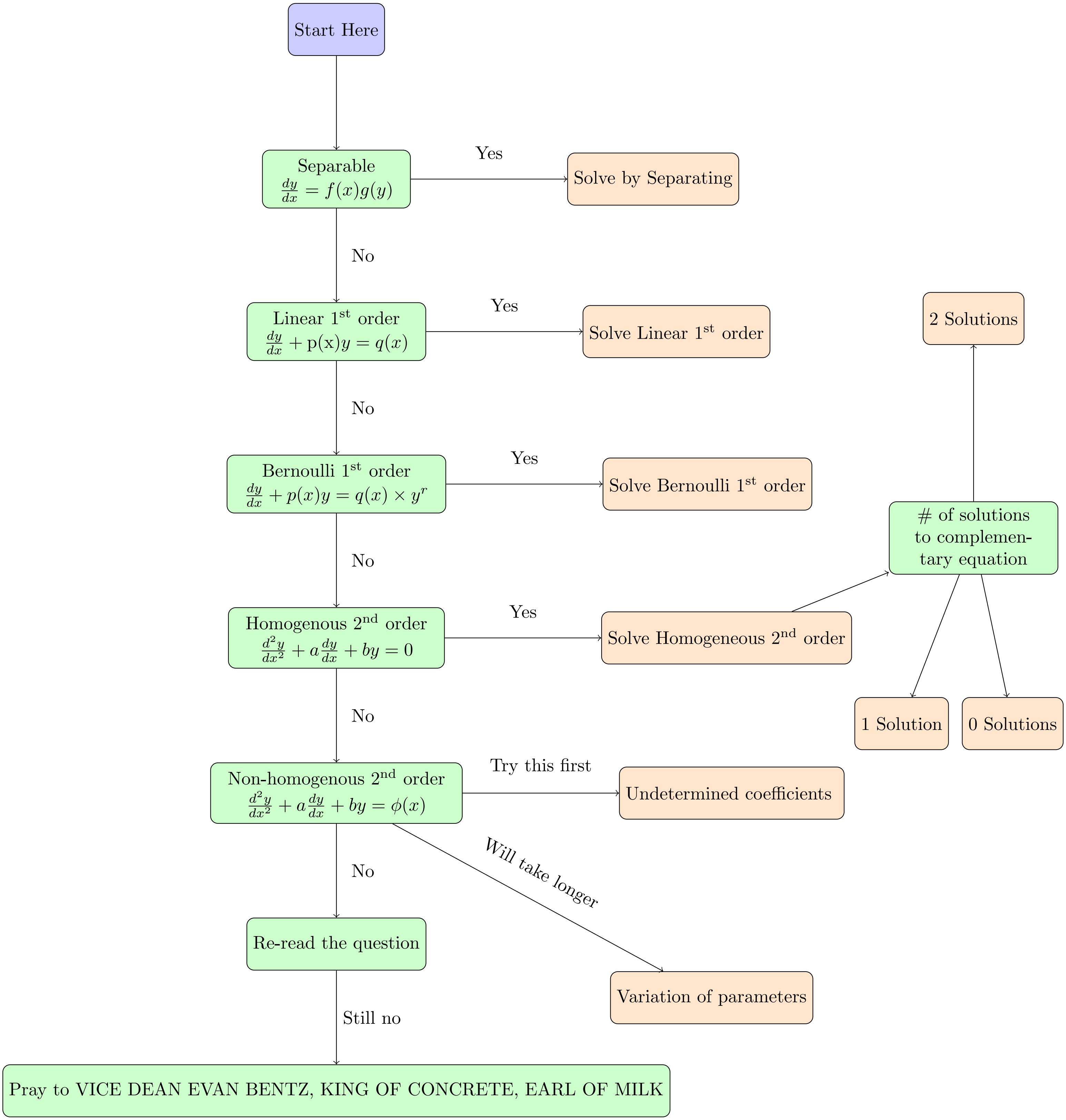
Flowchart inspired (first page mostly copied from) by: GOAT EngSci QILIN XUE!!!
Derivations and examples mostly taken from: lecture or Paul's Online Math Notes



Separable Differential Equations

Background info

If you have an equation like $\frac{dy}{dx} = xy$, pretend $\frac{dy}{dx}$ is a fraction. Rearrange so that you get all the y terms on one side and all the x terms on the other. Then integrate with respect to the variable on each side. The answer will probably be explicit, but could be implicit; if so just leave it like that.

