1 Arithmetic Series

An arithmetic series is when the difference between any 2 consecutive terms is the same. For example, $\{1,2,3,4,5,6,7\}$ and $\{-5,-3,-1,1,3,5,7...\}$ are arithmetic series because they have a common difference between terms.
If we want to predict, say 5 *more* terms in our sequence, we know we have to add 5 × the common difference between two terms. Notice we need a starting term to predict following terms. From this concept, we can derive that the nth term of a arithmetic sequence is::

$$t_n = t_1 + (n-1)(d) \tag{1}$$

where t_n is the nth term, t_1 is the starting term, and d is the common difference. The common difference is how much sequence increases each term. Note that we use $n-1$ in our formula and not n. This is because the nth term is $n-1$ terms away from starting term.
Example: What is the 12th term in the sequence $\{1,3,5,7,9,11\}$?

- We need to find the 12th term : t_{12}. The first term is 1 ($t_1 = 1$) so we need to find *11 terms after* $t_1 = 1$, and the common difference is 2 (the sequence increases by 2 each time)

$$\begin{aligned} t_{12} &= t_1 + (n-1)(d) \\ &= 1 + (11)(2) \\ &= 23 \end{aligned}$$

2 Sum of Arithmetic Series

The formula that predicts the sum of an arithmetic series is based off a clever observation. Let's derive the formula with an example. Consider the arithmetic sequence with a common difference of 1:

$$\{1,2,3,4,5,6,7,8,9,10\}$$

The sum of the series, S, is $1+2+3+4+5+7+8+9+10$. Let's rearrange terms so that terms on the "opposite" side (arrange least to greatest) are beside each other.

$$S = (1+10)+(2+9)+(3+8)+(4+7)+(5+6)$$

Each set of brackets has the same sum: 11. Since we are repeatedly adding 11, we can alternatively express this sum as:

$$S = 11 \times 5$$

11 is the sum of the first and last term. Since we group terms in pairs, we multiply 11 by $\frac{n}{2}$, where n is the number of terms in the series. From this observation, we derive a the sum of any n terms in an arithmetic series:

$$S_n = (t_1 + t_n)\frac{n}{2} \tag{2}$$

Furthermore, an alternate sum of arithmetic series formula can be derived. Remember the nth term of an arithmetic series can be expressed as $t_n = t_1 + (n-1)(d)$. Substitute this for t_n in equation (2) and we obtain:

$$S_n = (t_1 + (t_1 + (n-1)(d))\frac{n}{2}$$
$$S_n = (2t_1 + (n-1)(d))\frac{n}{2} \tag{3}$$

3 Geometric Series

In a geometric series, instead of adding a value to get the next term, we multiply some value. For example, $\{2,4,8,16,32\}$ is a geometric series because we multiply the previous term by 2 to get the next term. 2 is called the *common ratio* and is the quotient between any two consecutive terms in a geometric series.

Consider this geometric series: $\{3125, 625, 25...\}$ where the common ratio, r is $\frac{1}{5}$

To find the nth term in that geometric series, notice we have to repeatedly multiply by the common ratio, $r = \frac{1}{5}$. So if we want to find the 3rd term in our sequence, we multiply 3125 by $\frac{1}{5} \times \frac{1}{5} = \left(\frac{1}{5}\right)^2$. We multiply $\frac{1}{5}$ twice because the third term is two away from the first term.

$$t_3 = 3125 \times \left(\frac{1}{5}\right)^2$$

Replacing our values with letters lets us derive the general formula for the nth term in a geometric series

$$t_n = (t_1)r^{n-1} \tag{4}$$

4 Sum of Geometric Series

The sum of a geometric series is a bit tricky to derive.

First we must consider a geometric series of n terms. We'll call the first term of this series a. With a common ratio of r, we can list out the terms in this sequence.

$$\{a, ar, ar^2, ar^3, ar^4, ..., ar^{(n-3)}, ar^{(n-2)}, ar^{(n-1)}\}$$

The sum of this series, S_n is therefore:

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + ... + ar^{(n-3)} + ar^{(n-2)} + ar^{(n-1)}$$

Let's multiply each term in this series by r. We get the sum of rS_n

$$rS_n = ar + ar^2 + ar^3 + ar^4 + ... + ar^{(n-3)} + ar^{(n-2)} + ar^{(n-1)+ar^n}$$

Now, let's find the difference $S_n - rS_n$.

$$\begin{aligned} S_n &= a + \quad ar + ar^2 + ar^3 + ar^4 + ... + ar^{(n-3)} + ar^{(n-2)} + ar^{(n-1)} \\ -rS_n &= -(\quad ar + ar^2 + ar^3 + ar^4 + ... + ar^{(n-3)} + ar^{(n-2)} + ar^{(n-1)} + ar^n) \\ \hline S_n - rS_n &= a - ar^n \end{aligned}$$

Rearrange for S_n

$$\begin{aligned} S_n(1-r) &= a - ar^n \\ S_n &= \frac{a - ar^n}{1-r} \end{aligned} \tag{5}$$

5 Sum of Infinite Geometric Series

If a geometric series has a common ratio greater than 1 ($r > 1$), as the number of terms approaches infinity, so does the sum of that series. For instance, if we had to find the sum of this series with infinite terms:

$$5, 25, 125, 625, 3125...$$

You'll notice quickly that the sum approaches infinity if the series has an infinite number of terms. Similarly, if $r < 1$, you'll notice the sum approaches negative infinity. Interestingly, if $|r| < 1$, the sum is *not* infinite.

Here's an example: You want to move 1 meter. Your first step is $\frac{1}{2}$m and each step you take is half the previous step. Since each step is half the previous step, we can express the distance you travel as the sum of a geometric series. We just have to see if the series reaches 1.

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$$

Well, we have a general formula for the sum of geometric series. Since we want to see if the sum of the series ever reaches 1, we can put a very large value for n: the number of terms in our series.
Let's say n is 1000.

$$S_n = \frac{a - ar^n}{1-r}$$

Factor out a from the numerator

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \\ S_{1000} &= \frac{a(1-\left(\frac{1}{2}\right)^{1000})}{1-\frac{1}{2}} \end{aligned}$$

Recall that $a = \dfrac{1}{2}$ because the first step taken is 0.5m

$$S_{1000} \approx \frac{\frac{1}{2}(1-\cancelto{0}{\left(\frac{1}{2}\right)^{1000}})}{1-\frac{1}{2}}$$

If we choose 1000 for n, then r^n is effectively 0 (remember that $|r|<1$!)

$$S_{1000} \approx \frac{0.5}{0.5}$$
$$S_{1000} \approx 1$$

That's cool! We can conclude that if we choose an randomly large value for n, r^n is effectively 0 and the sum of the geometric series converges to some value. Let's write this a bit more mathematically:

$$S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$
$$S_n = \lim_{n\to\infty} \frac{a(1-\cancelto{0}{r^n})}{1-r}$$
$$S = \frac{a}{1-r} \tag{6}$$