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Chapter 1

Theory

1.1 Introduction

The present code solves the two-dimensional supersonic flow inside a nozzle. The shape of the nozzle is supposed to be known. Thus this code does not aim at designing a nozzle from desired requirements at the exit. Moreover the flow is steady and irrotational. The nozzle can be switched from planar to axisymmetric through a single parameter. The numerical method used in the code is a method of characteristics. This method offers the most accurate results among the marching-type numerical methods. The reader is referred to the two-volume books *Gas Dynamics* by Zucrow and Hoffman [1] for detailed explanations on the physics and the numerical method implemented.

Figure 1.1 provides an overview of the geometry of the nozzle. Only the supersonic part of the flow is of interest. Moreover it is supposed that a sonic condition is met at the throat of the nozzle. As a consequence only the region downstream of the throat is kept. The throat contour is joined tangentially to a second-order polynomial. The required geometric data are

- y_t the nozzle throat radius
- ρ_{tu} the radius of curvature of the upstream circular arc at the throat
- ρ_{td} the radius of curvature of the downstream circular arc at the throat
- θ_a the attachment angle between the downstream circular arc and the 2nd-order polynomial
- θ_e the exit lip angle
- x_e the nozzle length

The governing equations for steady two-dimensional irrotational flow are

$$\left(u^2 - a^2\right) u_x + \left(v^2 - a^2\right) v_y + 2uvu_y - \delta\frac{a^2 v}{y} = 0 \tag{1.1}$$

$$u_y - v_x = 0 \tag{1.2}$$

$$a = a(u, v) = \sqrt{\gamma RT - (\gamma - 1)(u^2 + v^2)/2} \tag{1.3}$$

where (u, v) are the velocity components along the x- and y-directions respectively. a is the sonic speed and is a function of (u, v) only. T is the stagnation temperature, P is the stagnation pressure, R is the gas constant and γ is the ratio of specific heats. $\delta = 0$ for planar flows and $\delta = 1$ for axisymmetric flows. The subscripts x and y denote the partial derivative along the

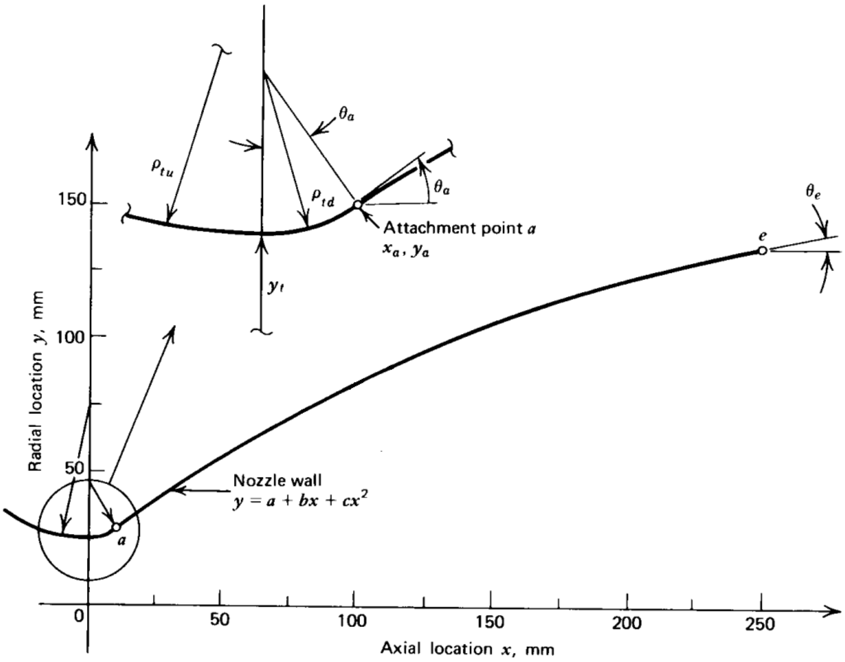


Figure 1.1: Schematic illustration of the geometry of the nozzle.

specified direction. The Mach number M is a function of the sonic speed a and the magnitude of the velocity:

$$M = a\sqrt{u^2 + v^2} \tag{1.4}$$

The static pressure p, static temperature t and density ρ are then evaluated by the relations

$$p = \frac{P}{\left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}} \tag{1.5a}$$

$$t = \frac{T}{\left(1 + \frac{\gamma-1}{2}M^2\right)} \tag{1.5b}$$

$$\rho = \frac{p}{Rt} \tag{1.5c}$$

The characteristic and compatibility equations corresponding to the governing equations are given by

