MAT185 – Linear Algebra
Assignment 2

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Student number: _____

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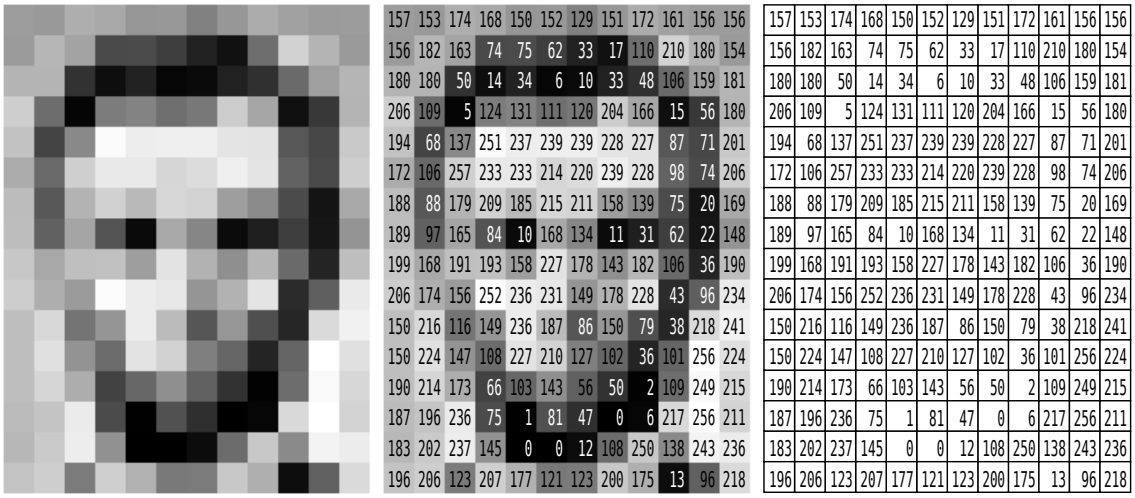
Preamble:

Image processing is the manipulation of digital images by applying mathematical tools and algorithms. A wide range of applications based on digital image processing are, for example, medical imaging, image optimization in consumer cameras, computer vision, and satellite imagery. Linear algebra, and the techniques you will learn during this course, plays a crucial role in that field by providing a mathematical foundation.

A typical digital image can be considered as a 2-dimensional matrix of *pixels* (abbreviation for *picture element*). Methods of digital image processing are manipulating this 2D-matrix. A pixel is the smallest element of a digitally acquired raster image, and can be considered as a colour sample at each point of an image. Typically, each pixel is represented by 3 positive integer values from 0 to 255 for each colour red, green, and blue: 0 is representing black and 255 ($= 2^8 - 1$) either red, green, or blue. In that case, 24 Bits are used to code 16,777,216 distinct colours for each pixel. This is called a 24 bpp (24 Bits-per-pixel) colour depth. If a grey-scale image is sampled, each pixel samples the light intensity. In that case, only 8 Bits (single integers from 0 to 255) are typically necessary, as shown in Figure 1, to store grey-scale images.



157	153	174	168	150	152	129	151	172	161	156	156
156	182	163	74	75	62	33	17	110	210	180	154
180	180	50	14	34	6	10	33	48	106	159	181
206	109	5	124	131	111	120	204	166	15	56	180
194	68	137	251	237	239	239	228	227	87	71	201
172	106	257	233	233	214	220	239	228	98	74	206
188	88	179	209	185	215	211	158	139	75	20	169
189	97	165	84	10	168	134	11	31	62	22	148
199	168	191	193	158	227	178	143	182	106	36	190
206	174	156	252	236	231	149	178	228	43	96	234
150	216	116	149	236	187	86	150	79	38	218	241
150	224	147	108	227	210	127	102	36	101	256	224
190	214	173	66	103	143	56	50	2	109	249	215
187	196	236	75	1	81	47	0	6	217	256	211
183	202	237	145	0	0	12	108	250	138	243	236
196	206	123	207	177	121	123	200	175	13	96	218

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156	182	163	74	75	62	33	17	110	210	180	154
180	180	50	14	34	6	10	33	48	106	159	181
206	109	5	124	131	111	120	204	166	15	56	180
194	68	137	251	237	239	239	228	227	87	71	201
172	106	257	233	233	214	220	239	228	98	74	206
188	88	179	209	185	215	211	158	139	75	20	169
189	97	165	84	10	168	134	11	31	62	22	148
199	168	191	193	158	227	178	143	182	106	36	190
206	174	156	252	236	231	149	178	228	43	96	234
150	216	116	149	236	187	86	150	79	38	218	241
150	224	147	108	227	210	127	102	36	101	256	224
190	214	173	66	103	143	56	50	2	109	249	215
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Figure 1: Grey-scale image represented by 8 bpp (Bits-per-pixel). [*source: stanford.edu*]

One application of linear algebra in the context of digital image processing is for a linear transformation of the images / 2D-matrices. We will discuss linear transformations in more detail later this term. Common transformations include scaling, rotation, and translation of the image. All of these linear transformations can be represented by a matrix multiplication.