Example:

Problem:

$y' = \frac{xy^3}{\sqrt{1+x^2}}, \quad y(0) = -1$

Steps:

y' is $\frac{dy}{dx}$, then separate the variables:

$$\begin{aligned} \frac{dy}{dx} &= \frac{xy^3}{\sqrt{1+x^2}} \\ \frac{dy}{y^3} &= \frac{x}{\sqrt{1+x^2}}\,dx \end{aligned}$$

Integrate! (u-sub for the right side)

$$\begin{aligned} \int \frac{dy}{y^3} &= \int \frac{x}{\sqrt{1+x^2}}\,dx \\ -\frac{1}{2y^2} &= \sqrt{1+x^2} + C \end{aligned}$$

Apply initial condition to find C:

$$\begin{aligned} -\frac{1}{2(-1)^2} &= \sqrt{1+0^2} + C \\ -\frac{1}{2} &= 1 + C \\ C &= -\frac{3}{2} \end{aligned}$$

Find explicit solution for y(x)

$$\begin{aligned} -\frac{1}{2y^2} &= \sqrt{1+x^2} - \frac{3}{2} \\ \frac{1}{y^2} &= 3 - 2\sqrt{1+x^2} \\ y^2 &= \frac{1}{3 - 2\sqrt{1+x^2}} \\ y &= \pm\sqrt{\frac{1}{3 - 2\sqrt{1+x^2}}} \end{aligned}$$

Note that given the initial condition, we only take the negative root. Also note that the domain of x is restricted.

Sample Problem: (Stewart 9.3.18)

$\frac{dL}{dt} = kL^2 \ln t, \quad L(1) = -1$

Linear 1^{st} Order Differential Equations

Background info:

The equation has to be in this form $\frac{dy}{dx} + p(x)y = q(x)$. Assume there is some function $\mu(x)$ (the integrating factor) has the property $\mu(x)p(x) = \mu'(x)$ (it later turns out that $\mu(x) = e^{\int p(t)dt}$).
Multiply through by $\mu(x)$ to get $\mu(x)\frac{dy}{dx} + \mu'(x)y = \mu(x)q(x)$. The left hand side of this equation is just the product rule of $(\mu(x)y(x))'$, so the equation becomes $(\mu(x)y(x))' = \mu(x)q(x)$. Integrate both sides **and keep the constant of integration from this step onwards**.
Isolate for $y(t)$ to get the solution to the equation:

$$y(t) = \frac{\int \mu(t)q(t)dt - c}{\mu(t)}$$

where $\mu(x) = e^{\int p(t)dt}$. Note the sign of the constant doesn't matter.

Example:

Problem:

$\frac{dv}{dt} = 9.8 - 0.196v, \quad v(0) = 48$

Steps:

$p(t) = 0.196$ and $q(t) = 9.8$ $\mu(t) = e^{\int p(t)\,dt} = e^{\int 0.196\,dt} = e^{0.196t}$
Multiply through to get $\frac{d}{dt}(e^{0.196t}v) = 9.8e^{0.196t}$
Integrate both sides and divide out to find explicit solution for v: $v(t) = 50 + Ce^{-0.196t}$
Use initial condition to find that $C = -2$, so final answer becomes: $v(t) = 50 - 2e^{-0.196t}$

Bernoulli 1^{st} Order Differential Equations

Background info

Given a differential equation in this form $\frac{dy}{dx} + p(x)y = q(x)y^n$, where $n \neq 0$ or 1 (because thats just a linear 1st order differential equation (see above). Divide through by y^n and make the substitution $v = y^{1-n}$. Note $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$, so the equation can be rearranged to

$$\frac{v'(x)}{1-n} + p(x)v = q(x)$$

which is just a linear differential equation (with a different variable!!).