Characteristic equation

$$\left(\frac{dy}{dx}\right)_{\pm} = \lambda_{\pm} = \tan\left(\theta \pm \alpha\right) \tag{1.6}$$

Compatibility equation

$$\left(u^2 - a^2\right) du_{\pm} + \left(2uv - \left(u^2 - a^2\right)\right) dv_{\pm} - \delta\frac{a^2 v}{y} dx_{\pm} = 0 \tag{1.7}$$

Figure 1.2 illustrates the C_+ and C_- characteristics and the angles θ and α. The $\pm$ in the previous equations correspond to the C_+ and C_- characteristics respectively. The subscript $\pm$ means that the differentials dx, du and dv are determined along the C_+ and C_- characteristics.

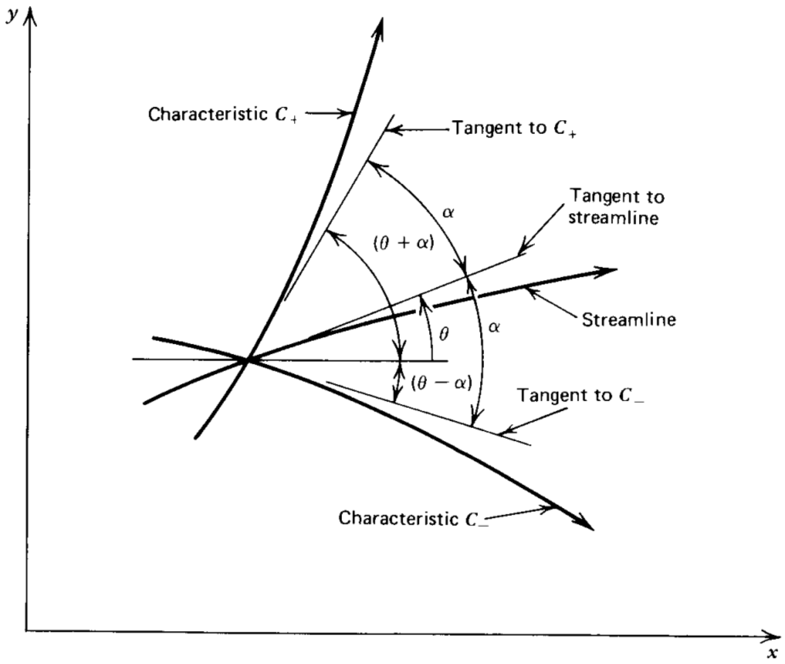


Figure 1.2: Schematic illustration of the characteristics in the two-dimensional space.

1.2 Unit processes

The method of characteristics is a marching-type numerical method. An initial-value line is thus required in order to find out the data in the domain downstream from that initial-value line. This line is prescribed in the throat region, where the flow is supersonic. Details about this line follow in Section 1.2.1.

Equations (1.6) and (1.7) are discretized in a finite difference form by a modified Euler predictor-corrector method. Sections 1.2.2 - 1.2.5 detail the implementation to determine the solution at an internal point, on the axis and on the wall.

1.2.1 Initial-value line

The initial-value line retained for the characteristic method is the locus for $v = 0$. Along this line the Mach number $M > 1$, which is a requirement for the method of characteristics. Figure 1.3 depicts the Mach line $M = 1$ and the initial-value line $v = 0$. The latter is located downstream of the line $M = 1$.

In the following ϵ is the origin of the coordinate system in the nozzle. This shifting value is equal to

