Mean Value Theorem

We begin by proving **Rolle's Theorem**

Theorem 1 (Rolle's Theorem)**.**

Let f be a continuous function over a closed interval $[a, b]$ and differentiable over (a, b) such that $f(a) = f(b)$. Then there exists at least one $c \in (a, b)$ s.t. $f'(c) = 0$.

Proof. Let $k = f(a) = f(b)$. We consider three cases.

1. $f(x) = k \; \forall x \in (a, b)$.
2. There is some $x \in (a, b)$ s.t. $f(x) > k$.
3. There is some $x \in (a, b)$ s.t. $f(x) < k$.

Case 1: If $f(x) = k$ then $f'(x) = 0$ for all $x \in (a, b)$ as required Case 2: f is continuous over the closed, bounded interval so we can apply extreme value theorem i.e. it has an absolute maximum. Since there is some $x \in (a, b)$ s.t. $f(x) > k$ then the absolute maximum must be greater than k. It follows that the absolute maximum cannot lie at either endpoint, so there is a maximum at some $c \in (a, b)$. By Fermat's theorem, the derivative of a maximum is 0, so there is some $c \in (a, b)$ where $f'(c) = 0$. Case 3: Same as case 2, except EVT guarantees there will be a minimum value since $f(x) < k$. The □
Rolle's theorem is a special case of the MVT. We can use Rolle's theorem to prove MVT, since it's simpler this way.

Theorem 2 (Mean Value Theorem)**.**

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. We prove MVT by constructing a new function based on $f(x)$ that satisfies Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. The slope of that secant line is

$$\frac{f(b) - f(a)}{b - a}$$

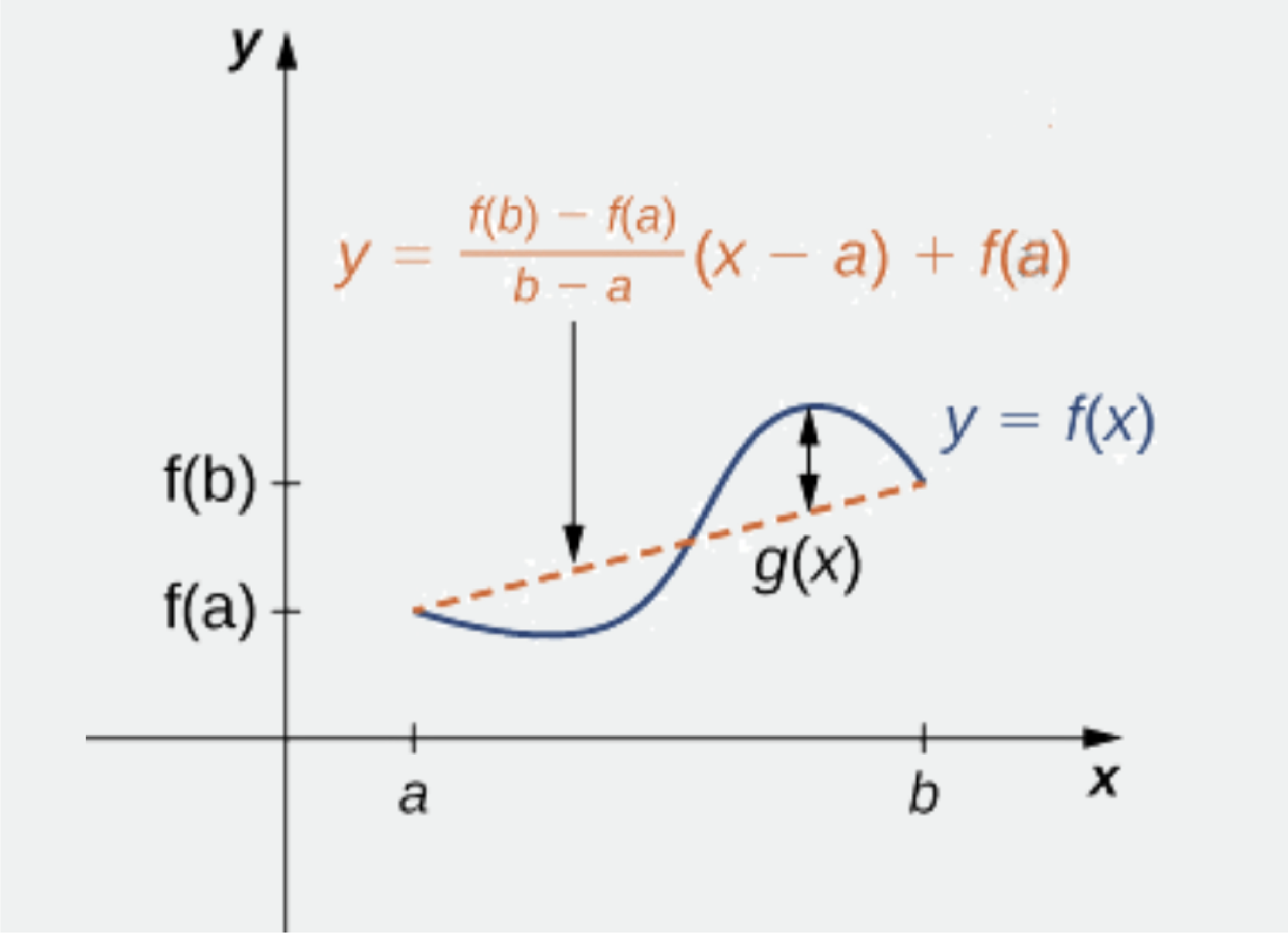
and the equation of that line is

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

(a way to see why this is true: take the graph and translate it a right, or towards the origin. Then the y-int of that line is $f(a)$).

Let $g(x)$ represent the vertical difference between the points $(x, f(x))$ and the points (x, y) on our line. Hence:

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right]$$



Note that $g(a) = g(b) = 0$. $g(x)$ might satisfy Rolle's theorem! Do our checks: $g(x)$ is differentiable since $f(x)$ is differentiable on (a, b). Since $f(x)$ is continuous on $[a, b]$ then $g(x)$ is continuous on $[a, b]$. It follows that there is some point $c \in (a, b)$ s.t. $g'(c) = 0$. If you differentiate $g(x)$:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

It follows that

$$g'(c) = 0 = f'(c) - \frac{f(b) - f(a)}{b - a}$$

so

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

□