(\sqrt{y}^2/2\right)$$

$$= y^2/2$$

$$\int_0^1 y^2/2 \, dy = y^3/6 \Big|_0^1$$

$$= 1/6$$

$$y \, x^2/2 \Big|_0^{2-y}$$

$$y/2 \, (2-y)^2$$

$$= y/2 \, (4 + y^2 - 4y)$$

$$= y^3/2 - 2y^2 + 2y$$

$$\int_1^2 \left(y^3/2 - 2y^2 + 2y\right) dy$$

$$= y^4/8 - 2y^3/3 + y^2 \Big|_1^2$$

$$= \left(16/8 - 16/3 + 4\right) - \left(1/8 - 2/3 + 1\right)$$

$$= \left(2 + 4 - 16/3\right) - \left(1/8 - 2/3 + 1\right)$$

$$= 5 - 1/8 - 14/3 = \frac{120 - 3 - 112}{24}$$

$$= 5/24$$

Total Volume

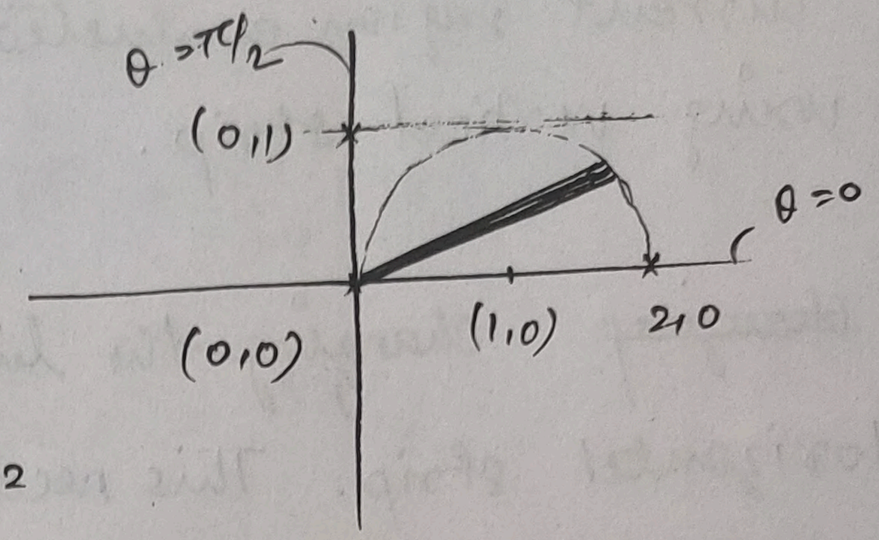
$$= 1/6 + 5/24$$

$$= 9/24 = 3/8$$

Ques 3 :- (a) $\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \left(\frac{x+y}{x^2+y^2}\right) dy\,dx$

$x = r\cos\theta$
$y = r\sin\theta$. → Need to replace.

Region R :-



$y = 0$ and $y = \sqrt{1-(x-1)^2}$

$r\sin\theta = 0$
$r = 0$

$r^2\sin^2\theta = 1-(r\cos\theta - 1)^2$

$= 1-(r^2\cos^2\theta + 1 - 2r\cos\theta)$

$r^2\sin^2\theta = 2r\cos\theta - r^2\cos^2\theta$

$r^2 = 2r\cos\theta$.

$r = 2\cos\theta$

So, r varies from $(r=0)$ to $(r = 2\cos\theta)$
θ varies from $(\theta = 0)$ to $(\theta = \pi/2)$

$$\int_0^{\pi/2}\int_0^{2\cos\theta} \left(\frac{r\cos\theta + r\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right) r\,dr\,d\theta$$

$$\int_0^{\pi/2}\int_0^{2\cos\theta} (\cos\theta + \sin\theta)\, d\theta\, dr$$

$$\int_0^{\pi/2} (\cos\theta + \sin\theta)(2\cos\theta - 0)\, d\theta$$

$$= \int_0^{\pi/2} (2\cos^2\theta + 2\sin\theta\cos\theta)\, d\theta$$

$$= \int_0^{\pi/2} (1+\cos 2\theta) + (\sin 2\theta) = \theta + \frac{\sin 2\theta}{2} - \frac{\cos 2\theta}{2}\Big|_0^{\pi/2}$$