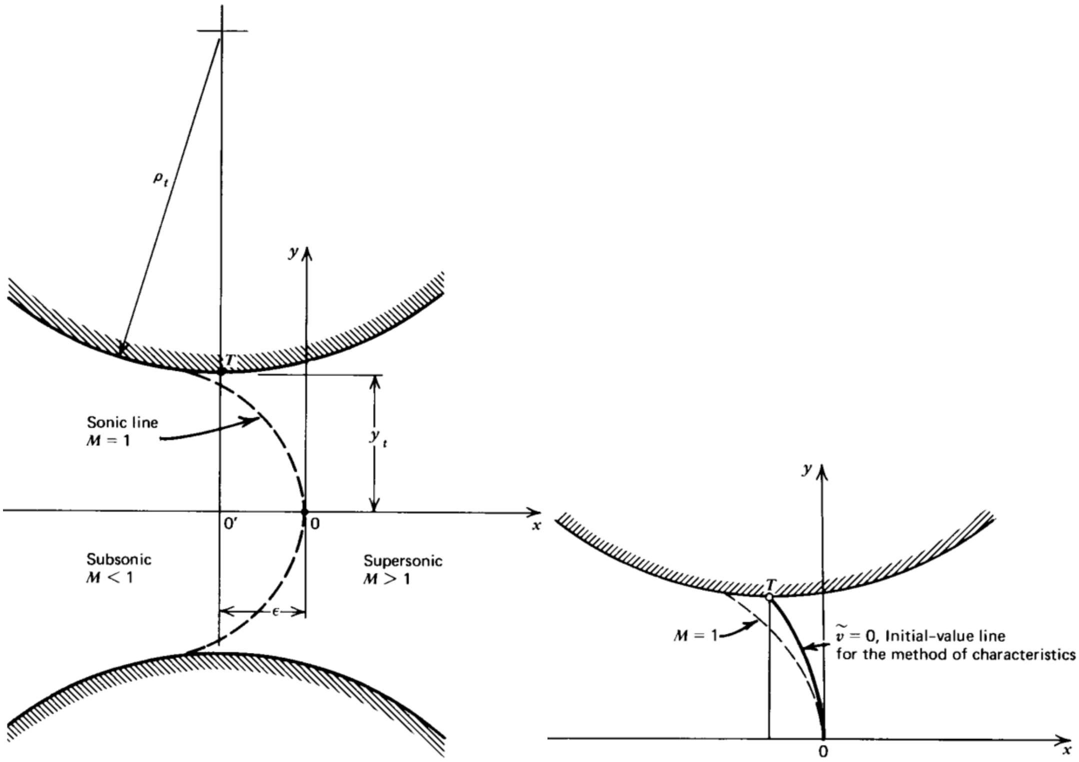
$$\epsilon = -\frac{(\gamma + 1)\alpha y_t^2}{2(3 + \delta)} \tag{1.8}$$

where α is a constant (termed the coefficient of the linear non-dimensional axial perturbation velocity).

$$\alpha = \sqrt{\frac{1 + \delta}{(\gamma + 1)\rho_{tu} y_t}} \tag{1.9}$$

The equation of the line $v = 0$ for $0 \leq y \leq y_t$ in the frame of reference specified in Fig.1.3b is

$$x = -\frac{(\gamma + 1)\alpha y^2}{2(3 + \delta)} \tag{1.10}$$



(a) Nozzle throat geometry and coordinate system **(b)** Initial-value line $v = 0$

Figure 1.3: Nozzle throat geometry and coordinate system for transonic flow analysis and initial-value line $v = 0$.

Along this line the u-velocity component is initiated to the value

$$u(x, y) = \alpha x + \frac{(\gamma + 1)\alpha^2 y^2}{2(1 + \delta)} \tag{1.11}$$

The flow being isentropic the sonic speed is function of the magnitude of the velocity $a = a(u, v)$:

$$a(u, v) = \sqrt{\gamma RT - (\gamma - 1)(u^2)/2} \tag{1.12}$$

Knowing the sonic speed and the velocity magnitude, the Mach number is easily calculated with Eq. (1.4). The static thermodynamic properties are then computed through the relations (1.5).

The x-coordinate must be corrected by the shifting ϵ in order to match the global system of coordinates centered at the throat (see Fig.1.1):

$$x_{\text{nozzle}} = x - \epsilon \tag{1.13}$$

1.2.2 Internal point

Figure 1.4 summarizes the situation. The solution is known at points 1 and 2 and must be calculated at point 4. For this purpose one computes the intersection between the left-running characteristic C_+ emanating from point 2 and the right-running characteristics C_- emanating from point 1.

