Cauchy's Mean Value Theorem

A generalized case of Mean Value Theorem

Theorem 1 (Cauchy's Mean Value Theorem)**.**

Assume that $f(x)$ *and* $g(x)$ *are continuous on a closed interval* $[a, b]$ *and differentiable on* (a, b)*. Assume that* $g'(x) \neq 0$ *on* (a, b)*. Then there must exist at least one* $c \in (a, b)$ *s.t.*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. First note that $g(x)$ satisfies the conditions needed to apply Mean Value Theorem. Hence, there is some $r \in (a, b)$ that satisfies

$$g'(r) = \frac{g(b) - g(a)}{b - a}$$

Our assumption is that $g'(x)$ is never 0, so $g'(r) \neq 0$ It follows that $g(b) - g(a) \neq 0$. Let's construct a new function $h(x)$ where Rolle's Theorem applies. We set

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x)$$

And note that

$$h(a) = h(b) = \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)}$$

Since $h(a) = h(b)$ we can apply Rolle's theorem which states there must be some $c \in (a, b)$ s.t. $g'(c) = 0$. Hence

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) = 0 \implies \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

□