Another important application of linear algebra in the context of digital image processing is filtering, which will be explored in Question 1 of this assignment. Filtering includes, for example, methods for noise reduction, blurring/sharpening, edge detection, white balancing, colour correction, and many more.

Lastly, images have to be stored in efficient ways. For a colour image of size $n \times m$ with a 24 bpp colour depth, $m \times n \times 24$ Bits of memory are necessary. An image of 8000×6000 Pixels (4:3 aspect ratio and 48MP resolution of modern smartphone cameras) with 24 Bit colour depth will lead to an uncompressed image filesize of 144 MB. To reduce the image filesize, lossy (permanently removing *unnecessary* information of the original image) and lossless compression methods are applied. For lossy compression algorithms, it is vital to identify unnecessary information of the image, which are not perceived by the viewer and can be removed. One approach for a lossy compression is based on the Singular Value Decomposition (SVD), which will be explored in more detail in Question 2.

Question 1:

For simplicity, we assume that each pixel of a grey-scale image is considered as a real value in Question 1(a) to (e). Let $G \in {}^{n}\mathbb{R}^{m}$ be a grey-scale image of $n \times m$ pixels. A collection of image filters $F_1, F_2, \ldots, F_k \in {}^{n}\mathbb{R}^{m}$ can be used to process this image, resulting in filtered images A_l represented as:

$$A_l = F_l \circ G, \quad l = 1, 2, \ldots, k, \tag{1}$$

where $\circ$ denotes the entry-wise product. The entry-wise product (also called *Hadamard* product) of two matrices of the same size is defined as the product, where each entry of the resulting matrix is the product of the corresponding entries of the original matrices. For example, if $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ and $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ are 2×2 matrices, the Hadamard product is defined as

$$P \circ Q = \begin{bmatrix} p_{11}q_{11} & p_{12}q_{12} \\ p_{21}q_{21} & p_{22}q_{22} \end{bmatrix}. \tag{2}$$

(a) Can the filter F_1 in Equation (1) be used to blur or smooth an image A_1? Unsupported answers will not receive full credit.

Yes. To smoothen or blur an image, neighboring pixels to some target pixel should be similar in value, the value representing light intensity. A filter F_1 can adjust this light intensity value of every entry in A_1 to provide the desired effect.
To illustrate this, if you set $F_1 = \mathbf{0}$ then $F_1 \circ A_1 = \mathbf{0}$, so the entire image appears black, or completely smoothened out.

(b) Name at least **three** applications in the context of digital image processing for a filter as defined in Equation (1). Additionally, give details how F_i would look like for your applications.

1. Blacking out parts of an image (i.e. faces)
   Every entry in F_i should be 1, except for the entries corresponding to the pixels that make up someone's face. Those entries should be 0 to create the black-out effect around a face.
2. "Flipping" an image
   To flip an image, F_i needs to retain some information about G. Specifically, for a flip across the center horizontal line of an image of dimensions $n \times k$ represented by the matrix G, set $f_{x,y} = g_{x,(k-y)}/g_{x,y}$, where $f_{x,y}$ and $g_{x,y}$ are the entries x^{th} row and y^{th} in F_i and G respectively. In essence, $F_i \circ G$ divides the (x, y) entry in G with itself, and multiplies by the entry on the other side of the image.
3. Vignette effect
   In a vignette effect, pixels towards the edges of an image are darkened. To achieve this, multiply all entries towards the edges in matrix G by some number < 1 to decrease the light intensity of the pixel. Construct F_i so that all its entries $f_{\text{x,y}} \leq 1$, with values smaller than 1 at the "edge" entries of F_i, with values gradually increasing towards 1 at the center entry of F_i (i.e. $f_{(n/2),(k/2)}$ for an $n \times k$ matrix G.)

