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# Z-Transform

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Soln: 1.a) $f(k) = 3^k$ where $-3 \le k \le 4$

By definition,

$f(-3) = 3^{-3} = \frac{1}{3^3}$

$f(-2) = 3^{-2} = \frac{1}{3^2}$

$f(-1) = 3^{-1} = \frac{1}{3}$

$f(0) = 3^0 = 1$

$f(1) = 3^1 = 3$

$f(2) = 3^2 = 9$

$f(3) = 3^3 = 27$ $\quad f(4) = 3^4 = 81$

The sequence whose kth term is $3^k$ is

$\{\frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, 81\}$ for $-3 \le k \le 4$

Soln: 1.b) $f(k) = \begin{cases} 2^k & k < 0 \\ 3^k & k \ge 0 \end{cases}$

For $k < 0$:

$f(k) = 2^k$

$f(-3) = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

$f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

$f(-1) = 2^{-1} = \frac{1}{2}$

$f(0) = 3^0 = 1$

$f(1) = 3^1 = 3$

$f(2) = 3^2 = 9$

The sequence is given by:

$\{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 3, 9, \ldots\}$

Sol$^n$ 1.c) $\{-5, -2, -1, 1, 3, 4, 7, 9, 10, \ldots\}$

$f(0) = -1$
$f(1) = 1$
$f(2) = 3$
$f(3) = 4$
$f(4) = 7$
$f(5) = 9$

$\therefore$ for $k = 5$, the term corresponding is 9.

Sol$^n$ 1.d) For a sequence $f(k)$, The Z-transform is defined as:

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

Solⁿ: 2.1)

$f(k) = \{-6, -5, \underset{\uparrow}{0}, 2, 4\}$

By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-2}^{2} f(k) z^{-k}$$

$$= -6z^{2} - 5z + 0 + 2z^{-1} + 4z^{-2}$$

$$Z\{f(k)\} = -6z^{2} - 5z + \frac{2}{z} + \frac{4}{z^{2}}$$

Solⁿ: 2.2)

$f(k) = \{2^0, 2^1, 2^2, \dots\}$

$f(k) = 2^k$ where $k \geq 0$

By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} 2^k z^{-k}$$

$$= 1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \dots$$

$$= \frac{1}{1 - \frac{2}{z}} \quad \text{where } \left|\frac{2}{z}\right| < 1$$

$$Z\{f(k)\} = \frac{z}{z-2} \quad \text{where } |z| > 2$$

Solⁿ: 2.3)

$f(k) = 5^k$ where $k \geq 0$

By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= 1 + \frac{5}{z} + \frac{5^2}{z^2} + \frac{5^3}{z^3} + \dots$$

$$= \frac{1}{1-\frac{5}{z}} \quad \text{where } \left|\frac{5}{z}\right| < 1$$

$$Z\{f(k)\} = \frac{z}{z-5} \quad \text{where } |z| > 5$$

Sol$^n$: 2.1) $\{f(k)\} = b^k$ where $k \leq 0$

by definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= 1 + \frac{b}{z} + \frac{b^2}{z^2} + \frac{b^3}{z^3} + \dots$$

$$= \frac{1}{1-\frac{b}{z}} \quad \text{where } \left|\frac{b}{z}\right| < 1$$

$$Z\{f(k)\} = \frac{z}{z-b} \quad \text{where } |z| > |b|$$

Sol$^n$: 3.1) $U(k) = 1$ for $k \geq 0$
$= 0$ for $k < 0$

By definition,

$$Z\{U(k)\} = \sum_{k=-\infty}^{\infty} U(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} U(k) z^{-k} + \sum_{k=0}^{\infty} U(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} z^{-k}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

$$= \frac{1}{1-\frac{1}{z}} \quad \text{where } \left|\frac{1}{z}\right| < 1$$

$Z\{u(k)\} = \frac{z}{z-1}$ where $|z|>1$

Sol: 3.2) $\delta(k) = 1$ for $k=0$
$= 0$ for $k \neq 0$

By definition,

$$Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} \delta(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \delta(k) z^{-k} + \sum_{k=1}^{\infty} \delta(k) z^{-k} + \delta(0)$$

$$Z\{\delta(k)\} = 1$$

Sol: 3.3) $f(k) = {}^nC_k$ where $0 \leq k \leq n$

By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k) z^{-k} + \sum_{k=0}^{n} f(k) z^{-k}$$

$$= \sum_{k=0}^{n} {}^nC_k z^{-k}$$

$$= 1 + \frac{{}^nC_1}{z} + \frac{{}^nC_2}{z^2} + \frac{{}^nC_3}{z^3} + \dots$$

$$Z\{f(k)\} = \left(1 + \frac{1}{z}\right)^n$$ $\forall$ z plane except at $z=0$

Sol: 3.4) $\{f(k)\} = k3^k$ for $k \geq 0$

By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} k 3^k z^{-k}$$

$$= 0 + \frac{3}{z} + \frac{18}{z^2} + \frac{81}{z^3} + \cdots$$

$$= \frac{3}{z}\left\{1 + \frac{6}{z} + \frac{27}{z^2} + \cdots\right\}$$

$$= \frac{3/z}{\left(1-\frac{3}{z}\right)^2} \quad \text{where } \left|\frac{3}{z}\right| < 1$$