$$= \pi/2 + 0/2 - \frac{(-1-1)}{2}$$

$$= \pi/2 + 1$$

JACOBIAN of TRANSFORMATION

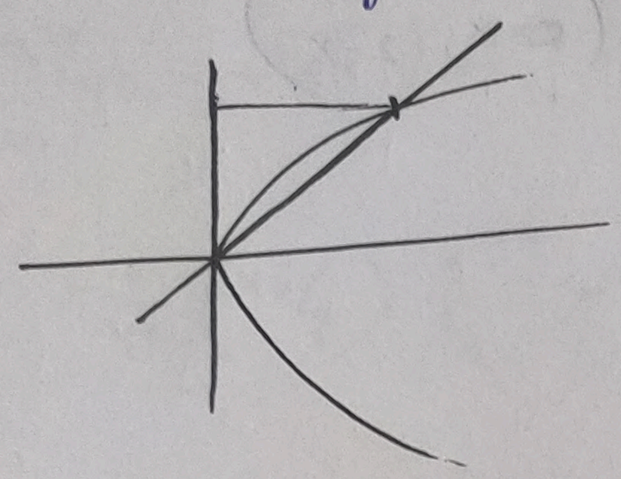
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}$$

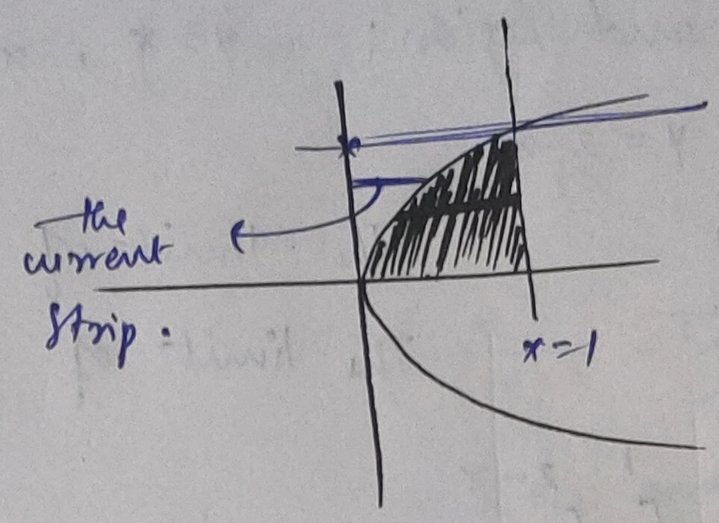
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

(b) $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dy\, dx$

Region R :-





Need to Replace :-

$x = 0$ and $y^2 = x$

$\downarrow$

$r\cos\theta = 0$

$\downarrow$

$r = 0$

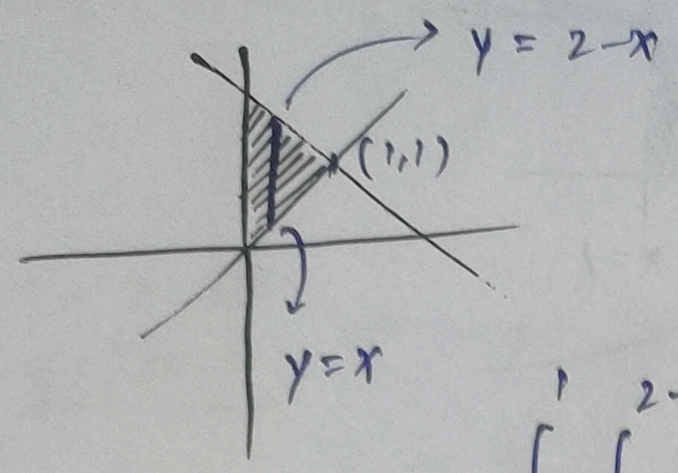
$\hookrightarrow r^2 \sin^2\theta = r\cos\theta$

$\downarrow$

$r = \dfrac{\cos\theta}{\sin^2\theta}$

$\int$

Q4 $z = x^2 + y^2$ and Region :- $y = x$, $x = 0$ and $x + y = 2$.