(c) Assume that the entries of the image G are all nonzero. Prove that the set of filtered images $\{F_1 \circ G, F_2 \circ G, \ldots, F_k \circ G\}$ is linearly independent if and only if the set of filters $\{F_1, F_2, \ldots, F_k\}$ is linearly independent.

First consider if $\{F_1, F_2, ..., F_k\}$ is linearly independent then $\{F_1 \circ G, F_2 \circ G, \ldots, F_k \circ G\}$ is linearly independent. From the definition of linear independence:

$$\sum_{i+1}^{k} \lambda_i F_i = \mathbf{0}, \lambda_i \in \mathbb{R}$$

only has one solution, all $\lambda_i = 0$. Now consider each entry in the matrix F_i. Since $\lambda_i F_i = \mathbf{0}$ it follows that any entry in the x^{th} row and y^{th} column in F_i, call it $f^i_{x,y}$, must be 0, so $\lambda_i f^i_{x,y} = 0$.
We now consider the set $\{F_1 \circ G, F_2 \circ G, \ldots, F_k \circ G\}$. If this set is linearly independent, then

$$\sum_{i+1}^{k} \gamma_i F_i \circ G = \mathbf{0}, \gamma_i \in \mathbb{R}$$

Once again, we consider the value of the x, yth entry in the matrix $F_i \circ G$. Since we constrain $F_i \circ G = \mathbf{0}$ to test for linear independence, by the definition of the Hadamard product, the x, yth entry of that matrix is simply $\gamma_i g_{x,y} f^i_{x,y} = 0$.
Performing algebraic manipulations:

$$\gamma_i g_{x,y} f^i_{x,y} = 0$$
$$\gamma_i f^i_{x,y} = 0$$

Note we can divide through by $g_{x,y}$ since all entries in G are nonzero.
But we showed earlier that for $\lambda_i f^i_{x,y} = 0$, $\lambda_i = 0$ was the only solution to $\sum_{i=1}^{k} \lambda_i F_i = 0$. It follows that if we set $\gamma_i = \lambda_i$ (they both are, after all some constant value), then $\gamma_i = 0$, as required for linear independence. It follows from this that the set $\{F_1 \circ G, F_2 \circ G, \ldots, F_k \circ G\}$ is linearly independent if the set $\{F_1, F_2, ..., F_k\}$ is linearly independent.
Next, we show that if $\{F_1 \circ G, F_2 \circ G, \ldots, F_k \circ G\}$ is linearly independent, then $\{F_1, F_2, ..., F_k\}$ is linearly independent. By definition of linear independence,

$$\sum_{i+1}^{k} \phi_i F_i \circ G = \mathbf{0}, \phi_i \in \mathbb{R}$$

so every x, yth entry in $\phi_i F_i \circ G$ equals 0 i.e. $\phi_i g_{x,y} f^i_{x,y} = 0$. Next we consider the set $\{F_1, F_2, ..., F_k\}$. This set is linearly independent if

$$\sum_{i+1}^{k} \theta_i F_i = \mathbf{0}, \theta_i \in \mathbb{R}$$

so every x, yth entry in $\theta_i F_i$ equals. But we know that:

$$\phi_i g_{x,y} f^i_{x,y} = 0$$
$$\phi_i f^i_{x,y} = 0$$

Dividing through by $g_{x,y}$ will not change the solution set, as learned in ESC103, so $\phi_i = 0$ is still the sole solution to that equation. If we let $\theta_i = \phi_i$ we get that $\theta_i = 0$ is the only solution to that equation so it follows that any θ_i in the equation:

$$\sum_{i+1}^{k} \theta_i F_i = \mathbf{0}, \theta_i \in \mathbb{R}$$

must equal 0, satisfying linear independence. Hence, if $\{F_1 \circ G, F_2 \circ G, \ldots, F_k \circ G\}$ is linearly independent, then $\{F_1, F_2, ..., F_k\}$ is also linearly independent.
We conclude that $\{F_1, F_2, ..., F_k\}$ is linearly independent if and only if $\{F_1 \circ G, F_2 \circ G, \ldots, F_k \circ G\}$ is linearly independent.

(d) Assume that the entries of the image G are all nonzero. Let $\mathcal{W}$ be the set of the filtered images $\{F_1 \circ G, F_2 \circ G, \ldots, F_k \circ G\}$. Suppose a new image filter F_{k+1} is introduced. Prove that the filtered image $F_{k+1} \circ G$ lies in $\operatorname{span} \mathcal{W}$ if and only if F_{k+1} is a linear combination of $\{F_1, F_2, \ldots, F_k\}$.

