Intermediate Value Theorem

Theorem 1 (Intermediate Value Theorem Lemma)**.**

If $f(x)$ is continuous on $[a, b]$ and we choose some $k \in (f(a), f(b))$, there there is some $c \in (a, b)$ where $f(c) = k$.

Proof. We begin by considering a special case, where $f(a) < 0 < f(b)$. Our goal is to prove there exists some c in $[a, b]$ s.t. $f(c) = 0$

We define a set $S = \{x \in [a, b] : f(x) < 0\}$. In other words, our set includes all negative values of f. Note that S contains a since $f(a) < 0$. Since $f(a) < 0$ and continuous by definition, it must be that $[a, a + \delta] \subset S$. You can deduce this by drawing a bunch of graphs. BUTTT!! we want to be RIGOROUSLY LOGICAL.

Since $f(x)$ is continuous to the right of a, then for all x in $a \leq x < a + \delta \implies |f(x) - f(a)| < |f(a)| = -f(a)$. You can convince yourself of this implication by drawing a possible graphs of $f(x)$ but to be more pedantic, we note that this looks like our $\varepsilon - \delta$ definition of a limit:

> If an $\varepsilon > 0$ is imposed, then find some $|x - c| < \delta$ s.t. $|f(x) - L| < \varepsilon$ holds.

$|f(a)|$ is like ε, and is certainly positive. If we rewrite the absolute value inequality without the abs. value, we obtain $f(a) - |f(a)| < f(x) < f(a) + |f(a)| \implies 2f(a) < f(x) < 0$. Since $f(x) < 0$, then $[a, a + \delta]$ in S must exist.

Since S is nonempty and has an upper bound, it has an least upper bound c. Since $a < c < b$, then f is defined at c. What is the value of $f(c)$? We determine this by considering 3 cases: $f(c) < 0$, $f(c) > 0$ or $f(c) = 0$.

Case 1: $f(c) > 0$ If $f(c) > 0$ then there is an interval to the left of f where f is positive. This comes from the definition of continuity; since f is continuous at c then there some δ s.t. $|f(x) - f(c)| < f(c)\ \forall |x - c| < \delta$ (note that $f(c)$ is taken to be ε here). This implies $0 < f(x)$ which means no number in the interval $(c - \delta, c]$ is in S (look back at the definition of S). Since c is an upper bound to S, it follows that each number in $(c - \delta, c]$ is also an upper bound, but that contradicts that c is the LEAST upper bound. Hence $f(c) > 0$ is not possible.

Case 2: $f(c) < 0$ If $f(c) < 0$ then there exists a $\delta > 0$ such that $f(x) > 0$ whenever $c \leq x < c + \delta$. You can show similarly to case 1 that this leads to a contradiction.

By elimination, $f(c)$ must equal 0

You can extend the above idea to show that □

Theorem 2 (Intermediate Value Theorem)**.**

Given f is a continuous real-valued function defined on the closed interval $[a, b]$ and C is a number for which $f(a) < C < f(b)$, then there exists a number $c \in (a, b)$ for which $f(c) = C$.

How to extend: define an auxiliary function $g(x) = f(x) - C$. Then the problem becomes equivalent to Theorem 1.

TODO: sqrt 2 yippie