The limit of $x \in (0,1)$

The limit of $y \in (x, 2-x)$

$$\int_0^1 \int_x^{2-x} (x^2+y^2)\, dy\, dx$$

$$\int_x^{2-x} (x^2+y^2)\, dy = x^2 y \Big|_x^{2-x} + \frac{y^3}{3}\Big|_x^{2-x}$$

$$= x^2(2-x-x) + \frac{1}{3}\left((2-x)^3 - x^3\right)$$

$$= x^2(2-2x) + \frac{1}{3}\left(8 - x^3 - 12x + 6x^2 - x^3\right)$$

$$= x^2(2-2x) + \frac{1}{3}\left(-2x^3 + 6x^2 - 12x + 8\right)$$

$$\int_0^1 (2x^2 - 2x^3) + \frac{1}{3}(-2x^3 + 6x^2 - 12x + 8)$$

$$2x^3/3 - 2x^4/4 + \frac{1}{3}\left(-2x^4/4 + 6x^3/3 - 12x^2/2 + 8x\right)\Big|_0^1$$

$$2/3 - 2/4 + \frac{1}{3}\left(-2/4 + 6/3 - 12/2 + 8\right)$$

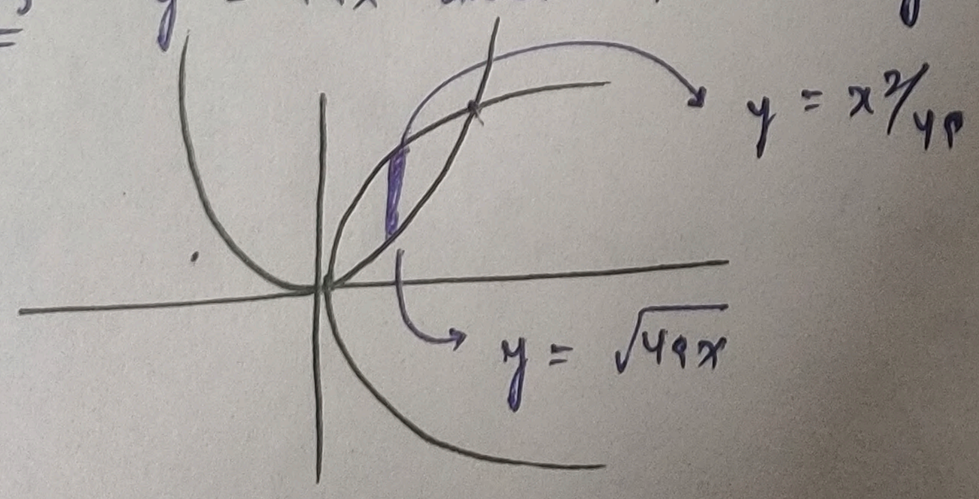
$$= 2/3 - 1/2 + 1/3\left(-1/2 + \cancel{2} - \cancel{6} + \cancel{8}\right) = 2/3 - 1/2 + 4/3 - 1/6$$

$$+4$$

$$= 2 - 1/2 - 1/6 = 3/2 - 1/6$$

$$= 8/6 = 4/3$$

Q5 $y^2 = 4ax$ and $x^2 = 4ay$.