A first location of the intersection is estimated by assuming that the two characteristics are straight line segments. This allow to determine a first estimation of the solution at point 4. This first step is termed the predictor step. A better guess for the location of point 4 is then

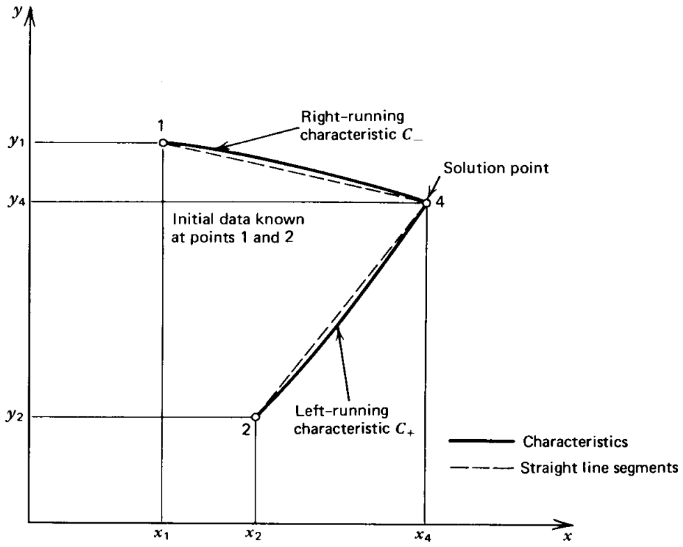


Figure 1.4: Schematic illustration of the characteristics in the two-dimensional space.

iteratively determined by averaging the solution of nodes 2 and 4. This is the corrector step. Equations (1.6) and (1.7) are discretized at points 1, 2 and 4 for this purpose. The following steps solved to get the location and solution at point 4:

$$\theta_\pm = \tan^{-1}\left(\frac{v_\pm}{u_\pm}\right) \tag{1.14a}$$

$$V_\pm = \sqrt{u_\pm^2 + v_\pm^2} \tag{1.14b}$$

$$a_\pm = a(u_\pm, v_\pm) \tag{1.14c}$$

$$\alpha_\pm = \sin^{-1}\left(\frac{a_\pm}{V_\pm}\right) \tag{1.14d}$$

$$\lambda_\pm = \tan\left(\theta_\pm \pm \alpha_\pm\right) \tag{1.14e}$$

$$Q_\pm = u_\pm^2 - a_\pm^2 \tag{1.14f}$$

$$R_\pm = 2u_\pm v_\pm - \left(u_\pm^2 - a_\pm^2\right)\lambda_\pm \tag{1.14g}$$

$$S_\pm = \delta\frac{a_\pm^2 v_\pm}{y_\pm} \tag{1.14h}$$

The definition of the sonic speed $a_\pm$ is given in Eq. (1.1). The location of point 4 emerges from the solving of

$$\begin{pmatrix} -\lambda_+ & 1 \\ -\lambda_- & 1 \end{pmatrix}\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} y_2 - \lambda_+ x_2 \\ y_1 - \lambda_- x_1 \end{pmatrix} \tag{1.15}$$

The velocity components at point 4 are then calculated by solving

$$\begin{pmatrix} Q_+ & R_+ \\ Q_- & R_- \end{pmatrix}\begin{pmatrix} u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} S_+(x_4 - x_2) + Q_+ u_2 + R_+ v_2 \\ S_-(x_4 - x_1) + Q_- u_1 + R_- v_1 \end{pmatrix} \tag{1.16}$$

1.2.2.1 Predictor step

As a first step the values for the flow properties $u_\pm$, $v_\pm$ and $y_\pm$ are set equal to their values at points 2 and 1 respectively. This provides a first estimation of the location for point 4 (x_4^0, y_4^0) and the velocity (u_4^0, v_4^0) at this point. This initial estimation then feeds the corrector step.