Example:

Problem:

$y' + \frac{4}{x}y = x^3y^2, \quad y(2) = -1$

Steps:

Let $v = y^{1-2} = y^{-1}$ and thus $y' = -v^{-2}v'$
Substitute to get $-v^{-2}v' + \frac{4}{x}v^{-1} = x^3v^{-2}$.
Divide through to get a linear first order equation which has an integrating factor of $e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x} = x^{-4}$
Multiply through to get $x^{-4}v = \int -x^{-1}dx$
Integrate both sides and divide out to find explicit solution for v: $v(x) = x^4(c - lnx)$
We need an explicit solution for y and the value of c, so we'll find both. The equation for y becomes $y^{-1} = x^4(c - lnx)$. Applying the initial condition, we get $c = \ln 2 - \frac{1}{16}$

Homogeneous 2^{nd} Order Differential Equations

Background info:

The equation should be in the form $ay'' + by' + cy = 0$.
The **Principle of Superposition** states that if $y_1(x)$ and $y_2(x)$ are two linearly independent "nice enough" solutions (the Wronskian doesn't disappear or something like that. Irrelevant for 194), then the general solution to the differential equation is $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where c_1, c_2 are some constants.

Complementary equation

Background info:

Set the polynomial to zero and make it a homogeneous equation. **Do NOT find the coefficients**

Homogeneous 2^{nd} Order Differential Equations: One Solution

Background:

If there's only one real root, then if we try the method with two roots we get $y_1 = y_2$, which doesn't really work. Magic happens and then the solution becomes:

$$y(x) = c_1 e^{-\,bx/2a} + c_2 x e^{-\,bx/2a}$$

Sample Problem:

Solve the differential equation $y'' - 4y' + 4y = 0$.

Homogeneous 2^{nd} Order Differential Equations: Zero Solutions

Background info:

If the roots are not real, use **Euler's formula:** $e^{i\theta} = \cos\theta + i\sin\theta$ (note you can (will) use the form where θ is negative, in that case, remember the even/odd behavior of sin/cos). The following equations use $\lambda = \frac{-b}{2a}$ and $\mu = \sqrt{b^2 - 4ac}$, where μ is just the real part (multiply by i).

$$y_1(x) = e^{(\lambda + \mu i)x} \quad \text{and} \quad y_2(x) = e^{(\lambda - \mu i)x}$$

$$y_1(x) = e^{\lambda x} e^{\mu i x} = e^{\lambda x}(\cos(\mu x) + i\sin(\mu x))$$
$$y_2(x) = e^{\lambda x} e^{-\mu i x} = e^{\lambda x}(\cos(\mu x) - i\sin(\mu x))$$

Add/subtract the two solutions together to get both solutions, and then you get:

$$y(x) = c_1 e^{\lambda x}\cos(\mu x) + c_2 e^{\lambda x}\sin(\mu x)$$

Homogeneous 2^{nd} Order Differential Equations: Two Solutions

Backgroud info:

A likely candidate for $y(x)$ is something along the lines of e^{rx}, where r is some constant. If that was the case, then by taking the derivative twice, the equation becomes $e^{rx}(ar^2 + br + c) = 0$. Since $e^{\text{something}}$ is never 0, the polynomial has to be zero at some point. Factor it to get the possible values of r, and then if they're real, plug in and you're done.

Method of undetermined coefficients

Background info:

$$ay'' + by' + cy = g(x)$$

Similarlyish to solving for homogenous equations, the solution is going to be something along the lines of the solution to the complementary equation + the actual solution. **1. Guess y_p based on $g(x)$:**

- **Case 1:** $g(x)$ is a polynomial. Guess: $y_p = Ax^n + Bx^{n-1} + \cdots + Cx + D$. Even if there are not all the polynomial terms in the original g(x), there still has to be all of them in the equation.
- **Case 2:** $g(x)$ is exponential (e^{kx}). Guess: $y_p = Ae^{kx}$.
- **Case 3:** $g(x)$ is sine or cosine ($\sin(kx)$ or $\cos(kx)$). Guess: $y_p = A\cos(kx) + B\sin(kx)$, or something easier if it clearly cancels.

Notes:

- If $g(x)$ combines functions, combine guesses.
- If any y_p term solves the homogeneous equation, multiply y_p by x (or x^2) until no term is a homogeneous solution.

Substitute y_p and its derivatives into the differential equation, then equate coefficients to find the undetermined coefficients.