(e) Assume $F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $F_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $F_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ are filters for a grey-scale image $G \in {}^2\mathbb{R}^2$. Verify whether the filters F_1, F_2, F_3 are linearly independent, and determine whether $F_4 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ lies in the span of $\{F_1, F_2, F_3\}$.

By the definition of linear independence, $\{F_1, F_2, F_3\}$ are linearly independent if

$$c_1F_1 + c_2F_2 + c_3F_3 = \mathbf{0}$$

for some $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1 = c_2 = c_3 = 0$.
Taking the sum of $c_1F_1 + c_2F_2 + c_3F_3 = \mathbf{0}$ shows that:

$$\begin{aligned} c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ c_2 & 0 \end{bmatrix} + \begin{bmatrix} c_3 & c_3 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} c_1 + c_3 & c_2 + c_3 \\ c_2 & c_1 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \implies c_2 = c_1 = 0 \text{ so } c_3 = 0 \end{aligned}$$

Since $c_1 = c_2 = c_3 = 0$ the set $\{F_1, F_2, F_3\}$ is linearly independent.

Next, consider the matrix $F_4 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$. We know some scalar combination of $\{F_1, F_2, F_3\}$ yields $\begin{bmatrix} c_1 + c_3 & c_2 + c_3 \\ c_2 & c_1 \end{bmatrix}$, $c_1, c_2, c_3 \in \mathbb{R}$. Solve for c_1, c_2, c_3 using the matrix entries. We get that $c_1 = 4$ and $c_2 = 1$ directly. This means $c_1 + c_3 = 2 \implies c_3 = -2$ and that $c_2 + c_3 = 1 \implies c_3 = 0$ Since $2 \neq 0$, the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ cannot be expressed as a linear combination of F_1, F_2 and F_3. Hence, $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ is outside the span of $\{F_1, F_2, F_3\}$

(f) For saving grey-scale images A of size $n \times m$ pixels in an efficient way, each pixel is stored as an 8 Bit integer, where 0 represents black and 255 represents white (see preamble). Let $\mathcal{V}$ be the set of these grey-scale images. Is $\mathcal{V}$ a vector space if one uses entry-wise vector addition and scalar multiplication?

$\mathcal{V}$ is not a vector space. It fails under closure over vector addition.

For instance, consider two $n \times m$ matrices, $A, B \in \mathcal{V}$ for some indices i, j, $1 \leq 1 \leq n$ and $1 \leq j \leq m$. Consider the i^{th} row and j^{th} column entry in A and B - call it $a_{i,j}$ and $b_{i,j}$ respectively. If $a_{i,j} = 255$ and $b_{i,j} = 255$ then when you compute the i, jth entry of $A + B$ to be $a_{i,j} + b_{i,j} = 510$ which cannot be stored as an 8 bit integer so it is not in $\mathcal{V}$. Hence, $\mathcal{V}$ is not a vector space.

Question 2:

Let $A \in {}^m\mathbb{R}^n$ be a grey-scale image of size $m \times n$, where each entry is representing a grey pixel. As you learnt in ESC103, the rank of an $n \times m$ matrix is the number of linear independent rows, which is the same as the number of the linear independent columns.

The Singular Value Decomposition[1] of a matrix A is a way of writing A as the product of three matrices:

$$A = U\Sigma V^T \tag{3}$$

with the matrices U, Σ, and V^T defined as

$$A = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \cdots & \mathbf{u_{n-1}} & \mathbf{u_r} \end{bmatrix} \begin{bmatrix} \sigma_1 & & \cdots & & 0 \\ & \sigma_2 & & & \\ \vdots & & \ddots & & \vdots \\ & & & \sigma_{r-1} & \\ 0 & & \cdots & & \sigma_r \end{bmatrix} \begin{bmatrix} \mathbf{v_1}^T \\ \mathbf{v_2}^T \\ \vdots \\ \mathbf{v_{r-1}}^T \\ \mathbf{v_r}^T \end{bmatrix} \tag{4}$$

where $\mathbf{u}_i$ are the columns of U and $\mathbf{v}_i^T$ are the rows of V^T. The matrix Σ is a diagonal matrix with the entries σ_i on the diagonal (the diagonal entries of Σ are nonnegative and all other entries of Σ are zero). Because Σ is diagonal, Equation (4) can be written as:

$$A = \mathbf{u_1}\sigma_1\mathbf{v_1}^T + \mathbf{u_2}\sigma_2\mathbf{v_2}^T + \cdots + \mathbf{u_{r-1}}\sigma_{r-1}\mathbf{v_{r-1}}^T + \mathbf{u_r}\sigma_r\mathbf{v_r}^T. \tag{5}$$