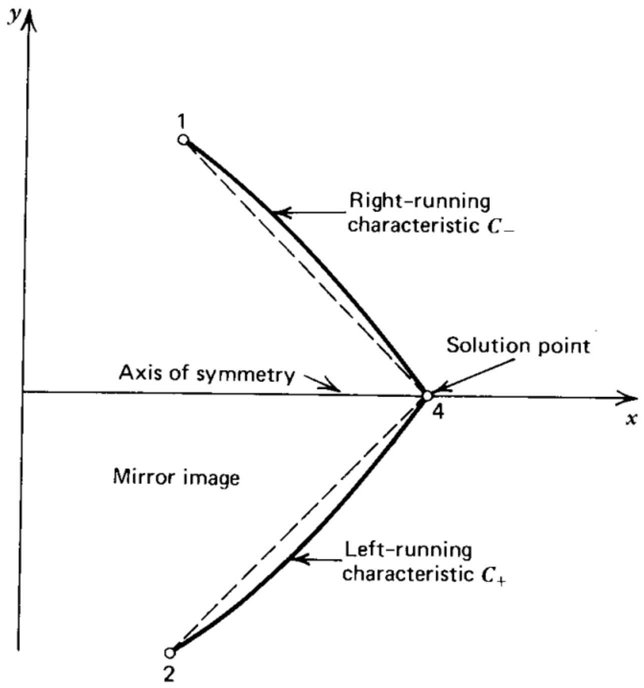


Figure 1.5: Schematic illustration of the characteristics in the two-dimensional space in the case of a point located on the axis of symmetry.

1.2.2.2 Corrector step

Each corrector step follows the same steps as described above except that the values are averaged between points 1, 2 and 4:

$$x_- = x_1 \qquad x_+ = x_2 \tag{1.17}$$

$$y_- = \frac{y_1 + y_4}{2} \qquad y_+ = \frac{y_2 + y_4}{2} \tag{1.18}$$

$$u_- = \frac{u_1 + u_4}{2} \qquad u_+ = \frac{u_2 + u_4}{2} \tag{1.19}$$

$$v_- = \frac{v_1 + v_4}{2} \qquad v_+ = \frac{v_2 + v_4}{2} \tag{1.20}$$

This provides the values for (x_4^n, y_4^n) and the velocity (u_4^n, v_4^n), which are required to estimate the solution (x_4^{n+1}, y_4^{n+1}) and the velocity (u_4^{n+1}, v_4^{n+1}) at the next iteration level. The iterative process is stopped once the difference in the position and velocity is smaller than a threshold or when the number of iterations exceeds a maximum value.

Function MOC_2D_steady_irrotational_internal_point.m implements the algorithm detailed hereabove.

1.2.3 Point on axis

The case where point 4 is located on the axis of symmetry is a particularization of the previous section. Indeed one sees on Fig. 1.5 that point 2 is located outside of the computational domain and is the mirror image of point 1. An additional constraint is that $y_4 = v_4 = \theta_4 = 0$. Thus the same algorithm is used as in the previous section, with the specification that $x_2 = x_1$, $y_2 = -y_1$, $u_2 = u_1$ and $v_2 = -v_1$.

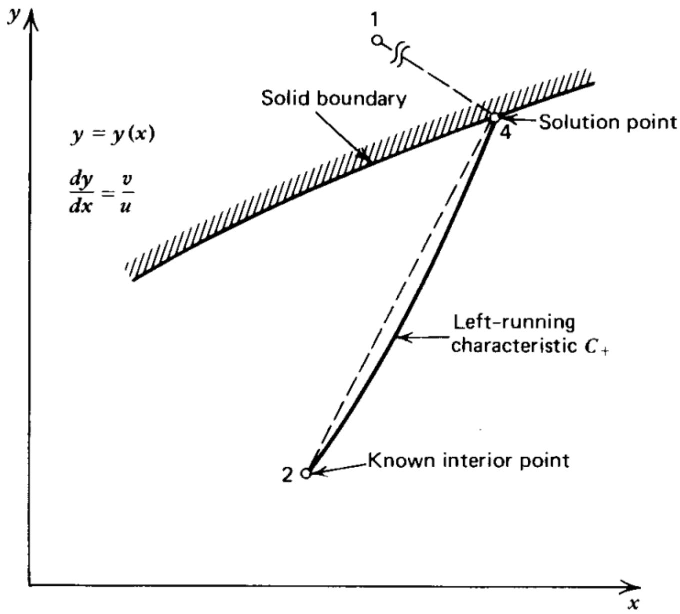


Figure 1.6: Schematic illustration of the characteristics in the two-dimensional space in the case of a point located on the wall, direct method.