Variation of parameters

Background info:

Same idea as undetermined coefficients. Start by assuming a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where $u_1(x)$ and $u_2(x)$ are functions to be determined. u is some made up thing so we can force it to follow $u_1'y_1 + u_2'y_2 = 0$ Plugging and chugging, we get.

$$\begin{aligned} y_p' &= u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2' \\ y_p'' &= u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' \end{aligned}$$

$$a(u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'') + b(u_1y_1' + u_2y_2') + c(u_1y_1 + u_2y_2) = g(x)$$

$$u_1(ay_1'' + by_1' + cy_1) + u_2(ay_2'' + by_2' + cy_2) + a(u_1'y_1' + u_2'y_2') = g(x)$$

Since $ay_1'' + by_1' + cy_1 = 0 \quad \text{and} \quad ay_2'' + by_2' + cy_2 = 0$ (solutions to complementary equation), we can simplify to $a(u_1'y_1' + u_2'y_2') = g(x)$ Integrate and substitute as neccesary.

Relevant exam question

(a)

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of

$$y'' + ay' + by = 0$$

where a and b are constants. By proving that $W'(x) = -aW(x)$, show that either $W(x) = 0$ for all x or $W(x) \neq 0$ for any x.

Proof.

1. **Derivative of the Wronskian:** The Wronskian is defined as $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$. Differentiating with respect to x, we get:

$$\begin{aligned} W'(x) &= y_1'(x)y_2'(x) + y_1(x)y_2''(x) - y_2'(x)y_1'(x) - y_2(x)y_1''(x) \\ &= y_1(x)y_2''(x) - y_2(x)y_1''(x) \end{aligned}$$

2. **Using the differential equation:** Since y_1 and y_2 are solutions to the differential equation $y'' + ay' + by = 0$, we have:

$$\begin{aligned} y_1''(x) &= -ay_1'(x) - by_1(x) \\ y_2''(x) &= -ay_2'(x) - by_2(x) \end{aligned}$$

   Substituting these into the expression for $W'(x)$:

$$\begin{aligned} W'(x) &= y_1(x)[-ay_2'(x) - by_2(x)] - y_2(x)[-ay_1'(x) - by_1(x)] \\ &= -ay_1(x)y_2'(x) - by_1(x)y_2(x) + ay_2(x)y_1'(x) + by_2(x)y_1(x) \\ &= -a[y_1(x)y_2'(x) - y_2(x)y_1'(x)] \\ &= -aW(x) \end{aligned}$$

3. **Solving the differential equation for $W(x)$:** The differential equation $W'(x) = -aW(x)$ has the solution: $W(x) = W(0)e^{-ax}$

4. **Conclusion:** Since $e^{-ax} \neq 0$ for all x, $W(x) = 0$ if and only if $W(0) = 0$. Therefore, either $W(x) = 0$ for all x, or $W(x) \neq 0$ for any x.

$\square$

(b)

Let $P(x)$ and $Q(x)$ be given, continuous functions. Let $y_1(x)$ and $y_2(x)$ be two solutions of

$$y'' + P(x)y' + Q(x)y = 0$$

such that their Wronskian $W(x)$ never vanishes. Show that between two consecutive zeros of $y_1(x)$, there is one and only one zero of $y_2(x)$.

Proof.

1. **Rolle's Theorem:** If a function $f(x)$ is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b) = 0$, then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

2. **Applying Rolle's Theorem to $y_1(x)$:** Let a and b be consecutive zeros of $y_1(x)$, so $y_1(a) = y_1(b) = 0$. As a solution to a second-order linear differential equation with continuous coefficients, $y_1(x)$ is continuous and differentiable. By Rolle's Theorem, there exists $c \in (a, b)$ such that $y_1'(c) = 0$.

3. **Using the Wronskian:** At $x = c$, the Wronskian is:

$$\begin{aligned} W(c) &= y_1(c)y_2'(c) - y_2(c)y_1'(c) \\ &= y_1(c)y_2'(c) \quad \text{(since } y_1'(c) = 0\text{)} \end{aligned}$$

   Since $W(c) \neq 0$ and $y_1(c) = 0$, we must have $y_2'(c) \neq 0$.

4. **Applying Rolle's Theorem to $y_2(x)$:** If $y_2(a) = y_2(b) = 0$, then by Rolle's Theorem, there would exist $d \in (a, b)$ such that $y_2'(d) = 0$. But $y_2'(c) \neq 0$ and $c \in (a, b)$, so $y_2(x)$ cannot have two zeros between a and b.

5. **Conclusion:** We have shown that $y_2(x)$ cannot have two zeros between consecutive zeros of $y_1(x)$. Since the Wronskian never vanishes, $y_1(x)$ and $y_2(x)$ cannot have a common zero. Therefore, there is one and only one zero of $y_2(x)$ between two consecutive zeros of $y_1(x)$.

□