The integral :-

$$\int_a^b \left(x^2/4a - \sqrt{4ax}\right) dx$$

$y^2 = 4ax \qquad x^2 = 4ay \longrightarrow \left(x^2/4a\right)^2 = 4ax$

$$\frac{x^4}{(4a)^2} = (4a)x$$

$$x\left(\frac{x^3}{(4a)^3} - 1\right) = 0$$

$x = 0$ and $x = 4a$

$$\int_0^{4a} \left(x^2/4a + \sqrt{4ax}\right) dx$$

$$= -\frac{1}{4a}\left(x^3/3\right)\Big|_0^{4a} + \sqrt{4a}\,\frac{(x)^{3/2}}{3/2}\Big|_0^{4a}$$

$$= -\frac{1}{4a}\,\frac{1}{3}(4a)^3 + \frac{\sqrt{4a}\,(4a)^{3/2}}{3/2} = -\frac{(4a)^2}{3} + \frac{(4a)^2}{3/2}$$

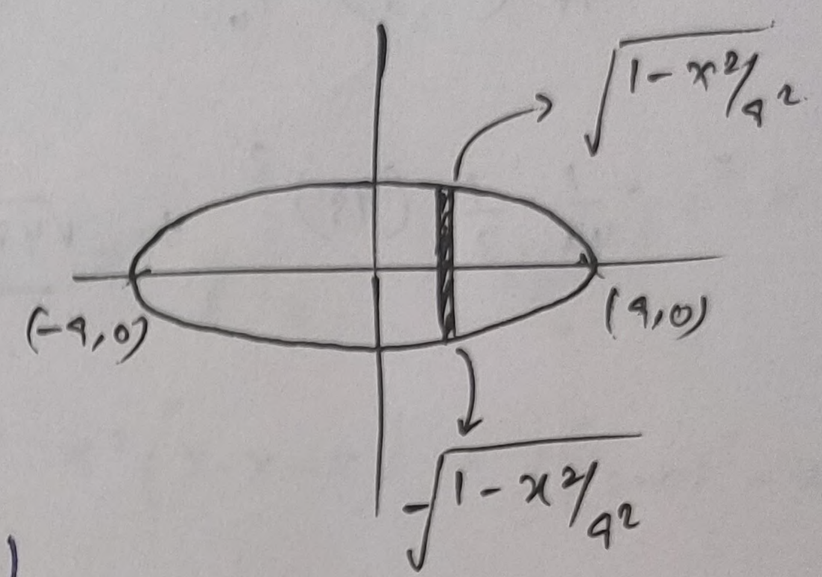
$$= \frac{(4a)^2}{3} = 16a^2/3$$

Hence proved

Q6 Given: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

to find : Volume of Ellipsoid.

$$z = f(x,y) = c\sqrt{1-\left(x^2/a^2 + y^2/b^2\right)}$$



limit of $x \in (-a, a)$

limit of $y \in \left(-b\sqrt{1-x^2/a^2},\ b\sqrt{1-x^2/a^2}\right)$

$$\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\left(x^2/a^2 + y^2/b^2\right)}\, dy\, dx$$

Let $\left(1-x^2/a^2\right) = p^2/b^2$

$\hookrightarrow p = \pm b\sqrt{1-\frac{x^2}{a^2}}$

and function is: $\sqrt{p^2/b^2 - y^2/b^2}$

$$\int_{-p}^{p} c\sqrt{p^2/b^2 - y^2/b^2}\, dy$$

$$\frac{c}{b}\int_{-p}^{p}\sqrt{p^2-y^2}\, dy$$

$$\left[\left(\frac{y}{2}\right)\sqrt{p^2-y^2} + p^2/2 \sin^{-1}\left(y/p\right)\right]_{-p}^{p} c/b$$

$$\left[\left(p/2\,(0) + p^2/2 \sin^{-1}(1)\right) - \left(-p/2\,(0) + p^2/2 \sin^{-1}(-1)\right)\right]\frac{c}{b}$$

$$= \left[p^2/2\ \pi/2 + p^2/2\ \pi/2\right] c/b$$

$$= \left(\frac{p^2\pi}{2}\right) c/b$$

$$\int_{-a}^{a} \frac{p^2\pi}{2}(c/b)\,dx = \int_{-a}^{a} \frac{p^2\pi}{2}(c/b)\,dx \qquad p^2 = b^2(1 - x^2/a^2)$$