1.2.4 Wall point - Direct method

At the wall point 4 the direction of the velocity vector must be identical to the local slope of the wall (see Fig. 1.6). The direct method makes no assumption on the location of the wall point and extends *directly* the C_+ characteristic from the known interior point 2 until it meets the wall. Point 1 does not exist in this case. The characteristic and compatibility equations from the C_+ are completed by two additional conditions for determining the location and flow properties at point 4:

$$y_4 = y(x_4) \qquad \text{specified on the wall} \tag{1.21}$$

$$\frac{dy}{dx} = \tan(\theta_4) = \frac{v_4}{u_4} \qquad \text{specified on the wall} \tag{1.22}$$

The geometry of the nozzle is discussed in section 1.1 but can be modified at wish in function MOC_2D_steady_irrotational_get_geometry.m. This function returns the ordinate y and the slope of the wall at a specified abscissae x and the values of the coefficients a, b and c for the second-order polynomial $y = a + bx + cx^2$.

Equations (1.14) can be reused to obtain the parameters of the left-running characteristic C_+. The location of point 4 (x_4, y_4) is obtained by intersecting the C_+ characteristic and the 2nd-order polynomial:

$$y_4 = \lambda_+ x_4 + y_2 - \lambda_+ x_2 \tag{1.23}$$

$$y_4 = a + bx_4 + cx_4^2 \tag{1.24}$$

which gives the only acceptable solution

$$x_4 = \frac{\lambda_+ - b - \sqrt{(\lambda_+ - b)^2 - 4c(a - y_2 + \lambda_+ x_2)}}{2c} \tag{1.25}$$

In the particular case where $c = 0$ (i.e. $\theta_a = \theta_e$, the nozzle is a straight line), then

$$x_4 = \frac{a - y_2 + \lambda_+ x_2}{\lambda_+ - b} \tag{1.26}$$

The velocity components at point 4 are then calculated by solving

$$\begin{pmatrix} Q_+ & R_+ \\ b + 2cx_4 & -1 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} S_+(x_4 - x_2) + Q_+ u_2 + R_+ v_2 \\ 0 \end{pmatrix} \tag{1.27}$$

1.2.4.1 Predictor step

As a first step the values for the flow properties u_+, v_+ and y_+ are set equal to their values at points 2. This provides a first estimation of the location for point 4 (x_4^0, y_4^0) and the velocity (u_4^0, v_4^0) at this point. This initial estimation then feeds the corrector step.

1.2.4.2 Corrector step

Each corrector step follows the same steps as described above except that the values are averaged between points 2 and 4:

$$x_+ = x_2 \tag{1.28}$$

$$y_+ = \frac{y_2 + y_4}{2} \tag{1.29}$$

$$u_+ = \frac{u_2 + u_4}{2} \tag{1.30}$$

$$v_+ = \frac{v_2 + v_4}{2} \tag{1.31}$$

This provides the values for (x_4^n, y_4^n) and the velocity (u_4^n, v_4^n), which are required to estimate the solution (x_4^{n+1}, y_4^{n+1}) and the velocity (u_4^{n+1}, v_4^{n+1}) at the next iteration level. The iterative process is stopped once the difference in the position and velocity is smaller than a threshold or when the number of iterations exceeds a maximum value.

Function MOC_2D_steady_irrotational_wall.m implements the algorithm detailed hereabove.

1.2.5 Wall point - Inverse method

The previous section focussed on the case where the location of the wall point is found by extending the characteristic from an internal point. This method can lead in too sparse wall points in regions where the gradients are extremely large. One can resort to the inverse wall point method to prespecify spatial spacing along the wall and employing the method of characteristics for computing the flow properties at the prespecified wall points.

This is illustrated in Fig. 1.7. Points 1 and 3 are known from previous calculations, point 4 is the prespecified wall point and point 2 is the intersection between the right-running characteristic C_- 13 and the left-running characteristic C_+ emanating from point 4. The flow properties at point 2 are then interpolated between points 1 and 3. The flow properties at point 4 are determined with the first line of the system of equations (1.16) and the condition (1.22).

The slope of the characteristic 24 depends on the unknown flow properties at points 2 and 4. A predictor-corrector method is thus required to locate point 2 during each step of the global modified Euler predictor-corrector algorithm:

- Initialize (x_+, y_+, u_+, v_+), see the next subsections on the global predictor and corrector steps.