As you can see, A is separated into the sum of (rank=1) matrices of the form $\sigma_i\mathbf{u_i}\mathbf{v_i}^T$. With that, the sum in Equation (5) can be expressed as

$$A = \sum_{i=1}^{r} \sigma_i \mathbf{u_i}\mathbf{v_i}^T \tag{6}$$

with r so called singular values $\{\sigma_i \in \mathbb{R} \mid \sigma_i > 0\}$ and both $\{\mathbf{u}_1, \ldots \mathbf{u}_r\} \subseteq {}^m\mathbb{R}$ and $\{\mathbf{v}_1, \ldots \mathbf{v}_r\} \subseteq {}^n\mathbb{R}$ are orthonormal sets of vectors.

What's an "orthonormal set of vectors"? In simple language, each vector has length one and any two vectors are perpendicular (also called "orthogonal"). More specifically, two vectors, $\mathbf{x}, \mathbf{y} \in {}^n\mathbb{R}$, are called *orthogonal* if and only if their dot-product is equal to zero: $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y} = 0$. The set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k\} \subseteq {}^n\mathbb{R}$ is an *orthogonal set of vectors* if and only if every pair of vectors is orthogonal: $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ if $i \neq j$. The set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k\} \subseteq {}^n\mathbb{R}$ is an *orthonormal set of vectors* if and only if every pair of vectors is orthogonal and each vector has length 1: $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ if $i \neq j$ and $\mathbf{x}_i \cdot \mathbf{x}_i = 1$ for every $1 \leq i, j \leq k$.

At the end of this course, you will have the tools you need to be able to find the SVD of a general matrix.[2] We can use Equation (5) to compress the image A by keeping the "most relevant" terms in the sum: As you can see from Equation (5), some matrices $\sigma_i\mathbf{u_i}\mathbf{v_i}^T$ could be neglected if σ_i is very small. Therefore, a typical strategy for compressing images is putting the singular values σ_i in nonincreasing order, $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > 0$ (as well as sorting the corresponding $\mathbf{u}_i$ and $\mathbf{v}_i$ in U and V, respectively) and only storing $k < r$ of these components in an image file (k is an integer of at least 1). The compressed image A_k is constructed as

$$A_k = \sum_{i=1}^{k} \sigma_i \mathbf{u_i}\mathbf{v_i}^T \tag{7}$$

This will reduce the image size dramatically, however, the image quality will be degraded if k was picked too small, see for example Figure 2 where 30 components were used ($k = 30$) for the compression of the initial grey-scale image.

[1] In this writing assignment, we are introducing you to the compact (or reduced) SVD, which is slightly different from the full SVD. To save space, we refer to it as the "SVD" throughout.

[2] *For the curious:* $\mathbf{u}_i$ are eigenvectors of the matrix AA^T and $\mathbf{v}_i$ are eigenvectors of the matrix A^TA.