$$\hookrightarrow \int_{-a}^{a} b^2(1 - x^2/a^2)(\pi/2)(c/b)\,dx$$

$$= \frac{\pi bc}{2}\left(x - \frac{x^3}{3a^2}\right)\Big|_{-a}^{a}$$

$$= \frac{\pi bc}{2}\left[\left(a - a^3/3a^2\right) - \left(-a + a^3/3a^2\right)\right]$$

$$= \frac{\pi bc}{\cancel{2}}\left[\cancel{2}a - \cancel{2}a^{\cancel{3}}/3\cancel{a^2}\right]$$

$$= \pi bc\left(a - a/3\right) = \frac{2\pi abc}{3}.$$

The function $f(x,y) = z$ is only positive.

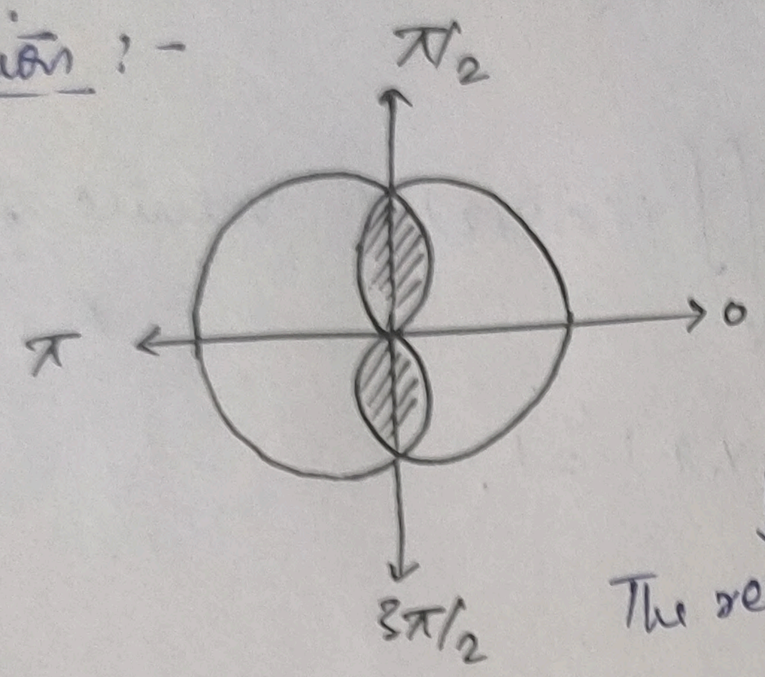
Similarly, for $-f(x,y) = z$ then you would get the same volume.

Thus, total $\longrightarrow$ $\dfrac{4\pi abc}{3}$

Q8// $\underbrace{r = 1+\cos\theta \text{ and } r = 1-\cos\theta}_{\text{given.}}$

to find :- area common b/w 2 curves.

solution :-



$$4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta$$

The region in all 4 quads is symmetrical
hence equal

$$4\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{1-\cos\theta} d\theta = 4\int_0^{\pi/2}\frac{(1-\cos\theta)^2}{2}\,d\theta$$