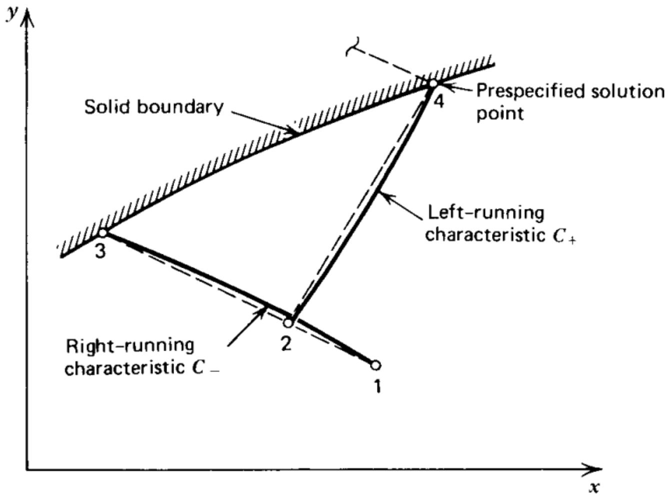


Figure 1.7: Schematic illustration of the characteristics in the two-dimensional space in the case of a point located on the wall, inverse method.

- While the values (x_2, y_2, u_2, v_2) did not converge, do the following steps:

$$\theta_+ = \tan^{-1}\left(\frac{v_+}{u_+}\right) \tag{1.32}$$

$$V_2 = \sqrt{u_+^2 + v_+^2} \tag{1.33}$$

$$a_+ = a(u_+, v_+) \tag{1.34}$$

$$\alpha_+ = \sin^{-1}\left(\frac{a_+}{V_+}\right) \tag{1.35}$$

$$\lambda_+ = \tan(\theta_+ + \alpha_+) \tag{1.36}$$

$$\lambda_- = \frac{y_3 - y_1}{x_3 - x_1} \tag{1.37}$$

Solve the system of equations

$$\begin{pmatrix} -\lambda_- & 1 \\ -\lambda_+ & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 - \lambda_- x_1 \\ y_4 - \lambda_+ x_4 \end{pmatrix} \tag{1.38}$$

Interpolate the data

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \frac{x_2 - x_1}{x_3 - x_1} \begin{pmatrix} u_3 - u_1 \\ v_3 - v_1 \end{pmatrix} \tag{1.39}$$

- Once (x_2, y_2, u_2, v_2) have converged, compute the data at point 4:

$$Q_+ = u_2^2 - a_2^2 \tag{1.40}$$

$$R_+ = 2u_2 v_2 - Q_+ \lambda_+ \tag{1.41}$$

$$S_+ = \delta \frac{a_+^2 v_2}{y_2} \tag{1.42}$$

Solve the system of equations

$$\begin{pmatrix} Q_+ & R_+ \\ \tan\theta_4 & -1 \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} S_+(x_4 - x_2) + Q_+ u_2 + R_+ v_2 \\ 0 \end{pmatrix} \tag{1.43}$$

1.2.5.1 Global predictor step

As a first step the values for the flow properties u_2 and v_2 are set equal to their values at point 3, which means $u_+ = u_3$ and $v_+ = v_3$. Following the procedure explained hereabove provides a first estimation of the location for point 2 (x_2^0, y_2^0) and the flow field $(u_2^0, v_2^0, u_4^0, v_4^0)$. This initial estimation then feeds the corrector step.

1.2.5.2 Global corrector step

Each corrector step follows the same steps as described above except that the values are averaged between points 2 and 4:

$$u_+ = \frac{u_2 + u_4}{2} \qquad\qquad v_+ = \frac{v_2 + v_4}{2} \tag{1.44}$$

This provides the values for (x_2^n, y_2^n) and the flow field $(u_2^n, v_2^n, u_4^n, v_4^n)$, which are required to estimate the solution at the next iteration level. The iterative process is stopped once the difference in the position and velocity is smaller than a threshold or when the number of iterations exceeds a maximum value.

Function MOC_2D_steady_irrotational_wall_inverse.m implements the algorithm detailed hereabove.

1.3 Solution strategy

1.3.1 Extent of the initial-value problem

1.3.2 Extent of the flow field determined by the initial-expansion contour

1.3.3 Complete flow field determined by the nozzle contour

Bibliography

[1] Zucrow, Maurice J. and Hoffman, Joe D., Gas Dynamics Volume 2: Multidimensional Flow, John Wiley and Sons, 1977.