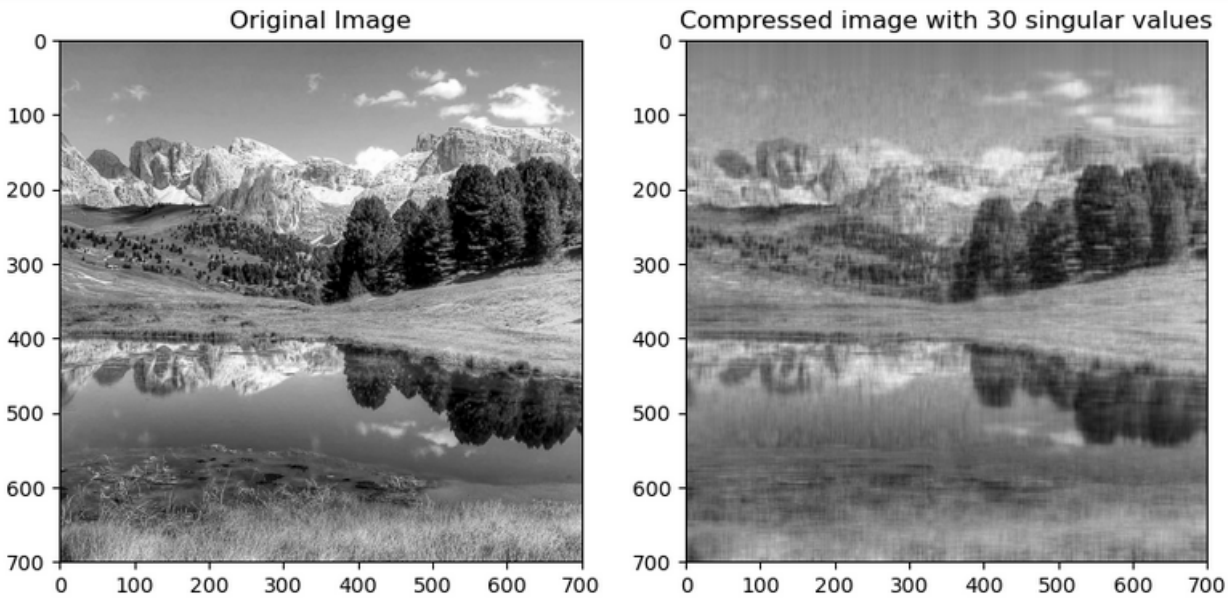

Figure 2: Lossy compression of a 700×700 pixel grey-scale image via Single Value Decomposition (SVD).

(a) For what values of k would A_k corresponds to a lossy compressed image? No additional explanation is necessary.

Any $k \in \mathbb{N}$ smaller than the rank of the diagonal matrix $\sum$.

(b) Let $A = \sigma \mathbf{x}\mathbf{y}^T$ where $\mathbf{x}, \mathbf{y} \in {}^n\mathbb{R}$ and $\sigma \in \mathbb{R}$ is nonzero. Prove that $\text{rank}(A) = 1$. *Hint:* Try constructing a 3×3 example, A, by choosing some nonzero $\sigma \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in {}^3\mathbb{R}$. This should give you the insight you need to answer this question for general $\mathbf{x}, \mathbf{y} \in {}^n\mathbb{R}$ below.

(c) Let $A = \begin{bmatrix} 36 & 9 & 12 \\ -48 & -12 & -16 \\ 144 & 36 & 48 \end{bmatrix}$. Determine the singular value decomposition $A = \sigma \mathbf{x}\mathbf{y}^T$.

(d) Assume $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set of nonzero vectors. Prove that the set is linearly independent.

(e) Consider the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ from part (d). What property of a 4th vector $\mathbf{v}_4 \in {}^4\mathbb{R}$ would ensure that the extended set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ spans ${}^4\mathbb{R}$? Give a short explanation. Note: your explanation does not have to include how you would find such a $\mathbf{v}_4$.

We showed in class that a set spanning ${}^4\mathbb{R}$ contains m linear independent vectors. Thus, a set spanning ${}^4\mathbb{R}$ contains 4 linearly independent vectors. So, we need to add an element $\mathbf{v}_4$ to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that $\mathbf{v}_4 \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, i.e. the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.

(f) Download the Jupyter notebook (see Quercus page of *Assignment 2*), which includes the python code of an SVD analysis. Please also download the three example grey-scale images *image1.png*, *image2.png*, and *image3.png* from the same Quercus page. You can run the code on the U of T Jupyter server, on your own local Jupyter installation, or on the ECF lab PCs. We recommend using the U of T Jupyter server, since all necessary packages are already installed: https://jupyter.utoronto.ca/hub/user-redirect/tree. For that, please login with your U of T credentials and click '*upload*' to upload the IPYNB file and images. After that, you can open the uploaded notebook and run the Python code.

Use the Jupyter Python code to analyze the provided three images. Plot the sorted singular values σ_i over the index i in one graph and discuss the results for all three provided images. Do you see a pattern of the result depending on each input image?

Insert your plot below. In addition, briefly discuss which of the 3 images would allow for the highest and which for the lowest compression ratio by singular value decomposition. *Hint:* The compression ratio is the uncompressed file size over the compressed file size. You don't need to calculate this ratio. However, think about which image would need the largest k and which the lowest k in Equation (7) for an accurate representation of the uncompressed image, Equation (6). For plotting σ_i, see the comments and plotting commands in the Jupyter file. Also, keep the logarithmic Y-axis for your graph.

(g) Calculate the SVD of *image1.png* with the given Jupyter code. For what choice of $k \in \mathbb{Z}$ would you expect a reduction in file size? To answer this question, estimate the memory needed to store the SVD-compressed image and compare it to the memory needed for the uncompressed 8 bpp version of the image. *Hint:* Estimate the memory size of the uncompressed image by using its size and given colour depth. Also note that each floating-point number uses 32 Bit memory when stored.