$$= \frac{3}{z}\left(1-\frac{3}{z}\right)^{-2}$$

$$= \frac{3}{z}\left(\frac{z-3}{z}\right)^{-2}$$

$$= \frac{3}{z}\left(\frac{z}{z-3}\right)^{2}$$

$$Z\{f(k)\} = \frac{3z}{z-3} \quad \text{where } |z| > 3$$

Sol$^n$: 3.5) $\{f(k)\} = \frac{5^k}{k!}$ for $k \geq 0$

by definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{5^k}{k!} z^{-k}$$

$$= 1 + \frac{5}{z} + \frac{25}{2z^2} + \frac{125}{6z^3} + \cdots$$

$$= -\left(-1 - \frac{5}{z} - \frac{5^2}{2z^2} - \frac{5^3}{6z^3} - \dots\right)$$

$$= \cancel{\log(1} \; 1 + \frac{(5/z)}{1!} + \frac{(5/z)^2}{2!} + \frac{(5/z)^3}{3!}$$

$$= e^{5/z} \quad \text{where } \left|\frac{5}{z}\right| < 1$$

$$Z\{f(k)\} = e^{5/z} \quad \text{where } |z| > 5$$

Sol$^n$: 3.i) $f(k) = \frac{1}{3^k}$ for $k < 0$

by definition

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k) z^{-k} + \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=1}^{\infty} 3^k z^k$$

$$= 3z + 3^2 z^2 + 3^3 z^3 + \dots$$

$$= 3z\,(1 + 3z + 3^2 z^2 + \dots)$$

$$= \frac{3z}{1-3z} \quad \text{where } |3z| < 1$$

$$Z\{f(k)\} = \frac{3z}{1-3z} \quad \text{where } |z| < \frac{1}{3}$$

Sol$^n$: 4.i) $f(k) = a^{|k|}$

$$f(k) = \begin{cases} a^{-k} & \text{for } k < 0 \\ a^k & \text{for } k \geq 0 \end{cases}$$

By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k) z^{-k} + f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^{k} z^{-k}$$

$$= \sum_{k=1}^{\infty} a^{k} z^{k} + \sum_{k=0}^{\infty} a^{k} z^{-k}$$

$$= (az + a^2z^2 + a^3z^3 + \dots) + \left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \dots\right)$$

$$= az(1 + az + a^2z^2 + \dots) + \left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \dots\right)$$

$$= \frac{az}{1-az} + \frac{1}{1-\frac{a}{z}}$$

$$= \frac{az}{1-az} + \frac{z}{z-a} \quad \text{where } |az|<1 \text{ and } \left|\frac{a}{z}\right|<1$$

$$Z\{f(k)\} = \frac{az}{1-az} + \frac{z}{z-a} \quad \text{where } |a|<|z|<\frac{1}{|a|}$$

Sol$^n$: 4.2) $f(k) = \left(\frac{1}{5}\right)^{|k|}$

$$= \begin{cases} \left(\frac{1}{5}\right)^{-k} & \text{for } k<0 \\ \left(\frac{1}{5}\right)^{k} & \text{for } k \geq 0 \end{cases}$$

By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$Z\{a^{|k|}\} = \frac{az}{1-az} + \frac{z}{z-a} \quad \text{where } |a|<|z|<\frac{1}{|a|}$$

For $a = \frac{1}{5}$

$$Z\left\{\left(\frac{1}{5}\right)^{|k|}\right\} = \frac{\frac{z}{5}}{1-\frac{z^2}{5}} + \frac{z}{z-\frac{1}{5}} \quad \text{where } \frac{1}{5}<|z|<5$$

$$= \frac{z}{5-z^2} + \frac{5z}{5z-1} \quad \text{where } \frac{1}{5}<|z|<5$$

Solⁿ: 4.5) $f(k) = c^k \cosh \alpha k$ for $k \geq 0$

by definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} c^k \cosh(\alpha k) z^{-k}$$

$$= \sum_{k=0}^{\infty} c^k \left(\frac{e^{\alpha k} + e^{-\alpha k}}{2}\right) z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{c^k e^{\alpha k} z^{-k}}{2} + \sum_{k=0}^{\infty} \frac{c^k e^{-\alpha k} z^{-k}}{2}$$

$$= \frac{1}{2}\left\{\sum_{k=0}^{\infty} \left(\frac{ce^{\alpha}}{z}\right)^k + \sum_{k=0}^{\infty} \left(\frac{ce^{-\alpha}}{z}\right)^k\right\}$$

$$= \frac{1}{2}\left\{\left\{\frac{1}{1-\frac{ce^{\alpha}}{z}}\right\} + \left\{\frac{1}{1-\frac{ce^{-\alpha}}{z}}\right\}\right\}$$

$$= \frac{1}{2}\left\{\frac{z}{z-ce^{\alpha}} + \frac{z}{z-ce^{-\alpha}}\right\}$$

$$= \frac{1}{2}\left\{\frac{z(z-ce^{-\alpha}) + z(z-ce^{\alpha})}{(z-ce^{\alpha})(z-ce^{-\alpha})}\right\}$$

$$= \frac{1}{2}\left\{\frac{z^2 - cze^{-\alpha} + z^2 - zce^{\alpha}}{z^2 - cze^{-\alpha} - zce^{\alpha} + c^2}\right\}$$

$$= \frac{1}{2}\left\{\frac{2z^2 - cz(e^{\alpha}+e^{-\alpha})}{z^2+c^2-cz(e^{\alpha}+e^{-\alpha})}\right\}$$

$$= \frac{1}{2}\left\{\frac{2z^2 - 2cz\cosh\alpha}{z^2+c^2-2cz\cosh\alpha}\right\}$$

$$Z\{f(k)\} = \frac{z^2 - cz\cosh\alpha}{z^2+c^2-2cz\cosh\alpha} \quad \text{where } |z|>|ce^{\alpha}| \text{ and } |z|>|ce^{-\alpha}|$$