$$= 2\int_0^{\pi/2}\left(2\sin^2\theta/2\right)^2 d\theta$$

$$= (4\times 2)\int_0^{\pi/2}\sin^4\theta/2\,d\theta$$

$$= 16\int_0^{\pi/2}\sin^4 t\,dt$$

$t = \theta/2$

$dt = d\theta/2$

$$2\left[\frac{\sin 2\theta - 8\sin\theta + 6\theta}{8}\right]_0^{\pi/2}$$

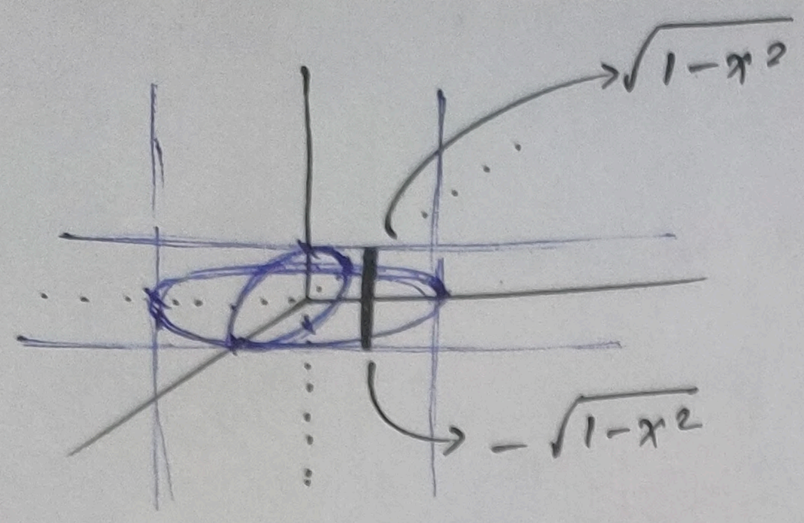
$$2\left(\frac{3\pi-8}{4}\right) = \left(\frac{3\pi-8}{2}\right)$$

ans

Q9 given cylinders : $x^2+y^2=1$
$x^2+z^2=1$

to find :- volume of region common.

solution :-

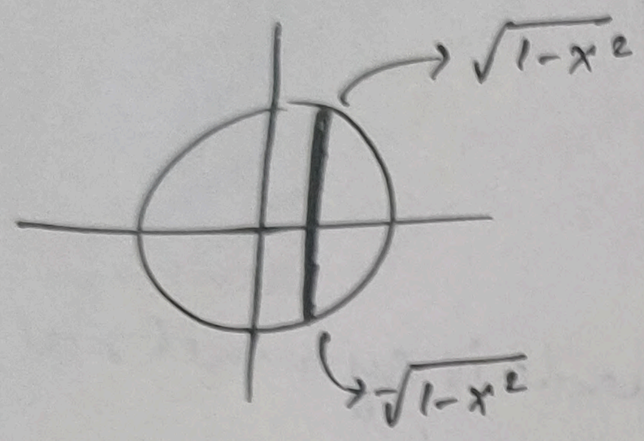


3D Diagram.

limit of x : $(-1,1)$

limit of y : $(-\sqrt{1-x^2}, \sqrt{1-x^2})$

limit of z : $(-\sqrt{1-x^2}, \sqrt{1-x^2})$



as the $\iiint_V f(x,y,z)\,dV =$ volume of Region R

if, $f(x,y,z) = 1$.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz\,dy\,dx$$

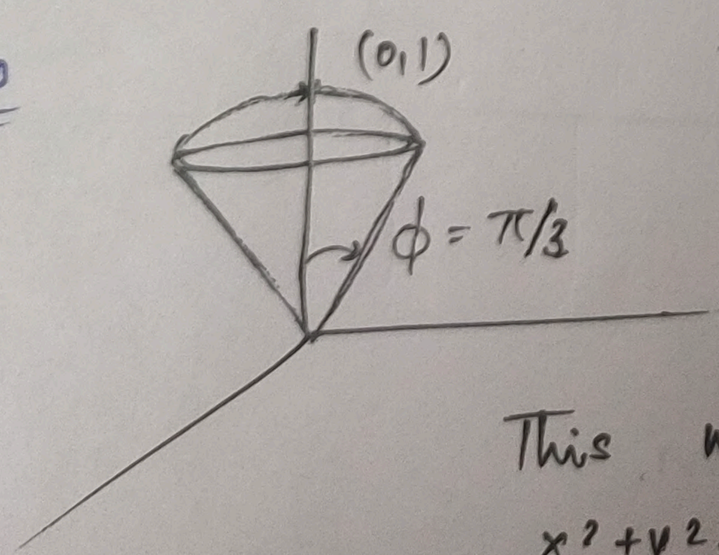
$$8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx = 8\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx$$

$$= 8\int_0^1 y\sqrt{1-x^2}\Big|_0^{\sqrt{1-x^2}} dx$$

$$= 8\int_0^1 (1-x^2)\,dx = 8\left(x - x^3/3\right)\Big|_0^1$$

$$= 16/3$$

Q10



a sphere of $x^2+y^2+z^2=1$ has a cone cut from it with $\phi = \pi/3$.

This means to integrate sphere of $x^2+y^2+z^2=1$ from $\theta \in (0, 2\pi)$ and $\phi \in (0, \pi/3)$

$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \tfrac{1}{3}\sin\phi \, d\phi \, d\theta$$

$$= -\tfrac{1}{3}\int_0^{2\pi} \cos\phi \Big|_0^{\pi/3} d\theta$$

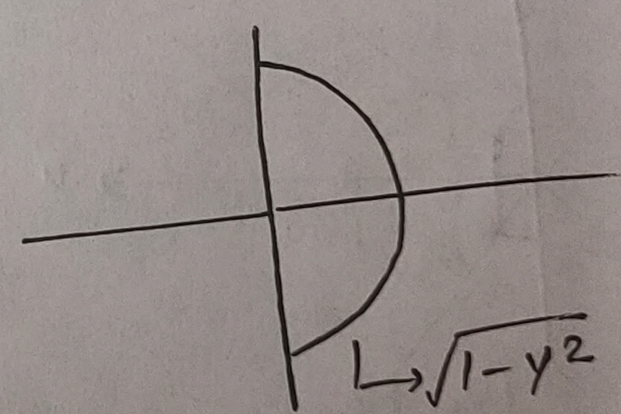
$$= \left(-\tfrac{1}{6}+\tfrac{1}{3}\right)\int_0^{2\pi} d\theta = \frac{1}{6}(2\pi)$$

$$= \pi/3$$

Q11 $$\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^{x} (x^2+y^2)\, dz\, dx\, dy$$

$x = r\cos\theta$ and $y = r\sin\theta$ and $dx\,dy \leftrightarrow r\,dr\,d\theta$.

$z = f(r,\theta) = r\cos\theta$.



$$\int_{-\pi/2}^{\pi/2}\int_0^{1}\int_0^{r\cos\theta} (r^2)\, r\,dr\,d\theta\,dz$$

$\theta \in (-\pi/2, \pi/2)$

$$\int_0^{r\cos\theta} dz = (r\cos\theta)$$

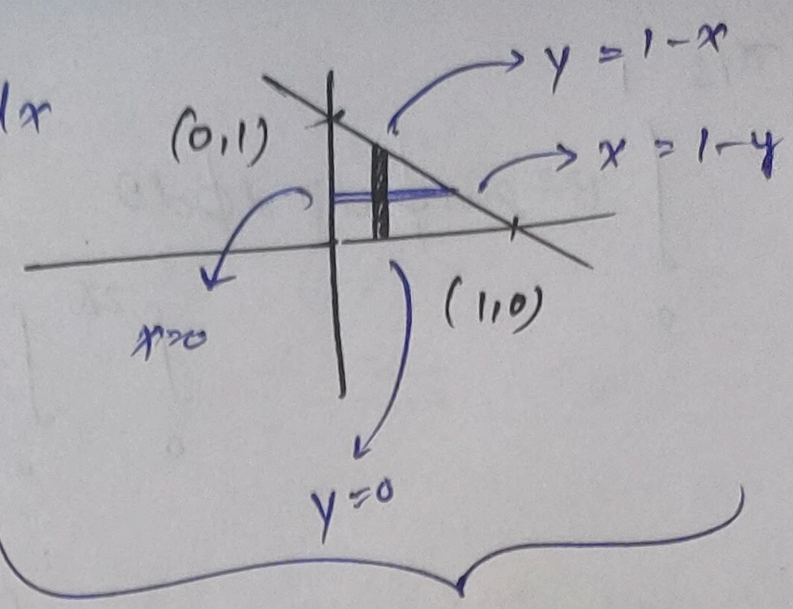
$\sqrt{1-y^2} = \cancel{\sqrt{1-(r\sin\theta)^2}}$

$$\int_0^{1} r^2(r\cos\theta)\, r\,dr = \left(\int_0^{1} r^4\,dr\right)\cos\theta$$

$$= (\cos\theta)\left(r^5/5\right)\Big|_0^{1} = (\cos\theta)/5$$

$$\int_{-\pi/2}^{\pi/2} \cos\theta/5 \, d\theta = 2/5$$

Q12 $\displaystyle\int_0^1\int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$



changing order of integration doesn't simplify here.

$y \in (0, 1-x) \rightarrow x+y=1$

and $x \in (0,1)$

$\rightarrow$ Try $x+y = u \longrightarrow x = (u-v)/3$

$y-2x = v \longrightarrow y = \dfrac{2u+v}{3}$

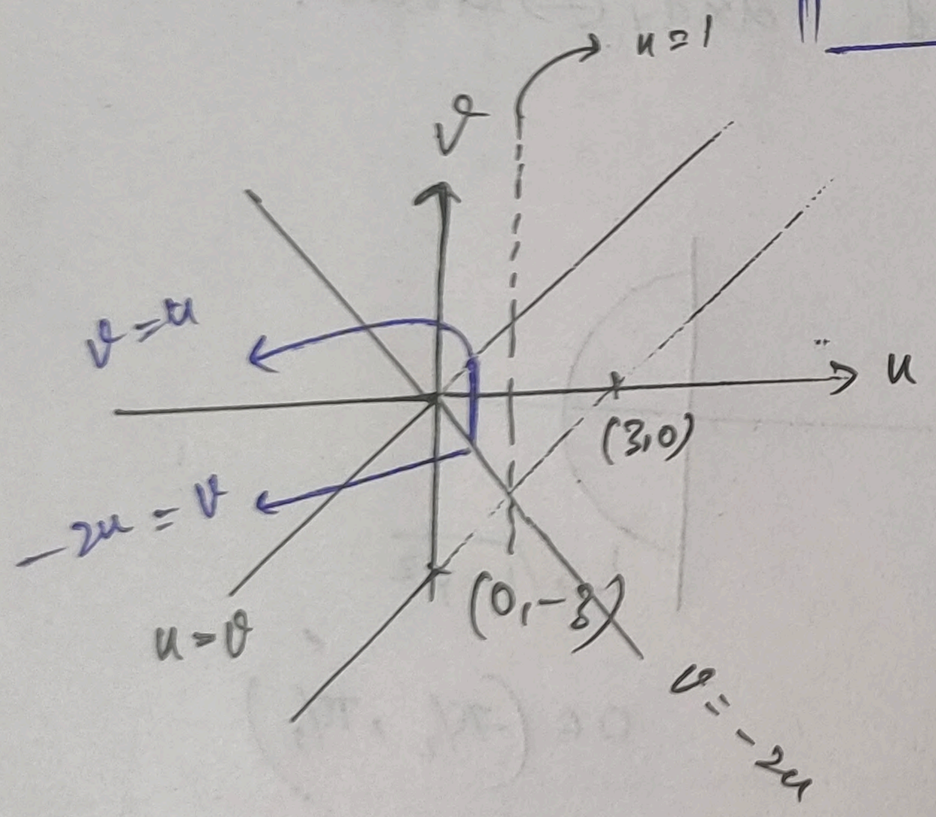
$x+y=1 \longrightarrow u=1$

$y=0 \longrightarrow v=-2u$

$x=1 \longrightarrow u-v=3$

$x=0 \longrightarrow u=v$

plotting graph of u-v.



Jacobian :- $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$= \begin{vmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{vmatrix}$

$= 1/3$

$$\int_0^1\int_{-2u}^{u} \sqrt{u}\, v^2\, \frac{1}{3}\,du\,dv$$

$$= \int_0^1 \frac{\sqrt{u}}{3}\left(v^3/3\right)\Big|_{-2u}^{u} du = \int_0^1 \left[\frac{\sqrt{u}}{3\cdot 3}\left(u^3 + 8u^3\right)\right] du$$

$$= \int_0^1 u^{7/2}\,du = \frac{u^{9/2}}{9/2}\Big|_0^1 = 2/9$$